The Divine Madness

Mathematics, Myths and Metaphors

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Albert North Whitehead:

"... to construct a history of thought without profound study of the mathematical ideas of successive epochs ... is certainly analogous to cutting out the part of Ophelia [from the play 'Hamlet'].

The simile is singularly exact. For Ophelia is quite essential to the play, she is very charming – and a little mad.

Let us grant that the pursuit of mathematics is a divine madness of the human spirit ..."

—*Science and the Modern World, 1925*

Definition of Myth

Webster: "A popular belief or tradition embodying the ideals and institutions of a society or segment of society."

The Basic Myth:

Mathematics is a Good Thing

Corollary:
Mathematicians are Good People

A Dissenting View

Saint Augustine:

"Quapropter bono Christiano, sive mathematici sive quilibet impie divinatium, maxime dicentes vera, cavendi sunt, ne consortio daemoniorum animam deceptam pacto quodam societatis inretiant."

(Good Christians should avoid mathematicians and all impious soothsayers, taking care not to consort with those demons and deceptive spirits whose society will entrap them.)

DeGenesi ad Litteram, 4th century

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But on the other hand,

Sir Arthur Eddington:

"I cannot accept Jeans's view that mathematical conceptions appear in physics because it deals with a universe created by a Pure Mathematician; my opinion of pure mathematicians, though respectful, is not so exalted as all that."

—*The Philosophy of Physical Science, 1939*

The World and Mathematics

Pythagoras' Myth:

"The world is number."

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A big number:

"I believe there are
15,747,724,136,275,002,577,605,653,961,181,555,468,
044,717,914,527,116,709,366,231,425,076,185,631,031,296
protons in the universe, and the same number of electrons."

—Sir Arthur Eddington, 1939

(The number is said to be 136×2^{256})

The Myths of Certainty

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1933:

"Our experience hitherto justifies us in believing that nature is the realization of the simplest conceivable mathematical ideas. . ."

"We can discover by means of purely mathematical constructions . . . the key to understanding natural phenomena. . ."

"Experience remains, of course, the sole criterion of the physical utility of a mathematical construction. But the creative principle resides in mathematics."

A modern descendant of the Myth of Certainty

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Computer Calculations are Reliable

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Example: A Maple calculation (circa 1990):

Input four numbers and test for primes:

$> A := 34816783 :$

$> B := 29698715047 :$

$> C := 120979604904878607889 :$

$> D := 1031956000233747418830001 :$

$>$ isprime (A); true

$>$ isprime (B); true

$>$ isprime (C); true

$>$ isprime (D); true

A, B, C, D are distinct primes

Compute products:

$> A * D;$ 359293881256863331582145720578 3

$> B * C;$ 359293881256863331582145720578 3

Compare products:

$> A * D - B * C;$ 0

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Therefore $AD = BC$

Prime Factorization is not unique!

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Hermite

"...the integers have an existence outside ourselves which they impose with the same predetermined necessity as sodium or potassium."

—*Letter to Stieltjes*

René Thom:

René Thom: "...there are formal structures, in fact geometrical objects, in biology which prescribe the only possible forms capable of having a self-reproducing dynamic in a given environment."

—*Structural Stability and Morphogenesis, 1975*

Are sets real?

Gödel:

- "The assumption of [sets] is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence."

  —*On Russell's Mathematical Logic, 1944*

- "They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions."

  —*What is Cantor's Continuum Problem?, 1947*

Lebesgue *vs.* Hadamard on Axiom of Choice, 1905

Lebesgue:

"The question comes down to this, which is hardly new: *Can one prove the existence of a mathematical object without defining it?*... it is impossible to demonstrate the existence of an object without defining it."

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But:

Hadamard

"The existence... is a fact like any other."

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(c) Every true mathematical statement has a proof in ZFC set theory (including this one?)

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H. Lenstra

"The problem with wrong proofs to correct statements is that it is hard to give a counterexample."'

— Berkeley colloquium, 1996

"A mathematics lecture without a proof is like a movie without a love scene"

—AMS lecture, San diego, 2002

G.H. Hardy:

"There is, strictly, no such thing as mathematical proof: we can, in the last analysis, do nothing but point.

