Logic in the Integers

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Yale Weiss
Saul Kripke Center, The Graduate Center, CUNY
New York, NY

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Preliminaries

Before getting into the substance of my talk, I'd like to spend a little time explicating the title. I assume 'in' and 'the' are self-explanatory, so I will focus on 'logic' and 'integers'.

What is *logic*?

For the purposes of this talk, *logic* may be understood to be the study of the correct forms and principles of deductive reasoning for some (suitably formalized) fragment of language.

Different logicians have focused on different fragments of language. Aristotle, in his *Prior Analytics*, was interested in a fairly narrow class of arguments, *syllogisms*, composed of three categorical propositions involving quantifiers such as 'all' and 'some'. On the other hand, Stoic logicians like Chrysippus focused on the logic of complex propositions built up using connectives like 'and' and 'or', that is, *propositional logic*.

What is *logic*?

Perhaps unsurprisingly, logicians have often disagreed with one another about what the correct, or *valid*, forms of reasoning are. These debates, especially over the conditional ('if ..., then ...'), have at times been widespread and intense, as Sextus attests in quoting an epigram of Callimachus (*M* I.309):

> *See there!*[1] *Doubtless, even the crows upon the rooftops are cawing about what sort of thing has followed [from what].*

[1]See where? Perhaps Alexandria, where Callimachus was active in the 3rd century BCE.

What is *logic*?

Today, if one is taught logic, they are most likely to be taught (the rather inappropriately named) classical first-order logic, which deals with both quantificational and propositional expressions. But debates over the valid principles of logic continue and the number of proposed *non-classical logics* continues to grow rapidly.[2] In this talk, I will be focused on non-classical logics, some old, some new.

[2] It is noteworthy that some of these modern debates have progressed little (except in technical respects) beyond debates which were already recorded in antiquity. See especially the debate on conditionals—material, strict, and relevant—recounted by Sextus, *PH* II.110ff.

What is *logic*?

Now then, there are different ways of specifying a logic. After fixing a formal language, you might specify the logic *proof-theoretically*, that is, by axiomatizing it, or by articulating rules of inference for it (for example, by presenting a natural deduction system or a sequent calculus). Alternatively, you might characterize the valid principles *model-theoretically* (i.e., semantically).

Any good logic (if I may editorialize) will in fact be articulated in both ways, which makes showing that the articulations are equivalent very important. A *soundness theorem* shows that the proof-theory "matches" the model-theory, and a *completeness theorem* shows that the model-theory "matches" the proof-theory.

What are the *integers*?

Well, I'm sure I don't need to tell any of you this!

Nevertheless, I will actually be interested only in certain narrowly circumscribed integer structures in this talk, so it would be good to spend a little time talking about these on their own terms.

What are the *integers*?

I take the *natural numbers* to be the positive integers, that is, $\mathbb{N} = \{1, 2, \ldots\}$.

Perhaps the most natural way of ordering $\mathbb{N}$ is by using the 'less-than-or-equal' order, $\leq$. It is clear that $\langle \mathbb{N}, \leq \rangle$ is a partially-ordered set ('poset'), for $\leq$ is *reflexive*, *anti-symmetric*, and *transitive*.

But there are other natural ways of ordering the natural numbers. Consider the 'evenly divides' relation $|$, that is, the relation such that $n|m$ iff $\exists p \in \mathbb{N}(n \times p = m)$. Then clearly $\langle \mathbb{N}, | \rangle$ is also a poset.

What are the *integers*?

The structures $\langle \mathbb{N}, \leq \rangle$ and $\langle \mathbb{N}, | \rangle$ are not only posets, but what are called *lattices*. Any two natural numbers have both a *join* and a *meet* with respect to $\leq$, namely, their maximum and minimum, respectively. But any two natural numbers also have a join and a meet with respect to $|$, namely, their least common multiple and greatest common divisor, respectively.

