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Classic for Tomorrow

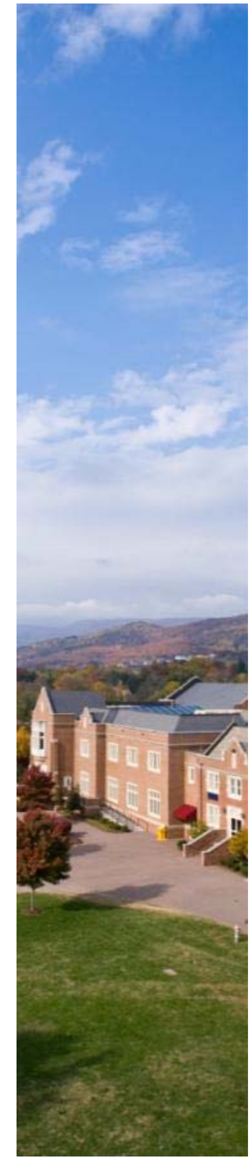
Combining Hands-On Probability with Calculations: Enhancing Quantitative Literacy through Textbook and Course Design

Dr. David G. Taylor

Today's Talk

- Part 1: About the "Mathematics of Games" Course
- Part 2: About Writing a Textbook that Supports this Course

- Course and Book Goals:
  - Promote quantitative literacy through seeing mathematics in action and learning about the mathematics needed to study games.
  - Eliminate students' asking "why are we learning this" in mathematics courses by focusing on question development first.



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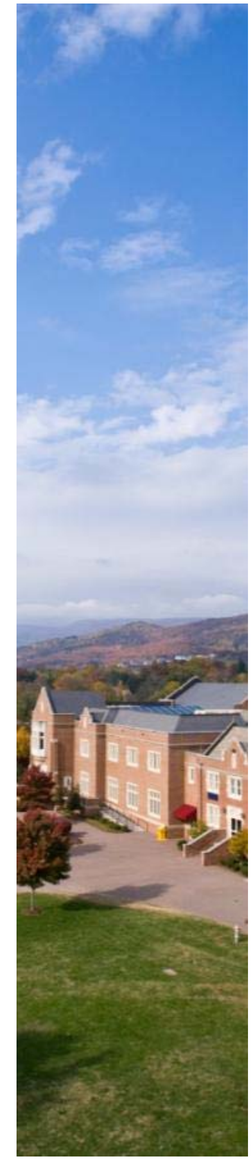
Part 1: About the Course

- Our Special Term at Roanoke College
  - We have an “Intensive Learning” term during May.
  - Courses emphasize learning opportunities “beyond the lecture” and focusing on a topic in ways that are not possible during the regular academic year.
  - Courses are fully-immersive in that students take only one “May Term” class at a time.
  - There are travel courses, field-trip courses, and on-campus courses available for students.
  - Courses must still adhere to 39 contact hours (the same as for a regular course at Roanoke College).



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Part 1: About the Course

- INQ 177 "The Mathematics of Games"
  - Three weeks, five days a week, for three hours each day.
  - First Half of the Course:
    - Study of games (roulette, craps, poker, blackjack, monopoly, scrabble, and more) preceded by a day of "coins, dice, and candy."
  - Midterm Exam, Group Presentation Topic Choices
  - Second Half of the Course:
    - Lecture/discussion of other games and time for groups to work.
  - Group Presentations and Final Examination
  - Note: I form the groups before May Term starts and balance out the groups so that each has at least one "quantitative" person in it.

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Part 1: About the Course

- First Half of the Course
  - Each day has an activity worksheet that students work on in groups of 4.
  - The activity has students
    - first play a game and keeping track of certain events guided by the worksheet,
    - then discuss observations in their groups, guided by the worksheet,
    - and finally learning and/or using mathematics to answer probability or combinatorics questions exactly.
  - This repeats for the activity, building up to "tougher" problems.
  - At the end, students reflect on "hands on" probabilities versus calculated probabilities.
  - Homework is given and discussed at the start of the next day. The worksheets and homework are graded.

