A Philosophical Account of Mathematics that Won’t Make You Hate Philosophers

Or: Autonomy Platonism and Intuition

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Joint Mathematics Meetings
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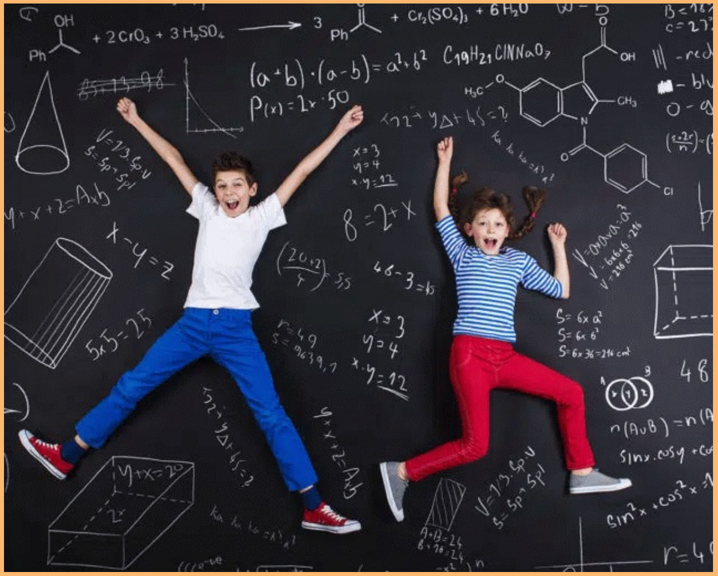
Overview for Today

An account of our knowledge of mathematics that

- Solves major philosophical challenges; and
- Should satisfy mathematicians, too.

I call this account an intuition-based autonomy platonism.

Happy Philosophers



Happy Mathematicians

Plan for Today

1. Why we need an account of mathematical knowledge

2. Platonism and objectivity

3. The insufficiency of axioms and derivations

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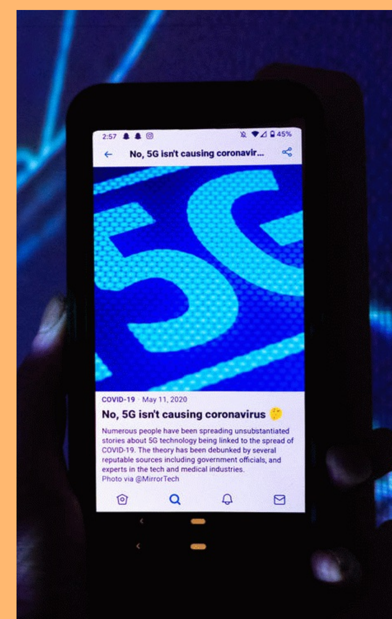
Sense Perception

- Mathematical objects are not visible (or otherwise sensible).
- Serious talk about invisible stuff is often and rightly received skeptically.
- Unlike quarks, for example, mathematical objects have no consitutional or otherwise causal relation to things we do experience.



The Benacerraf Problem, Generalized

- Our best ways of knowing about anything seem to require sense experience.
- Our best (most secure and lasting) knowledge (i.e. mathematics) seems to eschew sense experience.
- So, either we revise our views about knowledge or we revise our confidence in mathematics.
- Revising our confidence in mathematics seems unwarranted.
- So, we should reconsider (and perhaps revise) our views about knowledge.