For different emphasis, we might, for example, present the lattice $\langle \mathbb{N}, | \rangle$ as $\langle \mathbb{N}, \mathrm{lcm}, \mathrm{gcd} \rangle$, noting that the order is straightforwardly recoverable: $m|n$ iff $\mathrm{gcd}(m, n) = m$.

What are the *integers*?

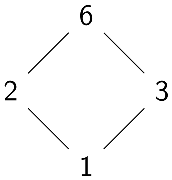

Figure: A sublattice of $\langle \mathbb{N}, | \rangle$.

What are the *integers*?

Lattices show up throughout mathematics and some of these are more exotic than others. Both $\langle \mathbb{N}, \leq \rangle$ and $\langle \mathbb{N}, | \rangle$ are *distributive*, both have a *least element*, neither is *complete*, and neither has a *greatest element*.

There are also important differences between $\langle \mathbb{N}, \leq \rangle$ and $\langle \mathbb{N}, | \rangle$. The first is a *total* or *linear* order, while the second is not. The first has only one *atom*, while the second has infinitely many.

Outline of the Talk

This is a talk about logics which can be modeled using integer lattices and which are, given suitable interpretations of the logical constants, sound and complete with respect to some such structures. I will mainly be interested in $\langle \mathbb{N}, | \rangle$ and the logics it gives rise to. I will also be keen to show how elementary arithmetical facts and properties correspond to or suggest philosophically interesting features of logics.

Outline of the Talk

The plan of the talk is as follows:

1. I will discuss a couple of historical highlights involving logics and the integers;
2. I will examine mathematical intuitionism and intuitionistic logic and connections to the integers;
3. I will offer a few remarks on relevance logic and the integers.

Lattices and Logic: A Love Story

Now, logicians can fairly be said to love lattices. The properties of the logical propositional connectives $\wedge$ and $\vee$ share much in common with the lattice operations meet and join, respectively. Nearly every propositional logic one encounters in the literature can be given a so-called *algebraic semantics* based on lattices.

But the history of lattice theory and logic goes back considerably farther than many logicians and mathematicians probably appreciate. Indeed, it goes back to a time when lattice theory wasn't a discipline and the received logic wasn't propositional.

Leibniz

In 1679, Leibniz, one of the greatest mathematicians and philosophers of his day, sought to present an arithmetical semantics (or model theory) for the then-prevailing logic, Aristotle's assertoric syllogistic.

'Why?', you might wonder. The motivation for producing some kind of arithmetical interpretation of reasoning appears to be intimately related to a long-held ambition of Leibniz to produce a *universal calculus* for expressing all thought as clearly as arithmetical notions and relations are expressed.

Leibniz

Leibniz—a sometime diplomat!—seems to have been a genuine believer in the power of symbolism, logic, and mathematics to resolve differences. In a famous passage (G VII 200), he writes:

> *To return to the expression of thought by characters, I feel that controversies can never be ended [. . .] unless we withdraw from complicated reasonings to simple calculations, from vague terms to definite characters [. . .] Then, when controversies arise, there will be no more need for dispute between two philosophers than between two mathematicians [Computistas]. It will suffice for them, pens in hand, to sit at their abaci, and say to each other [. . .] let us calculate.*

Leibniz

Let's take a closer look at Leibniz's proposal. First, however, I should note that Leibniz actually made several distinct proposals in 1679 (these are collected by Couturat [3, pp. 42ff.]). I aim to focus only on a few common features of these and my treatment here will be somewhat informal. For further details, consult Łukasiewicz [12] and Soritov [18].

The Syllogistic: Background

Now then, recall that Aristotle's (assertoric) syllogistic deals with the so-called 'categorical propositions', which are as follows:

(a) AaB (*universal affirmative*) is read: 'All Bs are As';

(e) AeB (*universal negative*) is read: 'No Bs are As';

(i) AiB (*particular affirmative*) is read: 'Some B is A';

(o) AoB (*particular negative*) is read: 'Some B is not A'.

A bit of notation: if p is a categorical proposition, $\overline{p}$ is the *contradictory* of p, as given by the traditional square of opposition (e.g., $\overline{AaB}$ is AoB).

