

An Ample Budget of Paradoxes

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Birthday paradoxes

- *Paradox 1.* Among 23 people, more likely than not that two have same birthday.
- *Paradox 2.* At 1:31 pm on March 21, 2025, in Northfield, Minnesota, exactly 74 years had elapsed since my birth. Yet my birthday is March 22. Why? Where?

What's a paradox?

- *“A truth that cuts so strongly against the grain of common sense that it is hard to believe even when you are confronted with the proof.”* (Martin Gardner, 1959)
- *“... any conclusion that at first sounds absurd but that has an argument to sustain it ...”* (WVO Quine, 1950s?)

Absolutely paradoxical?

Benjamin Peirce

ForMemRS HonFRSE



Peirce c. 1870

Born

April 4, 1809

Salem, Massachusetts, U.S.

Died

October 6, 1880 (aged 71)

Cambridge, Massachusetts,
U.S.

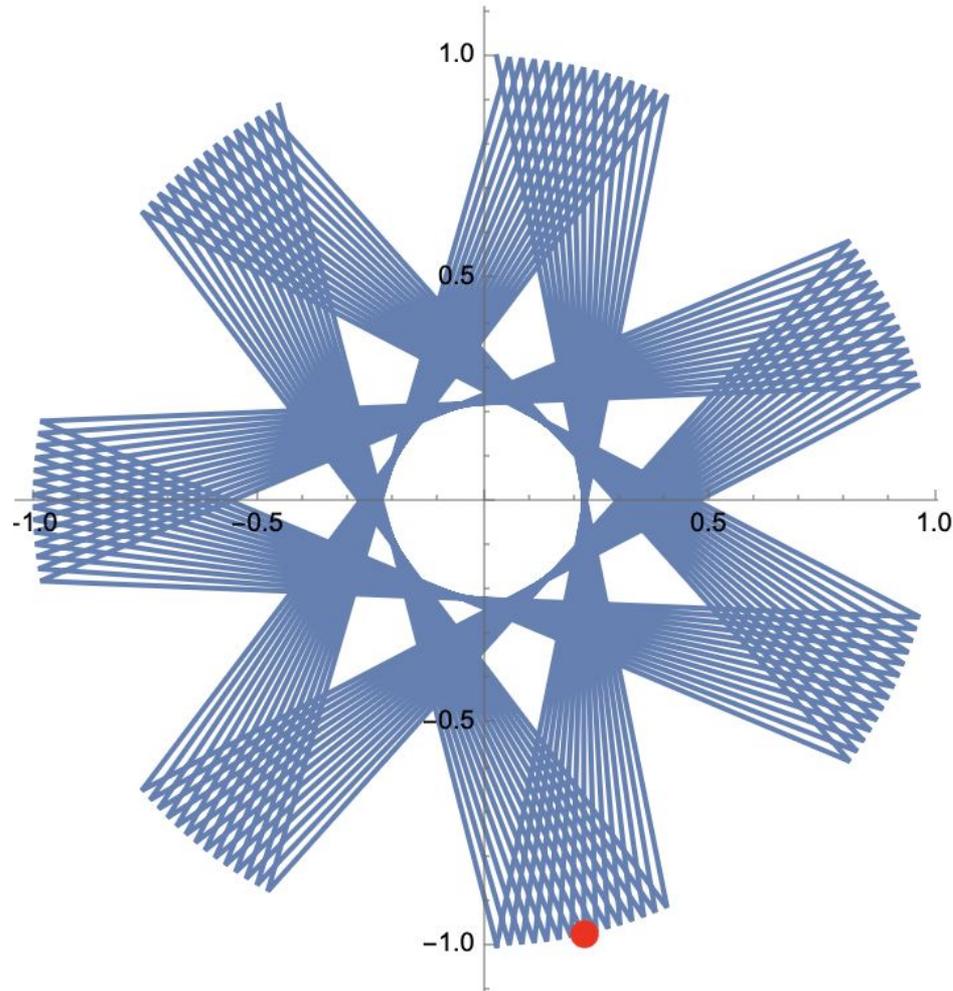
- Contemporary of De Morgan
- Taught Harvard, 50 years

