2021 Research in Undergraduate Mathematics Education Reports

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Preface

The past year since the 23rd Annual Conference on the Research in Undergraduate Mathematics Education was a difficult one for everyone around the world. The COVID-19 pandemic continues to rewrite what is considered to be normal in our collective personal and professional lives. In an effort to keep the members of our community as safe as possible, the SIGMAA on RUME Executive Committee made the decision to cancel the 2021 RUME conference.

The RUME community came together to support the research of the early-career RUME researchers. In lieu of our annual conference proceedings, early-career RUME researchers were invited to submit results of current research, contemporary theoretical perspectives and research paradigms, and innovative methodologies and analytic approaches as they pertain to the study of undergraduate mathematics education. These submissions underwent a rigorous double-blind peer review process. The ***2021 RUME Reports*** consists of 49 reports (44 Contributed Reports and 5 Theoretical Reports).

Many members of the RUME community volunteered to review submissions. We sincerely appreciate all of their hard work by the 106 reviewers.

Shiv Smith Karunakaran, RUME Organizational Director

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Contributed Reports

Theoretical Reports

Investigating the linkage between professional development and mathematics instructors' adoption of IBL teaching practices

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Inquiry Based Learning (IBL) professional development workshops are designed to increase participants' capacity to teach using IBL methods. This study used a sample of 312 participants from workshops held in 2010-2018 to examine the relationship between professional development participation, IBL capacity, and use of IBL teaching practices. We found that instructors' IBL capacity, meaning the beliefs, knowledge and skills that prepare them to use IBL, and use of IBL teaching practices increased after participating in professional development. Using the Theory of Planned Behavior as a conceptual framework, we used a structural equation model to explain the effects of workshop participation and other factors on the use of IBL teaching practices. Findings indicated that workshop participation, collegial support, prior IBL experience, class size, and course coordination influenced workshop participants' use of IBL teaching practices. These findings support the use of well-designed, intensive professional development as a means to change teaching practices.

Keywords: Inquiry-based learning, Professional development, Workshops, Teaching

Introduction

Student-centered, research-based instructional strategies (RBIS) such as inquiry based learning (IBL) improve learning and persistence in US undergraduate STEM education (Freeman et al., 2014; Ruiz-Primo, 2011). Recent studies in various STEM disciplines show that approximately 20% of instructors use these methods extensively (Stains et al., 2018; Eagan, 2016). Thus, most students do not experience active learning methods regularly: instructor adoption is a primary constraint to more widespread use of RBIS. There is expert consensus that professional development is one of the most influential factors in facilitating the use of RBIS in undergraduate STEM teaching (Khatri et al., 2013; Laursen et al., 2019). Only a few studies have linked teaching-focused professional development (PD) to teaching behavior of college instructors, and they are in science disciplines, not mathematics (Benabentos et al., 2020; Chasteen & Chattergoon, 2020; Manduca et al., 2017; Viskupic et al., 2019). The purpose of this study is to contribute to this limited literature by investigating the linkage between professional development and mathematics instructors' adoption of IBL teaching practices.

Theoretical Framework

The theory of planned behavior (Ajzen, 1991) has been applied to professional development (Patterson, 2001), teaching practice (Lee et al., 2010; Sadaf & Johnson, 2017), and in at least one study, both professional development and teaching practice (Chasteen & Chattergoon, 2020). In this study, we apply this theory to explain how participation in an intensive professional development workshop influences instructors' use of IBL methods.

The theory of planned behavior assumes that behavior is "planned" or rational and proposes that behavioral intention determine actual behavior (Figure 1). Behavioral intent is influenced by three components: attitude, subjective norm, and perceived behavioral control. Attitude refers to a person's favorable or unfavorable evaluation of the behavior of interest. Subjective norm refers to perceptions about whether their peers approve or disapprove of a behavior. Perceived behavioral control refers to an individual's perception that they have the ability to perform the behavior and that the behavior is under their control. Thus, it too moderates the relationship between behavior intention and behavior: according to the theory, strong perceived behavioral control is required for behavioral *intention* to manifest in *actual* behavior. The theory can be adapted to include other factors (e.g. demographics) that influence any of the core components.

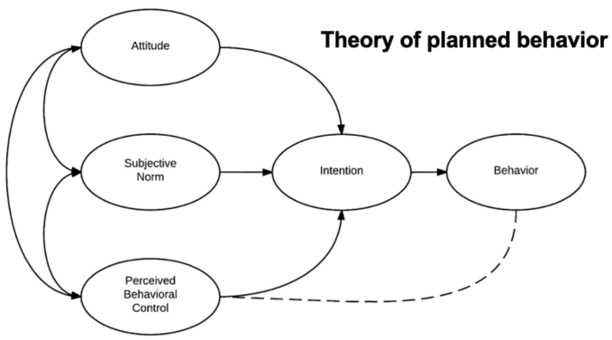

Figure 1. Theory of Planned Behavior, after Ajzen 1991

In applying the theory to this study, the behavior of interest is instructors' use of IBL teaching practices. This behavior is affected by their intent to use IBL teaching methods and their ability (e.g., knowledge and skill) to use IBL teaching methods. Instructors' intent to implement is affected by their attitude about IBL, their perceptions of peer support (subjective norm) and their IBL knowledge and skills (perceived behavioral control). While not portrayed in Figure 1, the theory accommodates other factors outside of the core model. Here we acknowledge that factors such as individual characteristics and the context of instructors' teaching may influence the degree to which they plan to implement or actually do implement IBL teaching practices.

Practical context for the study: IBL Intensive Workshops

Since 2010, 22 intensive workshops on IBL have served about 700 college mathematics instructors. The workshops offered instructors knowledge and skills to implement IBL in their own classrooms and sought to bolster their confidence and support their IBL decision-making. Four-day workshops were held in the summer at locations around the US.

While the workshop hosts and leaders varied over time, broad features of the workshops were consistent, including use of active and collaborative learning methods for teaching instructors about IBL. The current workshop model is grounded in research and practical experience with PD and incorporates video lesson study, educational research, IBL facilitation skills, and personal work time (Yoshinobu et al., 2021). Workshops accommodate instructors' diverse teaching settings because they do not promote a particular curriculum or classroom practice, but rather focus on pedagogy. The workshops have been described in some detail by Kogan and Laursen (2012); Hayward and Laursen (2014); Hayward and Laursen (2016); and Yoshinobu et al. (2021).

Research questions

The purpose of this research was to investigate the linkage between professional development and mathematics instructors' adoption of IBL teaching practices. To this end, this study addresses the following research questions:

1. To what degree do mathematics instructors' IBL attitudes, knowledge, and skill change after professional development?
2. How do teaching practices change 18 months after professional development?
3. What is the relationship between teaching practice, professional development, and other factors (e.g., individual, institutional, and teaching contexts)?

Methods

Data collection and sample

Workshop participants completed a pre-workshop survey about one month before their workshop, a post-workshop survey immediately after, and a follow-up survey about 18 months later. Analyses were limited to participants in the 2010-2018 workshop cohorts (n = 517) who completed all three surveys (n = 312), yielding a 60% completion rate. About half (49%) had IBL teaching experience. The sample included instructors from a range of institution types (Ph.D-granting 26%, Master's 24%, Four year 44%, 2 year 6%), and half of the participants were early-career instructors (50% with five years or less teaching experience). Over half (56%) were women (56%), higher than the general representation of women in mathematics (NSF, 2015).

Measures

Surveys included well-established measures (Hayward & Laursen, 2014; Hayward et al., 2016; Kogan & Laursen; 2012) from several categories related to our theoretical framework (Table 1). For constructs with more than one survey item, we calculated a measure of internal consistency. All constructs showed acceptable internal consistency (Chronbach's α ≥ 0.70).

Table 1.
Map of theoretical constructs, survey measures, categories, scales, survey administrations

Theoretical construct	Variable categories	Survey item	Scale	Survey admin (Chronbach's α)
Attitude	IBL capacity (belief)	Extent you believe inquiry strategies are effective learning method	1 = Don't know 2= Not very effective 3= Somewhat effective 4 = Highly effective	Pre Post Follow-up (single item)
Subjective norm	Departmental collegial support for IBL	Support from dept. colleagues to use IBL in teaching Support from dept head or chair	1 = Not at all supportive, 2= Mostly not supportive 3= Mixed or moderate support 4 = Mostly supportive	Follow-up (0.81)
Perceived behavioral control	IBL capacity (knowledge, skills)	How would you rank your current level of knowledge/skill in inquiry-based teaching?	1=None, 2=A little, 3=Some, 4=A lot	Pre (0.81), Post (0.71, Follow-up (0.72)

Behavioral intent	Intent to implement IBL	How likely implement in next academic year	1 = Not at all likely 2 = Somewhat unlikely 3 = Somewhat likely 4 = Rather likely 5 = Definitely	Post (single item)
Behavior	IBL intensity (core IBL teaching methods, -see Hayward et al., 2016)	Frequency of use of core IBL methods = student group work + student presentation + class discussion - lecture – instructor solving problems	1= Never, 2= Once or twice a term, 3 = About once a month, 4= About twice a month, 5= Weekly, 6= More than once a week, 7= Every class	Pre (0.75), Follow-up (0.70)
Other factors outside of the model	Individual characteristics	Career stage, teaching experience Gender, ethnicity, & race Prior IBL teaching experience		Pre, Follow-up
	Institutional characteristics	Institution type (highest math degree) Minority-serving institution		Follow-up
	Teaching contexts	Class size, course coordination, student majors, student level		Follow-up

Data analysis

RQ1 was answered by conducting a one-way ANOVA with repeated measures to check for differences in three IBL capacity measures over three time points. RQ2 was answered by using a paired samples t-test to test for differences in pre-workshop and follow-up IBL scores. RQ3 was answered by creating a structural equation model (SEM) based on the theory of planned behavior, to describe the relationships between corresponding theoretical components.

To maximize model parsimony, only three additional variables (prior IBL teaching experience, small class size, and course coordination) were included in the SEM used to answer RQ3. These variables were identified by conducting a series of preliminary analyses (e.g. ANOVA, regression) that checked for differences in three outcome measures (IBL capacity, intent to implement IBL, and IBL intensity) by all individual, institutional, and teaching context factors. These analyses yielded few statistically significant differences when controlling for all other factors; variables that did yield such differences were included in the SEM.

Results

First, we describe findings related to RQ1, addressing the degree to which IBL attitude, knowledge, and skill change after professional development. A one-way ANOVA with repeated measures was conducted to test for differences in measures of IBL capacity among three time points (pre, post, follow-up). We found statistically significant increases in all capacity measures from pre to post workshop (Table 2). Effect sizes (η^2) indicate the largest increases were in IBL knowledge and skill, with a smaller positive change in attitude about IBL effectiveness. From post workshop to 18-month follow-up, we found no difference in knowledge, a statistically significant decrease in attitude, and a statistically significant increase in skill. However, the small differences in attitude and skill were negligible considering the corresponding survey scales (see Table 1).

Table 2
Changes in IBL capacity: ANOVA Results for differences in mean scores by time point (N = 305)

	Pre-workshop		Post-workshop		18-month Follow-up		Omnibus Statistics		
IBL Capacity	*M*	*SD*	*M*	*SD*	*M*	*SD*	*F* (df = 2, 304)	*p*	η^2
Attitude	**3.36**	0.91	**3.84**	0.42	**3.67**	0.52	60.21	<0.001	0.17
Knowledge	**2.46**	0.70	**3.27**#	0.61	**3.20**#	0.60	248.06	<0.001	0.45
Skill	**1.95**	0.74	**2.53**	0.66	**2.77**	0.62	234.70	<0.001	0.44

Note. *M* = mean, *SD* = standard deviation, *F* =*F* statistic, p = probability, *df* = degree of freedom, η^2 = partial eta squared (effect size). Means within each row differ at p < 0.001 except for those marked with #.

To answer RQ2, we used a paired samples t-test ($n = 231$) to test for differences between pre-workshop and follow-up IBL intensity scores. Follow-up scores ($M = 6.78$, $SD = 4.67$) were significantly higher ($t(230) = 14.15$, $p < 0.001$), than pre-workshop scores ($M = 0.89$, $SD = 5.43$). A large effect size (*Cohen's d* = 0.93) indicated a strong meaningful change in IBL teaching intensity from pre-workshop to follow-up, as shown in Figure 2.

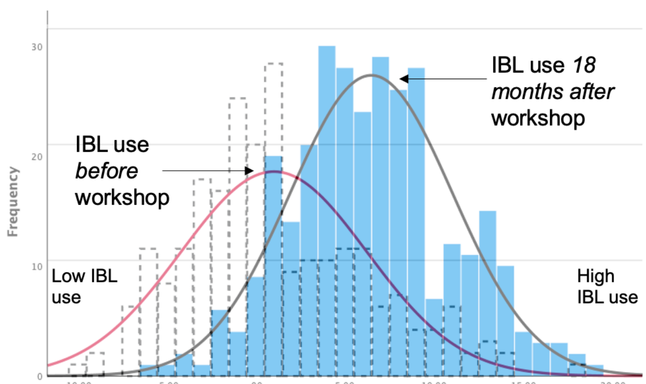

Figure 2. Change in IBL teaching practice: Distributions of IBL intensity scores before workshop and at follow-up

We used structural equation modeling to answer RQ3. Two fit indices (CFI and RMSEA) were used to assess model fit. CFI values range from 0-1, where values ≥ 0.95 are considered well-fitting (Hu & Bentler, 1999). RMSEA values ≤ 0.05 indicate close fit (Browne & Cudeck, 1993). The SEM (Figure 3) showed excellent model fit (*CFI* = 0.99, *RMSEA* = 0.01) and explained a moderate amount of variability ($r^2 = 0.21$) in IBL intensity, the intended behavior.

We see positive, statistically significant relationships between all specified components in the model. Of the four variables theorized to influence intent to implement IBL, two had a moderate effect, while two others had smaller effects. For actual implementation, IBL knowledge and skill had the strongest association, and three other variables were moderately associated.

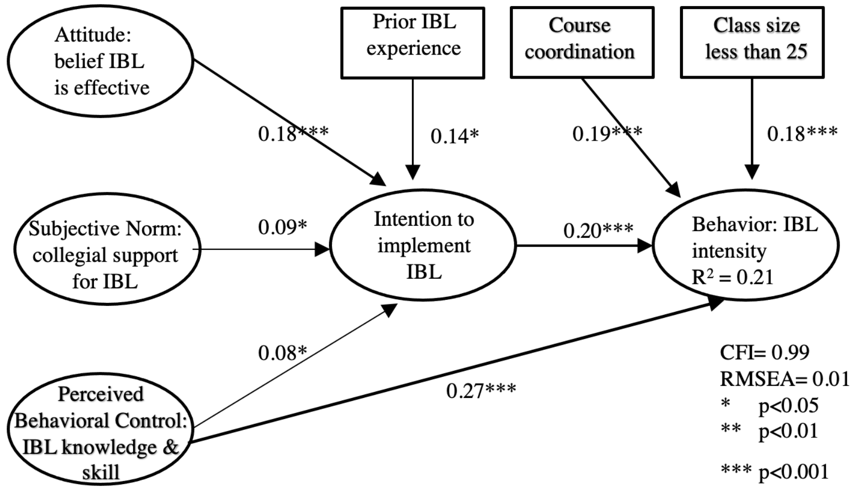

Figure 3. Structural equation model of the theory of planned behavior with standardized regression coefficients.

Discussion

The findings show strong linkages between professional development and use of IBL teaching practices. First, findings for RQ1 showed that participants' IBL capacity increased after participating in professional development. On average, faculty came to workshops with a strong belief about the effectiveness of IBL and thus had little opportunity for growth in this measure of capacity. Their initial strong positive attitude toward IBL did show a significant increase after attending workshops that persisted for 18 months after the workshop, indicating the workshop had a lasting effect. Larger gains in knowledge and skill from pre- to post-workshop were also sustained 18 months after the workshop. Participants reported greater gains in knowledge than skills: knowledge is gained quickly in a workshop, while skills develop over time with practice.

Strengthened positive attitudes, knowledge and skills may stem from workshops' emphasis on educational research about IBL teaching and on design principles and practical ideas for IBL courses. Research in physics has likewise shown that PD can both strengthen attitudes and improve knowledge of RBIS after workshop participation (Chasteen & Chattergoon, 2020).

Related to RQ2, we found that 18 months after taking part in PD, instructors' teaching became more IBL-intensive and less instructor-focused. The large effect size showed a substantial shift in IBL teaching intensity: most were not just dabbling with IBL practices but making substantial changes. Prior work in physics is mixed: some results show that many instructors abandon RBIS soon after initial adoption (Dancy & Henderson, 2010; Henderson & Dancy, 2009), and other work shows high initial adoption (Chasteen & Chattergoon, 2020). We suggest that IBL workshops promote implementation by providing time for instructors to plan their course and adapt methods to their own context (Hayward & Laursen, 2016).

Findings related to RQ3 were consistent with our theoretical framework. Model fit indices show that our data fit the model well, and observed statistically significant positive relationships demonstrate consistency with theorized relationships. First, the model specified four factors related to intent to implement IBL. The modeled relationships are all consistent with theory and are shown here in order from strongest to least strong factor according to our SEM:

- IBL attitude: As instructors' pro-IBL attitudes strengthened, intent to implement IBL also strengthened. (model coefficient 0.18)

- Prior IBL experience: Instructors with prior IBL experience had stronger intent to implement IBL than those without (model coefficient 0.14). Prior IBL experience has been identified as a factor that aided IBL implementation (Laursen et al., 2019).
- Perceived departmental support: Instructors reporting supportive peers were more likely to intend to use IBL, compared to those with less supportive peers (model coefficient 0.09). This finding corroborates prior research (McConnell et al., 2020).
- IBL knowledge and skill: Instructors with high knowledge and skill were more likely to intend to implement IBL than were instructors with less knowledge and skill (model coefficient 0.08).

The model specified four factors related to IBL intensity. The modeled relationships are all consistent with theory and are ordered here from strongest to least strong, according to our SEM:

- IBL knowledge and skill: On average, instructors with high knowledge and skill implemented IBL more intensively than did instructors with less knowledge and skill. (model coefficient 0.27)
- Intent to implement IBL: Instructors with high intent tend to use IBL more intensively than those with weaker intent. (model coefficient 0.20)
- Coordinated courses: Instructors teaching coordinated courses were more IBL-intensive than those whose courses were not coordinated, consistent with prior work linking course coordination to use of active learning strategies (model coefficient 0.19) (Bazett & Clough, 2020; Rasmussen et al., 2014).
- Small class size: Instructors who applied IBL in smaller classes were more likely to use IBL more intensively than those who taught larger classes (model coefficient 0.18). Some IBL teaching practices (e.g., student presentations) are harder to use in larger classes, especially for new users. Class size has been identified as one of the top factors making IBL implementation more difficult for new users (Laursen et al., 2019). Importantly, our finding reflects initial implementation in the first academic year after the workshop, and does not speak to whether small classes are important for long-term use of IBL. We know of many experienced instructors who use IBL effectively in large classrooms.

Connecting findings from RQ1 and RQ2, it appears that workshops increase IBL capacity, which in turn increases instructors' use of IBL teaching practices. Findings about RQ3 support this conclusion, as IBL knowledge and skill had the strongest association with IBL teaching intensity in this model. A positive attitude about IBL teaching, support by department colleagues, and prior IBL experience were important in supporting intentions to use IBL teaching, but skills and knowledge enabled instructors to actually implement IBL. Teaching contexts do matter: coordinated courses and class sizes can influence how intensively IBL is implemented.

These findings point to several practical implications. Investments should focus on intensive PD to strengthen instructors' attitudes and their knowledge and skill to enact RBIS. Targeted efforts to inform and train department leaders, faculty, and course coordinators to create supportive environments could also hasten uptake of IBL and other RBIS. Supporting initial implementation with small class sizes or team efforts could ultimately offer more students more research-aligned teaching practices, as instructors develop skills in more forgiving circumstances and then learn to adapt their practices to different teaching contexts.

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What's the Norm? Instructors Justify their Active Learning Moves and Decisions

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As VITAL faculty continue to teach more university mathematics courses, departmental efforts to improve instruction and shift toward active learning should align with instructors' teaching approaches and decisions. In this study, three instructors described their classroom norms and justified teaching moves via professional obligations. External factors, such as the physical learning spaces, interactions with students, and institutional constraints, influenced their pedagogy. This report emphasizes the support systems required within mathematics departments to sustain instruction improvement efforts among VITAL faculty.

Keywords: active learning, VITAL faculty, pedagogical decisions, classroom norms

Review of Relevant Literature

Recently, there has been a general effort in science, technology, engineering, and mathematics (STEM) departments across the nation to implement active learning strategies at the university level (Association of American Universities, 2017; Conference Board of the Mathematical Sciences, 2016). While there is evidence that active learning is more effective at facilitating student learning of mathematics than traditional lecture approaches (Freeman et al., 2014; Michael, 2006), the feasibility of implementing these interactive strategies is often challenged by faculty and instructors (Le et al., 2018; Michael, 2007). Instructors' hesitancy or resistance to implement evidence-based teaching practices, such as inquiry-based learning (IBL), can be due to their personal perceptions of these practices, lack of support, institutional factors such as course coverage pressure and classroom layouts, or qualities of the innovative approach (Keller et al., 2017; Lund & Stains, 2015; Shadle et al., 2017).

This paper focuses on a specific group of undergraduate mathematics instructors: VITAL faculty. The term "VITAL faculty" was established by MAA leaders (Levy, 2019) to refer to Visiting and Instructional faculty (full- and part-time), Teaching assistants (especially graduate-level), Adjunct instructors, and Lecturers. As universities move toward hiring more non-tenure-track faculty, mathematics departments rely increasingly on VITAL faculty to teach courses that are required of both STEM and non-STEM majors (Blair et al., 2018; CBMS, 2016).

In the past decade, researchers in undergraduate mathematics education studying IBL practices and outcomes have focused primarily on upper-level courses, such as linear algebra, differential equations, and abstract algebra (e.g., Johnson et al., 2020; Kuster et al., 2017; Larsen et al., 2013; Rasmussen et al., 2018). In the recent Compendium for Research in Mathematics Education, Rasmussen and Wawro (2017) dedicated a chapter to post-Calculus research, which includes a review of teaching approaches at that level. Thus, many of the understood active learning approaches and classroom norms are in upper-level courses, which are typically taught by non-VITAL, tenure-track faculty (Blair et al., 2018). There is a lack of literature about how non-tenure-track faculty members, especially novice instructors (Speer et al., 2010), teach introductory-level mathematics courses via an active learning approach.

The focus on upper-division courses is apparent in research on instructor preparation and professional development as well. Studies that examine the preparation of instructors of introductory-level courses (e.g., Pre-Calculus) tend to focus on graduate teaching assistants rather than instructional faculty (e.g., Beisiegel, 2017; Ellis, 2016; Milbourne & Nickerson, 2016). What is unknown is how specific groups of VITAL faculty members learn about, implement, justify, and improve on active learning practices in their classrooms. The lack of research about the *practice* of teaching (Speer et al., 2010) leads to limited research-based support for VITAL faculty and other instructors who are beginning to shift toward evidence-based teaching practices.

The practices and pedagogical knowledge of post-secondary instructors are influenced by a multitude of factors and external demands (Bennett, 2020). While time spent in the classroom can have positive effects on instructors' experiential learning of teaching, it can be influenced by the demands and culture of their institution (Keller et al., 2017; Oleson & Hora, 2014). Additionally, even when instructors are experienced lecturers, this does not necessarily translate to a sense of confidence or expertise with active learning practices (Bennett, 2020). VITAL faculty may have other backgrounds, constraints, and obligations that influence their teaching which differ considerably from their full-time or tenure-track colleagues (Laursen, 2019; Levy, 2019).

Theoretical Framework

Instructors make pedagogical decisions based on demands imposed on them by external people and structures. In mathematics education research, factors that influence teaching decisions, that is, compel teachers to abide by certain instructional norms while breaching others, have been previously answered with internally-based notions, such as knowledge and beliefs (e.g., Philipp, 2007; Speer, 2008). However, I view teaching college mathematics as existing within the context of a didactical contract (Brousseau, 1984), which shifts the justification of actions to the practice of teaching, rather than to individual teachers. This shift necessitates an examination of external factors and environmental constraints on pedagogical decisions.

Herbst and Chazan (2003, 2011, 2012) established the practical rationality of mathematics teaching framework by drawing on the instructional triangle (Cohen et al., 2003) as an activity system to describe how teachers make pedagogical decisions based on their institutional environment (see Figure 1). The novelty of the practical rationality framework is that it describes the practice of teaching, and thus shifts the unit of analysis to the external factors and justification sources that influence instruction, rather than internal factors of the instructor.

Contractual Norms within Teaching

Within the practice of teaching mathematics, there are norms, or routines and actions that are common and expected. Norms are implicit behaviors that exist in social settings, such as learning environments, that demand recognition and responses from the participants in the setting. Didactical contracts are differentiated and described by the norms expected by the actors as they play out their roles. Different expected norms create different contracts. "Thus, the norms of a contract can be a source of justification for actions in teaching" (Herbst & Chazan, 2012, p. 605).

Norms within a didactical contract are *contractual norms*. Contractual norms have a large grain size, meaning two teachers can abide by the same norms yet carry out those norms in what appear to be very different ways (Herbst & Chazan, 2012). Here, I take up Herbst and Chazan's supposition that a finer analysis might reveal nuances within norms. I denote the actualization of contractual norms by *teaching moves*, which are the actions performed by the person in the role of teacher during a lesson. Because of their smaller grain size, teaching moves describe instruction in greater detail and allow for disparities in contractual norms across instructors. Thus, I draw on rationality theory to understand instructional decisions and extend practical frameworks that explain specific teaching actions.

Professional Obligations of Teaching Mathematics

The role of teacher exists in institutions of schooling, surrounded by societal stakeholders in education. To understand teachers' justifications within these institutions, Herbst and Chazan (2011, 2012) described four professional obligations that act as implicit norms and constraints within the institutional environment: disciplinary, individual, interpersonal, and institutional.

The four obligations acknowledge that teachers are members of a common practice, rather than individuals who disregard external factors of teaching. College instructors, as members of the practice of teaching undergraduate mathematics courses, are assumed to consider similar factors that are reasonable and relevant to their practice (Herbst et al., 2011). The obligations may create tensions; a teaching action that aligns with one obligation may not align with another. In this way, the obligations help researchers explore why some innovations are taken up while others are viewed as not feasible or not in alignment with the practice of teaching mathematics (Herbst & Chazan, 2003). In the following paragraph, I describe each of the four obligations.

The *disciplinary* obligation ensures that the instructor accurately conveys and represents mathematics content, practices, and qualities, as deemed fit by stewards of the discipline. The *individual* obligation ensures that the instructor attends to individual students' behavioral, cognitive, emotional, and social needs, as a client of instruction and complete human being. In other words, there are two ways to attend to the individual: academically as a mathematics student and emotionally as a complete person. The *interpersonal* obligation ensures that the instructor attends to society, interactions between instructors and students, and the use of shared resources, such as discursive, physical, and social spaces and times. The *institutional* obligation ensures that the instructor regulates and abides by components of the schooling administration, such as curriculum, schedules, spaces, and policies.

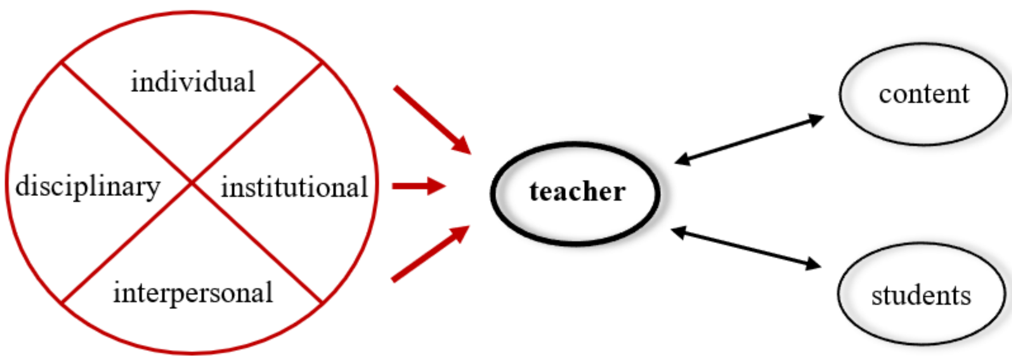

Figure 1. The four professional obligations influence the role of the teacher and the teacher's relationships

Research Questions

This paper addresses the following research questions:

1. What norms do instructional faculty use in their classrooms when teaching introductory-level mathematics via active learning?
2. How do professional obligations play a role in instructors' pedagogical decisions within an active learning approach to teaching mathematics?

Methods

The data presented here are part of a larger qualitative, dissertation study. This study took place at a large, public, research-oriented university that had recently made efforts to shift toward evidence-based teaching practices and create collaborative learning spaces (CLSs) as part of an initiative to improve undergraduate STEM teaching (AAU, 2017). CLSs are classrooms that promote student-centered approaches by incorporating flexible furniture, adaptive layouts, and interactive technologies (e.g., Baum, 2018; Odum et al., 2020).

This paper focuses on three participants, Maggie, Amber, and Charles, who were full-time instructional faculty members in a Department of Mathematics and are considered VITAL faculty. The instructors varied in age, education backgrounds, years of experience teaching, and approaches to active learning. Even though they had various experiences teaching mathematics at different grade levels, they did not have experience teaching via active learning (at any level). They chose active learning based on their recent arrival to this department or encouragement and inspiration from their colleagues.

Together, they taught three introductory courses: College Algebra, Pre-Calculus, and a data-based course intended for non-STEM majors, which focused on interpreting and representing data using a variety of graphs and tools (e.g., Microsoft Excel). Each instructor taught multiple sections of one or more of these three courses, and the sections took place in a variety of classroom spaces, including CLSs, large lecture halls, and smaller, traditional classrooms (see Table 1). They each sought different resources and mentorships in the department and across campus since the experience of teaching via active learning and in CLSs was new for them.

In the larger study, I observed each instructor's teaching six times using the Observation Protocol for Active Learning (OPAL) tool (Frey et al., 2016) and taking field notes. The OPAL tool focused on both student and teacher actions, such as asking and answering questions, working in groups, moving around the classroom, and resources used while lecturing and presenting. Observations informed lesson debriefs and subsequent interviews, all audio recorded. Lesson debriefs were casual discussions following observations that resembled informal interviews (Hatch, 2002). Semi-structured, in-depth interviews built on prior conversations and the observations to understand the teaching perceptions and rationale of instructors. A primary interview took place about halfway through the semester and a secondary interview took place at the end of the semester. The secondary interview also served as a member check for participants to review my interpretations of their lessons and of our prior conversations.

This paper focuses on the analysis of the norms observed during lessons or mentioned by instructors during lesson debriefs and interviews using multiple qualitative coding techniques. In particular, I used Saldaña's (2016) description of *concept coding* to highlight norms and teaching moves, which were guided by relevant research describing practices in undergraduate mathematics classrooms. The norms were then analyzed through the lens of the practical rationality framework (Herbst & Chazan, 2012).

Table 1. Key Information about Instructors and their Classes

Instructor	Course	Classroom	Class Size	Experience
Maggie	Pre-Calculus	CLS	35	Prior experience teaching high school math
	Pre-Calculus	Traditional Classroom	35	
Amber	Data-based Course	CLS	72	Prior experience as graduate teaching assistant
	Data-based Course	Lecture Hall	72	
Charles	College Algebra	CLS	72	Prior experience teaching Community College math
	Pre-Calculus	Traditional Classroom	35	

Findings

After coding the observations and instructors' discussions of their pedagogy, I found that there is a nested structure of norms and teaching moves, where contractual norms describe routines of larger grain size and teaching moves give finer detail about what the instructor implements. In other words, norms are comprised of various teaching moves. This is how, for instance, both Maggie and Amber established a norm of frequently asking their students questions during class, but it looked very different in their classrooms.

In her CLS, Maggie discretely selected table groups ahead of time to respond to her questions; whereas, in her traditional classroom, she let multiple students shout out the answers simultaneously. In her CLS, Amber created an elaborate protocol that used Excel to randomly select a table group and group representative to respond to each question; however, in her lecture hall, she simply asked questions to the whole class and waited for a volunteer to respond. All of these teaching moves correspond to the same norm of asking students questions, but opting for one over another caused the pace of instruction and types of interactions to appear very different.

Out of the 395 total teaching moves coded, 49 moves (about 12.4%) were coded as "rejected", meaning the instructor referred to it in a negative way or claimed to not use it in their instruction. Out of the 49 moves that instructors claimed to reject, 17 corresponded to the norm of lecturing. These rejected teaching moves used the term "lecturing" explicitly or mentioned classroom dynamics related to lecture. Indeed, observation data confirmed that the instructors sometimes (for Maggie and Charles) or always (for Amber) spent less time lecturing than doing activities with students during class. In this way, even though this was their first year trying active learning, they were successful with moving away from lecture-based norms and implementing more interactive teaching moves in their courses.

Discussion

In this section, I discuss how instructors used the norms mentioned above and how the obligations influenced their choice and justifications of these norms.

Discussion of Norms

In the context of teaching undergraduate mathematics, lecturing is typically viewed as the traditional or even expected approach to teaching mathematics at the undergraduate level (Stains et al., 2018); thus, it is the norm. However, the instructors in this study made decisions to alter this approach and breach this norm.

Lecturing was a nuanced norm, as all three instructors rejected it at times and found it to be beneficial other times. For instance, Maggie and Amber both rejected the norm of lecturing when it was used in excess, i.e., "lecturing for the entire class period". They instead implemented "mini-lectures" and "interactive lectures" throughout lessons, situated between collaborative activities. Similar to participants in Mesa et al.'s (2019) study, both Maggie and Amber saw benefits in lecturing only if it was used sparingly and at appropriate times, such as at the end of a lesson to summarize key concepts and restate mathematical definitions. On the other hand, Charles was the only instructor to not outright reject lecturing for an entire class period. He was "trying harder and harder to break that mode" of lecturing, but for most of the semester, he still felt the need to lecture for large amounts of some lessons, mostly due to external factors, such as pressures of content coverage and lack of classroom resources.

As the instructors learned about and experimented more with an active learning approach, they moved away from posing questions to the entire class and instead directed their questions to smaller, specific groups of students in an intentional way, such as Amber's protocol for selecting random tables in the CLS to call on groups of students. The instructors reflected that this not only made questioning norms more efficient by taking up less time but also encouraged equitable participation across all students and helped the instructors relinquish their authority and redistribute it to the groups (Cohen & Lotan, 2014).

Discussion of Obligations

Overall, the instructors attended to all four professional obligations, oftentimes to multiple obligations at once. Imbalances of obligations could often be explained by the instructors' backgrounds. For instance, Maggie taught mathematics for several years at a high school where she felt there were many structures and rules to follow, so she tended to emphasize the institutional obligation and its accompanying constraints. Whereas Charles came to the university after years of teaching in a community college setting and often talked about how he had much more autonomy there. This influenced him to downplay the institutional obligation at the university, since he was not used to such a structured environment.

The following two examples illustrate how Charles and Amber attended to the individual and interpersonal obligations, and how Amber and Maggie justified decisions based on the institutional obligation.

Attending to the individual and interpersonal obligations. Charles viewed lecturing as a way to attend to the individual obligation. He frequently told jokes and personal anecdotes to break up the lesson and not "overload" his students with information. He expressed his desire to lower students' stress levels through "distractions", such as humorous stories about his experiences or math-related puns. In this way, he was very aware of his students' emotions and purposefully crafted his lectures to ease their math anxiety.

Gradually, Charles became more comfortable interacting with student groups during class, rather than lecturing, but participation issues arose in his CLS. This was a large classroom with an unconventional, adaptive layout of furniture, which made it difficult to interact with all students. As other learning spaces researchers have documented, it created a "golden zone" for interactions in the front, center part of the room, and a "shadow zone" (Park & Choi, 2014, p. 757) in the back and along the perimeter, where students were too far from the designated front to interact with Charles while he was giving instructions or writing on the doc cam.

Attending to the interpersonal obligation, Amber also encountered issues with inequitable participation in her lecture hall. She reflected, "So what I found…was that when I just solicited responses last year, like 'somebody volunteer,' I had three students that would always respond." This dilemma led to the protocol for asking questions that she created for her CLS, where table groups were systematically and randomly called on. This satisfied the individual obligation as well, since the equitable participation norms created a low-stress environment where students could discuss questions with peers before answering.

Attending to the institutional obligation. For an example of the institutional obligation, I draw on Amber's and Maggie's experiences. Amber saw an alignment between the institutional obligation and teaching the data-based course via active learning. Since this was a terminal course for students and was designed to focus on problem-solving, it was better suited for active learning (than other introductory courses she had taught), which gave the appearance that Amber ignored some institutional obligations. In fact, these characteristics of Amber's course led to fewer constraints and less content coverage pressure. However, Amber did feel constraints at times as a VITAL faculty member with multiple sections to teach. She acknowledged her obligation to the math department to teach the courses she was assigned, but her assigned class times made it difficult for her to attend campus-wide workshops and seminars to learn about active learning and collaborative spaces.

Maggie frequently referred to tensions between her intended active learning norms and the institutional obligation. Keeping both sections of Pre-Calculus consistent was a constant dilemma, particularly because she was assigned two different types of classrooms. She reflected,

> I think [being in the CLS] was easier to get students to talk to each other about the math. …That collaborative space had the whiteboards, whereas the other more traditional space, there was no individual whiteboards. So it was like, well, if you do a group activity there, how can you get the group to show their work or share out?

Maggie considered the practical aspects of teaching and cited institutional factors and constraints as justifications for her decisions. She stressed how Pre-Calculus had a packed course schedule, which made it difficult to do active learning activities in class. Highly-coordinated, common exams in Pre-Calculus added extra course coverage pressure. Coordinated exams also meant that Maggie had to keep her two sections of Pre-Calculus at the same place in the curriculum, even though the differences in participation and activities meant that the two sections often progressed at different paces. This meant deciding how to utilize the resources in each classroom to best teach the course material.

Rather than mentioning the supports and benefits of coordinated courses and departmental structures (Martinez et al., 2020), the instructors in this study spoke more about the dilemmas and constraints that the institutional obligation placed on them.

Conclusion and Implications

In conclusion, the instructors in this study used a wide variety of classroom norms and teaching moves while teaching via an active learning approach. The practical rationality framework revealed a hierarchy of teaching moves that described each instructor's approach in greater detail, accounting for differences within specific contractual norms. The norms implemented were influenced by a variety of external factors, including student emotions and participation, features of the physical learning spaces, and institutional constraints on the specific courses. The norms were also influenced by the instructors' main teaching goals, their education backgrounds, and their teaching experiences.

The lens of professional obligations brought to light tensions that the instructors experienced while navigating an active learning approach within a mathematics department at a large, research-oriented university. The instructors responded to tensions by recognizing and attending to some obligations while downplaying others; this compartmentalization helped them manage the many factors that influence teaching. This study emphasizes the importance of understanding the teaching approaches and justifications of understudied groups of undergraduate mathematics instructors, namely VITAL faculty. Non-tenure track faculty face obligations that their tenure-track colleagues may not, as well as issues of status. Instructors' packed teaching schedules can prevent them from attending professional development seminars that are crucial for providing support and scaffolds for improving instruction.

The findings presented here suggest the need to study the pedagogy and justifications of VITAL faculty members to better support them as they transition toward evidence-based teaching practices such as active learning. Furthermore, qualitative descriptions of implemented teaching moves were necessary to highlight nuances within contractual norms, which illustrated the types of interactions and levels of collaboration occurring in active learning classrooms. As the number of VITAL faculty members increase while mathematics departments continue to shift toward active learning, it is crucial to align the efforts of those involved to improve undergraduate mathematics education.

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Students' Meanings for their Symbolization about Rate of Change Functions in Dynamic Geometric Contexts

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I present the results from a set of clinical interviews conducted during the 2019-2020 academic year to produce generative models of student thinking about rates of change towards the end of a university-level calculus sequence at a large State University in the Southwestern United States. The data presented here presents a case for supporting students' symbolization as emergent from their previous conceptualization and mental activity. I provide a conceptual analysis to illustrate ways of thinking that may be propitious for understanding rates of change in a calculus context and symbolizing with the intent to convey meaning.

Keywords: Student Thinking, Rate of Change Functions, Partial Derivatives, Calculus

The National Science Foundation (NSF) stated in its synopsis of the Improving Undergraduate STEM (Science, Technology, Engineering, and Mathematics) Education: Education and Human Resources (IUSE: EHR) Initiative that STEM-based job creation will outpace non-STEM based job creation in the coming years. Modeling dynamic systems, using derivatives as rate of change functions, and understanding the connection between rate of change and accumulation functions are crucial ideas for a first course in calculus (Carlson et al., 2015; Park, 2015; Rasmussen & Keene, 2019; P. W. Thompson, 1994; P. W. Thompson & Ashbrook, 2019; White & Mitchelmore, 1996). The sort of symbolization required to model dynamic systems (i.e., systems of ordinary or partial differential equations) is both nuanced and complicated. It is pertinent to help students' progression towards becoming scientifically literate and adept in the sorts of mathematical skills needed to succeed in their future careers. In the context of mathematics education research, aiding students in applying their mathematical skills to real-world scientific contexts is both relevant and necessary.

Rates of change are a ubiquitous topic in most mathematics and science courses. From understanding constant rate of change to understanding Maxwell's equations of Thermodynamics, the ideas of derivative and rate of change are prevalent in most upper-division STEM courses (Cannon, 2004, 2004; Henderson et al., 2017; Rasmussen & Keene, 2019; Roundy et al., 2014; J. R. Thompson et al., 2012; Wagner et al., 2012; Yeatts & Hundhausen, 1992; Zandieh, 2000). To further investigate the various ways of thinking students construct about rates of change across the calculus curriculum, I decided to examine multivariate calculus (MC) students' meanings for their symbolizations for rate of change functions in dynamic geometry contexts. I seek to investigate the following questions: (1) *How do students make sense of their own symbolization (geometry formulas, derivatives, function notation, differentials, etc.) in the context of modeling dynamic geometry contexts?* (2) *What ways of thinking do students exhibit as they attempt to make sense of multiple, possibly composed, rate of change functions?*

Theoretical Background

I leverage *radical constructivism* (Glasersfeld, 1995) as my theoretical lens for this study. One of the core tenets of radical constructivism is that individuals actively construct knowledge. Individuals' current cognitive structures (*schemes*) strongly impact how they are able to make sense of what we, as instructors, intend to teach them. Since researchers can't directly access each student's individual mathematical reality (*students' mathematics*), the best we can do, as researchers, is build models of the student's mathematics (*the mathematics of students*) (Steffe & Thompson, 2000).

Since I am curious about students' ways of thinking about rate of change functions after completing a course in MC, I leverage *quantitative reasoning* (QR) as an operational lens to explicate my model of each student's thinking (P. W. Thompson, 1990, 1993). QR can describe the ways of reasoning that pervades much of mathematics. To be clear, by *quantity*, I mean someone's conceptualization of a measurable attribute of an object. The process of *quantification* is a dialectic between someone's conceptualization of an object and the quality of the object they are imagining as measurable. The measure of the attribute is proportional to the unit (P. W. Thompson, 2011). A *quantitative operation* is the operation of thought by which one conceptualizes a situation quantitatively (P. W. Thompson, 2011). Someone's *quantitative structure* is their individual network of their various conceptualizations of quantitative operations and the quantities that comprise them (Tasova & Moore, 2020; P. W. Thompson, 2011).

I use Thompson and Harel's definitions of *understanding*, *meaning*, and *ways of thinking* (Harel, 2008a, 2008b; P. W. Thompson et al., 2014). To have *an understanding in the moment* is to *assimilate*. All understandings are technically understandings in the moment, relative to the student's current perception of the problem context, their individual *schemes*, and the meaning of their understanding. The student's meaning is the space of implications existing at the moment of understanding. Someone's *scheme* for a given concept is their web of meanings that can be interrelated and overlapping (P. W. Thompson et al., 2014). A student's scheme for rate of change is their meaning for rate of change. A student's *way of thinking* is a consistent anticipation of multiple meanings in reasoning which is attained by continually reconstructing that reasoning anew. Knowledge persists because it works for the student. For example, I conjecture that if a student's "differentiation" scheme is connected with their scheme for "composition of rates" and "function composition" that student will more readily anticipate situations where they could use the chain rule to construct a rate of change function in applied contexts (Jeppson & Jones, 2020; Jones, 2018). This way of thinking about the chain rule outlines by Jeppson and Jones may provide a strong foundation for meaning making and constructing mathematical models in applied scientific contexts.

I use O'Bryan's construct of emergent symbolization to describe a student's actions only if they think symbolization (the individual's, another student, their instructor, textbook author, etc.) is intended to convey meaning (O'Bryan & Carlson, 2016; O'Bryan, 2020). Though O'Bryan utilized emergent symbolization in the context of Precalculus, I believe this construct is useful for supporting students meaning making for the multitude of symbols throughout a university calculus sequence. Since symbolization is intended to convey meaning to another, it is useful to conceive of symbols as unalterable social constructs in the mathematics community. Zandieh et al. leverage the instructor as a *brokering agent* between a local classroom context and the larger mathematics community to discuss how instructors can use instructional tools to support the formalization of students' initial symbolization or mathematization of eigenvectors in the context of the Inquiry Oriented Linear Algebra (IOLA) curriculum. I use Zandieh and colleague's definition for *symbolizations* as a process of creating and using symbols to convey one's mathematical ideas (Zandieh et al., 2017). Our roles as instructors are then to support students in constructing productive meanings for the ideas we intend to teach them and helping students engage in normative symbolization activity with the intent to convey meaning.

Conceptual Analysis of the Growing Circle Task

Thompson provided four different ways to use a conceptual analysis for research. I intend to use the following conceptual analysis to accomplish: (1) building models for what students know in the context of the "Growing Circle" task, (2) describing ways that may be propitious for future learning about rate of change functions, (3) describing deleterious ways of knowing to students' understanding of important ideas in this context, and (4) analyzing the coherence and fit of various ways of understanding the "Growing Circle" task (P. W. Thompson, 2008).

Statement of the Growing Circle" Task:

1. Consider the area formula for a circle with a changing radius r, $A(r) = \pi r^2$, find $A'(r)$.
2. Suppose the radius of the circle is changing with respect to time, $r = r(t)$, find a function which represents area in terms of time t. [$r(t) = \ln(0.3t)$ if needed]
3. For the circle whose radius is changing with respect to time, find $A'(t)$.

A Productive Meaning for Constant Rate of Change

A student who has constructed a deep conceptual understanding of constant rate of change may symbolize the linear approximation of the variation in the circle's radius in the following manner: $r(t_0 + dt) = r(t_0) + r'(t_0)dt$. A student's quantitative structure may include their conceptualization of the following quantities: (1) The initial length of the radius of the circle ($r(t_0)$), (2) The number of seconds the circle's radius had been growing up to that moment (t_0), (3) the variation in the number of seconds (dt), (4) the rate of change of the radius-length with respect to the number of elapsed seconds at the moment t_0 seconds had elapsed ($r(t_0)$), (5) the linear approximation of the variation in the radius-length of the circle ($r'(t_0) * dt$), and (6) the approximation of the new radius-length of the circle after $t_0 + dt$ seconds has elapsed ($r(t_0) + r'(t_0) * dt$). The student may understand that this local, linear approximation well-approximates the variation in the radius-length for a small enough variation in the number of elapsed seconds.

Reconceptualizing the Radius of the Circle as a Quantity whose Value Varies

A student may construct a set of circles and imagine picking a particular radius-length and mentally create that particular circle from the set of all possible circles they previously conceptualized. Conversely, a student who has reconceptualized the radius-length of a circle as the value of a quantity may imagine a single circle whose radius-length varies smoothly and continuously through all possible values. I conjecture that a student who possesses the second image will likely think about the formula $A = \pi r^2$ as an equivalence statement relating the values of two quantities whose values vary: the value of the radius-length of the circle at any moment (r) and the corresponding area of the circle (A) for that particular radius-length.

Conceptualizing the Product of Rates as a Quantitative Operation

While many students have memorized the chain rule when first learning the derivative rules in calculus, thinking about whether to implement a mnemonic device to recall the chain rule and conceptualizing a composition of rates as a quantitative operation are very different cognitive operations. When multiplying two rates, many students will "cross out units" (i.e., crossing out the inches when multiplying $in^2/in * in/sec$). However, this does not imply the student has quantified the result of their calculation as a new rate, nor conceptualized multiplying rates together as a quantitative operation.

When constructing the rate of change function from the accumulation function $f(g(t))$, multiplying $f'(g(t))$ and $g'(t)$ is intended to convey a quantitative operation, i.e., quantifying the resultant product $(f'\big(g(t)\big) * g'(t))$ as a new rate. To use a rate of change function representationally in the "Growing Circle" context, $f'(g(t))$ represents the value of the rate of change of the area of the circle with respect to the radius at the moment t seconds have elapsed, $g'(t)$ represents the value of the rate of change of radius with respect to elapsed time at the moment t seconds have elapsed. Multiplying these two rates together creates a new rate, $f'\big(g(t)\big) * g'(t)$, which represents the rate of change of the area of the circle with respect to elapsed time at the moment t seconds have elapsed.

Methodology

Three students each participated in one to two generative *clinical interviews* (Clement, 2000) over the 2019-2020 academic year. Due to the burgeoning nature of the mathematics education literature on student thinking about multivariable calculus topics, these interviews were conducted by a single researcher analyzing students' responses across each task (Rasmussen & Wawro, 2017). An interview protocol was developed with the goal of forming initial models of how students think about derivatives and partial derivatives while enrolled in a MC course. James' first interview was conducted near the end of the Fall semester of 2019, while Malia and Alexis' interviews were completed toward the beginning of the Spring semester of 2020. All three students were enrolled in the same conceptually-oriented MC course with the same professor. James, Malia and Alexis were all three Mathematics Education majors, worked in the same Precalculus tutoring center, and were enrolled in the same conceptually-oriented DIRACC Calculus courses (P. W. Thompson & Ashbrook, 2019). My analysis of James' written work and verbal responses is presented here. James identified as a white man (he, him, his). The data was analyzed using a grounded theory approach (Strauss & Corbin, 1990).

Using Function Notation Representationally

As part of the DIRACC curriculum (P. W. Thompson & Ashbrook, 2019), students were introduced to various ways to write the function notation for a rate of change function (i.e., $r_f(x), f'(t), y'$, dy/dx, $d[f(x)]/dx$, etc.). The notation for the rate function r_f is intended to convey that the rate of change function is associated with a corresponding accumulation function f. Each of these notations is intended to convey different information about the value of a rate of change function or the relative size comparison of amounts of variation between an independent quantity and the corresponding amount of variation in the dependent quantity. For instance, dr/dt represents the value of the rate of change of the radius of the circle with respect to elapsed time at some moment. A student who has conceptualized differentiation as a process to construct a rate of change function from a given accumulation function may intend to use their rate of change function representationally.

Results

I present James' work on the "Growing Circle Task", which was part one of the third task in the initial clinical interview protocol. As my initial model for James' thinking was slowly built over the course of the interview, I draw from James' responses to the first two tasks to support my claims. I present James' novel symbolization when he intended to differentiate, and James' quantitative structure for the "Growing Circle" Task.

James' Novel Symbolization for his Intent to Differentiate

First, a quick note about some of James' unique symbolizations. James' symbolization for conveying his intent to differentiate a given function, expression, or equation differed somewhat from the normative method (Fig. 1). For instance, if James wanted to convey that he intended to differentiate a function f with respect to x, he would write $f'(x) = f(x)dx$ as opposed to $f'(x) = \frac{d}{dx}[f(x)]$ or $f'(x) = \frac{d[f(x)]}{dx}$. James' continual use and description of his "intent to differentiate" symbolization implied James' stable meaning for his novel notation. James' meaning for his symbolization for his intent to differentiate persisted throughout three semesters of a university-level calculus sequence.

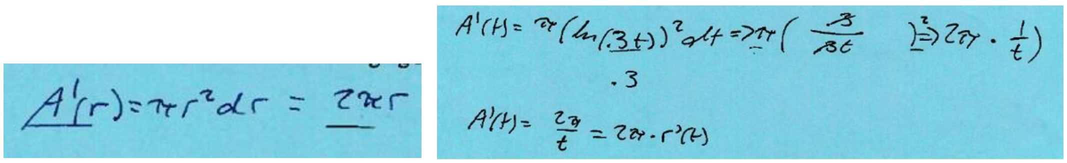

Figure 1. James' written work from the "Growing Circle" task conveying his intent to differentiate two functions with respect to r and t respectively.

James' Quantitative Structure for the "Growing Circle" Task

At first, when James was asked to differentiate the area function for a growing circle, he was very surprised by the result of his differentiation. Despite James' strong computational abilities, interpreting the result of his differentiation in the context of the growing circle task led James to believe his answer was incorrect.

James: What would A prime be? Okay so this is…so A represents the area with a changing radius so then A prime would represent how the area changes when the radius changes um…and you would just derive with respect to r. So this would be [writes $A'(r) = \pi r^2 dr$] so A prime r would equal pi r squared dr which is equal to 2 pi r…Is that right? See everything I said felt right and now this is the area or circumference and now I feel like I'm wrong.

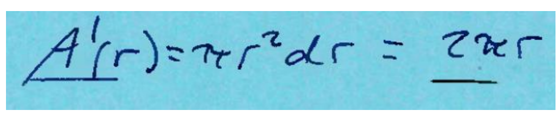

Figure 2. James' written work finding the derivative of the area function with respect to the radius-length.

After asking James to reiterate his thoughts about what each symbol meant to him, I showed James an animation for the growing circle task. James reiterated that he did not think the rate of change of the area of the circle should be equivalent to its circumference. James stated he believed it should be a "higher power", indicating perhaps that James thought the rate of change of the area of the circle with respect to its radius should be growing quickly since the equation for the area of a circle is quadratic in terms of its radius-length. James' eventual accommodation emerged from mentally comparing the radius-lengths of two different circles whose radii are almost numerically equivalent.

Interviewer: I'm going to let the radius change by 0.25. I'm going to let the slider play and the radius is going to change by 0.25 and then fill in 0.25 and then fill in. [plays the animation]…so if I stop that and I look at this…blue stuff. What does that kind of look like to you?

James: Ohh…it is right. It does look like a little other circle. Okay, I understand now. We sort of have this change in the radius [draws larger circle and shades in with blue]. In this case you have it as 0.25 and you can sort of see that is a thick circle almost. Its just another circle around it that has a really thick border. So if we were to make our change in r to be really, really small it might as well just be a circle. In that sort of sense since its just a circle, that area that it makes up would be the same as the circumference. When your r is really big this is kind of a really big circle…Its that change in area between this radius length and this radius length [draws another smaller blue circle]. When you make the radius length arbitrarily the same you get this circle which is our original circle [outlines first circle in blue] with original area and you get this sort of one [outlines second circle in red] that is barely outside of it to the point that it barely has no thickness…that's just the circumference of the circle with that radius.

Figure 3. James' sketch of the two circles with different radius-lengths after watching the "growing circle" animation.

James' quantitative structure for the remainder of the "Growing Circle" task emphasized three different quantitative relationships: (1) the radius determining values of the area of the circle, (2) the number of elapsed seconds determining the radius-length of the circle, and (3) the number of elapsed seconds determining the area of the circle. It was unclear whether James conceptualized the relationship between the number of elapsed seconds (t) and the area of the circle (A) as a composition of functions. Though James could think about the relationship between any pair of quantities, he did not appear to simultaneously hold in his mind his conceptualization of the relationship between the quantities of time, radius, and area. This could be why James did not correctly differentiate his area function with respect to time.

Though James substituted the function rule for r when he wrote his function rule as $A\big(r(t)\big) = \pi(r(t))^2$. James believed that due to the nature of what the prime notation represented (Fig. 4), he did not think his various derivatives of area with respect to time represented the same thing when he thought about the radius of the circle. As an artifact of James' unique way of symbolizing, James differentiated his $A(t) = \pi\big(r(t)\big)^2$ function as $A'(t) = \pi\big(r(t)\big)^2 dt = 2\pi r'(t)$. James had conceptualized the order of differentiation of $\pi\big(r(t)\big)^2$ in two parts: (1) differentiate $r(t)$, (2) differentiate what was left: the π and the square term. Since π was a constant, it simply "stayed" then he was supposed to bring down the exponent. It appeared James was operating from a previously memorized procedure that he had not thought about in a while. This is different than using the chain rule to convey a new rate of change function formed from composing two other rates ($\frac{dA}{dt} = \frac{dA}{dr} * \frac{dr}{dt}$). James attempted to reconcile his confusion using his prior work and reasoning as evidence that he was correct (James' earlier thinking about $A'(r) = 2\pi r$ implied that $A'(t) = 2\pi r'(t)$ was correct). Since James had conceptualized the rates as ratios, it made sense to him that if $2\pi r$ represented how the area changed if the radius was changed by a small amount, then $2\pi r'(t)$ represented how the area changed if you change time by a small amount.

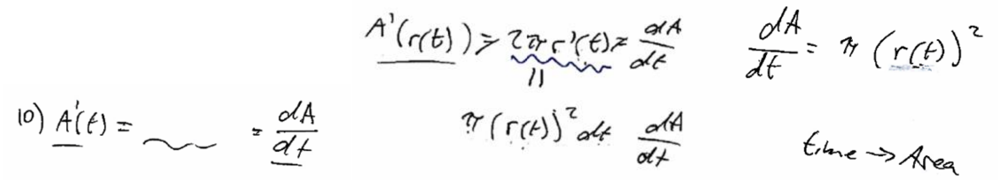

Figure 4. James' written work on the last part of the "Growing Circle" task demonstrating his symbolization emerging from his quantitative structure.

At the end of our second interview, James attributed his entire experience to forgetting the chain rule. Though James mentioned several times during the interview that he felt he was missing some foundational knowledge about the various ways mathematicians symbolize derivatives in calculus, James dismissed his experience from both interviews as "I forgot a rule." This comment suggests that this student believes the source of his difficulties were due to not remembering how to execute a procedure, as opposed to implementing the rule only after mentally constructing a composite rate of change function (Jeppson & Jones, 2020; P. W. Thompson & Carlson, 2017).

Discussion

The function notation associated with derivatives and rate of change functions is intended to convey a scheme of meanings. However, in James' case, he felt as if he lacked an explanation for why mathematicians use these notations in a normative manner. Though James' symbolization for his intent to differentiate was emergent, it was far from normative and was never corrected over three semesters of calculus. It may be crucial for students to first engage in quantitative reasoning before symbolizing to avoid falling into the trap of symbol pushing without thinking.

Perhaps by coordinating (1) an instructor's intent to support the formalization of students' burgeoning symbolization and (2) operationalizing the construct of emergent symbolization to support students in only symbolizing with an intention to convey meaning, we gain a clearer image of what it would mean to support students in first engaging in conceptualizing the relevant quantities before being asked to write symbols throughout the calculus curriculum. Students must be continually supported in symbolizing with the intent to convey meaning. Allowing students to reflect on their symbolizing activity may engender a habitual need to symbolize with the intent to convey meaning.

I suggest that future research should examine the longitudinal development of students' ability to use symbols to represent the meanings they have constructed. As a foundation for improving students' mathematical and scientific literacy, it seems reasonable that students at all levels should be supported in viewing and using symbols to express the quantitative relationships they have conceptualized. More work is needed to understand the mechanisms for coordinating students' symbolization activity with their conceptualized quantities in calculus.

Limitations

The work presented here used data from students enrolled in a conceptually-oriented calculus curriculum. As such, the generalizability of these results may be limited. The work and student thinking presented here was from the analysis of a single student. As such, more work is needed to support the strength of the claims presented here. Though student thinking about rate of change functions is not new or novel, this data emerged from a study designed to understand how students think about partial derivatives after completing multivariable calculus.

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Connecting Computation: Mediating Mathematical Knowledge Through Computational Modules

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This study examined the nature of Calculus II student's engagement with MATLAB modules and the ways in which the computational modules mediated student's understanding of Taylor series. Through analysis of observations and interviews, using an instrumental genesis framework, a pattern of student's views on the relationship between mathematics and computation developed in relation to student ability to conjecture and engage in mediated epistemic interactions with the MATLAB modules. This study highlights how the conceptualization of the two areas as distinct poses barriers and challenges for students, thereby questioning how to develop computational environments that support mediated epistemic interactions.

Keywords: Computation, Coding, Instrumental Genesis, MATLAB, Epistemic Mediated Interaction

New technological advances and the rise of data, coding, and computing have had drastic changes on the workforce infrastructure and the skills needed within the 21st century. There is an international shortage of those trained in the fundamentals of computational methods to fill jobs across academia and industry alike (SIAM Education, 2011), necessitating an examination of current educational methods to develop well-educated citizens in a computing intensive world (Cobb, 2015; CSTA, 2017). In preparing these students computational skills are needed, especially within mathematics (Lockwood et al., 2019). Computation is not only advantageous for student's future careers, but also has pedagogical motivation to deepen student's learning of mathematics by reinforcing concepts like abstraction, a key skill for mathematical proof and underpinning of computational thinking (Wing, 2011). Computation can reinforce desirable habits of mind such as communication and persistence (Lockwood et al., 2019). Despite the potential for the co-construction of mathematical and computational knowledge current studies in literature have focused on a proof by existence for this combination (Lockwood et al., 2019; Valdés-Fernández et al., 2019). Students not only have to grapple with the language of mathematics and the mathematical register but also comprehend the new grammatical strategies within programming simultaneously, resulting in the need to in essence translate between two technical languages (DeJarnette, 2019). Further, the blending of computation into mathematics does not always occur smoothly and can leave students feeling frustrated and confused as there is a disconnect between the two (Krause et al., 2019).

Universities have responded to this call for computing in a variety of ways including the new majors (Silvia et al., 2019) and new computational labs and modules (Chabay & Sherwood, 2008; Psycharis, 2016). The development of modules is a highly attractive as these can be paired with pre-existing curricula, are feasible for universities of any size, and have a low cost. Despite the potential promise, what is less explored in literature are the ways that computational modules can mediate mathematics learning. In order to gain a more nuanced understanding, I will address the following research questions: (1) What is the nature of student's engagement with computational modules employed during a Calculus II course? (2) How do the computational modules mediate student's mathematical learning regarding series during a Calculus II course?

Setting

This study focused on a specific MATLAB module developed for a large, midwestern R1 university as a part of the mathematical laboratory series for Calculus II courses. The course had no prerequisites for any computational or coding experience. The course met four times a week, where three days were lecture and the remaining day was a recitation. During the recitation, students alternated between taking a quiz or engaging with a MATLAB module. This course was chosen due to the pre-developed computational modules that had gone through years of section-focused reform and had been scaled up to a course-wide requirement during Fall 2019.

Theoretical Framework

In order to conceptualize the relationship between computational modules and students, an instrumental genesis framework (Béguin & Rabardel; Guin & Trouche, 1998; Vernillion & Rabardel, 1995) is utilized to focus on student learning in instrument-mediated activity. Students are able to either directly interact with the mathematics, or engage in a mediated interaction, as shown in Figure 1, where the computational module (the artifact) facilitates mathematical understanding (Lonchamp, 2012; Pargman et al., 2018). Through repeated mediated interactions, the artifact begins to reorganize the ways students engage with the mathematics, specifically that the nature of the knowledge a student generates has changed. As this continues, the artifact changes into an instrument, which does not exist in itself, but rather a student is able to appropriate and integrate it into their conceptions of mathematics (Verillion & Rabardel, 1995). During this process, which is referred to as instrumentation, students begin to understand the constraints, affordances, and the procedures linked to the artifact, and which leads to differentiated usages and ultimately to the psychological construct of the instrument (Guin & Trouche, 1998). Operationalization of this process is accomplished through instrumentalization levels. Longchamp (2012) delineates three levels, where the first is when the artifact is used for a particular action and under limited circumstances. The second level is when the artifact is linked to a class of situations, and the final level is when the artifact is modified substantially and permanently, as to perform a new function. These three levels provide insight into where a student is during the instrumentation process and relating the student's actions within the mathematical context.