The Syllogistic: Background

A deduction is, roughly, something of the form $p, q \Rightarrow r$, where p, q, and r range over categorical propositions. The 'good' (i.e., valid) deductions are called *syllogisms*.

Now, some valid deductions are better than others; Aristotle calls these 'perfect' (*APr.* I.1, 24b20ff.). We can think of these as being, roughly, self-evident axioms. These include, for example, Barbara (1-aaa):

$$AaB, BaC \Rightarrow AaC \tag{Barbara}$$

The Syllogistic: Background

Besides these axioms, there are also rules for deriving other syllogisms, including conversion rules like:

$$\frac{AiB, p \Rightarrow q}{BaA, p \Rightarrow q} \qquad (a\text{-Conversion})$$

And then there's the following rule:

$$\frac{p, q \Rightarrow r}{p, \overline{r} \Rightarrow \overline{q}} \qquad (\text{Antilogism})$$

Formally, then, a syllogism is just one of the axiomatic (or perfect) syllogisms, or any deduction derivable from them using rules such as these.

The Leibnizian Interpretation

What's arithmetic got to do with any of this? Consider AaB, where B, the subject term, is understood as Human and A, the predicate term, is understood as Rational. Then AaB says, colloquially, 'All Humans are Rational'.

What does it mean to say that all humans are rational? Us moderns tend to think of this *extensionally*: it means that the set of human-things is a subset of the set of rational-things. But this is, twice-over, not the way Leibniz thought about this.

The Leibnizian Interpretation

First, he wasn't thinking in terms of sets. Second, following Aristotle, he thought *intensionally*. Roughly, think of how Rational is part of, is contained in, the "real definition" of Human.

Now, following Leibniz, let's think of these disharmony-causing vague concepts (Human, Rational) as associated with precise characters (natural numbers). The idea is that for AaB to be true is for the A-number to be a part of (i.e., divisor of) the B-number, or in Leibniz's (translated) words:

> *The characteristic number of the subject can be divided exactly [. . .] by the characteristic number of the same sign belonging to the predicate. [3, p. 78]*

The Leibnizian Interpretation

We have here the very first steps of a full arithmetical interpretation of the *language* of the assertoric syllogistic. I will not go over the other cases (the *ei*-fragment raises complications, unfortunately), but Leibniz effectively shows how to give truth conditions for all the categorical propositions using the natural numbers, $|$, and gcd. That is, he (implicitly) articulates a semantics for the language with respect to the divisibility lattice $\langle \mathbb{N}, | \rangle$.

The Leibnizian Interpretation

So far, so good. But we haven't said anything yet about the most important thing: $\Rightarrow$.

Backing up for a moment, we can think of a *model* as $\langle \mathbb{N}, | \rangle$ together with an assignment (interpretation), ν, of logical terms into $\mathbb{N}$.[3] Then a deduction $p, q \Rightarrow r$ is invalid if we can find a model such that p and q are true, but r is false; and the deduction is *valid* if there are no such models.

[3] In his mature treatment, Leibniz in fact assigns terms to pairs of coprime natural numbers.

The Leibnizian Interpretation

This gives everything that is needed to start to see how facts about the divisibility lattice correspond to facts about the syllogistic. For example, recall the valid (perfect) syllogism Barbara: $AaB, BaC \Rightarrow AaC$. The validity of this straightforwardly corresponds to the fact that $|$ is a transitive relation. For suppose that Barbara was invalid; then we could find an assignment ν such that $\nu(A)|\nu(B)$ and $\nu(B)|\nu(C)$ but $\nu(A) \nmid \nu(C)$, which is clearly impossible.

The Leibnizian Achievement

Leibniz himself shows, or comes fairly close to showing, that the assertoric syllogistic is *sound* with respect to his (mature) arithmetical semantics. This was—for the time especially—a major achievement. He doesn't show *completeness*, but as a testament to his insight, the mature semantics he proposed is in fact complete, as logicians have subsequently shown (consult Słupecki [17] and Weiss [24]).