$$e^{\pi/2} = \sqrt[i]{i}$$

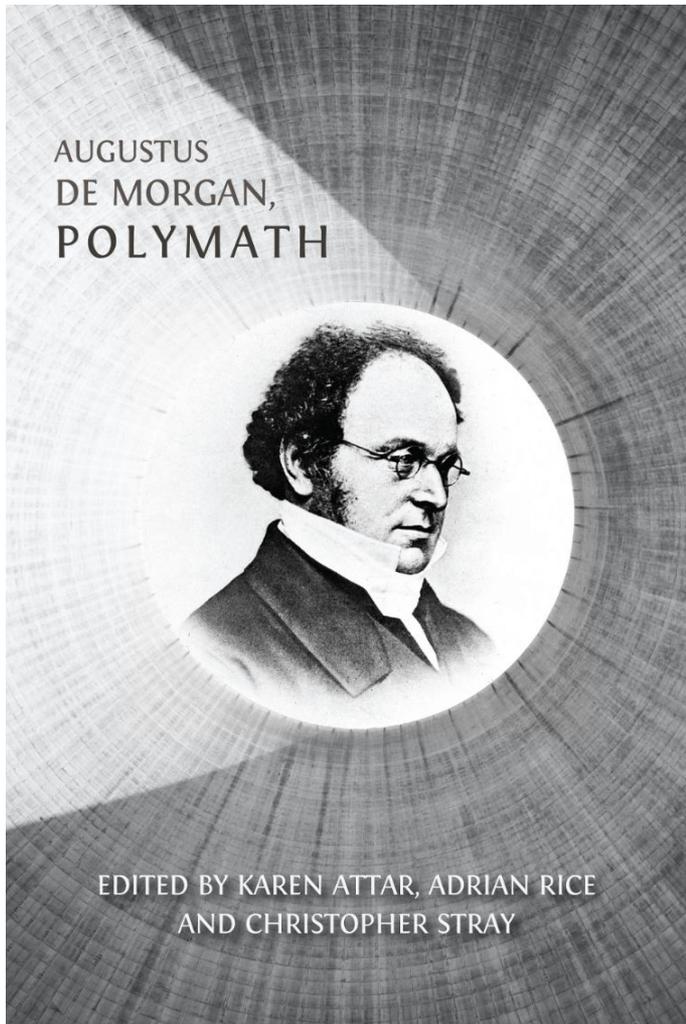
Peirce (1870s): ``Gentlemen, that is surely true. It is **absolutely paradoxical**, we can't understand it, and we don't know what it means, but we have proved it, and therefore we know it must be the truth."

Is this a paradox?