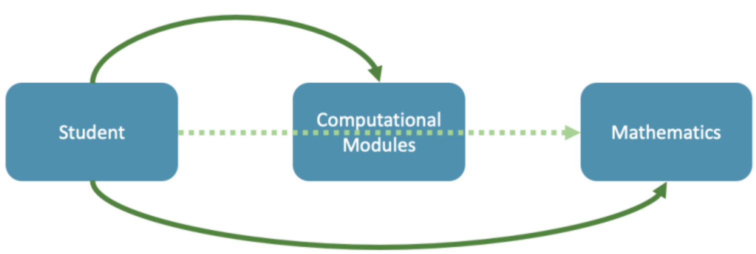

Figure 1. Students can engage with the mathematics and/or the computation directly, as indicated by the dark green solid arrows, or they can engage in a mediated interaction in mathematics through the computational module.

Methods

Participants and Data Context

Student participants were recruited across all sections, and a convenience sample of five students was selected. This sample spanned majors, computational experiences, and genders in order to bring a variety of perspectives and experience to the study. Due to Covid-19, all courses were moved completely online, including the MATLAB modules, so all data collection occurred through online video conferencing and screen recordings. It is important to note that since this change occurred mid-semester, students had been completing these modules with a group in person during a finite time period. After the switch to online, students were to complete the lab individually over the course of a few days, but still could get help through an online forum.

Taylor Series Module of Interest

The focal laboratory was the final MATLAB module focused on series. As students progressed through the course, the assumption was that there would be a greater proficiency with MATLAB. Therefore, since the familiarity with an artifact impacts the instrumentation process (Trouche, 1999), the final module was chosen in hope that there would be greater discussion about the mathematics, whether direct or mediated, rather than initial MATLAB environment questions. Additionally, during pilot interviews, students commented on enjoying seeing the applicability of series within the computational environment, rather than simply existing as an abstract mathematical concept. Students were provided with a MATLAB .mlx file which contained the bulk of code and instructions. They were responsible for modifying and potentially adding existing code in order to answer different mathematical questions.

Within the module, there were four tasks for the students to complete. The first task was to write a line of code that would calculate the maximum value of the difference between the fifth-degree approximation for ln(1+x) and the full Maclaurin series, then repeating this for the 200th degree approximation. Students were provided with a graph of the approximation and ln(2) plotted on the interval [-0.9, 0.9] with a slider to change the degree up to N=9. The second task focused on approximating ln(2). Students used a predefined function that calculated the first N terms of the approximation of ln(1+x), where x = 1 and then implement a trial-and-error strategy so that the difference between the Nth degree polynomial and ln(2) was no more than 10^{-8}. The third task had students rewrite the logarithm ln(2) = -ln(1/2) and then use the predefined function to calculate the Nth degree Maclaurin series polynomial at x = -1/2. The goal was to calculate the degree needed so that the error was no more than 10^{-8}. The code also generated plots to show the error plotted against the degree. Finally, students were to figure out why the second method needed fewer operations. Students were provided with the code to generate a plot of ln(1+x) plotted against the 10th order Maclaurin polynomial on the interval [-0.8,1.2].

Data Collection and Analysis

Each participant went through a POP cycle, which refers to a pre-observation interview, observation, and then a post-observation interview. The pre-interview focused on establishing a baseline for the participant, both with respect to their mathematical and computational backgrounds as well as to how they generally engage with the modules. Participants were also asked about Taylor series and their importance. This interview occurred 1-2 days prior to their observation. Observations lasted approximately one hour, in order to mimic the conditions prior to the transition to online learning and provide a time boundary for students. During this time, I was a passive observer, and participants could pursue any line of inquiry and use any resource to complete their module. Observational notes were taken in addition to the video and screen recordings to record any poignant points to clarify during the post-interview.

The post-observation was conducted within two days following the observation, allowing participants to reflect on their experience and ensure that the details were fresh in their minds. The same set of questions surrounding Taylor series were asked in addition to video-stimulated reflections. Short clips of interest from their observation were shown to clarify participant actions and motivation, allowing for illumination of student thinking during the observations.

After completing the POP cycles for all participants, general profiles of students were developed. Participant observations were coded so that each participant action was the unit of analysis, including, but not limited to, reading code, running code, modifying a variable, adding a new line of code, and scrolling. These actions were then coded as either direct or mediated as prescribed by the theoretical framework. Longchamp's (2012) dimensions of mediation were applied, in that during interactions, students would either engage in epistemic or pragmatic mediations. Epistemic mediation seeks to know the object (MATLAB module) and to transform it into an instrument in relation to the mathematical context, whereas a pragmatic mediation focuses on comprehending a specific task and context. Looking across the individual actions led to the grouping of actions by topic and/or participant focus; these were referred to as conversations. These conversations were then coded as either pragmatic or epistemic. A conversation could contain both epistemic and/or pragmatic actions despite the overall coding.

Participant Overview

Five students were a part of this study: Clint, Tony, Carol, Janet, and Margaret. These students had varied computational and mathematical experiences, as well as differing majors. An overview of the participants is presented in Table 1.

Table 1. A table summarizing the participant's majors, prior computational and mathematical experiences.

Participant	Major	Prior Computation	Prior Mathematics
Carol	Chemical Engineering	MATLAB & Python	Calculus I
Clint	Computer Science	Java, SQL, Python & C++	Calculus I
Janet	Chemistry	None	Calculus I
Margaret	Packaging	None	Calculus I & ITP
Tony	Mathematics	Calculus II Modules	Calculus II

Results

During the interviews and observations, what became increasingly apparent was that it was extremely difficult to mediate mathematics learning if participants viewed computation and mathematics as two separate entities. That is, when students tended to separate the mathematics into one domain and the computational interactions into another, it was difficult to engage in mediated epistemic interactions. Therefore, although the goal of the modules was to enhance student's understanding of Taylor series, when this divide existed, the computational activity was not necessarily developing student's understandings. It is important to note that this separation does not refer to the previous computational experiences, such as those that learned coding in the context of mathematics or physics versus those that took independent coding courses. Rather, the separation is referring to the in-the-moment actions and the ways in which students expressed their thought process during the interviews. Further, although this presentation is seemingly dichotomous, the reality is that the ways that each student viewed the relationship, or lack thereof, with mathematics and computation ebbed and flowed during the observations. Nonetheless, I contend that the separation of these two entities contributed to frustration and created barriers for some students to engage in mediated epistemic interactions.

In order to highlight the separation, I will initially focus on the work of Tony, who had previously taken Calculus II and had engaged in some prior MATLAB experience. For Tony, he began by exploring the pieces of code, and interacted with a built-in slider and visualization output, but he expressed some initial frustration in that the code was “really difficult for [him] to understand”. He switched to using a pen and paper since he felt that he could not “get it with the MATLAB code and was trying to cheat [his] way through with the MATLAB code”. He recorded his answer after some hand calculations and hoped that he did not “do all that math for nothing”, referring to his pen and paper approach instead of the code, and continued onto the next section. Both of these actions were direct and pragmatic interactions with computation and mathematics respectively. During the remainder of the observation, and the reflection, Tony continually made remarks such as the “code would have been helpful if I could have figured out what it had been telling me” or that he “honestly [found] this to be extremely confusing” when reading code. When working on the second exercise, Tony focused on a graph of error versus the degree of the polynomial and then stated:

> “Okay so I assume they mean it is some number in here. *[Pointing to the higher degrees and clicking on different points which display the number of terms and the corresponding error of the approximation]* Oh! Oh sweet. It is going to be way higher than 250 then. Can I get a better like approximation if I just change some code or something”

He began by changing a value in the code from 250 to 5,000, and then stated “I don't really feel like guessing and checking like they are asking me to. It seems tedious… Hmm only three digits in *[referring to the error]* so let’s try 50,000". He continued with this method but encountered roadblocks through some runtime errors and syntax issues, as there were different locations to potentially modify the existing code. After a brief pause, he reread the question, consulted some resources, and noted that

> “I thought I understood the question and I don't think I actually do. I am trying to see if that particular function, or the Taylor expansion follows the function ln(x) or ln(1+x) all the way to infinity and it doesn’t. It follows it to 1. I was trying to find its radius of convergence.”

He returned to the code and stated “I think that I might just do what I was doing before and try to find it when it is a huge number. I don't understand... It seems right somehow... but I don't really understand how." He continued to try different approaches and was met with new errors until the time of the observation had expired. During this part of the observation, Tony was primarily focused in on the coding portion and separated it from the mathematics. By viewing the coding as ‘cheating’ with MATLAB, this underscored an assumption that somehow the computational element was not genuine mathematics, or that there was some sort of deceit or fraud occurring. Rather than having computation as an environment for, or an extension of mathematics, this opposition formed a barrier for Tony. Even in his frustrations, he noted that it is the coding and after encountering errors, he then focused on the mathematics independent of the coding context. Tony continued to jump between the two different realms, rather than using one to mediate the understanding of the other and this transition brought about tensions.

This tension was not isolated to Tony, rather both Margaret and Janet had similar experiences with the separation. Margaret had encountered technical issues in the beginning and was unable to modify code, but she decided to read through the code and she stated that she was “just trying to read and understand the jargon” and followed up with “ I am trying to figure out the coding so that I can make it work for me so I can figure out uh the sum and figure out how I can change x and therefore change the term”. The observation ended early due to these difficulties, but before her follow up interview, she completed all the tasks by hand using pen and paper. Despite the difficulties, she emphasized that the goal was to make the coding work for her and in her follow up interview elaborated that she did not understand why the computational modules were required. She has in essence put coding and mathematics in distinct categories, mitigating the potential overlap and mediated epistemic interactions.

In her pre-observation interviews, Janet had noted that typically with the modules she walked away with “confusion and no real insight into calculus concepts” as many times she would “look for patterns in the code and change things”. During the observation, when comparing the two different approaches to approximations, she stated that “there must be something in the code which is why it probably makes sense” and then proceeded to work with the code in order to answer the question. During these encounters, Janet had separated into either looking at the code, or looking at the mathematics. When she used the different resources available to her, such as a group chat or online forum, she expressed that she was “just trying to figure out how do you know what you are doing” when modifying the code. Although there were periods for participants where the mathematics and computation were separate, there were also points which highlighted a connection between the two and using the computational environment to understand the mathematics.

For example, consider the case of Carol. During the third problem, she initially changed the code to try and center the Maclaurin series at .5; however, this resulted in a complex number, an unexpected result. She adjusted her array to have negative terms but commented “uhm that’s weird I have never summed something where N is negative… Well, we will go for it and see what happens”. This resulted in immediate MATLAB errors, and she commented “I feel like this is degrees because this is like an array that it is evaluating [approximation]”. After these MATLAB warnings, she discovered an error, reset her code using positive indices, and then fine-tuned her approximation. During this mediated interaction, Carol’s understanding of mathematics informed her approach to the code, in that it seemed irregular for her to have a negative N, and also her understanding of coding informed her mathematical understanding. She knew what an array did, and she began to link the terms of the approximation to this computational construct. The allowed overlap permitted mediated interaction. A key moment for Carol was during the final problem, where she reasoned:

> “A Maclaurin series is centered at 0 and radius of convergence is 1. So when x=1, you’re pretty much on the endpoint of where the function is converging *[long pause]* no could be converging since R=1, you have an open interval of convergence of (-1,1)... the question is does it converge on 1 because if [not] it won’t be very good at approximating”

After an initially reasoning, she carefully went through the lines of code and examined the outputs stating "okay I think I am right because it is telling you to look at this right here *[pointing at x=1]* and it really begins to really diverge here.. Out of curiosity though, what happens when you increase the number of terms… Oh it definitely diverges”. Carol was able to use the computational environment in order to develop a deeper understanding of the mathematics, namely what it means for a series to diverge. For her, the code was a realm in which she could explore the mathematics and conjecture, which led to a fundamental change in how she viewed diverging endpoints. She expressed that she had always wondered what a diverging endpoint meant and understood and could visualize it. This mediated epistemic interaction proceeded to develop a higher level of instrumentation as her understanding of diverging endpoints as a whole has been changed and she now expressed her understanding in relation to the MATLAB module.

Finally, Clint expressed statements where he would adjust the number of terms and related it back to the degree of a polynomial and error. When gesturing towards the error output, he stated the error is the difference between "the actual answer and the approximation and the lesser the difference, the better for us" and increasing the number of terms would "make the difference much much less… because it approximates it better". The ways in which he engaged in the computation related back to the mathematics he was investigating. He attributed this to his prior coding experience as he could "just figure out [the mathematics] at the start and get right to the point as opposed to figuring out the code itself". For him, he was able to use the computation as a way to investigate the mathematics. When adjusting the number of terms, Clint would conjecture a potential output and what the code would do, specifically surrounding the error and the quality of approximation.

Discussion and Conclusion

One of the primary results from this study was the challenge for students to engage in mediated epistemic interactions when viewing computation and mathematics as two distinct entities. This finding is reified in the literature as others have noted the perceived disconnect between the two (Krause et al., 2019) and the tendency to delineate struggles into either understanding the code or the mathematics (DeJarnette, 2019). Further, an interesting point arose in the mediated epistemic interactions, such as that of Carol and a diverging endpoint. Carol tended to view the computation and mathematics as connected, and then conjecture about the mathematics in this computational context. The act of conjecturing preceded the mediated epistemic interaction, and this occurred for Clint as well. This process of conjecturing, experimenting, and testing understanding of code and mathematics, are all in line with the practices detailed by Lockwood et al. (2019) that mathematicians use. Therefore, if there is an environment where computation and mathematics are viewed as connected, then potentially incorporating conjecture may lead to developing a deeper understanding of mathematics. Nonetheless, there is a key implicit assumption in that there is not a disconnect between the two. For example, Tony had engaged in conjecture and was beginning to engage with both the code, and the mathematics, but was missing the key link between the two that would have potentially allowed his conjecture to initiate an mediated epistemic interaction. If the perceived structure is that computation exists outside of the main lecture hall, as nearly all students commented that it was not mentioned during 'normal lectures, then this reinforces a view that mathematics and computation are separate, which in terms makes mediated epistemic interactions far more difficult. This delineation calls into question the notion of simply attaching labs to a mathematics class. If the goal is to understand the mathematics more deeply or be able to engage with it in a new context, then the dichotomy poses a serious threat. It seems as if an integrated computational component could better address the overlap and why computation is beneficial to mathematics, in turn providing an environment for mediated epistemic interactions to occur.

When designing computational experiences in the mathematics classroom, a key question that needs to be addressed is how to create an environment that foster's students understanding of both mathematics, computation, and their unique relationship. The danger of having a sole coding introduction at the beginning of the semester is that students are exposed to MATLAB in limited circumstances and linked to particular actions, thereby making it difficult for students to proceed to the second or third level of instrumentation. Further, if this initial introduction positions the coding and computation as distinct from the mathematics, then students may not be in an environment that promotes epistemic interactions with the mathematics. Further study on the relationship to the prior coding experiences and view on mathematics would be beneficial to further understand how students coordinate learning mathematics and computation simultaneously and use computing to explore the mathematics.

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Three Phases of Active Proof Reading Strategy for Comprehension of Mathematical Proofs: An exploratory study

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In advanced mathematics courses, proofs are instrumental in conveying mathematical knowledge. As a result, in many upper-division undergraduate math courses, proof comprehension is a crucial aspect of learning mathematics. This study examines proof reading strategies that eleven mathematics doctoral students profess to use to facilitate proof comprehension. This paper identifies 12 strategies that participants in this study employed to improve proof comprehension. I argue that those 12 proof reading strategies occur in three phases. In particular, doctoral students in this study engaged in three types of reading: preliminary reading, engaged reading, and reflective reading. The paper concludes with a brief remark on this research's implication for teaching proofs in undergraduate mathematics.

Keywords: Proof, proof comprehension, proof reading, undergraduate mathematics education

Students are expected to spend significant time reading proofs and glean mathematical knowledge from them in many upper-division mathematics courses. But recent studies show that proof comprehension is a notoriously difficult task for many students (Author, 2018). One way to support students' proof comprehension is to study how experts such as math professors read proofs. In an exploratory study, Weber and Mejia-Ramos (2011) identified three strategies research mathematicians generally employ when reading proofs: appealing to the authority of other mathematicians who read the proof, line-by-line reading, and modular reading. Moreover, in all three strategies, the authors noted that mathematicians used non-deductive reasoning. In another proof reading study, Weber and Mejia-Ramos (2014) surveyed 118 mathematicians and confirmed their earlier exploratory study findings. By modifying the research methodology used in Weber and Mejia-Ramos' (2011) study, this study seeks to further proof reading strategies that can facilitate proof comprehension.

Theory

Research on proof comprehension is continuing to uncover significant differences in proof reading strategies between novice (e.g., beginning undergraduate students) and expert (e.g., mathematics doctoral students) students. For example, Inglis and Alcock (2012) observed their participants' eye movement while reading a proof and concluded that undergraduate students, compared to the experts in their study, spend more time focusing on the "surface features" of a mathematical proof. Based on this observation, the researchers found that undergraduates spend less time focusing on proof's "big picture," such as the proof's logical structure. Inglis and Alcock's (2012) finding may explain why students often have difficulty understanding the logical structure of a mathematical argument, as evidenced elsewhere in the literature (e.g., Selden & Selden, 2003).

There is a growing body of research looking into strategies that can enhance comprehension of mathematical proofs. For instance, Weber and Mejia-Ramos' (2013) describes five proof reading strategies that undergraduates can use to facilitate their understanding. Their study observed four mathematics majors reading six proofs. The authors considered these students to be strong because they were successful in their content-based mathematics courses and on the follow-up proof comprehension test that the authors designed based on Mejia-Ramos et al.'s

(2012) proof comprehension assessment model. Their analysis revealed five proof reading strategies that the students used to facilitate their understanding of the proofs, which are (a) trying to prove a theorem before reading its proof, (b) comparing the assumptions and conclusions in the proof with the proof technique being used, (c) breaking a longer proof into parts or sub-proofs, (d) comparing the proof approach to one's own approach, and (e) using an example to understand a confusing inference. Weber and Mejia-Ramos (2013) followed up their qualitative study with a large-scale internet-based survey study that included mathematics majors and mathematicians from 50 large state universities in the United States. Their quantitative study's purpose was two-fold: (1) to explore whether mathematicians prefer mathematics majors to use these five proof reading strategies and (2) to investigate to what extent mathematics majors use these strategies. Their study's main finding is that the majority of mathematics majors do not use these proof reading strategies that mathematicians find useful for proof comprehension. This finding sheds light on why undergraduate students often gain little from proofs they read or see during lectures (e.g., Conradie & Frith, 2000; Rowland, 2001; Lew et al., 2016).

Research Methodology

Recall that this study's primary goal is to gain insight into how math Ph.D. students read proofs for understanding. The data to address the research question is obtained from interviewing eleven doctoral students in mathematics. I met with participants individually for a videotaped, semi-structured, hour-long, task-based interview. During the interview, participants were observed reading proofs of four distinct theorems. The content of three out of the four proofs is undergraduate mathematics; the other is from graduate-level mathematics. These interviews aimed to observe the strategies participants employed while reading proofs and examine how these strategies could contribute to proof comprehension. During the interview, participants were told that they could highlight or write on the proofs' text. Additionally, they were encouraged to think out loud while reading each proof.

After a participant completed reading the proofs, I asked several questions to explore their proof reading strategies. When participants said something I found confusing or interesting, I asked for clarification. The following questions are representative of my interview protocol:

- What are some of the things you were doing when reading these proofs to increase your understanding of them?
- Can you describe things you do to understand a proof? How do you know when you have understood a proof?
- When reading proof from a textbook or a journal article, what do you do when you get stuck?

When a participant did not mention any of the proof reading strategies described in Weber and Mejia-Ramos' (2013) study, I explicitly asked them how often, if any, they employed these strategies.

Data Analysis

Each interview was transcribed fully, and pseudonyms were assigned to the participants. Both the interview transcripts and analytic memos were studied carefully. The interview transcripts were initially analyzed using thematic qualitative text analysis (Kuckartz & Kuckartz, 2002; Maxwell, 2013). In the first pass through the data, any episode when a participant discussed something relevant to proof reading either prompted or otherwise was labeled as *a proof reading* strategy.

After identifying instances where participants talked about their proof reading strategies, I coded each instance using a semi open-coding scheme. Existing theoretical constructs from the literature on proof reading strategies such as Weber and Mejia-Ramos' (2013) study were used in the categorization process when I was confident that the literature's constructs agreed with the data. In contrast, when participants' responses were considered new or cannot be captured by existing constructs, I used *in vivo coding*—where terms used by participants are used as codes—and *process coding* to capture "observable and conceptual action in the data" (Saldana, 2013). I then organized similar codes into categories to develop a coding scheme, which was eventually checked by two experienced researchers, each with a Ph.D. in mathematics education and extensive research experience.

Finally, all interview transcripts were carefully examined and coded explicitly using the proof reading strategies identified in the coding scheme. Participants enacted behavior when reading a proof was cross-examined against a specific issue they were trying to address to facilitate their comprehension of the proof. For example, when I saw a participant underline a particular assertion or a definition in a proof or constructed a small sub-proof, I asked a follow-up question to understand their rationale for doing so.

Results

A careful analysis of the data suggests that participants in this study viewed mathematical proof reading as a non-trivial cognitive task that involves active engagement with the proof. Active engagement with proof is understood as any cognitive activity carried out to understand the proof. Taken together with previous literature on proof reading strategies (cf. Weber & Mejia-Ramos, 2011; Weber, 2015; Weber & Mejia-Ramos, 2013), the data in this study suggests that doctoral students engaged in three phases of reading cyclically: *preliminary*, *engaged*, and *reflective*.

Preliminary Reading

Preliminary reading is a theoretical construct that refers to a collection of strategies participants reported to employ to ascertain the *proof's framework* and overarching methods. Selden and Selden (1995) define *proof framework* as "the top-level structure of the proof, which does not depend on detailed knowledge of the relevant concepts" (p. 129). A key feature of *preliminary reading* is that participants are not interested in figuring out the proof's details; for instance, verifying the validity of an assertion or computation is less critical during *preliminary reading.* Instead, at this stage, the reader is interested in figuring out the overarching or "big picture" and motivating ideas of the proof. *Preliminary reading* can be considered an elaboration of what Weber and Mejia-Ramos (2011) consider *modular reading.* Table 1 below describes reading strategies participants may use during this stage of the active proof reading cycle.

Table 1

Table 1. Summary of Sub-categories or strategies included in preliminary reading, description of the content of the strategies, and the number of participants who mentioned them

Strategy	Content	Num

1. Understanding the statement of the theorem	Trying to understand the theorem by explicitly attending to concepts, objects stated or implied by the theorem (Weber, 2015)	3
2. Proving or sketching a proof for the theorem before reading its proof	Trying to construct a rough sketch or a complete proof of the theorem before seeing its proof (Weber, 2015)	10
3. Identifying the proof framework	Trying to understand how the proof proceeds by attending to what sub-claims are being proven, what proof technique is being used, what can be assumed, and what must be shown using that proof technique (Weber, 2015)	5
4. Skimming	Moving from one assertion to another without justifying them but to understand what is going on and how ideas in the proof are tied together to prove the theorem (Weber & Mejia-Ramos, 2011)	9

As shown in Table 1, the most frequently mentioned *preliminary reading* strategy is proving or sketching a proof for the theorem before reading its proof. Ten out of eleven participants said they used this strategy to enhance proof comprehension. For example, when I asked G7 if he tried to prove any of the theorems in the interview tasks before reading its proof, he responded:

> Some of them. I've seen [task] C before. [Task] B I did not try to prove it. Um [task] A I did try to prove it on my own and then [task] D I tried to prove it a little while, I thought about what is needed to be shown…I came up with one approach that was perhaps plausible and then I went to see what these authors had done. But I at least thought about it minimally. I think about what needs to be shown…

When I asked G7 how often he uses this strategy, he said:

> So you know ideally, if I were never in a rush, you know first I would try to prove it on my own and then um I want to make at least sure I know what needs to be shown and try to come up with some approach that seems like something that could work and then I'll look at the first line of the proof. And then maybe there's a key idea in the first line and say okay, now you know perhaps I can prove it from there.

It is important to note that when G7 uses this strategy—proving or sketching a proof for the theorem before reading its proof—he does not necessarily come up with a correct or complete proof; instead, an attempt or partial proof or laying out what the "landscape" of a proof may look like suffices.

Another participant, G8, also said he tried to prove the theorems before reading their respective proofs during the interview. When I asked him how frequently he uses this strategy, he said: "When I see a theorem, when I see a proposition or a statement, and I'm familiar enough with the ideas, and I will try and sketch out how I would understand it. If I'm unfamiliar with the ideas, I typically use the proof as learning material."

Participants in this interview discussed factors that influence *preliminary reading*: time and familiarity. For instance, G7 stresses the importance of time when deciding whether to sketch a proof of the theorem before reading its proof. As expected, G7 said he won't embrace the aforementioned *preliminary reading* strategy when he is in a "rush" or doesn't have enough time to execute the strategy. G8, on the other hand, pointed out that he makes a judgment call as to whether or not to use the strategy based on his level of familiarity with the underlying concepts of the theorem.

Engaged Reading

Engaged reading is a theoretical construct that refers to the set of strategies participants employed when trying to make sense of specific assertion (s) or an argument within a proof. Table 2 below describes seven strategies participants used during this phase of the active proof reading cycle.

Table 2

Table 2. Summary of sub-categories or strategies included in engaged reading, description of the content of the strategies, and the number of participants who mentioned them

Strategy	Content	Num
1. Justifying some assertions	Trying to fill in gaps to understand why a specific assertion is correct (note: some gaps could be filled mentally without having to write anything)	8
2. Partitioning the proof into smaller, manageable pieces	Trying to organize long proofs by breaking them into their modular structure	7
3. Using specific examples to understand a problematic assertion or series of assertions	Trying to use illustrations or diagrams to understand a complicated or confusing assertion	4
4. Zooming in	Trying to understand specific ideas in proof by conducting a unit analysis	2

5. Identifying, translating, and interpreting confusing assertions, key ideas, or notation using one's own words	Attending to higher-level concepts of the proof; making a note of key ideas or techniques employed in the proof	11
6. Forward-looking reading (Hide-and-seek)	Trying to complete the proof's remaining arguments after attending to the proof's *key ideas* and how they proceed.	4
7**.** Resource utilization	Relying on a book or a colleague for an explanation of a given assertion (s) within a proof	8

Engaged reading is by far the most active phase of the reading cycle. During this phase, participants were often observed underlining, highlighting essential ideas, and working out details of important assertions they found confusing. For example, when I asked G7 about some of the text he emphasized, he said: "… sometimes I underline something when I see that it's going to be used again later. So in this proof, we compare two computations. So we have this product over here, and then we have this product over here, so it's a visual guide." Another participant, G8, said:

> In proof B, I only underlined one thing. I probably should have highlighted two. I should have emphasized abelian group, and every non-identity element is of order two, right? That's the key idea, right? We have an algebraic structure…we have a group structure in particular. And everything is of order two, which means that it's a 2-group.

It is important to note that G8, in particular, said he underlines what he considers to be the proof's *key ideas*.

Reflective Reading

Reflective reading is a theoretical construct that refers both to a specific strategy and an analytical main coding category. As a strategy, *reflective reading* relates to instances where participants were observed reflecting on the whole or part of the proof. Evidence of this strategy include: (a) summarizing the entire or part of the proof using one's own words, (b) reflecting on specific takeaways such as a proof technique or *key ideas* from reading the proof, and (c) frequently asking oneself what a particular assertion in the proof says about the mathematical object under exploration within the proof. *Reflective reading* was a popular strategy utilized to improve proof comprehension. Nine out of eleven participants said they frequently use this strategy to enhance proof comprehension. The excerpt below illustrates introspective habits G8 engages in while reading a proof.

> If I think my approach is more elegant than the author, I will, which is not typically the case, think about the author. I'll try and read into what the author was trying to present. I'll try, and I'll ask myself if I'm using techniques, if I misunderstand something. If my approach is significantly more elegant. Am I imagining that something is nicer than it is? Am I imagining that certain facts are stronger than they are? So, am I cheating? When that's not the case, I will ask myself what is the author trying to present? Are they setting

me up for a corollary afterward that uses the ideas from this proof, which this particular approach makes more explicit? When the author's proof is better than something, I can sketch it out for myself. I will examine the methods, ideas that they used, and how those better encapsulate the ideas, better sum together, piece together the ideas that I'm trying to use. So were they able to assemble these landmarks a bit better? Were they able to connect them in a much more direct fashion?

Discussion and Concluding Remarks

This exploratory study is an attempt to support students' proof comprehension by carefully examining characteristics of proof reading strategies exhibited by experts, mathematics doctoral students. This study's key finding is that proof comprehension is an active cognitive process that involves several strategies that can be grouped into three reading phases: *preliminary*, *engaged*, and *reflective*. Typically, participants in this study would engage in a *preliminary reading* first and then proceed to the most active part of the reading phase: *engaged reading*. Typically, participants dive into *reflective reading* once they feel like they have an adequate understanding of the proof in its entirety, usually after *preliminary* and *engaged reading*. However, it could be the case that participants may elect to *reflect* on a small subset of assertions in the proof before understanding the whole proof. This study also revealed potentially effective proof reading strategies that were not documented in prior studies described earlier (e.g., *Identifying, translating, and interpreting confusing assertions or notation using one's own words*, *forward-looking reading*, and *resource utilization*).

Additionally, this research may shed light on a relationship between the proof comprehension assessment model proposed in Mejia-Ramos et al. (2012) and proof reading strategies. Recall that Mejia-Ramos et al. (2012) argue that proof comprehension can be assessed *locally* and *holistically*. A proof can be understood either locally by "studying a specific statement in the proof or how that statement relates to a small number of other statements within the proof, or holistically based upon the ideas or methods that motivate the proof in its entirety." (p.6) For instance, several of the strategies participants employed during *engaged reading* phase could support *local* comprehension; whereas, strategies employed during *preliminary reading* and *reflective reading* may enhance *holistic* comprehension of a proof.

To sum up, the difficulty associated with proof comprehension can be assuaged by exposing students to strategies that can enhance proof comprehension. If future studies confirm that the proof reading strategies presented here are effective in improving proof comprehension, mathematicians could incorporate them into their teaching. For instance, professors who teach proof-based mathematics courses, such as introduction to mathematical proof courses, can explicitly communicate these strategies to their students. A professor could model some of the aforementioned strategies by incorporating proof reading activity during a lecture. Moreover, a professor can develop proof comprehension questions using the Mejia-Ramos et al. (2012) model and foster these strategies' usage.

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How do undergraduate engineering students engage with puzzle problems related to first-order differential equations?

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This paper explores how undergraduate engineering students engage with puzzle problems related to first-order differential equations. One hundred and thirty- five undergraduate engineering students engaged with four puzzle problems related to first-order differential equations in self-selected groups of two or three *students while their communications were audio recorded. The findings related to one of these problems are reported here. The results show that many students struggled with identifying how DEs could be used for modeling real-world situations, and as a consequence of that, they solved the puzzle using their physics knowledge. The findings suggest that more focus should be paid to including modeling (or puzzle) tasks in DEs courses to help engineering students engage in higher-order thinking, develop thinking skills and problem-solving strategies needed for their future careers and advanced courses.*

Keywords: Puzzle-based learning, differential equations, high-order-thinking, problem-solving strategies, thinking skills.

Introduction

Many engineering students graduated with the ability of solving routine mathematical tasks, but are not successful in applying what they have learned in solving real-world problems that require critical and creative thinking (Kamsah, 2004; Falkner, Sooriamurthi, & Michalewicz, 2012). This might be due to many students are limited to textbook questions that are solved using the topics discussed in the classroom (Michalewicz & Michalewicz, 2008). Furthermore, university mathematical courses do not appeal to several engineering and mathematics first-year university students (Klymchuk, 2017).

A number of university engineering and mathematics students drop out from their study "not because the courses are too difficult but because, in their words, they 'are too dry and boring'" (Klymchuk, 2017, p. 1106). Engineering graduates need strong communication skills and team-work mindset, but some have not developed these skills and values (Kamsah, 2004; Mills & Treagust, 2003).

Puzzle-Based Learning (PzBL) has been recognized as one of the approaches that could be used to develop students' problem-solving strategies and thinking skills (e.g., critical thinking, creative thinking, and lateral thinking) (Badger, Sangwin, Ventura-Medina, & Thomas, 2012; Falkner et al., 2012; Klymchuk, 2017; Michalewicz & Michalewicz, 2008). In recent years, several studies have explored the use of PzBL in the teaching and learning of calculus (Badger et al., 2012; Klymchuk & Staples, 2013). However, a literature search revealed no study exploring the impact of using PzBL in the teaching and learning of Differential Equations (DEs).

DEs play an important role in engineering and mathematics (Maat & Zakaria, 2011; Rowland, 2006). DEs are frequently used in different disciplines, including engineering, to model real-world problems and understanding real-world phenomena (Arslan, 2010). This study seeks to fill the gap by exploring how undergraduate engineering students engage in solving puzzle problems related to first-order DEs. The research question sought to answer in this study is *How do undergraduate engineering students engage with puzzle problems related to first-order differential equations?*

Puzzle-Based Learning

PzBL refers to engaging students with puzzle problems to enhance students' problem-solving strategies and thinking skills (Michalewicz & Michalewicz, 2008). It could be considered as a sub-set of Problem-Based Learning (Thomas, Badger, Ventura-Medina, Sangwin, 2013). PzBL could help students develop their competency in solving real-world problems in their future careers (Falkner et al., 2010, 2012). Puzzle problems can activate higher-order thinking as typically they are non-routine problems that require a great deal of *analyzing* the given information, and students often need to *create* a new strategy for solving those problems (Radmehr & Vos, 2020). PzBL helps students learn mathematical concepts meaningfully and increase students' motivation to learn mathematics (Klymchuk, 2017). PzBL impacts the classroom environment by making it more entertaining and increasing students' curiosity and participation in classroom discussions (Klymchuk, 2017).

There are three types of puzzle problems: *Sophism*, *paradox,* and *puzzle* (Klymchuk, 2017). A sophism can be defined as an "intentionally invalid reasoning that looks formally correct, but in fact, contains a subtle mistake or flaw" (Klymchuk, 2017, p. 1106). A paradox is a "surprising, unexpected, counter-intuitive statement that looks invalid but in fact is true" (Klymchuk 2017, p. 1106). Puzzle refer to "non-standard, non-routine, unstructured question presented in an entertaining way" (Klymchuk, 2017, p. 1106). Puzzles are similar to mathematical modeling tasks, which is an important part of engineering and mathematics education (Klymchuk, 2017).

A task is called a puzzle problem if it has most of the four following criteria: *Generality*, *simplicity*, *Eureka factor*, and *entertaining* (Badger et al., 2012; Michalewicz & Michalewicz, 2008). In relation to the *generality,* puzzle-problems "should explain some universal mathematical problem-solving principles" (Michalewicz & Michalewicz, 2008, p. 3). Regarding *simplicity,* the puzzle-problems should be easy to state and remember (Michalewicz & Michalewicz, 2008). To meet the *Eureka factor* criterion, the solution of the puzzle-problems should not be immediately intuitive and finding the correct path to solving them may take time and energy. When students find the correct approach to solve a puzzle problem after spending much effort, the feeling that they have at that moment is called *Eureka* moment- Martin Gardner's Aha! - (Michalewicz & Michalewicz, 2008). Finally, regarding *entertaining*, puzzle-problems should be enjoyable activities that encourage students to keep looking for a correct solution for puzzle problems (Michalewicz & Michalewicz, 2008). There is no need for all the puzzle problems to have the above criteria (Michalewicz & Michalewicz, 2008). The best puzzle problems for teaching mathematics are those that have both a conventional and a lateral thinking solution (Thomas et al., 2013). Lateral thinking and creative thinking are closely related (De Bono, 1990), and therefore engaging in solving puzzle problems could improving students' creative thinking (Klymchuk, 2017; Thomas et al., 2013).

Differential Equations

DEs play an important role in engineering and mathematics students (Maat & Zakaria, 2011; Rowland, 2006). DEs are used in many disciplines, including engineering, to model real-world situations (Arslan, 2010). It is important that engineering students develop a strong knowledge of how different problems could be modelled using DEs and how different types of DEs could be solved (Kwon, 2002).

There are three typical approaches for solving DEs: algebraic (analytic), qualitative (graphical), and numerical (Rasmussen, 2001). Analytical methods are techniques for finding the symbolic form of DEs' solutions, while qualitative and numerical methods focus on finding the graphical and numerical forms of DEs' solutions, typically with the help of technology (Rasmussen, 2001).

The majority of students in traditional introductory DEs courses are inclined to use analytical methods for finding solutions to DEs (Arslan, 2010; Keene, 2007). However, these methods are not effective in solving all types of DEs (Rasmussen, 2001). Furthermore, previous studies pointed out students' competencies in using graphical and numerical methods for solving DEs are limited (Camacho-Machín, & Guerrero-Ortiz, 2015; Czocher 2017). Many students struggle with solving DEs tasks, and the relationship between DEs and their solution is not meaningful for them (Arslan, 2010; Rasmussen, 2001) due to having an instrumental understanding of DEs (Arslan, 2010; Rasmussen, 2001).

Engaging students with mathematical modelling tasks is an effective way to improve students understanding of how DEs can be used to solve real-world problems; develop students' problem-solving skills; and increase students' motivation for learning DEs (Czocher, 2017). Furthermore, modelling tasks could help students develop their competencies in communicating their ideas, educate them as independent learners, and prepare them for their future careers (Ärlebäck, Doerr, & O'Neil, 2013). However, many students in traditional DE courses encounter difficulties finding and formulating a meaningful DE when engaging with modelling tasks (Rowland, 2006). These difficulties could be related to students' understanding of DEs being procedural instead of conceptual due to heavy emphasis on analytical techniques in traditional courses for solving DEs (Arslan, 2010; Rasmussen, 2001).

Research method

In this study, a sequential mixed method study (Yin, 2014) was designed to explore how undergraduate engineering students engage in solving puzzle problems related to first-order DEs. The participants were 135 undergraduate engineering students from a public university in the east of Iran. Four puzzle-problem tasks (one sophism, one paradox, and two puzzles) were designed and were validated by two senior lecturers of DEs. Then, the tasks were piloted with eleven mathematics students. After minor changes to the wording of the tasks, the participants engaged in solving these tasks in self-selected groups of two or three students and audio-recorded their communication. In this paper, we report how students engaged with one of the puzzles. The data were analyzed using an inductive content analysis approach (Vaismoradi, Turunen, & Bondas, 2013) as previous studies have not explored how students engage with such tasks in DEs.

The puzzle problem

This task explores how students engage in constructing a suitable DE using their knowledge of mathematics and physics. Regarding the entertaining criteria of puzzle problems, integrating physics with DEs increases students' motivation to learn DEs (Rowland & Jovanoski, 2004). Additionally, many students are familiar with such physics problems in their high school education or when passing an introductory physics course in their first-year university study; however, they might not have the experience of how DEs could be used for solving such problems.

At 9 o'clock this morning, Asghar and Farzad flew from Tabriz to Ankara on a private jet with the weight of 265 kg. On the way to Ankara, on Van Lake, at the height of 6,000 meters from the lake's surface, the jet's engines failed. Consequently, the jet crashed vertically at a speed of 300 km/h. Eight seconds after the engines failed, Farzad pressed the emergency exit button. After pressing the emergency exit button, it usually takes 4 seconds for the pilot and the co-pilot to be ejected. Do Asghar and Farzad succeed in rescuing themselves? (Consider the air resistance coefficient as $5\ kg/km$ *and the total weight of Asghar and Farzad to be 135 kg).*

Regarding simplicity, this task is not very long and relatively easy to understand and cannot be shortened if one wants to keep its context. By considering air resistance and gravity force, students could solve this task and reach the eureka factor when they find out whether Farzad and Asghar were rescued. For instance, they could use the following DE $v' = g - \left(\frac{k}{m}\right) v$ (where *v* is the velocity of the jet, *g* is the acceleration of gravity, *k* is air resistance coefficient, and *m* is the weight of the jet), and find the distance between the jet and the ground after 12 seconds. Regarding generality, students by engaging in this task, could develop further how they could use their knowledge of DEs to solve physics problems related to the falling object and, more generally, how DEs could be used for calculating acceleration and velocity.

Results

The majority of students' communications (forty-one groups (87%)) were related to modeling the situation using their physics knowledge, and they could not find how DEs could be used for solving this task: "what is the relationship between physics problem and DEs. I have not seen any physics problem that could be solved using DEs". The remaining six groups engaged in modelling the situation using DEs, and one group was successful in doing so.

Of the forty-one groups, twenty groups (42%) realized that this task is similar to physic questions that can be solved using the Newton's second law (i.e., $\sum F = ma$); however, twenty-one groups (45%) used other physical formulas, such as $v^2 - v_0^2 = 2a\,\Delta x$ and $x = \frac{1}{2}\,gt^2 + v_0 t$.

Sixteen groups out of the twenty groups (34%) identified that the sum of the jet's vertical external forces is the product of the constant mass and the vertical acceleration of the jet. Furthermore, they realized that the resistance of the air influences the jet's motion in the opposite direction, and it is proportional to the velocity of the jet**.**

Nine groups (19%) first solved the task without considering the air resistance, and they found that Farzad and Asghar would succeed in rescuing themselves. Thus, they concluded that when they were rescued without air resistance, subsequently, Farzad and Asghar would be rescued by taking into account the air resistance because the air resistance impacts the motion, in the opposite direction.

The other seven groups (15%) considered air resistance from the beginning of their calculations (Figure 1)**.** Thirteen groups (28%) had difficulties considering the real-world factors, i.e., air resistance and its impact on velocity and acceleration of the motion. A sample response was: "We do not consider air resistance in physics; I do not know how to take it into account. So, let us solve it without air resistance".

$$\sum F = ma \longrightarrow \text{forces in the direction of motion} - \text{forces in the opposite direction of motion} = ma$$

$$mg - kv = ma \Longrightarrow (400 \times 9{,}8) - \left(5 \times 300\,\frac{km}{h}\right) = 400a$$

$$\Longrightarrow 3920 - \left(15 \times 300 \times 10^3\,\frac{m}{h}\right) = 400a \Longrightarrow a = -5{,}984$$

$$h = \frac{-1}{2}at^2 + v_0 t + h_0 \Longrightarrow h = \frac{-1}{2}(-5{,}984)(12^2) + 300 \times 10^5 \times 12$$

$$= 430{,}848 + 36 \times 10^7 \simeq 3600$$

Asghar and Farzad have been rescued because $6000 - 3000 > 0$.

Figure 1. A sample response to the task without using DEs

As highlighted earlier, six groups (13%) constructed a mathematical model using DEs. These groups used the Newton's second law to construct the following equation, $m\,g - k\,v = ma$. Then, by considering $a = \frac{dv}{dt}$, they formed a DE model and used the separable technique for solving the DE. They also calculated c using the velocity of the jet when the engines failed ($t = 0$). Afterwards, these groups computed the jet's velocity 12 seconds after the engines failed and used $v^2 - v_0^2 = 2a\Delta x$ to find Δx, in order to compare it with 6000m, the height of the jet when the engines failed. They concluded Farzad and Asghar were rescued after 12 seconds because the jet has not crashed in the lake by that time**.** An excerpt from a group is provided below:

Mahdi (The names used in this paper are pseudonyms): We had this type of question in physics, but we did not consider the air resistance. I think we should use the Newton's second law.

Kian: yes, I agree, but we should write the equation based on the Newton's second law, and consider air resistance...

Mahdi: [after 5 minutes] We have $mg - kv = ma$ from the Newton's second law. If we consider $\frac{dv}{dt} = a$, we can calculate velocity in each moment.

Parsa: It seems reasonable. Let us solve the equation. [they solved the problem together]

$$\mathrm{mg} - \mathrm{kv} = \mathrm{m}\frac{\mathrm{dv}}{\mathrm{dt}} \Longrightarrow 4000 - 5\mathrm{g} = 400\frac{\mathrm{dv}}{\mathrm{dt}} \Longrightarrow 800 - \mathrm{v} = 80\,\mathrm{v}' \Longrightarrow 10 - \frac{\mathrm{v}}{80} = \mathrm{v}' \Longrightarrow$$

$$\mathrm{v}' = \frac{800-\mathrm{v}}{80} \Longrightarrow \frac{800-\mathrm{v}}{80} = \frac{\mathrm{dv}}{\mathrm{dt}} \Longrightarrow \frac{\mathrm{dt}}{80} = \frac{\mathrm{dv}}{800-\mathrm{v}} \Longrightarrow \frac{1}{80}\mathrm{t} = -\ln(800 - \mathrm{y}) + \mathrm{c}$$

By considering the conditions: $\mathrm{t} = 0$, $\mathrm{v} = 300$.

$$-\ln 500 + \mathrm{c} = 0 \Longrightarrow \mathrm{c} = \ln 500 = 6.2$$

$$\frac{1}{80}\mathrm{t} = -\ln(800 - \mathrm{v}) + 6.2 \overset{\mathrm{t}=12}{\Longrightarrow} \mathrm{v} = 363$$

$$\mathrm{v}^2 - \mathrm{v}_0^2 = 2\mathrm{g}\,\Delta\mathrm{x} \Longrightarrow 364^2 - 300^2 = 20\,\Delta\mathrm{x} \Longrightarrow \Delta\mathrm{x} = 2088.45 < 6000]$$

Kian: so, they were rescued.

The remaining five groups (11%) were not successful in solving the DE they had formed. Three groups (6%) had difficulty in calculating $\int \frac{dv}{mg-kv}$. For instance, one group considered $u = mg - kv$, and calculated the integral as follows: $\int \frac{du}{-ku} = \ln u = \ln \frac{mg-kv}{mg}$. The fourth group found the general solution correctly; however, they made a mistake in calculating the particular solution when substituting the initial values ($t = 0$, $v \simeq 83.3\ {}^{m}/_{s}$). The fifth group found the following DE $v'(t)+\frac{5}{m}v=g$, and then represented it based on the derivative of y, i.e., $y''(t)+\frac{5}{m}y'(t)=g$. However, they were not able to solve it. They highlighted that after solving the DE and founding a general solution for $y(t),\ y(12)$ should be calculated, and if it is less than 6000, Farzad and Asghar were rescued.

Discussion and conclusion

This paper explores how students engaged in solving a puzzle related to first-order DEs. The findings indicate that many students struggled with identifying how DEs could be used for solving the puzzle and consequently used their physics knowledge to solve the puzzle. Puzzles are similar to modeling tasks (Klymchuk, 2017), and therefore, our findings could be compared with those studies that explored how university students engage with modeling tasks.

The findings are in line with previous studies (Britton, New, Sharma, & Yardley, 2005; Crouch & Haines, 2004; Klymchuck, 2010; Rowland, 2006) that reported many students are not capable of using their mathematical knowledge for finding an appropriate model for solving real-world problems. For instance, Klymchuck (2010) highlighted that “many students have difficulties in translating the word problem into the mathematical formula and then deciding which mathematics they should use” (p. 82). This might be because students did not have enough experience in solving modeling problems in the context of DEs (Crouch & Haines, 2004). As Czocher (2017) highlighted, in many traditional DEs courses, lecturers incline to emphasize more analytical techniques in their teaching.

Our anecdotal observations also showed that many lecturers in their introductory DEs courses do not focus enough on solving real-world problems and modeling tasks using DEs due to time restriction and intense curriculum. If lecturers discuss modelling (or puzzle) tasks in lectures more often and include such tasks in students’ assignments and possibly exams, students would have more opportunities and motivation to develop their problem-solving strategies needed for successfully engaging with mathematical modeling or puzzle tasks, and could also help students develop their conceptual understanding of DEs (e.g., Arslan, 2010; Rasmussen, 2001).

Engaging students with puzzle problems could create opportunities for students to engage in productive struggle, one of the main elements of powerful mathematical classrooms (Schoenfeld, 2014). Additionally, one of the main purposes of mathematical service courses such as DE is preparing students for the mathematics they will encounter in their future courses or careers (Czocher, 2017; Falkner et al., 2012). Challenge students to use their mathematical knowledge to solve modelling (or puzzle) problems could help them develop problem-solving strategies and thinking skills needed for future careers (Falkner et al., 2012) or advanced courses.

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Impact of COVID-19 on a Community of Practice of First Year Mathematics Instructors

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The COVID-19 pandemic has had unprecedented ramifications on higher education. In this paper we describe how the pandemic impacted a community of practice of instructors involved in the instruction of first year mathematics courses. In particular, we use qualitative data from interviews and open-ended survey responses to describe how members of this community responded to the changes in instruction brought about by the pandemic, and in what ways the community adapted to support instructors in engaging in this new instructional format in order to sustain itself through this transition.

Keywords: Community of practice, Graduate Teaching Assistant, COVID-19

Student experiences in introductory mathematics courses play a significant role in retention at U.S. institutions. These courses often act as a barrier for STEM-intending students, resulting in many students switching from a STEM major to a non-STEM major, or failing to matriculate at all. Such impacts are more pronounced with students from historically-underrepresented populations (Ellis et al., 2016; Griffith, 2010; Laursen, 2019; PCAST, 2012). In response to the concerning impacts of these courses, numerous mathematics departments have started initiatives to restructure their introductory mathematics courses. In particular, several departments are embracing the use of student-centered practices as a way to engage students in the learning process and support increased student success. Since such practices are often novel to instructors, these efforts necessitate a parallel initiative to increase the level of support instructors receive for enacting such practices. Research suggests that a community of practice model may be valuable in helping instructors use these practices (e.g., Thompson et al., 2015; Viskovic, 2006). By engaging in a community of practice, instructors can access communal sources of knowledge and resources that enable them to make changes to their instruction.

Unfortunately, like many change efforts, these communities of practice can devolve if members do not actively work to sustain the community. Sustainability may be especially tenuous given the unprecedented ramifications of COVID-19 on higher education. Over time, these communities have established ways of operating and visions for instruction that COVID-19 rapidly disrupted. Moreover, the pandemic has exacerbated existing socioeconomic inequalities at the collegiate level. In one survey of students at a large public institution, lower-income students were reportedly 55% more likely to delay graduation because of COVID-19 than other students (Aucejo et al., 2020). Given that many departments are relying on communities of practice to increase equitable teaching and therefore improve equitable student experiences, it is crucial that the field learns how such communities may have responded to, and sustained themselves, after this major disruption in their practice. In this paper, we report on how one community of practice within University X's mathematics department adapted in response to the sudden transition to remote learning in Spring 2020.

Theoretical Framework

A community of practice consists of three fundamental elements: "a *domain* of knowledge, which defines a set of issues; a *community* of people who care about this domain; and the shared *practice* that they are developing to be effective in their domain" (Wenger et al., 2002, p. 27).

The *domain* of a community of practice is "what gives a group its identity and distinguishes it from a club of friends or a network of connections between people" (Smith, Hayes, & Shea, 2017, p. 211). A domain "consists of key issues or problems that members commonly experience" (Wenger et al., 2002, p. 32). It also creates a shared sense of accountability and contributes to the development of a practice. While the domain is the broad topic that the community focuses on, the *practice* is "the specific knowledge the community develops, shares, and maintains" (Wenger et al., 2002, p. 29). The practice involves a set of common approaches and shared standards that evolves as a "collective product" (p. 39). The *community* refers to the group of people brought together by the domain, as well as "the quality of the relationships that bind members" (Snyder & Wenger, 2010, p. 110). By interacting with one another, members of the community of practice "learn together, build relationships, and in the process develop a sense of belonging and mutual commitment" (Wenger et al., 2002, p. 34). Moreover, a strong community "encourages a willingness to share ideas, expose one's ignorance, ask difficult questions, and listen carefully" (p. 28), thus creating a "social learning system that goes beyond the sum of its parts" (p. 34).

The First Year Mathematics Community of Practice

In this paper, we focus on a community of practice whose shared *domain* contains two key components: a desire to improve undergraduate mathematics education, specifically through increased student engagement and learning in first year mathematics courses, as well as a related focus on creating robust formal and informal professional development which "inspires members to contribute and participate, guides their learning, and gives meaning to their actions" (Wenger et al., 2002, p. 28).

This *community* is formed by those involved in the instruction of first year mathematics courses, primarily including graduate students and faculty coordinators. We use the phrase first year mathematics courses to refer to all precalculus courses, calculus, and a first-year liberal arts mathematics course. The majority of instruction of first year mathematics courses at University X is provided by graduate students, either as recitation instructors in large lecture coordinated calculus courses, or as instructors of record (e.g., most precalculus courses are taught by graduate students as instructors of record). For the remainder of this paper we use the term graduate teaching assistants (GTAs) to refer to individuals who are either instructors of record or recitation instructors. GTAs also play a critical role in the leadership of first-year courses. Calculus recitation instructors are overseen by an experienced GTA who conducts regular meetings and observations of recitations. Similarly, precalculus courses are coordinated by a GTA whose main responsibilities include leading regular course meetings, writing assessments, and conducting observations; they also typically teach the course they are coordinating. All GTAs have shared offices in the same building - encouraging interactions about teaching and learning. First year GTAs are all placed in a large office so that they can build personal and professional relationships more easily. The university has also hired dedicated faculty members to support this community of practice - their main roles include providing professional support for GTAs and overseeing the coordination of these courses. Since their main role is to support GTAs' growth as instructors, and ultimately the success of these courses, these faculty members have offices on the same floor as most instructors who teach first year courses.

The *practice* of this community includes pedagogical knowledge that is used to support student learning in first year mathematics courses, curriculum development, assessment methods, and other activities related to teaching first year mathematics courses. The practice involves a set of common approaches and shared standards that evolves as a "collective product" (p. 39). For example, all these first-year courses are taught using active learning methods. Further, GTAs are given explicit opportunities to shape this collective product. As an example, the department has employed GTAs on numerous projects to improve first year courses (e.g., writing open-access online textbooks for these courses, developing online homework problems, and contributing to a shared wikispace of detailed lesson plans).

Purpose and Research Question

This research is part of a mixed-methods study to understand how this first year mathematics community of practice adapted to support GTAs and students in first year mathematics courses that were impacted by the transition to remote instruction brought about by the COVID-19 pandemic. For this paper we analyze qualitative data collected as part of this study to examine how GTAs influenced and interacted with the community of practice as it evolved, and to what extent the community was able to sustain itself. To this end, we ask the following research question: *How was the first year mathematics community of practice impacted by the transition to remote instruction due to COVID-19, and how was it able to sustain itself through this transition?*

Methods

We gathered data in Summer 2020 by conducting 45–60 minute semi-structured interviews with 12 GTAs (four of whom were coordinators) who were involved in teaching first year mathematics courses. We also interviewed four faculty members who were actively involved in mentoring or overseeing the GTAs who were involved in these courses. These faculty members served as either course coordinators and/or program directors of the first year courses. Our research questions and interview questions were informed by prior attempts at categorizing this institution's community of practice, as well as literature on communities of practice and departmental and institutional change. Interview questions focused on how key dimensions relevant to a community of practice may have changed from the participant's perspective, including: norms for and ways of engaging with other members of the community, instructional resources, and instructional practices. Recordings of the interviews were transcribed and analyzed in Dedoose. An initial coding scheme of theory-driven codes was developed based on our previous work analyzing departmental change as well as the communities of practice literature (DeCuir-Gunby et al., 2011). Researchers used an iterative process to refine this coding scheme. Once the coding scheme was stable, each researcher used this scheme to independently code 11 of the interviews, and reconcile in groups of at least three researchers (Creswell & Poth, 2018). Based on this analysis the researchers believed that data saturation was achieved, and moved onto a second level of analysis. Individual researchers were assigned codes (e.g., communication) to summarize in a report that included emerging themes and supporting evidence (e.g., quotes, contextual data, etc.). Researchers specifically sought out competing pieces of evidence when developing themes related to each code (Stake, 1995).

Once we generated this report, we read through it to understand how each dimension of the community of practice (domain, community, and practice) may have been impacted by the transition. The report was then modified to include this analysis. In addition, we triangulated the interview data with open-ended responses from a climate and culture survey that was administered to all graduate students at the beginning of Summer 2020. These open-ended response questions asked GTAs to comment on how COVID-19 had impacted their interactions with members of the department in terms of who they go to for advice, who they go to for instructional materials, and who they go to in order to discuss teaching. GTAs were also prompted to describe how the change to an all-online teaching environment had impacted the community of practice of GTAs in the mathematics department at University X. The response rate for this survey was high amongst GTAs involved in first year mathematics courses (87%) and the majority of the open-ended response questions from this group were filled out (ranging from 69% - 82%). We read these open-ended responses and the remaining interviews for any evidence that supported or contradicted emerging themes.

Findings

Our findings are structured into two main sections: what changed in the community of practice and what did not change due to the transition to remote instruction. For each section, we describe overarching themes that emerged from our analysis of the interview and survey data and provide evidence of how the community of practice was impacted by changes due to COVID-19.

What Changed

Structure of courses. The abrupt transition to remote instruction necessitated rapid and significant structural changes to the department's first year mathematics courses. The most immediate change was that all introductory courses were to follow an asynchronous model, as mandated by university leadership. This model was chosen, in part, because of concerns about varying levels of student access to technology. This gave department members two weeks to re-envision how to preserve the active learning and student engagement that had been happening in these courses, while following the asynchronous requirement. Prior to the transition, all precalculus courses, as well as the first-year liberal arts mathematics course, were designed to support small group work on problems contained in a course packet, with small lectures interspersed throughout the class as necessary to guide students as they worked through these problems. In calculus, a similar model was followed in recitation sections. To replace this model most courses chose to implement discussion boards in an effort to engage students in peer conversations about mathematics. The discussion boards were based on content that was in the course packets, with the videos based on content that would normally be lectured on in the precalculus courses. Exams were also re-imagined to be distributed by online platforms. A precalculus coordinator shared, "we made our exams shorter in question length and we changed the type of questions we asked too. We tried to ask things [that] were more open-ended so they couldn't just steal it off of Chegg," with similar changes made in other courses.

To enact these plans, GTAs and coordinators shared the task of developing the new curricular materials to cover the remaining course content under the guidance of the faculty members who oversaw all coordination efforts. Some GTAs relied on community developed instructional resources, most notably the shared wikispace, to support their new role as curriculum designer. For example, one GTA shared that they "used the [wikispace] more after the transition while preparing the videos that I made. I found it helpful to use the structure of the lesson plans provided for structuring my lecture videos." To support student engagement, course instructors also took on the new role of managing and encouraging online discussions, rather than facilitating in-person discussions.

With the changing roles and responsibilities that followed the transition to remote instruction, so did the power structure in making course decisions. One GTA shared that in developing new curriculum materials to use across all sections, they lost control over individual instructional decisions. As one GTA described, before the transition,

> I had full control over how I'm going to present this material, how I'm going to utilize my [learning assistant], how I'm going to interact with students, but after the transition,

especially because like--I mean I'm thankful that I didn't have to create material for every single section, but like I didn't really have much control on how the students are getting this information.

Communication. After the transition, the ways GTAs communicated with other members of the community of practice changed dramatically. GTAs at University X had previously relied on informal, convenient interactions in hallways and shared offices to exchange teaching ideas and build personal relationships. Since this was no longer possible, the majority of GTAs reported interacting with fewer of their peers after the transition. One GTA explained that, "the move online has shifted us from a community of GTAs to various bubbles based on course." Another GTA stated that the community of practice had gotten "more segmented. We don't interact regularly with people outside of our immediate social spheres the way we would in-person." Overall, this led to a sense of isolation among many GTAs. As one GTA described,

Well it has become a whole lot more isolating. Before, I would run into random people and casually have a discussion about instruction, or other GTAs would pop into my office and have a chat with me. Now I only have these discussions with a few close friends in the department.

Informal ways of communicating were often replaced with more formal, less convenient modes of communication (e.g., emails, regular Zoom meetings). As one GTA coordinator described, communication became more scheduled, and "less impromptu. I feel like more people emailed me after than usual because you can always drop a quick email, kind of like how you can always just drop by an office." While the number of emails may have increased, there is evidence that the lack of convenient ways of communicating meant that some people were less likely to seek advice unless they felt they had to. As one participant said,

I haven't had the opportunity to drop by anyone's office, so I only contact them for advice if I have something I really need to ask; things can't come up in casual conversation and I'm more hesitant to email someone than drop by their office.

That said, it seems that overall GTAs felt supported by the community, in part due to regular, weekly formal communication in coordination meetings and emails. For most of the courses, the weekly meetings played an essential role in keeping instructors connected to each other. Several instructors mentioned that these meetings became longer and were more important after the transition to remote learning. However, Calculus did not continue regular meetings for recitation leaders. The GTA coordinator of this course noted how interactions between recitation leaders notably decreased and suggested that these weekly meetings should be reinstated for future semesters to check in with recitation leaders and support their instruction. Recitation instructors expressed how this decrease in interaction hindered their ability to learn from each other, "in my experience, I have interacted with fellow GTAs much less frequently after the transition. That is a bad thing as it prevents discussion about things that work and things that don't."

Identity. Another major component that changed after the transition to remote learning was how graduate students viewed their identities as teachers. Due to the university requirement that courses be held asynchronously, graduate students were no longer meeting with their students on a regular basis. As one GTA stated, "another thing...with asynchronous courses is it takes away, kind of like the feeling that I am actually teaching them because even if I produce some of the material, I don't see them learning or how they're using it." Moreover, under the coordination structure, GTAs shared the responsibility of course development tasks such as creating course videos and writing discussion board questions. In some courses, GTAs split into groups with some focusing on course videos, while others focused solely on writing exams. This distribution of curriculum development led some GTAs to view themselves as more of a content manager, rather than a teacher. As another GTA stated,

> GTAs teaching convened courses have really been hurt by this transition from an experience perspective. The role of each individual GTA in teaching their students has gone from being of central importance to having almost no impact, since most lessons are now other GTAs teaching the material.

Unfortunately, the asynchronous structure provided more opportunities for exam dishonesty. As a result, GTAs felt like they needed to police their students more and be on the lookout for cheating. There were frequent discussions in coordination meetings about how to mitigate cheating and administer exams online. As one GTA noted, “there were a lot of talks about how to properly implement an exam online that would be resistant to cheating.” This shift from a focus on teaching to preventing cheating was a noticeable change for some of the GTAs in terms of their identity as an instructor. As one GTA said,

> I also have not really been one to worry about cheating very much in the past because I never felt like I had to due to how the course was set up. Now that there are more opportunities for cheating, I feel like I have to actively do more to prevent it.

While this focus on cheating was overall a negative experience for GTAs, many GTAs expressed that because of the transition, they became more aware of equity issues. As one GTA said, “I believe it's heightened all of our awareness of the inequality in learning resources that our students have simply because of the difficulties some might have in procuring a good Wi-fi connection.” Another GTA mentioned that what he had learned from the transition would impact his practices in the future, saying that he thought Zoom office hours would be helpful in the future in order to encourage more students to attend.

What Stayed the Same

A major commonality throughout many of the interviews was the desire to preserve active learning in these courses. As one faculty member explained,

> So the first step was then to look at the University's requirements, and in particular the asynchronous requirement and to think through what it meant to do math asynchronously while prioritizing active learning. So that was sort of our, our motivation was that it's got to be asynchronous and it has to get students active.

Despite this, some GTAs felt that the transition to asynchronous instruction was incongruent with the established values of the community, and department as a whole. As one GTA said, “almost everything is asynchronous. I understand why this is. However, I feel that this goes against the teaching ideas of the department.” Another GTA said that the shift to remote instruction changed the focus of teaching to be on how content should be delivered, rather than focusing on interactions with students. When speaking of challenges brought about by the transition, almost all respondents expressed concerns about student engagement – reflecting their shared desire to continue to engage students in whatever ways they could.

Another theme throughout these interviews was the desire for GTAs to retain some ownership of these courses. During the initial weeks of the pandemic, a faculty taskforce and the faculty coordinators were forced to make radical changes to course structures within a period of two weeks, allowing for little input by GTAs. However, once the general structure of these courses was established, GTA coordinators resumed their roles in the decision-making process to put these structures into place. Many of the GTA coordinators we spoke with expressed that they continued to have quite a bit of agency in directing the course and, in some cases, they felt that their role had expanded. Moreover, coordinators relied heavily on feedback and support from

GTA instructors to make changes throughout the remainder of the semester. As one GTA mentioned, "a lot more of the instructional materials were entrusted to the GTAs." This underscores the level of trust placed on GTAs to support these courses. Although some GTAs felt like they lost control over individual instructional decisions, as a whole, GTAs retained a significant role in the operation of these courses.

