Measuring Imprecise Quantities: Degrees of Truth and Classical Logic

Puzzle in the metatheory of applied mathematics:
Mathematical terms don't appear to be vague and mathematicians use logical methods that rely on precision.
Show every number has property P by showing that a number has P if it satisfies $\varphi(x)$, then that it has P if it satisfies not-$\varphi(x)$.
Mathematical methods are applied in the empirical sciences, including the social sciences, where there is vagueness everywhere.
Perhaps a case of Wigner's observation about the unreasonable effectiveness of math in empirical sciences.

Bas van Fraassen: Distinguish the logical rule excluded middle (You may affirm $(\varphi \vee \neg \varphi)$ from the semantic thesis bivalence (Every sentence is true or false). Keep the former.
Supervaluations: Model theory gives us precise models for mathematical languages.
Semantics for vague language: Take a class of "acceptable"precise models.
A sentence is true (false, neither)if it's true in all (none, some but not all) of the acceptable models.

Compositional semantics: Degrees of truth.
Degree of truth of a disjunction is the sum of the degrees of the disjuncts. Likewise "$\wedge$," "$\neg$."
Most degree theories (Łukiesiewicz, fuzzy logic) assign numerical degrees. But "Lou is rich" and "Lou is bald" should be incomparable.

$\mathfrak{B}$ an algebra with signature +, ·, -. A *$\mathfrak{B}$-valuation* v sets $v(\varphi \vee \psi) = v(\varphi) + v(\psi)$, etc.
$\mathfrak{B}$ respects logical equivalence iff whever $(\varphi \leftrightarrow \psi)$ is valid, $v(\varphi) = v(\psi)$
TFAE: $\mathfrak{B}$ respects logical equivalence w.r.t. every language for the sentential calculus.
$\mathfrak{B}$ is a Boolean algebra.

When is the inference from Γ to φ a good inference?
Classically (Boolean algebra $\mathbf{2}$): Guaranteed to take true premises to true conclusions. Two ways to extend this to $\mathfrak{B}$:
Want at least this: Guaranteed truth preservation: For any v, if $v(\gamma) = 1$ for each γ in Γ, $v(\varphi) = 1$.
Maybe want this too: Guaranteed degree-of-truth preservation: For any v, $v(\varphi) \geq$ any lower bound on the $v(\gamma)$s for γ in Γ.

For any direct inference, TFAE for any Boolean algebra other than the trivial one-element algebra $\mathbf{1}$:
The inference is degree-of-truth preserving w.r.t. and $\mathfrak{B}$-valuation.
The inference is truth preserving w.r.t. any $\mathfrak{B}$-valuation.
The inference is classically valid.

Indirect rules of proof, like conditional proof, proof by cases, *reductio ad absurdum,* and contraposition:
If for each i, $1 \leq i \leq n$, you have derived φ_i with premise set Γ_i; you may derive ψ with premise set Δ.
The inferences from the Γ_is to the φ_is are *metapremises* and the inference from Δ to ψ is the *metaconclusion.*
Contraposition: Metapremise: You have derived ψ from $\Gamma \cup \{\varphi\}$. Metaconclusion: You may derive $\neg \psi$ from $\Gamma \cup \{\neg \varphi\}$.
"You have derived...": What methods of reasoning are allowed within the derivation?

Contraposition doesn't preserve truth preservation: {Lou is bald} $\vdash$ 2+2 = 5, but not {2+2 $\neq$ 5} $\vdash$ Lou is not bald,
Every classically valid rule preserves degree-of-truth preservation.

Why is natural deduction trustworthy within mathematics? In $\mathbf{2}$, direct rules are truth preserving, indirect rules preserve truth preservation.

Why are natural deduction rules reliable outside mathematics? Two explanations:
1st: Direct rules are degree of truth preserving, indirect rules preserve degree-of-truth preservation.
2nd: Direct rules are truth preserving, indirect ruler eliminable. This argument, reminiscent of Hilbert, works, but it's fragile..

Fine: $v(\mathrm{D}\varphi) = 1$ if $v(\varphi) = 1$, = 0 otherwise. D-introduction is truth preserving.
Use D-introduction with a conditional proof to prove bivalence.
Williamson: Classical logic requires bivalence: Without conditional proof, etc. we don't have full classical logic.
Because "validity is necessary preservation of truth," we don't have conditional proof unless $\mathfrak{B} = \mathbf{2}$.

"Orange" has an extension, an antiextension, and a buffer in between.
Tolerance principle: There aren't two things, visually indistinguishable in color, one in the extension and the other in the antiextension.
Sequence of 1000 tiles, 1st red, 1000th orange, adjacent pairs visually indistinguishable..
Inference from "Tile n is the first orange tile to "The moon is made of Wensleydale cheese" is truth preserving.
The disjunction "Tile 2 is the first orange tile $\vee$ ... $\vee$ tile 1000 is the first orange tile" is true.
Use proof by cases to conclude that the moon is made of Wensleydale cheese.

