Measurement and Truth in Set Theory

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Joint work with Shoshana Friedman

Overview

"Clarity should come at the end, not at the beginning of a mathematical endeavor."

–Gerald Sacks

Our motivating question

When we do set theory, what is it that we are doing?

Where set theorists agree

We are studying the concept of iterative set.

What is a set?

"By a 'set' we understand any gathering-together M of determined well-distinguished objects m of our intuition or of our thought, into a whole."

–Cantor

A six-slide history of set theory

Between 1868 and 1874, Cantor proved the foundational theorems about sizes of infinite sets, including that there are more real numbers than natural numbers.

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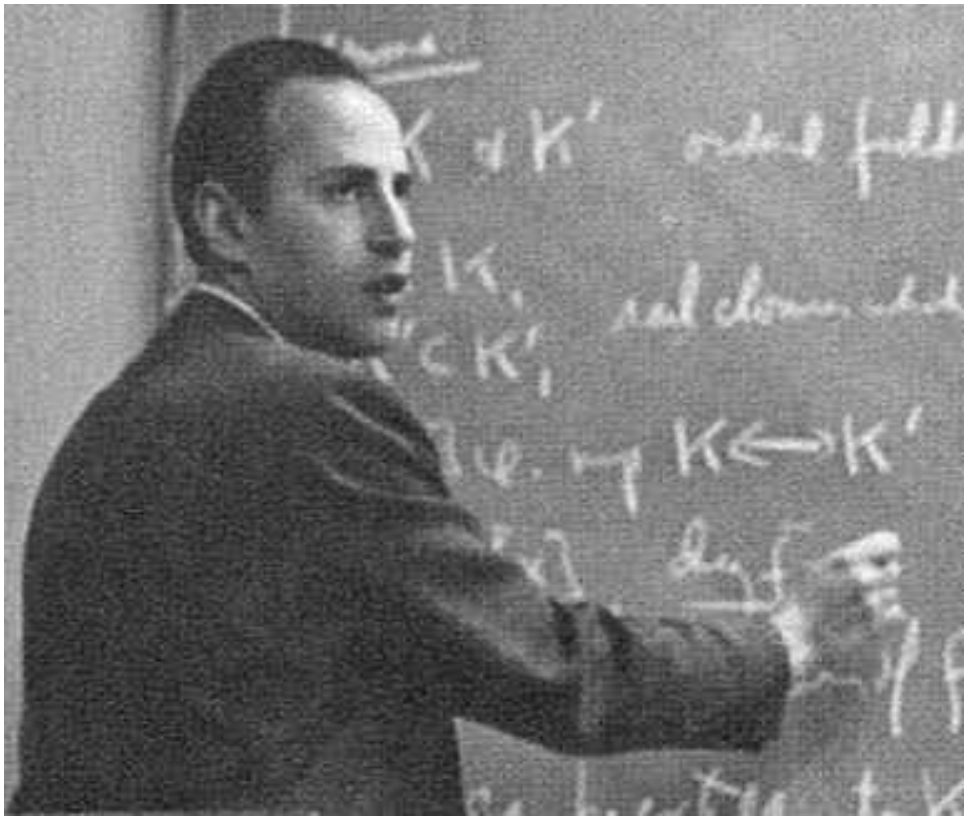
The hypothesis that they are—that there is no size of infinity between them—is now known as the Continuum Hypothesis, or CH.

The continuum hypothesis is compatible with the usual axioms of set theory

Gödel proved it is consistent with the usual axioms of set theory that there are no intermediate sizes of infinity between that of the naturals and that of the reals.

The negation of the continuum hypothesis is also compatible with the usual axioms of set theory



Inspired by Gödel's construction, Cohen invented the technique of 'forcing.'

In the face of incompleteness...

...what can we do?

Search for additional axioms

Is there a right axiom (or set of axioms) to add?