In fact, you don't need the full divisibility lattice for completeness. Weiss [24] has recently shown that you can get by with certain four-element sublattices thereof, e.g., $\langle\{1, 2, 3, 6\}, |\rangle$, and that four elements is the 'best of all possible results'.

Gödel Algebras

Leibniz's pioneering efforts in arithmetically modeling logic appear to have been almost completely unknown until the 20th century. Thus, our next stop, another logician famous for mixing logic and the integers, lived hundreds of years after Leibniz. No, I won't be talking about his eponymous coding scheme or famous incompleteness theorems; instead, I will just briefly touch upon one of Kurt Gödel's lesser known contributions, but the one which is most relevant to the topic of this talk.[4]

[4]Incidentally, Gödel was an enthusiastic fan of Leibniz—perhaps *too* enthusiastic. Menger [15, pp. 122–123] recounts how Gödel became convinced of a conspiracy to suppress or destroy Leibniz's work.

Gödel Algebras

Let G_n be the sublattice of $\langle \mathbb{N}, \leq \rangle$ consisting of the first n natural numbers. Each G_n is a *Heyting algebra* (roughly, bounded lattice residuated with respect to meet) where $\hookrightarrow$ is defined thus:

$$a \hookrightarrow b = \begin{cases} n & \text{if } a \leq b, \\ b & \text{else.} \end{cases}$$

These G_n are sometimes known as *Gödel algebras* since Gödel [9] used them to show that propositional intuitionistic logic (more on that in a moment!) is not a finitely valued logic.[5]

[5]Actually, Gödel [9] inverts the order so that 1 is always the greatest element, but this is not an important difference.

Gödel Algebras (and Beyond)

By extending $\langle \mathbb{N}, \leq \rangle$ with a greatest element ω, one can analogously obtain an infinite Gödel algebra, which Dummett [5] shows algebraically characterizes so-called Gödel-Dummett logic (**LC**), an important intermediate logic. (Interesting related results were obtained for **RM**, an important quasi-relevance logic; consult Anderson and Belnap [1].)

Intuitionistic Logic

At this point, you may be wondering, 'And what does this have to do with philosophy and logic again?'

Let's take a step back and take a closer look at intuitionistic logic (i.e., the logic Gödel proved this technical result about).

Intuitionistic Logic: Background

Intuitionistic logic is perhaps the best known of all non-classical logics today. It arose in the early 20th century out of *intuitionism*, a movement in the foundations of mathematics founded by L. E. J. Brouwer which primarily attracted adherents in Holland.

The specifics of intuitionism are beyond the scope of this talk, but the important thing, for our purposes, is that it is a species of *constructivism*. Hallmarks of constructivism are the assimilation of the notions of truth and proof and the rejection of proofs which rely on non-constructive principles (e.g., excluded middle).

Intuitionistic Logic: Background

The following is a classic example of a non-constructive proof:

Theorem

There are irrational numbers a and b such that a^b is rational.

Proof.

Either $\sqrt{2}^{\sqrt{2}}$ is rational or it is not. If it is, put $a = b = \sqrt{2}$. If it is not, put $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$, which suffices. $\square$

Intuitionistic Logic: Background

What is problematic about this proof, from a constructivist viewpoint, is its use of excluded middle. We are not told which of the two cases obtain—$\sqrt{2}^{\sqrt{2}}$ being rational or not—and it might even be, for all we know, impossible to say which.[6] Relying on excluded middle, the proof doesn't construct witnesses to the theorem, that is, it doesn't say which of two possible pairs are the witnesses.

[6]In this particular case it *is* possible to say which, but that doesn't change the philosophical point.

Intuitionistic Logic: Background

Intuitionistic logic is supposed to give a formalization of constructively acceptable reasoning. Let's be a little more rigorous and pin down the language. I use Π for an infinite set of propositional variables; formulae are built up from these and the connectives $\rightarrow, \wedge, \vee$, and $\perp$ using the standard formation rules. I use $p, q, \ldots$ for propositional variables and $\varphi, \psi, \ldots$ for formulae.