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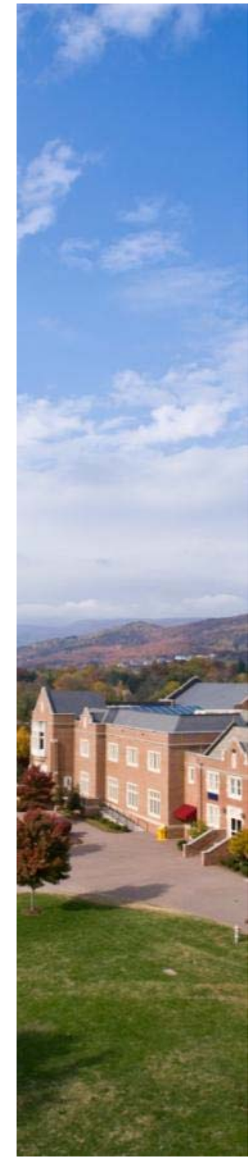
Part 1: About the Course

- Example: Craps Day (Day Three)
  - Students play through "rounds" of craps, recording whether the pass line (or don't pass line) wins.
  - With the entire class's data available, in groups, students discuss the odds for the pass line.
  - Students then learn advanced (for them) expected value calculations to find the exact odds.
  - Afterwards, they move on to other bet types in craps by playing, discussing, and calculating.
  - Bonus: Through calculations, students understand and appreciate the "oddities" in some bets (such as the don't pass line bet drawing on a roll of 12).



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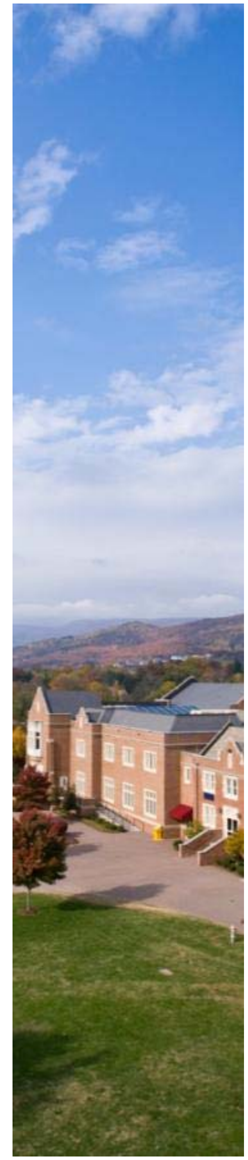
Part 1: About the Course

- Second Half of the Course
  - In their groups, students select a game to research and study intensely.
  - Class time migrates to more lecture and discussion and less on activities (groups get half of the class time to work on their presentations).
    - I use this time to choose games or topics that reinforce ideas the class may have struggled with.
  - Group presentations precede the final exam.
    - Groups present their game, have classmates play it with a worksheet, then present about the mathematics involved in the game.



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Part 1: About the Course

- Intended Learning Outcomes
  - By the end of this course, successful students will be able to
    - demonstrate knowledge of basic probability as it applies to gambling and games.
    - form connections between observed probabilities and actual probabilities.
    - assess a game of chance and analyze strategies related to betting.
    - communicate effectively the mathematics and ideas behind games of chance.
    - critically reflect on the role of mathematics in chance.

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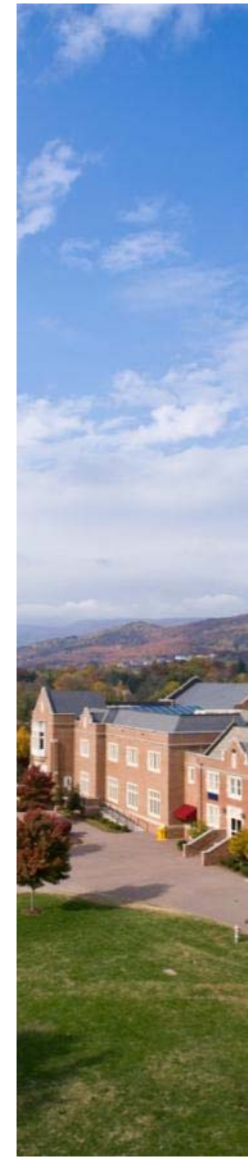
Part 1: About the Course

- Student Feedback
  - Aspect of the Course Most Helpful
    - the material
    - The daily activities were well thought out and helped us through the games and saved time.
    - talking about the subject matter at hand in class
    - Being able to apply the concepts with the daily activities and going over homework problems
    - Games activities
  - How did Course Experience Differ from Expectations
    - it was much more fun than expected
    - This course exceeded my expectations I learned a lot about the games themselves as well as the math that is behind them.
    - im much smarter at poker and know what to bet on in Vegas
    - I did not know how math would be applied to gambling and games.
    - This course is as fun as I expected