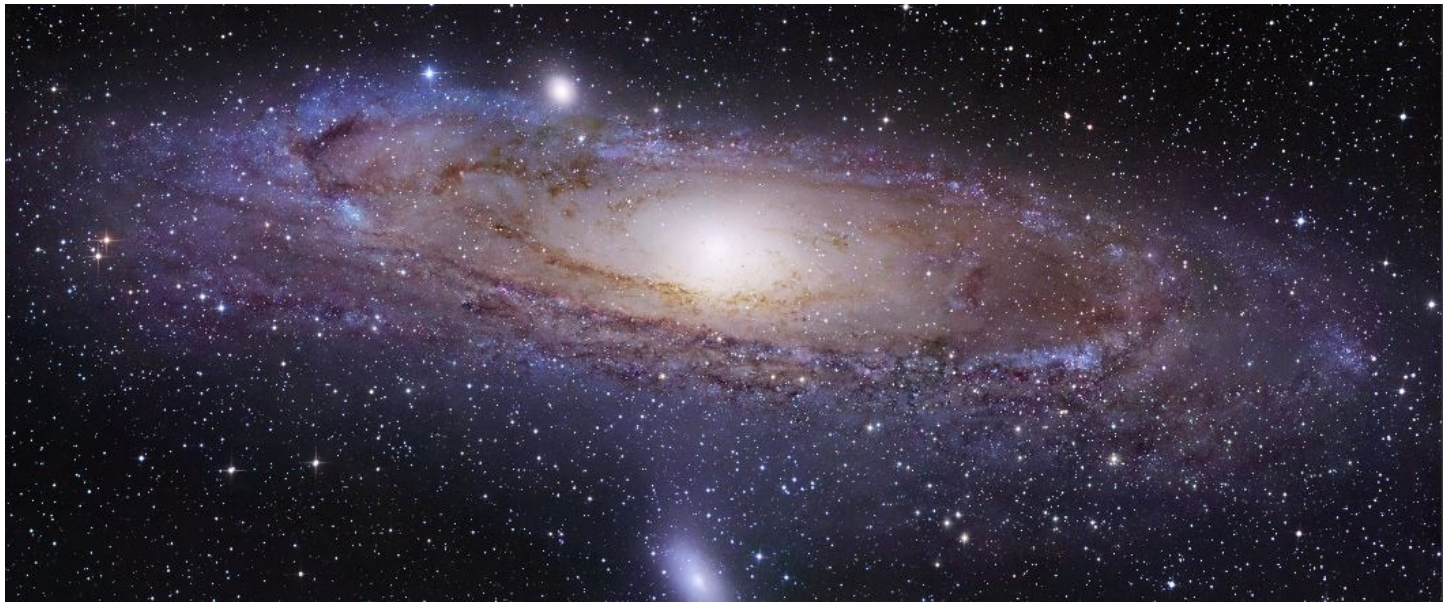
That depends on who you ask...

Natural to whom?

When we're doing set theory...

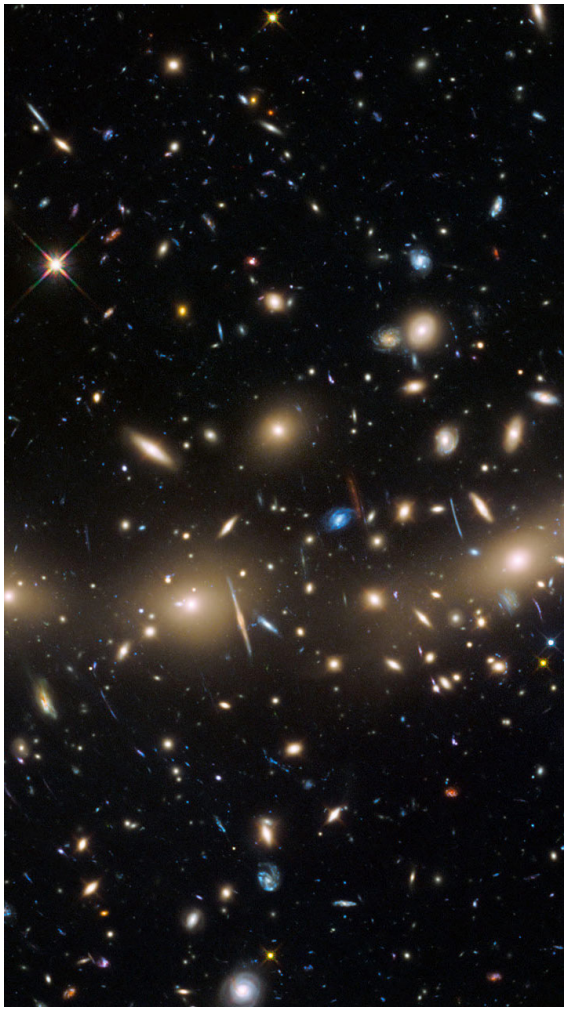
...what are we doing?