If tile n is in the buffer, the inference from "Tile n is the first orange tile" won't preserve degrees of truth.
Rival principles that might be justified by metatheoretic reflection:

Truth Preservation. From φ, together with the statement that the inference from φ to ψ is truth preserving, you may infer ψ.
Degree-of-truth Preservation. From φ, together with the statement that the inference from φ to ψ is degree-of-truth preserving, you may infer ψ.

The first gives us a proof of bivalence. The second is harmless.

Sequent calculus.
Truth preservation: If each of the antecedents is true, at least one of the succedents is true.
The calculus collapses $\{(\varphi \vee \psi)\} \vdash \{\varphi, \psi\}$ fails.
Degree-of-truth preservation. Every lower bound of the antecedents ≤ every upper bound of the succedents.
All the classically valid direct and indirect inferences are upheld.

Relaxed standard: **2** is the unique nontrivial algebra that uphold all classical rules, direct and indirect.
Rigorous standard: **2** is unremarkable. Any other nontrivial Boolean algebra would work as well, wherther or not the language is vague.

WARNING: The theorem that the classically valid indirect rules preserve degree-of-truth preservation assumes that the rules have finitely many metapremises.

A *state description* for Ω contains, for each sentence in Ω, either it or its negation.
Cut for sets (Shoesmith& Smiley): If you've derived θ from $\Gamma \cup \Lambda$ for each state description Λ for Ω, you may derive θ from Γ alone.
TFAE: Cut for sets preserves degree-of-truth preservation w.r.t. every $\mathfrak{B}$-valuation of a language.
$\mathfrak{B}$ is an atomic Boolean algebra.

Carol Karp: Introduce, for each set Γ of sentences, a disjunction $\bigvee\Gamma$ and a conjunction $\bigwedge\Gamma$.
Compositional semantics requires operations Σ and Π on $|\mathfrak{B}|$ with $v(\bigvee\{\varphi_i{:}i \in I\}) = \Sigma\{v(\varphi_i{:}i \in I)|$, $v(\bigwedge\{\varphi_i{:}\ i \in I\}) = \Pi\{v(\varphi_i){:}\ i \in I\}$.
If $\mathfrak{B}$ respects infinitary logical equivalence, Σ will have to be the lub, Π the glb.
For these operations to be defined, $\mathfrak{B}$ will have to be complete.

For given Ω, the disjunction of the conjunctions of the state descriptions for Ω must have value 1.
For assurance of this, $\mathfrak{B}$ will have to be atomic.

If $\mathfrak{B}$ is complete and atomic, every classically valid direct inference, finitary or infinitary, will preserve degrees of truth, and every indirect rule will preserve degree-of-truth preservation.

Turning to (finitary) predicate calculus.
Fixed domain *U*. For each member *u* of *U,* an individual constant c_u.
Precise proper names (within the object langaue): Speaker's use of tokens of the name picks a unique reference.
The c_us aren't in the object language and they don't have tokens. They are metalinguistic tools
No individual constants or function signs in the object language. Use definite descriptions instead.

For $\mathfrak{B}$ complete, a $\mathfrak{B}$-valued *U*-model is a function U assigning members of $|\mathfrak{B}|$ to atomic formula subject to the contraints:
$\mathfrak{U}(c_u = c_u) = 1$, and if $u \neq v$, $\mathfrak{U}(c_u = c_v) = 0$.
(Xinhe Wu sees prohibition of indeterminate identity as crucial to $\mathfrak{B}$-valued semantics as used here vs. its use in forcing.)
The value $\mathfrak{U}$ assigns to a disjunction (conjunction) is the sum (product) of the values assigned to components.
$\mathfrak{U}(\neg\, \varphi) = -\, \mathfrak{U}(\varphi)$.
Value of an existential (universal) sentence is the sum (product) of the values assigned to its instances.

If *a* is an atom of $\mathfrak{B}$, get a classical model $\mathfrak{U}_a$ by stipulating that φ is true iff $a \leq v(\varphi)$.
If $\mathfrak{B}$ is complete and atomic, $\mathfrak{U}(\varphi) = \Sigma\{\text{atoms } a{:}\ \varphi \text{ is true in } \mathfrak{U}_a\}$.
$\mathfrak{B}$ is isomorphic to the power set algebra on its atoms.
Declare the $\mathfrak{U}_a$s to be the *acceptable* classical models, and take the semantic value of a sentence to be the set of acceptable models in which it's true.
This is just the van Fraassen construction.

Usefulness of the model theory of precise languages in the semantics of vague languages isn't a lucky accident.
It's forced upou us by the logic.