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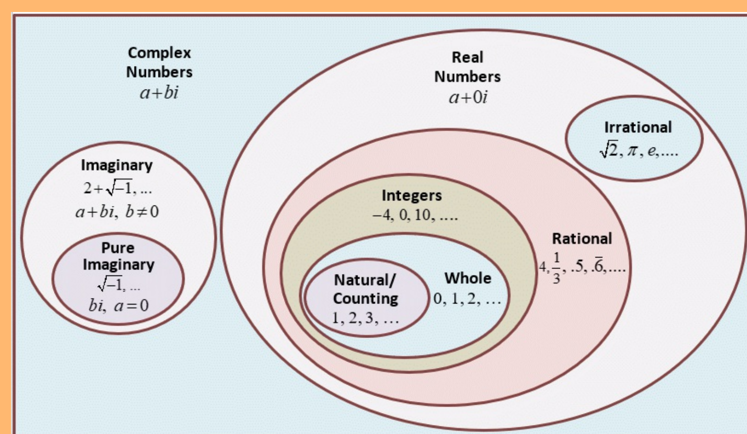
Platonism, Objectivity, and Access

- Platonism is the view that mathematics is objective.
  - Ontological thesis PO There are abstract mathematical objects, possibly including sets, numbers, shapes, and spaces.
  - Semantic thesis PS Some existential mathematical sentences are true and others are false. Universal and conditional mathematical claims may be non-vacuously true or false.
- Platonism faces challenges.
  - Critics of PO complain that it is impossible to justify knowledge of mathematical objects because we can not experience them.
  - Critics of PS complain that it is impossible for us to know any mathematical propositions since whatever makes them true or false is inaccessible to us.



Standard Responses Leave Us Unhappy.1

PO: There are abstract mathematical objects, possibly including (but not limited to) sets, numbers, and spaces.

PS: Some existential mathematical sentences are true and others are false. Universal and conditional mathematical claims may be non-vacuously true or false.

- Ignore the problem
  - Unhappy philosophers

Standard Responses Leave Us Unhappy.2

PO: There are abstract mathematical objects, possibly including (but not limited to) sets, numbers, and spaces.

PS: Some existential mathematical sentences are true and others are false. Universal and conditional mathematical claims may be non-vacuously true or false.

- Reject platonism
  - Fictionalism: there are no mathematical objects and contentful mathematical claims are false.
  - Unhappy mathematicians

Standard Responses Leave Us Unhappy.3

PO: There are abstract mathematical objects, possibly including (but not limited to) sets, numbers, and spaces.

PS: Some existential mathematical sentences are true and others are false. Universal and conditional mathematical claims may be non-vacuously true or false.

- Account for our knowledge of mathematics by its uses in science
  - The indispensability argument
  - Unhappy philosophers: Gets the priority of knowledge backwards
  - My view is instead autonomous:
    - Autonomy: Knowledge of mathematics does not depend on knowledge of science.

Standard Responses Leave Us Unhappy.4

PO: There are abstract mathematical objects, possibly including (but not limited to) sets, numbers, and spaces.

PS: Some existential mathematical sentences are true and others are false. Universal and conditional mathematical claims may be non-vacuously true or false.

- Appeal to a special faculty for perceiving mathematical objects or apprehending mathematical claims called mathematical intuition.
  - Unhappy philosophers: Intuition seems to be a mysterious or spooky faculty,
  - My view!

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Axioms and Theorems: A Common Attempt

- We might try to account for our mathematical beliefs by appeal to the derivability of theorems from axioms.
- We could then have a two-pronged epistemology.
  - Science is constrained by observation and logical consistency.
  - Mathematics is constrained only by logical consistency.
- Problem: We need an account of our beliefs about the axioms.

Axioms are Not Always Simple and Obvious

- Appeals to the immediacy and obviousness of the axioms are unsatisfying and often misleading.
- Euclid’s axioms for geometry, for example, are simple and relatively obvious, except for one.
  - Parallel postulate
- The Dedekind-Peano axioms for arithmetic are simple and relatively obvious, too, except for one.
  - Mathematical induction
- Some axioms systems are altogether neither immediate nor obvious.

Birkhoff's Postulates for Geometry

following James Smart, *Modern Geometries*

Postulate I: Postulate of Line Measure. The points A, B,... of any line can be put into a 1:1 correspondence with the real numbers x so that $|x_B - x_A| = d(A,B)$ for all points A and B.

Postulate II: Point-Line Postulate. One and only one straight line *l* contains two given distinct points P and Q.

