Logic, Intuition and Infinity

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More Questions than Answers

- Mathematics survived, even flourished, from ancient times through the late 19^{th} Century while for the most part the mathematical community of earlier times rejected the concept of an actual infinity. So how and why is the concept of completed infinity so prevalent and well-accepted in modern mathematics?
- Where do our intuitions about the infinite come from?
- Given we have no sensory access to infinite sets, how is the existence of infinity in mathematics justified?
- Are the cardinal and ordinal number systems, into the transfinite, intuitive? If so, what are these intuitions based on?
- Can we justify references to infinite sets without committing to them?
- Can infinitary mathematics be secured on finitistic grounds?

Where do our intuitions about the infinite come from?

- We count 0, 1, 2, ..., for any n we know there is a successor $n+1$, so we conclude this set is infinite.
- We can conceptualize finite sets and we extend/abstract this to a concept of infinite sets.
- (Cantor) We *count* some infinite sets – $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, the algebraic numbers – but we recognize others, e.g. $\mathbb{R}$, are uncountable.
- (Zermelo) Using the Axiom of Choice, we can well-order the uncountable sets, which gives us a kind of counting of those.
- (Cantor) In fact, AC gives us access to an entire infinite number system where every set is assigned a unique cardinality.

Are these last two bullets really grounded in intuition?

The human mind is finite and the set theoretic hierarchy is infinite. Presumably any contact between my mind and the iterative hierarchy can involve at most finitely much of the latter structure. But in that case, I might just as well be related to anyone of a host of other structures that agree with the standard hierarchy only on the minuscule finite portion I've managed to grasp.
– Penelope Maddy

In mathematics, as in any scientific research, we find two tendencies. . . the tendency toward abstraction seeks to crystallize the logical relations inherent in the maze of material in a systematic and orderly manner. On the other hand, the tendency toward intuitive understanding fosters a more immediate grasp of the objects. . . a live rapport with them. . . which stresses the concrete meaning of their relations.

– David Hilbert, Geometry and the Imagination

Where do our Intuitions about the Infinite Take Us?

Bizarre consequences:

- The unsettling and unsettled question of the size of the continuum…the independence of CH from ZF.
- A counter-intuitive arithmetic for cardinal numbers.
- A well-ordering of $\mathbb{R}$, but no way to describe it.
- Sets of reals that are not measurable, but no definable versions of them.
- (Banach Tarski) A decomposition of a ball into 5 pieces that can be reassembled through rigid motions to form two balls each the same size as the original.

> *No one shall expel us from the paradise that Cantor has created for us.*
> – David Hilbert

Hilbert's Program and Proof Theory

- (Hilbert) Find an axiomatic formalization of all of classical (infinitary) mathematics, together with a finitary proof that this axiomatization of mathematics is consistent.
- (Gödel) The Second Incompleteness Theorem shows Hilbert's Program (as stated) is impossible.
- (Gentzen) Finitary reasoning plus a limited amount of transfinite induction is sufficient to prove the consistency of Peano Arithmetic.

Proof Theory Approach

- Given a theory T, find the recursive ordinal associated with T, called the *proof-theoretic ordinal* for T, that can be used to measure the *proof-theoretic strength* of T.
- Using transfinite induction up to α develop a consistency proof for T by a kind of proof reduction.
- The search for proof theoretic ordinals has succeeded up through fragments of real analysis, but has failed to come anywhere close to ZF.

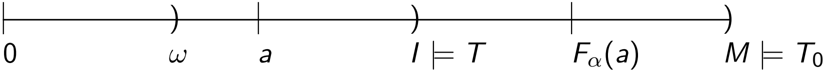
Model-theoretic Ordinal Analysis

(Paris & Kirby, S & Avigad) Consider

$$0 \quad \omega) \quad a \quad I \models T) \quad F_\alpha(a) \quad M \models T_0)$$

- T is an infinitary classical theory of mathematics– (e.g., Peano Arithmetic or a fragment of real analysis.– *we want to finitistically justify* T and we have found its *proof-theoretic ordinal*, α.
- T_0, a finitistic base theory, potentially a feasible version of Primitive Recursive Arithmetic.
- M is any *non-standard* model of T_0,
- ω is the set of natural numbers
- a is *non-standard*
- F_α is a fast-growing function that captures the combinatorial properties of α.
- I is an initial segment of M that falls between a and $F(a)$. Using combinatorial properties associated with F there is a feasible construction of I.

Make it all finite



- Take ω to be numbers we can count to (by any means). This set contains 0 and is closed under successor, but it is also finite and bounded.
- Let a be a number too big to count to.
- Working between a and $F_\alpha(a)$, there's a combinatorial process that can be used to define I.
- I provides a complete theory that settles all questions in the language, and is consistent with T as well as with our experience.
- But we can't know all of I since it goes beyond the "minuscule finite portion" of the universe that we've managed to grasp.

Conclusion

- The above provides a finitistic, in fact truly finite, viewpoint for a large segment of classical mathematics.
- This interpretation does not require us to change the language – i.e., we can still speak of infinite sets and work as we normally do. This interpretation is consistent with reference to infinite sets, within the context of analysis, geometry, number theory, and other branches of mathematics.
- However, it breaks down with the far reaches of the set-theoretic universe.
- What does this mean for Cantor's paradise? . . . , well, I'm going to continue to teach my Set Theory course as I always have. . .

Thank You!