Discussion

Despite the numerous challenges posed by COVID-19, University X's first year mathematics community of practice continued to operate through the transition to remote instruction, providing support for GTAs as they strove to engage students in an asynchronous environment. Based on the analysis from this study, it is clear the community focused on retaining structures that allowed for increased communication (e.g., regular weekly Zoom meetings), while also engaging GTAs in creation of new materials which allowed GTAs to continue to collaborate with another about teaching, even if the actual face of teaching changed dramatically. Moreover, this situation allowed for some unexpected growth in GTAs as teachers. Many GTAs reported thinking differently (and more often) about equity issues than they had in the past, and some GTAs discussed how these experiences may influence their teaching in the future. Further, GTAs gained experience with online tools to support student engagement, and developed materials that can be used in future semesters (both for in-person and online courses) to support different instructional models. As an example, since these data were collected, these courses have switched to synchronous models that were heavily influenced by instructional insights from Spring 2020. In particular, the instructional videos that were developed in Spring 2020 were used to inform new video development over the summer; these videos are currently being used to support a flipped-classroom model in several of these courses. While we have not conducted a formal study on the community of practice since Spring 2020, there is strong anecdotal evidence that the community is still thriving, despite the challenges it has faced in the last year. One possible factor contributing to the resiliency of this community, that we have not yet touched upon, is that the community had already been formed to accommodate radical changes. In particular, the *community* is always in flux due to GTAs graduating and shifting into new roles within the department on a semester basis. As such, in its original formation the community of practice was forced to accommodate this by developing permanent structures and roles (e.g., the faculty members who oversee coordination of these courses). Future studies should explore this relationship further to better understand how the community is sustained.

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Graduate Student Peer Mentoring Experiences as a Component of a Multi-Faceted Graduate Teacher Training Program

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Peer mentoring programs are one approach to improving the pedagogical development of mathematical sciences graduate students. This paper describes the peer mentoring experiences at three institutions that have implemented a multi-faceted GTA professional development program. Data was collected from surveys and focus groups conducted with graduate teaching assistants at each institution regarding mentees' ratings of their mentors, mentors' ratings of their impact on mentees, mentors' impressions of the benefits and challenges of peer mentoring, and mentees and mentors' ratings of program components related to support from mentors, their TA coach, program staff, and other graduate students. Most GTAs found value in participating in the peer mentoring program. While the mentees found their mentors to be significant to their own success and effectiveness, the mentors did not rate themselves as high as the mentees rated them with respect to their own significance in impacting the effectiveness of their mentee.

Keywords: graduate teaching assistants, peer mentoring, GTA training

Graduate Teaching Assistants (GTAs) in the mathematical sciences are often responsible for a significant portion of the entry-level undergraduate instruction at doctoral-granting universities in the United States (Eller, 2017). Consequently, departments in the mathematical sciences across the country have been implementing instructional and professional training programs to enhance GTAs teaching abilities; the goal is to improve existing and future undergraduate mathematics pedagogy (Gutmann et al., 2005; Ellis, 2014). In an assessment of a GTA professional development program at a large doctoral-granting institution, Ho and Pilgrim (2020) found that most of the participants envision teaching as a significant, if not, the main role in their future careers; furthermore, all participants pointed out that along with the positive outcomes already happening in their classes because of the pedagogical training, their communication skills and teaching confidence were meliorating as well.

Peer mentoring is one of the formal components that institutions have introduced in order to help develop GTAs teaching and research abilities (Flores-Scott & Nerad, 2012). Based on the classical mentor-apprentice model, Treston (1999) describes mentorship as a "relationship in which the mentor provides support, advice, feedback and guidance" to the mentee. Moreover, mentor-mentee conversations, especially after a class observation, are highly valued by GTA peer-mentors as a gateway to constructive criticism (Yee & Rogers, 2017). Additionally, Browne-Ferrigno and Muth (2012) explain that peer relationships in higher education consist of mutual support and sharing of resources. Fittingly, it has been found that peer-mentoring relationships among graduate students help them acquire knowledge and grow professionally (Lorenzetti et al., 2019). Considering the impact of peer-mentoring in GTAs' pedagogical training, it is imperative for universities to pursue its betterment and research its foundation.

Belnap and Allred (2009) reported the need for research that explores the reasons why graduate assistant development programs impact their participants and how it happens, and not just the outcomes of this impact. Hence, this paper explores mentees' and mentors' perceptions of the peer mentoring program during the first year of the program implementation and during the March 2020 transition to emergency remote instruction due to the COVID-19 pandemic. semester.

This study aims to address this need by examining the experiences of mathematics and statistics GTAs at the three institutions where PSUM-GTT operates, leading to the three research questions that guided this study.

1. How did mentees and mentors perceive the mentor's impact on the mentee's success as an educator and on their transition to remote instruction and/or tutoring?
2. What reasons did the mentees and mentors provide for their ratings?
3. What benefits and challenges did mentors identify in terms of their own development as an educator?

Theoretical Framework

The PSUM-GTT program was based on a social theory of learning by Wenger and Lave (1991). We claim that mentors and mentees learn and grow through their membership in intersecting communities of practice, namely the communities of mentors and of mentees, both of which are nested within the departmental teaching and learning community. Professional learning communities in educational settings have been shown to have a positive impact on both pedagogy and on student outcomes (Vescio, Ross, & Adams, 2008). As GTAs meaningfully engage with peers and faculty mentors in these communities, they develop their pedagogical expertise and refine their practices (Lave & Wenger, 1991; Chi, 2006). Consequently, GTAs become more active in the social-professional communities of educators within their departments, and they shape their identities in relation to those communities (Wenger, 1998). We also posit that the impact of participating in these communities of practice is magnified for those GTAs who serve as both mentees and as mentors, as the viewpoints obtained through their roles provides varied insights into the practice of teaching. Relevant here, we also believe that serving as a mentor contributes to the development of the GTAs as future professional educators.

The Context: The PSUM-GTT Program

The Promoting Success in Undergraduate Mathematics Through Graduate Teaching Assistant Training (PSUM-GTT) program is a comprehensive graduate teaching assistant (GTA) training program in mathematical sciences that was designed and refined at one institution and is being replicated at two peer institutions. The program components include a first-year teaching seminar, peer mentoring and support from a peer TA coach, a Critical Issues in STEM Education seminar, and K-12 outreach to inform the GTA's understanding of the paths that students take before entering the university. The program goal is to strengthen the teaching capabilities of mathematical sciences GTAs in order to improve the academic outcomes of the undergraduates that they teach. Intended outcomes include GTAs' increased preference for student-focused instruction, satisfaction with their teaching training and mentoring, increased attention to equity and inclusive pedagogy in the classroom, and decreased rates of their undergraduate students earning grades of D or F or withdrawing.

Peer Mentor Training

The peer mentor training offered in the PSUM-GTT program is conducted every fall semester and requires approximately 8-10 hours of time spread over two asynchronous training modules and up to 3 synchronous meetings. The mentor training goals are to (a) clarify the role and purpose of the mentor, (b) provide training that will support mentors in fostering a good working relationship with mentees, (c) provide tools that promote conversations and reflection about teaching and learning, (d) provide training in effective coaching tools and techniques, (e) train mentors to conduct teaching observations and provide effective feedback, and (f) provide support and mentoring to the mentors.

The coaching tools that are demonstrated in the training include: (a) supporting GTAs in considering and being open to new perspectives about teaching and learning, (b) bringing awareness to how people listen and to introduce the concept of listening like our lives depend on it, (c) teaching mentors how to seek permission before giving coaching, (d) training mentors in giving specific and concrete praise to their mentees, (e) teaching mentors how to hold space without giving coaching so that mentees can process their struggles, burdens, or frustrations, and (f) training mentors in effective goal setting.

Method

Data Collection

At the end of the spring 2020 semester, all 46 graduate students participating in the program completed an online survey in Qualtrics that addressed mentoring relationships, spring 2020 teaching experiences (including a focus on active learning and equity in the classroom), and overall impression of the program components. This paper focused only on the following survey items relating to mentoring: a pair of Likert survey items that asked mentees and mentors to rate the mentors influence on success as an educator and transition to remote instruction using a 5-point scale, the open-ended questions that immediately followed these Likert items that prompted the respondents' rationale for ratings, one open-ended item each about positive and negative impacts of being a mentor, an open-ended and a single Likert item about overall perception of the mentoring component of the larger PSUM-GTT program.

Participants

There were 11 graduate student mentors, 12 mentees, and 1 TA coach at Institution A; 8 graduate student mentors, 3 faculty mentors, 12 mentees, and 1 TA coach at Institution B; 10 graduate student mentors and 12 mentees at Institution C, all of whom chose to participate in the comprehensive training program. At Institution A, 56.5% of participants identified as male and 43.5% identified as female. Approximately 65.2% of participants were international students. At Institution B, 46.5% of participants identified as male and 43.5% identified as female. Approximately 13.0% of participants were international students. At Institution C, 45.5% of participants identified as male and 54.5% identified as female. Approximately 40.9% of participants were international students. All three universities are considered to be research-intensive.

Analysis

Descriptive statistics (e.g., means and percentages) were used to summarize Likert survey items. Thematic analysis (Braun & Clarke, 2006) was used to analyze the open-ended survey responses.

Results

Mentees' Ratings of Mentor's Influence

Approximately 67% of mentees at Institutions A and C and 83.4% at Institution B rated their mentor as being at least somewhat significant in their effectiveness as a GTA (i.e., instructor of record, recitation section leader, tutor) during the 2019-2020 academic year (see Table 1). At Institutions B and C, mentees' ratings with regards to their mentors' influence on their transition to remote instruction were lower. Approximately 50% of mentees at Institution B and 33% at Institution C rated their mentor as being at least somewhat significant in their transition to remote instruction during the Spring 2020 semester due to the Covid-19 pandemic.

Table 1. Mentees' Rating of Mentor's Significance on Mentees' Effectiveness

Response Category	**Overall, how significant do you believe your mentor was to your success ...**					
	as a GTA this school year?			**in your transition to remote instruction?**		
	Inst. A	Inst. B	Inst. C	Inst. A	Inst. B	Inst. C
Not at all significant	1 (8.3%)	1 (8.3%)	0	1 (8.3%)	4 (33.3%)	1 (8.3%)
Not very significant	3 (25.0%)	1 (8.3%)	0	2 (16.7%)	2 (16.7%)	3 (25.0%)
Somewhat Significant	3 (25.0%)	3 (25.0%)	1 (8.3%)	4 (33.3%)	2 (16.7%)	1 (8.3%)
Significant	3 (25.0%)	2 (16.7%)	6 (50.0%)	2 (16.7%)	3 (25.0%)	2 (16.7%)
Very significant	2 (16.7%)	5 (41.7%)	1 (8.3%)	2 (16.7%)	1 (8.3%)	1 (8.3%)

When reporting reasons for their ratings of their mentor's influence on their success as GTA during the 2019-2020 school year, positive impacts were attributed to the mentor giving advice/help (n = 9), providing encouragement (n = 5), and providing feedback on teaching (n = 3). While some mentees reported that there wasn't much communication with their mentor after the switch to remote instruction (n = 6) and/or that they relied on or were supported more by a course coordinator than their mentor during the transition (n = 2), others still found their mentor to be a source of advice (n = 4) or support and encouragement (n = 4). The three representative quotes below illustrate why mentees believed their mentors had a positive impact on their success as GTAs and in their transition to remote instruction. A mentee from Institution A stated, "I got help from my mentor at any time and also she always encouraged me." One mentee from Institution B reported, "My mentor was fantastic. [...] He also gave great feedback on how I can improve in the classroom." Another mentee from Institution C stated, "My mentor was a constant source of encouragement and reassurance, especially in the shift to remote instruction."

Mentors' Ratings of Their Own Influence on Mentees

While 83.3% of mentors at Institution C rated themselves as being at least somewhat significant in their mentee's effectiveness as a GTA during the 2019-2020 academic year, far fewer mentors rated themselves in a similar manner (58.4% and 33.4% at Institution A and B, respectively) (see Table 2). When discussing their reasons for how they rated their own influence, some mentors were more confident in their impact, such as one mentor at Institution B who replied, "I felt like I saw my mentee improve in concrete ways throughout the year." Some mentors reported that they specifically helped their mentees via answering questions and addressing concerns (n = 4), sharing ideas (n = 1), providing guidance about social or cultural aspects related to teaching (n = 1), and building the mentee's confidence (n =1). However, some (n = 4) believed that their mentees would have succeeded either way. For example, one mentor at Institution B stated, "My mentee is a good teacher, and I think she would still be a good teacher without me" and one mentor at Institution C reported, "My mentee was already a good teacher, but through our conversations, I believe we were both able to improve."

Table 2. Mentors' Rating of Their Own Significance on Mentees' Effectiveness

Response Category	**Overall, how significant do you believe your mentoring was to your mentee's success ...**					
	as a GTA this school year?			**in the transition to remote instruction?**		
	Inst. A	Inst. B	Inst. C	Inst. A	Inst. B	Inst. C
Not at all significant	1 (8.3%)	1 (8.3%)	0	5 (41.7%)	4 (33.3%)	0
Not very significant	2 (16.7%)	3 (25.0%)	0	1 (8.3%)	2 (16.7%)	7 (58.3%)
Somewhat Significant	4 (33.3%)	2 (16.7%)	8 (66.7%)	2 (16.7%)	2 (16.7%)	1 (8.3%)
Significant	3 (25.0%)	2 (16.7%)	1 (8.3%)	4 (33.3%)	0	1 (8.3%)
Very significant	0	0	1 (8.3%)	0	0	1 (8.3%)

Mentors' ratings of their own impact on their mentee's transition to remote instruction during the pandemic were lower, with approximately 50% at Institution A, 16.7% at Institution B, and 24.9% at Institution C rating themselves as being at least somewhat significant. Some mentors (n = 4) were able to see that they helped their mentees during this time, although the amount of credit they gave themselves varied. For instance, one mentor at Institution B said, "I was able to provide [mentee] with a little help and resources, but I don't feel I made a big impact." Another mentor at Institution A shared, "[The impact during the transition] is significant in terms of learning from each other by sharing experience, ideas, and some problems."

Some mentors felt they had less of an impact on their mentees during the transition to remote instruction. There were several reasons for this, including either the mentee or mentor not teaching during the spring semester (n = 4) or already teaching fully online (n = 1), the department faculty and staff providing the bulk of the support during the transition (n = 4), and diminished frequency of interaction (n = 6). For example, one mentor at Institution C reported, "[...] she did not have many issues, and she had a course coordinator who took over a lot of her duties, so she was able to make the transition smoothly without much help from me." Another mentor at Institution B said, "We didn't really meet very much once remote instruction started."

Mentors' Perceptions of Benefits and Challenges of Mentoring

Mentors shared that they believed that being a mentor in the program had a positive impact on them as instructors as they learned from their mentees (n = 10), reflected on their own teaching and/or growth (n = 13), grew their network (n = 2), and improved their leadership and/or listening skills (n = 3). A mentor at Institution A said, "I grew as a teacher. [...] this made me reflect on my teaching and make changes on it." One mentor at Institution B stated, "I always think there is a positive impact on my teaching by being a mentor. The act of mentoring and the process of thinking about teaching and how it could be better can always help." One mentor at Institution C reported, "It was fantastic. I was able to learn from someone by teaching."

Most mentors reported no negative impacts of the mentoring program. While two mentors mentioned the time commitment as a cost, neither indicated that the negatives outweighed the benefit. While one mentor said that having to assess why they taught the way that they did shook their confidence some, the other saw that same personal reflection as a positive. For example, one mentor at Institution C reported, "I was forced to question everything I did as a teacher, [...] In doing so I found habits that needed pruning, and also became more willing to attempt diverse strategies in the classroom."

Ratings of How Beneficial the Peer Mentoring Was to Mentors and Mentees

Mentees and mentors were asked to rate how beneficial the mentoring component of the training program was to them. Of those that answered the question, approximately 83.3% of the mentors at Institution A, 75.0% at Institution B, and 100% at Institution C rated the mentoring component as at least somewhat beneficial to them. Similarly, approximately 83.3% of the mentees at Institution A, 100% at Institution B, and 100% at Institution C rated the mentoring component as at least somewhat beneficial to them.

Discussion and Implications

Overall, the majority of mentors and mentees rated the peer mentoring component of the training program to be at least somewhat beneficial to them, which is consistent with previous literature about peer mentoring programs. The process of actively reflecting on their teaching in these intersecting learning communities, both independently and with their mentor/mentee counterpart, supported the refinement of the pedagogical practice of both the mentee and the mentor. Based on feedback from the mentees and mentors, the program had positive impacts on the mentees' success as GTAs and their transition to remote instruction when the COVID-19 pandemic started. The mentors additionally reported positive effects on their own growth as an instructor, as well as on their leadership and listening skills. While a practical concern related to GTA training and mentoring is the time commitment, few students mentioned this in their responses. Consequently, mentors and mentees alike belong to a departmental community where their teaching practices improve, and their growth occurs throughout their time as mentees and later as mentors, even during unexpected circumstances such as the sudden transition to remote instruction due to the COVID-19 pandemic.

While mentees felt that their mentors made significant contributions to their teaching effectiveness, many mentors, in particular at Institution B, didn't feel that their contribution was as significant as their mentees gave them credit for. All of the mentors from Institution B had been a mentee prior to serving as a mentor and all but two had served as mentors in prior years. It is possible that these prior experiences of being mentored and/or being a peer mentor over multiple years, which are different communities of practice, may be adding another filter through which some of the mentors view their contribution to the development of their mentees progress as an educator and their own identities as effective mentors. The issue of self-efficacy as a mentor is interesting and warrants additional research. How is prior experience in the program either serving as a mentor or participating as a mentee impacting mentor self-efficacy? In what ways are mentors with low self-efficacy as a mentor, engaging differently with their mentees? What impact, if any, is that having on mentees? What can be done to improve the self-efficacy of mentors?

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Sense of Belonging in Two First-Semester Calculus Pathways

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Retention in Science, Technology, Engineering, and Mathematics (STEM) continues to be a problem in the U.S. Prior research indicates that students leave STEM due to poor instruction and low sense of belonging in introductory STEM courses. Incorporating active learning into these courses, especially Calculus, has potential to support students in developing a greater sense of belonging and staying in STEM. This study investigates students' sense of belonging in two versions of introductory Calculus – a standard course and a non-standard course infused with active learning. Results indicate that students in the two courses recognized differential opportunities to engage in active learning. While there was no significant change in students' sense of belonging over the course of the semester in either course, students in the active learning course reported significantly higher sense of belonging than students in the standard course, both early in the semester and at the end of the semester.

Keywords: undergraduate calculus, active learning, sense of belonging

Problem

The U.S. has a retention problem in undergraduate STEM. Less than half of all STEM-intending students complete a STEM degree, and the students who leave STEM are disproportionately women and students of color (Chen et al., 2013; PCAST, 2012; Seymour & Hunter, 2019). To expand and diversify the pool of STEM graduates, the PCAST's 2012 report called for adoption of empirically validated teaching practices at the undergraduate level, especially in the first two years of STEM coursework. And yet almost 10 years later, we continue to lose students throughout the STEM pipeline.

Students who leave STEM often cite poor instruction in introductory STEM courses as a reason for leaving — citing Calculus in particular (Ellis et al., 2014; Seymour & Hunter, 2019). In fact, students are less tolerant of poor teaching now than they were 20 years ago (Seymour & Hewitt, 1997; Seymour & Hunter, 2019). Students reported "good teaching" as engaging and interactive and "bad teaching" as boring and non-interactive (Seymour & Hunter, 2019).

In addition to concerns about pedagogy, students who leave STEM also report a low sense of belonging in (Rainey et al., 2018; Seymour & Hunter, 2019; Shapiro & Sax, 2011). A student's sense of belonging is their "sense of being accepted, valued, included, and encouraged by others (teachers and peers) in the academic classroom setting and of feeling oneself to be an important part of the life and activity of the class" (Goodenow, 1993, p. 25). Students with low sense of belonging are more likely to leave STEM, and these students are disproportionately women and students of color (Rainey et al., 2018; Seymour & Hunter, 2019).

Calculus can be an especially critical leak in the STEM pipeline. It is required of most STEM majors, and is usually taken in students' first year. Thus, Calculus is often one of students' first experiences with undergraduate STEM coursework. In addition, it is often a prerequisite or corequisite for other STEM courses, and so students who perform poorly may be prevented from taking other courses in their major. Consequently, students' experiences in Calculus can be influential in their decision to persist in STEM (Ellis et al., 2014; Seymour & Hunter, 2019).

Relation to Literature and Conceptual Framework

This study investigates relationships between students' sense of belonging and opportunities to engage in active learning in Calculus. In this section, sense of belonging and active learning are defined and a theoretical argument linking active learning and students' sense of belonging is presented.

Students' Sense of Belonging

Sense of belonging refers to the level at which one feels connected to a particular environment, or feels accepted and appreciated by others in that environment (Rosenberg & McCullough, 1981). According to Strayhorn (2019), sense of belonging is a basic human need and other processes like listening to a lecture and studying for a test cannot be completed until the basic need is met. Thus, sense of belonging can influence how students interact in a classroom. This study adopts Goodenow's (1993) definition of sense of belonging as the "sense of being accepted, valued, included, and encouraged by others (teachers and peers) in the academic classroom setting and of feeling oneself to be an important part of the life and activity of the class" (p. 25).

Researchers have studied students' sense of belonging in math or STEM courses using a range of approaches. For example, Good et al. (2012) developed a Mathematical Sense of Belonging (MSoB) instrument to measure students' sense of belonging in mathematics. A factor analysis revealed five distinct factors contributing to one's sense of belonging: Acceptance (e.g., "I feel like I fit in"), Affect (e.g., "I feel calm"), Desire to Fade (e.g., "I wish I could fade into the background"), Trust (e.g., "I trust my instructors to be committed to helping me learn"), and Membership (e.g., "I feel that I belong to this class"). Rainey et al. (2018) conducted interviews with students to investigate their sense of belonging in STEM. Students' reasons for feeling they did or did not belong were categorized into four central themes: interpersonal relationships, science identity, personal interest, and competence. Lahdenperä et al. (2020) conducted a study in which students completed the MSoB survey and responded to two additional items asking about positive and negative factors contributing to their sense of belonging in math. Based on their analyses, Lahdenperä et al. expanded Rainey et al.'s framework, identifying learning environment as a distinct category influencing students' sense of belonging. Here, learning environment refers to "student perceptions of elements in the teaching practices, such as lectures and small group discussions" (p. 484).

Research indicates that students with a higher sense of belonging are more likely to persist in STEM, and that women and students of color are less likely to develop a strong sense of belonging in STEM than their white male classmates (Rainey et al., 2018; Seymour & Hunter, 2019; Shapiro & Sax, 2011). Good et al. administered their survey to students at a highly selective university and found sense of belonging in math to be a significant predictor of intent to study math in the future. Rainey et al. (2018) found students' sense of competence and ability to form interpersonal relationships to be most influential for students' decisions to persist in STEM.

Active Learning and Its Potential Connection to Sense of Belonging

Active learning can be broadly conceived of as any instruction that engages students in the learning process (Prince, 2004). More specifically, active learning can be defined as engaging students in "the process of learning through activities and/or discussion in class, as opposed to passively listening to an expert" (Bonwell & Eison, 1991, p. iii). Examples of ways instructors can provide active learning opportunities are facilitating whole-class discussions, engaging students in group work, and using student response systems such as clicker polls. Research indicates that students who have opportunities to engage in active learning attain higher levels of achievement, self-efficacy, sense of mastery, and persistence than students who do not have such opportunities (Freeman et al., 2014; Lahdenperä, Postareff, & Rämö, 2017; Michael, 2006; Prince, 2004; Rasmussen et al., 2019; Salomone & Kling, 2017; Watkins & Mazur, 2013).

Instruction that incorporates active learning provides students with a more engaging and interactive learning experience and is linked to higher achievement and self-efficacy, supporting students' sense of competence. Instruction that incorporates active learning provides opportunities for students to interact with their instructors and/or classmates and thus supports students in forming interpersonal relationships. Thus, by bolstering students' sense of competence and ability to form interpersonal relationships, active learning has the potential to sustain or strengthen students' sense of belonging. Specifically, this study hypothesizes that having multiple opportunities to engage in active learning in Calculus will be linked with a greater sense of belonging.

This study examines students' sense of belonging in two versions of Calculus I offered at the same post-secondary institution. One version is a standard large-lecture Calculus course taught primarily by lecture; the other is a non-standard, integrated Calculus course taught using active learning instructional strategies. The purpose of this study is to better understand students' sense of belonging in Calculus, and how it may be related to opportunities to engage in active learning. Specifically, this study addresses the following research questions in the context of these two versions of Calculus: (1) How do students in each course characterize the instruction they experience? (2) To what extent does students' sense of belonging change over the course of a semester?, and (3) How does students' sense of belonging in the two courses compare?

Research Methodology

Setting and Participants

This study was conducted at a mid-sized R1 research university in the mid-Atlantic region of the U.S. during the Fall 2020 semester. The university offers two pathways for Calculus I. One is a standard one-semester Calculus course. Each semester, multiple sections of this course are offered, taught by a mix of permanent and temporary faculty. The course is coordinated in the sense that there is a common textbook (Stewart, Clegg, & Watson, 2021) and common exams. To enroll, students must earn a certain minimum score on the university's math placement test. Students in the course are typically freshmen intending to major in STEM. In the Fall 2020 semester, eight sections were offered, taught by four different instructors, with each section capped at 100 students. Typically, this course is taught in a large auditorium with lecture as the primary means of instruction. However, due to the COVID-19 pandemic, this course was taught in a synchronous virtual format over Zoom during the Fall 2020 semester. For the remainder of this paper, this course will be referred to as C-T.

The second pathway through Calculus I is a two-semester Integrated Precalculus and Calculus course designed to incorporate multiple opportunities for active learning. The first semester of the course, which is the focus of this study, develops differential Calculus, weaving in necessary pre-calculus topics as they arise. The second semester (which is not the focus of this study) develops integral Calculus, again weaving in necessary pre-calculus topics. Each semester, multiple sections of the course are offered, taught by permanent faculty. The course also uses the Stewart et al. textbook, as well as a second textbook designed specifically for integrating calculus and pre-calculus topics. The course is highly coordinated. In addition to common textbooks and exams, the instructors teach from common lesson plans specifying which problems students work on each class, and whether they will be discussed as a whole class or in small groups. To enroll, students must earn a certain minimum score on the math placement test; however, this minimum is lower than that required for C-T. Students are typically freshmen who need Calculus for their intended major but have not mastered all of the pre-calculus prerequisites. In the Fall 2020 semester, two sections were offered, taught by two different instructors, with each section capped at 50 students. Typically, this course is taught in a special classroom in which students sit at circular tables in groups to enable opportunities to work on math problems together during each class session. However, due to the COVID-19 pandemic, this course was taught in a synchronous virtual format over Zoom during the Fall 2020 semester. To maintain opportunities for group work, Zoom's breakout room functionality was used in most class meetings. For the remainder of this paper, this course will be referred to as C-A.

Participants were students enrolled in C-T or C-A in the Fall 2020 semester. Only one C-T instructor (who happens to use some active learning strategies in her lectures, especially student-response polls) was able to have students participate. All 198 students enrolled in her two sections of C-T and all 91 students enrolled in the two sections of C-A received an email inviting them to participate. All participants were asked to complete two surveys, one administered early in the semester and one near the end of the semester. For this study, only students who were freshmen and completed both surveys were considered, resulting in a final sample size of 114 C-T students and 69 C-A students. Table 1 provides the demographic background of the final set of participants. Most C-T students (~90%) reported they had taken Calculus in high school, whereas 60% of C-A students reported they had never taken Calculus before.

Table 1. Student demographics for C-T and C-A.

Characteristic	C-T		C-A	
	n	%	*n*	%
Gender				
Female	44	39	37	54
Male	70	61	31	45
Non-binary/Third Gender	0	0	1	1
Ethnicity				
Hispanic, Latinx, or Spanish	10	9	11	16
Not Hispanic, Latinx, or Spanish	93	82	54	78
Race				
White/Caucasian	79	69	57	83
Black or African American	6	5	8	12
Asian	18	16	5	7
American Indian or Alaskan Native	1	1	2	3
Middle Eastern	0	0	1	1
Other	5	4	2	3

Data Collection

Students from two sections of each course completed two surveys during the 14-week semester. The first survey took place in the third week of the semester, just after the drop/add deadline to attempt to avoid participant mortality. The second survey was distributed in the eleventh week of the semester. The two surveys were nearly identical. The first survey was designed to capture students' early sense of belonging, after just a few weeks of instruction. The second survey was designed to capture students' sense of belonging towards the end of the course, and to collect information about the instructional strategies they had experienced in the course.

The surveys were distributed and completed electronically using Qualtrics, a web-based survey tool. Good et al.'s (2012) MSoB instrument was incorporated into each survey. The MSoB portion consists of 30 Likert items, forming five factors that contribute to students' sense of belonging in a mathematics community. Additionally, students were asked to indicate how often they experienced each of eight instructional strategies, and to identify the three most frequently used strategies. The instructional strategies were chosen based on typical active learning instructional practices (e.g., group work), as well as strategies known to be used in at least one of C-T or C-A. See Table 2 for the full list of instructional strategies. In C-T, 80% (N=158) completed the first survey and 77% (N=153) completed the second survey. In C-A, 94% (N=86) and 88% (N=80) completed the first and second survey, respectively.

Data Analysis

To capture students' perceptions of the instructional strategies used in each course, frequencies were computed to determine the percentage of students who reported experiencing each strategy. These proportions were tested with a two-sample proportion z-test with $p<0.05$ to determine significant differences between strategies used in the two courses. Remaining analyses were conducted using IBM SPSS Statistics 27 software. Mean responses were calculated for overall sense of belonging and for each factor of the MSoB. Dependent samples t-tests were conducted to determine significant differences between responses on the first and second surveys; independent samples t-tests were used to determine differences in mean responses between C-T and C-A students with $p<0.05$. The effect size of each significant difference was calculated using Cohen's (1988) benchmarks for d (≤ 0.20 small effect, 0.50 medium effect, ≥ 0.80 large effect).

Results

Instruction

To determine how students perceived their Calculus instruction, students were presented with eight instructional strategies and asked to identify the three most frequently used. Table 2 presents the percent of students in each course who chose each strategy as frequently used.

Table 2. Percent of C-T and C-A students selecting each instructional strategy as one of the most frequently used.

	C-T	C-A	z	Cohen's d
Students work on math in groups	21%	97%	9.81***	1.84
Students work on math individually	87%	9%	-10.19***	1.79
Instructor asks for student responses	16%	80%	8.40***	1.39
Instructor lectures	86%	45%	-5.73***	0.9
Instructor encourages students to ask questions	45%	48%	0.24	0.06
Instructor encourages students to talk to each other	3%	14%	--	--
Students present their work to the class	4%	3%	--	--
Students answer in-class polls	25%	4%	--	--

Asterisks are used to denote the p-values (* for $p<0.05$, and *** for $p<0.001$ significance levels). Entries with "—" denote tests that could not be conducted due to a small sample size.

C-T students were significantly more likely to report lecture and working individually on math as the predominant strategies. In contrast, C-A students were significantly more likely to report working in groups and being asked to share their responses as the predominant instructional strategies. These data indicate that students perceived very different types of instruction in the two courses, with C-A students reporting more opportunities for active learning.

Sense of Belonging

Students' responses to the MSoB portion of the survey were used to measure their sense of belonging. Table 3a presents the mean scores on each survey for each course. There are two main results. First, there was no significant change in students' sense of belonging or its associated factors between the first and second survey for either course. Second, C-A students reported significantly higher sense of belonging than C-T students, both early in the semester and at the end of the semester, with medium effect sizes. Additionally, C-A students had significantly higher averages on each of the five factors associated with sense of belonging. Table 3b presents the results from the tests for significance and effect size.

Table 3a. Mean scores (and standard deviations) on the MSoB items for C-T and C-A.

	C- T Survey 1	C-T Survey 2	C-A Survey 1	C-A Survey 2
Sense of Belonging	5.95 (1.21)	5.95 (1.34)	6.72 (0.99)	6.86 (1.03)
Acceptance	6.19 (1.24)	6.18 (1.41)	6.96 (1.02)	7.10 (0.98)
Affect	5.75 (1.59)	5.76 (1.60)	6.34 (1.41)	6.53 (1.42)
Desire to Fade (-)	5.52 (1.74)	5.55 (1.66)	6.09 (1.31)	6.25 (1.50)
Trust	6.18 (1.45)	6.23 (1.52)	7.16 (0.98)	7.22 (0.99)
Membership	5.88 (1.64)	5.85 (1.62)	6.95 (1.15)	7.09 (1.12)

Table 2b. Dependent and independent T-test comparisons.

	Dependent Samples *t*-Test				Independent Samples *t*-Test			
	C-T		C-A		Survey 1		Survey 2	
	t(113)	Cohen's *d*	*t*(68)	Cohen's *d*	*t*(181)	Cohen's *d*	*t*(181)	Cohen's *d*
Sense of Belonging	0.08	0.01	2.13*	0.14	4.68***	0.71	5.17***	0.79
Acceptance	-0.09	-0.01	1.88	0.14	4.38***	0.67	5.20***	0.79
Affect	0.09	0.01	1.83	0.14	2.52*	0.39	3.30***	0.50
Desire to Fade (-)	0.26	0.02	1.41	0.11	2.52*	0.38	2.88*	0.44
Trust	0.49	0.04	0.55	0.06	5.45***	0.83	5.31***	0.81
Membership	-0.27	-0.02	1.05	0.12	5.20***	0.79	6.14***	0.94

Asterisks are used to denote the *p*-values (* for $p<0.05$, and *** for $p<0.001$ significance levels).

Discussion

This study examined students' sense of belonging in two different Calculus learning environments at the same institution. Results indicate that there were no significant changes in sense of belonging or its associated factors in either course. However, C-A students consistently reported a higher sense of belonging than C-T students, both early in the semester and at the end of the semester. Several hypotheses and areas for further investigation emerge from these results.

What factors might account for the higher sense of belonging in C-A students? Although the design of this study does not allow for causal claims, several hypotheses emerge. C-A students did report experiencing more frequent use of active learning strategies than the C-T students. Specifically, C-A students reported the two most common instructional strategies were working on math problems in groups and being asked by the instructor to share their thinking. Regular group work ensures students have opportunities to get to know one another and discuss their ideas, thus enabling them to form interpersonal relationships. Having opportunities to share their thinking with the instructor supports their ability to form a relationship with their instructor. Both instructional strategies have the potential to foster a safe learning environment in which students feel accepted and "belonging." In contrast, the two most common instructional strategies reported by C-T students were lecture and working on math individually, suggesting that students had fewer opportunities to form relationships with their peers and their instructor. The smaller class size in C-A may have further enabled students to form relationships with their instructor and their peers. Additionally, the fact that C-A incorporated prerequisite Pre-calculus topics and only covered differential Calculus may have contributed to a higher sense of competence in the C-A students. Further research is needed to tease out the differential effects of instructional and curricular features of the courses. However, the findings do align with Lahdenperä et al.'s (2020) argument that learning environments that enable social interaction support students' sense of belonging in the mathematical community.

The majority of C-T students reported having taken Calculus previously; the majority of C-A students reported never taking Calculus before. Given this, one might assume that C-T students would report a higher sense of competence, and thus sense of belonging in Calculus, than C-A students, at least early in the semester. Interestingly, the results indicate the opposite was true — C-T students reported significantly lower sense of belonging than C-A students, both early in the semester and at the end of the semester. Perhaps having taken Calculus before but needing to take it again in college had a negative impact on C-T students' sense of competence. Similarly, given that C-A students had placed into a "lower" version of Calculus, one might assume they would have a lower sense of belonging, at least early in the semester. Again, the opposite played out. Thus, the findings from this study highlight the importance of considering the learning environment when studying students' sense of belonging.

This paper presents results on all students who completed both surveys. Further research is needed to determine the differential impact of different instructional approaches, including active learning strategies, on women and students of color, as these students disproportionately leave STEM.

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Semiotic Conflicts in Students' Collective Proof Comprehension Activity

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Making sense of proofs and statements is a fundamental part of advanced mathematics classes; however, researchers have established that students have limited approaches to reading proofs and may struggle to comprehend them. Converting between representation systems can play an essential role in comprehending formal mathematics including proofs and statements. While navigating representation systems, students are likely to evoke an array of personal meanings that can lead to semiotic conflicts in communication. In this study, we examine what conflicts arose as a group of students collectively worked to comprehend the Fundamental Homomorphism Theorem. Our results show that the students had conflicts related to functions and quotient groups that arose when converting between the formal and other representation systems. Although these conflicts can be problematic, we believe that with a productive discussion and instructor intervention (when necessary) these conflicts can be resolved.

Keywords: proof comprehension, representation systems

Proof comprehension is a fundamental activity in advanced mathematics classrooms where students encounter proofs in lecture and textbooks. However, research points to students' passive consumption of proof presentations as inadequate for students' understanding of them (e.g., Lew, et al., 2016). In Mejia-Ramos and Inglis' (2009) overview of mathematics education literature in proof-based settings, they pointed to a lack of studies related to proof comprehension. Since this review, researchers have developed an assessment framework for proof comprehension (Mejia-Ramos, et al., 2012) and suggested and studied productive strategies for proof comprehension (Samkof & Weber, 2015; Weber & Mejia-Ramos, 2013). These approaches share the commonality of comprehension as multidimensional – focusing both on holistic and local line-by-line understanding.

These studies suggest that proof comprehension involves transforming all or some aspects of a formally represented proof into other representation systems (such as informally summarizing key ideas or illustrating a portion of the proof through an example.) In fact, Mejia-Ramos et al. (2012) point to the use of examples (and diagrams) as a key component of their comprehension framework. Other studies have pointed to the role of examples and diagrams in developing better comprehension of proofs for both students and mathematicians (e.g., Weber & Mejia-Ramos, 2013, Samkoff & Weber, 2015, Weber, 2015) at both the level of a high-level generic proof (Lew, et al., 2020), to make sense of inferences (Weber & Mejia-Ramos, 2013), and even the theorem itself (Samkoff & Weber, 2015).

Because of the significant role of representation systems (and their constituent signs) in proof comprehension, theories informed by semiotics can provide a useful lens to analyze proof comprehension. In this study, we focus on proof comprehension as a collective activity – analyzing a group of four students making sense of the *Fundamental Homomorphism Theorem* (*FHT*) statement and proof. We identify *semiotic conflicts* (Godino, et al., 2007) apparent in their discourse as they both *convert* objects between the formal representation system and other mathematical registers and *treat* objects within registers (Duval, 2006). From analyzing these episodes, we share both challenges that may be common to understanding a syntactically and symbolically dense proof, and ways that students' disparate meanings may be resolved.

Theoretical Orientation and Analytic Framing

In the context of abstract algebra, focal mathematical objects are abstract and general. As noted by Presmeg, et al. (2016), engaging with and communicating about such mathematical objects requires the use of *signs* that "are not the mathematical objects themselves but stand for them in some way" (p. 9). As a result, theories of semiotics, that is, theories related to study of signs, have become a prevalent way to analyze mathematical activity and discourse. In this study, we take a view of semiotics influenced by Duval's (2006) *theory of registers of semiotic representation* and Godino et al.'s (2007) *onto semiotics*—two theories that can serve a complementary role (Pino-Fan, et al., 2017). We focus specifically on Duval's (2006) notion of representation systems (registers). Mathematical meaning is inherently constrained and shaped by the available signs and rules of a representation system (natural language, symbolic, figures/drawings, and diagrams/graphs.) In our context, we add the representational system of proof, which we label *formal*: "generalized symbolic statements which can be combined into permitted configurations via the rules of, for example, predicate calculus, propositional logic and acceptable proof frameworks" (Alcock & Inglis, 2008, p. 114). We then consider the activities of *conversion* and *treatment,* "to substitute one semiotic representation for another, only by changing the semiotic system mobilized; and to substitute two semiotics representations within the same semiotic system" (Pino-Fan, et al., 2017, p. 101), respectively.

We then use *onto semiotic* (Godino, et al., 2007) notions of primary objects – focusing on the subset of objects that are not directly observable (concepts/definitions, propositions and procedures, Font, et al., 2013). Mathematical objects contain dualities that reflect that there is not "one 'same' object with different representations." (Font, et al., p. 7). Rather, meaning and representation of an object are subject to dualities including: the *expression* (the sign) and *content* (the meaning referent) and correspondingly what is *ostensive* (observable) and *non-ostensive* (imagined). Objects are also understood both *personally* (to the individual) and *institutionally* (which can refer to the local community of students or a larger community). An object can also be treated as *unitary* (a single thing) or *systematic* (something to be decomposed into a system). Finally, an object can be *intensive* (general) or *extensive* (a specific example). While these dualities are presented as dichotomies, we argue for a spectrum where something like a *generic example* (Font & Contreras, 2008) can serve as a bridge. These dichotomies can lead to *semiotic conflicts,* "disparity or difference of interpretation between the meanings ascribed to an expression by two subjects, being either persons or institutions" (Godino, et al., 2007, p. 133) as students discuss mathematical objects in different representational systems. Such conflicts, while observable by a researcher, are often unnoticed by students. These conflicts can serve to limit student communication or progression but can also serve as a space for students to negotiate new meaning and make mathematical progress.

Background on the Fundamental Homomorphism Theorem

The Fundamental Homomorphism Theorem (FHT) describes an essential relationship between complex group theory ideas including quotient groups, homomorphisms, kernels, and isomorphisms. Each of these topics individually can be quite challenging for students to develop robust and normative meanings (see Dubinsky, et al., 1994; Leron, et al., 1995; Melhuish, 2019; Melhuish, et al., 2020). As Asiala, et al. (1997) documented, the algorithm for creating cosets often supersedes other conceptions. Further, to understand the FHT, students need to conceive of cosets as both sets and objects themselves that can be elements within a group structure. Indeed, Nardi's (2000) study of a tutor and student engaging with the FHT points to the struggle associated with decontextualized nature of learning quotient groups. This study also suggests a fundamental challenge in "co-ordination and understanding of the link between and, as well as the clarification about the definition of" (p. 184) the relevant functions: isomorphisms and homomorphisms. Research related to isomorphism and homomorphism point to reliance on algorithmic processes and the need for a sophisticated understanding of function (Leron, et. al, 1995). Melhuish, et al. (2020) have further documented that students' treatment of homomorphisms is tied to their coordination with the concept of function and that the meaning associated with function can serve as a support or hindrance in productive engagement with homomorphisms, kernels, and the FHT. In sum, the literature points to complexity involved in understanding the FHT and its constituent parts. In this study, we focus on:

(1) What meanings do student evoke from different representation systems as they engage with the FHT and its proof? (2) What semiotic conflicts arise as students engage in comprehension tasks related to the FHT and to what extent are they resolved?

Theorem 2 (The First Isomorphism Theorem). *If $\phi : G \to H$ is a group homomorphism, then*

$$\frac{G}{\ker\phi} \cong \phi(G).$$

Proof. First, we note that the kernel, $K = \ker\phi$ is normal in G.

Define $\beta : \frac{G}{K} \to \phi(G)$ by $\beta(gK) = \phi(g)$. We first show that β is a well-defined map. If $g_1K = g_2K$, then for some $k \in K, g_1k = g_2$; consequently,

Figure 1. The theorem and beginning lines of the proof.

Methods

As part of a larger project, we have conducted a series of task-based interviews centered on proof in abstract algebra in a large, public university in the United States. In this paper, we focus on a group of four undergraduate mathematics majors who had recently completed an abstract algebra course. Students were first provided the FHT statement and asked to dissect the important terms within the statement. Next, they were given example groups and homomorphisms and asked to identify parts of the example that correlate with the statement of the FHT. Lastly, the students were given the proof of the FHT and prompted to make sense of the theorem globally and locally in conjunction with their prior statement dissection and examples.

The focal transcript (along with video and student work) was analyzed through several passes. First, the three members of the research team independently read through the data and created a set of memos broadly identifying student activity and representational systems at play. From this initial pass, the team arrived at the set of relevant representation systems and initial objects. After this holistic pass, the lead researcher chunked the transcript into a series of 16 episodes. For each episode, the lead researcher created a narrative, first identifying mathematical objects, representation systems, and student meanings ascribed to objects. This narrative was then expanded to describe conversion, treatments, relevant object dualities, semiotic conflicts, and resolution paths for these conflicts. For each episode, other team members read each narrative and compared it to the existing transcript with the aim of challenging initial interpretations. Disagreements were resolved through discussion. Here, we provide a brief overview of pervasive conflicts, and share two illustrative episodes.

Results

An essential part of understanding in the FHT and its proof is understanding the meaning of function and well-defined. Students' personal meanings for function appeared consistent when referencing a *procedure* from the graphical system (vertical line test), but their meanings were in conflict in the diagram representation system (with a subset of students producing 1-1 rather than well-defined), and in verbal descriptions where two of the four students focused on everywhere-defined ("mapped each element to something"). Throughout the tasks, students would bring different personal meanings for the properties and often reach resolution (such as coming to an agreement over the contradicting diagrams), only for the conflict to reemerge in a different representation system (such as identifying well-defined within the proof).

We also documented consistent conflicts related to quotient groups. Of note, when describing what a quotient group is, some students attended to a "list of cosets" whereas others attended to a group structure. A second conflict emerged in relation to the meaning of "factor group" with students in disagreement as to whether the normal subgroup would be "factored out" evoking a meaning of factor related to removal rather than a meaning consistent with factoring as partitioning. A third conflict emerged in relation to the meaning of the elements in the quotient group where some students treated the elements as sets (H as the identity in G/H) whereas others treated the elements as singleton ("e" is the identity G/H). Finally, a related conflict emerged in relation to the representative symbolic notation for cosets (aH) and symbolic set notation ($\{a_1, a_2, a_3, \ldots\}$) where the representative notation was personally meaningful to some students (and connected with coset formation) but did not appear to have meaning beyond the symbols for other students.

Converting the FHT between the formal and diagrammatic representation system

The two pairs of students were each given a homomorphism example (ϕ: $\mathbb{Z}$ to $\mathbb{Z}_4$ where $\phi(x)=x \ mod\ 4$ and ϕ: $\mathbb{Z}_{12}$ to $\mathbb{Z}_3$ where $\phi(x)=x \ mod\ 3$, respectively) and tasked with creating a function diagram and identifying where the FHT can be seen in the diagram (a task to convert from the formal representation system to the diagrammatic system). Moreover, this task involved the duality between intensive (general statement) and extensive (diagram example). Both pairs converted between the formal statement and informal symbolic by identifying G and H, respectively. However, at this point, several of the students voiced uncertainty about the quotient group connection and where the isomorphism can be seen in the diagram.

In order to address this issue, the teacher-researcher asked the students to put their diagrams on the board. Student C explains that they began by listing "dozens of elements" but simplified to "just the four unique cases" (a treatment). Student A and D agreed that they had a similar process. Student C noted, "at least I'm starting to really see the coset groups forming individual elements." We interpret this comment as reflecting some resolution around the elements of quotient group conflict. Although, we acknowledge there is still some language inconsistency with the cosets being referenced as groups.

The teacher-researcher then had the students concretely connect the parts of the formal statement with the diagrammatic representation on the board - focusing on the finite example.

The students converted between the systems, and navigated between the exemplar-type duality to identify the kernel, *G*, and *H*. They are then prompted, “where is our isomorphism? Where is our quotient group?” There are two viable ways to create the cosets of the quotient group. The cosets can be built by identifying the kernel and using the coset formation procedure or through creating pre-image subsets. In this case, one pair of students appeared to have used the latter version, but voiced uncertainty about whether these created the quotient groups, “We did that right? […] We don’t know. We think we could have possibly started” identifying the kernel as the identity (further evidencing resolution around the element-set coset conflict.) In contrast, Student C used the coset procedure sharing, “So, I just started with what we have here, kernel of ϕ […] from the left I added the next operation of our H.”

The teacher-researcher had previously requested representative notation by asking for a “name” for the cosets (a more standard notation in the formal representation system). At this point, Student A recognized they have the same cosets, but states “we just didn’t know how to name them.” However, when the teacher-research prompted about the + operation, Student A and D identify the operation from $\mathbb{Z}_3$ (the operation in the image) rather than from $\mathbb{Z}_{12}$ (the operation in the coset) further evidencing that there is a conflict in the personal meanings attribute to the coset formation process. After some discussion, the teacher-researcher advocated that the representative elements are from $\mathbb{Z}_{12}$, but it is unclear whether this conflict was resolved for all the students. However, we do have evidence that the students were all seeing the cosets (regardless of formation process) as the quotient group elements as they are easily able to answer how many elements are in the quotient group itself (3).

At this point, the teacher-researcher focuses on where the isomorphism is. Student B notices “the order of the image of phi is the same as your quotient group.” With this comment, the teacher-researcher prompts the students if the cosets are clear from the original function diagram (Figure 2a). To which Student B and D indicate that “If you got all the purple lines” (B) and “They would regroup it like that” (D). With the teacher-researcher as the scribe, the students produced the following function diagram, a treatment in the diagrammatic representation system (Figure 2b - without the red markings).

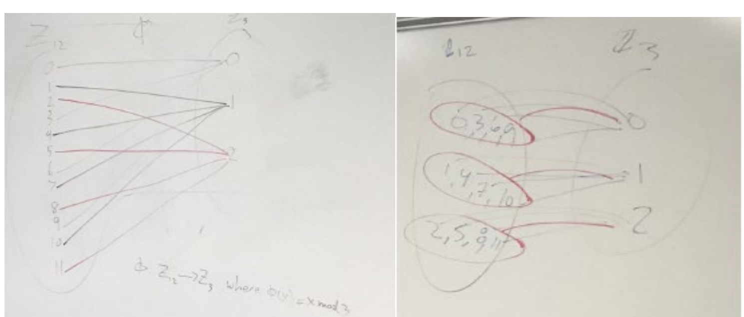

Figure 2. (a) Function diagram of the homomorphism phi from ***Z****12 to* ***Z****3. (b) Reorganized function diagram*

After the creation of this diagram, Student B makes the comment that “you see it's one-to-one and onto” evidencing conversion between the formal meaning of isomorphism and the function diagram instantiation. However, Student B took back their suggestion when asked for further explain indicating a potential conflict between what they see (ostensive) and corresponding non-ostensive mathematical concepts. Student C continues, “we have that grouping, if we change the mapping...” suggesting creating “$\mathbb{Z}_{12}$ so you divide by zero three six nine slash kernel” with Student D suggesting, “bracketed off and that becomes the one element” with the teacher-researcher using their suggestions to add the red circles/lines to the diagram. We see evidence of resolution around both the quotient groups and isomorphism through the use of the diagram instantiation. Student B asked, “so now the red is your factor group?” and student D agreed. Student B also commented, “and now it’s 1-1 and onto” with Student D elaborating, “and there’s your isomorphism.” At this point, the students seemed to have successfully converted from the formal representation to a diagram.

Converting Between a Line in the Proof and a Symbolic Example

In this episode, the students were tasked with making sense of the line if $g_1K = g_2K$, then for some k in K, $g_1k= g_2$ in the FHT proof. The teacher-researcher prompted the students to convert this line to the symbolic/ diagrammatically illustrated example (Figure 2). Recognizing that these objects are cosets, Student B suggested “kernel of phi [as] one of em' and then four plus kernel of phi is another one?” Here we see the student using distinct cosets rather than distinct elements from the same coset. This conflict is recognized and resolved by Student D who suggested changing the second coset, “It’d be 1+ *ker* ϕ.” to which Student B agreed, “Oh, I'm talking about the wrong one. Okay, yeah.” The students continued converting from the formal representation to the informal symbolic example with Student C suggesting another coset equivalence, “so kernel phi would be equal to three plus kernel phi.”

The teacher-researcher returned focus to the formal representation asking, “But how do we know that $g_1k= g_2$ for some k?” Student B made two suggestions: “because it’s normal” and “I thought it was because it was not nonempty, so you have identity in there.” As only one student is suggesting, the teacher-researcher prompted for students to explain what “$g_1K = g_2K$” means. Student B focused on the entire set: “two cosets that are the same” and while Student D attends to the elements, “These sets repeat down in a row?” reflecting uncertainty in their personal meaning. The teacher-researcher then introduced the use of a symbolic example prompting students to create g_1K and g_2K. The teacher-researcher and students constructed the expanded versions (Figure 3) converting between the condensed version of cosets typical of formal proof representation into an expanded symbolic representation. While it was not clear that the meaning shared for the equivalence was in conflict for Student B and D originally, the conflict is explicated in this new representation. Student B suggested the cosets being the same means “all the elements.. They're matching” further explaining “g_2, g_1 are the same g_2k_1, g_1k_1.” Student D responds, “Not necessarily those... all the elements in g_2K there's an element that matches them somewhere in g_1K.” Student B’s personal meaning focused on the element level, while Student D’s personal meaning was at the set level (aligned with the normative meaning for set equality).

Figure 3a and 3b. The cosets of K with representative g1 and g2.

To see if other students understand Student D’s reasoning, the teacher-researcher asked someone to revoice what Student D said. In response, Student A stated, “that's like continuing? So somewhere along the line there would be g_2k in the g_1k function” Here we see that Student A shared their meaning with Student D. However, Student C returned to the formal proof line and asked, “But there's only like g_2 so what does just g_2 look like?” This question may reflect a desire to know the exact match. Student D then explains the idea that “g_2 has to be somewhere in here” because the cosets are equivalent and introduces a “k_m” such that “$g_1k_m = g_2$.” Student C responded that they are finding a “specific one” and Student A suggested “arbitrary.” In the expanded cosets, specific k_i's are listed and Student D introduces k_m. In the formal proof, the line has just “k.” Student C’s personal meaning attributed k_m as more specific than k while Student D and A are seeing k_m as representing an equally arbitrary element.

After some continued conversation, we see an additional layer of complexity and some resolution. Student B commented, “I've been thinking that g_2 is some like a group.” With Student C agreeing, “That's what I was thinking, too.” This may evidence a conflict around the meaning for representation notation: when an element is the referent and when a set is the referent. Student D commented that “g_2 is like another element” with Student B explaining they went from “what is that?” and identified “the whole revelation I had was when you put k_m.” At this point, the conflict seemed to be resolved for the students, although this claim relies on weak evidence for Student C as they had stopped voicing questions.

Discussion

In this paper, a variety of conflicts arose as students conveyed meaning for different objects as they converted and treated them across and within representation systems. The students in the task had difficulty with functions and converting functions from a formal representation to a diagrammatic representation. They had conflicts with quotient groups such as, notational representation, set vs groups, and elements as cosets vs elements as singletons. We saw that students had conflicts in converting a formal representation of the isomorphism in the FHT theorem to a diagrammatic representation of an isomorphism in a given example. We also saw students having conflicts when converting between the line of the formal proof and symbolic concrete and generic representations.

From a research and teaching perspective, we found the identification and, in some cases, resolution of these conflicts to be a useful lens. The conflicting means students evoke may account for some of the disconnect between instructor intentions and students’ comprehension. We found students brought different meanings for functions and quotient groups, and different notation and representation systems often shaped the meanings involved. Prompts to convert between representation systems appeared productive to terms of explicating conflicting meaning and allowing for discussion space to resolve conflicts. Further, instructors may find it to useful to identify the representation systems in which students’ personal meanings may diverge and strengthen the connections between objects symbolized in different systems.

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An Instrumental Approach Towards Graduate Teaching Instructors' Use of Applets

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This study investigated how mathematics instructors' instructional goals for teaching constant rate of change (CROC) influences their perception and use of applets in mathematics instruction. This report presents results from a clinical interview with graduate teaching instructors (GTIs) to illustrate the degree to which their mathematical meanings for teaching (MMT) impact their image for how an applet should be used and how the GTIs' use of the applet influences their MMT. I use instrumentation theory as a lens for explaining the relationship that develops between an instructor and particular features of the applet. I conclude with a discussion of how GTIs MMT for constate rate of change impacts their intended use of an applet's feature and how they imagine they can use an applet to advance students' mathematical understanding of CROC.

Keywords: Applets, Instrumentation Theory, Teachers Mathematical Meanings, Technology

Introduction and Literature Review

As technology advances, digital technologies such as graphing software and applets are being created and used in elementary, secondary, and undergraduate mathematics education (Eason and Heath, 2004) to support students' mathematical learning. Applets are interactive computer-based objects that enable students to visualize attributes of specific mathematical ideas (Gadanidis, Gadanidis, & Schindler, 2003). The dynamic abilities of an applet can promote discussions that focus on quantities and their relationship within a problem context as they vary in tandem (Moore, 2009).

Although applets are frequently included in mathematics curricula as resources to support students' learning and performance, few instructors are using applets in their teaching (e.g., Engelbrecht & Harding, 2005; Daher, 2009). Teachers' pedagogical beliefs, mathematical knowledge, classroom management, and comfort with applets have been identified to influence their perception of applets and how they incorporate applets into their instruction (Gadanidis et al., 2003). When solving mathematical problems, some teachers perceive applets as tools for visually representing quantities within a problem context, thus potentially supporting students in constructing an appropriate image of the problem context (Daher, 2009). However, a teacher's perspective on teaching and her view of what is involved in learning a particular idea or making sense of a problem context might influence whether or not they use the applet.

Over the years, researchers (Bowers, Bezuk, & Aguilar, 2011; Moore, 2009; Guy, 2020) have demonstrated the benefits of applets as tools for supporting student mathematical learning when utilized as a *didactic object* (Thompson, 2002). An object (e.g., a visual aid, a graph, or an applet) is not considered didactic until the teacher or researcher views the object as a tool for producing reflective classroom discussions focused on a particular theme or way of thinking (Thompson, 2002). Moore (2009) revealed that an applet implemented as a didactic object could enable and sustain discussions that focus students' imagery on coordinating changes in the vertical distances' values above the horizontal diameter and the changes in arc-lengths' values as they varied in tandem. An important feature of his implementation of the applet was his decision to use the enactment of the applet's dynamic features as a way to verify and/or modify students' predictions of how the quantities' values may vary in the context of trigonometry. His implementation of having students anticipate allowed them the opportunity to imagine how the quantities varied and reflect on their conjectures relative to their conception of the dynamic contextual situation. Moore's use of the applet to support students' conception of trigonometric function was driven by his *mathematical meanings for teaching* angle measure.

Thompson (2016) explained that a teacher's *mathematical meanings for* teaching (MMT) consist of the teacher's images of the mathematics they teach and the intended meanings that they hope their students to have. A teacher's mathematical meanings are described as productive when the meanings she holds are productive for students' mathematical learning, in the long run, if they were to hold them as well (Bylerley & Thompson, 2017). Thompson, Philipp, Thompson, & Boyd (1994) stated that teachers' images of the mathematics they teach influences how they implement curricula in their classroom(s). These researchers characterized and defined these images as teachers' orientations towards mathematics teaching. Thompson et al. (1994) claimed that a teacher with a *conceptual orientation* tends to focus their students' attention away from the thoughtless application of procedures and provide opportunities for students to conceptualize quantities in problem contexts and explain the rationale for claims. In contrast, a teacher whose actions are driven by their image of mathematics as the application of calculations and procedures for producing numerical results ("solving task to just get an answer") is described to possess a *calculational orientation* towards teaching mathematics.

Theoretical Perspective

Prior research on the instrumental approach has proven to be fruitful for investigating how technology can be used and what shapes its use during an interaction between the learner and a digital tool. Instrumentation literature (e.g., Vérillon & Rabardel, 1995; Artigue, 2002; Thomas & Holton, 2003; Drijvers, Kieran, Mariotti, ... & Meagher, 2009; Drijvers, 2019) has focused on describing the interaction between the student and digital tool. In this study, I use instrumentation theory as a lens to examine the initial interactions between a teacher and an applet that they are viewing for the first time.

An instrumental approach consists of one's transition in viewing an object as an artifact (e.g., a tool) to an instrument. When an object is viewed as an instrument, there exists a bilateral relationship for the use of the tool, in which, on the one hand, the user (and their mathematical meanings) shapes the techniques for using the tool, and on the other hand, the tool shapes and transforms the user's mathematical practice (Drijvers, 2019). An *artifact* is described as an object that is used as a tool[1]. Whereas an *instrument* is a conceptualization that entails the uses of an artifact with a specific goal in mind (e.g., GTIs' uses of an applet with their teaching goals in mind). The use of an artifact as an instrument entails the researcher's (or teacher's) *schemes* and *techniques* for how they imagine using the artifact. I define a *scheme* as the organizations of mental activity (or actions) that includes a set of conditions that trigger the scheme and an expectation of the results of having applied the scheme (Piaget, 1971). *Human activity* (e.g., a set of gestures, ways of thinking, and actions both physical and mental) by an individual in order to perform a given task (Trouche, 2005) is referred to as the individual's use of the applet. Thus, as a GTI experiences one or several features of an applet, I (the observer) will speak of it as a *conceptualized use of the applet* or an *instrumented technique*.

The process of building an instrument from an artifact is called *instrumental genesis* (see Figure 1). This process consists of two interconnected components: One component is

[1] An applet described in instrumentation theory is already assumed as a tool (Russell, 1997) and not as an object that one must come to view as a tool. The latter view of an applet as a tool stems from a radical constructivist (Glasersfeld, 1985) stance that I adopt.

instrumentalization, which is a process where the user (e.g., GTI) constructs an image of what the artifact is designed for and how it should be used. For an applet, this entails the activity of selecting and discovering relevant features (e.g., clicking on boxes or moving sliders) and unplanned attempts to transform certain functions of the applets (e.g., the interface) with the goal of modifying and personalizing the applet to one's liking. Instrumentalization can be described as the process of a GTI differentiating between their MMT for a mathematical idea and their image of the purpose of the applet's capabilities and features.

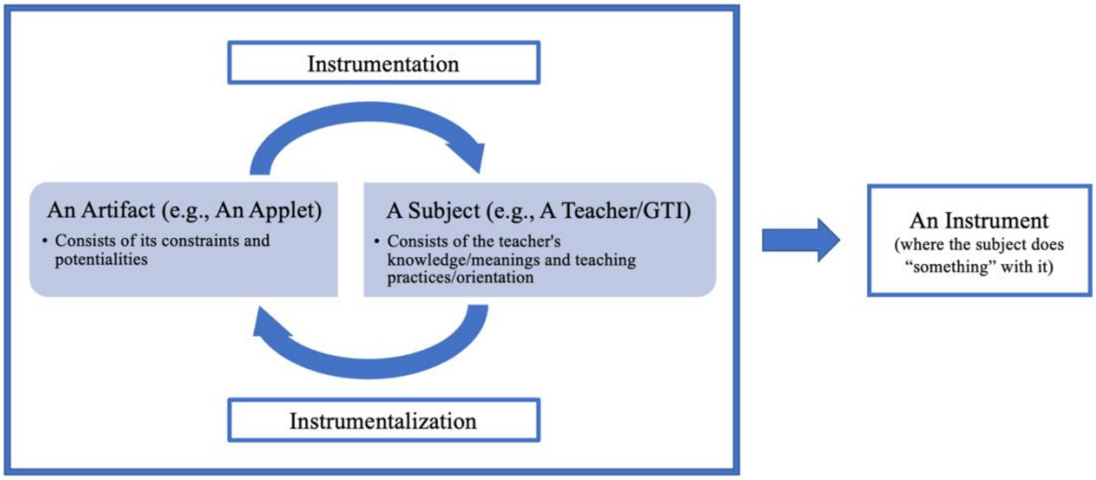

Figure 1:The transition from an artifact to an instrument through instrumental genesis.

The second component is *instrumentation*. This is the process through which the constraints and potentialities of an artifact shape the subject and how they may use the tool during their instruction. During this process, one goes through the emergence and evolution of schemes while performing tasks of a given type. Defouad (2000) described this process in two phases, which I will explicate in relation to this study. The first phase, which happens at the beginning, entails the GTI attempting to find a balance between their former teaching practices and their imagery for conveying mathematical ideas with the newfound capabilities from the tool. The second phase is where a state of equilibrium is established (rather the subject is aware of it or not) in the use of the applet features and capabilities that align with the GTIs instructional goals and what they perceive the applet to be useful for. Hence, this report investigates instructors' conception of the applet's usefulness in supporting students' understanding of constant rate of change and answer the following questions:

- *How do GTIs' mathematical meanings for teaching influence the ways in which they spontaneously incorporate applets in preparation for instruction?*
- *In what ways are GTIs' instructional goals for the concept of constant rate of change aligned with how they imagine incorporating applets in their teaching?*

Methodology

This study involved two graduate teaching instructors, Ebony and Armando, at a large public university in the southwest United States. The GTIs teaching experience of the *Pathways Precalculus* (Carlson, Oehrtman, & Moore, 2018) curriculum ranged from 2-3 years, with each completing one year of a professional development seminar called TUME.[2] (Teaching

[2] This seminar included mathematical tasks designed and sequenced to support the GTAs in constructing productive meanings for key mathematical ideas in the precalculus curriculum.