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Part 2: About the Book

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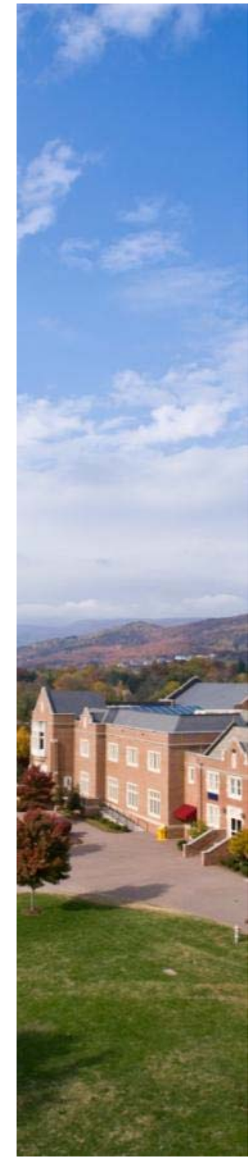
Part 2: About the Book

- Most textbooks introduce a mathematical topic, give formulas, then focus on examples. This gives students the impression that we “force” mathematics on the “real world.”
- Instead, I wanted to develop mathematics in response to very specific questions about the “real world.”
- The goal for the book is to first introduce a topic (game), explain the topic in detail (rules), then ask a natural question about the topic for which some mathematical concept needs to be developed (expected value).



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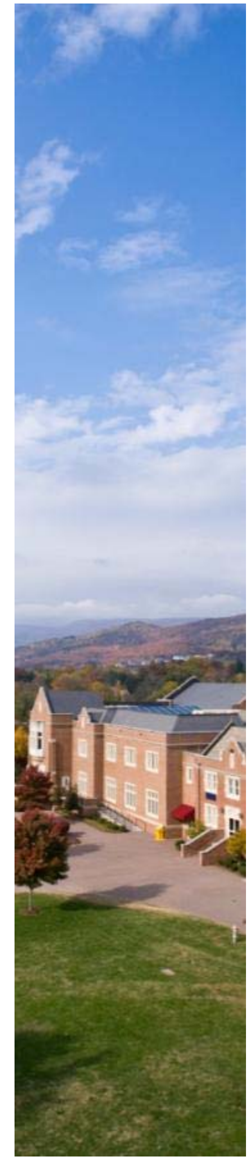
Part 2: About the Book

- Development
  - The book was developed to be used in this course but isn't made to be used solely for the course!
  - Writing such a book made me think about the course "backwards" and "forwards" in terms of what mathematical topics should the book have, then what games best illustrate those topics, and then what questions for those games are natural.
  - Starting with wanting to always introduce mathematics as a solution to natural questions and still be something readable and understandable was not always possible.
    - The very notion of what probability is was better done using dice and coins rather than trying to connect to a natural question to a true game (flipping coins and rolling dice are understandable fairly easily already).
    - I did not want the same game to appear in two places unless it was to show neat connections.



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Part 2: About the Book

- Chapter 1: Dice, Coins, and Candy
  - Games: Coins, Dice
  - Mathematics: Probability Definition, Rules, and Shortcuts
- Chapter 2: Wheels and More Dice
  - Games: Roulette, Craps
  - Mathematics: Expected Value
- Chapter 3: Counting the Pokers
  - Games: Poker (Five- and Seven-Card)
  - Mathematics: Counting, Binomial Coefficients, Game Theory (Bluffing)
- Chapter 4: Windmills and Black Jacks?
  - Games: Blackjack, Blackjack Switch
  - Mathematics: Probability Trees, Decision Making