101 values of i^π , joined by lines

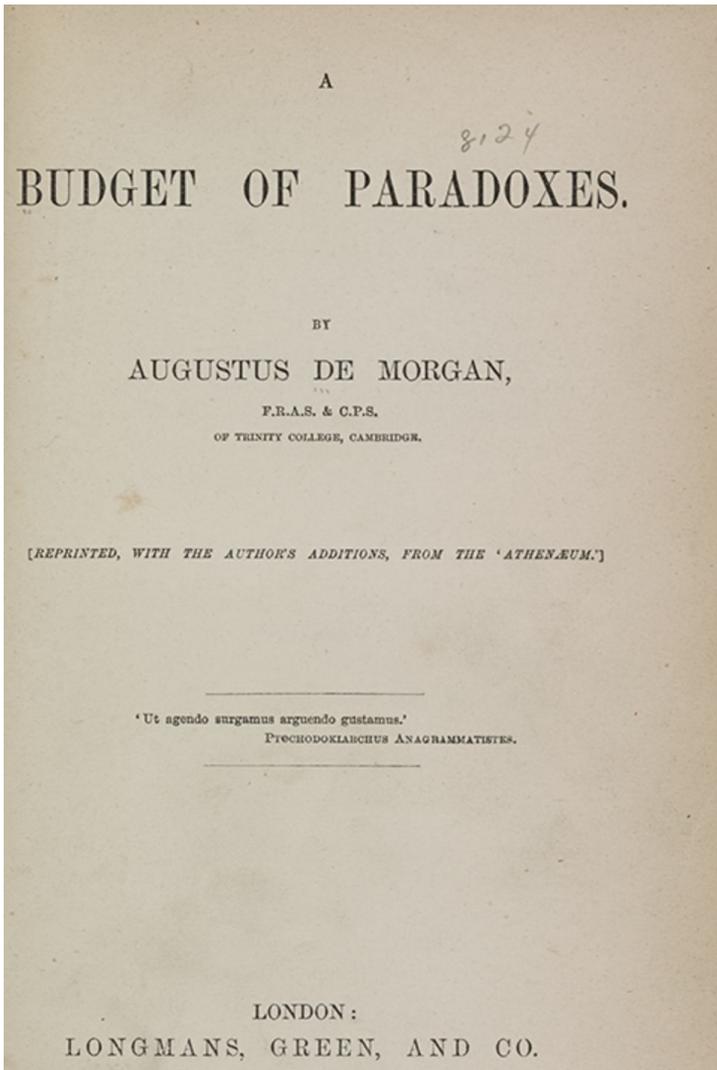


Augustus de Morgan (1806–1871)



- “laws” on conjunction, disjunction, negation
- born in South India; parents were “colonials”
- 2200+ publications, including 600+ for *Penny Cyclopaedia*
- wrote a calculus text

Augustus de Morgan (1806–1871)



- posthumous in book form
- “budget” \approx “collection”
- “paradoxers” \approx “cranks”
- ≥ 150 “paradoxers”, not all math
- angle trisection, values of π , ...
- values of π : < 3 , $25/8$, $22/7$, $\sqrt{10}=3.162$
- De G’s advice to paradoxers:
do your homework

Simpson's paradox(?)

Batter \ Year	1995		1996		Combined	
	Derek Jeter	12/48	.250	183/582	.314	195/630
David Justice	104/411	.253	45/140	.321	149/551	.270

- “paradox” persisted in 1997
- *A Mathematician at the Ballpark*, by Ken Ross. “No justice for Justice”
- Ross: This sort of thing happens about once/year for “interesting ballplayers”. Hmm ...

Simpson's paradox ... how often?

Ask Mathematica

```
battingavederek = .300;  battingavedave = .350;
```

```
derekhits1 = RandomVariate[BinomialDistribution[200, battingavederek]];
```

```
derekhits2 = RandomVariate[BinomialDistribution[500, battingavederek]];
```

```
derekave1 = derekhits1/200;
```

```
derekave2 = derekhits2/500;
```

```
derekave = (derekhits1 + derekhits2)/700;
```

```
davehits1 = RandomVariate[BinomialDistribution[500, battingavedave]];
```

```
davehits2 = RandomVariate[BinomialDistribution[200, battingavedave]];
```

```
daveave1 = davehits1/500;
```

```
daveave2 = davehits2/200;
```

```
daveave = (davehits1 + davehits2)/700;
```

etc.

Simpson's paradox ... how often?

1000 *Mathematica* runs, batting ave Derek .300,
batting ave Dave .350, 17 “Simpsons”