We are studying the universe of sets, V



Andromeda Galaxy (Photo Credit: NASA)

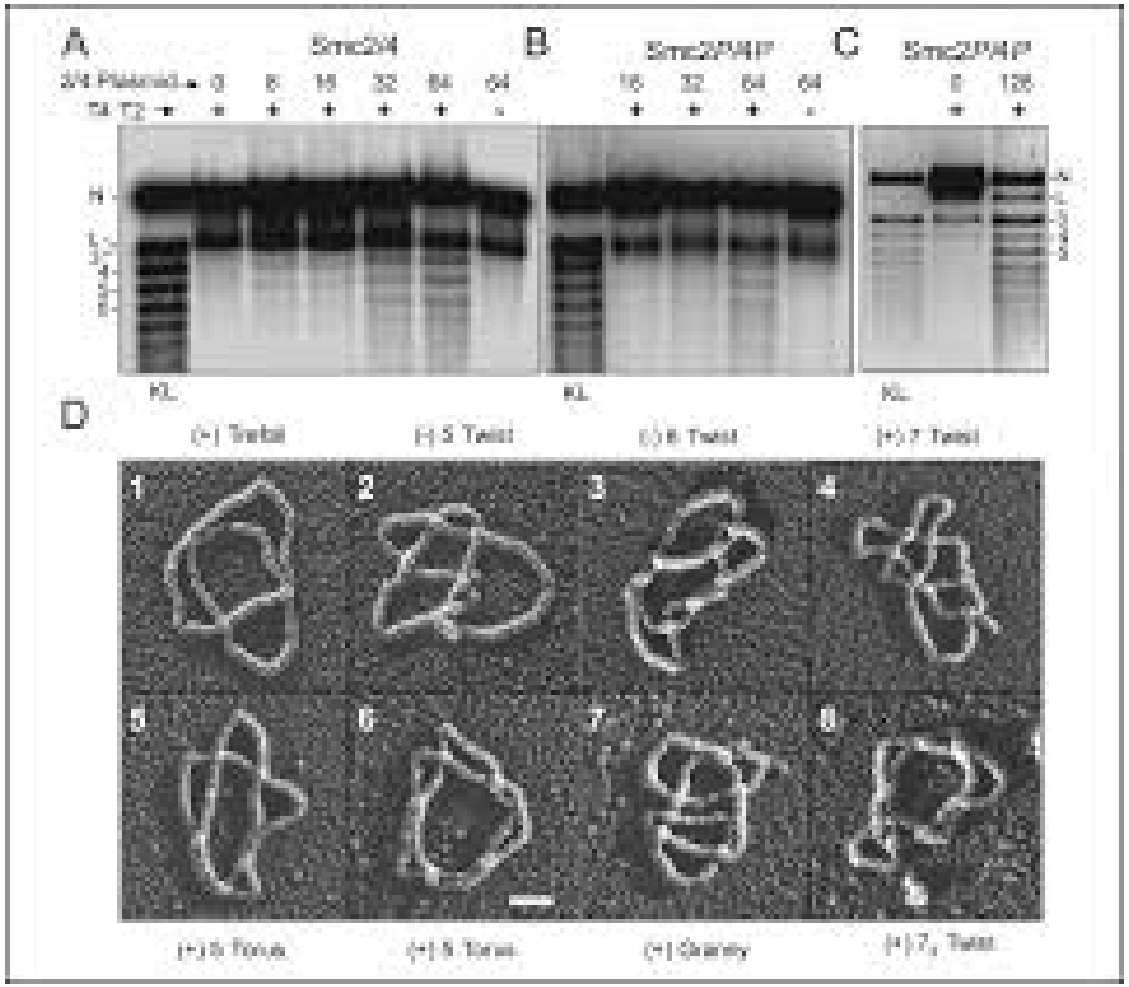
We are studying models of set theory



A relatively nearby galaxy cluster, MACSJ0416.1–2403. The thin streaks and arcs, mostly blue, are galaxies in the far background whose images are warped due to gravitational lensing by the foreground cluster. NASA / ESA / J. Lotz (STScI)