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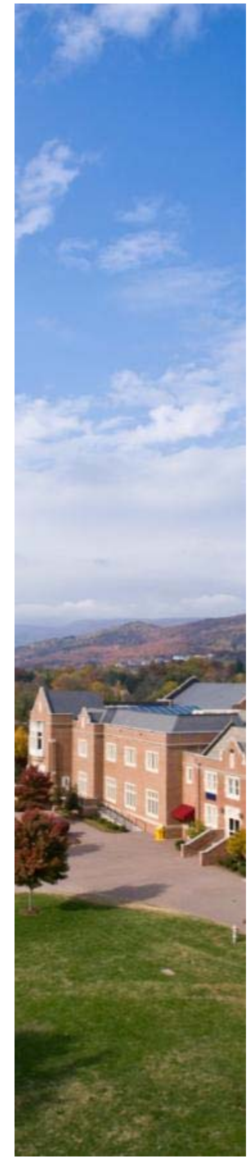
- Chapter 5: More Fun Dice!
  - Games: Liar’s Dice, Yahtzee, Zombie Dice
  - Mathematics: Binomial and Multinomial Distributions, Permutations, Summations
- Chapter 6: Board Games, Not “Bored” Games
  - Games: Yahtzee, Pay Day, Monopoly
  - Mathematics: Probability Matrices, Steady-State Equilibria
- Chapter 7: Can You Bet and Win?
  - “Games”: Betting Systems and Gambler’s Ruin
  - Mathematics: Modeling, Recursion
- Chapter 8: There Are More Games!
  - Games: Lottery, Bingo, Baccarat, Farkle, Backgammon, Memory
  - Mathematics: Advanced Versions of Prior Things



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Part 2: About the Book

- Example (Chapter 2, Expected Value)

Chapter 2

Wheels and More Dice

Roulette

The game of roulette has been around for over 200 years; several games throughout Europe in the 17th and 18th centuries contributed to what has been standardized in the last century. Historians regularly credit mathematician (and inventor, writer, philosopher, and physicist) Blaise Pascal's experiments with perpetual motion as the impetus behind the wheel itself, and the game that most resembles current roulette started to become famous in Paris during the late 1700s. Indeed, the name itself comes from the French word for "little wheel."

FIGURE 2.1: Roulette

The game is rather easy to play, and the mathematics behind the bets will give us a glimpse of exactly how a casino is able to stay in business. If you're lingering around the floor of a casino in Las Vegas, Atlantic City, or anywhere else, it will be very hard to not notice the roulette table! As the ball rolls around the outside of the wheel, the anticipation builds at the table, and as the ball begins to settle into the middle, bouncing around several times and hitting a few spokes, thousands of dollars can be won or lost! Some grumbles and perhaps a lot of excitement can be heard as the ball finally settles into a slot as the croupier yells "thirty-three black!"

We will be working with what we will call American roulette; the exercises at the end of the chapter will explore variations of roulette (in particular European roulette is traditionally slightly different than American roulette, as you will see in the exercises). As you can see in Figure 2.2, there are the numbers 1 through 36, separated into half red and half black numbers, and two additional (green) spaces, one called 0 and the other 00 (double zero). Note for the purposes of the "even" bet we'll discuss below, neither 0 nor 00 are considered even during a game of roulette (despite the fact that 0 is an even number to mathematicians and pretty much everyone in the world *but* roulette players). As another historical note, the original game in Europe did include both the 0 and 00, with the 0 space colored red and the 00 space colored black; however, both spaces were considered the "bank's numbers"

and did not pay out for the player, even when betting on red or black. The spaces were colored green very shortly after to avoid confusion (the current European version, where 00 is not included, was introduced at the time to compete with casinos that offered roulette with both 0 and 00; the reason that this strategy worked for casinos will become apparent later).

Playing roulette is fairly simple, and the payouts are fairly easy to understand as well, so this early casino game will be a great introduction to some of the mathematical tools we will end up using for most of our game analysis. Roulette is played in rounds, and a round starts by the croupier indicating that it is time to place bets. Shortly after, he or she will give the wheel a spin, start the ball rolling in the opposite direction, and very soon call out "no more bets" at which time the fate of your bets is about to be determined. After the result has been determined, the croupier will take away the losing bets, and pay out on the winning bets. Then a new round will shortly start.

Odds

One example bet that we can use to explain payout odds and typical casino behavior is a bet on the single number 23 (or any other number). What will become especially important in our study of casino games (and the single source of the "house advantage") will be the difference between true mathematical odds and casino payout odds for the various bets.

Definition 2.1 The *true mathematical odds* or *true odds* of an event or wager is a pair of numbers, separated by a colon, as in $X : Y$, where the first number X represents the number of ways to not win the wager and the second number Y represents the number of ways to win the wager.