Postulate III: Postulate of Angle Measure. The half-lines *l*, *m*... through any point O can be put into 1:1 correspondence with the real numbers a(mod 2π) so that if A≠0 and B≠0 are points on *l* and *m*, respectively, the difference $a_m - a_l$ (mod 2π) is angle∡AOB. Further, if the point B on *m* varies continuously in a line *r* not containing the vertex O, the number a_m varies continuously also.

Postulate IV: Postulate of Similarity. If in two triangles △ABC and △A'B'C', and for some constant k>0, d(A', B') = kd(A, B), d(A', C')=kd(A, C) and ∡B'A'C'=±∡BAC, then d(B', C')=kd(B,C), ∡C'B'A'=±∡CBA, and ∡A'C'B'=±∡ACB.

Axioms are Not Accounts of Knowledge

- Axiom systems are not designed to account for our mathematical beliefs.
- They are designed to demonstrate elegant relationships between minimal assumptions and powerful theories.
- We construct axiomatizations in order to capture what we already know to be true.

Bertrand Russell on the Problem

When pure mathematics is organized as a deductive system—i.e. as the set of all those propositions that can be deduced from an assigned set of premises—it becomes obvious that, if we are to believe in the truth of pure mathematics, it cannot be solely because we believe in the truth of the set of premises. Some of the premises are much less obvious than some of their consequences and are believed chiefly because of their consequences.

("The Philosophy of Logical Atomism" 1924: 325)

Our Most Secure Mathematical Beliefs are Not Our Best Axioms

- Our account of our knowledge of mathematics may not recapitulate our axiomatic systems.
- Indeed, it would be a surprise if it did.
- For the philosophers:
  - Compare Descartes's so-called analytic *Meditations* with his synthetic presentation of the same results in the Second Replies.
- We need a different kind of account of our mathematical beliefs.
- Let's think about how mathematicians actually work and how your methods compare to those of philosophers.

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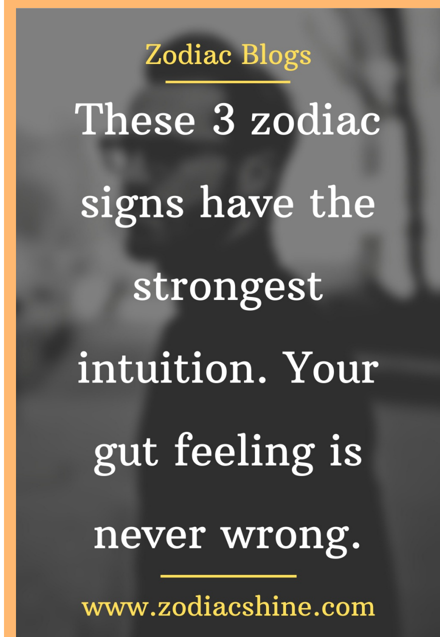
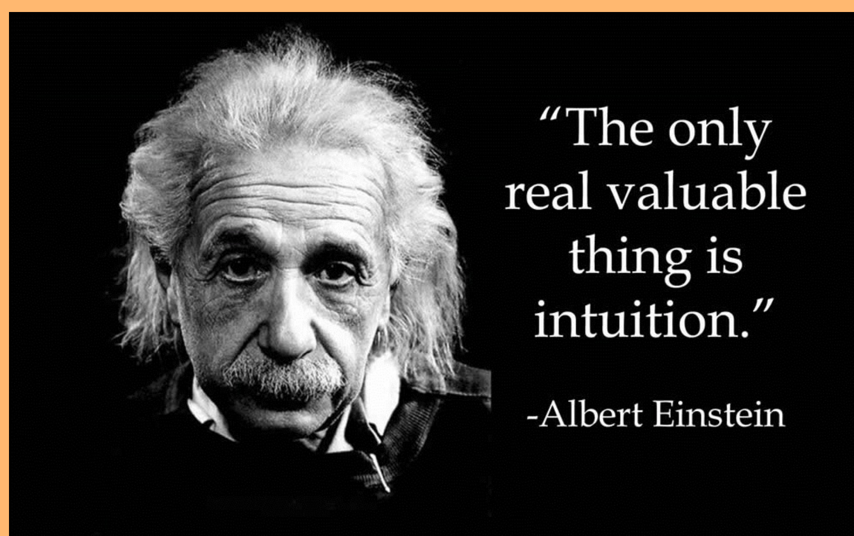
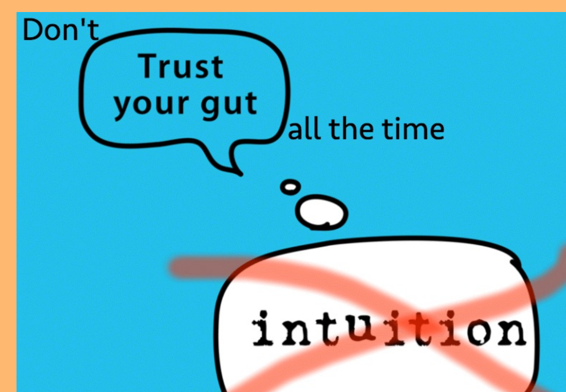
Intuition in Popular Imagination