Undergraduate Mathematics Education). The GTIs participated in one clinical interview (Clement, 2000) where the researcher asked the GTIs to complete two tasks designed to reveal their thinking and meanings for CROC. Following the task, I presented the feature of an applet designed to support student understanding of the idea of CROC to the GTIs and asked them to think aloud as they interacted with the applet. This consisted of open-ended questions that focused on how the GTIs intended to use the applet to support their instructional goals for the task & CROC. Interviews lasted 1.5 hours and were recorded using Zoom Video Communication technology (2020) to capture facial gestures, audio, and written responses with transcripts.

The interviews were analyzed in two phases following an open coding approach (Strauss & Corbin, 1990). The first phase entailed analyzing GTIs' mathematical meanings for teaching CROC. The second phase entailed analyzing teachers' interactions with the applets through the lens of the instrumental approach. This consisted of identifying moments in which GTIs leveraged their MMT and instructional goals for CROC when using specific features of the applet.

A candle is 8.5 inches tall when it has been burning for 6 hours at a rate of ¼ in. per hour. How tall was the candle before it was lit?

Figure 2: The Candle Burning task (Carlson et al., 2018).

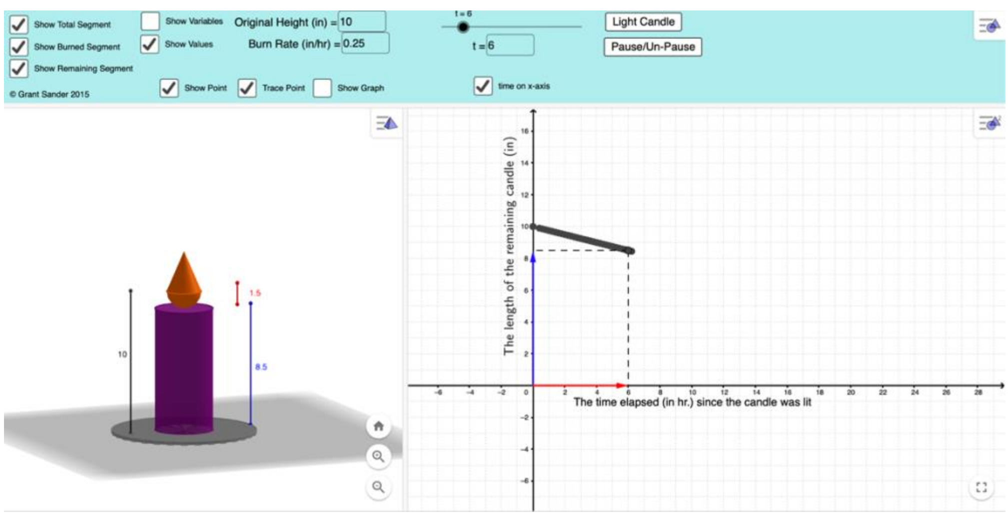

Figure 3: A screenshot of the candle burning applet (Rendered in GeoGebra by Sander, 2015)

Results

I share the following results from the GTIs' interview with Task 1 (Figure 2) and the Candle Burning Applet (CB-Applet, see Figure 3) from the Pathways Precalculus course materials. I first highlight the instructors expressed meanings for CROC and their image of the ideas they hope to convey to students. Next, I illustrate the GTIs orientation to teaching mathematics by describing their image of the pedagogical task (e.g., the goal it serves) and the questions they intended to ask when using the applet during discussions with their students. Lastly, I share how the GTIs expressed meanings and orientation toward teaching CROC impacted their image of how they intend to use the applet and how this use is aligned with their instructional goals.

GTIs' Meanings and Orientation for Teaching CROC

Ebony began the interview by reading the prompt (Task 1) aloud and explicated how she would determine the original length of the candle (see Figure 4, left panel). Next, I asked Ebony how she would use this task in her class. She indicated that she would lead a discussion aimed at getting her students to attend to the quantities and how they are related. She explained that she wanted to support the students in conceptualizing an appropriate image of the problem context. She later described this conversation as being centered around a key way of thinking that she later defined as *quantitative reasoning* (Thompson, 2008). Next, she further explained that she would focus students' attention on the variations in the remaining length of the candle and burnt lengths of the candle with respect to time elapsed. She also described *covariational reasoning* (Thompson & Carlson, 2017) as another way of thinking that she imagined students must be engaged in to support their understanding of CROC as she intended. She stated that with the help of prior conversations about the quantities, their relationships, and the construction of tables of the covarying values that students meaning of CROC (in the context of the problem) may express "the change in the length of the candle burned is always negative $1/4^{\text{th}}$ times as large as any change in time." During her interaction with the applet, Ebony expressed the goal that her students understand CROC as entailing a constant ratio between two covarying quantities. This evidence suggests that Ebony's image of this task included a conversation centered around CROC as a way for describing how the changes in one quantity's values change relative to the changes in another quantity's values in the context of a burning candle.

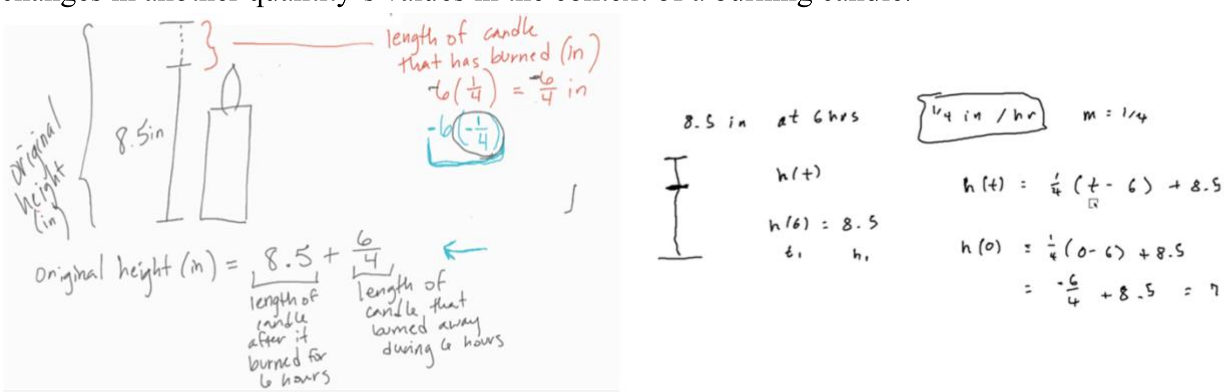

Figure 4: Ebony (left panel) and Armando's (right panel) written work for Task 1.

In contrast to Ebony, Armando read the prompt aloud and began to write a function formula to determine the initial height of the candle in terms of time (see Figure 4 right panel). Armando stated that if he presented this problem to his student, he would prefer to go right into generating a formula to represent the situation. But for the prescribed curriculum, it was important to attend to quantities as stated in the TUME seminar because this was key in supporting students in constructing meaningful formulas. For this task, Armando stated that the intended meaning for CROC that he imagines his students to develop is that the expression $\Delta h/\Delta t$ is a ratio that represents a single number "no matter what two points you pick" (referring to his drawn graphical representation of CROC where there exists a CROC between the remaining length, h, and time, t). He further expressed that by showing his students the definition (see Figure 4 right panel), the goal of the task is to support students in understanding the general case, which can be represented by constructing the point-slope formula. In addition, he explained that CROC "is a way for characterizing linear relationships and as far as the intuition and sort of the pictorial reasoning and physical reasoning goes… You know, I hope that they (the students) think that the constant rate of change is all about linear relationships. All about lines." This statement suggests that Armando's image of CROC entailed a computational and geometric representation of a line that is comprised of a linear relationship.

The GTIs' Use of the Applet

As Ebony first interacted with the CB-Applet, she voiced her appreciation for the interface of the applet as the image of the candle burning in the left panel aligned with her goal of supporting students' image of the problem context. Ebony clicked on each of the boxes and moved the sliders to discover the capabilities of the applet. She mentioned that the variations of the blue and red vectors on the axes correspond with the line segments on the candle (both on the interface of the applet and in her constructed image). Next, Ebony clicked on the following boxes in this particular order: *ShowPoint*, *ShowTrace*, then *ShowGraph*, then proceeded to describe her intended use of these functions. She verbalized that the use of these three features supported her goal of engaging her students in coordinating the changes in the two quantities values as they vary together (see Excerpt 1).

Excerpt 1

Ebony: Okay, so that's nice, that it doesn't just immediately show the graph. So, I like this because I imagine using these three features [*referring to ShowPoint, ShowTrace and ShowGraph*] to help students...to help the graph emerge in students' thinking. So we could, like, really have a conversation about covarying quantities where I would like to turn on this point [*only leaves the ShowPoint box checked*] and have conversations about like, okay, so like. What does it... why is that, what does it mean for that point to be on the *y*-axis [Ebony *had moved the t-slider to 0 with the original height to 10 in*.]? Now let's imagine that the candle has burned for one hour, where do you imagine you're going to see that point on the graph and we could actually like, show them [*moves t-slider to 1 hour and circle's her cursor around the point*] and -have a conversation about if they're correct. And then, the nice thing is then having a conversation about how we can use this tracepoint and how all of the values on this line. We can help the graph emerge in students' thinking [*moves t-slider to illustrate the trace feature on the graph*] that's nice. I like that.

As Ebony continued interacting with the applet, she explained to me that she wanted a line segment on the graph to represent the change in the burnt length as she increased the *t*-slider. In response to her perceived constraint of the applet, Ebony constructed a new technique for conveying her intended image for the change in the burnt length by remembering the capabilities of Zoom and its annotation feature. She described that if she were to use this applet during instruction, she would press play on the applet and have students watch the emergence of the tracing of the graph in order to verify the numerical values on the constructed table that represents changes in the remaining length and burnt length relative to the changes in time. She used the annotation feature to draw horizontal vectors to represent changes in time and vertical vectors to represent the changes in the burnt length, which would result in changes in the remaining height of the candle.

Armando's initial interaction with the CB-Applet entailed trying to discover what the applet could do in relation to the task by typing in the original height of the candle that he produced and the CROC from Task 1 (e.g., 1/4 in per hr., but mistakenly typed 1.4). He began to demonstrate that if he used the applet during instruction, he would use the trace function to plot the points (6, 1.6) and (5,3) on the line and then determine the value of the CROC based on the definition. He suggested that if one would like to know the CROC between these two points (if the burn rate box was hidden), then one can compute the CROC by using the ratio ($\Delta r/\Delta t$ where r and h represent the remaining length of the candle). He explained that once students are able to calculate the CROC to be -1.4, then the mathematical expression $\Delta r/\Delta t$ can support their construction of a linear line through the idea of building a "staircase" (see Figure 5). His goal of building a staircase was to determine corresponding changes in r when the change in t is known as it can be used to produce the original height of the candle. Throughout his interactions with the applet, Armando appeared to be interested in the calculations needed to determine a CROC and its values to construct a graph, rather than thinking of how a student might understand their

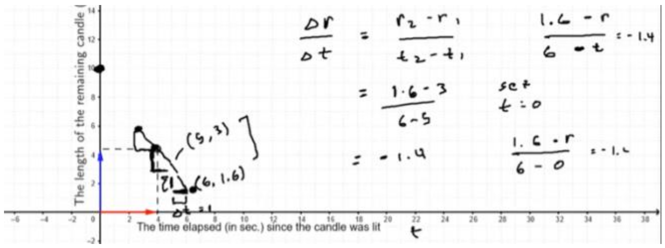

Figure 5: Armando's illustration of building a staircase

conveyed meanings for CROC. In addition, the questions he described that he would pose to his class when using the applet all focused students' attention on interpreting static information about the graph.

Discussions

Ebony's MMT with regard to Task 1 focused on engaging her students in conversations centered around thinking about the quantities, their relationship between the quantities, and how those quantities vary together. Then, when describing her use of the ShowTrace feature in the context of a classroom discussion, Ebony focused on a conceptual understanding of CROC as a way to describe the proportional relationship that exists between the changes in two quantities' values as they vary together. She used the dynamic visualization of tracing a point as representing a record of the relationship between covarying quantities (e.g., a graph that represents the time elapsed since the candle was lit and the remaining length of the candle as they vary together at a CROC of -¼ in. per hr.). Hence, her intended use of the applet as an instrument to support students understanding of CROC aligned with her expressed instructional goals for teaching CROC.

In contrast, Armando's MMT and orientation for Task 1 focused on expressing his meaning for CROC as a procedure for calculating other values that satisfy the relationship of the problem context. For him, key attributes of CROC were that the ratio $\Delta r/\Delta t$ expresses the slope of a line and indicates the graph's steepness. As he was describing his use of the ShowTrace feature in the context of teaching, Armando focused on the ratio $\Delta r/\Delta t$ as a way for computing coordinate points that can be connected by a straight (e.g., a way for illustrating the slope) and used the building of "staircases" to construct a linear graph. His perception of the applet was an instrument that can visually verify that the ratio for CROC as a secant line between two points on the graph of a function. Rather, Ebony perceived the intended use of the applet as an instrument that can support and sustain a conversation centered around student's conceptualization of the problem context.

The results from this study suggest that a teacher's conceptualization of how an applet's features can support students' mathematical learning is guided by their MMT. The mathematical meanings a teacher holds guide their instructional decisions and actions, which impacts how they use the applet during instruction. One limitation to this study is that only two instructors were presented. I intend to conduct a similar study on a larger scale, where further teacher research should focus on the teachers' conceptualization of the applet(s) as an instrument which impacts their implementation of the applet to support students' mathematical learning.

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Collaborative Creativity in Proving: Adapting a Measurement Tool for Group Use

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Creativity is central to mathematics and mathematics education. One hindrance to research in mathematical creativity is the complex nature of defining and measuring creativity. Some efforts have been made to study mathematical creativity at the K-12 level, but just recently researchers have begun exploring mathematical creativity at the undergraduate level. Proof is essential to an undergraduate mathematics education, and as university mathematics classrooms evolve to incorporate more active and collaborative learning, it is imperative to understand the relationship between collaboration and creativity in proving. This study seeks to apply the Creativity-In-Progress Rubric (CPR) on Proving (Savic et al., 2017) to two collaborative small-group proving episodes and to evaluate its ability to present a holistic image of a group's creative proving process. Findings of this evaluation led to the three suggestions for future use of the CPR on Proving in collaborative settings.

Keywords: mathematical creativity, proof, creativity in proving, collaboration

Creativity is a central tenant of mathematics, and mathematical creativity is a rapidly growing research area. Between the years 2000 and 2021, there are over 6000 items on Google Scholar that contain the phrase "mathematical creativity," and about 1300 of these were published in the years 2019 or 2020. Mathematical creativity has been recognized as an important part of mathematics education. Neglecting mathematical creativity in mathematics curricula denies students from truly understanding what mathematics is and from seeing the beauty of mathematics (Mann, 2006), and encouraging students to think creatively is essential to successful mathematics major courses (Schumacher & Seigel, 2015).

This study investigates the assessment of collaborative creativity in proving. I seek to answer the question: how can creativity in proving be assessed in a collaborative setting? I wish to examine the ability of the existing individual Creativity-In-Progress Rubric (CPR) on Proving (Savic et al., 2017) to depict a holistic representation of the creative process of a group. This study's theoretical framework aligns with that of Savic et al. (2017); I assume that mathematical creativity is domain specific, relative in terms of context and background, and assessed as a process rather than a product.

Background Literature

Defining mathematical creativity has been a difficult task for scholars in mathematics and mathematics education (Karakok et al., 2015; Mann, 2005, 2006; Nadjafikhah et al., 2012; Sriraman, 2004). Definitions of creativity vary depending on whether creativity is characterized as: domain-general or domain-specific, a process or an end product, and relative or absolute (Savic et al., 2017). Lack of a central definition of mathematical creativity has created a challenge in measuring creativity and thus pursuing research in mathematical creativity.

Despite difficulties in defining creativity, there have been several attempts to create a measure of both domain-specific and domain-general creativity (e.g., Kattou et al., 2013, Leikin & Elgraby, 2020; Levav-Waynberg & Leikin, 2012; Savic et al., 2017; Siswono, 2010; Torrance, 1966, 1974). Guilford's (1959) characterization of creativity aimed to present testable factors of creativity and has been commonly used as a framework for measuring and defining creativity.

The four aspects of creativity, as suggested by Guilford, are fluency, flexibility, originality, and elaboration. This characterization inspired the Torrance Tests of Creative Thinking (TTCT; Torrance, 1966, 1974) to measure domain-general creativity, the similar Mathematical Creativity Test (MCT; Kattou et al., 2013) to measure domain-specific mathematical creativity, and many others (e.g., Leikin & Elgraby, 2020; Levav-Waynberg & Leikin, 2012; Siswono, 2010).

The Creativity-In-Progress Rubric on Proving

MAKING CONNECTIONS:	**Beginning**	**Developing**	**Advancing**
Between Definitions/Theorems	Recognizes some relevant definitions/theorems from the course or textbook with no attempts to connect them in their proving	Recognizes some relevant definitions/theorems from the course and attempts to connect them in their proving	Implements relevant definitions/theorems from the course and/or other resources outside the course in their proving
Between Representations[1]	Provides a representation with no attempts to connect it to another representation	Provides multiple representations and recognizes connections between representations	Provides multiple representations and uses connections between different representations
Between Examples	Generates one or two specific examples with no attempt to connect them	Generates one or two specific examples and recognizes a connection between them	Generates several specific examples and uses the key idea synthesized from their generation
TAKING RISKS:	**Beginning**	**Developing**	**Advancing**
Tools and Tricks	Uses a tool or trick that is algorithmic or conventional for the course or the student	Uses a tool or trick that is model-based or partly unconventional for the course or the student	Creates a tool or trick that is unconventional for the course or the student
Flexibility	Attempts one proof technique	Acknowledges the possibility of different proving approaches, but attempts no further examination	Acts on different proving approaches
Perseverance	Begins to engage with proving	Continues to engage with surface level features but not with the key ideas	Continues to engage with the key ideas
Posing Questions	Recognizes a question should be asked, but does not formulate a question	Poses questions clarifying a statement of a definition or theorem	Poses questions about reasoning within a proof
Evaluation of the Proof Attempt	Checks work locally	Recognizes a successful or unsuccessful proving attempt	Recognizes the key idea that makes the proving attempt successful or unsuccessful

[1] We define a *mathematical representation* similar to NCTM's (2000) definition. It includes written work in the form of diagrams, graphical displays, and symbolic expressions. We also include linguistic expressions as a form of lexical or oral representation. For example, a student can use the lexical or oral representation, "the intersection of sets A and B"; a Venn Diagram to depict his/her mathematical thinking; a symbolic representation $A \cap B$; or set notation $\{x | x \in A \text{ and } x \in B\}$ (which is also a symbolic representation). Note the last two representations are in the same category, e.g. symbolic, but they are still considered two different representations.

Figure 1. The Creativity-In-Progress Rubric. From "Formative Assessment of Creativity in Undergraduate Mathematics: Using a Creativity-in Progress Rubric (CPR) on Proving" by Savic M., Karakok G., Tang G., El Turkey H., Naccarato E., 2017 In R. Leikin & B. Sriraman. (Eds) Creativity and giftedness (pp. 23–46). https://doi.org/10.1007/978-3-319-38840-3. *Copyright 2017 by Springer International Publishing.*

The CPR on Proving, which can be seen in Figure 1, assumes creativity to be domain-specific, a process, and relative. The most recent published version of the CPR on proving (Savic et al., 2017) presents two categories: making connections and taking risks, which are split into subcategories. Student work is assessed on a continuum to characterize the proving process of an individual student. The CPR on proving has been used as both a formative assessment tool (e.g., El Turkey et al., 2018) and a student reflection tool (e.g., Omar et al., 2019; Tang et al., 2017). The Creativity Research Group (Creativity Research Group, 2020) has suggested future research on the influence of socialization and collaboration on creativity in proof (Savic, 2016) using the Creativity-In-Progress Rubric.

Group Creativity

The Creativity-in-Progress Rubric on Proving was designed to measure the creative efforts of an individual student, yet students in a classroom do not necessarily act on their own. Students' ideas and creative contributions are influenced through collaboration with peers and instructors. Many mathematics classrooms employ group work as a means to improve learning; Slavin (1996) even called research on collaborative learning, "one of the greatest success stories in the history of educational research" (p. 43). However, in my review of undergraduate mathematical creativity research, studies have almost exclusively investigated the creativity of an individual rather than the creativity of a collaborative group. Some research exists on group mathematical creativity in K-12 students (e.g., Aljarrah, 2020; Levenson, 2011; Sarmiento & Stahl, 2008). Existing studies with undergraduate students have compared group and individual creativity (Molad et al., 2020) and investigated how students construct every-day linear algebra examples in an interview setting (Adiredja & Zandieh, 2020).

Group creativity is the generation of creative ideas by groups when the interactions and inputs of several people are considered (Levenson, 2011). Investigations on group creativity have indicated that a creative product may not necessarily be improved by collaboration (Paulus et al., 2000). Paulus and Yang (2000) claimed that poor outcomes of group creativity may be a result of inattentiveness of group members, difficulty generating ideas while listening to the ideas of others, and lack of incubation time during collaborative work. In collaborative settings wherein diverse individuals work toward a common goal, a variety of backgrounds and knowledge may contribute to the creative product, but it is also possible for diversity of perspectives to be too wide and deter agreement on a common solution (Kurtzberg & Amabile, 2001).

The purpose of the following study is to explore the relationship between creativity in proving and group creativity. This study specifically investigates the assessment of collaborative creativity in proving and seeks to answer the question: how can creativity in proving be assessed in a collaborative setting?

Methods

Participants and Data Collection

Data was collected from an undergraduate Introduction-to-Proof course at a large public southeastern university in the United States. The data used in this study consisted of 25-minute video recordings of two small-group proving episodes during a class meeting in the first week of the course. Group 1 consisted of three students: Nathan (math major), Ganesh (non-degree seeking graduate student), and Matthew (math major); group 2 also consisted of three students: Susan (math major), Mason (computer science major; math minor), and Craig (computer science major; math minor). All names used are pseudonyms.

This course was facilitated in a collaborative, inquiry-based learning environment where small groups of students would work together to prove instructor-provided mathematical conjectures. The two groups investigated in this study were selected using a matched-comparison sampling technique, where each group worked independently to prove the same conjecture: the product of consecutive twin primes is one less than a perfect square. When the class was presented with the proving task, the instructor (a) presented students with definitions for the terms prime number, twin prime, and perfect square, and (b) encouraged students to use alternative representations such as diagrams, pictures, language, and symbols in their group proof. Students worked in their groups of three to prove the given statement and occasionally asked questions of the instructor or graduate assistants. A notable feature of this task is that the statement is also true in a more general context (e.g., for any two integers that differ by two).

This setting and sampling technique provided unique insight to answer my research question as (1) the given task was likely novel to students and thus allowed for an openness in students' approaches to proving the conjecture, (2) the group work occurred during week one of the course, so students were probably unfamiliar with proving and most contributions in terms of proving strategy could be assumed to be novel, or at the very least un-shared, by all members of the group, and (3) both groups were asked to prove the same statement and this provided the opportunity to analyze two different group processes in completing the same task.

Data Analysis

The two video recordings were first transcribed and then coded according to a protocol coding method (Saldana, 2013).The codes were predetermined according to the seven subcategories of the CPR on Proving: *Making Connections* between (1) Definitions/Theorems, (2) Representations, and (3) Examples; and *Taking Risks* according to (4) Tools and Tricks, (5) Flexibility, (6) Perseverance, and (7) Posing questions. Each episode was coded for instances within the group proving pertaining to the subcategories as described in Savic et al. (2017).

The goal of this coding strategy was to identify moments in which the group engaged in creative proving as defined by the CPR. This allowed the group to be viewed as a cohesive unit of analysis and assess their proving process in each subcategory on the spectrum of beginning, developing, and advancing. Descriptions of each of these levels can be found in Figure 1 above.

The final stage of analysis aimed to assess the ability of the CPR on Proving to tell the story of each group's creative proving process. In this stage, I reconsidered the video data and observed important moments and themes which influenced the group's process and may not be conveyed by the CPR on Proving.

Findings

In this section, for each group I present first a holistic general summary of the group's interactions as they attempt to prove the conjecture "the product of consecutive twin primes is one less than a perfect square." I then describe the placement of each group on the CPR on Proving. Then, in the Discussion section of this paper, I compare the findings from my holistic summaries of each proving episode to the impression given by the CPR on Proving.

Group 1 Holistic Summary

Group 1 worked very well together and constructed their proof by challenging, recognizing, and building upon the ideas of their fellow group members. The group began the team's efforts by verifying examples of the statement. Ganesh first challenged the group by proposing that simply generating examples would not be enough to prove the statement and suggested the use of more general principles. Despite this objection, Matthew noticed through the group's list of examples that the perfect square referenced in the statement is the result of squaring the "middle" number of the twin primes used, and he wrote an algebraic representation of the statement: $\mathrm{n}^2 - 1 = (\mathrm{n}-1)(\mathrm{n}+1)$. Ganesh and Matthew argued over how to restrict this equation to be true only for twin prime numbers, and Matthew questioned what would happen if they plugged in a non-prime. Prompted by this, Nathan tested an example using $\mathrm{n} = 10$. Matthew then noted that if the statement is actually true for all natural numbers, then the statement would have to hold for the smaller subset of the twin primes; Ganesh did not initially understand Matthew's claim, but Nathan and Matthew explained it to him together. The group proceeded to collectively generalize the statement (i.e., questioning and checking whether the statement would hold for natural numbers, negative integers, real numbers, etc.) until the group work time concluded.

Group 1 Assessment via the CPR on Proving

Figure 2 below provides an aggregation of my coding for group 1's proving attempt in the 25-minute span. Figure 1 provides specific descriptions of each subcategory level.

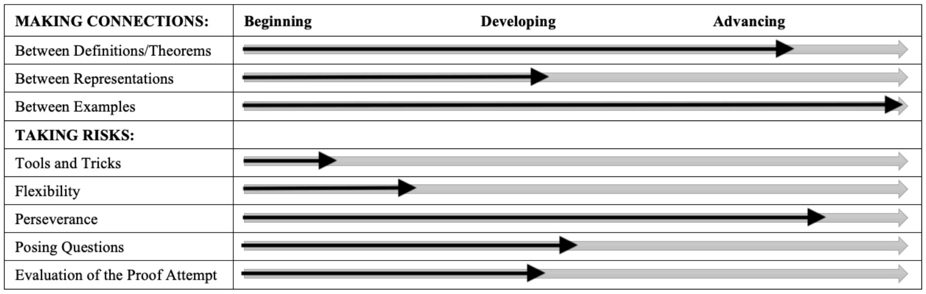

Figure 2. Levels of Group 1's work according to the CPR on Proving.

Using Figure 2, one can observe in which categories group 1 demonstrated strength or weakness. For example, consider the "Between Representations" category. First, group 1 used an algebraic representation when Matthew said:

> If we look at it and n squared is a perfect square, we have n minus 1 times n plus 1, which would be our two prime numbers here, F.O.I.L. it out, and we get n squared minus one.

Group 1 also attempted to use symbolic representation by looking up appropriate symbols for "element of" and "subset" to use in their proof; however, the group did not seem to make use of a conceptual connection between the symbols and their other work. Since group 1 attempted different representations and made some incomplete connections, I coded their work as "developing" in the Between Representations subcategory.

Group 2 Holistic Summary

In contrast to group 1, the students in group 2 spent more of their group time working independently. The first few minutes in group 2 were spent trying to understand the vocabulary in the statement. When Susan attempted to generate an example of the statement, she considered 3 and 5 and observed that the number used to generate the perfect square creates the Pythagorean triple 3, 4, 5. While Susan worked on her idea, Craig was focused on using mathematical symbols and language to rewrite the statement, and Mason worked to verify more examples of the statement. Mason's engagement with the examples led him to observe that the number in the "middle" of the twin primes being the square root of the perfect square mentioned in the statement. While Mason contemplated his idea, Susan admired Craig's rewriting of the statement using symbols for "for all," "therefore," etc. Mason worked to represent the statement algebraically and produced the equation: $n(n+2) = (n+1)^2 - 1$. Mason finally made the conclusion that his equation would work for any number, and hence it would prove the statement for twin primes. Susan challenged this approach, and while she was eventually able to revoice Mason's ideas, she remained confused about the proof at the end of the group's time together. Later during class, Mason presented the group's work to their classmates.

Group 2 Assessment via the CPR on Proving

Figure 3 below provides a summary of the coding for group 2's proving attempts in the 25-minute group work session.

Using Figure 3, one can observe group 2's performance in each category throughout their proving process. For example, consider the "Posing Questions" category. Throughout their collaboration, the members of group 2 posed several clarifying questions. For example, while generating examples, Craig asked, “What’s the prime number after 7?” and later Susan asked her group members and the instructor of the course, “Does one count as a prime number?”

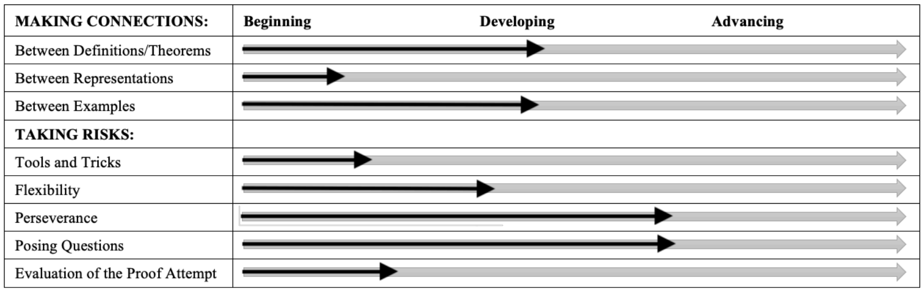

Figure 3. Levels of Group 2’s work according to the CPR on Proving.

Group 2 also demonstrated posing questions while considering possible generalizations of the statement. Mason asked the group, “What if that's true for any two numbers that are two apart? Like...what if that's true for any two numbers?” After posing this question, Susan and Craig largely ignored this idea; nonetheless, Mason proceeded to test his question independently and conclude that the statement could apply to any two numbers with a difference of two.

Although most questions posed were to clarify the given statement and definitions, Mason’s question about generalizing the statement somewhat propelled a conversation regarding the reasoning of the proof. Thus, I coded group 2 between the “developing” and “advancing” levels in the Posing Questions subcategory.

Discussion

Does the CPR on Proving tell the whole story?

The Creativity-In-Progress Rubric on Proving was designed to measure the creativity displayed in an individual’s proving attempt, yet in this study I have attempted to assess a group of students as a unit. Here I describe some observations made through a comparison of the holistic summary of each groups’ work to their respective placement on the CPR.

When reviewing the CPR on Proving assessment of group 1, there are only a few moments neglected in consideration of this group’s holistic story. The coaction of group 1 is not evident by viewing the CPR on Proving assessment of their work. Each student’s individual creative contributions to the proof were deeply intertwined with their teammates, and this should be represented in a rubric of their creative proving process. In contrast to group 1, group 2 did not construct their ideas collectively. I imagine that the CPR on Proving would have been a more appropriate tool for assessing each individual group member of group 2 rather than viewing the group as a unit. Students in both groups frequently posed suggestions or conjectures to their classmates, and this is not acknowledged on the CPR.

Suggestions for Use of the CPR on Proving in Collaborative Settings

With the above observations in mind, I have developed three suggestions for use of the CPR on Proving when applied to a group. My suggestions below are motivated by improving the rubric to enhance its use in revealing mathematical creative abilities in tertiary students through formative assessment and/or reflection. These suggestions may allow the CPR on Proving to help students overcome common roadblocks in proving by reflecting upon their creative process in a group setting.

First, I suggest an additional category titled "Collaboration." The addition of this category to the rubric will require reflection upon the group's ability to recognize and build upon one another's contributions. In the continuum of evaluating collaboration, at the *beginning* level the group members' attempts of the proof are individual. At the *developing* level, group members acknowledge one another's ideas and attempt to produce a cohesive proof. Finally, at the *advancing* level, the group constructs a proof collaboratively and successfully incorporates the contributions of each group member cohesively. For example, in the cases examined in this paper, group 1 would be placed at the "advancing" level, while group 2 would be placed at a high "beginning" level.

The second suggestion for modification of the CPR on Proving is to expand the "Posing Questions" subcategory. I observed in my analysis that students working in a group frequently posed suggestions or conjectures where they may have been posed as questions in an individual setting. I would expand this category to be "Posing Questions/Making Suggestions." For example, in group 1's attempt to generalize the statement, Matthew said:

> "It's a positive number. I'm going to say positive because I don't think it would work with negative... But I don't know, because there's only... Ugh! If we include negatives, then it could be like one and zero. So maybe we can say any nonzero number."

This suggestion prompted a conversation between Matthew and Nathan regarding whether or not the statement would apply to negative numbers similar to how the conversation could have proceeded if Matthew had asked "can we include negative numbers?" The CPR on Proving did not capture this moment because Matthew did not phrase his thought as a question even though his body language and tone indicated he was seeking the groups input on his suggestion.

Finally, when implementing the CPR on Proving in a collaborative setting as a reflective tool, I suggest that students evaluate the proof attempt by first viewing the group as a unit and then using the CPR on Proving to assess their individual contribution to the group. For example, in group 1, Nathan contributed a majority of the examples, Ganesh was strong in making connections to other theorems and flexibility, and Matthew posed important questions to the group. In group 2, Susan and Craig would have had lower scores overall in most categories, while Mason would have had higher scores in most categories. Encouraging group members to examine their contributions may allow them to notice areas for improvement, appreciate contributions of their group members, and learn how to approach proofs more collaboratively.

Future Research

Mathematical creativity is a central tenant of tertiary mathematics education, especially in proof. For this reason, future research should focus on determining how to foster creativity in tertiary mathematics students. Problem posing (Silver, 1997), Model-Eliciting Activities and phenomenon-based learning (Asahid & Lomibao, 2020), and multiple solution tasks (Leikin & Elgraby, 2020) have been conjectured to improve mathematical creativity skills. A clear direction for future research is to examine if collaboration in proving fosters mathematical creativity.

The CPR on Proving has offered a wonderful contribution to the world of undergraduate mathematical creativity in proof by providing a reflective and formative assessment tool to encourage and emphasize creativity. In my experience, as students progress in their mathematics education, they increasingly look to their peers for support and engage in collaboration. My suggestions for future use will support the continued development of the use of the CPR on Proving in a collaborative setting.

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Examining the Distribution of Authority in an Inquiry-Oriented Abstract Algebra Environment

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Recent large-scale research points to evidence of inequitable outcomes between women and men in inquiry-based mathematics education (IBME) courses. One explanation for differing outcomes may be that women are having different experiences in these courses than men. Specifically, the ways in which students garner mathematical authority and leverage their authority in both whole class and small group contexts may differ between students. Framing authority as a relation between people determined by their mathematical activity, we present an exploratory analysis of the authority relations between students as they engage with tasks developed for an IBME abstract algebra course. Findings suggest there are indeed discrepancies in the amount of time students have mathematical authority. We present examples of situations in which discrepancies are visible to begin examining the underlying nature of these discrepancies.

Keywords: Abstract algebra, inquiry, mathematical authority

Many researchers have positioned inquiry-based mathematics education (IBME) (Laursen & Rasmussen, 2019) as means to support student-centered undergraduate classrooms. These paradigms share commonalities: "student engagement in meaningful mathematics, student collaboration for sensemaking, instructor inquiry into student thinking, and equitable instructional practice to include all in rigorous mathematical learning and mathematical identity-building" (p. 129). Until recently, the first three pillars have dominated research in this paradigm with equity largely underexplored at the undergraduate level (Adiredja & Andrews-Larson, 2017). Consider the context of abstract algebra—there are several research-based curricula (e.g., Larsen, et al., 2013) and studies exploring student and teacher activity in inquiry classes (e.g., Fukawa-Connelly, 2016). Such work reflects a positive reality: students are engaging in meaningful mathematics. However, recent large-scale research points to differing outcomes for students such as Laursen, et al. (2014) and Johnson et al.'s (2020) contradictory results when looking at the outcomes for women and men in these types of courses.

It is possible that students are having different experiences in these courses in terms of their ability to leverage their authority in both whole class and small group contexts. Possessing mathematical authority can aid students in developing deeper conceptual understanding of concepts and developing more productive identities and dispositions toward mathematics (Langer-Osuna, 2017). In addition, mathematical authority can result in improved motivation and engagement because of increased choice and responsibility. However, typical mathematics classrooms can possess structures that remove authority from students (Wagner & Herbel-Eisenmann, 2009). Although IBME courses may differentiate in several ways from these typical classrooms, IBME courses may still possess structures that serve to restrict or remove mathematical authority from students. While the restriction of authority is typically framed as teachers (or other institutional figures) removing authority from students by limiting their opportunities to engage with mathematics, it is possible that students themselves may restrict authority from others. Given that IBME courses provide more opportunity for students to engage with mathematics and with each other, a related question then is whether and how student-to-student authority relations might work to restrict authority from some students.

While authority relations have been analyzed to some degree in the K-12 setting (e.g., Amit & Fried, 2005; Cobb, Gresalfi & Hodge, 2009), they remain largely implicit in the undergraduate setting. As we conjecture that authority relations may provide an explanatory mechanism for differing student experiences in IBME classrooms, developing a theory of authority relations in the advanced undergraduate setting is important. Thus, in this paper, we focus on a group of four students engaged in task-based interviews designed to mimic an IBME classroom in order to carefully analyze their authority relations. We focus on a lab setting because (1) some of the highly studied curricula at this level have been developed based on student thinking/activity in similar settings (e.g., Larsen, et al., 2013) and (2) the lab setting provides opportunity to explore in depth the ways that students are interacting in a group without extraneous classroom influences. This exploratory analysis focuses on the following research question:

How do mathematical authority relations manifest among participants as they engage with abstract algebra tasks?

Theoretical Framing

A key distinction of authority within a mathematics classroom is between pedagogical and mathematical authority (Wilson & Lloyd, 2000). In this paper, we primarily focus on the latter. In other words, we focus on describing authority that is derived from engaging with mathematics rather than sources of authority that are perhaps institutional in nature (e.g., the pedagogical power of a teacher to determine a discussion topic). We assert that authority is visible by observing people's actions. Symbolic interactionism describes relationships between individuals and asserts that communication is how people make sense of and act upon their world. We adopt the theoretical lens of symbolic interactionism and use Goffman's (1981) notion of authorship and animation to describe authority relations as interactions between people and the activities in which they are allowed to take part. In this paper, mathematical authority is defined as a dynamic and negotiated relationship between people (or groups of people, or organizations, etc.) where one party defers to another in a mathematical situation (Lambert, Hicks, Koehne & Bishop, 2019). This definition stands in contrast to definitions of authority that rely on characteristics or immutable traits of an individual, such as perceived status due to age or possessing a PhD, and definitions that consider authority to be an attribute of an individual that may evolve over time (e.g., Engle et al., 2014). Instead, we conceptualize authority as a dynamic, moment-to-moment feature of interactions that is not carried beyond the situation in which the authority is manifested (i.e., a student with a "dominant" personality does not necessarily maintain mathematical authority from one class session to another.)

Table 1. The AAA Authority Framework for Authority Relations

Authorship	Refers to the significant contribution to the mathematical ideas under consideration.	Asks: Who is the primary source, or author, of the mathematical ideas?
Animation	Refers to who is publicly communicating or extending mathematical ideas.	Asks: Who is communicating mathematical ideas in the classroom?
Assessment	Refers to judgements/evaluations made about an idea.	Asks: Who is assessing mathematical ideas in the classroom and who is being assessed?

To operationalize authority relations, we utilize the AAA (read "triple A") framework (Lambert et al., 2019). AAA consists of three components of activity: authorship, animation, and assessment of ideas. Table 1 broadly outlines the AAA framework; we discuss specifics of the components of AAA in the methods below. Aligned with the AAA framework, we draw upon two concepts from Goffman's (1981) Forms of Talk: participation frameworks and footing. Participation frameworks offer an assignment of participation types to everyone involved in an event. In the context of AAA, the possible types of participation are the author, the animator, or the assessor of mathematical ideas. Footing refers to a participant's alignment with others within a participation framework as well as the obvious or subtle shifts in that alignment from moment-to-moment. Communication is then accomplished by attending to everyone's participatory status during an event. In the context of AAA, this amounts to identifying who holds the status of authorship, animator, or assessor from one event to another. In the next section, we describe how these concepts assist in coding our data.

Methodology

The data in this study were collected as part of a larger project focused on adapting student-centered teaching practices from the K-12 setting to the undergraduate setting. The data consists of two 2-hour long task-based interview sessions with a group of four undergraduate students (one woman and three men) at a large, public university in the United States. To begin analyzing our data, the first author segmented each interview by attending to footing: (a) shifts in participation structure, (b) shifts in the mathematical content focus, or (c) the introduction of a new task or activity. Three types of segments were identified: segments that emulated whole-class (WC) activity (i.e., everyone engaged in focused discussion with the interviewer facilitating), segments that emulated small group (SG) activity (i.e., participants worked in pairs; one or more participants worked without the interviewer present), and segments that were either not mathematical or involved independent work with no student interaction.

The AAA framework was originally developed to attend to the collective authority of all students (contrasted with the teacher) rather than the authority of individuals in a given moment. In other words, if one student authors an idea in a class discussion, then students collectively receive credit for authorship. In this paper, we adapt AAA to track authority for individuals. Each segment that was labelled as WC or SG was assigned codes of which individuals were the holders of authority for each of authorship, animate-speak, animate-scribe, and assessment. Authorship is based on opportunities to contribute which are constrained by the overall focus of a segment. Authorship involves establishing the focus or making a significant mathematical contribution to the overall focus of a segment. It is assumed that the interviewer will direct classroom activities and functions, such as assigning a worksheet or posing a question to the class, and thus, these activities do not count toward authorship.

The component of animation is divided among speaking and scribing activity. Speaking is the public oral communication of mathematical ideas. Speaking may include (a) revoicing another's idea, (b) expressing disagreement/confusion with another's ideas, (c) sharing new ideas/content, and (d) asking probing questions to elicit another's ideas. Scribing includes the communication of ideas which occurs publicly through (a) inscribing mathematical content on a board/document camera, (b) and gesturing to written work. A key aspect of scribing is board control, which often involves placing oneself at the front of the classroom.

Finally, an assessment is defined as an explicit statement that indicate the correctness, validity, or sensibility of a given mathematical contribution (such as a statement, argument, or strategy). Any disagreement or argumentation between students is considered as explicit assessment. Acts of assessment may or may not be followed up with a justification. Who makes a justification is not necessarily the same individual who makes the initial assessment and justification can be presented with varying degrees of quality.

For each segment, we determined which students held authority for the activities of authorship, animate-speak, animate-scribe, and assessment by assigning a code to each student that participated in each activity. Segments that were particularly difficult to code were brought to the group to resolve coding and interpretations. Once each segment was coded, the length of time for which each student was involved in a segment in which they held authority for each activity was determined for both of whole class and small group time. In the next section, we present the results of this analysis.

Results

There was a total of 69 coded segments across the interviews: 55 WC and 14 SG. Figure 1 below represents the lengths of time (in minutes) for which each of the participants successfully took the opportunity to participate across both interviews in both WC and SG contexts.

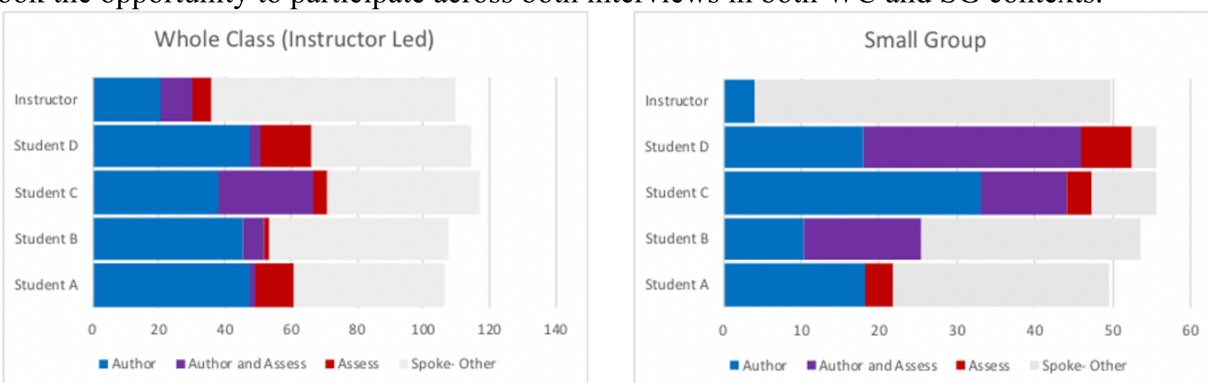

Figure 1. Lengths of time (in minutes) participants held authority.

There is clear unbalanced participation across all AAA activities. Across WC segments, Student C was moderately dominant in both of authorship and assessment. More striking, however, is the discrepancy in time spent as a holder of authority when shifting the participation structure to SG settings. In particular, Student A was a holder of authority for each of author, speak, and assess the least amount of time across all SG segments. In contrast, Students C and D spent the greatest amount of time as holder of authority for each of author, speak, and scribe, while Student D's participation in assessing increased dramatically from WC to SG. Thus, the dominant holder of authority for an activity shifted depending on the participation structure. In the sections that follow, we seek to begin disentangling the underlying reasons for such difference and explore possibilities for these disparities by examining a variety of excerpts.

Authoring Public Ideas

Within the two interviews, Students C and D were found to be the author of mathematical ideas more than A and B, especially during SG time. In contrast, Student A was found to be an author for the least amount of time compared to the other students across the total time. The following segment illustrates a particular example of Student A failing to achieve authorship despite making utterances that contribute to the mathematical conversation. In this segment, the teacher-researcher (TR) is prompting students to share out their thoughts on how they would begin proving a theorem:

TR: So what are the types of things that we think about when we're going to prove something?
Student B: What you're given.
Student A: <concurrently with Interviewer> …assuming.
TR: What you're given. All right, so let's go ahead and keep track of these things. Let's say ... "what you're given." <writes phrase on board> All right, and what are the things that we're given?
Student B: G and H are isomorphic.
Student A: < concurrently with Student B > …are isomorphic.
TR: G and H are isomorphic. <*writes phrase on board*>. Okay.
Student B: G is abelian.
TR: G is abelian. <*writes phrase on board*>. Anything else?
Student B: They're groups.
TR: G and H are groups. <*writes phrase on board*> Anything else?
Student C: That's about it.
Student D: That's all that I see.

Although Student A does in fact contribute independent mathematical ideas in response to the interviewer's questions, Student A's contributions are either: (a) made concurrently with Student B who speaks up over Student A, or (b) spoken while the instructor is already acknowledging and making public Student B's contribution. Further, the instructor only animates Student B's contributions (through re-voicing and scribing on the board.) Because Student A never successfully puts forth an idea that is clearly recognized or acknowledged by the group, she does not receive authorship credit in this segment. It is also noteworthy that Students C and D's turns of talk at the end effectively halt further opportunity for authorship from others.

Refusing the Opportunity to Speak

The length of time in which students took opportunities to speak were seen to be relatively high for all students across all situations. This is to be expected due to the usual ease of which individuals can contribute by simply making a statement relative to the mathematics. However, we found it worth investigating small discrepancies to make sense of why a student may not have spoken in a segment. In the following segment, Students B and D have just finished giving a presentation at the board while Students A and C are asked to respond to the presentation with a comment and a question:

TR: So, something that makes sense, something that you have a question about?
Student C: Yeah, something that made sense, I really thought you did a great job explaining the definition well in tandem using the diagram. I understood everything you said, as you were showing the mapping from *a* ... is phi of *a* equal to phi *b*, *a* equal to *b* <*referring to 1-1 property*> you did everything very well in sync, with the instruction and explanation-wise. **Looks to Student A who silently nods her head in agreement**
TR: Question?
Student C: **Looks back at Student A who shakes her head** Just when we were talking about the operations, I thought it was really well-done, writing down the operations, what you were using, although this equation at the

bottom ... you all did switch back and forth of operation, or dot, or star, your saying of it.

Examples of students not speaking can appear in two ways: passively and actively. Passive non-speaking may occur when a student does not contribute verbally, but without acknowledgment or invitation by others. In contrast, this segment illustrates active non-speaking. Although, Student C immediately begins answering the first question posed to the pair, Student A refuses Student C's invitation (nonverbal cue by looking back at her) to both elaborate on Student C's initial comment, and to initiate a verbal response to the second prompt – indicating through gesture her choice to turn down the invitation.

Examining Who Assesses and How

Assessment can be tied to other authoritative activities, chiefly by taking up authority by speaking, but also through authoring and scribing. Who assesses has implications for who the main actors are within a mathematical discussion. However, the form of assessment may also influence who is taking up authority among the other activities. In this excerpt, the students are engaged in a debate following a prompt by the interviewer about making a modification to a diagram on the board that was scribed as part of a presentation by Students A and C (authors.) In particular, Student B has pointed out a flaw of the diagram not explicating the mapping of an element a in a group G to its image: $\varphi(a)$.

Student C:	We don't know for... in G, what *a* itself maps to. We know what *a* operation *b* maps to, but we don't know what *a* maps to.
Student B:	I don't know why not. You wrote "phi of *a*"
Student C:	This <*points at board*> is in G, though. Or, I'm sorry, this is in H, this is not in G. I only think ... because we only work in terms of elements in G when we're using them in tandem with another operation ... if that makes sense.
Student A:	But the diagram is confusing.
Student C:	Yeah. Well, I think ... I'm just saying I think the operation is necessary, because, unless we're working with a kernel. If we find a kernel, then we could say… *a* is in kernel, and it maps to *e*. We could say that for sure, but we don't know any single element that maps to another element in H.
Student B:	Even though we know that it's onto and one-to-one.
Student C:	Well, yeah, but we don't know the exact element. We know there is a kernel.

Note the variety of assessment that occur in this excerpt and who makes it: (a) assessments of an idea intended to significantly alter another authored idea, thereby establishing new authorship; (b) detailed assessments either for (or against) an authored idea put forth by oneself (or another); and finally, (c) assessments that either confirm or deny the contribution of another without making new contribution. The first type of assessment is primarily made by Student B who has begun the argument with Student C. The second of these types is done only by Student C as he is working to defend a diagram drawn during his presentation. His assessment of Student B's critique allows for him to maintain authorship, as well as give him an opportunity to scribe (by gesturing at the board.) Finally, Student A contributes a brief assessment that serves only to affirm up Student B's argument, thereby limiting her opportunity to receive authorship credit.

Discussion

Inquiry approaches to mathematics instruction often emphasize "supporting students in maintaining authority and ownership of the mathematics" (Kuster et al., 2018). However, the quantifier (all students) is often left implicit in these stated principles and the ways we analyze student activity in such settings. Through our analysis of just four students in a relatively open setting, we were still able to document discrepancies in authority relations. The AAA authority framework allowed for observation of which activities students had the opportunity take part in and provided a lens to further parse individual segments. In contrast with prior work using AAA, we examined authority at the individual student level illustrating important differences across our students including: differences in authority within small groups versus whole class segments, substantial differences in who assesses in small group, and differences in participation at a base level as reflected by speaking. The chosen examples of authority relations allow for insight into potential explanations for the differences in participation in authoritative activity: student interactions may prevent others from holding authority (either inadvertent or not), students may simply choose to not participate, and finally, authority may be limited or extended dependent on the type of assessment one partakes in. If a primary aim of IBME is for students to have authority and ownership of mathematics, then we suggest that it is vital to explore authority relations in these settings. If students leave these courses with different experiences (as suggested in Johnson, et al. (2020), and we argue further evidenced by this analysis), we may need to revisit the implicit "all students" quantifier to better understand the experience of "some students" in terms of authority relations.

The current form of AAA provided an insightful global view of authority. Based on our current analysis, we have several conjectured routes of adaptation to the framework that can further support its use in relation to students individually rather than collectively. One limitation of our current analysis is that authorship (and other activities) for a segment may vary in degree and contribution. Future research can move away from a binary approach to analyze both nature and degree (such as primary author for idea #1). Further, we conjecture that tracking particular mathematical ideas that are the focus of authorship, animation, and assessment can provide richer illustrations. This is particularly salient when looking at small groups where conversation often ebbs and flows in less directed ways than in whole class instruction. Finally, we note the important role invitations and barriers played in shaping authority relations. Future analysis can expand on the nature of these contributions.

The results of this work point to suggestions for improving equitable teaching practice. Gerson and Bateman (2010) argue that the limiting of teachers' mathematical authority is one way of helping to ensure shared authority in the classroom between students and the teacher. However, teachers may also consider ways of utilizing their pedagogical authority to assist students in having more opportunity to author, speak, scribe, and assess. For example, the shutdown of further responses at the end of the first example presented above could have been mitigated by an intervention from the teacher-researcher through extending another invitation to respond and further authored ideas may have become public. Thus, striking a delicate balance between limiting and leveraging the types of authoritative power held by a teacher can assist in creating a more equitable distribution of mathematical authority.

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Promoting Student Agency and Engagement Through Student Partnerships: A Case of an Introduction to Analysis Course

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Most advanced mathematics courses are taught via lecture, in which it is hard to foster meaningful student engagement, agency, and community. In this paper, we discuss an academic support resource called The Students as Partners Program that was implemented in an Introduction to Analysis course with 26 students at a private institution in the Northeastern United States. This model of support positioned three currently-enrolled students in this course as "Student Partners" who were tasked with communicating information to the instructor about student experiences that the instructor could then use to respond dynamically to student needs. In this exploratory study, we examined how student-partnerships can promote student agency, which allowed instructors to implement instructional design structures that supported and facilitated student engagement in- and out-of-class and classroom community.

Keywords: agency, engagement, advanced mathematics, undergraduates, student partnerships

Introduction

Students often have unpleasant experiences in upper-level, proof-based mathematics courses like real analysis. Most advanced mathematics courses in the United States and internationally primarily utilize lecture-style instruction (Fukawa-Connelly et al., 2016; Artemeva & Fox, 2011) in upper-level courses. Most of these courses employ a "definition-theorem-proof" instructional style, during which students tend to be passive during class time, participating primarily through note-taking (Weber, 2004). Paoletti, et al. (2018) found that while lecturers in advanced mathematics courses asked students a number of questions, lecturers failed to provide students with authentic opportunities to construct or co-construct mathematical content and reasoning. Learning mathematics requires student engagement, but most advanced mathematics courses do not allow much opportunity for engagement; and the normative passive nature of students in advanced mathematics classes can contribute to decreased outcomes for students.

Trowler and Trowler's 2010 meta-analysis on student engagement in higher education found that student engagement is correlated with student achievement. Student engagement is also correlated with particular positive outcomes for students including critical thinking, self-esteem, productive racial and gender identity formation, improved grades, and persistence. We agree with Trowler and Trowler's claim that student engagement is a shared responsibility among students and institutions and their staff. While ultimately, students must exercise agency in choosing to engage and the quality of engagement, institutions and staff are responsible for instituting structures that allow for, encourage, and incentivize student engagement.

In this paper, we discuss one way to construct productive structures for student engagement called student partnerships. In these partnerships, select student(s), called Student Partners (SPs) are invited to be collaborators in the educational process. Each week, they meet with their instructor to provide ongoing feedback on teaching practices, discuss the learning needs of the students in their class, and directly engage in pedagogical conversations surrounding student learning. Positioning students as active participants in the shaping of the course design lifts the curtain to allow students to see the process of developing and delivering a course, promotes metacognitive and communication skills, and encourages active community participation.

Moreover, student-faculty partnerships provide students a means to have anonymous (and thus, hopefully authentic) input on the ways they can engage with the course material and with each other in the classroom which, in turn, and can help the instructor construct and refine structures for students that promote greater engagement in, and responsibility for, their own learning (Cook-Sather and Luz, 2014).

This is an exploratory case study about student partnerships and their impact on student engagement and agency. The central goal of this paper is to illustrate how student partnerships can promote student agency, which allows instructors to implement instructional design structures that support and facilitate student engagement through interaction and participation, time and effort, and belonging.

Theoretical Frameworks

Our study employs two theoretical frameworks: student agency and student engagement. Our conception of student agency draws on Emirbayer and Mische's (1998), who define agency as a response to structure - the set of forces that either facilitate or constrain an individual's progress towards a goal. Given an element of structure, whether that structure is facilitating or constraining an individual's goal, agency encompasses the individual's choice to respond and the quality of response to that structure. Agency is not a quality individuals have or do not have, rather agency is something an individual achieves in particular situations. In an advanced mathematics class, many students have the goal of learning mathematics or participating in meaningful mathematical activity. Traditional class structures may limit the ways students can achieve agency by constraining the ways in which they can respond to structures that facilitate or constrain their learning, but alternative structures may give students more opportunity to achieve agency. In this study, we are focused on student agency as a student's response to instructional design structures (classroom structures over which an instructor has control) towards the goal of learning mathematics or participating in meaningful mathematical activity.

Our framework for student engagement is adapted from Kahu's (2013) framework for student engagement in higher education. There are six dimensions to this framework: the sociocultural context; the structural and psychosocial influences; engagement; and the proximal and distal consequences. We have modified this framework to focus on the aspects of student engagement salient to our study (see Figure 1). As we are focused on the classroom level and only collected data from one semester, have only retained psychosocial influences, engagement, and proximal consequences. In our framework, psychosocial influences include teaching, teachers and staff, and support at the university, along with student identity and skills. In this context of this study, staff includes the university employees in the Center of Teaching and Learning (CTL) who helped facilitate the student partnership and support includes teaching assistants assigned to this course. We have also added student agency to this dimension, by which we mean the degree to which they respond - and the quality of these responses - to instructional design structures; the agency a student can achieve influences their engagement. In Kahu's framework, a student's individual experiences are influenced by qualities of both the student and the institution to emphasize that student engagement comprises "more than just an internal static state," (Kahu, 2013, p. 13).

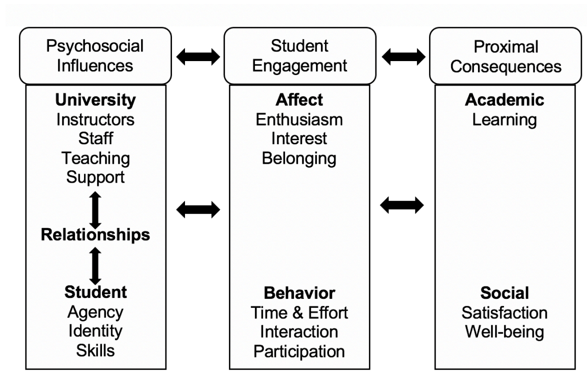

Figure 1: Student Engagement Framework; adapted from Kahu, 2013.

Methodology

Background

In spring 2020, the CTL at a small northeastern liberal arts college in the United States of America introduced a pedagogical program that was modeled after the Students as Learners and Teachers program at Bryn Mawr and Haverford Colleges. Each instructor was teamed up with at least one student partner and had autonomy in selecting their partners. Student Partners (SPs) met with instructors weekly and with CTL staff and other student partners every other week. SPs were paid for their time. In this paper, we highlight the partnership between the mathematics faculty and three SPs in an upper division introduction to analysis course that is required for all students majoring in mathematics at the institution and was supported by two undergraduate teaching assistants who had already completed the course.

Study Participants

Participants were 26 undergraduate students enrolled in an introduction to real analysis course that met for four 50-minute class periods per week, three of whom served as SPs. One of the researchers was also the instructor of the course, Dr. T, and served as a participant reflecting on their experiences. We gave all interviewees pseudonyms: Nico (a student partner), Milo, Rory, Sun, and Jaime.

Data Collection

Data was collected in the spring 2020 semester. The data consisted of 1) five synchronous classroom observations over Zoom, 2) five 30-minute student interviews, 3) Dr. T's reflections, and 4) SP reflections. Generalized Observation and Reflection Platform (GORP) (Center for Educational Effectiveness, n.d.) was used to observe five classes towards the end of the semester. In March 2020, the course transitioned to remote teaching due to the COVID-19 pandemic and observations occurred after this transition. The two researchers who observed were muted and their videos were turned off and they did not interfere with the class. Of the five student interviewees, one was a SP. Interview questions reflected student experience in- and out-of-class, their interactions with the instructors and peers, and student perception of themselves in the context of mathematics and real analysis.

Analysis

Using our framework for student agency, we used open and axial coding to analyze interview data for references to structures implemented by the instructor control (instructional design structures - IDS) and student responses to these structures (Braun & Clarke, 2006). Classroom observation data and student interview data were used in the analysis of student engagement within our framework for engagement described above. In order to examine student and faculty interaction in the classroom, the researchers noted the number of times a student interacted with the faculty and used an open coding thematic analysis to categorize the different types of faculty-student interactions, namely answering a question, asking for clarification, asking follow-up questions, stating conjectures, and other, occurred during a 50-minute class time. We used open and axial coding to analyze interview data for evidence of student engagement, as defined in our theoretical framework. Dr. T's and SPs' reflections were used to triangulate and validate results.

Results

We first present evidence of IDS referenced in the student interviews and student responses to these structures. Next, we present themes of student engagement in the course and connect these to themes of IDS.

Themes of Instructional Design Structures

Assignments. One element of IDS referenced in several student interviews was a Theorem-Definition sheet assignment. Nico (a SP) described this assignment: "you basically just type up or write up all of the theorems and definitions that we had done that week." While Dr. T implemented this assignment without input from the SPs, after implementation, SPs validated the value of this assignment for him. Nico reported that this "mandatory assignment" helped them study for the course and Milo reported that even though this assignment was "extra work," it helped them understand the "ideas more concretely." Jaime initially thought this assignment "would be annoying," but "in the end, … it out-weighed any drawbacks. And I thought that was really helpful." The student partnership provided a mechanism through which students could respond to the class structure. This feedback helped reaffirm for Dr. T the value of this assignment for students' learning.

Class Format. When the onset of the COVID-19 pandemic interrupted physical instruction, the instructor had to make decisions about adjusting the instructional design to the remote learning environment. In their interview, Nico described how the instructor leveraged the student partnership to help make these decisions. Nico reported that "before the switch to remote learning, our [SPs] main focuses were making sure that we could cater to the needs of everyone in the class," while after the switch, "it was more focused on structuring the class in a remote setting." They reported that they discussed different formats for exams, live versus recorded instruction, and the use of breakout rooms with Dr. T. Through the student partnership, students had an opportunity to respond and provide input on the class format during an uncertain time. This direct student feedback helped Dr. T make informed decisions about structuring class in the new remote modality.

Themes of Student Engagement and Connections to Instructional Design Structures

From the five student interviews, we identified three themes of student engagement: interaction and participation, time and effort, and belonging.

Interaction and Participation. Nico noted that participation in class after the switch to remote learning was noticeably different than before the switch, "... most people before remote learning would feel free to jump in and propose different ideas. ... now it's mainly just getting questions answered based on the material and clarifications on stuff that we're doing right now." This suggests that the class format before the switch to remote promoted student participation in class. Even after the switch to remote, students still regularly asked questions during class time. Sun reported that they typically do not ask questions during class time, and while they did not ask questions via audio during remote learning, they were able to ask more questions using the chat feature in Zoom than they would have during physical instruction. During all five class observations, Dr. T used a tablet to write and lecture, something he had used for instruction even before the switch to remote, and frequently paused to ask and answer questions. Classes that were observed mostly consisted of lecture and some Desmos demonstrations. We observed a high amount of student participation during these observations, especially for a remote environment at the onset of virtual instruction. On average, there were 13 student interactions in a 50-minute class. We also observed that students were actively listening and taking notes as most of the student videos were available. The SPs were highly involved during class; their interactions comprised almost half of the total number of interactions (asking or answering questions) during class time.

The class format, informed by conversations between Dr. T and the SPs, allowed for students to ask questions in-the-moment and helped promote interaction and participation, aspects of the behavioral dimension of our framework for student engagement. Thus, through the student partnership, students achieved agency by responding to the class format. From this information, Dr. T was able to make choices about the class format that increased student engagement.