It is easy to think of true odds as (# of ways to lose):(# of ways to win), which will be correct as long as there are no opportunities for the wager to be a push (tie).

Example 2.1 For a bet of 23 at roulette, since there are 38 spaces on the wheel and only 1 way to win, there are 37 ways for us *not* to win; the true mathematical odds for a single number bet at American roulette is 37:1 (in Definition 2.1, X is 37 and Y is 1).

Spoken out loud, you would read "37:1" as "thirty-seven to one." One important note here; even though the probability of getting a 23 on one spin of the wheel is indeed $\frac{1}{38}$ (one physical outcome gives success, out of a possible 38 outcomes), the true mathematical odds are 37:1 since, while the ideas are very, very closely related, the meaning is slightly different. To convert true mathematical odds of 3:2 to a probability of winning, note that since there are 3 ways of losing and 2 ways of winning, there are 5 possible outcomes. Then the probability of winning is 2/5 and *not* 2/3.

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Part 2: About the Book

Example (Chapter 2, Expected Value)

Theorem 2.2 An event with true mathematical odds $X : Y$ can be converted into a probability of winning as

$$\frac{Y}{X+Y}$$

and an event with probability A/B can be converted into true mathematical odds of $(B - A) : A$.

It's also worth mentioning here that the true mathematical odds of say 4:2 and 2:1 are identical for all purposes that we will use them for (use Theorem 2.2 to convert each of those odds to probabilities and you will get the same value). At this point, you might wonder where the house gets its advantage? If you wager \$1.00 on 23 at roulette, and it actually does come up, how much money will you now have? The following definition will help.

Definition 2.3 The *casino payout* or *casino or house odds* for a wager or event, expressed also in the form $X : Y$, means that the casino will pay you winnings of $\$X$ for each $\$Y$ you wager.

A quick study of roulette payouts will tell you that the casino payout is listed at 35:1, or 35 to 1. This means, using Definition 2.3, that on your \$1.00 wager, the casino will give you an additional \$35.00 for winning; you also get your original \$1.00 back, giving you a total of \$36.00 returned on that \$1.00 bet. There are two important things to mention here; *first, unless otherwise specified, it is assumed that your original wager will be returned to you in addition to your winnings.* It should also be common sense, for example, that if instead you wager \$2.00 on number 23 and win, that the 35:1 house odds means that you will receive \$70.00 in winnings, since, as per Definition 2.3, you receive winnings of \$35.00 for each \$1.00 that you bet.

Let's now take a look at all of the types of bets and payouts for American roulette as depicted in Figure 2.2. A bet on a single number, as we discussed above, is pictured as item ① and has a 35:1 payout. Item ② is a pair of adjacent numbers, such as betting on 19 and 20, or 7 and 10, or, as pictured, 2 and 3. The payout for a pair of

FIGURE 2.2: Roulette Bets

numbers is 17:1 as is often called a "split" bet. A wager of three numbers in a row, such as ③ in the figure, has a payout of 11:1 and is called taking the "street." Choosing a bet on four numbers of a square, such as 17, 18, 20, and 21 pictured as bet ④ is called a "corner" bet and has a 8:1 payout from the casino. Item number ⑤ is the "first five" bet, with house odds of 6:1, and, as we will see, is the single worst bet that can be made at the game of roulette. Bet ⑥ is an example of taking a pair of adjacent streets, often called a "six line" bet that pays 5:1. Two types of bets, ⑦ and ⑧, have a 2:1 payout. These wagers consist of first dozen, second dozen, or third dozen (bets identified by ⑦), and each of the three columns of numbers depicted as ⑧ (one example is the collection of numbers 1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, and 34). Finally, there are the three different types of even money bets (defined as having a payout of 1:1): first 18 or last 18 (⑨ and ⑩), red or black (⑪), and even or odd (⑫). A summary of this information is provided in Table 2.1.

