Related Rates and Right Triangles: Developing Intuition in a Calculus Course

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Definition

A related rates problem is (typically) a word problem where two or more variables are changing at the same time.

In the course of answering the question, students must *relate* the *rates* of change of these variables. This is frequently the first class of word problems students run into in a Calculus I course, and can be a major challenge for students used to the more formulaic derivative rules.

Example Problems:

- A helicopter is descending to land on a helipad 15 feet away from you. It's descending at a rate of 5 feet per second, and is currently at an altitude of 20 feet. How fast must you turn your head to keep the helicopter in view?
- A 10 foot tall ladder is propped up against a vertical wall. At the time the foot of the ladder is 6 feet from the base of the wall, it is sliding down the wall at a rate of 3 feet per minute. How fast is the base of the ladder moving at that time?
- A 6 foot tall man is walking towards a streetlight that is 20 feet tall. He is walking at a rate of 4 feet per second. How fast is the length of his shadow increasing or decreasing when he is 10 feet from the light?

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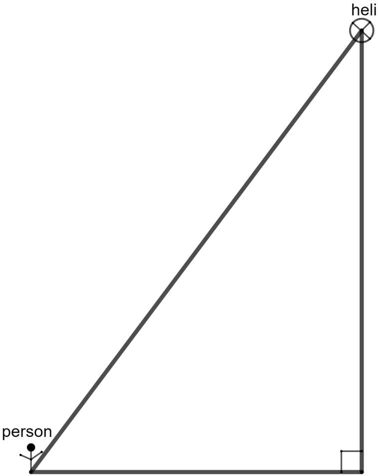
Picture 1



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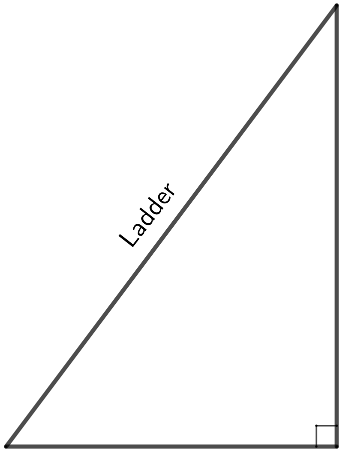
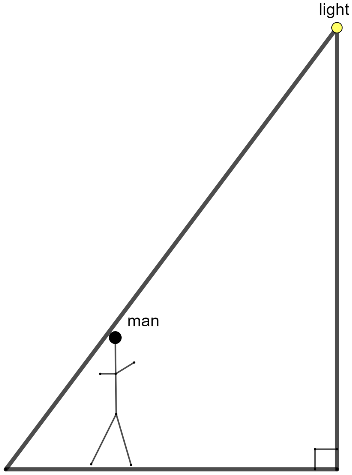


Picture 3



Student Quote from Spring 2022

“So do we just have to get lucky in what equation we use to start?"

The Problem

- Even with a picture accurately illustrating the problem, students must make a choice about how to use that picture.
- And when that picture involves right triangles there can be decision paralysis where students have too many options about what to do next.
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Backing Up

- Let's rewind a bit. Immediately prior to related rates, students are learning derivative rules.
- When faced with a question like 'Find the derivative of $\sin(x^2)$' there's no doubt what to do; a student who knows their rules will apply the chain rule and find the solution!
- With these related rates problems, that's no longer the case! There's no 'rule' to identify.

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Logic vs. Intuition

- These word problems cannot be approached in the same algorithmic manner as those derivative questions. Students need to decide on what approach to take.
- While logic can and should be used to guide this decision, in many ways it is a first chance for students to develop their own intuition as they decide what approach to take.
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Logic Vs. Intuition

For example, let's look at the way these problems are assessed.

- In a significant number of books and classrooms, students are not asked to provide any rationale for **why** they chose the equation relating the variables they begin with.
- The logic behind that choice, arguably the most important step in reaching an eventual solution, is not evaluated at all.
- This is in stark contrast to the rest of the problem where students are often evaluated on their ability to demonstrate in detail their taking a derivative, working algebraically, and stating in words their solution.

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Developing Intuition

- A key part of learning how to approach these problems is developing the understanding that we don't just 'get lucky' in the equation we use to start...
- There are clues, sometimes subtle, that point us in a particular direction.
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Some Methods for Developing Intuition

- At first, provide a list of some commonly used methods to relate quantities (Area/Volume formulas, Similar Triangles, Trigonometry, etc.). If students don't remember a formula exists, they can't try it.
- Provide ample time in class to work on this first step, and discuss with peers.
- Have students share their thought process about that first step with the class. Then have another student repeat it in their own words.
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Examples from Earlier

- A helicopter is descending to land on a helipad 15 feet away from you. It's descending at a rate of 5 feet per second, and is currently at an altitude of 20 feet. How fast must you turn your head to keep the helicopter in view?
- The question asks how fast you must turn your head, meaning we need the rate of change of an angle. The picture makes clear it is an angle in a right triangle. That combination of facts tells us to try using a trigonometric function, and looking at which sides are changing and which are constant can help us decide which one to try.

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- A 6 foot tall man is walking towards a streetlight that is 20 feet tall. He is walking at a rate of 4 feet per second. How fast is the length of his shadow increasing or decreasing when he is 10 feet from the light?
- With the person and the light in the same picture, we can see two similar right triangles in the image. Since we're not asking anything about angles, a good starting point might be the proportionality of the sides of right triangles.

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That's so Imprecise!

And that's the point!

- The goal is not for students to determine the 'best' approaches for these problems, its for students to have an approach, and then check if it is able to lead to the conclusions asked for.
- These are skills that students will be able to use for the next problem that does not look exactly like any of these (but looks close to all three of them...)
- It's not about memorization or a checklist, its about getting a feel for this new class of problems, so that they can start on the right foot.

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A construction crew is moving a large streetlight to illuminate an area where they will be working. The 30 foot tall lamp is being moved towards a bulldozer, to eventually be set up 20 feet to the right of the 10 foot tall stationary bulldozer. They are moving the lamp left at a constant rate of 1.5 feet per second. **At the time the lamp is 25 feet to the right of the bulldozer, determine how quickly the length of the shadow cast by the bulldozer is increasing or decreasing.**

1. **What mathematical concept** can we use to relate the length of the shadow to the position of the light.
2. Give an equation relating the length of the shadow to the position of the light.
3. Using implicit differentiation, and then plugging in current values, determine how fast the length of the shadow is changing at the time asked about.

Looking Ahead

While this may be one of the first times students are required to use their intuition in Calculus, it is far from the last. In Calculus II, intuition is again crucial.

- Deciding what integration method to use.
- Deciding how to test a series for convergence.
- Even choosing what to make u and dv in an integration by parts problem.

Each of these are frequently areas of significant student difficulty. Word problems provide an early opportunity for students to develop that 'gut feeling' of what approach to use, check to make sure it works, and not give up if they need to try again!

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Why Use Intuition?

- Going down a dead end can be frustrating to students, and if there is a rule like ‘Always try the Pythagorean theorem when you see a right triangle’, there will always be exceptions where it does not work.
- It helps to combat the decision paralysis... a first guess might NOT be the right approach, but seeing where it fails to do what’s needed can help with the next one.
- In the context of a timed exam, where time pressure adds to student stress, developing a sense of how to approach these word problems makes the first attempt more likely to succeed, helping to avoid a cycle of anxiety.

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Caveats

- It is essential to be clear that intuition is NOT a replacement for showing work/explaining the process.
- There can be push-back at the lack of formula/strict rules for deciding how to approach a question.
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- Any questions?