Proofs are what Littlewood and I call 'gas':

—rhetorical flourishes designed to affect psychology, pictures on board in the lecture,

—devices to stimulate the imagination of pupils."

—A Mathematicians's Apology, 1929

Richard Hamming:

"Some people believe that a theorem is proved when a logically correct proof is given; but some people believe a theorem is proved only when the student sees why it is inevitably true. The author tends to belong to this second school of thought."

— *Coding and Information Theory, 1980*

Quoting Hilbert: "When rigour enters, meaning departs."

On non-trisection of angles: "It is not a theorem of the real world".

— *MAA Monthly, 1998*

The Myths of Proof

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Necessity:

Arguments unsupported by correct proofs are not accepted.

Proofs and Formality

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Can we use:

- the Axiom of Choice?
- the Continuum Hypothesis?
- the Law of the Excluded Middle?
- Transfinite Induction?

Which Formal System?

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Which Formal System?

The Myths of Consistency

- Mathematics is consistent
- Every true statement can be proved in a consistent formal system.
- We can find such a system . . . . . . **and** prove it consistent.

Corollary to Consistency

The Myth of Universality: The truths of mathematics are universally and eternally valid.

- Is our Mathematics valid on Mars?

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- "Commensurability of lengths" was true for the Pythagoreans, but not for Plato or us: It implies $\sqrt{2}$ is rational.

Perhaps theorems should have *expiration dates*, requiring them to be reproved periodically?

Does Mathematics Have a Philosophical Foundation?

Some Philosophical Myths

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- **Formalism:** Mathematics is the study of formal systems of symbols whose interpretation is irrelevant. Its correctness depends only on syntax.
- **Humanism:** Mathematics is a human activity, hence we cannot expect consistency or timelessness.

These cannot all be right!

Mathmematics without Philosophy?

Hilary Putnam:

"I do not think that the difficulties that philosophy finds with classical mathematics today are genuine difficulties; and I think that the philosophical interpretations of mathematics that we are being offered on every hand are wrong, and that 'philosophical interpretation' is just what mathematics doesn't need."

—*Mathematics Without Foundations, 1972*

Why do we think Mathematics is real?

Quine and Putnam's Realism:

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- **—but we do not have to define "real"!**

Must Mathematics be rigorous?

Paul Halmos:

- "Applied mathematics is bad mathematics"
- "It isn't really . . . but it's different."
- "But usually, applied mathematics is bad mathematics just the same."

—in 'Mathematics Tomorrow', 1981

Much Mathematics is nonrigorous:

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- Heaviside, Dirac, Feynman, ...
- The Journal of Experimental Mathematics

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- **In many scientific papers the mathematics is only a Narrative, Model, or Metaphor. Rather than proving something, it tells a story about the world.**

Mathematics as Metaphor

What is a Metaphor?

1. A figure of speech in which a word or phrase literally denoting one kind of object or idea is used in place of another to suggest a likeness or analogy between them ("drowning in money").
2. An object, activity, or idea treated as a metaphor. From Greek *metapherein*, to transfer.

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Metaphors preserve important inferences.

"A man's reach should exceed his grasp, or what's a metaphor?"

—*Not from Matthew Arnold*

Some Metaphors:

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- population is continuous function of real variable (!)

Mathematics as Models

Tentative definition:

A Model is an explicit, extended metaphor.

- Arithmetic is a model for Counting
- Algebra is a model for Arithmetic
- Geometry is a model for Seeing, Measuring, Drawing, Space-time ...
- Group Theory is a model for Permuting, Moving, Transforming ...

More Models:

- The field of Real Numbers is a model for distance, velocity, acceleration, heat, pressure, curvature, density, ...
- Measure Theory is a model for Probability
- Probability Theory is a model for Quantum Mechanics
- Set Theory is a model for ...what we do when we think about sets—
- Set Theory is a model for Mathematics
- Category Theory is a model for Set Theory

Mathematics is a collection of Models.

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Reality is just another model

—Graffito in Berkeley Math Department, 1970