TABLE 2.1: Roulette Bets

Bet	Payout
Single Number	35:1
Adjacent Numbers	17:1
Three Numbers	11:1
Four Numbers	8:1
Five Numbers	6:1
Six Numbers	5:1
Dozens	2:1
Columns	2:1
First/Last 18	1:1
Red/Black	1:1
Even/Odd	1:1

What questions can we ask about roulette? Determining the probabilities of any of these bets should not be difficult at this point. With only 38 spaces on the American roulette wheel, all of these can be computed directly without the need of the additive rule or other "heavy" machinery from Chapter 1. However, as you hopefully have experienced in previous mathematics classes, practice is the key to true understanding. It is definitely worth the time to make sure that computing the probability of any of these bets is straightforward and simple. We will do one together, and the rest are left for you as an exercise at the end of the chapter.

Example 2.2 To determine the probability of the "dozens" bet for American roulette, we recall that there are 12 possibilities for our bet to be successful (for example, for the "first dozens" bet, any of the numbers from 1 to 12 will make us happy) out of a possible 38 slots on the wheel. This gives us a $\frac{12}{38} \approx 31.58\%$ chance of success.

At this point, we are ready to ask the next question of interest to us in our study of roulette, and a question that will be of interest to us for any casino game we may study.

Question: What is the house advantage for a bet on a single number in roulette?

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Part 2: About the Book

Example (Chapter 2, Expected Value)

Expected Winnings (Expected Value)

Before we answer our specific question, imagine yourself in a scenario where you are about to flip a coin, and before you do so, a friend offers you a "friendly" bet. If the coin comes up heads, he will pay you \$1.00, and if the coin comes up tails, you have to pay him \$1.00. Is this a fair bet? Most people would look at this and quickly agree that it is indeed a fair bet, but the better question is: why? When I ask my students, I usually get a response such as "since half of the time you win a dollar and half of the time you lose a dollar, in the long run no one makes (or loses) money." If you had a similar thought, you're well ahead of the game! The idea here is that we could easily compute the *average* winnings, which tells us a lot about this bet.

$$\text{Average Winnings: } \underbrace{\left(\frac{1}{2}\right)\cdot \$1.00}_{\text{heads}} + \underbrace{\left(\frac{1}{2}\right)\cdot(-\$1.00)}_{\text{tails}} = \$0.00$$

This tells us that the *expected winnings* (or *expected value*) for this bet is \$0.00, meaning no advantage for you or for your friend. How do things change when the situation isn't evenly balanced (as with a coin)? Instead of using a simple average, we can use a *weighted average* to determine the expected winnings. My guess is that you are already familiar with this concept but perhaps the name doesn't ring a bell; in the classroom, when computing grades, oftentimes "exams" will count more than "participation," so your final grade average in the course is computed by applying a higher weight for exams than participation. We do the same here to answer our question.

Example 2.3 To determine the house advantage on a \$1.00 single number bet, say 34, we need to know the probability of winning and losing. Since only the number 34 will lead us to a win, the probability of hitting that single number (along with any other single number) is $\frac{1}{38}$. Any of the other 37 results on the wheel will cause us to lose, with probability $\frac{37}{38}$ (or, as a faithful reader from Chapter 1 will remember, this is also $1 - \frac{1}{38}$ since in this scenario, it is only possible to win or lose; there are no ties). If we win, the payout is 35:1, so we will win \$35.00. If we lose, we simply lose our \$1.00 wager.

$$\text{Expected Winnings: } \underbrace{\left(\frac{1}{38}\right)\cdot \$35.00}_{\text{win}} + \underbrace{\left(\frac{37}{38}\right)\cdot(-\$1.00)}_{\text{lose}} \approx -\$0.0526$$

From our calculation, it appears that, in the long run, we lose about 5.26 cents per single number wager if we bet \$1.00. Looking at the expected winnings calculation, you should notice that if we wagered \$10.00 instead of \$1.00, each number in the formula would be multiplied by 10, including the answer. We can then safely say that the house advantage on a single number bet in roulette is \$0.0526 per dollar, or 5.26% if you prefer.

Question: How does a casino maintain an edge on new games or wagers?

As a small example, we will consider the idea of adding a new bet to roulette that wins whenever a prime number comes up on the wheel. You may remember that a prime number is a positive integer greater than one whose only divisors are itself and one. From the collection of numbers 1 through 36, the set of prime numbers consists of 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, and 35, for a total of 11 numbers. How much should the casino pay out for this wager?

