Metaontology in Light of the Frege-Hilbert Controversy

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AMS JMM 2024

SAN FRANCISCO, CA

JANUARY 5TH, 2024

Motivation

- The philosophy of mathematics has figured more prominently in metaphilosophical disputes pertaining to purportedly *a priori* domains of knowledge
- Drawing parallels between this ongoing metaphilosophical work and prior disputes in the philosophy of mathematics may be informative
- Two such (interrelated) disputes:
  - The Frege-Hilbert controversy
  - The 'categorical foundations' dispute

Motivation (cont.)

- A recent trend: bringing considerations from the philosophy of mathematics to bear upon metaphilosophical disputes about other domains of *a priori* knowledge
- Notable examples:
  - Justin Clarke-Doane: *Morality and Mathematics* (2020), *Mathematics and Metaphilosophy* (2022)
  - Jared Warren: *Shadows of Syntax* (2020), *The* A Priori *Without Magic* (2022)

Motivation (cont.)

- Similarities and differences between Clarke-Doane and Warren:
  - Both characterize their positions with Rudolf Carnap as a significant reference point
  - However, while Clarke-Doane advocates for radical metaphysical pluralism that departs from Carnap, Warren defends a revitalized version of logical and mathematical conventionalism akin to Carnap's
- Goal: to lay the groundwork for a defense of conventionalism and a critique of radical metaphysical pluralism
  - Specifically, to respond to an argument put forward by Clarke-Doane that I term the *metalogical indispensability argument*

I. A Tale of Two Realisms

Reliability of Mathematical Knowledge

- An issue typically associated with mathematical platonism: Benacerraf's (1973) epistemological challenge
  - Also generally considered an issue for other purportedly *a priori* domains of knowledge, such as moral and modal knowledge
- As Clarke-Doane puts it, the following criterion must be satisfied:
  - *Mathematical Reliability* – It does not appear impossible to explain the reliability of our mathematical beliefs, realistically construed (Clarke-Doane 2020, 130)

Mathematical Knowledge (cont.)

- A prominent answer: adopt some version of ontological pluralism where "every consistent theory is true of its intended subject, independent of human minds and languages (*Ibid*, 153)
  - A helpful analogy: Euclidean and non-Euclidean geometries
    - Pluralism: the Parallel Postulate is true *of the lines*$_{Euclidean}$ and false *of the lines*$_{hyperbolic}$
    - Contrast with relativism: the Parallel Postulate is true in the framework of Euclidean geometry and false in the framework of hyperbolic geometry
  - On this view, realism and objectivity *come apart*
- Examples include:
  - Balaguer's (1998) *full-blooded platonism*
  - Hamkins's (2012) multiverse view of sets
  - **Clarke-Doane's** (2020, 2022) *radical metaphysical pluralism*

Realist Carnapianism

- Clarke-Doane extends his radical metaphysical pluralism to other domains:
  - "If one rides the pluralist train to the last station, then one arrives at realism with a pragmatist spin. The world – including its mathematical, modal, or logical aspects – is out there, independent of us. We do not make up the pluriverse of sets, possibilities, or consequence relations, despite their perspectival and indefinite character. Realism is true of all three subjects. **Nevertheless, for practical purposes, it is *as if* realism about them were false and conventionalism were true**." (Clarke-Doane 2022, 43, bolded for emphasis)
- Thus, this pluralism mimics Carnap's (1950) pragmatism:
  - Holds true to the principle of tolerance
  - However, *not* by adopting an antimetaphysical stance; e.g., by bifurcating between framework *internal* and framework *external* questions
  - Rather, by positing a mathematical pluriverse so vast that it satisfies *Mathematical Reliability*

Logical and Mathematical Conventionalism

- Logical conventionalism: the conventions of a language *fully explain* logical truth and necessity
- Undergirded by *unrestricted inferentialism*, which is the conjunction of:
  - **Logical Inferentialism**: the meaning of a logical expression is fully determined by (some of) the inference rules according to which the expression is used (Warren 2020, 56)
  - **Meanings Are Cheap (MAC)**: Any collection of inference rules that can be used for an expression can (in principle) be meaning determining rules for the expression (*Ibid*, 64)
- The **MAC** principle amounts to a restatement of Carnap's principle of tolerance:
  - "*In logic, there are no morals*. Everyone is at liberty to build up his own logic, i.e., his own form of language, as he wishes. All that is required of him is that, if he wishes to discuss it, he must state his methods clearly, and give syntactical rules instead of philosophical arguments" (Carnap 1937, §17)

Logical and Mathematical Conventionalism (cont.)