Characterizing ‘Intuition’

- ‘Intuition’ is a bloated and ambiguous term.
- I take intuition to be similar in philosophy and mathematics.
  - We’ll see if you agree!
- Intuition:
  - A belief-forming cognitive process which may yield belief in a claim.
  - An intuition is an immediate inclination to belief.
  - It’s an ‘aha’ feeling.
  - An intuition can be about objects or concepts.
    - I can intuit that a circle is the locus of all points equidistant from a given point .
    - I can intuit that even numbers and odd numbers have the same cardinallity.
    - I can intuit that an act is immoral or that a distribution of goods is unjust.
  - We can intuit *modal* properties unavailable to sense experience.
    - Whether some claim holds necessarily or is impossible
  - “Reason is rationality in application to deductive structures and intuition is the same faculty in application to elements of such structures. We can think of intuition as reason in the structurally degenerate case.” (Katz 1990: 381)
  - Lastly: Intuition is fallible.
    - E.g. I can intuit that every property determines a set.

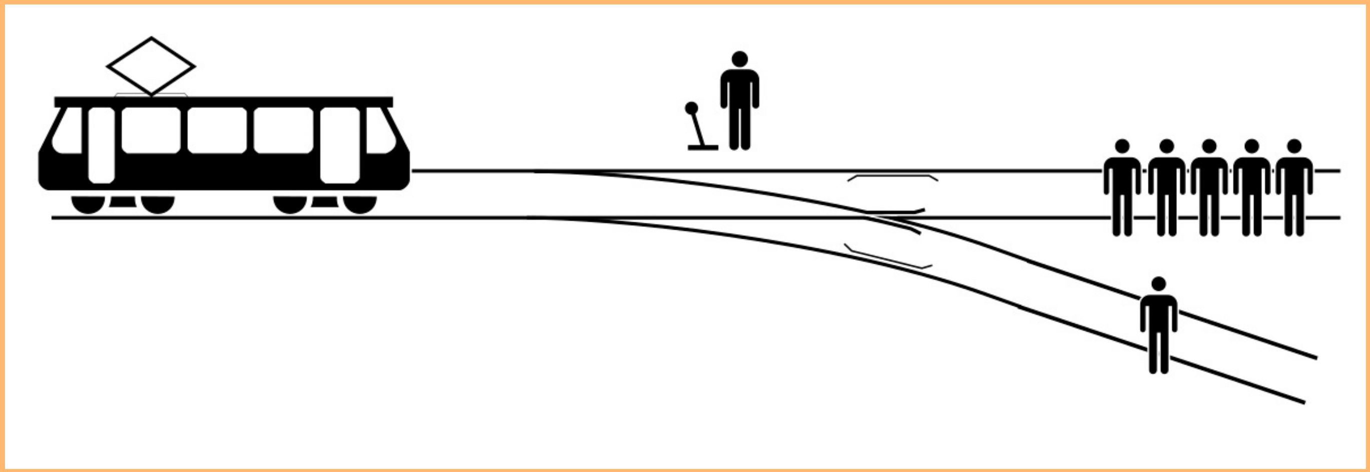
Intuitions in Metaphysics

Locke’s Prince and Cobbler

“Should the *soul* of a prince, carrying with it the consciousness of the prince's past life, enter and inform the body of a cobbler, as soon as deserted by his own soul, every one sees he would be the same *person* with the prince, accountable only for the prince’s actions.” (*Essay* II.27)

Intuitions in Ethics

Philippa Foot's Trolley Problem



Gödel on Intuition in Mathematics

"[D]espite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e. in **mathematical intuition**, than in sense perception, which induces us to build up physical theories and to expect that future sense perceptions will agree with them, and, moreover, to believe that a question not decidable now has meaning and may be decided in the future." (Gödel 1964: 268)

Worries About Intuition 1: Gender Differences

- Suppose Scientists figure out the exact state of the universe during the Big Bang, and figure out all the laws of physics as well. They put this information into a computer, and the computer perfectly predicts everything that has ever happened. In other words, they prove that everything that happens, has to happen exactly that way because of the laws of physics and everything that's come before.
- **Is a person in such a universe free to choose whether to murder someone?**
  - 63% of women responded that a person in this situation is free to choose whether or not to murder someone.
  - Only 35% of men gave that response. (Buchwalter and Stich, reporting research by Geoffrey Holtzman, in Knobe and Nichols 2014: 314)