Intuitionistic Logic: Background

There is an informal semantics for the language of intuitionistic logic, the so-called *BHK semantics*, which brings out the proof-centric approach to truth. Here are the conditions for the propositional fragment:[7]

(i) A proof of $\varphi \wedge \psi$ is a combination of a proof of φ and a proof of ψ;

(ii) A proof of $\varphi \vee \psi$ is given by giving either a proof of φ or a proof of ψ;

(iii) A proof of $\varphi \to \psi$ is a construction which, given a proof of φ, returns a proof of ψ;

(iv) There is no proof of $\bot$.

[7] There is some variation in how these conditions are presented; cf. Artemov [2, pp. 1–2] and Fine [7, p. 550].

Intuitionistic Logic: Background

This semantics is, I emphasize, *informal*. I will look at a formal version shortly. Nevertheless, one can already see intuitively how excluded middle fails. For if neither φ nor $\neg\varphi$ ($\neg\varphi := \varphi \to \bot$) has a proof, by the BHK condition for disjunction, $\varphi \vee \neg\varphi$ doesn't either. Thus, it is not generally guaranteed that $\varphi \vee \neg\varphi$ holds.

Intuitionistic Logic: Background

Intuitionistic logic is supposed to codify intuitionistically acceptable reasoning as intuitively characterized by the BHK semantics. Propositional intuitionistic logic (**J**) can be axiomatized thus:

$$(\varphi \to \psi) \to ((\theta \to \varphi) \to (\theta \to \psi)) \qquad \varphi \to (\psi \to \varphi)$$
$$(\varphi \to (\psi \to \theta)) \to (\psi \to (\varphi \to \theta)) \qquad (\varphi \wedge \psi) \to \varphi$$
$$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi) \qquad (\varphi \wedge \psi) \to \psi$$
$$\varphi \to (\psi \to (\varphi \wedge \psi)) \qquad \varphi \to (\varphi \vee \psi)$$
$$(\varphi \to \theta) \to ((\psi \to \theta) \to ((\varphi \vee \psi) \to \theta)) \qquad \psi \to (\varphi \vee \psi)$$
$$\bot \to \varphi \qquad \frac{\varphi, \varphi \to \psi}{\psi}$$

Intuitionism in the Integers

Now, I have already touched upon one result concerning **J** and the integer structure $\langle \mathbb{N}, \leq \rangle$. But the points made thereby were really negative. **J** is not (algebraically) complete with respect to that structure or any of the related structures I mentioned.

However, **J** *is* complete with respect to $\langle \mathbb{N}, | \rangle$, in a sense to be spelled out more fully below. Not only that, but the semantics corresponds in a fairly natural way to the BHK semantics which I briefly described above.

Intuitionism in the Integers

The central intuitive idea is as follows: we can think of the natural numbers as proofs or constructions which, when combined, yield other proofs and constructions. We think of combination as a generally accretive operation. Thus, we shall interpret combination, mathematically, as join—more specifically, lcm.

Intuitionism in the Integers

Let's be a little more rigorous. The formal semantics adapts ideas mainly due to Urquhart [19]. Define a *hereditary model* over $\langle \mathbb{N}, | \rangle$ as follows:

DEFINITION (Hereditary Model)

A *hereditary model* is a structure $\mathfrak{M} = \langle \mathbb{N}, |, V \rangle$ where $V : \Pi \cup \{\bot\} \to \mathcal{P}(\mathbb{N})$ is subject to the following conditions:

1. $j \in V(p)$ implies $\text{lcm}(j, k) \in V(p)$;
2. $j \in V(\bot)$ implies $\text{lcm}(j, k) \in V(\bot)$;
3. $j \in V(\bot)$ implies $j \in V(p)$ (for all p).