- Mathematical conventionalism needs some additional argumentative steps due to the ontological commitments of mathematics. For this, Warren develops a metaontological position termed *trivial ontological realism*:
  - "According to mathematical inferentialism/conventionalism, the existence of numbers is not a substantive and controversial fact about our world, **but is instead a *trivial byproduct* of our using language in the way that we do.** … It can't be overstressed that the ontological realism here is *trivial* – it doesn't involve metaphysically suspect *Platonic* assumptions about the nature of numbers and other abstract objects." *(Ibid,* 213, bolded for emphasis)
- Since the existence of mathematical entities is explained by use of language, Warren's trivial ontological realism violates what Clarke-Doane terms **Mathematical Independence** (that mathematical truths are independent of human minds and *languages*)
- Hence, Warren's conventionalism is *not* a species of **Mathematical Realism** (in Clarke-Doane's sense of the term)

II. Prior Disputes in the Philosophy of Mathematics

Elsewhere in the Philosophy of Mathematics

- Early 2000's/2010's: dispute over whether category theory provides a foundation for structuralism
  - Feferman (1977): early cornerstone of the dispute
  - Hellman (2003): the 'home address' problem for categorical foundations
  - Shapiro (2005): pushes debate to meta-mathematics
    - First mention of the Frege-Hilbert controversy in the context of this dispute; sides with Frege
  - Landry (2011, 2013): responses to Shapiro, Feferman
    - Also invokes the Frege-Hilbert controversy, this time on the side of Hilbert
- Parallels with the Frege-Hilbert controversy
  - Esp. the relationship between existence and consistency, of philosophical theorizing and mathematical practice, the role of axioms, etc.

The Frege-Hilbert Controversy (cont.)

- Patricia Blanchette's understanding of the dispute:
  - "By 1899, Frege had a well-developed view of logical consequence, consistency, and independence, a view which was central to his foundational work in arithmetic and to the epistemological significance of that work. Given *this* understanding of the logical relations, I shall argue, Hilbert's demonstrations do fail. Successful as they were in demonstrating significant metatheoretic results, Hilbert's proofs do not establish the consistency and independence, in Frege's sense, of geometric axioms." (Blanchette 1996, 318-19)
- This understanding does not attribute a simple cognitive error on Frege's end
  - Rather, it reassesses the Frege-Hilbert controversy as a disagreement over who's conception of the relevant logical notions is best suited for the purpose at hand
  - According to Blanchette, is not merely a verbal dispute since Frege thinks there are important philosophical ramifications tied to which logical notions are utilized

Frege on Logical Relations

- Consistency, independence, etc. do not hold between sentences (as they do on Hilbert's understanding)
- Rather, they hold between nonlinguistic propositions (or Fregean thoughts) expressed by those sentences:
  - "We have to distinguish between the external, audible, or visible that is supposed to express a thought, and the thought itself. It seems to me that the usage prevalent in logic, according to which only the former is called a sentence, is preferable. Accordingly, we simply cannot say that one sentence is independent of other sentences, for after all, no one wants to predicate this independence of what is audible or visible." (Frege 1906, cited in Blanchette 1996, 322-23)
- In other words: content matters for Frege-consistency

Frege on Logical Relations (cont.)

- Thus, syntactic (Hilbertian) consistency is a necessary but insufficient condition for Frege-consistency
- This also ties into their different conceptions of the relationship between existence and consistency
  - Frege – consistency is *necessary* for existence
  - Hilbert – consistency is *sufficient* for existence
- Hence for Frege, method must answer to metaphysics, and for Hilbert, no appeal to metaphysics is needed to justify method (at least with respect to object-level mathematics)

III. From Metamathematics to Metaontology

Tying the Disputes Together

- Some important disanalogies out of the way:
  - Clarke-Doane is a radical pluralist about logical consequence, whereas Frege believes in a privileged notion of logical consequence
  - While Hilbert does not think a metaphysical account needs to be given at the level of mathematics, he does impose philosophical constraints at the metamathematical level (in virtue of his philosophical formalism, Landry 2011)
  - Warren need not impose such philosophical constraints at the metamathematical level (in virtue of his unrestricted inferentialism)

Tying the Disputes Together (cont.)