Time and Effort. For Nico (a SP), the "Theorem-Definition Sheet" assignment provided a structure which incentivized spending time and effort on course material. They noted that continuously refreshing course content is an important practice, but "it's always hard to kind of remind myself to do it on my own. So [the theorem-definition sheet assignment] was a super helpful way to do it no matter what." Despite its value, Nico doubted that they would continue this practice in the future as they were unlikely to complete an assignment that was not mandatory (i.e. part of their grade). This suggests that this assignment increased Nico's general course engagement by incentivizing the expenditure of time and effort in refreshing course material. Even though Jaime they thought this assignment "would be annoying" or "hand-holding, in a way," in the end, they decided that it was really helpful and provided a lot of value in their learning. This assignment provided a structure that promoted this behavioral component of engagement. While this assignment originated from Dr. T without input from the SPs, SPs helped reaffirm this assignment's value for Dr. T.

Belonging. From the outset, Dr. T put a lot of effort into establishing a community-oriented class. During instruction, he invites students into the process of constructing mathematical knowledge through scaffolding questions that encourage and build on student contributions. In their reflections, SPs noted that Dr. T emphasized that student participation was valuable and welcomed, despite the "correctness" of answers or suggestions, and that this helped them feel more comfortable participating in class. Dr. T was also using a tablet for instruction before the switch to remote, which allowed him to face his students during instruction and establish a stronger connection with students during class than if he were writing on the board with his back to students. The atmosphere in this course promoted a sense of belonging for Nico so that, in spite of the switch to remote learning, they maintained connection with classmates and the instructor. They reported that "Dr. T always stressed in this class is it's important to build a sense of community, and so try to keep that." As a student partner, Nico worked closely with Dr. T in structuring the course. In their statement above, Nico reaffirmed Dr. T's valuing of classroom community. While the transition to remote learning certainly impacted the atmosphere of class, other interviewees reported that Dr. T worked to maintain the classroom community after the switch. While building community and belongingness are core teaching values that Dr. T had at the start of the course, feedback from his conversations with SPs helped reaffirm these pedagogical decisions. Data from student interviews and Dr. T's reflections indicate that many students experienced a sense of belonging, especially before the switch to remote learning. Thus, through the student partnership, students could respond to Dr. T's choice to emphasize classroom community. This choice promoted a sense of belonging for students, which is reflected in the affective dimension of our engagement framework.

From interviews with students, classroom observations, and Dr. T's reflections, we identified three themes of student engagement from our framework: interaction and participation (behavioral), time and effort (behavioral), and belonging (affective). We connected these themes of engagement to their structural influences. Interaction and participation were influenced by the class format Dr. T established, which was informed by the student partnership. Time and effort were influenced by assignments Dr. T chose, such as the Theorem-Definition sheets, which was reaffirmed through the student partnership. Belonging was influenced by the classroom climate Dr. T established through his teaching, which was reaffirmed by the student partnership.

Conclusion and Limitations

Student partnerships can increase agency by providing a mechanism through which students can respond to IDS. Using these responses, instructors can make choices that increase student engagement. Dr. T used SPs "as a bridge between the students and the professor" (a quote from Nico) to gather authentic and dynamic feedback about class structures. In their weekly meetings, Dr. T was able to discuss feedback with SPs, ask clarifying questions, and understand the reasoning behind the feedback SPs provided. Knowing the "why" behind this feedback helped him to make meaningful pedagogical choices to address student struggles. SPs served as a humanistic liaison between the students and instructor; this human connection was especially important for students during the transition to remote learning when in-person learning was interrupted. The SP program provided a continuous feedback loop (see Figure 2) of meaningful information that Dr. T could use to reaffirm, revise, or generate new IDS. Given this feedback, Dr. T could also, of course, choose to make no changes to IDS.

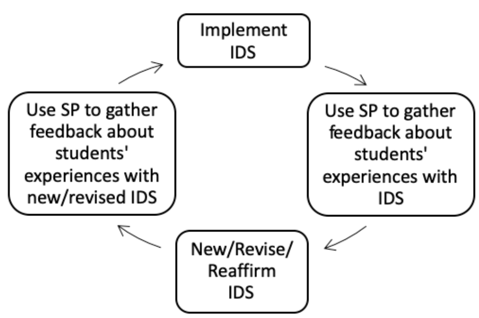

Figure 2: Feedback Loop Provided by SP Program

Some limitations include a small data set and only one interview with a SP (Nico). Other data was collected for this study, including student surveys, that is not reported in this paper as they have not been analyzed yet.

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The Presentation of Composition Across the Curriculum

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Research has shown that college students struggle to understand the operation of composition and the compositive structure of functions. A study of the treatment of composition in written US curricula identified the opportunities that were provided to teachers and students. This report presents the types of functions and representations used to communicate composition in mathematics curricula across the transition from secondary mathematics to college calculus. The presentation of composition was overwhelmingly algebraic and less than four percent involved multiple representations. The results also revealed that composition was mainly presented with linear binomials, monomials, and the inverse cancellation of transcendental functions. The opportunities in written curricula were similar to the types of tasks on which researchers have reported students being successful. Diversifying textbook problems and examples has potential to increase student and teacher knowledge on composition and the compositive structure of functions.

Keywords: Function Composition; Precalculus; Representation; Curriculum

The operation of composition is essential to the function concept and is foundational to the structure of complex functions, calculus, and advanced mathematics (Carlson et al., 2010). Research has shown that students are proficient solving composition tasks represented algebraically (Carlson et al., 2010; Hassani, 1998; Meel 1999), use multiplication instead of composition (Engelke et al., 2005; Horvath, 2010; Meel 1999), and routinely determine the algebraic rule for the composite prior to addressing any other aspect of the problem (Horvath, 2010; Uygur & Ozdas, 2007).

College students who have successfully completed composition tasks have commonly used one of two main methods. One way is to view composition as a recursive sequence of functions where substitution is used to evaluate the innermost position function at a particular numerical value of the independent variable and then relays this function value to next function in the sequence and evaluate it in the same pointwise manner. This sequence of recursive substitution was illustrated by a calculus student that evaluated the composition of $f(x) = \begin{cases} |x-1| & \text{if } x > 1 \\ 2 & \text{if } x \leq 0 \end{cases}$ and $g(x) = x - 2$ at the point $x = 2$ by first determining $g(2)$ to be 0 and then evaluating $f(0)$ to obtain the result of 2 (Vidakovic, 1996). Another common approach has been to view composition as a substitution operation. The algebraic formula of the composition of $f(x)$ and $g(x)$ is determined by substituting the algebraic formula of the first function $g(x)$ into the independent variable of the second function $f(x)$.

Most of the literature on function composition has focused on students' performance on composition tasks. There is little literature on the designing of instruction on the topic of composition (i.e., Ayers et al., 1998; Webster, 1978) and no studies analyzing the presentation of composition in textbooks. Building on the knowledge of college students' solution strategies of composition tasks, this study investigated the treatment of composition in written curricula from Algebra 1 at the secondary level to calculus at the college level. The data presented in this paper came from this larger curricular analysis. This report concentrates on the types of functions and representations that were used to communicate composition and on the mathematical shifts that existed in the written curriculum between the secondary and post-secondary level.

Conceptual Framework

The larger study categorized the composition content in the mathematical texts according to Van Dormolen's (1986) categories of theoretical, algorithmical, and communicative. The theoretical category included definitions, descriptions of composition, and the properties of composition. The algorithmical category included procedures and the actual performance of composition. The communicative category included language, symbols, and mathematical notations used to present composition. Similar categories have been used in other studies of mathematics written curricula (i.e., Smith et al., 2016).

This short report is comprised of data only from the communicative category. In particular, the types of functions and representations used by textbooks to present composition is discussed. Whenever there were multiple function types or representations in a single textual element, every type of function and representation were coded. This paper specifically reports on the following research questions: (1) Which types of functions were used to present composition and how did that change across the curriculum? (2) Which types of representations were used in the composition content of mathematics textbooks?

Method

Twelve US textbooks in four series of mathematics curricula were selected for analysis. A textbook series was defined as a collection of subsequent course texts marketed by a single publisher with common authors. Two series were selected at the college level and the other two series at the secondary level. Selecting two series at each level provided one perspective of composition that was widely used, and a second perspective that differed from the first. Together they provided a wider view of composition across the curriculum than a single series. The secondary level was included in this study to provide background information on the mathematics students encountered prior to their college experience. Additionally, comparing the data at both levels could depict any mathematical shifts students may experience as they transition into college mathematics.

At the college level, Stewart's *Calculus: Early Transcendentals* (2012) and *Precalculus: Mathematics for Calculus*, by Stewart et al. (2007) published by Thompson Brooks/Cole was selected as the most widely adopted series (Horvath, 2012). *Calculus* (2009) by Hughes-Hallett et al. and *Functions Modeling Change: A Preparation for Calculus* (2011) by Connally et al. published by Wiley was selected for the second college series because it approached the topic of composition differently than the widely adopted texts. It included a variety of representations and utilized a mixture of justifications and proofs (Askey, 1997), while the Stewart texts contained fewer representations and more formal proofs.

At the secondary level, each series consisted of textbooks for Algebra 1, Geometry, Algebra 2, and Precalculus. *Glencoe/McGraw-Hill Mathematics* (2010) was selected as widely adopted series (Dossey et al., 2008). This series presented its content by an exposition-examples-exercises format and the composition content focused on substitution. The other secondary series was the *CME Project* (2009) published by Pearson. It arranged its content by mixing the exposition with examples and problems. Composition was introduced as the linking of function machines and formally defined composition as an operation on functions.

This paper reports on the types of functions and representations used to communicate the principles and properties of composition and not to compare curricula. Differences between courses will be reported, but the differences between textbooks for the same course will not be discussed.

Every page in each secondary and in each college precalculus textbook was examined to identify all instances of composition. Every page in the calculus texts were analyzed from the beginning through the section on the chain rule. There were 4040 sentences, figures, and problems involving composition identified as composition elements. The type(s) of functions and the type(s) of representations that appeared in each element was recorded. A second coder applied the coding scheme to a random sample of 300 elements. There was 100% agreement between the two coders in their coding of the types of function and the types of representation for each data element.

Results

Types of Functions

The composite structure of functions builds complex functions from simple ones. Of the 2597 elements that contained at least one type of function, 93% were the composition of two functions, 6% were the composition of three functions, and the remaining 1% were compositions of four or more functions. Half of these elements included a polynomial function. Trigonometric, root, and exponential functions each appeared in 12% of the elements, logarithmic functions and affine transformations were each included in 10%, geometric transformations in 8%, and rational functions in 7%. Ordered pairs, piecewise-defined functions, and absolute value functions each appeared in less than 2% of the elements. These numbers do not sum to 100% because one-fourth of the elements contained more than one type of function. For example, finding the composition of $f(x) = \frac{x^2+1}{3x}$ and $g(x) = x^3$ would be coded as both a rational and polynomial functions. The distribution of the number of elements and the number of codes for each course is shown in Table 1. As seen in the table, the data revealed a significant increase in the number of multiple codes in the college texts compared to the secondary texts. This indicated that the secondary texts were predominately compositions of homogeneous types of functions and the college texts contained a greater number of compositions of diverse function types.

Table 1. The number of Types of Functions elements and codes per course.

Course	Elements	Codes	Multiple Codes
Algebra 1	111	133	20%
Geometry	184	184	
Algebra 2	841	925	10%
Secondary Precalculus	448	568	27%
College Precalculus	799	1155	45%
Calculus	214	319	49%

While most types of functions occurred throughout the curriculum, the composition involving the geometric transformations of translations, rotations and reflections mainly appeared in Geometry and compositions involving affine transformations and ordered pairs only appeared in Algebra 2. In Geometry, all 184 elements were geometric transformations. In Algebra 2, affine transformations occurred in 29% of the codes. The distribution of the remaining types of function codes within each course is shown in Figure 1. Polynomials had the highest frequency in every course. Trigonometric and root functions were next most prevalent in secondary precalculus and the exponential and logarithmic functions were more emphasized in the college courses.

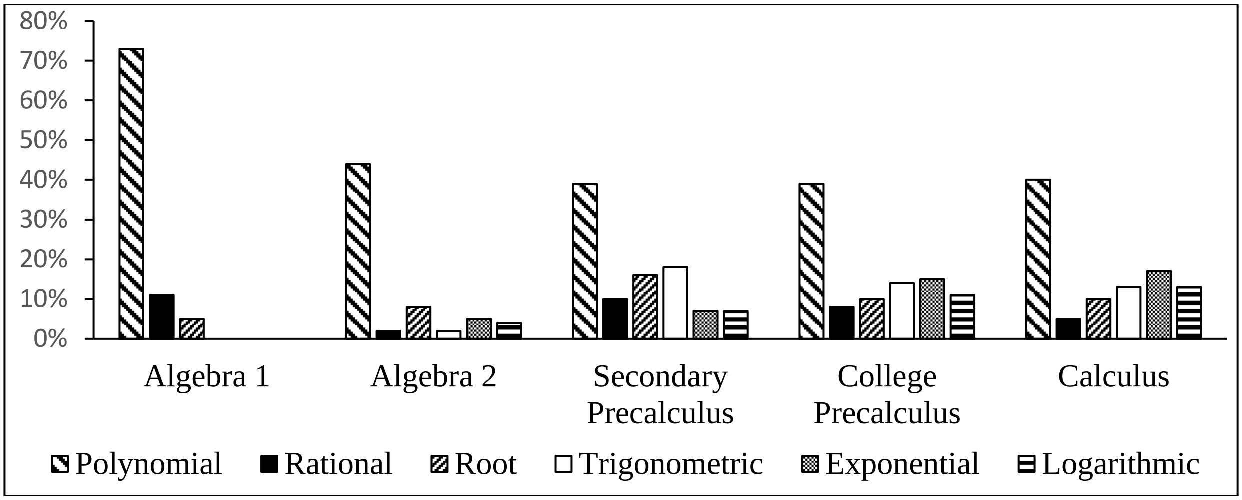

Figure 1. The distribution of the codes for the Types of Functions within each course.

The polynomial functions included polynomials of various degree and number of terms. Linear polynomials appeared in 62% of the elements, quadratic functions in 32%, cubic polynomials in 4%, and 1% had degree higher than 3. Regarding the number of terms 25% were monomials, 66% were binomials, and 8% were trinomials. In every course, linear binomials occurred more than any other type of polynomial. The distribution shown in Figure 2 illustrates the emphasis of linear binomials within each course. In the graph, 100% is the total of Function Type Codes within a respective course. Quadratic binomials received more attention in the college courses and quadratic trinomials in the secondary courses. Across the entire curriculum, half of all polynomials being composed were linear binomials and another 25% were monomials.

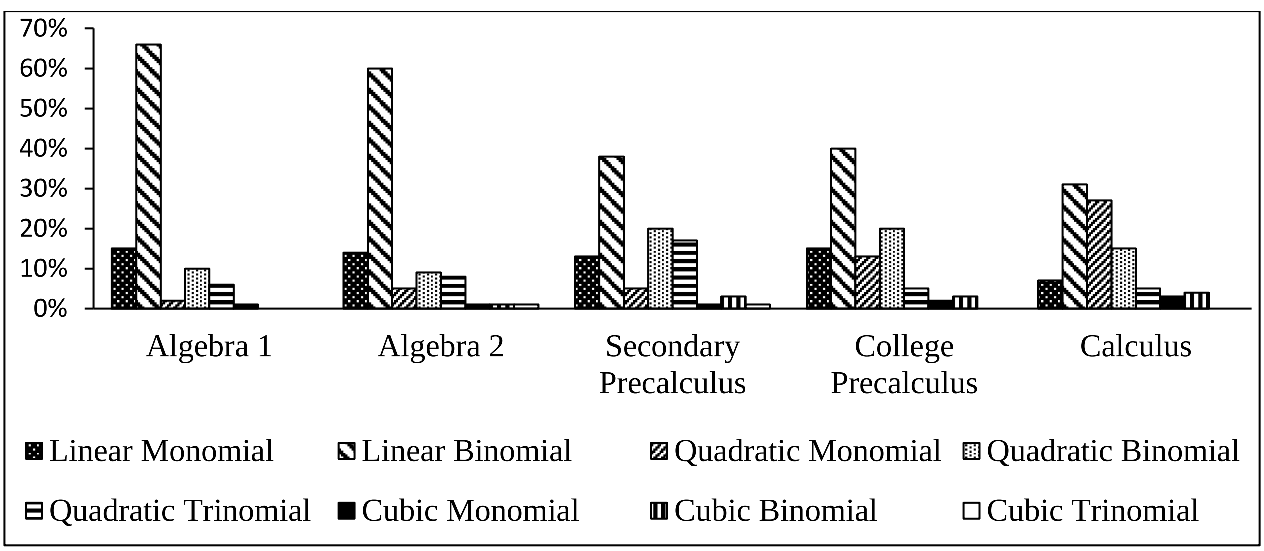

Figure 2. The distribution of the types of polynomials within each course.

The compositions of transcendental functions predominately involved the composition of inverse functions. Seventy-one percent of the trigonometric functions compose trigonometric and inverse trigonometric functions. Similarly, 62% of the compositions involving exponential functions and 77% of those involving logarithmic functions are the composition of exponentials and logarithms with a focus on the cancellation property of inverse functions. Overall, 17% of the elements across the curriculum contained the composition of a transcendental function with its inverse.

Representations

The composition content in these textbooks was represented by algebraic symbols, graphs, tables, verbal descriptions (Example A), numerical functional correspondence such as ordered pairs and function notation (Example B), illustrations such as function machines and mapping diagrams, and geometric figures. The geometric figures representation included geometric shapes and pictures that were transformed via composite transformations.

Example A: f is the rule "square" and g is the rule "subtract 3." The function $f \circ g$ *first* subtracts 3 and *then* squares; the function $g \circ f$ *first* squares and *then* subtracts 3. (Stewart, 2012, p. 193)

Example B: Suppose that $j(x) = h^{-1}(x)$ and that both j and h are defined for all values of x. Let $h(4) = 2$ and $j(5) = -3$. Evaluate, if possible, $j(h(4))$. (Carter, Cuevas, Day, et al., 2010, p. 411)

Of the elements with a representation, 81% contained an algebraic equation or expression, 7% contained geometric figures, 6% were graphical, 5% were verbal, 3% were tabular, 2% were numerical correspondences, and 1% was illustrations. Algebraic, graphical, and verbal representations appeared throughout the curriculum. Numerical correspondence representations were predominately found in secondary texts, and tables predominately appeared in college texts.

Multiple representations appeared in 4% of the elements with a representation. Of these, 30% occurred in Algebra 1, 8% in Algebra 2, 12% in secondary Precalculus, 26% in college Precalculus, 24% in Calculus. Of the multiple representation elements, 44% contained both algebraic and graphical representations and 43% contained both algebraic and verbal representations. Of these multiple representation elements 30% occurred in Algebra 1, 7% in Algebra 2, 12% in secondary Precalculus, 27% in college precalculus, and 24% in calculus. The college precalculus course contained more than double the number of multiple representation elements than the secondary precalculus course.

Discussion

Researchers have reported that college students have performed well on basic algebraic composition tasks and have struggled on tasks that focus on the compositive structure of functions. This study has identified that monomials, linear binomials, and the inverse cancellation property of transcendental functions accounted for 55% of the types of functions that appeared in the composition content of these secondary and early college textbooks. Additionally, the representation with the highest frequency in every course was algebraic.

While there have not been studies focused on students' performance on compositions tasks of various function types, researchers have reported statistics on students solving composition tasks represented algebraically, graphically, and tabularly. Carlson, et al. (2010) reported student success rates as 94%, 50%, and 47%, respectively. Hassani (1998) reported rates of 84%, 10%, and 50%, respectively. In an interview with a student in a developmental algebra course, DeMarois & Tall (1996) reported that the student needed guidance to complete a composition task using the table, unable to begin a graphical composition task, and successful with minimal guidance on an algebraic composition task. This analysis of written curricula revealed the same representation with which students were successful was the same as representation predominately in mathematics textbooks.

This study also looked for the mathematical shifts between the secondary and college curriculum. The data revealed that the college texts were similar to the secondary texts in the dominance of the algebraic representation and the high frequency of linear binomials. A shift did occur with the college curricula having more quadratic monomials and having a two-thirds increase in the number of elements that composed multiple types of functions. Being aware of these changes in the mathematical demand, teachers and curriculum authors can better support students' transition into college mathematics.

While it is true that written curricula do not determine what teachers teach and what students learn, it does contribute to both (Remillard et al., 2009; Stein et al., 2007). Teachers look at the adopted textbook as they prepare what they will teach and the students themselves have direct access to the written curriculum. Written curricula are involved at both ends of the teaching and learning experience.

Even though these findings are limited to a small sample of twelve texts, we can gain valuable information. The problems that students performed well on were common types of composition tasks in textbooks. More research on composition is needed to better understand how to improve students' understanding of composition and the compositive structure of functions and the similarities between textbooks and student knowledge.

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GTAs' Experiences of Switching to Remote and Online Teaching and Tutoring during the Start of the Covid-19 Pandemic

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The abrupt switch from in-person instruction and tutoring to remote or online instruction and tutoring as a result of the COVID-19 pandemic in March 2020 was difficult for even the most experienced instructor. In this paper, we explore how graduate teaching assistants (GTAs) at three different institutions responded to and experienced this change. Data was collected from surveys and focus groups conducted with graduate teaching assistants at each institution, as part of our ongoing collaborative NSF-funded project focusing on equipping mathematical sciences GTAs to become better teachers. In their responses, the graduate teaching assistants discussed topics ranging from what they did in their remote classrooms to the challenges they faced and supports they received from their department, university, and fellow classmates and faculty.

Keywords: graduate teacher training, remote instruction and tutoring, teaching assistants

Much of the undergraduate instruction in mathematics and statistics is provided by graduate teaching assistants (GTAs), who often have only limited training and experience in teaching (Blair et al., 2015; Ellis, 2014; Speer et al., 2005; Speer & Wagner, 2009). Responding to this need, Promoting Success in Undergraduate Mathematics Through Graduate Teaching Assistant Training (PSUM-GTT) is a multi-institution National Science Foundation-funded program that supports GTAs in developing knowledge, skills, resources, and mindsets to become effective instructors. The goal of PSUM-GTT is to improve academic outcomes of undergraduates in mathematical sciences courses by strengthening teaching capabilities of GTAs. PSUM-GTT aims to provide immediate benefits for the undergraduate students that GTAs currently teach. In addition, because many of today's GTAs are the "faculty of the future" (Saxe & Braddy, 2015, p. 27), we expect the program will provide lasting benefits to the post-secondary students current PSUM-GTT participants will teach for decades to come.

While there is extensive literature related to instructional practices and professional development for online teaching, Hodges et al. (2020) and Ho, Cheong, & Weldon (2021) distinguish between emergency remote teaching due to an emergency or crisis and extended planning for online learning, which might be a 6–9-month process. While some recent work in mathematics education has focused on remote and online instruction and tutoring practices during emergency remote teaching necessitated by the Covid-19 pandemic (Dumbaugh & McCallum, 2021; Johns & Mills, 2021), these studies focused on best practices, not the lived experiences of mathematical experiences GTAs and the training and support they received to support the sudden transition to remote teaching and learning in March 2020. This study aims to address this need by examining the experiences of mathematics and statistics GTAs at the three institutions where PSUM-GTT operates, leading to the three research questions that guided this study.

1. How did the rapid transition to remote learning and instruction impact and change the instruction of GTAs and their approach to teaching?
2. What challenges did the GTAs face during the transition to remote learning and instruction?
3. What supports did the GTAs receive during the transition to remote learning and instruction?

Theoretical Framework

The theoretical foundations of communities of practice (Lave & Wenger, 1991; Wenger, 1998) have become a model for growth and change in higher education (McDonald & Cater-Steel, 2017) and have been studied in mathematics and mathematics education faculty (Sack et al., 2016). Recent work has included a focus on infusing active learning into foundational science, technology, engineering, and mathematics (STEM) courses (Tomkin et al., 2019) through communities of practice. The elements of the PSUM-GTT program (described below) interact to foster an instructional community of practice among the mathematical sciences GTAs in the program.

Study Context: The PSUM-GTT Program

PSUM-GTT is a multi-faceted approach to supporting GTAs' growth as mathematical sciences educators. It was launched at one institution in 2015 and was expanded to two other institutions in 2019. At all three institutions, beginning GTAs are expected or encouraged to participate in PSUM-GTT. The program includes a seminar on teaching, held weekly for PSUM-GTT participants in their first year of the program, which examines best practices in classroom instruction and assessment, equity and inclusion, active learning, and student engagement. A Critical Issues in Undergraduate STEM Education seminar series features scholars and practitioners for seminars which provide deep dives on significant topics. Additionally, each GTA in the first two years of the program (mentee) is paired with an experienced GTA or faculty member who serves as a mentor, and one GTA at each institution serves as a TA coach to provide additional non-evaluative support to all GTAs. Finally, PSUM-GTT participants learn about the K-12 mathematics pipeline by visiting local middle or secondary schools or volunteering in STEM programs for students hosted at the university campus or in the community.

Method

Data Collection

The PSUM-GTT project includes data collection at the end of each semester for the purpose of program refinement and broadly relevant research. At the end of the Spring 2020 semester, all 67 of the GTAs in the program across the three universities completed an online survey via Qualtrics. The survey included matrix-formatted items that asked GTAs to indicate (from a provided list) which instructional techniques they employed before and after the switch to remote learning and instruction. Open-ended survey items asked the GTAs to report the challenges they faced, and the support they received. Additionally, 35 GTAs agreed to participate in focus groups to further discuss their teaching experiences that semester.

Participants

There were 11 graduate student mentors, 12 mentees, and 1 TA coach at Institution A; 8 graduate student mentors, 12 mentees, and 1 TA coach at Institution B; 10 graduate student mentors and 12 mentees at Institution C, all of whom chose to participate in the comprehensive training program. At Institution A, 56.5% of participants identified as male and 43.5% identified as female. Approximately 65.2% of participants were international students. At Institution B, 46.5% of participants identified as male and 43.5% identified as female. Approximately 13.0% of participants were international students. At Institution C, 45.5% of participants were identified as male and 54.5% identified as female. Approximately 40.9% of participants were international students. All participants had some undergraduate teaching experience, either at their current institution or a different institution, prior to the Spring 2020 semester. All three universities are considered to be research-intensive.

When all three schools shifted to remote learning in March 2020 because of the COVID-19 pandemic, 16 participants at Institution A were serving as instructors of record, as were 20 participants at Institution B and 10 participants at Institution C. Of these 46 who were instructors of record, only 6 had taught online previously and 21 had taken a prior online class.

Analysis

Descriptive statistics (e.g., means and percentages) were used to summarize items related to instructional strategies. Thematic analysis (Braun & Clarke, 2006) was used to analyze the open-ended survey items and focus group transcripts.

Results

What Happened During Instructional Time

After the suspension of in-person classes, 39.5% of GTAs who were instructors of record or recitation leaders reported using a synchronous online format that met at the usual class time. Approximately 18.6% reported a mixture of formats that was mostly synchronous, while 16.3% reported a mixture that was mostly asynchronous. Approximately 14% reported using an entirely asynchronous format. The remaining 11% chose “Other”, and listed details such as posting recorded videos, using a flipped classroom approach, or live lecturing for those students that could attend and archiving lectures for those who could not attend synchronously.

Instructors were asked to identify the instructional approaches they used before and after the transition to remote instruction. As indicated in Table 1, all active learning instructional techniques were reported being used more frequently when teaching face-to-face compared to teaching remotely. For example, student brainstorming was used by 80.4% of participants for face-to-face instruction but just 43.4% during remote instruction. Similarly, the use of teacher questioning decreased from 87% for face-to-face instruction to 69.6% for remote. Related to teacher questioning, the use of wait time after questions decreased from 89.1% during face-to-face to 65.2% when remote.

The decreased use of active learning and student-focused approaches was also seen in techniques such as jigsaw, students find the teacher’s mistake, peer review, informal assessment and student feedback at the end of class. All of these techniques had reported decreases of over 50% with the shift to remote instruction and 45.3% of instructors reported spending less than 20% of class time on active learning. The decrease in student-focused teaching also led to a decline in perceived student engagement with 88.1% of instructors identifying a moderate or great decline in engagement during remote instruction. Finally, the GTAs’ perception of their quality of instruction declined with 61% describing their instruction as “not as good” as before shifting to remote instruction, 34.1% reporting their instruction about the same, and 4.9% indicating better instruction during remote.

Table 1. Instructional Techniques Usage Prior to and During Remote Instruction

Use of Instructional Techniques (n=46)	**Face-to-face**	**Remote**	**Difference**
Direct instruction/lecture	40 (86.9%)	31 (67.4%)	9 (-22.5%)
Students brainstorming	37 (80.4%)	20 (43.4%)	17 (-45.9%)
Peer review of work	22 (47.8%)	6 (13.0%)	16 (-72.3%)
Informal assessment – end of class	19 (41.3%)	5 (10.9%)	14 (-73.7%)
Pop quiz	8 (17.4%)	6 (13.0%)	2 (-25%)
Using wait time after questions	41 (89.1%)	30 (65.2%)	11 (-26.8%)
Students posing questions in class	36 (78.3%)	32 (69.6%)	4 (-11.1%)
Teacher questioning	40 (87.0%)	32 (69.6%)	8 (-20%)
Role playing	5 (10.9%)	4 (8.7%)	1 (-20%)
End of class feedback – (e.g. minute paper)	14 (30.4%)	6 (13.0%)	8 (-57.1%)
Have students find teacher's mistake	29 (63.0%)	16 (34.8%)	13 (-44.8%)
Group quiz	11 (23.9%)	7 (15.2%)	4 (-36.4%)
Students share work on board	23 (50.0%)	5 (10.9%)	18 (-78.3%)
Inquiry-based learning tasks	16 (34.8%)	11 (23.9%)	5 (-31.3%)
Game-based approaches (e.g. Jeopardy)	5 (10.9%)	0 (0.0%)	5 (-100%)
Jigsaw	4 (8.7%)	1 (2.2%)	3 (-75%)
Think-pair-share	22 (47.8%)	3 (6.5%)	19 (-86.4%)
Using clickers or other response options	3 (6.5%)	3 (6.5%)	0 (no change)

Note: In the survey given to GTAs, this item was a multiple response item as GTAs could use multiple different instructional techniques in their courses.

In addition to the surveys, selected GTAs participated in focus groups regarding their remote classrooms. This focus group data provides further insight and information into how GTAs made the transition to remote instruction. Several mentees and mentors converted their course to a flipped classroom, with recorded lectures and synchronous discussion, problem-solving, and collaboration. They used features in their learning management system and breakout rooms in their video-conferencing software to create smaller groups for students. Some GTAs at all three institutions mentioned uploading notes and pre-recorded lectures for students to view prior to class. At Institution B, mentors described how course coordinators for some large service classes had pre-recorded lectures to both assist the GTAs in preparation and provide more uniformity in the instruction across that course. The majority of mentees and mentors across all three sites conducted their courses synchronously, but several GTAs, particularly at Institution A, changed to asynchronous instruction in response to poor student attendance, internet connectivity issues, etc., and reserved class time for problem-solving sessions and active learning activities. Other items mentioned that were used to facilitate instruction included iPads for drawing, computer software such as Desmos and GeoGebra for graph and figure construction, and virtual whiteboards.

The GTAs discussed that during remote instruction they worked to provide appropriate accommodations for students who reported varying “life issues” affecting their performance in the class. Sample accommodations made included GTAs conducting varied office hours for students with work and/or family obligations, flexibility with due dates, and providing extensions where necessary. Several also reported following up with missing students via email or phone. GTAs were also aware of equity issues impacting student participation and performance related to a lack of software, insufficient technology and/or lack of reliable internet. GTAs themselves reported spending more time in the preparation, delivery and grading components of their course.

Challenges and Supports Reported by GTAs

Teaching and learning remotely.

Low attendance and participation. The most common challenge identified by GTAs in focus groups was students not attending class. A number of GTAs also expressed concern about students who logged on but whose audio and video were turned off and who did not participate, so the GTAs could not tell whether or not students were following along with class. In addition, when GTAs did not see their students regularly in the classroom, it was more difficult to follow up with students who were struggling. Several GTAs noted that some students did not respond to their email outreach or that it is harder to communicate effectively via email.

Instructional design and active learning. Many GTAs conveyed a desire to engage their students in active learning, but expressed difficulty doing so in the remote context. For instance, one GTA noted that students were more reluctant in the remote context to share their work, and others noted that active learning activities were more difficult in the remote context. While breakout rooms offered an analogous experience to think-pair-share and group work, GTAs commented that students were less likely to participate in remote learning breakout groups than their in-person equivalent.

Assessment, and recognizing and responding to student understanding. Numerous GTAs expressed concern about formative and summative assessment and responding to student questions during remote instruction. GTAs accustomed to in-person instruction missed the opportunity to circulate the classroom and see students’ individual and group work. For instance, one GTA described missing “being able to walk around the classroom and see what's on your paper.” Numerous GTAs also lamented not being able to see students’ faces so that they could perceive students’ confusion, understanding, and “aha! moments.”

With less ability to gauge students’ understanding, GTAs were less able to adjust their instruction to address student questions or confusion. One GTA explained, “It was just kind of a lot of guessing and hoping like, hey, I'm putting this out there. I think it should be good enough. But I don't really know until they take your test.” Another GTA noted it was more intimidating for students to ask questions in the remote context since they had to ask them in front of everyone rather than being able to ask other students sitting near them. When students did use the chat function during class for questions, it was hard for the GTA to notice and respond to them while continuing to lead instruction. A GTA also observed that students were more reluctant to attend office hours via Zoom. Thus, it was more difficult for GTAs to gauge and respond to student understanding in the remote context than in the classroom context.

With regard to tests and exams, many GTAs expressed concern that students were receiving unauthorized aid, whether from web resources or other students. In addition, they described their own and students’ concerns about the proctoring services used by the institutions to monitor students while they took their exams.

Time required for instructional planning, provision of special help, and assessment in remote context. Support from other GTAs and from faculty and staff helped GTAs address the increased time demands of remote/online instruction. GTAs named specific graduate teacher training personnel as providing them with individual support in addition to supporting the larger body of student instructors. GTAs in some highly coordinated courses received substantial support from faculty course coordinators, such as the sharing of videos or loading of content into course management systems, in addition to regular "check-ins" or meetings. GTAs reported other graduate students as a source of support. These peer interactions via email, text and Facebook included discussion of details provided in university and department emails, answering of specific questions related to a class, sharing of resources from those who had taught online before, and assistance with creating assessments, such as writing exam questions.

However, a number of GTAs commented on the increased time demands of online teaching. Indeed, across the three schools, 70% of instructors of record reported spending more or much more time on remote instruction than they previously had for in-person instruction. Focus group respondents identified specific time-consuming activities such as recording and editing videos, preparing for class, grading student work, and responding to student questions.

Technology resources and know-how.

GTA technology know-how. A number of GTAs who had not previously taught online (i.e., asynchronous) or remotely (i.e., online synchronous) reported needing to identify appropriate technology tools that were available to them, learn to use these tools and troubleshoot issues, and determine by "trial and error" which worked best. The mathematical sciences context made it particularly important to find a technology tool to fill the purpose served by a chalkboard or whiteboard in a classroom because instructors wanted students to be able to see them working a problem (as on a chalkboard) at the same time that the instructor was able to see and be seen by the students.

GTA access to technology. Emails from each university and department provided details regarding resources for moving to remote teaching, such as training on Zoom or Bluejeans and course management systems. Additionally, two universities provided some students with university-owned technology for teaching, such as laptops, iPads or tablets, and microphones, or reimbursed them for individual purchases. However, some students indicated that the process for getting this technology was not clear to them, so not all GTAs initially had the technology they needed for remote teaching.

Student access to technology. Numerous GTAs reported that students contended with weak Internet connections and/or a lack of helpful hardware like a camera, microphone, stylus, iPad, or whiteboard.

Pandemic as context.

Practical difficulties faced by GTAs. The pandemic created a number of practical difficulties for GTAs. GTAs who were parents reported difficulty working from home with their young children, and another GTA who is a parent relocated to live with family. A couple of GTAs experienced increased demands in their own academic work, with one describing the workload as "unforgiving." Several also noted that no longer having access to a workspace on campus made it more difficult to focus or to get help from other GTAs or faculty, and one described being "overwhelmed" by the volume of university e-mail.

Personal struggles faced by GTAs. GTAs identified personal struggles they faced as a result of the pandemic, including diminished focus or motivation, mental health challenges, low energy, financial hardship, and not enjoying teaching in the remote context as they had in person.

Obstacles to student participation and learning. In addition to student challenges related to technology and teaching and learning, as described above, a number of GTAs reported that they had students who had difficulty attending class because of changes in their work schedules, the need to care for young children, or time zone differences. GTAs were faced with limiting synchronous interactions or knowing that some students would not be able to participate. One GTA commented that some students had gone home to another country and mused, “is it equitable to ask them to log in at 2am in the morning math class where they're expected to participate?”

Additional support that GTAs wished they had.

When asked what additional support could have helped them, GTAs identified two main areas. In terms of support for teaching, GTAs indicated that they could have used more direction on how to request technology and how to create their own questions in WebAssign, as well as more guidance on how to teach effectively in the remote environment. GTAs without course coordinators noted that they wish they’d had the same level of support as those with highly coordinated classes. In addition to support for teaching, GTAs indicated the need for more support for themselves as students and individuals.

Conclusion and Implications

The training program created a network of support and collegiality among GTAs and between GTAs and PSUM-GTT faculty and staff that was essential for the transition to remote/online instruction. While this support network was beneficial in fostering the community of practice around mathematical sciences instruction and learning, the difference among GTA assignments within departments resulted in GTA reports of varying support levels. Overall, PSUM -GTT faculty, staff, and students at the three institutions came together around a shared commitment to undergraduate education in the mathematical sciences, learned from each other, and contributed to the growing knowledge base of remote teaching and learning (Smith, Hayes, & Shea, 2017).

Active learning, student engagement and equity are foci of the program. Active learning approaches were used less often remotely, and instructors reported declines in student engagement. While many GTAs found different methods of reaching their students and learned to use different technologies that they might not have in a traditional setting, departments can support GTAs in their teaching by providing strategies and resources for active learning, student engagement, and assessment in online and remote courses and tutoring experiences. GTAs attention to equity issues was greater during the pandemic than prior to the transition. GTAs were more likely to make accommodations and follow-up with students.

GTAs were simultaneously contending with increased time related to remote teaching and learning and their own coursework and/or research as necessitated by the remote context. Due to GTAs receiving communication from multiple sources (e.g., dean’s office, department chair, course coordinator), streamlining of communication from within the department specifically about resources would have been helpful.

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Are We Sharing the Same Interpretations For "Prove" And "Show?": Comparison of An Instructor's and Students' Interpretations

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Proof is a medium of mathematical communication and has distinct characteristics related to it. Among various characteristics, we focus on the words in proving tasks prompts (PTP), such as "prove," "show," "justify," or "explain" because students' arguments can be changed depending on the keyword, which can impact assessing their learning when the students and the instructor do not share the same interpretation for the same keyword in PTP. Thus, in this study, we observed an Introduction to Proofs course to see which and how the keywords in PTP are presented to students and interviewed the instructor and a focus group interview of her three students. The finding indicates that the instructor's interpretation and students' interpretations were different. Although students are adjusting their pre-existed meaning for those words to the context, this study suggests that instructors cannot assume that students share the same meaning of the keywords in PTP with them.

Keywords: proving, proof, argumentation

Learning a discipline is deeply related to the discourse used therein so that one can fully participate (Lemke, 1990; Lew & Mejía-Ramos, 2019; Schleppegrell, 2007)**.** Specifically, to communicate using mathematical arguments, it is important for students to learn the discourse used and to understand the expectations of them in such situations. However, upon examining typical types of problem statements used in Introduction to Proof (ITP) courses, we find a variety of words used to prompt proving tasks such as "prove [a claim]," "show [a claim]," "give an explanation," or "justify your answer." Thus, we posed the following questions: "Are these words prompting the same task? Are they interchangeable?" Students are asked to create or evaluate mathematical arguments in response to varying prompts such as "prove/show/justify/explain a claim." What often occurs is that instructors enter the discursive moment with their own interpretations of the prompts for proving tasks while students enter with their own (possibly differing) interpretations. Hence, we spotlight the keywords in proving task prompts (PTP), such as *prove, show, justify*, or *explain*, as an important area for getting students to think about what they are asked to do.

Of course, one might ask why the issue of such a small aspect of proving tasks is of concern in undergraduate mathematics education. Depending on the prompt, students' interpretations affect the logical structure of the arguments and how they present their work. Moreover, if one's goal is to assess how students understand proving, assessing their understanding of proving becomes difficult if students think that the words for prompting proving are interchangeable. For example, depending on how students interpret *show*, some might think that they can provide some examples or can use graphs to make their point, whereas others may view *show* as a synonym to *prove*. As these keywords in PTP can change the structure of the arguments, we think it is important to focus on how people interpret these words. Especially, we focus on whether a teacher and their students would interpret the words similarly. Therefore, in this study, we investigated how an instructor and her three students interpret the keywords, *prove, show, justify*, and *explain*, in PTP and whether they interpret similarly or differently.

Background and Theoretical Framework

Several researchers have raised questions about the meanings of keywords in PTP, such as "show," "prove," "justify," "convince," or "explain," used to prompt proving tasks (e.g., Dreyfus, 1999; Hersh, 1993; Hwang & Karunakaran, 2019). For instance, Alcock (2013) posited that mathematicians consider "prove" and "show" to be synonyms when they prompt proving tasks. Dreyfus (1999), however, questioned the word "show" when it is used to prompt argumentation as it asks for students to provide a formal proof or an inductive argument. This question is supported by a study of Calculus I students' interpretation of the words "prove" and "show" used to prompt proving tasks (Hwang & Karunakaran, 2019, 2020**.** Hwang and Karunakaran (2019, 2020) suggested that students might think the word "prove" necessitates a narrower set of appropriate and formal options than the word "show". Their finding suggested that students think the prompting word "show" could prompt an inductive argument even if they think that argument is not appropriate for "prove." Moreover, they found that responses to the prompting word "show" need to focus on demonstration and can include visuals or the measuring of figures.

As another example, Mejía-Ramos and Inglis (2011) studied student engagement with the words "proof" and "prove" used to prompt proving. They used semantic contamination as their framework, and the research was based on their assumption that meanings from daily use of the words might be able to influence students' engagement with tasks. Their finding suggested that students responded more casually to the verb "prove" than the noun "proof," which aligns with their research on the usage of the two words in English. The finding also suggested people's usage of words in daily life affects their responses to mathematical questions.

Although very few researchers have examined the meaning of keywords in PTP, the need for further research about the prompting words could be found from some of the existing studies. Researchers questioned and/or reflected on how they worded their prompts for interviews or survey questions. For instance, Liu and Manouchehri (2017) reflected that changing wordings of questions might help to get a better result regarding the gap of student understanding concerning roles of proofs; thus, they called for further research on how these wordings affect the research. Also, sometimes researchers did not explicitly reflect on wordings to prompt students' responses during the interview tasks, but their results indicated that students might interpret those words differently. Bieda and Lepak (2014) purposefully avoided using the prompt "prove" and used "convincing" as a word to study students understanding, considering their population, middle school students. Their research results implied that students distinguish "show" and "tell (explanation)" when they are asked to choose "convincing" arguments. As another example, a student in Healy and Hoyles (2000) indicated that "examples could be a "proof," although not an "explanation" (p. 418), which again implied that students distinguish between proofs and other types of justifying arguments. This result possibly means that students would have differently responded to prompts that were worded differently. Hence, again, we claim that understanding linguistic aspects of the prompts in proving tasks are important because it is one of the starting points in solving the problem and building their arguments (Mejía-Ramos & Inglis, 2011).

In this study, we use the framework by Stein et al. (1996) as our theoretical framework that guides our study design and data collection. According to the framework by Stein et al., (1996) a task evolves throughout three different stages: when a teacher or task designer develops the task; when the teacher sets up the tasks to the classroom; and when the task is enacted in the classroom. They claim that the sequence of this evolution eventually influences students' learning. We consider that teacher's interpretation as a kind of teacher subject matter knowledge that can influence the task set-up as well as viewing students' interpretation as a factor influencing the task implementation. Therefore, according to the framework, if a teacher and their students do not share the same interpretation of the keywords in PTP, then it can lead students to possible misunderstanding of mathematical proving.

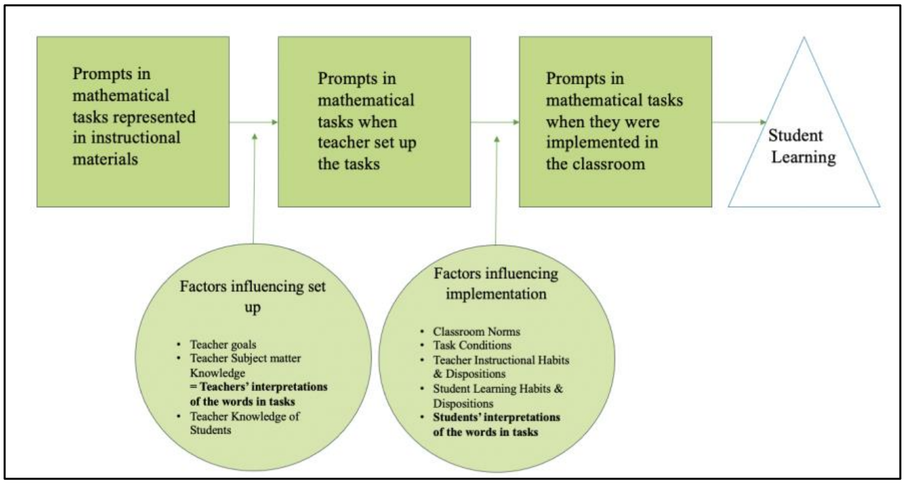

Figure 1. The modified mathematical framework by Stein et al. (1996)

Methods

The setting of the reported study is an ITP course in a large public midwestern university. We chose the ITP course because instructors and students in the course need to engage with proving tasks frequently. Therefore, an instructor of the course would have some interpretations of the keywords in PTP as well as at least some understanding or expectation of how students would react to the keywords. For that reason, Alycia (all names are pseudonyms), who was one of the six instructors teaching the course in spring 2020, participated in the study. Alycia was a mathematics doctoral student teaching this course for the first time as the instructor of record, but she had served as a teaching assistant for this course during the previous semester. The data collection with her consisted of two cycles of a half-hour-long pre-interview, two classroom observations, and an hour-long post-interview. During the pre-interviews, the first author asked her about her teaching plan for the week such as proofs that she would present to students or the problems that students would do. Alycia's voice was recorded during the classroom observation and the first author took notes on what she wrote on the board. Then, during the post-interviews, there was a brief discussion of what she did and whether there was a relevant case related to keyword in PTP as well as two interview tasks–one for each post-interview.

After the data collection with Alycia was finished, three students (Ryan, Leo, and Tyler) from Alycia's class participated in a focus group interview. The reason we chose the focus group interview was for participants to have a similar situation to their classroom because the ITP course in the university used groupwork as one of its major practices. During the focus group interview, they solved one proving task, discussed some questions related to PTP, and then collectively tackled the task used in Alycia's second post-interview. In this second task, participants were asked to sort six already generated arguments for the same statement "$x^2 + 4x + 18 > 0$ for all real numbers *x*." to one or more of the keywords in PTP, *prove, show, justify, explain,* or *none of the above*. The six arguments had different characteristics such as using a graphing calculator (A3), using empirical examples (A4), using algebraic expressions

(A1), or using different mathematical contents (A2, A5, and A6) (some examples in Figure 2). All interviews were video-recorded, and the observations were audio-recorded.

Once the data collection was finished, all of the interviews and relevant parts of observations were transcribed. In the first round of coding, sentences were categorized depending on the focus keywords (*prove, show, explain,* and *justify*). At this stage, for Alycia's data, we distinguished between her responses that were about how she would herself interpret the prompts and those that were about her expectations for her students' responses. Under each keyword, descriptive codes were developed. After the second round of coding, narratives of each keyword in PTP for an instructor and the focus group were created. To compare an instructor's interpretations and students' interpretations, we categorized relevant codes with three characteristics of mathematical arguments from Stylianides (2007): *set of accepted statements, modes of argumentation,* and *modes of argument representation*. Set of accepted statements refers to mathematical statements accepted by the mathematical community where the author meant to present the argument; modes of argumentation refer to the logical structure of an argument such as using inductive reasoning or deductive reasoning; and the modes of argument representation indicates the form of argument such as linguistic, graphic, or symbolic aspects. We posit that the keywords in PTP can influence on each of these three characteristics, especially the latter two.

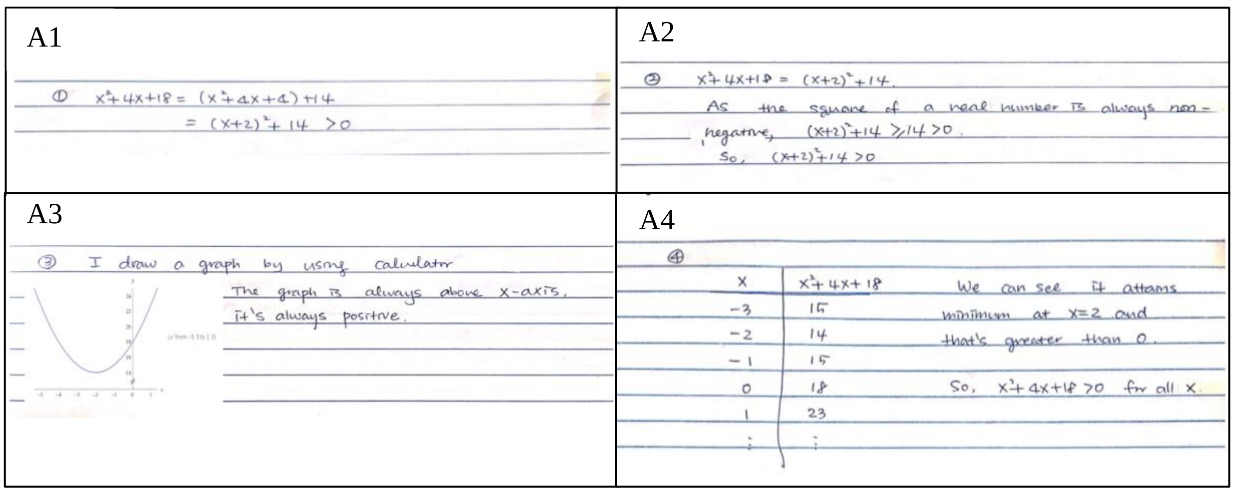

Figure 2. Some Examples from The Task in Which All the Participants Engaged

Findings

In this paper, because of the space constraints, we focus on *prove* and *show* which are the most common keywords in PTP used in proof-based mathematics courses. From our data analysis, we found that Alycia refers to *prove* and *show* as synonyms, but her students do not consistently agree. First, we present the data from the instructor, Alycia, and then we present the students' data, which we compare to Alycia's responses.

Alycia's Interpretations

Alycia herself interprets *prove* and *show* as synonyms as commonly used in the professional mathematical community and wanted her students to consider them that way as well:

> Alycia: if someone said, "*show* me that this is true," I feel like I still need to go through every scenario. And part of that is just probably just because of my training [as a mathematician].

This quote shows that due to mathematical experience, she became to think that *prove* and *show* are synonyms, because both of the words ask her to consider all possible scenarios. In the first post-interview, she claimed that she had tried to imply the message to students by switching up the words. But in the later interview, she said she will state the synonymity explicitly to students. And in the observation, we found that she vocalized the following expectation to her students:

> Alycia: So, *show* or *prove*. So, in your everyday life, these might mean different things, other professors that you encounter in your life, they may mean different things, *show* or *prove*. But if I ask you to *show* something, I'm asking you to *prove* it. [If] I ask you to *prove* something, I'm asking you to *show* it.

As well as wanting her students to know the synonymity of *prove* and *show*, Alycia also expected them not to use example-only based argument (empirical arguments) or calculators. In addition, she wanted her students to have good proof writing skills where she viewed a proof as a response to *prove* and *show*.

Alycia's Anticipation of Her Students' Responses

Although she emphasized and wanted her students to know the synonymity of the two words, she anticipated that students might answer differently to *prove* and *show*. Alycia thought that responses to *prove* would be more formal than responses to *show*. As part of it, she thought that students might use various forms, such as tables or graphs, when answering to *show*. In addition, she also wondered if students would feel less intimidated when they saw *show* or more stressed because they are not sure what *show* meant compared to when they saw *prove*:

> Alycia: …when people think of a proof or something like that in the mathematical sense, I think that just, I think, I'm sure plenty of people's blood pressure spikes when they hear that [*prove*] …. is if I asked them to *show* something, will that give them anxiety because they don't know what I'm asking for? You know? So, so it is kind of like this double-edged sort of for like, sure. Proof might sound more intimidating … So then if I, so, so if I ask them to show something but maybe they don't know what that means, well then give them more anxiety, then if I asked them to prove something and that just sounds more intimidating.

Hence, although Alycia viewed and expected students to know that *prove* and *show* are synonyms, she expected some students to struggle in interpreting and responding to different PTP words.

Students' Interpretations

From the focus group interview with her students, it seemed her anticipation of how students would respond to those words has some relationship with how a subset of her students interpret the words. Students were aware that *prove* and *show* are similar and that similarity being consistent in the class. However, they needed to remind themselves that those are asking the same things:

> Leo: So, if it says, just like, "Show why this isn't true," then you can ... Your first reaction, a lot of times, is going to be just like, "Okay, I'm just going to write down the counterexample and move on." But we really, then you have to like, step back and like, "Wait a second. They're not asking for that. They're asking for a proof.

Descriptions similar to this one appeared repetitively. Student participants were reminding themselves that the class they are in is "Transitions," where their instructor told them *prove* and *show* are synonyms. The following is the description by Leo when the interviewer asked them what the word *show* means to them:

> Leo: Sometimes it's, show is, in some cases, *show* can mean like, give an example of something, and then, in some cases, like in [course number of Transitions], a lot of times it ["show"]'s going to mean *prove*. So, you have to just make the judgment, yourself, of what they're really trying to ask.

Leo again referred to the course and mentioned that they are asked to produce the same arguments to *show* as if they are responding to *prove* in the context. Ryan and Tyler agreed with this description. However, as Leo indicated, they viewed *show* to mean something else in different contexts, which is related to how Alycia anticipated students would respond to *show*.

Also, related to Alycia's wondering related to affective aspects of the two words *prove* and *show,* students claimed that they felt different for both of them. During the interview, I brought up the case when Alycia mentioned in the class that *prove* and *show* are synonyms, and the following were their reactions:

Interviewer: … I remember Alycia, one time, said, "Show and prove are the synonyms in the class."

Leo: Yeah, but like-

...

Leo: Yeah, you hear that, and, but it's like, you see it on the paper, you don't think of them the same way.

Ryan: Yeah.

Interviewer: Mm-hmm (affirmative).

Tyler: If somebody says show, or prove, you're like, "Well, all right. I'm going to be doing pretty much the same thing," but there's a different feel, I think, to both of them.

Ryan: Mm-hmm (affirmative).

Although the three students did not say and I did not ask further at this point how they felt differently, during the interview they interpreted that *show* is more casual or provided "a little bit of breath of relief" than *prove* did. However, in general, it seems that they were adjusting their reaction to the word *show* as what the mathematical community that they belong to interprets it as.

Comparison of Alycia's and The Three Students' Interpretations

Although students are being enculturated to the professional mathematical community and both populations' interpretations looked similar, the responses to the task indicated some differences. When looking at Alycia's interpretation of the four keywords, deductive argumentation was the base requirement for any of them because she chose none of the above for non-deductive argument (A3 and A4; see Table 1). However, students included A3 as a possible answer for the *show* PTP, which means that deductive argument was not necessarily the requirement, but they focused on the graphical mode of argument representation. Similar to this, student participants generally paid lots of attention to the mode of argument representation such as focusing on defining variables, explaining every step, or making a conclusion, when they were deciding if something can be a response to the *prove* in PTP. This finding is related to why students' choice and Alycia's choices were different for the choice "prove," and seems to be related to Alycia's standpoint that she wanted to teach them to have a good habit of writing proofs.

Also, when comparing the choices for *show*, two of them are common A5 and A6, but the others do not overlap with each group's responses–A5 and A6 were deductive arguments using different mathematical knowledge. Considering A1 is an argument with only algebraic work and A3 is with graphs, it seems students still think concepts that *show* can have a variety of forms. Interestingly, the choices did not overlap with what Alycia anticipated her students might choose. Therefore, although (or as) students are learning how the PTP words are used in professional mathematical communities, their interpretations of the words were different than the instructor's interpretations and expectations.

Table 1. Choices of the prompts

	Prove	Show	Justify	Explain	None of the above
Alycia for herself	A2, A5, A6	A2, A5, A6	A1	A2	A3, A4
Alycia for students' possible responses	A2, A5, A6	A2, A4 (general students), A5, A6	A1, A3 (only first line), A4	A3, A4, A5 (nervous students)	
Students	A6	A1, A3, A5, A6	A2, A5, A6	A2, A3, A5, A6	A4

Discussion

Our study indicates that some students' interpretations of keywords in PTP were different than how their instructor interpreted the words. During the interview without the tasks, it seemed their meanings were similar, but their choices were different when we asked them to decide a proper PTP. More specifically, students distinguished *prove* and *show*, and thought that *show* can have a variety of meanings, unlike *prove*. This result can relate to Mejía-Ramos and Inglis's study (2011) when they found that student responded differently to the subtle difference between "*prove"* and "*give a proof*." As students have seen these keywords in PTP in their lives other than mathematics and even in other mathematics, the meanings can be transferred to the new mathematical contexts that they face.

However, in our study, we could observe that the students are adjusting their pre-existing meaning before the ITP course to what their instructor interprets it as, based on what the student participants mentioned as possible meanings in the other contexts such as calculus courses. But, given that this data collection is a month after the course started, students need some time to adjust their meaning of the keywords in PTP. Our study indicates, therefore, that interpreting the keywords in PTP as an instructor wants them to do might not be tedious for students requiring some extra time or work. This is because students need to learn if and so how three characteristics of mathematical argument (Stylianides, 2007) change depending on the PTPs. In addition, we also found that students' choices and their instructor's anticipated choices of the PTPs were different, which means that although the instructor is aware of the possible difference, it is not easier for them to anticipate how students would think about the responses or how students might respond to a given PTPs. That is, instructors also need some time and work to understand how their students are viewing the keywords in PTP and communicate with them about the possible issue.

These results suggest that the keywords in PTP can lead to students' responses being different than what task designers want, which can influence the assessment of student work; hence, task designers should be careful about the word choice in PTP especially when they do not have chance to share their meanings with task-doers such as in the case of national standardized tests or certain research settings. The data set in this study is not large, but generalization of the result was not the goal of the study, although the result that students see the words *prove* and *show* as different was found from our previous study as well (Hwang & Karunakaran, 2020). In the future, how these keywords in PTP influence students' responses in test settings on a large scale or how these keywords have been used in research settings might help the mathematics education community to see the importance of PTP.

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The Circle Schema: An Example of Schema Interaction in College Geometry

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There is a need to address student understanding of the role of definitions in undergraduate mathematics, and research is needed to determine pedagogical strategies that facilitate development of this understanding. In geometry, research shows students can better develop their understanding of concepts and their definitions by observing properties and making conjectures in non-Euclidean geometry. In this study, students learned concepts in Taxicab geometry in a College Geometry course in which theory in Euclidean geometry is the primary focus. The triad of stages of schema development and a model of schema interaction were used as frameworks in the analysis of student responses to a questionnaire and follow-up interview about the definition of circle in Taxicab and Euclidean geometry. As a representative illustration of a level of schema interaction, insight into the thinking of one student is presented in this report. Pedagogical suggestions are provided as a result of this data analysis.

Keywords: Definitions, Geometry, Taxicab, Circle, Schema

Introduction

Studies show that many preservice and even in-service teachers have insufficient understanding of the high school mathematics curriculum (Speer et al., 2015) and, more specifically, mathematical definitions (Chesler, 2012; Moore-Russo, 2008; Leikin & Winicki-Landman, 2001; Zazkis & Liekin, 2008). Edwards and Ward (2008) claim this is true for many mathematics majors and that this should be addressed in undergraduate mathematics, as research is needed to determine pedagogical strategies that help facilitate student's understanding of the concept of definition. Research shows by exploring concepts and definitions in non-Euclidean geometry, students can better understand Euclidean geometry (Dreiling, 2012; Hollebrands, Conner, & Smith, 2010; Jenkins, 1968). One example of a non-Euclidean geometry in which students can explore concepts is Taxicab geometry, which is the geometry that is the result of measuring distance as defined by the L_1 norm. Siegel, Borasi, and Fonzi (1998) and Dreiling (2012) encourage the introduction to Taxicab geometry before other non-Euclidean geometries since the simpler space makes it easier for students to reason and thus abstract concepts. For this report, we present results and discussion on the following research questions pertaining to student understanding of **Circle**: (a) How do students adapt their understanding of relationships among concepts associated with the definition of *circle* from Euclidean geometry to Taxicab geometry and vice versa? (b) How can using a model of schema interaction for the *circle in Euclidean geometry schema* and *circle in Taxicab geometry schema* help to describe the overall, underlying structure of the *circle schema* in students' minds?

Theoretical Framework

In APOS Theory, a constructivist framework based on Piaget's *reflective abstraction*, there are four different stages of cognitive development: Action, Process, Object, and Schema (Arnon et al., 2014; Dubinsky, 2002). A description of these levels of cognitive development in a general sense are omitted for the sake of space and detailed descriptions of the different levels of cognitive development associated with particular concepts in Euclidean and Taxicab geometry (like **Distance, Circle,** and **Midset**) are provided in Kemp (2018) and Kemp and Vidakovic

(2018, 2019, 2021). Arnon et al. (2014) stated that to describe certain learning situations, considering the schema structure may be necessary. As a result of the progression of APOS-based research, analyzing this structure may help to explain "why students have difficulty with different aspects of a topic, and may even have different difficulties with the same situation in different encounters," (p. 110). Considering the development of a student's schema has proven to lead to a deep understanding of how he or she reasons when confronted with a mathematical problem situation - how a student uses certain evoked components of a schema and relates them to one another when presented with these situations can reveal the structure of this schema and its development. Arnon et al. (2014) state that there is a need to investigate the development of schemas and how they are applied in mathematics.

To investigate the development of the *circle schema*, we utilize "the triad" of stages of schema development proposed by Piaget and García (1989). In studies such as Clark et al. (1997), Cotrill (1999), McDonald et al. (2000), Baker et al. (2000), and Trigueros (2000, 2001), researchers found the addition of the triad to their analysis helped to paint a better picture of how the components of schemas work together in certain circumstances. In particular, the reader's attention is focused to Baker et al. (2000) and how the authors define an overall calculus graphing schema in terms of the interaction of two schemas, as it is the first model of schema interaction described in detail. The authors describe the relationship between what are named the *interval schema* and the *property schema* and analyze common student errors when solving "an atypical calculus graphing problem," (p. 558). It is also noted Trigueros (2004) provides a second model of schema interaction for the solutions of systems of differential equations.

Below, descriptions are provided of the stages associated with the triad of development of the *circle in Euclidean geometry schema (cEg)* (which is the *Euclidean geometry schema* as it is evoked within the *circle schema*) and the mental constructions believed to be necessary to achieve these stages. Similarly, the evoked *Taxicab geometry schema* within the *circle schema* will be referred to as the *circle in Taxicab geometry schema (cTg)*. A genetic decomposition is defined as a "description of how the concept may be constructed in an individual's mind," (Arnon et al., 2014, p. 17). For this study, a genetic decomposition was developed to identify the different ways the *cEg* and *cTg schemata* interact with one another in a student's mind. The *circle schema* is described as its components may be evoked within the interaction of the *Euclidean geometry schema* and *Taxicab geometry schema,* using the framework and genetic decomposition presented in Baker et al. (2000) as a model. This genetic decomposition was used in data analysis and will be referred to as the *cEg-cTg schema interaction*. The genetic decompositions presented in Kemp (2018) and Kemp and Vidakovic (2018, 2019, 2021) were also used in analysis to help inform the researchers of the underlying structure of students' *circle schema* and how this schema is rearranged, or *accommodated,* to form new relationships among its components. For the representative example presented in this report, a brief description of the level of schema interaction will be provided. This example is chosen as an illustration of the interaction of schemata since the student appeared to accommodate her *circle schema* during the interview. This student provided insight as to how connections can be made to facilitate the development of the *circle schema* and ways to deepen student understanding of the role of mathematical definitions.