Intuitionism in the Integers

Given a hereditary model $\mathfrak{M} = \langle \mathbb{N}, |, V \rangle$ and $j \in \mathbb{N}$, the relation $\models_j^{\mathfrak{M}}$ ('j is a proof of') is defined as follows:

1. $\models_j^{\mathfrak{M}} p$ if and only if $j \in V(p)$;
2. $\models_j^{\mathfrak{M}} \bot$ if and only if $j \in V(\bot)$;[8]
3. $\models_j^{\mathfrak{M}} \varphi \wedge \psi$ if and only if $\exists k, l \in \mathbb{N}$ such that $j = \text{lcm}(k, l)$, $\models_k^{\mathfrak{M}} \varphi$ and $\models_l^{\mathfrak{M}} \psi$;
4. $\models_j^{\mathfrak{M}} \varphi \vee \psi$ if and only if $\models_j^{\mathfrak{M}} \varphi$ or $\models_j^{\mathfrak{M}} \psi$;
5. $\models_j^{\mathfrak{M}} \varphi \rightarrow \psi$ if and only if $\forall k \in \mathbb{N}$, $\not\models_k^{\mathfrak{M}} \varphi$ or $\models_{\text{lcm}(j,k)}^{\mathfrak{M}} \psi$.

[8]This is really the only case that doesn't fit the usual BHK semantics neatly; one might say various things here [20, 7, 22], but note that the usual condition yields a Jankovian $\bot$ [10].

Intuitionism in the Integers

We say that φ is true in a hereditary model $\mathfrak{M}$ if the 'null-proof' makes it true, i.e., if $\models_1^{\mathfrak{M}} \varphi$. And we say that φ is *valid* if φ is true in all hereditary models. This may be extended to define the validity of arguments in the obvious way: the inference from Γ to φ is valid if the conditional formed by the conjunction of the elements of Γ as antecedent and φ as consequent is valid.

Integer Insights

The first thing to note is that the foregoing semantics exactly characterizes propositional intuitionistic logic (for the details, consult Weiss [22, 23]):

THEOREM (Completeness)

J *is sound and complete with respect to hereditary models.*

Integer Insights

With that result in hand, it becomes very intuitive to obtain various technical results about intuitionistic logic by relying on familiar features of the divisibility lattice.

For example, $(\varphi \to \psi) \vee (\psi \to \varphi)$ is not a theorem of **J**. We can easily specify a countermodel $\mathfrak{M}$ to an instance of it by putting $V(p) = 2^{\uparrow}$ and $V(q) = 3^{\uparrow}$ (define $j^{\uparrow} = \{k \in \mathbb{N} : j|k\}$). This, in effect, works because 2 and 3 are incomparable with respect to $|$.

Integer Insights

But maybe more important than any specifically technical point is that the structure of the divisibility lattice suggests further philosophical insights via its connection to the informal BHK semantics.

For example, the lattice $\langle \mathbb{N}, | \rangle$ is not complete, in the sense that it is not guaranteed that a join (lcm) exists for arbitrary subsets of $\mathbb{N}$. But *should* a constructivist hold that an arbitrarily large (even infinite) combination of proofs or constructions is itself, necessarily, a proof or construction? Aren't these supposed to be finitary, surveyable objects? So, here is a way in which an integer structure might be used to help elaborate and fill-out an underspecified intuitive concept.

Relevance Logic

Before wrapping up this talk, I would like to say a few words about relevance logic in the integers. Relevance logic, though it has (modern) roots in early 20th century [4], really did not develop into its own until the 1960s and later.

Relevance logic is really a family of logics (like intermediate logics), all of which are supposed to be motivated by considerations of—get this—*relevance*.

Relevance Logic

The easiest way to get a grip on what relevance logic is about is to look at the sorts of things relevance logicians don't like (I'll focus just on formulae here). If what is most characteristic of intuitionistic logic is its rejection of excluded middle, perhaps what is most characteristic of relevance logic is the rejection of weakening:

$$\varphi \to (\psi \to \varphi) \tag{K}$$

In the presence of otherwise fairly unobjectionable axioms, this straightaway leads to 'irrelevant' theorems like $\varphi \to (\psi \to \psi)$ and $\varphi \to (\psi \vee \neg\psi)$.