FIGURE 2.3: Expected Winnings on Prime Number

Example 2.6 Given that there are 11 ways to win, so out of the 38 possible outcomes for American roulette, there must be 27 ways to lose. This gives us true mathematical odds of 27:11, or, by dividing both numbers by 11, equivalent true mathematical odds of about 2.45:1. To determine the expected winnings, consider a payout of $\$W$ on a \$1.00 wager for this bet. We then get

$$\text{Expected Winnings} = \underbrace{\left(\frac{11}{38}\right)\cdot \$W}_{\text{win}} + \underbrace{\left(\frac{27}{38}\right)\cdot(-\$1.00)}_{\text{lose}} = \frac{11W - 27}{38}$$

where we drop the dollar sign on the right to keep the formula from looking too ugly. Pictured in Figure 2.3 is this quantity, plotted for values of W ranging from \$0.50 to \$4.00. Do you find it rather curious that the expected winnings are \$0.00 when $W = 2.45$? I would imagine by now that you don't find that surprising! If the casino wants to guarantee a house advantage for a game, any payout that is below the true mathematical odds will do.

To see how the casino can really affect their bottom line, we try two values of $\$W$ that seem plausible. Note again that for true mathematical odds of X:1, as long as the casino payout is set at Y:1 where $Y < X$, the casino will make money. For a payout of 2:1, we get

$$\text{Expected Winnings } (W = 2) = \underbrace{\left(\frac{11}{38}\right)\cdot \$2.00}_{\text{win}} + \underbrace{\left(\frac{27}{38}\right)\cdot(-\$1.00)}_{\text{lose}} = -\$0.132$$

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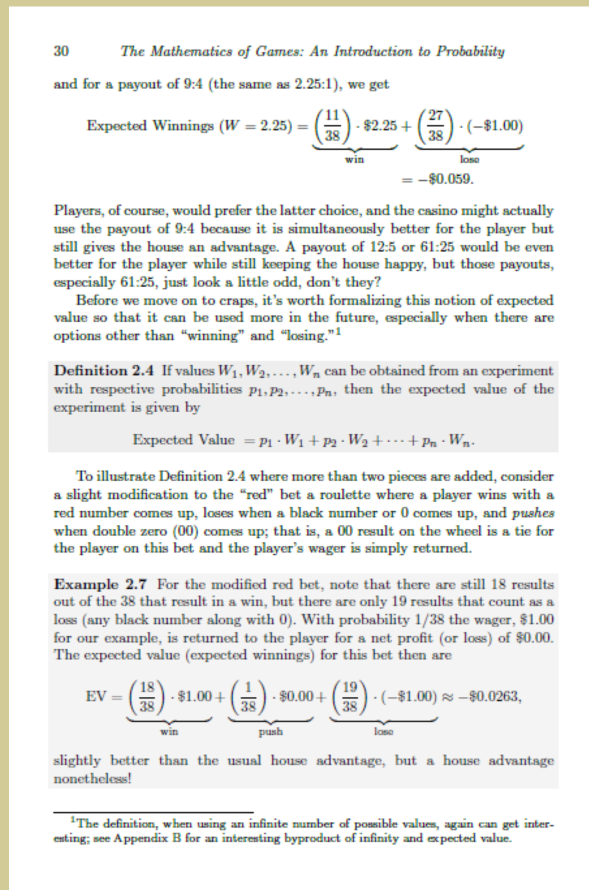
Part 2: About the Book

- Example (Chapter 2, Expected Value)



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and for a payout of 9:4 (the same as 2.25:1), we get

$$\text{Expected Winnings } (W = 2.25) = \underbrace{\left(\frac{11}{38}\right) \cdot \$2.25}_{\text{win}} + \underbrace{\left(\frac{27}{38}\right) \cdot (-\$1.00)}_{\text{lose}} = -\$0.059.$$

Players, of course, would prefer the latter choice, and the casino might actually use the payout of 9:4 because it is simultaneously better for the player but still gives the house an advantage. A payout of 12:5 or 61:25 would be even better for the player while still keeping the house happy, but those payouts, especially 61:25, just look a little odd, don't they?