The three stages of schema development are prefixed Intra-, Inter-, and Trans- and each of these stages is described in general in Arnon et al. (2014). For length, we only provide descriptions of these stages in relation to the *cEg schema*. The stages of schema development are similar for the *cTg schema*. A student operating at the **Intra-cEg** exhibits this by viewing the components of the circle schema as isolated structures. A circle in Euclidean geometry is analyzed in terms of its properties either geometrically or algebraically (e.g. - it is round). Explanations of properties are local and particular. At least an action conception of the concepts of **Euclidean Distance, Radius, Center**, and **Locus of points** are necessary mental constructions to operate at this stage. A student operating at the **Inter-cEg** can exhibit this by forming relationships among the isolated ideas from the Intra-cEg stage and making connections between geometric and algebraic properties of a circle in Euclidean geometry. At least a process conception of some of the concepts of **Euclidean Distance, Radius, Center,** and **Locus of points** (and evidence that he or she can coordinate at least two of them) are necessary mental constructions to operate at this stage. A student exhibiting that she is operating at the **Trans-cEg** has constructed an awareness of the completeness of the circle in Euclidean geometry schema and can "perceive new global properties that were inaccessible at the other levels," (Baker et al., 2000, p. 559). The student coherently understands and can describe the construction of a circle and the structure of the equation for a circle in Euclidean geometry and how these are a result of the definition of a circle. It is necessary for a student to have at least a process conception of all **Euclidean Distance, Radius, Center,** and **Locus of points** concepts and has coordinated all combinations of these processes (which coordinates all geometric/algebraic representations).

Due to length, we provide here only a partial genetic decomposition for the *cEg-cTg schema interaction.* In general, this model includes nine levels of schema interaction that result as the two schemata of *circle in Euclidean geometry (cEg)* and *circle in Taxicab geometry (cTg)* interact with one another at various stages. These nine stages are shown in Table 1 below and each of these is described in detail in the full genetic decomposition. As an example, in the genetic decomposition, the mental structures necessary for a student to operate at an Inter-*cEg* stage of schema development and Intra-*cTg* stage of schema development, and what the interaction of these schemata look like at these different stages (which we refer to as the *Inter-intra* level of schema interaction) are described.

Table 1. The nine levels of cEg-cTg schema interaction.

		circle in Taxicab geometry schema (cTg)		
		Intra-	Inter-	Trans-
circle in Euclidean geometry schema (cEg)	Intra-	Intra-intra	Intra-inter	Intra-trans
	Inter-	Inter-intra	Inter-inter	Inter-trans
	Trans-	Trans-intra	Trans-inter	Trans-trans

Methodology

This research study was conducted at a large university in a College Geometry course during a Fall semester, which has an introduction to proof course as a prerequisite. Since it is a cross listed course, there were both undergraduate and graduate students enrolled in the course. The textbook used in the course was College Geometry Using the Geometer's Sketchpad (Reynolds & Fenton, 2011), written based on APOS Theory and the ACE Teaching Cycle (Asiala et al., 1996). The material of the course covered concepts and theorems in Euclidean geometry often seen in a College Geometry course and included Taxicab geometry for four class sessions at the end of the semester. Written work from the semester and videos from the in-class group work and discussion during the Taxicab geometry sessions were collected as data. After the semester but before final exams, semi-structured interviews were conducted with 15 of the 18 students enrolled in the course who voluntarily signed up to participate in the interviews. The following questions from the questionnaire were relevant to this data analysis, and are a subset of the questions asked before and during the interview:

1. Define and draw an image (or images) that represents each of the following terms however you see fit: Circle, Distance.
2. For any two points $P(x_1, y_1)$ and $Q(x_2, y_2)$
   (i) Euclidean distance is given by $d_E(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
   (ii) Taxi distance is given by $d_T(P,Q) = |x_2 - x_1| + |y_2 - y_1|$
   Using the grids below, illustrate each of these two distances. Be as detailed as possible in labeling them.
3. Is the following definition true in both geometries? Explain. "The circle (Euclidean or Taxi) is a set of points in the plane equally distant from a fixed point."
4. Using the grids below, sketch the following circle in both geometries: Circle with center at $C(3,3)$ and radius $r = 2$.

In general, there is no unique ideal response for most of these questions since the main goal of these interviews was to get students to elaborate on their thought processes as they responded to these questions. However, Figure 1 below shows how a student could illustrate what is asked in Question 2 and Question 4. For Question 3, this definition holds in both geometries, but the circles will visually appear different as a result of the way in which distance is defined.

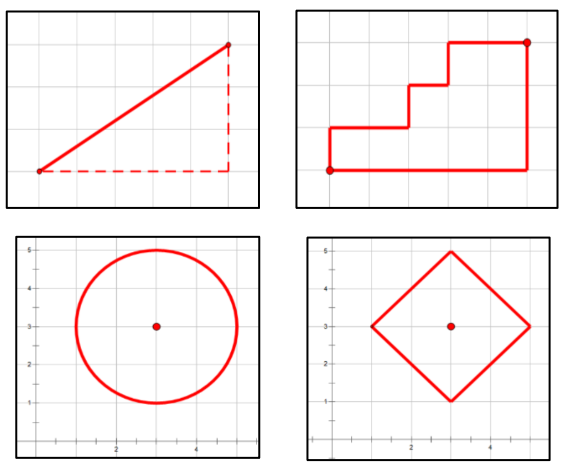

Figure 1. Visual representations of Euclidean and Taxicab distance (top row) and a Euclidean and Taxicab circle each with center (3,3) and radius 2 (bottom row).

By asking questions about both Euclidean and Taxicab geometry on the questionnaire as well as probing from the interviewer, students were presented with an atypical task, just as the participants in Baker et al. (2000). For analysis of data from these interviews, using the idea of schema interaction allowed for a deeper analysis of each student's make up, or structure, of the *circle schema*. Considering each of the concepts within this schema (**Distance, Radius, Center,** and **Locus of points**) helped to describe these structures. There was also a need to consider each of these concepts within both Euclidean geometry and Taxicab geometry, how each student understood these concepts within these geometries both algebraically and geometrically, and how this resulted in their overall understanding of the definition of **Circle.** From this information and using the genetic decomposition for the *cEg-cTg schema interaction*, all 15 students' work and responses were analyzed. Due to space limitations, only one case of one level of schema interaction is presented in this report, and more information about schema interaction can be found in Kemp (2018). What follows is a description and example of a student exhibiting the *trans-trans* level of schema interaction between the *cEg* and *cTg schemata* associated with the *circle schema.* This analysis will focus on what components of one student's *circle schema* were evoked to justify the responses she provided on a questionnaire prior to a follow up interview and her exploration of these concepts during the interview.

Results

A student is exhibiting operating at the trans-*cEg*, trans-*cTg* (trans-trans) level of schema interaction when the student has generalized the concept of **Circle** completely and has a coherent understanding of how the construction of a circle and the equation of a circle is a direct result of the definition of a circle. That is, the student has generalized the structure of the equation for a circle geometrically and algebraically in both Euclidean and Taxicab geometry. In terms of APOS Theory, the student has successfully constructed new processes geometrically and algebraically by coordinating all of her **Distance, Radius, Center,** and **Locus of points** processes. There is a complete coordination of the student's *cEg* and *cTg schemata*, and the student would be able to evoke any necessary components of these schemata to coherently talk about a circle in both geometries. There were three students out of the 15 who participated in interviews who exhibited evidence of operating at the trans-trans level of schema interaction. As a representative illustration for this level, we provide the APOS Theory and schema-interaction based analysis of Parker's responses to the questionnaire and prompts during her interview. We present a description and representative example of this level of schema interaction as this student provided evidence that she was able to accommodate her *circle schema* during the interview to begin operating at a trans-trans level of schema interaction. Through this accommodation, she provided unique insight into her thinking.

Parker was both a graduate student enrolled in the MAT program and a secondary mathematics teacher and was interviewed with two other graduate students. Parker's definition of a circle written on the questionnaire was the "set of all points equidistant from a given point known as the center." On the part of the questionnaire that asked if the definition of a circle provided held in both Euclidean and Taxicab geometry, Parker responded "Yes, the given distance just looks different in reference from the center." This provided evidence that while evoking her geometric representation of **Circle** and saying, "distance just looks different," Parker was aware that the choice of metric is the distinguishing factor between the visual appearance of a circle in Euclidean and a circle in Taxicab geometry. Parker's illustrations of a Euclidean and Taxicab circle from the questionnaire are shown in Figure 2, below. Although she was not asked to explain how she constructed her geometric representations of the circles, many details can be deduced from her illustrations.

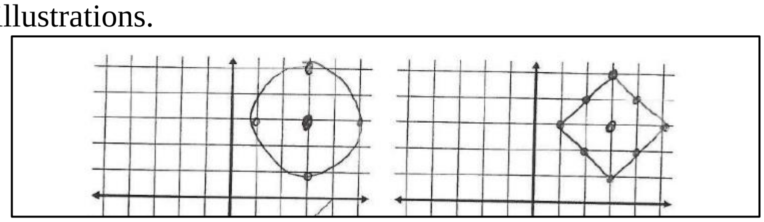

Figure 2. Parker's illustrations of a Euclidean and Taxicab circle on the questionnaire.

In her sketch of the circle in Euclidean geometry, she plotted the four points that were two units away from the center on the vertical and horizontal from the center then most likely "connected" these points based on how she knew the circle would look. For her sketch of the circle in Taxicab geometry, in addition to plotting the same four points on the vertical and horizontal from the center, she also plotted four more points that were two units away from the center and used these eight points to construct her circle. This is an indication that she likely used her understanding of the definition of a circle and the Taxicab metric to construct this circle, and that she had formed a coherent understanding of the geometric representation of a circle. In each interview, the students were prompted to try to write the equation of the circles they had drawn in Euclidean and Taxicab geometry. During conversation between the interviewer and the other two students in this group interview, Parker was writing down the equation for Taxicab distance under her illustration of a circle in Taxicab geometry, seen in Figure 3. It is noted that anything in red ink indicates it was written during the interview (and black writing was written while filling out the questionnaire prior to the interview). She interrupted the conversation between the interviewer and the other students and said the following.

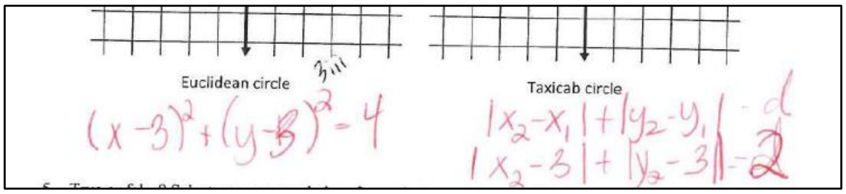

Figure 3. Parker's equations for the Euclidean and Taxicab circles she drew, written during the interview.

Parker: This might be wrong… I wrote the formula for…calculating distance for Taxicab, and then I know…this has to be compared to the center which is (3,3) so… if my distance is 2, doesn't that make it?...Is equal to 2. So let me change that…'Cause then like this point up here is [(3,5)] and if we plug in 3 and 5…3 minus 3 is 0, 5 minus 3 is two, so that would mean our distance is 2, which means it's on the circle.

Note that she used the variable d, and then substituted the value of 2 (the radius) for d, which is what indicated she understood how distance and the measure of the radius was involved in the equation of a circle. Parker implied here verbally that she knew the distance formula should be related to the radius when she says, "this has to be compared to the center". Specifically, she indicated she knew if the distance from the center to all of the points on this circle is the same, then this distance should be equal to the radius of the circle. She then used an arbitrary point to verify that a point she knew *geometrically* was on the circle, would also satisfy her *algebraic* representation of the circle, saying, "…our distance [between the center and (3,5)] is 2, which means it's on the circle". In this moment, we believe Parker had accommodated her *circle schema* and made connections between the geometric and algebraic representation of a circle in Taxicab geometry. It appeared Parker had generalized her understanding of the definition of a circle and was aware how the representations of a circle geometrically and algebraically are a result of the definition of circle. Parker provided evidence here that she was coordinating her **Distance, Radius, Center,** and **Locus of Points** processes across her *cEg* and *cTg schemata* to articulate her reasoning for writing the equation of her Taxicab circle in this way.

The interviewer then asked Parker if she could she draw a circle using a different metric if she were given one (and to describe how she would approach that task). Parker responded confidently, "yeah… figure out what the metric is asking…and what it's… well we eventually get our radius, and then we can…," and she began moving her finger in the air like she was starting at a center point and drawing outward. In further elaboration, she said "…the unit circle…and the radius needs to be one, so we have to figure out from the metric…what would yield an answer of one and if we do that then we can sketch the circle." Without explicitly stating she would construct multiple radii in order to sketch the circle, she implied this by her motions in the air with her finger, since it is interpreted that she was constructing multiple radii using some arbitrary metric. Further, she described that for a unit circle, she would use the new metric to find the points that would be one unit away from the center ("would yield an answer of one"). During this exchange, Parker provided evidence she had successfully constructed new processes geometrically and algebraically by coordinating all of her **Distance, Radius, Center,** and **Locus of points** processes. She appeared to have a complete coordination of her *cEg* and *cTg schemata* and was able to coherently talk about a circle, not only Euclidean and Taxicab geometries, but in an arbitrary space. For these reasons, Parker provided evidence that by the end of the interview she had formed a complete, coherent understanding of the underlying structure of the *circle schema* and was operating at the trans-trans level of schema interaction.

Discussion and Concluding Remarks

Parker exhibited evidence that she had generalized the definition a circle geometrically and algebraically in both Euclidean and Taxicab geometry. She also indicated she would be able to draw a unit circle using an arbitrary metric and, based on her reasons for writing the equation of the circle in Taxicab geometry, we believe she would be able to also write the equation for that circle. All three students who exhibited evidence of operating at the trans-trans level of schema interaction were able to explain the structure of the equations of a circle in both geometries in general terms and how it related to their illustrations and the definition of a circle. This is the main difference between the students operating at the inter-inter level and the trans-trans level. Some of the students operating at the inter-inter level of schema interaction could recall the Euclidean circle equation and could write the equation of a circle in Taxicab geometry, but mainly relied on "copying" the format of distance formulas. In contrast, the students operating at the trans-trans level showed they could clearly discuss how the distance formula is involved with this equation as a result of the definition of a circle. By making connections between the algebraic and geometric representations of concepts in both Euclidean and Taxicab geometry, students were able to generalize their understanding of this definition. There appeared to be a positive relationship between the amount and depth of these connections and the extent to which students were able to generalize the definition of a circle. In the example provided here, by visualizing and algebraically defining a circle in an atypical context, such as Taxicab geometry, Parker continued to construct and re-construct relationships between components within her *circle schema*. This is consistent with Dreiling (2012), Hollebrands, Conner, & Smith, (2010), and Jenkins (1968) in regard to the development of her understanding. This is evidence that more activities meant to help students form these connections between geometric and algebraic representations in geometry should be used in classroom instruction.

There is a lack of research that uses of the stages of the triad in APOS Theory (for examples, see Clark et al. (1997) and McDonald et al. (2000)), but even fewer studies that use the idea of the "double triad" or schema interaction. A model of schema interaction was used in this study and one level of schema interaction was described in detail in this report, using the original model by Baker et al. (2000) as a guide. Baker et al. (2000) determined that analyzing their data using APOS Theory without the triad or schema development hindered the researchers from understanding the thought process of many of the participants. Similarly, by involving a genetic decomposition for a model of schema interaction into the analysis of this study, the researchers were able to better describe the structure of the participants' *circle schema* and their overall understanding of the definitions of various concepts in geometry. It is believed that the genetic decomposition of this schema interaction developed for this study could be used as a model for how students can assimilate any metric into their *circle schema*, although further research would need to be done to validate this.

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Influences on Problem Solving Practices of Emerging Mathematicians

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Mathematical problem solving research that focuses on the development of problem solving practices can provide mathematics educators the tools with which to foster the use of such practices in their students. This study hopes to identify influences on the development of emerging mathematicians' problem solving technique. Emerging mathematicians from an undergraduate real analysis course and first-semester graduate course in linear algebra were invited to a sequence of two interviews, during which they completed problem solving tasks and reflected on their growth as mathematical problem solvers. In particular, they were asked to expound on formative experiences that affected the development of their problem solving strategies. Participants reported ways that both teaching mathematics and learning mathematics from skilled instructors influenced the way they solved novel problems.

Keywords: Problem solving, Teaching, Graduate students

The body of research on mathematical problem solving is multifaceted. Studies in this field, for example, might hope to explain the nature of problem solving, generate a framework for analyzing problem solving, or catalogue the problem solving behaviors of expert mathematicians (cf. Schoenfeld, 1989; DeFranco, 1996; Selden et. Al., 1999; Carlson & Bloom, 2005). Resourceful mathematics educators could leverage any one of these studies, regardless of its stated purpose, in service of second, implicit goal: to better understand how to foster problem solving growth in their classrooms.

One might suspect that, to help mathematical novices grow as problem solvers, one might only have to teach them to replicate the problem solving strategies employed by experts. But as noted by Silver (1985),

> This is specious reasoning and such advice may be counterproductive to learning. Surely the behavior of experts is highly efficient and often elegant, but the polished, mature problem solving behaviors of experts fail to reveal the successive approximations that preceded these mature procedures (pg. 251).

The study described in this report aims to capture these successive approximations (and thereby better understand the expert problem solving methods in which they culminate) by asking emerging mathematicians to identify which experiences contributed to the ongoing development of their problem solving strategies. Here, we define an *emerging mathematician* as a student of mathematics who understands foundational content knowledge but has had limited opportunities for refinement of their mathematical practice. Over the course of a semester, participants self-reported experiences both as teachers and as students of mathematics that influenced their process for solving novel problems.

Background and Theoretical Perspective

One hallmark of classrooms that successfully encourage problem solving is the breadth and connectedness of the mathematics knowledge they impart (Arcavi et al., 1998; Collins et al., 1988); another is that this knowledge is also robust and thoroughly justified (Arcavi et al., 1998; Schoenfeld, 2014). While these qualifications might be lumped together in order to assert that expert problem solvers simply have more content knowledge, this is an oversimplification.

Certainly, experts have more content knowledge (DeFranco, 1996). However, content knowledge cannot be leveraged for problem solving if it is *inert* (Whitehead, 1929; as cited in Renkl et al., 1996), that is, confined only to applications that are similar to the specific instructional circumstances under which it was introduced. The difficulty for some mathematics students (primarily, those with a preponderance of inert knowledge) to apply familiar concepts to novel situations is sometimes referred to as the *transfer problem*; researchers have developed many pedagogical strategies that hope to address this issue. Those that fall under the *situated learning* model of instruction (e.g. Bransford et al., 2012; Spiro et al., 1991; Campione et al., 1988, Collins et al., 1988) hope to overcome the transfer problem by introducing mathematics concepts already embedded in a novel, real-world context from the outset.

Problems of transfer, however, are not only relegated to students who cannot solve real-world problems with abstract mathematics. Research has also shown that even otherwise successful calculus students have significant difficulty applying their textbook knowledge of the subject to completely abstract (but entirely novel) problem solving situations (Selden et al., 1999; Selden et al., 1994). In response, some pedagogical models describe instructional strategies that teach both content knowledge and powerful thinking strategies, allowing the application of knowledge to solve unfamiliar problems (e.g. Liljedahl, 2016; Schoenfeld, 2014; Fiorella & Mayer, 2015). Illustrations of such classrooms are multifaceted and necessarily describe not only of the type of content delivered but also the method of delivery, type and frequency of corresponding assessment, layout of the classroom, and appropriate questioning strategies. One belief embodied by these models is that students are more likely to develop successful patterns of thought and effective problem solving strategies when they are actively engaged both with the content and with each other (see also Silver, 1985; CBMS, 2016). A notable example of such a classroom is Schoenfeld's problem solving course, taught at UC Berkeley and analyzed, in part, in Arcavi et al. (1998). Through an authentically collaborative mathematical environment, this course emphasized effective modes of mathematical thought that included instruction on both metacognitive strategies and the application of specific heuristics.

Methodology

Setting and Participants

The study took place at a large, urban, public, and Hispanic-serving university. In order to recruit emerging mathematicians, participants were solicited from the population of upper-division undergraduate mathematics majors and first-semester mathematics graduate students. Undergraduate participants came from an undergraduate analysis course that has an "introduction to proofs" course prerequisite; that is, these undergraduates had experience with foundational techniques for mathematical justification but had not had extensive experiences refining them through prolonged or advanced application. Graduate participants came from a first-semester graduate course in linear algebra. Their status as first-semester graduate students acknowledges a baseline experience and competency with a broad spectrum of mathematical technique; however, as beginning graduate students, they had not yet encountered the demands of mathematics graduate courses, mathematical research, and exploration of unsolved problems.

In the first week of the Fall 2020 semester, both groups of participants responded to a short questionnaire (administered at the beginning of a class period corresponding to one of the respective courses described above) in which they were asked to rate certain problem solving techniques using a Likert scale (from "1 – Strongly Disagree" to "6 – Strongly Agree"). For an example of such questionnaire items, see Figure 1 below.

	When solving a difficult math problem, it is helpful to...						
1.	Read the problem carefully	1	2	3	4	5	6
2.	Try and imagine what a solution might look like	1	2	3	4	5	6
3.	Consider multiple different ways in which the problem might be solved	1	2	3	4	5	6

Figure 1. Select Likert scale items from the preliminary questionnaire.

Participants also indicated their interest in participating in a sequence of two interviews. The first interview was scheduled as soon as possible so that it would take place close to the beginning of the semester; the second interview occurred at the end of the same semester. For interview protocol and procedure, see the following section. A total of 11 undergraduate students and 20 graduate students responded to the initial questionnaire, and of these, 4 undergraduate students and 8 graduate students participated in the interview sequence. Due to the small number of respondents, the interviewed participants necessarily represented a convenience sample. Information on interview participants is provided in Table 1, below.

Table 1: Interview participant information

Pseudonym	Gender	Age Bracket	Population
Will	Male	18-22	Undergraduate
Riley	Female	18-22	Undergraduate
Brady	Male	18-22	Undergraduate
Verna	Female	33+	Undergraduate
Anne	Female	18-22	Graduate
Greg	Male	23-28	Graduate
Taylor	Other	28-33	Graduate
Mia	Female	18-22	Graduate
Sara	Female	18-22	Graduate
Frank	Male	33+	Graduate
Julie	Female	18-22	Graduate
Carol	Female	23-28	Graduate

Data Collection and Analysis

Each interview was recorded and lasted between 45 and 90 minutes. During an interview, participants completed a sequence of three tasks intended to prompt problem solving engagement. They were also asked to comment on their decisions and thought processes while working on the task sequence. For an example of an interview problem, see Figure 2 below.

Problem 2. *What value(s) of α maximize the number of solutions to the following system of equations?*

$$x^2 - y^2 = 0$$
$$(x-\alpha)^2 + y^2 = 1$$

Figure 2. The second task given during the first interview, adapted from Schoenfeld (1998, pg. 98).

At the end of the first interview, participants also elaborated on their responses to the initial questionnaire. This process was facilitated by the interviewer, who asked them to expand on initial questionnaire responses that were sufficiently different from the average response for their population. During the second interview, after completing the requisite sequence of three tasks, participants also completed the same questionnaire to which they had already responded at the beginning of the semester. This time, the interviewer prompted them to respond to questionnaire items for which their second response was significantly different than their first, i.e., if the response was two or more points different or when it indicated any shift from agreement to disagreement (or vice versa).

The goal of the questionnaire analysis conducted during interviews was to provide participants with a chance to reflect on the development of their opinion towards particular problem solving strategies; to supplement this goal, a number of interview protocol questions were asked at the end of each interviewer. For example, these questions included: (1) How has your approach to problem solving changed (over time/ in the past semester)? (2) Has any singular (class/ person/ event) significantly affected your approach to problem solving (in the past semester)? How? (3) Describe the particular qualities of your favorite math course that made it stand out.

Responses to interview questions and any discussion of questionnaire items were initially transcribed verbatim. The select quotes provided in the section below have been lightly edited for clarity by removing speech disfluencies, except when doing so would alter the tone or meaning of a sentence. The transcripts were then analyzed using open and axial coding (Strauss & Corbin, 1998) that aligned with techniques from thematic analysis (e.g., Braun & Clarke, 2006; Nowell, Norris, White, & Moules, 2017). An initial coding pass was conducted by both researchers, who then met to compare their work; codes were expanded, subsumed, or redefined appropriately until both researchers were in agreement.

Results

Below, we introduce two prominent themes that emerged during interviews: participants recognized when development of their problem solving strategies had been instigated by their experiences both while teaching and while learning from influential teachers.

Growth while Teaching

Across interviews, participants identified that teaching experiences served to augment their originally surface-level understanding of mathematics. For example, Mia explained how teaching had increased her proclivity for sense-making:

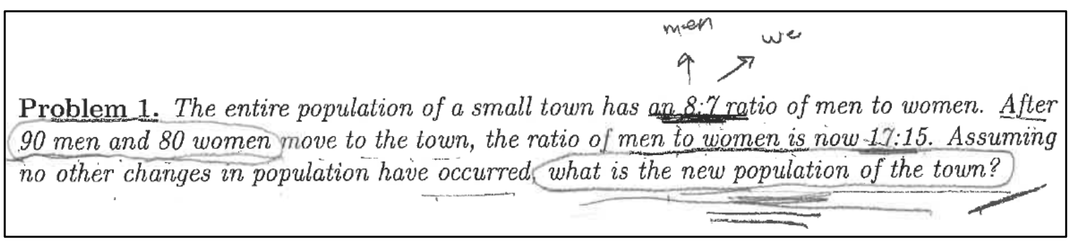
> Before, when I used to do math but I didn't understand something, I feel like I would just skip it. And then move on, take it as: okay, this is fact. […] Teaching my calculus course, since I have to go *so* in-depth with the students because they ask *so* many questions, I feel like—now, I feel like I get more caught up on the things that I don't understand as well. […] Now when I understand things, I *understand* them.

Brady identified this phenomenon as having to "over-learn" a topic that he planned to teach, a practice which made him aware of an important pedagogical dichotomy tied to sense-making: "You can memorize a formula, or you can actually understand what the formula's doing." Anne echoed Brady's sentiment, explaining that adopting the former strategy was "the reason why people need tutoring, I feel: they don't understand why you're doing what you're doing. And so, I can't just be like, well, use this formula. I don't know why it works, but it does. It just confuses them more. So it's made me have to figure out why I was doing things in previous courses as well." Several participants found that preparing for teaching not only increased the depth of their mathematical understanding but also its breadth; especially when dealing with students of varying mathematical competencies, participants recognized that, as teachers, they needed to consider multiple ways to approach one problem or explain one concept. Carol observed that "While going for the teaching, I see sometimes, like: what are the other options to solve this? What can be the different methods?" and that this practice of planning ahead and weighing multiple options carried over into her own problem solving.

Additionally, some participants found that teaching led them to discover specific techniques or explanations that could be applied to their own areas of study. Anne noted that teaching "things to help them [her students] figure out what to do on a problem, when they see it, helps me to do that." She was not the only participant who adapted her own problem solving strategies to reflect what she recommended to her students: Frank noted that he was "more attuned to" making sure his work "flows in a logical way" after interacting with students whose work was disjointed and not well-supported with mathematical justification. Verna taught her students the importance of carefully reading and interpreting the text of a given problem:

> I want to help them to understand the question. So I—like, for me, I read it but go with them over the question and ask them: look what they ask here. They figure out, oh, this is what they ask? Like this. So I make them to understand the question and what they ask about. And they have the knowledge, but they didn't know. That's why—that's helped me.

Appropriately, Verna appeared to be especially conscientious of the need to read interview tasks carefully. Figure 2 below illustrates a sequence of annotations indicative of her approach to interview tasks; this included identifying important numerical data, isolating the question being posed to better anticipate the form of her solution, and clarifying any unfamiliar mathematical notation.


men
we

Problem 1. *The entire population of a small town has an 8:7 ratio of men to women. After 90 men and 80 women move to the town, the ratio of men to women is now 17:15. Assuming no other changes in population have occurred, what is the new population of the town?*

Figure 3. An example of Verna's propensity for carefully attending to the text of a given task. Problem 1 preceded Problem 2 (see Figure 1) in the first interview.

Riley described tutoring a student to whom she would sometimes give the solution to a problem. Then, she would prompt the student to "justify why I got that, exactly. What am I doing, and why does it relate to this problem?" She mentioned, in her second interview, that thinking about her own work from a similar perspective (especially when that work was proof-based) was sometimes helpful when approaching a difficult problem. In both interviews, Riley lived up to this claim. For example, she attributed her difficulty solving the task pictured in Figure 1 to the fact that she was "not completely certain what the answer's supposed to look like," and so could not apply this strategy. On the other hand, in a different task that required her to inscribe a square in an arbitrary triangle, Riley successfully applied an approximation of her working backwards strategy by starting with a square and instead attempting to verify whether she could circumscribe an arbitrary triangle about it.

Growth while Learning

Many participants also commented on those of their own instructors whose pedagogical decisions they perceived as influential in their problem solving development. Some instructors contributed directly to this development by giving the participant new strategies or heuristics for tackling novel problems. For example, Riley described the lasting impression of a favored instructor's teaching strategy:

> It's one thing for a professor to give you a function and give you a problem, give you an example, and show you how it works. You could mimic that, a lot of the time. But to have a professor that gives you a problem that you haven't seen before, and gives you certain tools that will help you solve the problem, and so you know that those tools are there, you'll be able to solve the problem a lot easier than someone that's mimicking. They won't take that away.

Other participants were more specific about the problem solving heuristics that they had learned. Verna learned the value of providing thorough mathematical justification. With respect to one influential professor, she said, "When she give us a problem, she told us to go step by step and don't, like—don't ignore *this*. This [has] some meaning. *This* [has] some meaning. You must understand what you [are] talking about." Brady, in reference to the same instructor, remembered the same lesson: "But with most of our proofs, it felt like it was pretty much line by line, you know? […] And I felt like, in the beginning, that that was very important. Because now, I don't just skip a few lines of work without really thinking about it first." This behavior was in evidence during the first interview, wherein Brady solved a task that required him to cut an arbitrary right triangle into two isosceles triangles. There, he drew an arbitrary rectangle and attempted to appeal to the mathematical properties of its diagonals. Brady was able to indicate places during this justification process where he relied on visual intuition to substantiate a claim; he explained that a truly rigorous explanation would not be so informal.

Riley explained how her previously tentative approach to novel problems was affected by one instructor who encouraged her to, "even if you're not sure about going about a problem, draw something. Put *something* down. Because it's probably worth it in the end. Because you're thinking of it." Carol had also been encouraged to draw diagrams by several instructors who valued visual representations, and she noted that this was particularly helpful for reframing familiar proofs in more interesting, intuitive ways; as an example, she recalled one professor who used diagrams to illustrate the relationship between Cauchy and convergent sequences. During her work on the task pictured in Figure 1, Carol drew ten discrete visualizations as tools for sense-making and justification. This was more than the rest of the participants combined; a selection of Carol's visualizations are provided in Figure 4 below.

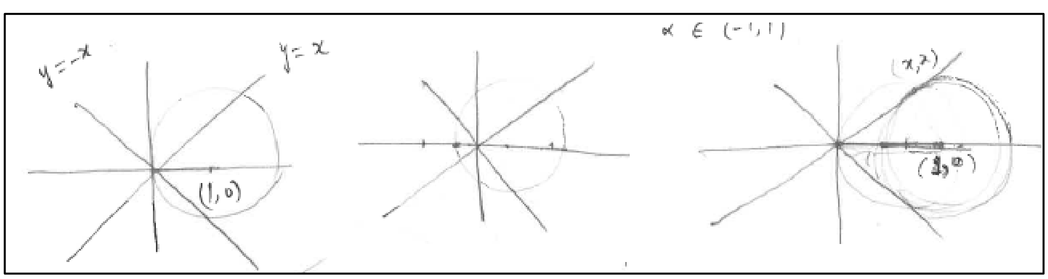


Figure 4. Some of Carol's diagrams, depicting three different cases corresponding to different values of α.

Carol also noted that her professor's visualizations were more engaging for students: "It was not just writing on the board and we have to copy that." Other participants also found previous instructors memorable for their ability to engage students by giving them agency in the creation of mathematics. Brady remembered an instructor who called on the class directly to answer important mathematical questions, including having "us do proofs up on the board and stuff. The students of the class. And that kind of, I guess, added a community feeling to the class." The epistemological responsibility of students to generate their own explanations even extended to one of Frank's professor's office hours, and Mia recalled one professor who "challenged you to think by yourself. […] Like, he didn't *tell* you the answer. He made you think it through." Sara explained the value of classrooms in which students themselves were involved in the creation of mathematics when she said that students who are "part of the conversation" are better equipped to explain the phenomena that they have learned. To substantiate this claim, she drew upon her experience with a recent professor who encouraged students to generate their own definitions for important concepts; from a problem solving perspective, this challenged Sara's preconception that mathematics was a rigidly defined subject wherein problems had prescribed methods of solution.

Conclusion and Implications

From participant interviews, a number of themes arose with respect to problem solving development. Graduates and undergraduates alike thought that teaching mathematics helped them to recognize and leverage more effective problem solving techniques. In particular, participants saw increased value in drawing diagrams, justifying their methodology, and carefully making sense of the problem text. Most graduate students also felt that their teaching experiences gave them a deeper understanding of the material, both because of the pedagogical requirement that they know several effective explanations and because of the opportunity to reacquaint themselves with older content. As observed in Selden et al. (1999), incremental growth in facility with content over a period of repeated exposure is not unusual; students in constant contact with calculus, for example, can still take years to be able to apply that content in creative ways. Furthermore, repeated exposure from a teacher's perspective might foster a deeper understanding of mathematical content. Such deeper understanding is a fundamental aspect of many pedagogical frameworks aimed at improving problem solving (e.g. Schoenfeld, 2014). In participants' work, these conceptual advantages allowed them to consider multiple approaches to a given problem and apply a diverse array of heuristics; furthermore, they felt comfortable justifying why their chosen methodology was both appropriate for the circumstances and mathematically sound.

On the other hand, undergraduates were more likely than graduate participants to recall specific problem solving strategies taught to them by instructors. These strategies were identical, though, to two of the previously noted strategies learned through teaching: visualizing to make sense of difficult concepts and justifying every claim. As students, both graduate and undergraduate participants also recognized that their most influential instructors tended to give them the opportunity to discover and take ownership of the mathematics content. This agency engaged students, mirroring recommendations from literature (Schoenfeld, 2014; Liljedahl, 2016) and giving participants better tools with which to justify their mathematical reasoning. It also instigated an affective shift; participants who constructed their own mathematical meanings felt more confident in constructing their own approaches to novel problems. That is, participants reported feeling less compelled to remember and apply a "correct" method after experiencing the flexibility of mathematics first-hand in an influential classroom environment. Such instruction may thus promote the ability of students to transfer content knowledge from the original introductory environment to novel problem solving situations.

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Bridging-Course Students Extract Square Roots: Commognitive Account of Seeming Conflicts Within Students' Discourses

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This study explored the grasp of square roots among 11 students in a bridging course, with a special focus on instances where the same student generated seemingly conflicting responses. Building on the commognitive framework, the analysis indicated that individual students square-rooted differently in a range of situations, such as cases where roots were extracted from square numbers and from squared radicands, where roots "stood alone" and where they were incorporated in an exercise. Differences were found in the procedures that students employed and the tasks that they pursued. A theoretical account was offered, suggesting that what may appear as a conflict within a student's discourse could be a sensible difference of actions taken in task situations that this student construed as incompatible.

Keywords: commognitive framework, roots, algebra, bridging courses

Rationale and Aim

The study of roots accompanies students all the way from middle school through to university mathematics. Typically introduced in the context of integers, roots are gradually extended to fractions, parametric expressions, equations, and functions. Through the Pythagorean theorem and the law of cosines, roots find their way into geometry and trigonometry lessons. In university courses, roots are often used to exemplify inverse functions in calculus and multi-valued functions in complex analysis. Then, it is barely surprising that school curricula and bridging courses dedicate dozens of teaching hours to develop students' proficiency with this cross-curricular mathematical concept (Kontorovich, 2018a).

Research has been interested in the learning and teaching of roots, with a special focus on students' difficulties and mistakes (e.g., Cangelosi et al., 2013; Kontorovich, 2016, 2018a, b, 2019). For instance, Crisan (2014) asked a group of pre-service secondary teachers to *square-root* from $25y^2$ (i.e. to simplify $\sqrt{25y^2}$). This request divided the group between the proponents of "$\sqrt{y^2} = y$", and those who adhered to "$\sqrt{y^2} = \pm y$".

In Kontorovich (2019), I explored the unconventionality of such answers among school graduates, with a focus on the tensions that may have been evoked. Indeed, if a student was to assume that square-rooting from squared radicands yields the initial input (e.g., "$\sqrt{(-6)^2} = -6$"), then would this same individual endorse the chanted property of even roots as outputting non-negative numbers? Alternatively, if a student was convinced that square roots generate two opposite results (e.g., "$\sqrt{x^2} = \pm x$"), would they entertain the idea that $\sqrt{x^2} - \sqrt{x^2}$ is not only 0 but also $\pm 2x$? When students' answers to such questions were consistent with 'no', I interpreted it as a conflict in their mathematical knowledge. In this article, I revisit my previous analysis with the aim of *generating a theoretical account of situations in which students produce conflicting responses as part of their square-rooting.*

Offering theoretical accounts for the described situations is not straightforward. For instance, when introducing their seminal notion of concept image, Tall and Vinner (1981) acknowledge that "[a]t different times, seemingly conflicting images may be evoked" (p. 152). The researchers explain such conflicts with learners' unawareness of them but do not elaborate on the mechanisms that conceal the conflict from the learner. I capitalize on the commognitive framework (Sfard, 2008) to make sense of the conflict phenomenon.

Theoretical Framework

The commognitive framework is concerned with human communication and discourses (Sfard, 2008). The latter are distinguished through their *keywords* (e.g., “square roots”), *visual mediators* (e.g., “$\sqrt{\blacksquare}$”), verbal and symbolic *narratives* (e.g., “$\sqrt{x^2} = |x|$”), and recurrent *routines* (e.g., square-rooting).

Communication is a patterned and rule-driven endeavor, which allows people to be efficient in situations that they consider as similar. Lavie et al. (2019) contextualize routines in *task situations* – settings where one considers themselves bound to do something. Commognition explains one’s capability to act in a new situation by harking back to *precedents* that this person views as sufficiently similar to the present one. This justifies recreating patterns that are familiar from the person’s own experience or from the experiences of others. The choice of specific precedents occurs with the help of *precedent identifiers* – features of a current task situation that a person considers sufficient to make the link to past events.

Lavie et al. (2019) suggest that identifying relevant precedents would be infeasible without a pre-selection, which gives rise to a *precedent-search-space*. The researchers explain that this space is often created before any specific task situation is set based “on the unarticulated assumption that precedents for whatever happens in this setting should come from the same discursive, material, institutional, and historical context” (ibid, p. 160). I suggest that such spaces can be further deconstructed into “smaller” *precedent pockets* – sets of precedent events on which a person draws when coping with a series of task situations assigned in the same setting. For instance, if asked whether a specific number is a perfect square, my actions would depend on the number: for some numbers, I can retrieve their square roots from memory; for others, I would first check whether they end with 0, 1, 4, 5, 6, or 9. In this sense, my replications would come from different pockets in my precedent-search-space.

Lavie et al. (2019) define a *task*, as it appears to a person in a given task situation, as “the set of all the characteristics of the precedent events that she considers as requiring replication” (p. 9). Due to differences between the precedents and the current task situation, replication preserves some past actions while changing others. A *procedure* is subsequently described as “the prescription for action that fits both the present performance and those on which it was modeled” (p. 9). Eventually, Lavie et al. suggest that a *routine* “performed in a given task situation by a given person is the task, as seen by the performer, together with the procedure she executed to perform the task” (p. 9).

Commognitive conflicts feature in an “encounter between interlocutors who use the same mathematical signifiers (words or written symbols) in different ways or perform the same mathematical tasks according to differing rules” (Sfard, 2008, p. 161). The divide described by Crisan (2014) epitomizes such a conflict since the teachers from different square-rooting “camps” applied different routines and arrived at contradictory outcomes. Such conflicts stem from mathematical discourses that do not share a well-defined set of rules that would allow their participants to resolve their disagreements deductively. Sfard (2008) terms such discourses as *incommensurable* and explains that “two narratives that originate in incommensurable discourses cannot automatically count as mutually exclusive even if they sound contradictory” (p. 258).

While commognitive conflicts are rooted in interpersonal communication, the primary unit of the commognitive interest is discourse. It seems more consistent, then, to position commognitive conflict as an inter-discursive phenomenon (Kontorovich, 2019). This opens the door to consider conflicts within one's mathematical discourse, i.e. between its sub-discourses (e.g., arithmetic and algebra) and even between their "smaller" components, such as narratives. Furthermore, a person does not necessarily need to experience a conflict for an external observer to argue for its presence. As such, I suggest that the notion of "same mathematical task" used by Sfard in her definition of a commognitive conflict is open for researchers' operationalizations and interpretations.

The Study

In the presented commognitive terms, the question underlying this study is "*what are the characteristics of task situations, in which students employ square-rooting routines that, on the face of it, entail conflicting narratives?*"

Participants and context

The data was collected from a class of 11 eighteen- and nineteen-year-olds enrolled in a so-called *bridging course* affiliated with a large technological university in Israel. These students had finished school with the minimal mathematical requirements of the national educational system. They enrolled in the course to improve their school qualifications and achievements and thus gain acceptance into the universities and faculties of their choice. The particular course re-engaged its students with high-school mathematics and additional topics useful for first-year tertiary courses. At the time of data collection, the participating students were at the end of an instruction sequence that covered topics in arithmetic and algebra where roots were discussed.

Research has noted that canonical discourses on roots do not always align with each other (e.g., Kontorovich, 2016, 2018b). Thus, let me delineate a discourse that may be considered as canonical in the context of this study. Israeli curriculum defines b as a square root of a if $b^2 = a$. The signifiers "the square root" (in Hebrew "השורש הריבועי") and '$\sqrt{\ }$'-symbol are used for non-negative roots only. This might smooth students' transitions between arithmetic, algebra, and calculus. For instance, the narrative "$\sqrt{9} = 3$", is correct through the lens of algebraic operations as well as when it is approached as a function $f(x) = \sqrt{x}$ at $x = 9$. In this way, while being capable to draw on several precedent pockets, as long as the radical symbol was involved, students could be expected to generate non-contradictory narratives. Nevertheless, 'root' is also used to refer to all the values that satisfy an equation. Then, 3 is the square root of 9, but ± 3 are roots of 9 as well as being roots of the equation $x^2 = 9$.

Data collection and the questionnaire

Sfard (2008) notes that each communicational medium – verbal, gestural, iconic, symbolic – "has its own discourse that supports its unique set of narratives" (p. 156). This makes some discursive steps "easier to perform than some others, depending on the type of 'materials' in which they are implemented" (ibid, p. 156). The written medium has been chosen for this study as it is a key communicational channel, through which mathematical discourses of mature students are captured, developed, and assessed.

The data was collected through a specially designed questionnaire. Two parts of it contained questions asking students to square-root from a variety of numbers and parametric expressions squared. Specifically, the questions involved square numbers, integers, monomials, binomials, and their combinations. The emphasis was placed on square-rooting from squared radicands so as to remove the calculational burden and to allow students to complete more questions. Each question was followed by a request to explain the generated response, which was expected to provide wider access to students' routines. The last part of the questionnaire was concerned with the definition of square root and the meanings of the radical symbol, and it is analyzed elsewhere. The students' course teacher confirmed that the questions in the questionnaire were not very different from the ones that were discussed in the classroom. The questionnaire was distributed in a regular lesson and the students worked on it individually without using calculators, as was usual for this course. Students' work was not time-limited, but all of them submitted their work in less than 25 minutes.

Analysis

The questionnaire of each student was analyzed in three stages. The first stage aimed at indicating responses where each student employed the same routines. For this aim, each response was construed as an implementation of some routine as a response to an assigned question, i.e. a task situation. Implementations were categorized as similar based on their words and symbols, number of steps involved, and resulting narratives. In the second stage, similar implementations were examined to delineate the procedure and the task of their underlying routine. Having no access to students' interpretations of the assigned task situations, the task was deduced based on the steps that a student took, while paying special attention to the end-result that they obtained. The procedure was constructed by generalizing the steps that each student captured in writing. Comparing the responses of different students highlighted nuances and changes in the emerging routines. In the last stage, task situations in which each student demonstrated the same routine were characterized, aiming to identify their common features, such as formulations and the mathematical objects involved.

Results and Commognitive Account

I start by taking a close look at Anna's responses to introduce the commognitive account for apparent conflicts in her square-rooting. This is followed by additional examples of students' routines.

When the radicands in the questionnaire were presented as perfect squares (e.g., 9, 16, 169), Anna copy-pasted the prompt preceding it with the '±'-symbol and responded with two opposite roots encapsulated under the '±'. For instance, as a response to the request to simplify $\sqrt{169}$, she wrote "$\pm\sqrt{169} = \pm 13$". In the questions where the radicands were presented in a squared form, she started with converting the radical to the power of half, followed by reducing the powers to 1, and concluded with the initially squared input (e.g., "$\sqrt{11^2} = (11^2)^{\frac{1}{2}} = 11^1 = 11$"). More or less the same procedure was observed when Anna square-rooted from squared negative numbers and parametric expressions (e.g., "$\sqrt{(-6)^2}$" and "$\sqrt{x^2}$"). In some of the questions, she explained that "square root of x^2 is x itself". This description attests to the difference of routines that Anna employed when square-rooting from perfect squares and squared radicands.

From the perspective of canonical mathematics, the described patterns of Anna's square-rooting appear as a source of potentially conflicting narratives. Indeed, if "square root of x^2 is x itself", then what is the square root of 169? Anna maintained that it is "± 13", but if acted on in accordance with the procedure associated with squared radicands, she is likely to end up with 13 as the only root. Before labelling these two narratives as mutually exclusive, let me attend to assumptions that justify considering the narratives together in the first place. For an algebraically versed person, it may seem self-evident that square numbers, squared numbers, and squared parametric expressions are instances of the same object. Then, when asked to square root from it, this person could turn to the *same* precedent pocket and replicate the *same* actions. Alternatively, by drawing on different precedent identifiers, this person might resort to different pockets; for instance, one involving square numbers and one with squared radicands. After all, it is not rare for people to recall square roots of some numbers quickly but stumble over the power notation. The point, in this case, is that the actions associated with the pocket of squared radicands are still applicable to the task situation with square numbers, and these actions lead to the same outcome. The coherence of mathematics ensures that manipulations with powers, such as 13^2, will yield the same outcome as recalling the root of a square number 169 right away. In this sense, I suggest that the pocket of squared radicands *subsumes* the pocket of precedents with square numbers (see Sfard, 2008 for a similar relation between subsumed and subsuming discourses).

There seem to be no grounds to assume that in her square-rooting Anna resorted to the same or subsumed precedent pockets. To the contrary, the consistence of her routines suggests that she turned to (at least) two *incommensurable* pockets of precedents; for example, one involving perfect squares and one with squared radicands. It is possible, then, that each task situation was compared to the relevant precedent pocket, which entailed different actions. I use "incommensurable" to stress that the precedent pockets can be construed as different to the extent that Anna would not see a reason to think about them together and would not consider looking for connections between them. Within this account, claiming the existence of conflicts between actions that Anna employed in task situations that she saw as different, would be like arguing that having different breakfasts on the weekdays and the weekends creates a contradiction.

Anna was not the only student to consistently employ different routines in different task situations. For instance, Betty's acted differently when the assigned roots "stood alone" and when they were part of more compound calculations. In the former cases, she generated two opposite results, while in the latter instances, only the positive roots were considered (e.g., "$\sqrt{169} = \pm 13$" vs ""$\sqrt{81} - 8 = 9 - 8 = 1$"). Charlie showed similar patterns when parametric expressions were involved.

The next two examples showcase that some routines differed not only in their procedures and outcomes, but also in their tasks. The analysis of Daisy's questionnaire showed that her square-rooting changed for monomial and binomial expressions. In the former task situations, she realized the radicands in a single step producing simplified expressions that were free of the radical sign (e.g., "$\sqrt{a^2} = a$"). In the latter pocket, she expanded the brackets by squaring the binomials and keeping the result under the root symbol (e.g., "$\sqrt{(-a+1)^2} = \sqrt{(1-a)^2} = \sqrt{1^2 - 2a + a^2}$"). And while some may argue that no roots were extracted in the latter cases, I refer to them as square-rooting procedures as they consistently featured in task situations asking Daisy to simply expressions with roots.

The second example (shown in Figure 1) comes from Ella. The left part of Figure 1 illustrates that her procedure was consistent with the task of simplifying the assigned roots when the parameters were named with letters from the beginning of the alphabet (e.g., "a", "b"); however when faced with "x", "y", and "t", she equated the assigned expressions to zero and attempted to solve the emerging equation.

To explicate, by referring to a student's actions in sets of task situations with shared features, I do not imply that these features were *the* precedent identifiers on which each student drew when employing their routines. For instance, it is possible that Ella's square-rooting depended not on the name of the parameter but on the structure of the assigned expression. Indeed, she may have been sensitive to situations where the roots were extracted from monomials and where they were incorporated with addends that did not require simplification (like $2x$ in the right part of Figure 1).

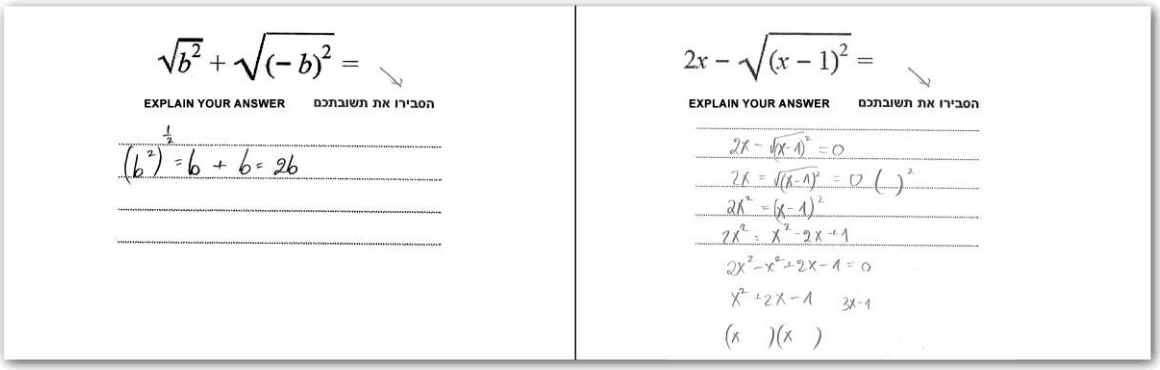

Figure 1. Illustrations of Ella's square-rooting

Summary and Discussion

This study enriches the body of literature on learning and teaching of roots (e.g., Crisan, 2014; Kontorovich, 2016, 2018a, b). While some previous studies focused on the correctness of students' mathematizing (e.g., Cangelosi et al., 2013), this investigation was concerned with seeming conflicts within students' square-rooting. The key contribution of this study to this body of knowledge is in showing that at the end of an instructional sequence, students can operate with roots differently across, but consistently within, collections of task situations with common features. Specifically, the analysis showed that students' square-rooting can change between task situations involving perfect squares and squared radicands, roots that stand alone and those featuring in more compound exercises, monomials and binomials, and when parameters are named with different letters. Some of the identified routines differed not only in their procedures, but also in their tasks. Notably, none of the students square-rooted numbers differently from parametric expressions – something that might have been expected given the extensive literature on the transition from arithmetic to algebra (e.g., Carr, 2012). The fact that these differences did not emerge may be related to the fact that, as part of the instructional sequence, arithmetic and algebra were taught in close proximity one to another and the connections between the two were stressed.

This study offers some insights to the commognitive strand of research. Currently, this strand has been mostly attentive to interpersonal conflicts (e.g., Lavie et al., 2019), when the study at hand has concerned apparent conflicts within one's mathematical discourse. Following in the footsteps of Lavie et al. (2019) and others, the construct of conflict has been dis-objectified with the aim of considering it from the perspective of mathematics learners, i.e. those who are claimed to demonstrate it. Specifically, I suggested that in different task situations a student can draw on different precedent identifiers and hark back to incommensurable pockets of precedents. The narratives emerging from replicating different actions may seem conflicting to those who juxtapose them. However, the juxtaposition is viable as long as the assigned task situations are construed as elements from the same collection; a collection that invites resorting either to the same precedents or to precedent pockets that subsume one another. However, if a person associates different task situations with incommensurable precedent pockets, there seems to be no reason for this person to consider either the replicated actions or their outcomes together in the first place. Within this account, acting differently under circumstances that are deemed as different is the sensible thing to do.

So how can students like those who participated in this study be supported to ensure that their mathematizing to become canonical? This question invites further research into teaching and learning aimed at students revising their already developed discourses. Practice-wise, this endeavor requires teachers to become familiar with routines that their students develop. This draws attention to the diagnostical potential of engaging students with carefully-varied task situations. That said, the sensitivity to task situations that the students exhibited in this study suggests that it may be not easy to come up with diagnostically informative variations. This is because newcomers to mathematical discourse – like students – and discourse oldtimers – like teachers – often have conflicting ideas regarding what is similar and what is different in mathematics.

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The Development of Critical Teaching Skills for Preservice Secondary Mathematics Teachers Through Video Case Study Analysis

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Using social learning theory with the central concept of a community of practice, we situate this work within a secondary mathematics methods course to unpack preservice secondary mathematics teachers (PSMTs) development through the use of video case studies. We analyzed six sessions of the course in which PSMTs engaged in discussions about video segments of mathematics teaching rooted in the Teaching for Robust Understanding (TRU) framework for high-quality instruction. Analysis of this data showed opportunities for PSMTs to develop critical skills for teaching (Hiebert et al., 2007). Our contribution includes the addition of a new skill to Hiebert's framework, Understanding the Mathematics, as an important component of PSMT learning that may precede the original four skills. Future research should focus on connections between PSMTs' evolving mathematical understandings and analysis of video to better understand the impact of their content knowledge on developing the critical skills for teaching.

Keywords: Preservice Secondary Mathematics Teachers, Video Case Studies, Mathematics Methods Course

Introduction

Hiebert and colleagues (2007) suggested four teaching skills that preservice teachers should develop within a teacher education program: (1) specifying the learning goals for the instructional episode; (2) conducting empirical observations of teaching and learning; (3) constructing hypotheses about the effects of teaching on students' learning; and (4) using analysis to propose improvements in teaching. These skills were centered on the notion that the goal of teaching is to support student learning, and therefore, to acquire these skills, preservice teachers needed learning experiences embedded in actual classroom settings (Hiebert et al., 2007). Teacher education programs have provided preservice teachers with experiences in the form of supervised clinical teaching experiences, or practical teaching courses (practicums). However, these experiences are varied depending on individual supervising teachers whose practices might not be aligned with best practices and theories discussed in university methods courses (Cavanagh & Prescott, 2007). Based on these inconsistencies, preservice teachers might rely on their own experiences as students (Feiman-Nemser & Buchmann, 1985). Thus, there is concern that preservice teachers might not have the opportunity to build the critical skills needed to become effective teachers (U.S. Department of Education & Office of Postsecondary Education, 2013).

To build the critical skills for teaching, Hiebert and colleagues (2007) advocated for providing opportunities for preservice teachers to learn to teach from teaching not only within classrooms, but before entering a classroom. Methods courses can provide preservice teachers with such learning experiences where they develop the skills needed for field-based teaching through analyzing teaching in terms of student learning. One powerful type of learning experience that has been shown to provide preservice teachers a way to view teaching and learning in a classroom setting is the use of video case study (Star & Strickland, 2008). In a methods course, guided discussions and analyses of video-recorded classroom teaching can help preservice teachers focus on important aspects of a lesson, which can provide opportunities for preservice teachers to develop and understand their teaching skills in real contexts (Star et al., 2011). Moreover, new research calls for video case study analysis need to be rooted in a framework for teaching (e.g., Bonaccorso, 2020). In our work, we partnered video case study analysis with the Teaching for Robust Understanding (TRU) framework, which has allowed us to investigate how preservice secondary mathematics teachers (PSMTs) attend to and develop teaching skills necessary to learn how to teach from studying teaching (Hiebert et al., 2007). In particular, this study seeks to answer the following research question: *In what ways does video case study analysis rooted in a teaching framework provide PSMT opportunities to develop the critical skills of teaching?*

Conceptual Framework

Communities of Practice

To understand PSMTs' learning, we adopted a community of practice (CoP) model for our conceptual framework. A CoP is defined as a group of people "who share a concern, a set of problems, or a passion about a topic, and who deepen their knowledge and expertise in this area by interacting on an ongoing basis" (Wenger et al., 2002, p. 4). A methods class consists of preservice teachers and the instructor(s) working together to deepen knowledge of teaching and learning during regular course meeting times across a semester. In our study, PSMTs and the methods instructors analyzed video case studies of mathematics teaching in order to foster sustained teacher learning about high-quality instruction using the TRU framework. Therefore, in the context of this study, the methods course acted as a CoP because the group tackled a set of problems around the nature of high-quality instruction and deepened their knowledge regarding the skills needed for successful implementation.

Video Case Studies in Preservice Teacher Education

Researchers have found that video analysis has the potential to focus preservice teachers' attention on particular aspects of teaching and learning. PSMTs typically have difficulties in attending to issues of mathematics teaching and learning when they begin teacher training (e.g., Star & Strickland, 2008). Research has shown that analyzing classroom videos provides preservice teachers a chance to broaden their perspectives regarding what to observe, and direct preservice teachers' attention to critical aspects of mathematics teaching and learning. In particular, Martinez et al. (2015) found preservice teachers' ability to anticipate student mathematical thinking could be enhanced by partnering video watching with a research-based framework. Other researchers (Santagata & Angelici, 2010; Santagata et al., 2007) placed particular emphasis on the use of an observation framework as a lens for analyzing a videotaped lesson. Findings from this research suggested that an observation framework designed to guide teachers' analyses of classroom videos has a positive impact on improving preservice teachers' ability to attend to important elements of the teaching and learning of mathematics. Therefore, PSMTs in our study analyzed video case studies with a specific lens to better understand the teaching of mathematics.

Teaching for robust understanding (TRU). We used video case studies rooted in the TRU framework to help PSMTs align their understanding of mathematics teaching and learning to dimensions of high-quality instruction that support deep mathematical learning opportunities for students. The TRU framework is categorized by five interrelated dimensions of high-quality mathematics instruction: (1) the mathematical content; (2) cognitive demand; (3) equitable access to content; (4) agency, ownership, and identity; and (5) formative assessment (Schoenfeld, 2017). In this study, the purpose of using video case studies rooted in the TRU framework was to focus PSMTs' discussions on particular aspects of powerful mathematics classrooms and provide them with a common language to examine the teaching and learning of mathematics.

Methods

Context

The participants of the study include 12 PSMTs enrolled in two separate sections of an undergraduate mathematics methods course that met on campus. This course is the second of two required methods courses that PSMTs took prior to their practicum experience. Each of the six 75-minute sessions studied was facilitated by the course instructor. On two occasions, an additional instructor familiar with the teaching materials and framework provided support.

Over the course of each semester, the instructor scheduled three sessions for video case study analysis. For these course meetings, PSMTs engaged with the mathematics in the lesson, viewed a video clip from a real classroom setting, and collectively analyzed the video through whole class discussions. When engaging with the mathematics PSMTs completed tasks from the student perspective and discussed the core mathematical ideas embedded in the lesson. After watching the video of the enacted lesson PSMTs responded to reflective questions and utilized tools aligned to the teaching framework to analyze the mathematics teaching and learning. One of the tools, the *On Target*, is shaped like a bullseye and helps PSMTs situate teaching moves within different TRU dimensions. Each *On Target* provides a reflection question and lists teaching moves associated with one TRU Dimension. The outer rings provide examples of possible teaching moves less aligned, and the inner rings showcase teaching moves more aligned with the high-quality instruction defined by the TRU dimension. By using this tool, the PSMTs could focus their observations and use common language to reflect on what they saw in the video. All materials for the video case study analysis were grounded in a particular dimension of the TRU framework (Schoenfeld, 2017) to further assist PSMTs in developing teaching hypotheses aligned with powerful mathematics classrooms. This *On Target* tool was modified with permission from materials in development by the TRU group at the University of California, Berkeley (Schoenfeld & the TRU Project, 2018).

Data Collection and Analysis

Each of the class meetings in which PSMTs engaged with video case study analysis was video and audio recorded. After transcribing each meeting, we analyzed the data using thematic analysis (see Herbel-Eisenmann & Otten, 2011; Lemke, 1990). The course instructor, a graduate research assistant, and a second graduate student with expertise in the materials, began by open coding each session independently and then met to discuss and define emerging themes. From our thematic analysis six themes emerged: Understanding the Mathematics, Understanding TRU Tools/Dimensions, Understanding the Lesson, Recognizing TRU Dimension in Video, Understanding Teacher Moves, and Connecting Teacher Moves a TRU Tool. The three researchers then coded the transcripts by identifying episodes related to the six themes. An episode is a portion of the transcript that focused on one particular topic. An episode ended when the focus of the talk changed. We then noted the number of episodes of each theme occurring over the course of the six sessions. Finally, the three researchers collaboratively engaged in a secondary analysis connecting the themes to Hiebert et al.'s (2007) skills to investigate the opportunities the PSMTs had to develop these critical skills of teaching.

Results

Through our secondary analysis, we identified relationships between our themes and the four critical skills for teaching. Table 1 shows this alignment and the frequency of each theme. The frequency of episodes suggested opportunities for the PSMTs to develop the critical skills for teaching while analyzing video case studies in their methods class. However, our most frequently observed theme did not align with the previously defined skills (Hiebert et al., 2007).

Table 1: Theme alignment to Hiebert et al. (2007) and number of episodes

Theme	Alignment to Hiebert et al. (2007)	Number of episodes
Understanding the Mathematics	No defined skill	33
Understanding TRU Tools/Dimensions	Skill 1 - Specifying the learning goals for the instructional episode	9
Understanding the Lesson	Skill 1 - Specifying the learning goals for the instructional episode	7
Recognizing TRU Dimension in Video	Skill 2 - Conducting empirical observations of teaching and learning	2
Understanding Teacher Moves	Skills 3 & 4 - Constructing hypotheses about the effects of teaching on students' learning; and (4) Using analysis to propose improvements in teaching	16
Connecting Teacher Moves a TRU Tool	Skills 3 & 4 - Constructing hypotheses about the effects of teaching on students' learning; and (4) Using analysis to propose improvements in teaching	19

Connecting Teacher Moves to a TRU Tool and Developing Skill 3

Our first example comes from PSMTs' discussions around a video case titled *Defining and Describing Quadrilaterals* that highlighted the TRU dimension of cognitive demand. After watching the video, the PSMTs were prompted to discuss their observations with the support of the related *On Target* tool (Figure 1) with the following reflection question: *To what extent are students supported in grappling with and making sense of the mathematical concepts*?

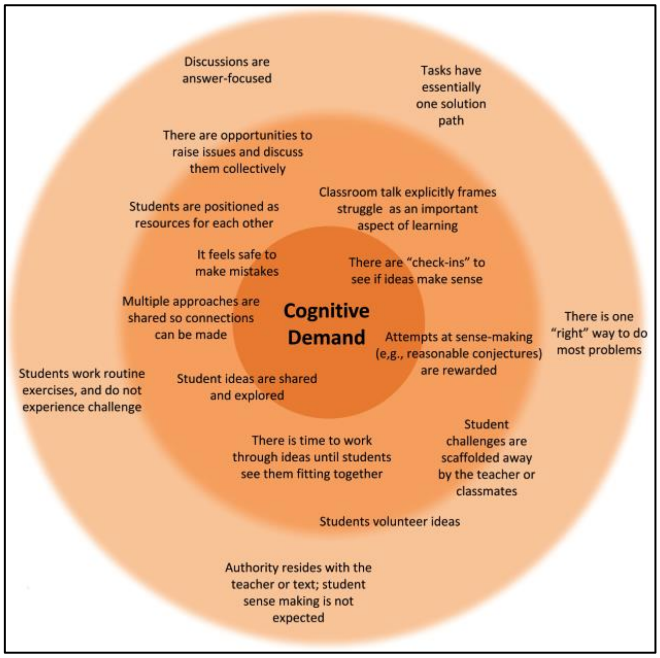

Figure 1. A version of the On Target tool discussed by PSMT-1 and PSMT-2 (recreated by authors)

PSMT-1: So we thought he was good because he stayed away from that outside things. Uh, one of the things we mentioned was how, um, for scaffolding moves students in productive directions without giving answers away, was like when the student said par…, like gave the definition for parallel, and instead of just like shooting him down, and being like, no that's wrong, he like, kind of like opened it up for everyone, you know, and like what does that really mean? And he kept moving forward that way, so that was good.