Worries About Intuition 2: Cultural Differences

- Pat is at the zoo with his son, and when they come to the zebra cage, Pat points to the animal and says, “That’s a zebra.” Pat is right—it is a zebra. However, given the distance the spectators are from the cage, Pat would not be able to tell the difference between a real zebra and a mule that is cleverly disguised to look like a zebra. And if the animal had really been a cleverly disguised mule, Pat still would have thought that it was a zebra.
- **Does Pat know that the animal is a zebra?**
  - Subjects from the Indian subcontinent were evenly split on knowledge ascriptions.
  - Westerners were more than twice as likely to withhold ascriptions of knowledge.
  - People with low SES were about twice as likely as not to withhold knowledge ascriptions.
  - People with high SES were about eight times as likely as not to withhold knowledge ascriptions. (Weinberg, Nichols, and Stich 2001)

(Some) Philosophers of Mathematics are Wary

- “Someone *could* try to explain the reliability of these initially plausible mathematical judgments by saying that we have a special faculty of mathematical intuition that allows us direct access to the mathematical realm. I take it though that **this is a desperate move**...” (Field 1989: 28)
- “The naturalism driving contemporary epistemology and cognitive psychology demands that we not settle for an account of mathematical knowledge based on processes, such as *a priori* intuition, that **do not seem to be capable of scientific investigation or explanation**.” (Resnik 1997: 3-4)
- Appeals to intuition are, “**[U]nhelpful as epistemology and unpersuasive as science**. What neural process, after all, can be described as the perception of a mathematical object? Why of one mathematical object rather than another?” (Putnam 1980: 10)

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Mathematical Intuition: Thick and Thin

- Philosophers have often invoked a thick concept of intuition.
  - Infallible
  - The highest form of knowledge
  - Descartes, Spinoza
- I call mathematical intuition thin because of the gap between the intuition and the beliefs which may be formed on its basis.
  - First we have an intuition.
  - Then we think about what it shows.
  - Eventually, we may settle on a firm belief.
- The process is the same for beliefs based on sense experience and intuition.

Sense Experience and Belief

- When I have a sense experience, I might form a belief based on that experience.
- On seeing an apple in my hand I may form a belief that there is an apple in my hand.
- I might not form that belief.
- I might for example be considering whether I am in a dream state.