We are measuring attributes of sets

Knots in DNA



River Meanders

The Three Body Problem

Orbit Perturbations

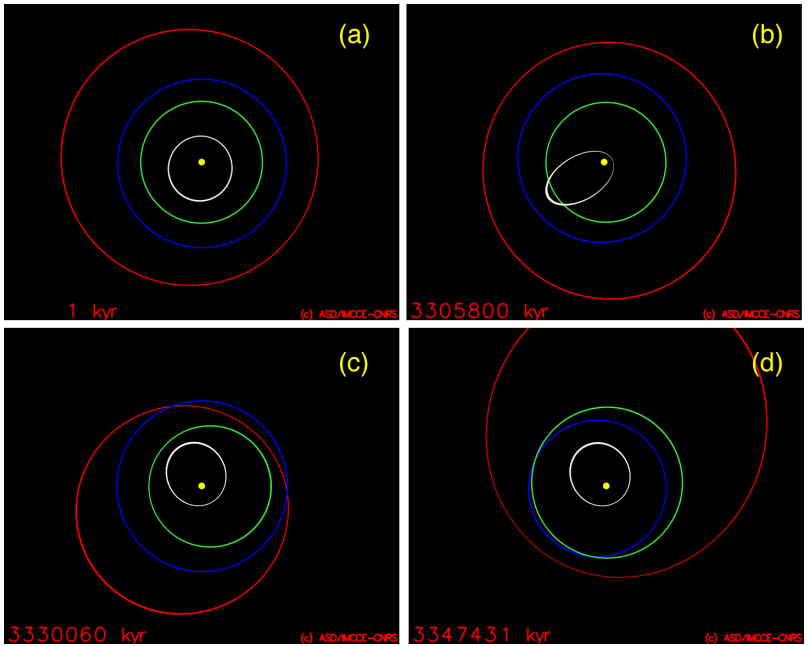


Figure 7 of Stability of the Solar System (Laskar): Example of long-term evolution of the planetary orbits: Mercury (white), Venus (green), Earth (blue), Mars (red). Time is indicated in thousands of years (kyr). (a) In the vicinity of the current state, the orbits become distorted under the influence of planetary perturbations, but without allowing close encounters or collisions. (b) In about 1% of cases, the orbit of Mercury may be distorted enough to allow a collision with Venus or the Sun in less than 5 Gyr. (c) In one of the trajectories, the eccentricity of Mars increases sufficiently to allow for a close encounter or collision with Earth. (d) This leads to a destabilisation of the terrestrial planets that also allows a collision between Venus and Earth. Adapted from (Laskar and Gastineau, 2009)

Quantum Superposition

$$|\Psi\rangle = \frac{|\ \text{  }\ \rangle + |\ \text{  }\ \rangle}{\sqrt{2}}$$

Frank Rioux, `http://www.users.csbsju.edu/~frioux/workinprogress.html`

What is a measurement?

"...any method by which a unique and reciprocal correspondence is established between all or some of the magnitudes of a kind and all or some of the numbers, integral, rational or real."

Bertrand Russell (1903)"

"...what is the objective meaning of expressing through denominate numbers the relations of real objects as magnitudes, and under what conditions can we do this?"

Hermann von Helmholtz (1887)

Two questions

What can be measured?

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What can be measured?

How do we measure it?

A (too) short history of measurement

- Quality versus quantity (Aristotle)

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- Model-theoretic accounts of measurement (Mari, Giordani, Tal)

Common set theoretic functions, such as cardinality, are measurements of sets.

Those measurements give different results in different models.

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Still—

- Measurements bound what *can* be true and help us direct our search for new information and new methods

Thank you!