PSMT-2: And he also, he didn't scaffold them so much that it just took all the demand away. You know, it was, like he didn't just, you know, like PSMT-1 said, go off that one wrong answer and just tell everyone what it was.

This example shows two PSMTs unpacking Skill 3: constructing hypotheses about the effects of teaching on students' learning (Hiebert et al., 2007). By stating "stayed away from the outside things," PSMT-1 referred to teaching moves from the outer rings of the *On Target* that lowered the cognitive demand of a lesson. PSMT-1 stated that the teacher maintained cognitive demand through questioning moves that privileged and responded to students' mathematical thinking and did not funnel students toward one particular way of thinking. PSMT-2 built upon PSMT-1's observation and concluded that students are being provided the opportunity to grapple with the mathematical ideas in the lesson. This suggests that both PSMT-1 and PSMT-2 connected the teaching moves of maintaining cognitive demand to students' learning, the main focus of Skill 3.

Understanding the Mathematics and Defining A New Skill

During another session of the methods course, the PSMTs analyzed a video case titled *Representing Quadratic Functions Graphically* focused on the TRU dimension of formative assessment. PSMTs were asked to sketch two quadratic curves that "look quite different from each other." PSMT-3 drew a standard quadratic curve with its vertex at the origin, and the same quadratic curve rotated 90 degrees (see Figure 2), which prompted the following discussion:

Figure 2. PSMT-3's two different quadratic curves

I-2: The other one's not a function, right? That's a difference. Quadratic relation. Not a function.

I-1: [PSMT-3], do you understand what [I-2] just said there? About it not being a function?

PSMT-3: Yeah.

I-1: So what would be the difference between the equation you would have for those two then? Like you said the first one was $y = x^2$, what would be an equation for the other one?

PSMT -3: I don't know from the top of my head, to be honest.

PSMT-1: It would be a root of x, right? A square root? But that would only be positive. Half a circle.

I-2: Did you say half a circle? Can you say, what do you mean by half a circle?

PSMT-1: I have no idea how you would represent that, but it looks like half a circle. So it would make me think the equation of a circle, centered at whatever the center of that would be.

After viewing the video clip, the conversation returns to discussing the prior graphs. In the final minutes of the class, the instructor brings the students back to two graphs created by PSMT-3. The instructor challenges the class to come up with a quadratic equation that could represent the graphs. After a bit of discussion, the PSMTs continue to grapple with the content.

I-2: Can you think of an equation that defines this relationship? If this is y equals x squared, then can you think of an equation for this [referring to rotated graph].

PSMT-2: [To PSMT-1] What is that? $y^2=x$? Is that what you just wrote there? $y^2=x$?

I-2: Is this? Does that work?

PSMT-2: Yes. I was on Desmos trying to graph what I was doing but I made something really ugly. So no I mean what I was doing before, it was like just a whole bunch of lines.

I-2: Is that, is $y^2=x$ a quadratic relation?

PSMT-1: Yes.

I-2: Is there anybody who's not convinced that that curve is defined by y squared equals x?

PSMT-2: I'm convinced.

PSMT-1: Why are you convinced PSMT-2?

PSMT-2: Because I graphed it and it makes sense.

I-1: It looks like it.

PSMT-2: It makes complete sense.

These mathematical discussions suggest that the PSMTs were developing their own understanding of mathematics. Before watching the video, we see the PSMTs struggling with the task itself. They talk about the square root and a half circle, thus illustrating uncertainty with what would constitute a quadratic relationship. After watching the video lesson, the PSMTs continue to unpack the meaning of a quadratic relationship and again return to the graphs created earlier. In this discussion, we continued to observe uncertainty around some of the mathematical concepts (e.g., inverse, function, relation).

As we analyzed this discussion, we recognized that our interpretation did not directly align with any of the four critical skills for teaching (Hiebert et al., 2007). This led to our defining a Skill 0, Understanding the Mathematics, defined as making sense of the mathematics for yourself, unpacking different ways others might solve a problem, and/or unpacking students' understanding of the mathematics. We contend Skill 0 to be an important component of PSMT learning that may precede the original four skills.

Discussion and Conclusion

The aim of this study was to explore opportunities for PSMTs to develop the critical skills of teaching in a methods class using video case studies rooted in a framework for high-quality instruction. Through our thematic analysis and subsequent alignment to the skills, we found several episodes of PSMTs' discussion that aligned to the four critical skills for teaching (Hiebert et al., 2007), suggesting an opportunity for PSMT development. Our most frequently occurring theme, Understanding the Mathematics, did not align with any of the previously defined skills as determined by Hiebert and colleagues. This finding suggests that PSMTs may require opportunities to think and interact with their own mathematical understandings since they must first grapple with these before understanding and hypothesizing about teaching moves and their impact on students' mathematical thinking.

Other researchers have also found that interrogating personal mathematical thinking is an important precursor for PSMTs developing skills for teaching. For instance, Graeber (1999) suggests that PSMTs must be able to provide alternative representations of mathematics for students as well as have the ability to recognize and analyze students' alternative solutions and approaches to solving mathematical tasks. She suggests that methods classes focus on providing opportunities for PSMT to explore various models and approaches to mathematical concepts in order to achieve this. Hine (2015) and Şahin et al. (2016) also note that PSMTs' mathematical understanding influences their ability to notice student misconceptions, offer alternate approaches, and assist students with deepening their understanding of mathematics. Therefore, we propose the addition of Skill 0 to the critical skills for teaching framework (Hiebert et al., 2007). Adding this skill to the existing framework has the potential to support PSMTs in developing the other four critical skills for teaching, such as constructing hypotheses about the effects of teaching on students' learning and proposing improvements in teaching.

One limitation of this study is that it existed in only two sections of a methods course taught in the same year with a small number of participants. This context does not allow us to make general claims about PSMTs' skill development. We therefore suggest future research to examine PSMTs use of video case study on a larger scale. Such research should focus on studying the connections between PSMTs' evolving mathematical understandings and their analysis of video cases to suggest how PSMTs are learning about the critical skills of teaching.

Even so, our work indicates that when designing a mathematics methods course, instructors should incorporate video cases with discussions rooted in a framework for high-quality instruction. These additional activities should also include time to focus on PSMTs understanding of their own mathematics. This focus is the avenue through which instructors can open up opportunities for PSMTs to develop the critical skills for teaching (Hiebert et al., 2007).

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A Student Perspective on Excellence in Undergraduate-Mathematics Instruction

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The question of what high-quality undergraduate-mathematics instruction entails has been examined in prior studies mainly by directly analyzing lecturers' teaching practices. The current exploratory study aims to provide a complementary angle by inquiring into a student perspective on this topic. Based on student-centered data consisting of interviews and questionnaires, the findings point towards 8 characteristics of "excellent lecturers" as perceived by undergraduate students. These provide student-driven support for some of the prior lecture-centered research claims, though also highlight the importance students attribute to the affective dimension in teaching, which has not yet received sufficient research attention.

Keywords: Teaching practices, High-quality instruction, Students' perceptions, Affect

> *"During the studies of an average student, he or she will encounter different types of lecturers [...], yet there are only a few lecturers that the mere mentioning of their name puts them at a different league above the rest. They are considered mythological warriors who stand side by side with students in their struggle to succeed at the [university] despite all difficulties." (a student quote from an article on an excellent undergraduate mathematics lecturer, published in a student-run university magazine; Reuven, 2014)*

Recent years have shown a growth of interest in undergraduate mathematics education within the research community (Biza, Giraldo, Hochmuth, Khakbaz, & Rasmussen, 2016; Nardi & Winsløw, 2018). In particular, one topic that has received research attention regards the characterization of quality undergraduate-mathematics instruction, aiming to provide a practical contribution to the improvement of undergraduate teaching (e.g., Biza et al., 2016; Speer, Smith, & Horvath, 2010). Studies addressing what it means to be a good undergraduate lecturer have mainly done so through empirical data stemming from lecture observations and interviews with lecturers, as well as by theorizing what constitutes high-quality teaching (e.g., Bergsten, 2007; Schoenfeld, Thomas, & Barton, 2016; Wood & Su, 2017). However, there seem to be relatively few studies that inquired into a *student perspective* when attempting to characterize the teaching practices of good lecturers. As illustrated by the quote above, students may share strong opinions when asked to evaluate their lecturers, where "only a few lecturers" would be considered by students as "a different league above the rest". Accordingly, the current study aims to provide another "brick" in the growing corpus of studies on undergraduate teaching by focusing on the characteristics and traits students attribute to those lecturers they regard as "excellent teachers".

Characterizing High-Quality Undergraduate-Mathematics Instruction

Previous studies have offered various (yet often overlapping) frameworks for characterizing what it means to be an excellent lecturer. In one line of studies addressing this topic, lecture observations served as the main source for data analysis. For example, Bergsten's (2007) study, examining a Calculus lecture, provided a list of criteria accounting for high-quality instruction: *information delivery*, *connections*, *rigor-intuition*, *algebraic-imagistic modes*, *gestures*, *socio-mathematical norms*, *mathematical mind*, *inspiration*, *personalization*, and *general criteria*. Most of these criteria focus on the mathematics-related dimensions of teaching, such as making connections across different topics (the criterion “connections”) or balancing between formal and informal mathematical registers (e.g., “rigor-intuition”). However, several of the criteria also emphasize the importance of lecturers serving as role models for students, providing motivation and support (e.g., “inspiration” and “personalization”). In another study, Movshovitz-Hadar and Hazzan (2004) analyzed an Algebra lecture and offered six principles characterizing an excellent lecturer’s rhetoric. For example, Principle 1 suggests first providing a “top-down” description of the main ideas and only then a “bottom-up” construction of the details; and Principle 6 suggests explicitly distinguishing between important and less important ideas of the lecture.

A more theoretical approach was taken in another line of studies by conceptualizing which dimensions could be used to describe, analyze, and improve undergraduate-mathematics instruction. For example, through a review of prior research, Speer et al. (2010) suggested different cognitively-oriented dimensions for analyzing undergraduate teaching practices, among them: *selecting and sequencing content*; *motivating specific content*; *allocating time within lessons*; and *asking questions, using wait time, and reacting to student responses*. Also Pritchard (2010) utilized a literature review to offer a framework that accounts for effective lectures based on three interrelated dimensions: *communicating* mathematical ideas, which requires clarity and flexibility in the delivery while responding to instantaneous student feedback; *modeling* mathematical reasoning while highlighting different heuristic approaches to problem solving; and *motivating* students by expressing enthusiasm and cultivating a “learning community”.

However, as noted earlier, relatively few studies have explicitly included a student perspective when examining and analyzing undergraduate-mathematics instruction. Within these, a particularly interesting case is presented in Lew, Fukawa-Connelly, Mejia-Ramos, and Weber’s (2016) study, which involved not only an observation of a lesson by a reputable lecturer, but also subsequent interviews with students in order to examine their understanding of the lesson. Though the direct analysis of the lecture identified characteristics of quality instruction, the interviews surprisingly revealed that the students did not comprehend the main ideas of the lecture. One of the explanations provided by Lew et al. regarded the students’ prioritization of the written formal content on the board, whereas the main ideas were expressed by the lecturer only orally. Another study incorporating a student perspective on undergraduate teaching is offered by Marmur and Koichu (2018), who identified 8 types of lesson-events the students found conducive to their learning, characterized in relation to their lecturers’ instructional acts (see also the notion of *key memorable events*; Marmur, 2017, 2019).

The importance of incorporating a student perspective in research on effective teaching practices has been repeatedly acknowledged in studies situated at the school level, recognizing the mathematics classroom as a place of meaning-negotiation between teachers *and* students (e.g., Kane & Maw, 2005; Kaur, 2008). Taking these ideas to research at the tertiary level, I suggest that an inquiry into a student perspective may offer lecturers an opportunity to reflect on, and possibly re-evaluate, their teaching practices according to a hierarchy of which of these teaching practices students value most. Accordingly, the current study focuses on students’ perceptions of undergraduate teaching by addressing the following research question: *How do university students characterize high-quality undergraduate-mathematics instruction?*

Method

The data for this exploratory study are comprised of semi-structured interviews with 5 students and questionnaires filled by another 18 students at a highly-ranked research university in Israel. Due to this paper’s length constraints, while nonetheless wishing to discuss the findings in detail, in what follows, I focus only on the data from the interviews. The interviewed students were in their second (n=1) and third (n=4) year towards a BSc in electrical engineering. The choice of interviewing students in a later stage of their degree was based on the assumption that students who had already experienced different mathematical courses with various lecturers would be able to provide a more informed opinion on what high-quality instruction entails.

The participants were asked to prepare for the interviews by contemplating how they would characterize an excellent lecturer while thinking about specific examples in advance. The notes they made in preparation for the interview were collected as part of the data. During the interviews, the students were first asked to describe their general experience of mathematics lectures thus far and were subsequently guided in sharing detailed accounts and characteristics of good (and for contrasting purposes, also "bad") lecturers they had studied with. The duration of the interviews ranged from 30 to 50 minutes, and they were audio-recorded and transcribed.

For the data analysis, I employed an inductive approach, where categories emerged from dominant themes through a process of iterative engagement with the data (Yin, 2011). The themes were based on the students' perceptions of what it means to be an "excellent lecturer", a "bad lecturer", and perhaps most interestingly, nuanced distinctions the students made between lecturers who were "good" and those who were "excellent". While acknowledging the study's limitation in making inferences from a small number of students, I suggest that the following analysis considerations offer support to the validity of the findings: (a) each of the themes characterizing excellent lecturers was identified based on statements by all interviewed students, with the exception of one at most; (b) statements in support of the identified themes were expressed *repeatedly* by the students throughout their interviews; (c) the significance of the themes to the students was further supported by a strong affective dimension in the students' statements, inferred either directly through expressions of emotion (e.g., "something I really *loved* about these lecturers") or implicitly through the use of intensifying adverbs (e.g., "Dr. B is *very very very very* didactic"); (d) additional support in the other collected data (i.e., questionnaires and students' notes); and (e) examining and contextualizing the themes in relation to the research literature, thus aiming to offer *analytic generalizability* rather than *sample-to-population generalizability* (Firestone, 1993), as further addressed in the Discussion section.

Findings

The analysis stage revealed 8 themes characterizing high-quality undergraduate-mathematics instruction according to a student perspective. These are described and exemplified below.

Adopting a Student Perspective

A repetitive theme characterizing excellent lecturers was that they were able to see the mathematical material and its possible difficulties from a student perspective. The ability to "get into the student's head" was described by the participants as a feature that distinguishes between lecturers who are good and those who are excellent. For example, Student D said:

> "I had lecturers that I can say were good but not excellent. I think the difference maybe goes beyond the teaching of the material. It's that if a student asks a question, then they know how to answer it in a satisfactory way, as if they entered the student's head."

According to the interviews, a lecturer that adopts a student perspective can support students' learning in both cognitive and affective terms. From a cognitive perspective, the teaching should address and incorporate the ways in which students construct mathematical meaning and develop their understanding. From an affective perspective, the acknowledgment of the student perspective also provides emotional support by creating an environment in which the learning process is a journey the students do not have to walk alone; as for example shared by Student A:

> "You feel that he [Dr. B] is walking with you hand in hand […]. Like he's coming from inside you, from within the audience [of students]. It's like he's learning the material with you all over again, while also showing it to you along the way."

Performativity

"A good lecture acts a site for 'awe and wonder'." (Rodd, 2003, p. 16). The data here support this claim and suggest that an excellent lecturer demonstrates performative abilities, including attributes such as charisma, sense of humor, and leadership qualities. It is a person who has a passion for the subject, and moreover, is able to connect students to this passion as well. For example, Student E described the difference between an excellent and a good lecturer based on their performative abilities in keeping students involved and interested:

> "Part of the reason there's such a consensus about Dr. B [being an excellent lecturer] is because of his theatrical abilities and his charisma. There's no doubt this is part of it. […] There are lecturers whose complete lack of charisma is the reason why students will fall asleep. Last semester I studied Probability with Dr. T, who, as I said before, is a good lecturer. But people would simply lose him on a regular basis because he's so monotonous. In the end, there's a tidy notebook […] because that's how he teaches the course. But on the other hand, if you were sitting like a mannequin all semester long, it's probably going to be very hard for you to prepare for the exam."

Furthermore, the interviews suggest that this teaching characteristic also supports students' emotional well-being and raises their motivation to understand the taught mathematics. For example, Student E stated that Dr. O's "humor helped lift people's spirits and pick them back up. […] When you take something heavy and difficult and deflate it with a stab of humor, then suddenly things are not scary anymore." And Student A shared the following:

> "I think that a trait that characterizes a good lecturer is leadership, […] it's a matter of who excites students, who succeeds in making them feel connected. […] And then, even if students don't understand something, then they will ask. […] It means that I'll care enough to come and ask the lecturer about those things I didn't understand."

Making Connections and Providing Directionality

This characteristic of excellent lecturers regards making mathematical connections in the following ways: (a) connections between past, present, and future ideas of the course; and (b) connections between concepts learned in the course and their utility both within and outside the mathematical domain. For example, student C discussed the importance of connections both at the local level of the material and at the global level of how ideas combine to a "bigger picture":

> "There are lecturers that at the beginning of each lecture say a sentence or two about what we did last time and then make the connection to what we'll do with this now – which is something I really loved about these lecturers. […] I really felt that lecturers who truly taught well showed how to look at everything in the context of the big picture. This was really helpful. And it was also very interesting seeing how different things connect."

Moreover, the participants reported that when such connections are made, this gives them a sense of directionality in their studies, which addresses both their cognitive and affective needs:

> "It's about clearing the fog for students, so they would know a bit more where this [material] is heading towards, as well as why. […] We are curious people, so the 'why' is

important to us. When we don't know, then there's this fog along the path […]. So this freaks us out. You don't know what's going to happen. It drives you crazy." [Student A]

Positive Attitude Towards Students

"A good lecturer is a lecturer who cares" [Student D]. The importance of lecturers caring about their students' understanding was a repeating theme: "Caring. This is a trait that really influences the atmosphere in class, in the sense of how much you'll feel comfortable asking questions. […] And also caring in the sense that they [lecturers] will really speak with you until they see you understand" [Student C]. According to the interviews, this could be achieved if the lecturer were to use inviting language that encourages students to ask questions:

> "I understand that the bigger the group is, the harder it is to give personal attention. […] But at least the lecturer should give the students the feeling and confidence of 'you can come to me, you can approach me, we'll help you'." [Student A]

Furthermore, the data suggest that lecturers' attitude may have a substantial impact on students – "The attitude of the lecturer can really make you love or hate the course" [Student A]. Whereas lecturers who demonstrate a positive attitude towards students may raise students' involvement, lecturers who behave dismissively towards students may cause students to dislike the course – even when taught by knowledgeable, organized lecturers; as shared by Student B:

> "Dr. E is really organized and knows the material excellently. But if you ask a question in her lesson, then you'll receive a very cold shower, you'll be humiliated. […] Like, serious humiliation. If she had done it to me, I would have been really insulted. […] Like, yelling at you to go over the material again in front of a class of several-hundred students […]. On the other hand, Dr. M, her attitude is amazing. She answers questions calmly."

Keeping It Simple

The mathematics that undergraduate students need to learn and deal with is often complex and challenging. However, based on the interviews, an excellent lecturer is able to thoroughly *simplify* the material. One of the methods mentioned by the students was using "extremely simple examples" to demonstrate definitions and proofs. The students emphasized that it is the deliberate *choice* of which examples are presented that can make even complicated material become simple, understandable, and clear. For example, Student A shared the following:

> "Dr. Z, what Dr. Z does, she brings the material to amazing simplicity. […] She explains it by bringing it to such a simple level. She simplifies everything and gives the most basic examples in the world, that not understanding them […] is just impossible."

Other ways of simplifying the material included the use of illustrations, gestures, simple language, and daily situations to explain the material. For example, Student B stated:

> "He [Dr. B] really brings every theorem to the level of a person on the street, like even they could understand what he's talking about. [...] He takes a complex idea and really simplifies it. [...] I'll give you an example. When he explained the definition of a limit of a function, I think everyone understood it. […] He used words that are very intuitive. You can imagine it, it's words that create an image in your head. He uses a lot of words like that, it's very very easy to follow, very easy to connect. […] So suddenly even this scary definition feels much more understandable."

Calm Time Management

While the data suggest students acknowledge the time constraints on lecturers, they nonetheless characterize excellent lecturers as lecturers who are able to manage their time in a calm manner. This as opposed to lecturers who seem to be "running against the clock" in their teaching in order to cover the material. As articulated, for example, by Student D: "What I less liked about this lecturer was that she was super stressed and super quick – if you blinked for a second, then you lost her." Also Student E shared a similar stance: "If a lecturer is speeding through the material, this means that something in the planning of the schedule is not right."

It should be noted, however, that according to the students, *calm* time management does not mean *no* time management. For example, Student E claimed that a good lecturer knows when to insert pauses, which "can even be just a few seconds long" – allowing the students to finish copying from the board or "the time needed for an understanding to kick in". Student E explained this could prevent students from feeling "lost" and "entering a whirlpool of frustration". Another repeating claim was that good time management not only regards allocating time for answering students' questions calmly, but also knowing when lecturers need to say "no" to additional questions and instead continue in the lesson; as, for example, shared by Student C:

> "It's really nice when a lecturer answers everyone's questions and does it really patiently. But many times I encountered the situation where the lecturer answered too many questions, and then we simply didn't have the time to learn what we were supposed to."

Meticulous Preparation and Presentation

Another characteristic of excellent lecturers is an extremely orderly and thorough presentation of the material, which also stems from being rigorously prepared for each lesson. For example, consider the following excerpt by Student A, who described excellent lecturers as having high pedagogical awareness in both planning and executing the lesson:

> "Everything is very thorough and everything is very orderly, and you see that there's an orderly lesson plan that they thought about in depth: what comes before what, where should the example go, where should the theorem go? I think that at this level, Dr. B is very very very very didactic, and like, really, he's at the highest level I have seen."

According to the interviews, other aspects in lecturers' meticulous preparation and presentation of the material include a very deliberate choice of examples in the preparation stage, as well as making sure to not only explain the material orally, but also write all their explanations on the board (cf. Lew et al., 2016). As, for example, claimed by Student C: "I think it's really good when lecturers write down *everything* they explain. Especially if these are important things. That they really write in an orderly manner on the board, so we could look at it afterwards."

Creating a Hierarchy of Importance in the Mathematical Material

While the students acknowledged that attention to detail is crucial in mathematics, they also wished to be guided on which of these details they should concentrate and focus on first, before attempting to make sense of others. The characteristic "creating a hierarchy" thus regards lecturers helping students distinguish between more important and less important aspects of the material. As illustrated by Student A, not knowing what is vital to comprehend and what is less crucial to understand may result in students experiencing anxiety and stress:

> "When you don't know what to focus on, then you don't know what is important [and] you don't know what's not important. […] When you don't know what to focus on, everything puts you into some kind of panic mode, and this is what makes it so difficult."

According to the interviews, one way in which this hierarchy can be created is by lecturers repeating and summarizing only the most important and relevant information at the end of each lesson or topic. Another way regards lecturers first providing students with the general structure of the proof and only subsequently filling in the details (see also "structural proof"; Leron, 1983). Student B described this teaching practice, as well as its affordances for student learning:

> "Every good instructor does this. […] They create […] a fragmentation of the solution into blocks [and explain] 'this is what I'll to do [in each block]'. […] This is good for a lot of reasons. […] If you lost some part of the proof, a block in proof, you can say, 'well, I don't understand what the lecturer is doing right now, I'll wait for the next block, I'll accept the previous block as a given'. This might sound a bit funny, but in the end, it's the difference between understanding and not understanding anything."

Discussion

The motivation for this study stemmed from a gap in the research on undergraduate-mathematics instruction, which has not sufficiently incorporated a student perspective when inquiring into this topic. Addressing this gap in this initial, exploratory study, 8 characteristics of high-quality instruction were identified based on student-centered data, offering the following contribution to undergraduate-mathematics research.

First, I suggest that the findings herein provide important initial support to prior research on undergraduate teaching by adding a student perspective to the discussion of the topic. For example, the characteristic "Positive attitude towards students" could be regarded as a student validation to one of Jaworski's (2002) elements in her teaching triad, *sensitivity to students* (regarding their affective needs); "Calm time management" resonates with Speer et al.'s (2010) theoretical discussion on *allocating time within lessons*; "Making connections" across topics, a notion that has been widely discussed in prior lecturer-focused studies (e.g., Bergsten, 2007; Yan, Marmur, & Zazkis, 2020), seems to indeed motivate students; and "Creating a hierarchy of importance" supports prior characterizations of effective teaching, such as Leron's (1983) notion of *structural proof* and Movshovitz-Hadar and Hazzan's (2004) claim that good instruction requires distinguishing between important and less important ideas of the lecture.

Secondly, and perhaps more importantly, the student perspective on teaching as reflected here might challenge us, as a research community, to widen our focus of attention towards the affective dimension that is inherent to any teaching-learning interaction. As can be observed in the findings above, there is a strong affective dimension in the students' descriptions of which teaching qualities they found significant to their learning experience in lectures. This affective dimension was expressed not only regarding characteristics that explicitly attend to students' affective needs (e.g., "Positive attitude towards students"), but also regarding characteristics that are often viewed through a purely cognitive lens – for example, the characteristic "Making connections", which when not met, "drives you crazy", as argued by Student A. Furthermore, the affective dimension in the students' interviews was often inseparable from the cognitive one, supporting a non-dualistic approach to affect and cognition in mathematics education (e.g., Roth & Radford, 2011); for example, consider the characteristic "Adopting a student perspective", which tends to both students' cognition *and* affect. While we may find some discussion of affect-related characteristics of high-quality undergraduate teaching in prior studies, this dimension is often presented as secondary and subordinate to the cognitively-oriented teaching characteristics (e.g., Movshovitz-Hadar & Hazzan, 2004). Accordingly, I suggest not only that further empirical research into the student perspective is needed, but also that the data analysis of such empirical studies would benefit from further development of theoretical frameworks that highlight the affective dimension of undergraduate-mathematics teaching and learning.

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Not Choosing is Also a Choice

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Combinatorics is a growing topic in mathematics with widespread applications in a variety of fields. It has become increasingly prominent in both K-12 and undergraduate curricula. There is a clear need in mathematics education for studies that address cognitive and pedagogical issues related to students' conceptions of combinatorial ideas. I investigate students' perceptions of the option of not choosing while solving counting problems. In this report, as part of a larger study, I focus on experiences of one undergraduate student who was interviewed in the larger study. The interview was conducted as they solved combinatorial tasks that included the possibility of not choosing a particular attribute when enumerating choices. The data analysis highlighted detecting choices as one of the factors contributing to students' success in solving counting problems. I suggest an extension of a model of students' combinatorial thinking that has been introduced in mathematics education literature.

Keywords: combinatorics, counting problems, detecting choices

Counting problems comprise an essential part of combinatorics and are often used to introduce this area of mathematics. Although counting problems are easy to state there is much evidence that students struggle with understanding and solving them (e.g., Eizenberg & Zaslavsky, 2003; Godino, Batanero, & Roa, 2005, Batanero, Navarro-Pelayo, & Godino, 1997). The existence of abundance evidence might be one of the reasons that draw the field's attention to the importance of combinatorics in mathematics curricula (e.g., Lockwood, Wasserman & Tillema 2020). As counting problems are currently part of K-12 and undergraduate curricula, there is a necessity to study factors that might have affected students' success. To this end, addressing students' ways of thinking at a level that benefits a deeper understanding of how students conceptualize counting problems is significantly essential. Lockwood (2013) conceptualized students' combinatorial thinking by considering students' ways of thinking about counting problems based on three components—formulas/expressions, counting processes, and sets of outcomes. In solving a counting problem, errors may come from not correctly detecting options/choices which the problem specifies. In fact, detecting all the choices in solving a counting problem is one of the factors of students' success. However, options/choices can be presented in different ways. In this study, I narrow the focus to students' understanding of the particular option of not choosing as a choice and how students may consider it in their counting activity.

Theoretical Framework

Lockwood (2013) presented a theoretical framework that provides a model of students' combinatorial thinking. The focus of this model was on the ways students conceptualize counting problems. In fact, this model facilitates a conceptual analysis of students' thinking in facing counting problems and it provides a language to address and describe different aspects of students' activities related to combinatorial enumeration (counting). It has equipped educators with ways of elaborating on how students might think in solving counting problems, and it has provided a powerful lens for analyzing different facets of students' counting activity. This framework has three components, formulas/expressions, counting processes, and sets of outcomes. Any mathematical expressions that results some numerical values can be considered as a formula/expression. While solving a counting problem, a counter may engage in an enumeration process or series of processes. The counter either performs some steps or procedures or even imagines doing them. Both cases are considered as enumeration processes which describe counting processes as a component in this model. Sets of outcomes refer to the sets of elements that a counter wants to count. Any collection of objects that the counter tries to generate or calculate by a counting process, is considered as a set of outcomes. The relationship among the three components of this model is illustrated in Figure 1.

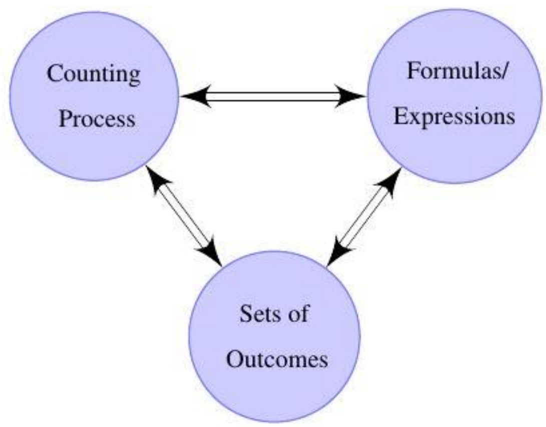

Figure 1: Lockwood's model of students' combinatorial thinking

Methods

In this study, as part of a larger study, I report to an interview with one of the students who was involved in the study that focused on students' perception of choosing. The participant, Harry, was an undergraduate student who attended a discrete mathematics course (Discrete mathematics II). The course focused on counting problems, graph theory, trees, inclusion-exclusion, generating functions, recurrence relations, and optimization and matching. The student participated in an individual semi-structured interview, one hour long. The interview instrument consisted of two problems that aimed at probing students' understanding in how they consider the possibility of not choosing and comparing students' understanding of "choosing" to "not choosing". I designed the two problems with the goal of highlighting students' judgments in validating the possibility of not choosing as a choice.

Problem 1: There are 9 international students and 5 domestic students to fulfill English language requirement. International students have to take one of three English courses (ENGL 100A, ENGL 100B, ENGL 100C). There is no requirement for domestic students to take any of these three courses. Taking one of these three courses is an option for them. How many possibilities are there for students' choices. (for example, one possibility is 3 international students take ENGL100A, 3 international students take ENGL 100B, 3 international students take ENGL 100C and all domestic students take ENGL 100B)

Problem 2: A group of 8 people had a meeting. Three of them had a presentation in this meeting. After their presentations they desperately needed a cup of coffee. The people who did not present can choose to have coffee. There were 11 different types of coffee to choose from. How many possibilities are there for these people's choices? Amy wrote the following answer:

$$\binom{5}{0}\times 11^3+\binom{5}{1}\times 11\times 11^3+\binom{5}{2}\times 11^2\times 11^3+\binom{5}{3}\times 11^3\times 11^3+\binom{5}{4}\times 11^4\times 11^3+\binom{5}{5}\times 11^5\times 11^3$$

Do you agree with her? What was she thinking about? Nikki's answer was $11^3\times 12^5$. What do you think about Nikki's answer?

The problems were designed such that a possible solution—$3^9\times 4^5$ for Problem 1, and $11^3\times 12^5$ for Problem 2—includes consideration of not choosing a particular attribute. The option of not choosing is present in different formations in each problem.

Results

Although the participant's work shows he is familiar with the fundamental principles of counting, his solutions indicate he was consistent in not considering the option of not choosing in the two problems. Although different ways of presenting this option did not affect his consideration of the option of not choosing, it affected his combinatorial thinking.

Organizing the Data by the Three Components of Lockwood's Model

After highlighting noteworthy parts of the participants' responses, I categorized them into the three components of Lockwood's model. I have assigned each highlighted situation to the components of Lockwood's model that have played an important role in making the situation significant. Pushing students to initiate their counting activity by one of the three components does not necessarily keep the rest of their counting activity in the same component.

The option of not choosing and formulas/expressions. The two numerical expressions in the second problem pushed Harry to start his combinatorial thinking with the formulas/expressions component. To be able to evaluate $11^3\times 12^5$ as an answer, the extra option of not having a coffee needed to be numerically explained. Harry started working on the second problem by making a connection between the addition principle and the plus signs in Amy's answer.

Harry: So, what she has is that there is eleven cubed that's constant among each of her expressions and it looks like she also separated it by case by case basis which you can tell by rule of sum from the plus signs.

Harry perfectly interpreted Amy's counting thought and precisely made sense of each counting step he had defined based on Amy's expression. His consideration of Nikki's answer started by interpreting 11^3 correctly and continued as follows:

Harry: For the five optional people the ones who don't have to have coffee she gave each of them twelve choices instead of eleven for the coffee, so what I believe she did was, we know that there are eleven types of coffee, for each person, she added one more choice, which is whether they are going to have coffee or not. So, say this is person number one here, they have eleven different types of coffee to choose from if they choose coffee, she thought of the option where that person does not choose coffee as an option. So, it goes from eleven to twelve options.

He continued by writing out 5 dashes, writing 12 in each of the dashes. He indicated each 12 refers to the 11+1 options. Although $11^3\times 12^5$ brought up the idea of considering the option of not choosing as an option, supportive counting processes were needed to convince Harry. I discuss Harry's hesitation in acknowledging the option of not choosing in the following subsection.

The option of not choosing and counting processes. Although multiplication and addition principles which are the two foundations of counting processes presented in Harry's counting activity, detecting the option of not choosing appeared problematic. After Harry explained what might be going on in Nikki's answer by clarifying that 12 in her answer comes from 11+1, I asked him if he agreed with her and he answered as follows:

Harry: No. then I am also biased because this is [pointing to Amy's answer] exactly how I approached the problem the first time and it's just the way I think of it when I like splitting up the cases deciding by rule of sum and the rule of product to determine the stages, that's just how I think. But then hers logically doesn't seem fundamentally sound because it seems even though she umm. I don't know! It just doesn't feel right.

Although He initially felt it may make sense to consider the option of not choosing as a choice, he was not convinced to validate this possibility as a choice. He explicitly mentioned, "it just doesn't feel right". I suspect that he had not seen/considered this kind of choice in his previous counting activity and/or in his routine life. After I asked him to explain why he thinks considering the twelfth flavor as "no coffee" is not right, he answered,

Harry: for twelfth option she decided to have no coffee. I am trying to see if, um, I think the thing that might be confusing here is that there is no rule of sum differentiating the cases. So, the way I have been approaching problems and the way I have seen the most of the answers usually is if there are cases involved there is always the rule of sum that separating them. What is giving me confusion here is that we only have a rule of product. Yes, she has added an extra option to account for there being no coffee chosen, but intuitively it feels wrong to me. If that makes sense. I can't put words to it but if that speaks about my own understanding if I can explain why it is not correct. I feel it is not correct, but I cannot give an argument as to why she isn't right. Because I totally understand where she is approaching the problem from.

Harry could interpret Nikki's expression, but it seemed he was confused if Nikki's answer was correct. Harry's discussion points to "met-before" construct introduced by Tall (2004). The notion of met-before concerns how prior knowledge and previous experiences affect learning in a new context. A "met-before" is a personal mental structure in our brain as a result of experiences met before and can act both positively and negatively when a new concept is introduced (Tall, 2004; 2008). Harry's previous experience seems to build a connection between the existence of cases in counting problems and the necessity of applying the addition principle to solve the problem. This connection is an obstacle for considering any other approach that does not attend to the cases. In fact, it seems his met-before experience affects negatively on accepting the option of not choosing as a choice. Interestingly, to evaluate Nikki's answer, he tried to overcome this obstacle by justifying that multiplication can be seen as repeated addition, so having just multiplication in Nikki's expression does not necessarily make the answer wrong. However, the equality of these two suggested answers (expressions) come from the equality of their sets of outcomes (based on proper choices and counting processes), and not a specific manipulation that rewrites repeated additions as multiplication. I suggest that not validating the possibility of not choosing as a choice with supporting counting processes directed Harry's attention to expressions.

He began working on the first problem by reading it and underlining some words and phrases such as "9 international", "5 domestic", "one of three", "no requirement", "one of these three courses", and "option". Although he highlighted the keywords in this problem, his first interpretation of the problem was incorrect as follows:

Harry: So, it is possible for me to just only distribute the nine international students among the three courses but do nothing with the five domestic students.

In solving a problem, having an intended interpretation of what the problem asks is essential. Particularly in counting problems, detecting all the options/choices correctly would be the main part of interpreting the problems. Harry's interpretation of having the possibility of not choosing any English course for domestic students made him think he does not need to make them involved in his counting activity. The option of not choosing has been missed in his first consideration. After he correctly counted all the possible ways for international students' choices and reached 3^9, I asked him to think about domestic students. Although he initially eliminated domestic students from his counting activity, he started thinking about them by making an interesting distinction between the two principles. He made the distinction to explain where to apply the two principles as follows:

Harry: What I remember in MACM101 [Discrete mathematics I] we learned the shortcut was that, one shortcut was when you are doing anything involving "and", like saying student one takes this and student two takes that, is the rule of product, but if you have "or", either student one takes this, or student two takes this, you do the rule of sum.

His explanation about the two principles points to what Tucker (2002) mentioned, "Remember that the addition principle requires disjoint sets of objects and the multiplication principle requires that the procedure break into ordered stages and that the composite outcomes be distinct" (p. 170). Harry's statement highlights two correspondences between, 1) "and" and the multiplication principle and 2) "or" and the addition principle. His language seems to suggest he was looking for keywords in distinguishing the proper operation in his solution. However, he went on to articulate why considering different cases would be legitimate as follows:

Harry: this combination or number of possibilities [pointing to his solution that was 3^9] will stay the same for the international students, regardless whatever the domestic students are doing. So, it does not have any effect on the international students at all. So that's ok if we try cases.

He began to list the cases systematically according to how many domestic students get an English course. In addition to the interesting distinction he made between the two principles I discussed above, he made the second distinction as follows:

Harry: My intuition is saying that in this case [only one domestic student takes a course] it is still the inside the case and not across the cases and inside the cases is still the rule of product, because with one domestic student taking a course really you are adding a tenth stage. The international students they have their nine stages of choosing, now you add tenth stage and it would be three to the ten, because now there is three more choices, how will that domestic student choose out of the three courses. I do the rule of sum across the cases, because this is where it is like either you have the situation where no domestic students take any course, or one student takes it or two domestic, or three or four.

When he talked about "cases" above, his language of "inside the case" and "across the cases" indicate that he is aware of how to use the addition and multiplication principles. Although he made some insightful comments about how he decided to apply the two principles, his language suggests some keywords for each principle. It seems not validating the possibility of not choosing as a choice makes him consider several cases for domestic students and use the addition principle. Harry continued by trying to make Problem 1 fit into a combination with repetition model, usually called the "stars and bars" technique. He applied the model to count all the possible ways for both international and domestic students identically. It seemed he applied one formula for both international and domestic students regardless of the possibility of not choosing to take an English course for domestic students. After I asked him what the difference is between international students and domestic students, he read the problem several times and ended up crossing his answer. He was ultimately not able to realize the option of not choosing can be one option for domestic students and the problem can be solved without considering different cases.

The option of not choosing and sets of outcomes. While Harry initially was not assured that the option of not having a coffee in the second problem can be considered as the twelfth flavor of coffee, his attempt to consider one possible case was convincing for him.

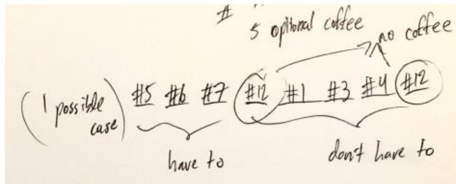

Figure 2: Harry's example of an element of the set of outcomes

Harry: When I think about it and I break it down this way [pointing to the first three dashes in his solution] this is just one case, here I add the other part as well [adding five more dashes]. These are the three people that they have to. These are the five people who don't have to. And we say this person have coffee number five, coffee flavor six, coffee flavor seven. This is one possible way that everything is distributed. If these two people no coffee, it is the same. I believe it is logically valid by adding the twelfth choice. And that choice is no coffee.

Harry's explanation shows his awareness of the overall conditions of the problem. Interestingly, considering one element of the set of outcomes helped him to convince himself that the option of not choosing is an option. In fact, the case he examined facilitated his insight into picturing the option of not choosing in the setting of the problem to validate this type of option. I continued the interview as follows:

Interviewer: So, you also agree with Nikki?

Harry: Yeah!

Interviewer: You agree with both?

Harry: But I usually approach problems like this [pointing to Amy's answer] and this is the only way I have seen it, I agree with Amy's answer more. I am much more hundred percent about Amy's but when I break Nikki's option dawn into one case, I can see that should be nothing wrong with adding the twelfth option.

Interviewer: So, you are not hundred percent agree with Nikki?

Harry: Yeah! Let's go with that. I can say, I agree with Amy hundred percent, Nikki I can see where her logic coming from but since I cannot fully endorse Nikki's answer. I understand it, I see what she did, I can break it dawn. But I don't understand the problem to the point where I can say confidentially that she is right. So, I go with Amy.

Although he reached the point to acknowledge the option of not choosing as an option, due to lack of supportive counting processes, he could not make himself convinced with Nikki's answer.

Discussion and Conclusion

Although detecting the option of not choosing is not a counting process, it definitely affects the process of counting. This research confirms rejecting the option of not choosing as an option may cause a detour in students' paths of solving counting problems by increasing the number of processes. In addition, not having it as an option may force counters to go through different structures of counting processes. Both scenarios make counting processes become more complicated. In addition, while detecting the option of not choosing cannot be considered as any combinatorial thought related to the other two components of the model, not detecting it as an option may affect both. Indeed, detecting the option of not choosing belongs to a broader aspect of students' combinatorial thinking which I call *detecting choices*. My data shows by including this aspect of students' combinatorial thinking the model will benefit from a refinement. Detecting choices would be a necessity for any counting process. However, while in Lockwood's model detecting choices appears to be part of counting processes, my data brings the conclusion that detecting choices is a separate element connected to all the three components in Lockwood's model.

In solving a counting problem, the numbers of choices that the problem specifies are needed to be detected in order to make any enumeration process (counting process) begun. The data shows detecting choices is not part of counting processes but it is definitely needed in counting processes. Lockwood implicitly considered students' combinatorial thinking related to choice detection as part of counting processes. However, the data demonstrates detecting choices carries an important role in students' combinatorial thinking that makes it deserved to have more attention. It appears merging the two important aspects of students' combinatorial thinking—detecting choices and counting processes—decreases the facilities this model could have brought for educators and teachers in this field. In fact, Lockwood's model does not equip researchers to determine how students might detect choices in their counting activities while their combinatorial thinking does not involve in any counting process. This is important as it demonstrates a need to refine the framework to include this aspect of students' combinatorial thinking.

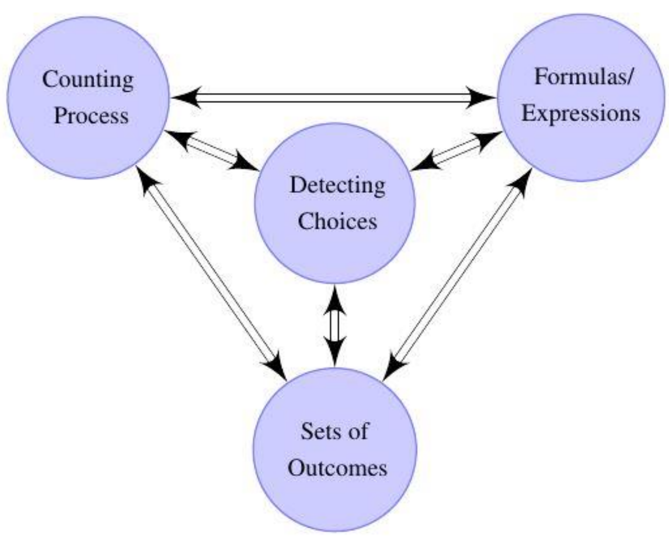

Figure 3: An extension for Lockwood's model of students' combinatorial thinking

The data shows detecting choices tremendously affects the other three components. Hence, detecting options/choices not only could be part of students' combinatorial thinking but also could be the core of students' counting activity. So, I locate the fourth component in the core of Lockwood's model.

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Math Anxiety: A Case Study on the Individualized Nature of Math Anxiety

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This study qualitatively explored the effects of math anxiety (MA) on undergraduate college students. Four undergraduate students were selected using the Abbreviated Mathematics Anxiety Rating Scale (A-MARS) and then interviewed about their beliefs about MA and the symptoms they experience from the anxiety—including the origins of the anxiety and methods with which they cope with (or attempt to cope with) the symptoms of MA. Students reported the onset of their anxiety occurred around elementary school and reported some common experiences such as strong feelings and a physical response. However, participants also showed the individualized nature of MA with different effects and coping mechanisms. This has implications for undergraduate instructors who want to help math anxious students.

Keywords: Math Anxiety, Undergraduate Education, Math Education

MA can lead to lower performance and a tendency to avoid majors that require mathematics (Ashcraft, 2002). Thus, it is important to better understand undergraduate students' MA and how to help them cope with it. We present a qualitative study that examined four undergraduate students and their beliefs about the origin of their MA and how these feelings have impacted their lives both in- and outside of the mathematics classroom.

Literature Review and Conceptual Framework

An individual is said to have math anxiety if that person experiences "feeling[s] of tension, apprehension, or fear that interferes with math performance" (Ashcraft, 2002, p. 181). There are three components to this definition—anxiety, mathematics, and performance. That is, in order to be considered math-anxious, an individual must exhibit feelings of *anxiety* that specifically relate to *mathematics* (whether that be seeing, doing, or being evaluated over mathematics); and these feelings of anxiety must somehow affect the individual's ability to do mathematics. MA is also often characterized by decreased performance in mathematics (Lindskog, Winman, & Poom, 2016) as well as physical symptoms typical with anxiety, such as sweaty palms or even crying (Ashcraft, 2002). This study will focus specifically on MA in undergraduate students.

There is some division within the literature about the differences between test anxiety and MA, with some researchers finding support for differences between the two constructs and others reporting no such support (Dowker, Sarkar, & Looi, 2016; Kazelskis et al., 2000; Ho et al., 2000). For this study, we follow Ho and colleagues (2000) in assuming that MA and subject-specific test anxiety are separate constructs and can therefore be examined individually.

Origins of MA

The origins of MA have not been a focus of much empirical research, though researchers agree that there is no single cause for MA (Ashcraft, 2002). Possible contributors to MA have been suggested to include genetics, parent feelings about mathematics, peers'/siblings' feelings about mathematics, teacher enthusiasm, classroom instruction, school tracking, mathematics experiences, mathematics ability, and other anxieties such as test anxiety (Ashcraft, 2002; Quan-Lorey, 2017; Wheeler & Montgomery, 2009; Sloan, 2010; Stoehr, 2017). Quan-Lorey (2017) writes, “mathematics anxiety is a complex phenomenon that is usually not caused by an individual factor” (p. 20).

One qualitative study examined three preservice teachers and found that each experienced MA differently, and that the origins of each participant’s anxiety were different (Stoehr, 2017). One participant developed MA as a result of being separated from the advanced students in elementary school. Another participant always operated with the understanding that she was simply not a math person. The third participant’s anxiety seemed to stem from her perception of a lack of a practical need for the subject. The case study presented here builds on this work, further exploring how undergraduate students experience MA differently.

Performance Effects of MA

MA is important to understand because it has been found to increase difficulty in student learning and decrease understanding (Wheeler & Montgomery, 2009). Students that experience high MA also “tend to avoid college majors and career paths that depend heavily on math or quantitative skills” (Ashcraft, 2002, p. 182). Students with high MA may avoid taking mathematics or quantitative reasoning courses in general. The literature suggests this could also contribute to lower mathematics performance skills. Furthermore, this could lead to less participation in class and lower grades (Ashcraft, 2002; Lindskog, Winman, Poom, 2016).

Classroom Practice Remedies

Researchers have suggested multiple ways to help remedy MA or help students cope with MA. Some suggested remedies for MA include teaching students to use cognitive skills to control their anxiety response (Lyons & Beilock, 2012), implementing math methods courses for preservice teachers (Sloan, 2010), and teaching ‘why’ a problem solving approach works rather than memorizing a formula (Wheeler & Montgomery, 2009). Other suggestions for remedies that help to control emotional responses include creative writing and a change in teaching methods. Geist (2010) suggests that a change to the teaching methods used in elementary classrooms could help to prevent the onset of MA altogether. Several researchers have found that asking students to write about their feelings can help lower MA and decrease the performance gap between low and high math anxious students (Maloney & Beilock, 2012; Park, Ramirez, & Beilock, 2014). Overall, researchers suggest that a focus on helping students decrease their anxiety is a better solution than creating new courses to simplify, or water-down, the mathematics (Lyons & Beilock, 2012).

Holes in the Literature and Project Rationale

A majority of studies on MA have used a quantitative approach, including multiple that use anxiety rating scales (Ashcraft, 2002; Cates & Rhymer, 2003; Goolsby, et al., 1988; Lindskog, Winman, & Poom, 2016; Lyons & Beilock, 2012, Sloan, 2010). Less research has used a qualitative approach to examine MA, though some recent research has used participant interviews, written participant samples, focus-group interviews, and the creation of mathematical timelines (Sloan, 2010; Stoehr, 2017). However, more research using a qualitative approach is needed to better understand how MA develops, how it differs from one individual to the next, and how it can be remedied within the classroom. The goal for this project was to qualitatively explore the symptoms, origins, and effects of MA on highly math anxious undergraduates enrolled in college mathematics, as well as the student’s coping mechanisms. The research questions were:

1. How do math-anxious undergraduates experience symptoms of math anxiety?
2. What do math-anxious undergraduates view as the source(s) of their MA?
3. Do math-anxious undergraduates believe math anxiety has affected their life and schooling? If so, how?
4. Do math-anxious undergraduates have ways they cope with their math anxiety? If so, how and are the methods effective?

Method

Participants were chosen from two different college mathematics courses. The first of these courses, which we will call Math Modeling, is not algebra-based and is designed for students that are not in business or STEM programs; it focusses on modeling mathematics found in everyday life activities, such as financial management. The second course is a traditional, algebra-based pre-calculus course and is typically taken by students that are interested in business or STEM-related degrees as a prerequisite for either calculus or business-calculus.

Participants

While the analysis methods of this study were qualitative in nature, a more quantitative measure was used to locate participants that experience symptoms of MA. For this, a survey was used—the Abbreviated Mathematics Anxiety Rating Scale (A-MARS)— which is a 25-point likert-scale (Alexander & Martray, 1989). A-MARS is scored on a 1 – 5 scale with 1 indicating little to no MA and 5 indicating high MA. Students in six mathematics classes were invited to take the A-MARS survey and 121 did so. Four students with some of the highest scores on the survey, all first-year students, were invited to participate in interviews. These participants were not told they ranked the among the highest levels of MA for the set of surveys returned but were told that their scores seemed to indicate that they experienced MA.

Data Collection and Analysis Procedures

Semi-structured interviews were used to collect qualitative data from participants. Participants were asked about their feelings regarding mathematics, why they believe they have those feelings, if they believed they were anxious about math, and what processes (if any) help them to cope with their math anxiety. Participants were also asked about how MA has affected their lives, and specifically their college experiences (if at all). These interviews were audio recorded, transcribed, and then analyzed using an open coding technique to develop codes inductively from the data. Coding led to creating profiles for each participant, with the goal of comparing and contrasting participants to address the research questions.

Results

Reported below are the results to each of the above research questions. For each research question, there were commonalities among the participants as well as a few stark differences.

Symptoms

The participants in this study had a wide range of reactions to their MA. All four participants expressed feelings of frustration and stress. However, each participant had other strong reactions when it came to discussing their anxiety. For example, Jacob frequently used the term “freak out” when talking about his feelings during mathematics class. When talking about his experience in high school, Jacob commented,

> …I [would] literally just freak out. It was really bad for me. … I felt dumb. I really was just like 'oh my freak. Why can I not get this?' I used to freak out really badly.

For Jacob, these "freak out" moments would sometimes get so bad, that his friends would ask, "are you okay? You look like you're going to burst a hole through the wall or something". This indicates a level of stress that is not only felt but shown. Other participants did not express quite this level of frustration or visible "freaking out". However, others did express a similar discomfort. When asked about how she typically feels in a mathematics classroom, Karly responded, "Anxious. Like really ready to go. Even if I am understanding, it's still not a comfortable feeling. I just want to leave". Participants also reported a physical response to MA. Both Karly and Laura reported that they cry when doing mathematics. Karly cried during the interview and expressed that simply the act of talking about her anxious feelings toward mathematics caused negative emotions. According to her, crying when thinking about or participating in mathematics is not something that was new to her. In fact, she indicated that she cries a lot when it comes to mathematics. Laura also said that she feels sick to her stomach when it comes to high stakes assignments such as tests in the mathematics classroom. Both Jacob and Macy talked about sweating as a physical reaction to the anxiety.

Despite the strong feelings described by each of the participants, Laura was the only participant that indicated her MA was significantly worse with mathematics tests than with mathematics in general. Each of the participants agreed that their MA was elevated during testing situations, but Laura was the only participant to indicate that her anxiety elevates with math tests no matter how comfortable she is feeling with the material.

Origins

All four participants indicated that the onset of their anxiety symptoms either began in elementary school or that they had MA feelings for as long as they could remember. Macy recalled doing mathematics homework in elementary school with her father, saying:

> …there was one story problem I remember. I had my math book, and I had my story problem right there and I remember I cried. We did it so many times, and I still couldn't get it right. He's a good guy because he was so patient, but I was crying. I just remember saying to him, 'I'm going to fail at life'. Fifth grade. I remember distinctly being so distraught and upset because, I just couldn't get it right.

Peer humiliation was a strong factor in the development of Jacob's, Macy's, and Laura's anxieties. Macy recalled timed multiplication tests that were checked by peers and how she was afraid of failing in front of peers. Jacob recounted experiences in elementary school of his classmates being called to the board to work out a problem for the class. He explained how he refused to go to the board out of fear of embarrassing himself in front of his classmates because he did not understand the material as quickly as his peers. Laura's anxiety onset also seemed to stem from this peer humiliation. When asked why she thought she had these feelings about mathematics, she answered,

> I’ve always struggled with it and been slower than everyone else and everyone else kind of made me feel bad about myself because I wasn’t as fast as them or I didn’t learn as fast as they did.

While peer humiliation was a factor in the development of MA for multiple participants, Laura was the only participant that indicated peer humiliation was still a strong component of her anxiety toward mathematics. When asked why she had chosen not to utilize a mathematics resource center on campus, she explained that she would not be comfortable going because she would risk other people “judging, or staring, or watching”.

On the other hand, Karly did not express these feelings at all. In fact, she does not give any reason for her anxiety. Instead, she simply states that sometime around second grade she realized that mathematics was not her strong suit and her anxiety continued from there. While Karly feels she cannot relate to her peers because she believes mathematics comes more naturally to them, she gave no indication that her anxiety is rooted in this belief.

Participants talked about parents and other role models in the interview. Both Jacob and Karly indicated that neither of their parents exhibited strong mathematics skills, though Jacob talked about a mentor of his that loves mathematics and encourages Jacob. Macy and Laura both had one parent they perceived as good at math and another parent that they related to because of the parent’s lack of desire to do mathematics. However, none of the participants attributed their MA to their role models. Each of the participants argued that they formed their math anxieties on their own.

Effect

Participants reported mixed feelings about how their MA has affected their lives and their college experience. Macy, Karly, and Laura all indicated that their MA had not affected their choice of college major but had affected either their choice of math class or their experience in their college math class. For example, Macy reported that her MA has negatively affected her college experience by making the entire experience more stressful. Laura spent some time talking about how her MA has caused her to not want to attend school and how she avoids mathematics as much as possible. Both Karly and Laura reported taking the Math Modeling course because of their MA. Karly had previously enrolled in a different college math course and explained:

> I was there and [the professor] was just talking and showing something on the board and I thought ‘I can’t stay here’. I’m not understanding. I’ve been here all semester long and haven’t been understanding. I even went to the math center and I’m still not understanding, so, I can’t stay here. …and I never came back for the semester.

Karly said the experience she described above was the reason she chose to take the Math Modeling course. Her statement not only shows how her MA played a role in her course decision, but also showcases how Karly’s MA has led to her avoiding mathematics where possible. Laura showed even stronger avoidance tendencies than Karly. When asked about how she reacts to her MA, Laura responded, “if I get frustrated, then I just completely shut down and I don’t do it. I just give up”.

In contrast, Jacob claimed his MA was a factor in his decision to major in social work. In fact, he claimed he was upset to learn he needed to take a mathematics course for competency credit. He said, “if someone had told me that I had to take a math class, I probably would have reconsidered.” For Jacob, the MA feelings are strong enough that they have influenced important decisions in his life.

Coping Mechanisms

The most prominent coping mechanism among the four participants was seeking outside help in some form. Macy talked at length about calling her father for assistance when she was struggling. Laura and Jacob indicated that contacting friends or classmates for assistance helped them to relax their anxieties. Jacob and Karly each indicated that the mathematics resource center on campus was a useful tool.

Another interesting coping mechanism that was brought up by Laura and Jacob was listening to music while working on mathematics. For both participants having a quiet environment with music in the background was one way they were able to cope with MA while working to complete mathematics homework assignments. Interestingly, these were the same two participants that indicated they would reach out to friends for help when they were stuck on a homework problem. However, both argue that they cope best with the anxiety when they are alone to work on their homework.

Karly was the most unique in this study because she indicated that aside from visiting the mathematics resource center once in a while, she does not have any coping mechanisms for her MA. She said that the most she has been able to cope with the anxiety is by trying to detach herself from the mathematics by intentionally not thinking about the fact that she is doing mathematics when she is in class. The way she described this process was:

> I’m present, but I’m not really thinking too hard about it. Like, Yeah, I’m doing math but I’m not like really caring that I’m doing it. But if I think, ‘oh yeah I’m doing math. This is annoying me’, then I will start to have those feelings. But other than that, if I just detach myself from it, then I don’t really [have the anxiety]

Aside from the detachment she described above, Karly stated she has no other way of coping with her MA.

Discussion

The results of this study show the individualized nature of MA, which is not developed nor experienced in the same way by different individuals. While many of the participants showed similar characteristics, no two participants in this study were similar enough to formulate a pattern for individuals experiencing MA. The most prominent similarity, the origin timeline for the participants, is the only similarity among all four participants beyond general feelings of stress and frustration that can be expected with any form of anxiety.

Researchers have hypothesized multiple origins for MA (Ashcraft, 2002; Quan-Lorey, 2017; Wheeler & Montgomery, 2009; Sloan, 2010; Stoehr, 2017). However, the participants interviewed for this study showed no evidence of most of these hypothesized origins, such as genetics or role models. The only supported hypotheses were previous math experiences (e.g. timed tests) and participants’ perception of their abilities in mathematics.

Implications

The current study provides a lens through which to view MA. Rather than looking at MA as a whole, this study allows researchers and educators to begin to explore how MA affects students on an individualized basis. Hopefully, seeing how students describe their MA will help educators recognize signs of MA and have better insight into the experiences of math-anxious students. For example, instructors might underestimate how intense the feelings and physical reactions of students experiencing MA can be (as shown by the results of this study). This study also shows how students are coping with their anxiety, including the importance of support from teachers, mathematics resource centers, and family and friends. Undergraduate instructors have an important role to play in providing coping mechanisms for students. Finally, this study again shows that students with MA may avoid majors or classes that are highly mathematics based. Undergraduate advisors may need to pay attention to students' MA, particularly for those students who might leave their programs due to the mathematics requirements. This study opens the door for future research (an example of which is described below) to further explore MA and how it is experienced at the undergraduate level.

Future Research

The current study suggests individuals experience the effects of MA differently. However, the current study is narrow in scope and only focuses on four individuals. A larger study is currently being developed to dig deeper into three aspects of MA—the origins of MA, the effects MA has at the collegiate level, and coping mechanisms that may help to alleviate the symptoms associated with MA at the undergraduate level.

Other future work could examine how instructors could incorporate coping mechanisms such as support from family and friends into their classes. It would be interesting to examine how encouraging students to form cohorts or other networks for support in college mathematics might affect MA in the classroom. Music might also be an interesting coping mechanism to examine. Future work could examine detachment, as experienced by Karly, in greater detail.

Conclusion

The results of this study indicate a need for a more individualized approach to the study of MA. Understanding MA at an individual level can help to shape the way mathematics is taught and how MA is dealt with in the classroom. By creating a more complete picture of MA, educators and researchers will be better equipped to create tools for prevention and intervention.

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A Preservice Teacher's Experience of Mathematical Research

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As part of a summer research project, an undergraduate named Anabella (pseudonym) and I (author) collaborated in efforts to prove an unsolved conjecture from the field of graph theory. Anabella is a Hispanic mother who one day desires to be an elementary or middle school teacher. The researcher collected and analyzed extensive data in order to understand what Anabella learned about the nature of mathematics through the research experience as well as how she perceived this new knowledge is relevant to her future as a teacher of mathematics. Anabella learned that her own mathematical ideas are valuable and reflected that it will be important to entertain and consider (rather than dismiss) the ideas of her future mathematics students. She also gained a new appreciation of the importance of mathematical communication.

Keywords: Nature of Mathematics, Mathematical Research, Preservice Teachers, Beliefs and Conceptions

Scholars in mathematics education have found that elementary and middle school teachers often have naïve views of the nature of mathematics (Pair, 2017; Thompson, 1992; Watson, 2019). It is important for teachers in particular to have informed views of the nature of mathematics so that they may support their own students in developing informed rather than naïve conceptions of the nature of mathematics. The university, with its role in teacher development, plays a substantial role in providing opportunities for future teachers to enhance their perspectives of the nature of mathematics.

This study explores the opportunities afforded for a preservice elementary/middle school teacher to learn about the nature of mathematics through research mathematics—in this case, collaborating to prove an unsolved conjecture from the field of graph theory. Anabella (pseudonym) was a student in a content course for preservice elementary and middle school teachers that the author was teaching at a large public university in Southern California. Anabella stood out in the class for her willingness to share her solutions and engage with mathematics. On the first day of class she was drawing complete graphs to solve the handshake problem. Anabella also expressed interest in the discipline of mathematics after learning that there were still unsolved mathematics problems. The author wrote an application to fund Anabella with a student research assistantship and obtained approval from the university's institutional review board to collect data during the project in order to understand what Anabella learned about the nature of mathematics and its relevance to her future teaching through the research.

Literature Review

While many mathematics education scholars have argued that misconception of the nature of mathematics is a major problem in mathematics education (Boaler, 2016; Hersh, 1997; White-Fredette 2010), little research has been conducted that explicitly addresses how to nurture positive conceptions of the nature of mathematics. As Hudson and colleagues (2020) recently noted, "There is still much to learn about what students and teachers think about the nature of mathematics and how these views impact their work" (p. 55). There are some recent research articles and dissertations that address the conceptions of mathematics held by both preservice teachers and undergraduate mathematics majors. Kean (2012) developed a survey instrument designed to form a profile of teacher's conceptions of the nature of mathematics. However, she concluded that her first attempt needed much improvement; and that developing a good survey instrument would be more difficult than anticipated. To date, Kean's survey has not been used in subsequent research.

Pair (2017) argued that in order to conduct meaningful research into students' views of the nature of mathematics, the field would need to develop a list of goals for students' understandings of the nature of mathematics. Through an extensive data-driven project that included 1) interviews with mathematicians, educators, and mathematics majors; 2) collaboration with a pure mathematician in conducting mathematical research; and 3) experience co-teaching an inquiry-based transition-to-proof course with a mathematics educator, Pair developed the IDEA framework for the nature of mathematics. The framework drew inspiration from humanistic philosophy of mathematics (e.g. Ernest, 1991; Hersh, 1997; Lakatos, 1976) and emphasized the human discipline of mathematics and its dynamic, rather than its static nature. This framework will be discussed further in the theoretical framework section.