Intuition and Belief

- When I have an intuition of a basic mathematical fact, I might form a belief based on that intuition.
- I might have an intuition that twice two is four.
- That intuition will ordinarily lead to or confirm my belief about that sum.
- In skeptical cases, I might withhold my belief; I might wonder whether arithmetical claims are merely fictional.

From Mathematical Intuition to Mathematical Theory

- Intuition yields some mathematical beliefs.
  - Basic sums or facts about shapes at first.
  - As we learn more mathematics, we develop better and more sophisticated intuitions.
    - E.g. whether the continuum hypothesis is true or false, or whether $V=L$.
- Because intuition is fallible, we need methods to systematize and check our beliefs.
- That's what mathematical theories, especially formal ones, are for.

Mathematics: Intuitions and Theories

- We balance our intuitive apprehension of elementary mathematical truths with our evaluations of the systematizations of our mathematical knowledge.
- The intuitions are constraints on the system-building, as well as on the basic claims.
- The systems are constraints on the intuitions.
- The processes of refining and extending the mathematical beliefs generated by intuition are just the natural and well-refined methods of mathematics.
- Philosophers call this process *seeking reflective equilibrium*.
- It's how we iron out our (individual, gender, cultural, and other) differences.

Simple, intuitive claims

Broad, sytematic theories

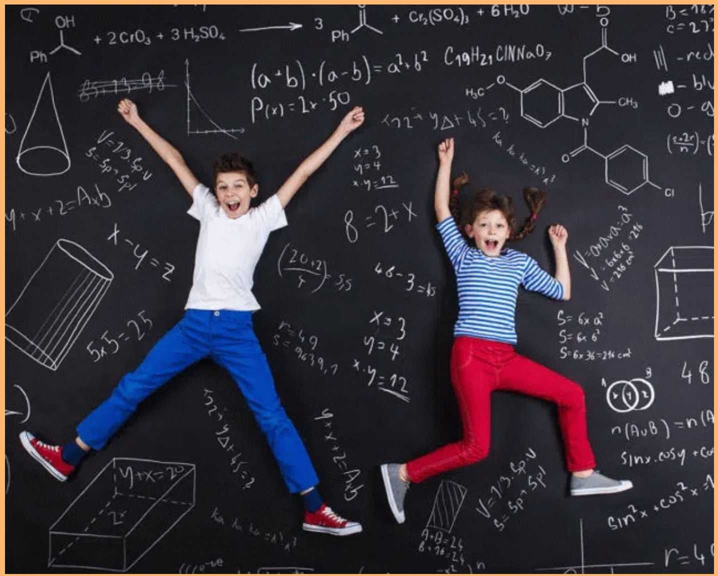
Mathematics with Fallible Intuition

- We can give up claims which appear intuitive but turn out to be false.
  - Unrestricted comprehension axiom
- We can adopt claims which appear counter-intuitive but turn out to be true.
  - Leibniz's work with infinitesimals and Newton's work with fluxions
- We might find that certain systematizations better organize mathematical phenomena than others.
  - Various axiomatizations, e.g. arithmetic with Dedekind-Peano axioms, by those axioms modeled within set theory, by those axioms modeled within category theory, or within second-order logic

Intuition-Based Autonomy Platonism

- An account that aligns our mathematical epistemology with our methods in other domains, like philosophy.
- An account that preserves objectivity (platonism) in mathematics.

Happy Philosophers



Happy Mathematicians

Intuition-Based Autonomy Platonism

Some Questions

- Is this really a justificatory account?
  - It’s not just a genetic fallacy.
- How does this account yield knowledge of inaccessible mathematical objects?
  - All objects are theoretical posits.
- Does this account make math and science too similar?
  - Scientists use intuition, too, but are constrained by observation in different ways than mathematicans and philosophers.

Some Excellent Recent Work on Intuition

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