- The relevant similarities:
  - For Frege and Clarke-Doane, the reliability of method must be certified by a complementary metaphysical account
    - Frege – logicist program
    - Clarke-Doane – radical metaphysical pluralism
  - For Hilbert and Warren, no such account is necessary
    - Hilbert – consistency implies existence
    - Warren – trivial ontological realism

The Metalogical Indispensability Argument

- Traditionally, indispensability arguments (e.g., the Quine-Putnam version) appeal to applications of mathematics to and mathematical explanations in scientific practice
- By contrast, what I shall term Clarke-Doane's *metalogical indispensability argument* appeals to the indispensability of metalogic to any formal theorizing whatsoever
  - "[S]ome mathematical truths seem to be indispensable to *metalogic* – the theory of what follows from what. But we seem to be committed to metalogical truths in virtue of being committed to any theories whatever … But while the theory of what follows from what officially concerns sentences – or more exactly, strings of symbols – the theory of strings is bi-interpretable with the theory of natural numbers. Hence, there is a way to "say" anything that we want to say about strings of symbols in the language of arithmetic." (Clarke-Doane 2020, 81-82)
- I.e., arithmetization commits us to *some* arithmetic truths (or truths just like them)

The Conventionalist Response

- We see an analogous sort of problem raised by Shapiro (2005) for the status of metamathematics – who concludes that either 'categorical foundations' cannot serve the proper metamathematical role, or mathematical structuralism is in need of a philosophical gloss
  - Underpinned by the assumption that the metamathematical theory must be *assertoric* (a la Frege) and not *schematic* (a la Hilbert)
    - Assertory theory – expresses putative truths about a particular subject matter (e.g., the Dedekind construction of the continuum)
    - Schematic theory – implicitly defines the mathematical objects in question (e.g., the definition of a category)
  - Landry (2011) demonstrates how, taking Hilbert's approach to object-level mathematics, we can be structuralists *all the way down* (by carefully disambiguating the roles of **ETCS**, **CCAF**, and higher categories at the mathematical and metamathematical levels)
- Taking Warren's unrestricted inferentialist approach, we can also be inferentialists *all the way down*

The Conventionalist Response (cont.)

- The conventionalist's unrestricted inferentialism is a *metasemantic* theory
  - Semantic theory – answers *what* logical connective X means (e.g., deflationism)
  - Metasemantic theory – answers *how* logical connective X comes to get its meaning (e.g., by the implicit or explicit linguistic conventions of a community)
- An inferentialist explanation of this sort has advantages over radical metaphysical pluralism
  - The radical metaphysical pluralist is motivated by indispensability considerations, but fails to provide a precise explanation as to how radical metaphysical pluralism addresses concerns about the applicability of mathematics
  - In contrast, the conventionalist can do so by appeal to an inferential conception of mathematical application (Bueno and Colyvan 2011) – but more work needs to be done to flesh out the details of this account in the context of mathematical conventionalism

Concluding Remarks

- The current landscape in metaontology (esp. regarding recent work appealing to the philosophy of mathematics) bears some relevant similarities to both the 'categorical foundations' dispute and the Frege-Hilbert controversy
- Understanding Clarke-Doane's and Warren's positions from this lens helps to delineate potential advantages the latter's conventionalism has over the former's radical metaphysical pluralism

Works Cited

- Blanchette, Patricia A. (1996). Frege and Hilbert on Consistency. *Journal of Philosophy* 93 (7):317-336.
- Bueno, Otávio & Colyvan, Mark (2011). An Inferential Conception of the Application of Mathematics. *Noûs* 45 (2):345-374.
- Carnap, Rudolf (1950). Empiricism, Semantics and Ontology. *Revue Internationale de Philosophie* 4 (11):20-40.
- Carnap, Rudolf (1937). *The Logical Syntax of Language*. Routledge. Edited by Amethe Smeaton.
- Clarke-Doane, Justin (2022). *Mathematics and Metaphilosophy*. Cambridge: Cambridge University Press.
- Clarke-Doane, Justin (2020). *Morality and Mathematics*. Oxford, England: Oxford University Press.
- Feferman, Solomon (1977). Categorical Foundations and Foundations of Category Theory. In Robert E. Butts & Jaakko Hintikka (eds.), *Logic, Foundations of Mathematics, and Computability Theory*. Springer. pp. 149-169.

Works Cited (cont.)

- Frege, Gottlob. "On the foundations of geometry: Second series." *Collected papers on mathematics, logic, and philosophy* (1906): 293-340.
- Hellman, Geoffrey (2003). Does category theory provide a framework for mathematical structuralism? *Philosophia Mathematica* 11 (2):129-157.
- Landry, Elaine (2013). The genetic versus the axiomatic method: Responding to Feferman 1977. *Review of Symbolic Logic* 6 (1):24-51.
- Landry, Elaine (2011). How to be a structuralist all the way down. *Synthese* 179 (3):435 - 454.
- Shapiro, Stewart (2005). Categories, Structures, and the Frege-Hilbert Controversy: The Status of Meta-mathematics. *Philosophia Mathematica* 13 (1):61-77.
- Warren, Jared (2022). *The A Priori Without Magic*. New York, NY, USA: Cambridge University Press.
- Warren, Jared (2020). *Shadows of Syntax: Revitalizing Logical and Mathematical Conventionalism*. New York, USA: Oxford University Press.