{{Derek:,0.310,0.356, 0. 343 },	{Dave:,0.324,0.370, 0.337}}
{{Derek:,0.295,0.348, 0. 333 },	{Dave:,0.322,0.350, 0.330}}
{{Derek:,0.285,0.380, 0. 353 },	{Dave:,0.330,0.400, 0.350}}
{{Derek:,0.335,0.362, 0. 354 },	{Dave:,0.342,0.365, 0.349}}
{{Derek:,0.275,0.334, 0. 317 },	{Dave:,0.306,0.340, 0.316}}
{{Derek:,0.305,0.360, 0. 344 },	{Dave:,0.316,0.410, 0.343}}
{{Derek:,0.325,0.376, 0. 361 },	{Dave:,0.330,0.380, 0.344}}
{{Derek:,0.295,0.358, 0. 340 },	{Dave:,0.322,0.365, 0.334}}
{{Derek:,0.290,0.342, 0. 327 },	{Dave:,0.296,0.350, 0.311}}
{{Derek:,0.325,0.376, 0. 361 },	{Dave:,0.342,0.400, 0.359}}
{{Derek:,0.325,0.354, 0. 346 },	{Dave:,0.330,0.355, 0.337}}
{{Derek:,0.320,0.374, 0. 359 },	{Dave:,0.328,0.385, 0.344}}
{{Derek:,0.310,0.346, 0. 336 },	{Dave:,0.318,0.375, 0.334}}
{{Derek:,0.330,0.368, 0. 357 },	{Dave:,0.332,0.385, 0.347}}
{{Derek:,0.290,0.374, 0. 350 },	{Dave:,0.314,0.385, 0.334}}
{{Derek:,0.280,0.362, 0. 339 },	{Dave:,0.324,0.365, 0.336}}
{{Derek:,0.295,0.366, 0. 346 },	{Dave:,0.306,0.375, 0.326}}

Types of paradox (channeling Quine)

Veridical paradox: a surprising claim is *true*

- *Example.* Frederic, from *Pirates of Penzance*, is 21 years old after only 5 birthdays.
- *Example.* There are different infinities (Cantor, ca. 1900)

Types of paradox

Falsidical paradox: a plausible argument is *false*

- *Example.* Achilles and tortoise (Zeno)
- *Example.* Motion is impossible (Zeno's arrow is still at every instant).
- *Example.* Proper subsets are “smaller” (Cantor)

Paradox platinum

AnTINomy: higher bar ... “produces self-contradiction by accepted ways of reasoning” (Quine)

Example: **Grelling’s paradox**

- “*autological*” adjective: applies to self, like “short”
- “*heterological*” adjective: does not apply to self, like “long”

Question: Is “heterological” heterological?

Russell's antinomy

Fact: Some sets contain themselves as elements; some don't.

Examples:

1. If $S = \{\text{all things that are not teacups}\}$, then $S \in S$.
2. If $T = \{\text{all teacups}\}$, then $T \notin T$.

The paradox: Let $R = \{x \mid x \notin x\}$. Is $R \in R$?

Answer: “Naive” set theory needs help.

Other antinomies

Strange properties of infinite sets (Cantor et al., from 1870s)

- countability of \mathbb{Q}
- uncountability of \mathbb{R}
- Banach–Tarski paradox
- axiom of choice, Zorn’s lemma . . .

Voting: an Arrow to the heart (1951)

Voting procedure: Each *individual* voter ranks all candidates. Procedure gives *society's* rankings, without “cycles.”

It's hard ...

<i>Preference order</i>	<i>Votes</i>	<i>Voting method</i>	Winner
Alice > Bob > Carol	4	Plurality	Alice
Alice > Carol > Bob	2	Anti-plurality	Bob
Carol > Bob > Alice	5	IRV	Carol
Bob > Carol > Alice	4	Borda Count	Bob

Arrow to the heart of the matter

Two reasonable requirements:

1. *Unanimity*, aka Pareto: If every voter prefers A to B, then B should not be elected over A.
2. *Independence from Irrelevant Alternatives (IIA)*: If A beat B without C, then A still beats B if C enters—C can't cost A an election unless C wins.

Arrow to the heart of the matter

Arrow's theorem (1951): With ≥ 3 candidates, only one voting method satisfies both Unanimity and IIA: *Dictatorship*.

Question: Is this a paradox? An antinomy? Discuss.

Cantor, Kronecker ...

Irrational numbers

what are they?

how many?

are they legit?