Relevance Logic

Different relevance logicians have different stories to tell about what the operative notion of relevance is and what sort of connection(s) logical reasoning should preserve. But almost all relevance logicians agree that a necessary condition on a logic being relevant is that it satisfies the following condition:

DEFINITION (Variable Sharing Property)

A logic **L** satisfies the *variable sharing property* if $\varphi \to \psi$ is a theorem only if φ and ψ 'share content' in the sense of having at least one propositional variable in common.

Exact and Inexact Proof

The approach to relevance logic that I favor (which is—full disclosure—a bit nonstandard and yields nonstandard logics) is to think of it and intuitionistic logic as belonging to the same broadly constructive family. Where relevance logic differs from intuitionistic logic is, fundamentally, in resting on an *exact* rather than *inexact* notion of construction or proof (cf. Fine [7]).

Exact and Inexact Proof

You will recall that in the hereditary models for **J** I gave above, among other things, it was required that $j \in V(p)$ imply $\mathrm{lcm}(j,k) \in V(p)$. Colloquially, this is to require that, given any construction j establishing p, any combination of j with any other construction gives a construction establishing p. In fact, we can prove a fairly general heredity result for the hereditary models: for any formula φ, if $\models^{\mathfrak{M}}_{j} \varphi$, then $\models^{\mathfrak{M}}_{\mathrm{lcm}(j,k)} \varphi$ (cf. the analogous result in Kripke semantics for **J** [11]).

Exact and Inexact Proof

But think about the following example: the usual proof of the Fundamental Theorem of Arithmetic (that every positive integer greater than 1 has a unique product of primes factorization) really has two component proofs, one establishing the existence of a unique factorization, and the other establishing its uniqueness. The proof may fairly be regarded as a combination of those two proofs. Is the proof of the theorem also a proof of the existence claim? Of course it *implies* the existence claim, but the proof contains much that is extraneous to establishing that claim. In other words, it's not an exact proof of that result, and is, so to speak, largely irrelevant to it.

Exact and Inexact Proof

Speaking a bit loosely, relevance logic—anyway, the relevance logic **S** and its neighbors—is what you get when you throw out the heredity conditions from the hereditary models and keep the truth conditions (at least mostly) the same.[9] **S** and its neighbors are like **J**, but *exactified*.

[9] There are some subtle issues here and different decisions on these matters result in slightly different logics; consult Weiss [23]. I should remark that **S** as standardly formulated requires a different condition for $\wedge$ than that I presented above for **J**, and does not include $\bot$.

Completeness

Now, the logic **S**, originally proposed by Urquhart [19], is also complete with respect to $\langle \mathbb{N}, | \rangle$, but of course its models differ from the models for **J** mainly by not imposing heredity conditions.

Variable Sharing Property

And we can use the integers, once again, to obtain interesting results, like that (positive) **S** has the variable sharing property. The idea of the proof is simple and can be described very briefly. Suppose φ and ψ share no variables. For every variable p in φ, put $V(p)$ equal to the set of even numbers; for every variable p in ψ, put $V(p)$ equal to the set of odd numbers. It is readily shown that $\varphi \to \psi$ fails in this model—the 'key observation' is that $\mathrm{lcm}(j, k)$ is odd if and only if j and k are—whence $\varphi \to \psi$ is not a theorem of **S**, and hence, **S** has the variable sharing property. For more details, consult Weiss [21].

Concluding Remarks

I hope to have conveyed to you today that there are a number of interesting connections between integer structures and logic, and that the divisibility lattice, in particular, is an especially rich structure from the point of view of logic.

As a final remark, I would like to emphasize that, while many of these logics can be (and are) studied from a more abstract perspective, certain benefits attach to focusing on concrete structures like $\langle \mathbb{N}, | \rangle$. Not only are they more intuitive to work with technically, but by virtue of having additional structure, they can also suggest insights which a purely abstract approach might not.

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