Before we move on to craps, it's worth formalizing this notion of expected value so that it can be used more in the future, especially when there are options other than "winning" and "losing."[1]

Definition 2.4 If values $W_1, W_2, \ldots, W_n$ can be obtained from an experiment with respective probabilities $p_1, p_2, \ldots, p_n$, then the expected value of the experiment is given by

$$\text{Expected Value } = p_1 \cdot W_1 + p_2 \cdot W_2 + \cdots + p_n \cdot W_n.$$

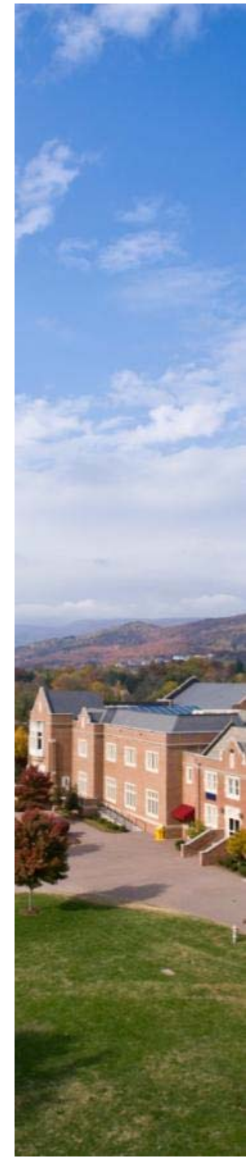
To illustrate Definition 2.4 where more than two pieces are added, consider a slight modification to the "red" bet a roulette where a player wins with a red number comes up, loses when a black number or 0 comes up, and *pushes* when double zero (00) comes up; that is, a 00 result on the wheel is a tie for the player on this bet and the player's wager is simply returned.

Example 2.7 For the modified red bet, note that there are still 18 results out of the 38 that result in a win, but there are only 19 results that count as a loss (any black number along with 0). With probability 1/38 the wager, \$1.00 for our example, is returned to the player for a net profit (or loss) of \$0.00. The expected value (expected winnings) for this bet then are

$$\text{EV} = \underbrace{\left(\frac{18}{38}\right) \cdot \$1.00}_{\text{win}} + \underbrace{\left(\frac{1}{38}\right) \cdot \$0.00}_{\text{push}} + \underbrace{\left(\frac{19}{38}\right) \cdot (-\$1.00)}_{\text{lose}} \approx -\$0.0263,$$

slightly better than the usual house advantage, but a house advantage nonetheless!

[1] The definition, when using an infinite number of possible values, again can get interesting; see Appendix B for an interesting byproduct of infinity and expected value.



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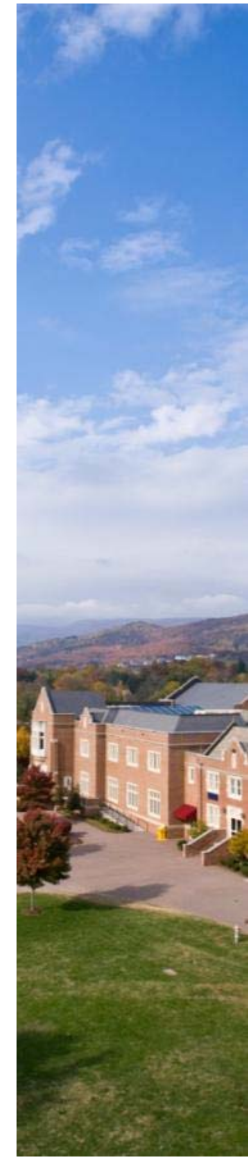
Part 2: About the Book

- “Mini-Excursions” into interesting side topics are included in each chapter after the exercises!
  - Chapter 1: “True Randomness?”
  - Chapter 2: “Three Dice `Craps’”
  - Chapter 3: “Counting `Fibonacci’ Coins Circularly”
  - Chapter 4: “Compositions and Probabilities”
  - Chapter 5: “Sicherman Dice”
  - Chapter 6: “Traveling Salesmen”
  - Chapter 7: “Random Walks and Generating Functions”
  - Chapter 8: “More Probability!”



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Part 2: About the Book

- Feedback has been very positive, from my own reviewers, to the faculty members who piloted a version in teaching INQ 177 “The Mathematics of Games,” to the students in those classes, and to the external reviewers and my editor.
- Shameless (or shameful) plug:
  - “The Mathematics of Games: An Introduction to Probability” by David G. Taylor, CRC Press/Taylor & Francis Group, ISBN 978-1-4822-3543-2, Catalog Number K23047

- Thank you for coming! Are there any questions?

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