In studying the nature of mathematics conceptions of preservice elementary teachers, Watson (2019) found that many possessed a view of mathematics as "a static-unified body of knowledge" (p. 148). In interviews with some of these preservice teachers, she found that those who experienced mathematics learning through memorization and practice tended to view mathematics as a bag of tricks rather than a dynamic problem-solving discipline. However, there were some cases in which the preservice teachers from Watson's study experienced mathematics creatively. This led them to see mathematics as more of a creative, problem-solving discipline.

> Tabby's experience with one mathematics task and a teacher who valued her ideas in the classroom allowed Tabby to experience mathematics as a problem-driven dynamic discipline. Maggie, Tatum, Margot, and Millie also recounted experiences with teachers who allowed them to view mathematics as more than a bag of tools and instead as a discipline where creation and freedom to create could be valued. (p. 171)

Watson's findings align with those of Thompson (1992), who three decades earlier noted that that when a teacher views mathematics as "a discipline characterized by accurate results and infallible procedures" this "can lead to instruction that places undue emphasis on the manipulation of symbols whose meanings are rarely addressed" (p. 127). Watson's (2019) study suggests that providing students with opportunities for creativity through problem solving can lead to a more productive view and positive attitude towards mathematics.

While there is some older literature relevant that addresses the nature of mathematics (Dossey 1992; Lampert, 1990; Thompson, 1992), the topic still has yet to be systematically studied by mathematics education researchers (Hudson et al. 2020). While there are researchers who have recently begun to refocus on the nature of mathematics in their research, research findings are tentative. How do we best support students and future teachers in developing informed and healthy conceptions of the nature of mathematics? The object of the current study is to explore how an undergraduate research experience may impact a future teachers' understanding of the nature of mathematics.

Theoretical Framework

The foundational framework that guided this research was developed is the IDEA Framework for the Nature of Pure Mathematics (Pair 2017). The IDEA acronym refers to four characteristics of the nature of mathematics that are important for students to understand so that they are supported in developing a humanistic conception of the nature of mathematics. These characteristics are now described with a brief description. First, mathematical ideas are part of both our human cultural and individual personal **I**dentity; we inherit mathematical ideas from past mathematicians and cultures, and form our own personal mathematical ideas and techniques when engaged in doing mathematics. Second, mathematical knowledge is **D**ynamic and forever in the making; the history of mathematics is a story of creation and revision of ideas. Furthermore, new mathematical results are published every day. Thirdly, pure mathematics is an emotional and creative **E**xploration of ideas; students should have the opportunity to explore such ideas for their own sake in ways similar to that of mathematicians and understand that such exploration is essential to mathematics. Fourthly, mathematical **A**rgumentation is required for the social validation of mathematical knowledge; mathematicians validate and knowledge claims through verification and proof, leading to robust knowledge. The characteristics in this list may serve as learning outcomes for instructors wishing to promote a humanistic view of mathematics and as a conceptual framework to conduct research on students' views of the nature of mathematics. As these characteristics were inspired by my own experience conducting research mathematics, these were characteristics of mathematics that I anticipated Anabella would learn about in research.

Research Design and Methodology

Anabella is a Hispanic mother and was a traditional undergraduate senior at the time of the study. Anabella was compensated for participating in a summer assistantship; and one of her responsibilities was to keep a research journal in which she wrote about her experience and the new things she was learning about the nature of mathematics and how this new knowledge was relevant to her future as a teacher of mathematics. This journal was part of the data collected for the study. Additionally, every meeting was audio-recorded (with Anabella's permission as part of the study) in order to capture Anabella's experience so I could identify meaningful moments that contributed to her evolving conceptions of mathematics. The research experience lasted 2 months; and I collected approximately 60 hours of audio recordings of our meetings, and approximately 70 pages of Anabella's mathematical work and reflections in her journals. Other sources of data include photographs of our mathematical work and my mathematical notebooks in which I worked on the conjecture and took notes of Anabella's growth and expanding knowledge of the nature of mathematics. The goal in collecting so much data was to be able to later summarize the evolution of Anabella's conception of the nature of mathematics and describe in detail, through stories, the experiences that had the most impact on Anabella's growing understanding of the nature of mathematics and uncover how Anabella perceived the research experience was relevant to her future as a teacher.

After transcribing the data, I use qualitative approaches such as open coding and sorting in order to find the recurring themes in her reflections (Patton, 2015). The goal was to be able to paint a picture of the experiences Anabella had that led to transformations in her understanding of the nature of mathematics and also describe how Anabella believed those experiences and new understandings are relevant to her as a future teacher. All sources of data are brought together to tell stories about Anabella's experience and how that experience led to a change in her understanding of the nature of mathematics and its teaching and learning.

Results

I here describe the early stages of the research experience in detail so the reader can get an idea of the nature and scope of the mathematical research involved. After providing this background, I then provide a summary of the remainder of the experience and some of Anabella's takeaways from the research.

To begin the research, I shared with Anabella a paper from a mathematics journal which discussed our open problem—what is the chromatic number of graphs with girth 5 and no odd holes of length greater than 5? Anabella had several questions about the terminology and symbology of the paper, and on the first official day of our research collaboration, we discussed mathematical concepts such as girth, chromatic number, clique number, complete graph, cycle, chord, and hole. I asked Anabella to conjecture what was the relationship between the clique number and the chromatic number; and she conjectured that they were the same. We examined several graphs for which it was true that the chromatic number equaled the clique number, but ultimately found some examples for which this was not true. I took this as an opportunity to explain that it would be common for us to make and revise conjectures:

> So this is, I think, the first lesson we have had since we started. You are going to make conjectures that you are going to think are true; and you are going to find out they are not true, pretty quickly. It happens all the time [in mathematics] actually. I think that is important.

As we continued to think about the relationship between the clique number and the chromatic number, we discussed the journal article, which laid out the special conditions for which the chromatic number would be one more than the clique number. As we did this, Anabella reflected on how there are always new mathematical questions one can pursue. "This can go on forever and ever and ever," she said. Immediately, she got a new idea, a process of constructing new graphs from complete graphs which she described as "working backwards." Anabella took a complete graph, removed the "outside cycle," added vertices equal in number to the outside cycle, then connected the remaining vertices. See Figure 1 for some images of the process conducted on a few graphs. The Peterson graph is created through this process; and I believe Anabella invented the process based on her observations of the structure of the Peterson graph.

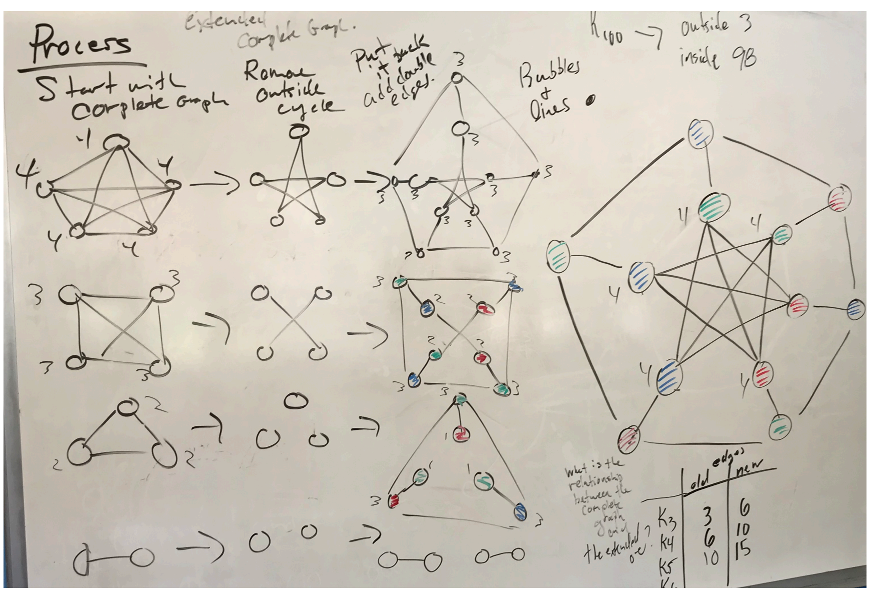

Figure 1. Anabella's Invented Process of Creating New Graphs from Complete Graphs

For the rest of the meeting, we worked with Anabella's process, creating new graphs from complete graphs, finding the chromatic number, and thinking about how many vertices and edges the original and new graphs would possess. I had never read about this kind of process in any graph theory text, and I was not sure if other mathematicians had done it before; I shared this with Anabella. She was excited at the prospect of exploring mathematics in a way that others had not. At the end of our first session she wrote,

> I am not fully convinced that no one else in the entire world, in its history, has ever came up with what we did today. However, if this is the case that only a few other people have ever done so, I feel a sense of great accomplishment because now I can say that I am one of those few. I have come up with this all on my own and it is only my first official day of work. I can't wait for what I will achieve tomorrow. Even if my biggest achievement is only in my head (or in this case, my notes). […] I have also begun to see the world (in the area of mathematics) as infinite. Just today, both Dr. Pair and I had tons of questions on each new thought/discovery we made. This is pretty exciting because I really enjoy being able to think "outside the box" and ask questions that would otherwise never even be considered taking a second look at. The sense of knowing things in general is fun to me and from today, I know it is also infinite. This is unlike any other subject because, let's say, with English or history, there are only so many things to know/learn.

On this first day, I believe that Anabella experienced the creative side of mathematics and learned more about its dynamic and infinite nature. She recognized that mathematical activity has no end, as there are always more mathematical questions one can ask and pursue. She enjoyed "thinking outside the box" which I have found students tend to associate with their first experiences of mathematical research (Pair & Calva, 2020).

On the second day of the research, Anabella, based on her own internet research, had many questions about concepts such as planarity and girth. The conjecture we were working on concerns the chromatic number of graphs with girth 5 and no odd holes of length greater than 5; Anabella created the name "high five class" for the set of such graphs. The properties of graphs in this class are generalized from the Peterson graph, and Anabella and I spent a substantial amount of time discussing the cycles of the Peterson graph in detail. We then proved an important (but simple) lemma for the high five class, that no graph in the class has a 7 cycle. We then discussed partitioning a graph in terms of levels, or distance from some arbitrary vertex. We took many graphs that we had already drawn and partitioned them into levels, also observing how the levelling process allowed us to systematically color the vertices of the graphs. We went through proof that each level is bipartite, and we also went through a proof that for any graph in the class with two levels, the chromatic number is 3. We accomplished a lot on Day 2, and after the day was over, Anabella wrote:

> My head hurts. I just crammed this new info into my brain in just a few hours. But I think we made more accomplishments today. For example, I helped Dr. Pair figure out that subgraphs in level 3 cannot connect. […] I learned about levelling and I was pretty fast at figuring those out. I learned that level 1 is always independent for any high-5 graph. The most important thing I think isn't understanding the new vocab, but understanding the concepts. I can always rename them like Dr. Pair lets me do. It is only my 2nd day with

Dr. Pair, but already have come to understand his dilemma, why he is stuck and cannot solve the problem. Still, we got many steps closer to solving it.

In this reflection, Anabella realizes what is important are the mathematical concepts, the ideas, rather than the vocabulary and the name of the concept. As she says, we can rename these. She often referred to vertices as "bubbles." This is an important notion for teachers to understand—the name we give to mathematical concepts is not as important as understanding the concepts themselves. It is also enables one to stamp their own identity into mathematics as one chooses how to name mathematical objects.

On the beginning of the third day, Anabella took a lead in suggesting what we should work on in approaching the conjecture. She said, "I was thinking what if we focused on like the options, like focus on level 2. Because we know level 1 always has to be a color, and level 3 can alternate between two colors." We spent most of the rest of the day imagining we had some even cycles in level 3, then looking at how the parents of those cycles were connected in level 2. Near the end of the session, I asked Anabella if she saw any similarities between the type of mathematics we were working on and what she experienced in school. At first, she said no "not at all." But then a few minutes later, reflecting on our thinking process during the day, she said:

Anabella: So it's kind of like the overall idea. The bigger idea connects.

Interviewer: There is a bigger idea that connects to the school somehow?

Anabella: Yeah. It's like. It's kinda like those Common Core, you read those Common Core. Have you seen the video where they talk like, they kinda say that the Common Core sucks? And they want to get rid of it. It just makes it hard for kids? But it makes sense why it would be a good thing; because if kids can do this, this way, then they can do anything else.

Here Anabella reflected on how if we can engage children in the high-level mathematical processes similar to those we engaged in during research, for instance problem solving tasks like those promoted by the Common Core, then the students will have the capability to do the lower-level mathematical tasks expected of children in traditional classrooms.

On the fourth day of our collaboration we began by discussing how different 5 cycles can be connected in our graph class; and then we began writing out a proof of our progress thus far. I was writing the proof in what I understood to be the standard format as conventional for mathematicians. But Anabella was not familiar with the conventions of proof writing. She had to ask several clarifying questions to understand the language. Throughout this process Anabella contributed her thoughts on how we should word the proof. At the end of the day, Anabella wrote in her journal:

Unlike in school there is no right or wrong answer yet. Our math research is 100% based on our knowledge, so we may come up with conjectures that are not true. Human error is our only flaw. This is normal, but it is the reason we may take longer to solve problems.

Through the next few weeks, we had a breakthrough in our work, finding a proof that for graphs in our class with three levels or less, the chromatic number is always three. Yet it took us many days to verify this proof. Anabella was convinced our argument was correct, and yet I needed to mend all the holes that I knew existed in the argument. Anabella learned a lot about the social validation of mathematical knowledge and mathematical communication through this process and throughout the research experience.

As the culmination of our work together, Anabella presented her work twice, both at a local mathematics conference as well as at a student research competition. At those events, Anabella described her big takeaways from the project. She was proud of all the different graphs that she created, inspired by such famous graphs as the Peterson graph and the Grötzsch graph. Anabella said that she learned that “my ideas are important.” She spoke about how she was free to explore her own ideas, and how I (as her research instructor) always took them seriously and considered them. She recognized that as a future teacher, it would be important to “Let mathematics students speak and be heard!” Recognizing how important her own ideas contributed to the research process, she realized that this should carry over to her teaching, as she would need to recognize that her students’ ideas were important and let them express and explore their ideas in the classroom. She also reflected that “communication is key, but the key to communication is knowledge.” She learned about the importance of precise communication of mathematical ideas and the foundational mathematical knowledge necessary to engage in such communication.

Beyond the current study, I have more evidence of what Anabella learned through another study (Pair & Calva, 2020). In that study, students in my transition-to-proof course worked on either the Twin Primes conjecture or the Collatz conjecture, documenting their reflections on the process. Anabella, as part of her pursuit of a supplementary authorization to teach middle school mathematics, was a student in that course. Her reflections showed a sophistication beyond many of the mathematics majors in the course and she was the only student that demonstrated an understanding of the importance of the social validation of mathematical knowledge. Anabella wrote,

> A mathematical proof is a consensual, regarded answer to a mathematical problem. Ideally, it should have no errors (holes) and must convince other mathematicians that it is correct and true for its purpose. [...] If you have a conjecture, the only way that you can safely be sure that it is true, is by presenting a valid mathematical proof. A proof has to be well thought out and tested before being accepted.

At the end of the semester she identified mathematical research as her own and expressed a desire to keep researching: “Although this class has finished, my research has not. I will continue it and the best part is that I will do so at my own pace—wherever I feel like it!”

Conclusion

I presented the results in a story format to paint a picture of Anabella’s research experience, capture what I as the researcher aimed to teach Anabella about the nature of mathematics, and to capture her impressions of mathematical research and what she learned about mathematics and mathematics teaching through the process. Anabella first displayed her skill and interest in mathematics in a content course for preservice elementary and middle school teachers. The current study may be seen as an existence proof that some preservice elementary/middle school teachers are capable of meaningful engagement in research mathematics—and that engaging in research can be a powerful learning experience that can lead to the opportunity to develop their own mathematical identity. Anabella learned about the importance of her own ideas as starting places for mathematical exploration and reflected that it would be valuable for her future students to have similar opportunities. Anabella experienced mathematical communication in a way that also positions her to be a strong mathematics teacher, willing to engage in mathematical discourse with her future students.

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Conceptualizing and Representing Distance on Graphs in Calculus: The Case of Todd

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Numerous calculus concepts rely on the ability to conceive of and represent distances in the Cartesian plane. Yet, research has found that undergraduate students, even those who have studied calculus, may not readily do so (Parr, 2020). This study reports on findings from the development of a hypothetical learning trajectory aimed at supporting students in representing distance in the Cartesian plane. We hypothesized that this skill would require (a) the ability to represent a distance on a number line as a difference and (b) drawing on the Cartesian connection to represent a horizontal or vertical distance in terms of x and in terms of y. We analyze the types of reasoning used by a Calculus I student while working on these tasks, as well as obstacles she faced relative to these two target understandings.

Keywords: Calculus, Graphical Representations, Distance, Hypothetical Learning Trajectory

Consider the following task. A generic point (x, y) is indicated on a graph of $y = \sqrt{x-1}$. A horizontal segment is drawn from this point to the line $x = 2$. Represent the length of the segment (a) in terms of x and (b) in terms of y.

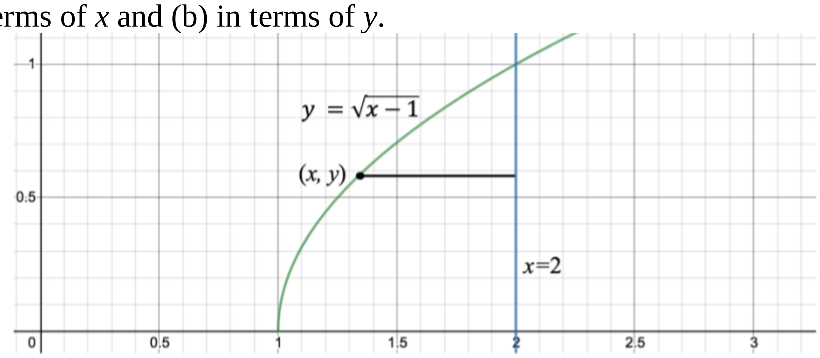


Figure 1. Distance between two curves task.

The skill reflected in this task is the ability to represent distances in the Cartesian plane in terms of input and output values of a graphed function. This skill is crucial in Calculus; it is used when creating integrals that represent areas bounded by curves and volumes of rotation, and also for representing a difference quotient for a function in the definition of the derivative. This skill relies on conceptualizing a point's coordinates (x, y) as describing the horizontal and vertical (signed) distances between the point and the origin. Yet, research has shown that students may not conceive of coordinates of points in the plane as expressing distances when reasoning about related expressions (Parr, 2020). In fact, when the second author gave this task to the students in his Calculus I class, only 17.4% of students answered both (a) and (b) correctly, and only 37% answered one of the parts correctly. This was, alarmingly, <u>after</u> the chapters about finding areas between curves and volumes of solids of revolution using integration.

Our broader goal for this study is to investigate how students can learn this important skill. We conducted a teaching experiment (Steffe & Thompson, 2000) with a hypothetical learning trajectory (Simon & Tzur, 2004) involving a sequence of tasks intended to develop this skill. This study focuses on several elements of student reasoning that emerged during that teaching experiment.

Theoretical Background

In this study, we focus on two target understandings that we hypothesized to be important for correctly solving tasks like the one above. The first is conceptualizing a difference of the form $b - a$ as representing the distance (along a one-dimensional continuum) between a and b. Since this understanding is grounded in interpreting notation, it helps to frame it, and the other related concepts we found in the study, using the language of semiotics (Barthes, 1957). A sign is made of a symbol (signifier) and a signified. The signified can be a mental object or another mark or collection of marks (which itself could be a signifier of something else). An interpretation is then the association between the symbol and the signified. With this in mind, we describe this first understanding as a *composed magnitude interpretation* of distances on number lines.

In the context of representing a segment's length, we define a composed magnitude interpretation as treating an expression of the form "*b–a*" as representing the following object: the length (a measurable attribute) of the segment measured from a to b. This requires the construction of the mental object of a quantity (e.g., Thompson, 1994). This interpretation of *b–a* is consistent with Parr's (2020) description of a *magnitude* interpretation. We extend this definition to include the notion that the distance is *composed* of the difference of two distinct distances. The *b* is the distance from the origin to one endpoint, the *a* is the distance from the origin to the other, and *b–a* is the difference between these distances, thus the length of the segment. Finally, we contend that this interpretation involves appealing to a difference model of subtraction (5–2 is the difference between 2 and 5) as opposed to the more prevalent take-away or counting down model of subtraction (5 take away 2 leaves you with 3) (Saxe et al., 2013).

The second understanding we hypothesized to be important for tasks like the one above involves using the Cartesian connection (Knuth, 2000) to represent the distance between a general point on a function's graph and the *x*- and *y*-axes, in terms of *x* and in terms of *y*, flexibly. The Cartesian connection refers to the relationship: points lie on a graph if and only if the coordinate pairs (*x*, *y*) satisfy its equation. To use this relationship to represent distances in terms of *x* and *y* involves conceiving of a coordinate point (*x*, *y*) as a multiplicative object (Saldanha & Thompson, 1998) of a pair of distances from the axes, rather than as the name of a location. Although interpreting coordinates as a pair of distances from the point to respective axes may seem elementary, undergraduate students may instead interpret coordinates nominally, as labels for locations on an axis or the graph (David et al., 2019; Parr, 2020).

In terms of these two understandings, our research question for this particular study is: What types of reasoning emerge, and what obstacles are encountered, when students engage in tasks meant to develop their ability to express distances on graphs in the Cartesian plane?

Methods

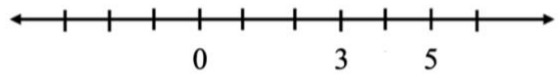
As part of a hypothetical learning trajectory (HLT) (Simon & Tzur, 2004), we designed six tasks and corresponding learning goals to support students in coming to conceive of and express distances on graphs such as the one in Figure 1. In fact, this exact task was given as both a pre-assessment and as the final task in the sequence. We aimed to sequence the tasks such that each subsequent task built off the previous, with a single new conception or step added. The goal of the first task of the sequence is to develop the composed magnitude interpretation of a difference. As such, this task asked students to represent and label a difference (5–3) on a number line by first drawing a segment from the origin of length 5 (a), a segment from the origin of length 3 (b) (Figure 2).

1. Draw the segments described using the x-axes provided.

a. Draw a segment with a length of 5 on the x-axis below, beginning at 0.

b. Draw a segment with a length of 3 on the x-axis below, beginning at 0.

c. Draw a segment to represent the difference 5 – 3 on the x-axis below and label it "5 – 3."



d. What is the length of the segment you drew in part c.?

Figure 2. A portion of Task 1 of HLT to develop composed magnitude interpretation.

The tasks progressed to asking students to represent various difference expressions as segments, involving whole numbers, integers, rational numbers, and variables as distances on both horizontal (Task 1) and vertical number lines (Task 2). Task 3 reversed this and provided students with horizontal or vertical segments, asking students to express the length of the given segment as a difference. Task 4 advanced from number lines to the Cartesian plane, giving students a generic point (x, y) and asking them to construct a segment from each axis to the point and label its distance. This task sought to make the Cartesian connection explicit in students' reasoning. Task 5 was similar, giving a point on a graph of a linear equation, but this time asking students to label distances both in terms of the input variable x and the output variable y (See Figure 3). Task 6 asked the same questions as Task 5 for a non-linear equation.

We conducted a teaching experiment (Steffe & Thompson, 2000) using our HLT with four groups of two students and one individual student (whose classmate was absent) from an undergraduate Calculus I course in Fall 2020 in their recitation section (held remotely) during the first week of the course. Students worked through the activity in breakout rooms on Zoom, with the researchers observing and offering assistance only if students could not resolve an issue in a timely manner. When these obstacles arose, we asked a series of questions intended to scaffold the necessary understandings we intended for the task. Often, these scaffolds took the form of asking a similar question with a numerical value rather than a variable, clarifying the intention of the task, or referring students back to previous tasks. We then conducted thematic analysis of the transcripts of these videos to identify and characterize students' interpretations relative to the development of conceptions we intended.

Results

Our goal in this study was to identify types of reasoning students use and obstacles they face in coming to conceptualize and represent distance between two curves. Using the case of one student, Todd, we describe two related obstacles that arose in many of the groups of students, which required additional support from the interviewers beyond the tasks to overcome. The first is related to students' conceptions of distance of a segment of endpoints "a" and "b." The second is related to students' ability to express the same distance in different ways using relationships between input and output variables via the Cartesian connection.

Interpretations of Distance of a Segment

While working with students through these tasks, we noticed three distinct types of student interpretations of distance emerge that were distinct from the composed magnitude interpretation we were targeting:

1. *Endpoints interpretation.* Students using an 'endpoints' interpretation of distance referred to a segment's length, when asked, as "from 0 to *a*" or "from *a* to *b*," giving the endpoints of the segment. In these instances, students reported where the segment began and ended but seem to be describing the segment itself rather than imagining a quantity of distance.

2. *Template interpretation.* Students using a template interpretation of distance may represent the segment length as 0–*a*, but "–" symbol is considered a dash, not the operation of subtraction. In this way, they apply a template, transferring "from *a* to *b*" into "*a*–*b*" or "*b*–*a*" (and may not be sure when or why the order matters). If they use "*b*–*a*," it serves as a (single) sign for the following object: the segment with endpoints at b and a.

3. *Magnitude interpretation.* Students using a magnitude interpretation refer to the distance of the segment as *b*–*a*, which signifies a quantity, consistent with Parr (2020). It is a sign for the following object: the (measurable attribute) of the length of the segment. Students give the difference "*a*–0," and treat the segment's distance as a measurable attribute of the segment with a numerical value that corresponds to the difference. This interpretation requires *b*–*a* to represent a single amount, even if it is unsimplified or unsimplifiable. This interpretation does not necessarily draw upon the subtraction as representing the difference between two magnitudes of length *a* and *b* themselves; in this way it differs from the composed magnitude interpretation we were targeting (described earlier in the theoretical background section).

We note that the simplification of the difference is available, as a small syntactic move, to students who use the magnitude interpretation. The distance "*a*–0" is the same as "*a*" and the distance "5–2" is the same as "3." On the other hand, students who did not use the magnitude interpretation did not readily recognize that "*a*"is the distance from 0 to *a*, nor readily invoke that it is the same as *a*–0.

We use the case of Todd (female student who chose her own pseudonym) to illustrate how the distinction in these interpretations became apparent to us. As context, we describe an earlier exchange between Todd and her classmate Walter that occurred with Task 2. Todd had asked whether the expressions were "additive" or "all just differences." Walter "assumed" they were all "differences," but was unclear about whether to interpret the "–" symbol as subtraction or a dash. She said,

> So I thought it was like above [Task 1] where we, I guess that's where I might have gone wrong up above…I assumed…that it was asking from 3.5 *to* 2, like the subtract was, or the dash was *to* instead of, or is it supposed to be a subtraction thing?

In this instance, Walter appears to be using an endpoints interpretation of the distance between 2 and 3.5 as "from 3.5 to 2" and is unsure about how to interpret the "–" symbol. This endpoints interpretation becomes relevant in Todd's later work on these tasks. While working on Task 5a, Todd was at first confused about the intention of the question, and thought x indicated a location at x=0 on the y-axis, the left endpoint of the horizontal segment. To clarify, Interviewer 1 explained that the point represented a generic point (x, y) on the line. This exchange follows:

Int 1: Is there a general way of saying how long that [green horizontal segment from the y-axis to a generic point on the line y=2x+1] is in terms of just the, the point itself which is (x, y)?

Todd: I feel like there is but I, I don't really have any ideas on that like I just maybe again it would just kind of <u>go back to the whole zero to the *x*-coordinate</u> like because <u>if I don't</u>

want to say exactly what length it is, I just go from like let's say every time, doesn't matter what length the segment...it just it stayed on the *y*-axis and it never went past it you know so it always in relation to the *x*-axis be at zero if I didn't want to say the length of the horizontal segment, I would simply just say from zero to *x* wherever that *x* and the (*x*, *y*) would be.

5. The graph of $y = 2x + 1$ is shown to the right.

a. Represent the length of the horizontal segment (green) in terms of *x*. (Your expression should have "*x*" in it).

b. Represent the length of the horizontal segment (green) in terms of *y*. (Your expression should have "*y*" in it).

c. Evaluate your expressions at the point where *x* equals 1, and confirm that you get the same result from a. and b.

d. Represent the length of the vertical segment (blue) in terms of *y*.

e. Represent the length of the vertical segment (blue) in terms of *x*.

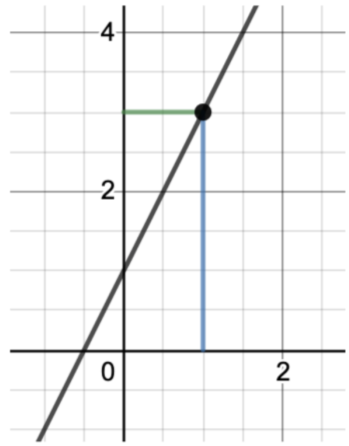


Figure 3. Task 5 of the HLT- represent the horizontal and vertical segments in terms of x and y

In this exchange, Todd first appeals to an endpoints interpretation, stating the endpoints of the segment rather than the length of that segment. We also infer that Todd recognized that the length of the segment was varying ("doesn't matter what length the segment is"), and that the location of *x* was varying ("wherever that *x*…would be"). However, Todd did not use a magnitude interpretation, and was even unable to express the distance as the difference *x*–0, in spite of completing the previous 4 tasks, with the help of prompting. When asked the same question with a numerical value, Todd was quick and confident to answer correctly:

Int 1: How about this: If you have a segment that goes from zero to two, how long is it?
Todd: It's two
Int 1: If you have a segment that goes from zero to five how long is it?
Todd: 5
Int 1: If you have a segment (Todd: Sorry) go ahead
Todd: I was just gonna say so for this one from zero to it's clearly one, it would be one.
Int 1: Yeah, and so if it goes from zero to *x* how long is that?
Todd: *x*
Int 1: That's it.
Todd: So it's just *x*? For [part] a?

At first glance, it might seem that the only obstacle in Todd's reasoning is in moving from a specific value (2 or 5) to an abstract value (*x*), or that Todd had difficulty conceiving of *x* as a varying value. However, on the graph, Todd is treating *x* as a varying *location*, an endpoint of a segment of varying length. It was step of associating "*x*" with the varying *length* of the segment that appeared to be the obstacle. Thus, we take this obstacle as more than an issue of moving between a numerical value and varying value in the symbolic register—this required a shift in how she was interpreting how variables were assigned in the graphical register. Because of this obstacle, Todd reverts to an endpoints interpretation rather than a magnitude interpretation.

Signifying a Variable Quantity in Multiple Ways

Another obstacle we noticed in two groups of students was in expressing a variable distance in another way (e.g., in terms of *y*, rather than *x*). This obstacle became apparent when students did not interpret a given equation of a function as relating distances within a graph; in other words, these students failed to make the "Cartesian connection" (Knuth, 2000) between the given equation and the graph. As an example, we continue to look at Todd's work on Task 5, part b. Todd explains her thoughts for this question, which asks her to represent the same segment's length, now in terms of *y*.

> *Todd*: Okay so this is where my head is going but …when I look at it in relation to numbers uh you know the blue line would be y right so that one's three *x* is one…I think if it is kind of like <u>*y* over 3</u> would represent the horizontal segment because it's *y* divided by three so three divided by three would be one and since *x* is one *y* over three…
>
> *Int 2*: Oh, it sounds like you're connecting in some way the one and the three or the *x* and the *y*, right? (Todd: yeah) Is there a way that *x* and *y* are connected that were told…?
>
> *Todd*: Well are you talking about like the, the linear function or are you I don't know um
>
> *Int 2*: So yeah, what does the linear function tell you about *x* and *y*?
>
> *Todd*: (*reading equation*) $y = 2x+1$, and then when I think about *x* and *y* in relation to the graph it tells you how it's going to look, so the slope would be 2, the *y*-intercept would be 1, yeah.

In this exchange, Todd recognizes the need to represent the horizontal distance from the *y*-axis to the graph in terms of *y*, and she attempts it with $y/3$. Todd uses the static distances of the point at (1,3) to devise this relation and appears to go through the process of how to operate on 3, a stand-in for *y*, to obtain 1, the stand-in for *x*. Todd knows what the task is asking of her, but does not use the given equation $y=2x+1$. When directed to the equation, she does not seem to think this gives her the relation she was searching for earlier with her conjecture of $y/3$. Instead, the equation represents instructions for the visual features of the graph ("how it's going to look") including values of the slope and *y*-intercept.

As the discussion continues, Todd does see the equation as useful (perhaps because she's been directed to) when asked to find lengths at specific non-variable values.

> *Int 2*: Would there be a way to for you to figure out what the horizontal length would be at that point where y would be 19?
>
> *Todd*: Yes
>
> *Int 2*: How would you do that?
>
> *Todd*: uh you would do the, basically find, I don't want to call it the inverse, but you would plug 19 in for *y* in the original equation and do $2x$ plus 1, subtract 1, to be 18 equals $2x$, so then your *x* would be 9, right? 18 over 2, 9? Yeah so, your horizontal segment would be 9 if your *y* segment was 19.
>
> *Int 2*: So if I gave you any *y*-value would you be able to figure out the horizontal length then?
>
> *Todd*: Yes, oh, okay, okay wait, wait, wait, wait uh actually I don't know if I'm going the right way but I'm gonna do it anyway two yeah okay so if I were to plug in 3 for *y* there *x*

would wind up being 1 which is what it is, I guess I just I'm getting kind of confused on how to include *y* in there what I just kind of would I uh *y* equals two *x* plus one could I do *y* minus one equals two *x* divided by two, so *y* minus one over two equals *x*, would that be the horizontal component trying to express like…?

Int 2: okay so would that work um that *y* minus one over two?

Todd: Oh my god, it would work, it would work, okay.

It is notable that Todd quickly and confidently answers the question of what the horizontal length would be when *y* is 19. When given a specific *y*-value that is not shown, she sees the equation as useful for finding the corresponding *x*-value, and she recognizes it also to be a length. But she had not immediately generalized this process to work for *y* as a variable length. She retraces the same operations she used on 19, this time using *y*, to get the equation's inverse relationship $x = \frac{y-1}{2}$, and is then confident in her response. We theorize that this general expression of the Cartesian connection was previously unavailable to her because it relies on interpreting *x* and *y* as variable magnitudes, an interpretation she has not previously developed.

Discussion

We reported two obstacles in coming to develop an ability to represent distance in the Cartesian coordinate system through examples from one student, Todd. Todd's endpoints interpretation of distance was an obstacle on Task 5 when using *x* to represent a varying length of the horizontal segment. Another obstacle for Todd was using the Cartesian connection to represent the same segment in terms of *y*. We conjecture that both of these obstacles are rooted in a lack of a composed magnitude interpretation of differences, yet more research is needed to investigate the relationship among these conceptions.

Further research is needed to better understand the development of the conception and representation of distance in the Cartesian plane to complete tasks like the ones in this study. For instance, future research may investigate how students come to develop a composed magnitude interpretation of differences. We intend to refine our tasks and continue to conduct teaching experiments to better characterize and support the cognitive processes involved in coming to understand a composed magnitude interpretation of differences.

We conclude by returning to the composed magnitude interpretation we described earlier as the foundational understanding involved in representing distances in the Cartesian plane. At the heart of the composed magnitude interpretation is the underlying notion that each value represented on a number line can be conceived of as a distance from 0. When we use a difference to represent a distance between two values, we are utilizing two distinct conceptions of a value *x*: (1) *x* is a location on a number line and (2) *x* gives the distance from the location on the number line called *x* and the origin. In practice, we as instructors may switch between these two conceptions seamlessly and without acknowledgment that they are fundamentally distinct conceptions. For instance, when we say that *b–a* gives the distance between *a* and *b*, we are treating *a* and *b* as locations, yet the reason that the difference gives that distance relies on unpacking *a* and *b* as also giving the distance from the origin to each of those locations. When we use *b–a* to represent this distance, we may take for granted the underlying mental actions involved with understanding why a difference yields this distance. The results of this study indicate that the process of developing a composed magnitude interpretation is non-trivial and should be considered central to the goals of teaching calculus conceptually.

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Introducing An RME-Based Task Sequence to Support the Guided Reinvention of Vector Spaces

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In this paper, we introduce an RME-based (Freudenthal, 1991) task sequence intended to support the guided reinvention of the linear algebra topic of vector spaces. We also share the results of a paired teaching experiment (Steffe & Thompson, 2000) with two students. The results show how students can leverage their work in the problem context to develop more general notions of Null Space. This work informs further revisions to the task statements for using these materials in a whole-class setting.

Keywords: Linear Algebra, Inquiry-Oriented, Vector Spaces, Realistic Mathematics Education

Linear Algebra is a critical course for many majors in Science, Technology, Engineering, and Mathematics: students learn important computational methods, as well as begin to work with mathematical definitions, proofs, and theory. A survey of research on the teaching and learning of linear algebra identified significant bodies of recent research in the areas of span, linear independence, basis, and eigentheory (Stewart et al., 2019). Notably less work has been done in the importantly related areas of vector spaces and subspaces such as null and column spaces.

Literature Review

Researchers who have written about student learning related to the topic of vector spaces and subspaces have highlighted the challenges inherent and somewhat unique to teaching the topic, which was "not introduced to solve a specific open problem but rather to solve different problems with the same tools in an economic formal way" (Grenier-Boley, 2014, p. 439). A central theme in this literature focuses on the importance of how students reason about sets, binary operations, closure, and linear combinations of vectors (Britton & Henderson, 2009; Maracci, 2008; Mutambara & Bansilal, 2018; Parraguez & Oktac, 2010; Wawro et al., 2011). This is perhaps unsurprising when one considers vector spaces to basically be sets of vectors that are closed under linear combinations. In our work, we follow the recommendations of the Linear Algebra Curriculum Study Groups in not taking on an abstract treatment of vector spaces as a focus for a topic in a first course, but rather focus on subspaces of $\mathbf{R}^n$ (Carlson et al., 1993; Stewart et al, 2021).

Our curricula are based on Realistic Mathematics Education (RME), a curriculum design theory that relies on design heuristics, specifically, didactical phenomenology, guided reinvention, and emergent models (Freudenthal, 1991; Gravemeijer, 1999; Van den Heuvel-Panhuizen, 2020). In this paper, we describe our development of a sequence of tasks designed to support students' reinvention of vector spaces and present results from a paired teaching experiment implementing those tasks. Consistent with the RME design heuristic of didactical phenomenology, during our development of the tasks, we identified key conceptual goals for supporting students' conceptualization of vector spaces: understanding vector spaces and subspaces as spans of sets of vectors and understanding vector spaces and subspaces as sets closed under linear combinations.

We also sought to identify an approach to these views of vector space and subspace that connected to students' prior experiences in the Inquiry-Oriented Linear Algebra [IOLA] materials. Accordingly, we identified null spaces and column spaces of linear transformations as appropriate entry points into reasoning about vector spaces and subspaces as sets closed under linear combinations (or, equivalently, as the span of a set of vectors).

With this in hand, our general strategy for these materials is to approach the content objectives of Subspaces of $\mathbf{R}^n$ via an exploration of null spaces and column spaces. In this unit, students' solutions to problems will benefit from the development of special sets of vectors (which correspond to null spaces), which we anticipate generalizing to the broader notion of subspaces as: (a) sets that are closed under scalar multiples and vector addition and (b) sets that can be written as the span of some subset of $\mathbf{R}^n$. Our development team identified a problem context and iteratively refined a task sequence to support core conceptualizations for null spaces that we saw as promising because: (a) it would extend closed loops reasoning about linear dependence from prior tasks for making sense of null spaces in a new context and (b) this reasoning might then be extended to identify generalized solutions for non-homogeneous systems within the problem context (affine spaces). Our team conjectured that the notion of closure under linear combinations and closed loop reasoning would be useful for students, but we were unsure if and how they would engage in that reasoning. Given this approach to the guided reinvention of vector spaces and our current iteration of the task sequence, we developed the following Research Question:

What meanings do students develop for null spaces and subspaces from their engagement in the task sequence that we designed?

Methods

This study was conducted as part of a broader NSF-funded grant focused on expanding research-based curricula for inquiry-oriented linear algebra. Our data is in the context of subspaces of $\mathbf{R}^n$, with particular emphasis on the idea of null spaces. Our paired teaching experiment (PTE; Steffe & Thompson, 2000), or experiment involving interviews with one teacher-researcher and two participants, was organized around a sequence of four central tasks. PTEs can be helpful in seeing how students learn and reason about a concept. The teacher-researcher's role is to elicit and test participants' ideas. This approach is also helpful in designing and refining tasks; in this case, the task sequence leverages hallway closures in a school to better understand subspaces (null spaces), which we will detail in a following section. PTEs can be useful regarding guided reinvention (e.g., Lockwood & Purdy, 2019; Swinyard, 2011), where the goal of our PTE was for students to reinvent ways of organizing or thinking about subspaces.

Participants, Data Sources, and Analysis Methods

The participants in this study were two white male undergraduate students (which we have given the pseudonyms Carson and Drew) at a predominantly minority public institution in the Southeastern United States. All students who completed this class were invited to participate. However, Carson and Drew were the only students to volunteer at the end of the semester who also met the age constraints of the IRB protocol. Both Carson and Drew had just successfully completed a semester of inquiry-oriented linear algebra that did not include explicit instruction about subspaces. The course content did include the topics of span, linear independence, matrices as linear transformations, composition and inverses, eigenvectors, and eigenvalues. The two students participated as a pair in four, 90-minute problem-solving sessions that took place across four different days within a one-week timeframe. The first author was the instructor for the course as well as the teacher-researcher for the interviews.

The sessions were conducted and video recorded via Microsoft Teams, with the interviewer screen sharing a presentation of problem statements, annotating student ideas on that shared screen, and students typing additional work, responses, and ideas into the chat or holding their written work up to their cameras. During each interview session, a second member of the research team was present to ask additional questions about the participants' thinking, to witness student work, and to provide additional insight into how to plan for subsequent interviews. After each interview, the teacher-researcher, the observing team member, and at least one other research team member met virtually to discuss the day's progress and revise the planned materials for the next interview session.

In addition to the audio-video recordings of the meetings and collection of participant work and responses, the team members kept concurrent notes of the interview sessions and recorded thoughts shared during the debriefing sessions after each interview. Through this process, we developed areas of focus for better understanding the participants' reasoning as they worked through each task in the interviews. We specifically identified one key construct that continually emerged throughout the participants' discussions. Based on these conversations, we identified instances of the participants using the construct throughout the recordings and documented the evolution of how the students leveraged the construct from each task to the next. The field notes, post-interview discussions, and recordings provide triangulation to support the construct's importance throughout the series of interviews.

Task Sequence

Drawing on the design heuristic of didactical phenomenology (Freudenthal, 1991; Van den Heuvel-Panhuizen, 2020), the research team worked to identify a context to draw out aspects of subspaces, especially considering students' anticipated mathematics at the point in the semester at which these materials are planned to be implemented. As an entire research team, we have organized the instructional units so that these tasks would occur after students have learned about linear in/dependence, span, solving systems of equations, and matrices as linear transformations, including composition and inverses of linear transformations. The authors of this paper developed the following task sequence based on the idea of One-Way Hallways, which we think provides an experientially real starting point that is consistent with directed graphs, or graphs made of edges and vertices where the edges have an associated direction to specific vertices (Figure 1). Students are first presented with a diagram of the west wing of Ida B. Wells High School, with an arrow drawn along each corridor and an explanation of the diagram.

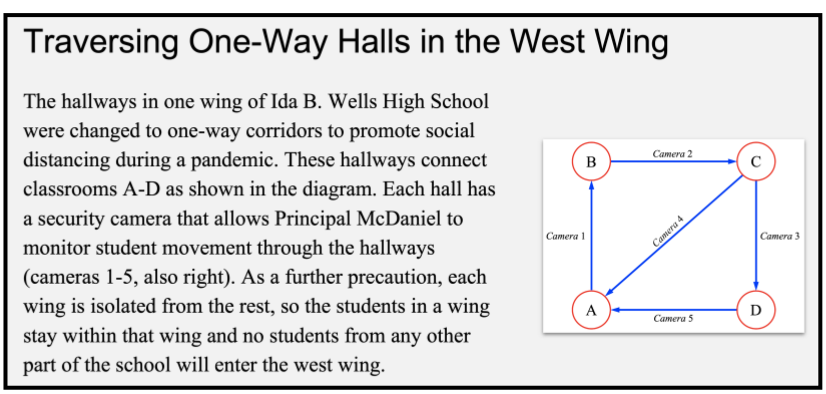

Figure 1. The setup for the first task in the sequence.

Task 1. The task sequence begins by focusing on one individual student's possible paths between two rooms and from one room, back to that same room. Students are asked to represent paths with column vectors that show how many times a student passed by a camera in each of the five hallways. So, for instance, the vector <1, 1, 0, 0, 0> represents a student passing by camera 1 and camera 2, but no other cameras. Students are first asked to identify all routes that a student could possibly take to start a journey in Room A and end the journey in Room C and write these possibilities as efficiently as possible. There are an infinite number of such paths if the student in the task repeats their trip down some of the hallways. For instance, the journeys described by the vectors <3,3,1,1,1> and <17,17,12,5,12> would also result in the student traveling from Room A to Room C. After this, students are asked to find routes that describe all journeys one student could take from Room C back to Room C while also considering the change in populations for each room. Students then are asked to consider both journeys (from A to C and from C to C) for 5 students traveling the hallways, once again also considering the population changes for each room. At the end of the task, students are prompted to consider the set of vectors they have developed for each of the four trips and any comparisons they can make between the trips.

Tasks 2. Task 2 is intended to extend the students' reasoning toward an understanding in which the hallway diagram encodes a mapping from "camera vectors" (5-tuples in which the k^{th} entry is the number of students who pass the k^{th} camera) to "classroom change vectors" (4-tuples, in which the first, second, third, and fourth entry is the change in student population for room A, B, C, and D, respectively). To promote this shift, we ask students to consider the effect that given "camera vectors" would have on classroom populations as well as identify possible "camera vectors" that would result in given changes in the four classrooms. This activity builds from the first activity by abstracting the vectors from being associated with any specific journey within the problem context and instead focuses on the input/output relationship between the camera vectors and classroom vectors. Specifically, Task 2 presents students with a room capacity constraint that requires none of the room populations change throughout the day. This anticipates a homogeneous system within this problem context.

Task 3. In Task 3, students are asked to develop a matrix that corresponds to the mapping defined by the hallway diagram and extend their reasoning about the input/output relationship by connecting it to their existing understanding of linear transformations. This task then asks students to reason about the set of vectors that result in no change in room populations as well as the set of all possible vectors that could describe changes in the room populations. In other words, this task leverages the problem context to prompt students to reason about the null space and column space of the linear transformation. Specifically, students are asked whether these sets are closed under scalar multiplication and whether they are closed under linear combinations. At this point, an instructor using these materials would define vector spaces and subspaces as sets of vectors that are closed under both operations and, equivalently, as sets of vectors that can be described as the span of some subset of the set.

Task 4. To generalize students' activity up to this point, Task 4 presents them with the matrix in Figure 2. This matrix represents students passing through hallways to a different set of classrooms in another wing of the school (the East Wing). Students are first asked to identify how many hallways and classrooms must be in the East Wing. Students are then prompted to figure out all possible hallway flows that would leave the populations unchanged, all possible hallway flows resulting in a particular vector, and all possible changes in population for each classroom in this wing. At the end of the task, students must decide if this latter set of vectors is a subspace of $\mathbf{R}^5$. The goal of the last part of this task is to formalize ideas in the context by having students think about them in terms of a matrix equation and set of vectors, while also linking these to the concept of subspaces.

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & -1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \end{pmatrix}$$

Figure 2. A matrix representing students passing through hallways of the East Wing of Ida B. Wells High School.

Results and Discussion

As conjectured, Carson and Drew developed robust reasoning during the initial task that extended into the later tasks. Specifically, the two participants identified closed loops within the hallway diagram as a helpful construct to make sense of multiple solutions within the problem context. The participants relied on this loop reasoning when discussing solutions to the homogeneous system as well as to non-homogeneous systems. Stated formally, this conceptualization of the problem allowed the participants to reason about the null space as well as an affine space that mapped to a non-zero vector.

When solving the Task 1, Carson and Drew took time to make sense of the given diagram and the vector <1,1,0,0,0>. The participants discussed how to interpret the vector as a journey through the hallways, eventually agreeing that the vector did represent a path from classroom A, down the hallway containing camera 1, past classroom B, and down the hallway containing camera 2 to arrive at classroom C. Following this discussion, the teacher-researcher asked the participants, “Can you think of another vector that might show a path from A to C?” Drew responded by saying,

> Well, the only other way you could get a path from A to C is if you go back to A from C and then you could just have like a loop going. So you could have it where it's like 221 - or no, 2201 [sic]. So it goes through that 4th camera goes through that 4th hallway diagonally and then back up and to the right and back to C.

The teacher-researcher asked Drew to clarify what he meant by the description, which prompted Drew to trace a triangle in the air while describing the journey, adding, “And then you can repeat that as many times as you want.” When asked to explain his last point, Drew said, “Yeah, so you could put a scalar before the vector and you can make it whatever you want and you would still get you to A or - from A to C every time.”

The teacher-researcher then asked the two participants to each write their understanding of what Drew had just described. Rather than writing anything, Carson stated that Drew was just describing going in a bunch of loops around A, B, and C. Then Carson said, “What's like, it's kind of like the homogeneous vector you go back to where you are. When we talk about linear transformations - that's basically what you're doing. You're going, you're go in an entire loop where you go somewhere, and then you go back.” This imagery references existing IOLA materials, specifically the “Getting Back Home” task (Unit 1 Task 3 of IOLA materials), which is reasonable considering that Drew and Carson had recently completed those materials.

At this point, the second researcher in the interview asked the participants to provide two different vectors that they though would be in that set. The teacher-researcher then elaborated on this by suggesting the participants provide generic vectors or other solutions. In response to this, Carson explained that he could see another path, but did not know how to describe that as a vector, saying, “You can go A, B, C, D and then back to A.” In this description, Carson is identifying a different loop than the one Drew suggested before. Meanwhile, Drew had typed into the meeting chat the vector expressions “x<1,1,1,0,1> + <1,1,0,0,0>” and “x<1,1,0,1,0> + <1,1,0,0,0>,” adding that his first expression represented the journey that Carson described. The two participants eventually determined that both loops could be used in combination to describe all possible journeys of one student traveling from room A to room C.

Throughout the remainder of the task and, indeed, the subsequent sessions of the teaching experiment, the participants continually referred to the loops as ways to generate additional solutions to problems in which they were asked to identify paths that would result in given room changes. This included later in the first interview when they were asked to find all possible journeys that would result in one student traveling from room C back to room C, five students traveling from room A to room C, and five students traveling from room C back to room C. This culminated in the development of four sets of solutions, which the participants compared. Drew and Carson correctly identified that the set of solutions for one student passing from Room C back to Room C was equivalent to the set of solutions for five students passing from Room C back to Room C. Similarly, the participants identified parallels for the affine sets they had found for the 1-person and 5-person journeys from A to C, noticing that the constant vector <1,1,0,0,0> would be scaled by 5 for the 5-person journey, but any combination of loops could be used.

During Tasks 2 and 3, the participants extended this loop reasoning to apply to any relationship between camera vectors and room change vectors. They also developed the appropriate matrix that is consistent with the vector mapping from $\mathbf{R}^5$ to $\mathbf{R}^4$ and explored the row-reduced echelon form of that matrix to further reason about solutions both for given camera vectors and for given room change vectors. We further identified an interesting result in response to Task 4 when the participants described and subsequently completed two different solution approaches to making sense of the East Wing matrix. Carson suggested row-reducing the matrix to identify solutions to the system. Drew, on the other hand, suggested constructing a map of the East Wing based on the given matrix and identifying loops in that diagram to find solutions to specific systems. Throughout the task sequence, the participants’ loop reasoning continually proved useful, including when considering new systems. In future implementations of the task, we anticipate further revising the materials to better support these conceptualizations. We are also transitioning to generating instructor support materials that will incorporate results from this PTEs as well as planned whole-class implementations.

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Incorporating Generalization into University Classrooms:
An Emerging Distinction from Instructors

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Abstract: In this paper we share findings from six interviews with instructors on whether, and how, they attend to generalization in their teaching. In particular, the interviews highlighted a distinction between generalizing in the classroom as being teacher or student generated. This study furthers our understanding of generalization beyond student activity and opens us to further questions such as how, if at all, students understand this distinction in the classroom.

Keywords: generalization, instructional practice, teaching

Introduction

Generalization is a foundational mathematical activity, and there is much agreement in the field that generalization is an important practice for students to learn (Davydov, 1972/1990, Kieran, 2007; Mason, Stacey, & Burton, 2010; Mata-Pereira & da Ponte, 2017). While studies have explored students' generalizing activity through interviews (e.g., Jones & Dorko, 2015) and small group teaching experiments (e.g., Reed & Lockwood (2021)), much less has been investigated about generalization occurring in classrooms. In particular, although generalization is clearly a fundamental and important mathematical practice, little is known about the extent to which generalization is explicitly or implicitly discussed, taught, or presented in university classrooms (if at all). The work presented in this paper is part of a larger study that seeks to explore both how generalization is currently presented in K-16 classrooms and how generalization might be effectively taught and incorporated into K-16 classrooms. We focus here on the postsecondary setting, examining ways in which university instructors may attend to and incorporate generalization into their teaching. In this paper, we share findings from interviews with six university instructors, in which they discussed the practice of generalization and ways in which they think about and incorporate generalization in their teaching practice. These interviews revealed important insights about the nature of generalization, highlighting different ways in which practicing instructors think about and treat generalization in their classrooms. Given that relatively little has been explored about the teaching of generalization, especially at the postsecondary level, our broad research goal is to engage in initial, exploratory work to investigate how generalization is actually presented and taught in postsecondary classrooms. With this broad goal in mind, in this paper, we seek to answer the following research questions:

1. To what extent do instructors incorporate generalization in their teaching at the university level? In particular, to what extent did the instructors we interviewed attend to generalizing in their classrooms?
2. What insights about the nature of generalization emerged from conversations with practicing postsecondary instructors?

Background Literature

Generalization is a fundamental mathematical practice, and researchers (Ellis, 2007; Lannin, 2005; Peirce, 1902; Rivera & Becker, 2007, 2008), curriculum designers (e.g., Hirsch et al., 2007; Lappan et al., 2006) and policymakers (Council of Chief State School Officers, 2010) emphasize generalization as important. However, even given its status as a valued mathematical practice, there is ample evidence that students face difficulties in engaging in generalizing; for example they are not always successful in stating and evaluating general statements or in recognizing and articulating patterns (Blanton & Kaput, 2002; Čadez & Kolar, 2015; English & Warren, 1995; Lannin, 2005; Mason, 1996; Moss, Beatty, McNab, & Eisenband, 2006; Orton & Orton, 1994; Rivera & Becker, 2008). Much work has been done in the field to help students develop a deeper understanding of generalization and to gain proficiency with engaging in generalizing activity. In some cases, such attempts have taken the form of conducting theoretical (e.g., Harel and Tall, 1991) and empirical (e.g., Ellis, 2007; Ellis et al., 2017) explorations of different ways of characterizing generalizing activity, or of designing domain-specific tasks to help students engage in rich generalizing activity (e.g., Reed & Lockwood (2021)).

Many of the studies that have explored students' generalizing activity have focused on students' generalizing in interview settings but not within classrooms themselves (e.g., Dorko & Jones, 2015; Ellis, 2007; Reed & Lockwood, 2021). Further, such work has concentrated on *students'* engagement and has not interrogated or investigated teachers' experiences with thinking about or promoting generalizations; neither has it explored occurrences of generalization within classrooms. A goal of this study, then, is to begin to examine instructors' perspective of generalization, particularly within the context of their teaching practice. Compared to the wealth of literature on students' generalizing activity, considerably fewer studies have explored generalization in teaching. Indeed, the relatively limited research on teachers' support of students' generalization suggests that it is challenging for teachers to fostering correct generalization among students (Callejo & Zapatera, 2017; Cockburn, 2012; Mouhayar & Jurdak, 2012). We thus see value in trying to better understand ways to support teachers in promoting generalization in their classrooms.

Theoretical Perspective – Characterizing Generalization

There are a number of ways in which generalization has been defined in the literature, and in this section, we clarify how we characterize generalization in this study. Generalization has been presented both from an individual, cognitive perspective (e.g., Carraher, Martinez, & Schliemann, 2008; Davydov, 1990; Harel & Tall, 1991; Kaput, 1999) and as a social construct that involves discourse and activity (e.g., Doerfler, 1991; Jurow, 2004; Latour, 1987). We broadly adopt this social perspective by framing generalization within contexts and interactions. We follow Ellis (2007) and Ellis et al. (2017) in characterizing generalization as involving any of the following actions: (a) identifying commonality across cases, (b) extending one's reasoning beyond the range in which it originated. and/or (c) deriving broader results from particular cases.

There are a number of ways in which researchers have distinguished between aspects of generalizing. For example, Harel and Tall (1991) differentiate between expansive, reconstructive, and disjunctive generalization, and Harel (2001) has offered distinctions between result pattern generalization and process pattern generalization. Ellis also created a generalization taxonomy to focus on different kinds of generalizing actions (2007). Others have also built upon Ellis' (2007) taxonomy to articulate different kinds of generalization actions that emerged for students in a variety of domains and age ranges, such as describing the *Relating-Forming-Extending (RFE) Framework* (Ellis et al., 2017), which distinguished three major categories of generalizing activity: Relating, forming, and extending. When we consider students' generalizing activity, we often view such activity in terms of these generalizing actions. We mention these constructs here to demonstrate the many ways in which researchers have attempted to characterize salient aspects of generalizing. Each of these distinctions and characterizations are designed to shed some light on a feature of generalization, be it ways in which students might engage in generalizing activity, or ways in which we as researchers can identify, understand, and interpret generalization activity. We seek to understand relevant distinctions among generalization that might inform our understanding of generalization in the undergraduate classroom in particular, and our work contributes to these existing distinctions in the literature.

Many of the perspectives shared to this point have focused on the generalizing actions of students. However, as we will see in this paper, the results from the interviews shed light on additional distinctions that we find noteworthy and that are related to classroom contexts. So, while we frame our work in existing literature and characterizations of generalization, our findings will reveal what we think are novel distinctions that our instructors attended to and viewed as relevant. These provide additional insight into how we might think about generalization broadly and how we might encourage students to engage in generalization.

Methods

Data Collection

We report on individual interviews conducted with university instructors. These interviews are the first phase in a larger project in which we aim to create a professional learning community among 3-4 instructors over the course of the academic year, engaging them in professional development about the nature and practice of generalization in their teaching. We sent recruitment emails to 15 instructors at a large public university, inviting them to participate in an interview. Six instructors responded, and we interviewed all of them. The participants represented a wide range of the department (information is found in Table 1). We conducted individual, ~90 minute interviews over Zoom with the participant and both authors of this paper.

Table 1. Pseudonym, position title, experience, and courses typically taught for each participant.

Pseudonym	Position Title	Years teaching	Courses typically taught
Participant A	Senior Instructor	9	calculus, differential equations, linear algebra
Participant B	Associate Professor	9	discrete math, advanced calculus
Participant C	Professor	32	upper-level geometry courses
Participant D	Senior Instructor	10	calculus, differential equations, discrete math, linear algebra
Participant E	Senior Instructor	9	calculus, differential equations, linear algebra
Participant F	Instructor	1	college algebra, pre-calculus, calculus

Our goal with these interviews was to get a sense of the instructors' approaches to teaching, to identify their initial views and definition of generalization, and to understand if and how they incorporated generalization into their teaching. To this end, we asked demographic questions, general questions about their teaching, their initial characterizations of generalization, and questions about teaching generalization, including whether and how they attended to generalization in their own classrooms. After they presented their own initial characterization, we provided them with our working characterization of generalization from Ellis (2007): "Generalization involves engaging in at least one of three activities: a) identifying commonality across cases, b) extending one's reasoning beyond the range in which it originated, or c) deriving broader results about new relationships from particular cases" (p. 444).

Data Analysis

The interviews were transcribed. The first author wrote summaries of each interview to get a better understanding of each participant's views of generalization in their teaching by identifying each of the participants' initial characterizations of generalization and comparing/contrasting them against each other. Using MAXQDA, she engaged in an open coding of the interviews for all the examples of generalization provided in the interviews (the final categories were: examples from courses taught, participant's own mathematical experiences, research experiences, and hypothetical examples) and she then used the coded sections to create a table of examples of generalization that the participants described. To identify themes and the results in this paper, both authors had ongoing conversations about what was appearing in the data through the participants' characterizations and examples of generalization, and together we engaged in initial open coding based on the different ways in which the participants described generalization in their classrooms. Several key distinctions emerged from this initial open coding, and from this we articulated the results given below, drawing on examples from the data, verifying our initial coding by repeatedly reviewing the transcripts, and having ongoing conversations about the examples given by instructors.

Results

Overall, the interviews yielded new insights about postsecondary instructors' views of and incorporation of generalization in their classrooms. We found that these instructors had a rich repository of characterizations and examples of generalization that went beyond our current understanding of generalization. We saw a variety of ways in which teachers engaged, or did not engage, in generalizing activity in the classroom, and they varied in the extent to which they had thought about generalization prior to the interviews. Indeed, it is noteworthy that some instructors were in fact quite attentive to generalization in their teaching, and they shared thoughtful insights about generalizing in their teaching.

Due to space, we can only present one key finding from these interviews. We describe one main distinction about generalization in the postsecondary classroom that emerged from the interviews, and we share examples and discuss it. This distinction, and the participants' discussion about it, informs our understanding of generalization in postsecondary classrooms, highlighting a variety of ways in which generalization might be incorporated and taught.

An Emerging Distinction between Generalizations in the Classroom

In analyzing interview data, a key distinction emerged about generalization in the classroom as to whether a generalization is instructor or student generated. As noted, many of the ways in which previous research has characterized generalization has been focused on students' generalizing actions – that is, how we might foster student engagement in particular generalizing actions such as relating, forming, and extending. While we maintain that such a focus on students' generalization is important, in our conversations with instructors we gained insights into broader ways in which they were conceiving of generalization, and we identified additional ways in which generalization might actually arise in classroom settings. We acknowledge that our current understanding of this distinction is not comprehensive (and we discuss limitations below). However, this distinction sheds light on nuanced ways in which generalization might emerge in a postsecondary classroom, and we found it compelling and enriching to consider.

As the instructors described generalization in their classrooms, it became clear that there was variety in the extent to which students themselves had opportunities to engage in generalizing activity. This distinction was most clearly articulated by Participant C. The following exchange shows Participant C initially describing a distinction he saw between two different ways in which generalization might be presented – whether he or the students were doing the generalizing. This exchange occurred toward the beginning of his interview when he was reflecting on characterizations of generalization:

> *Participant C:* … the tension is between generalization as a pedagogical strategy, which is what I think about most of what I do in the classroom to bring students to the stage of understanding more general idea framework context by starting with simpler such things. And there is separately teaching students to generalize themselves. In one case I'm doing the generalizing –
> *Int. 1:* Yeah.
> *Participant C:* – in order to give students something, to anchor their understanding of the more complicated scenario too. In the other case, you're trying to get students to recognize some new situation as a generalization of some old situation. Those are, those are at least subtly different because in one case I'm doing the generalizing for them and the other case they're doing it.

This underscores the idea that there may be two separate objectives for generalization in the classroom – one is to help students think about and handle general ideas or concepts, and the other is to teach students to generalize themselves. Participant C was clearly thoughtful in his own consideration of these different objectives, and he went on to say that in much of his teaching he tends to do more of the former and less of the latter.

This distinction was borne out in other interviews, too (not as explicitly, but participants did share whether they or their students engaged in generalizing activity). Overall, two participants said that they attempted to give students opportunities to engage in generalizing, two said that they as instructors generalized or discussed generalization but their students themselves did not generalize, and two indicated that they did not bring up generalizing in their classes at all. We now provide three examples from different participants (one that is instructor generated, one that is student generated, and one in which both parties engage in generalization) to highlight this distinction between instructor generated and student generated generalizations.

Example 1: instructor generated. The following example with Participant B demonstrates an instance of the instructor, and not the students, doing the generalizing activity. In this example, Participant B described a conversation she had with students during lecture about the nature of equivalence classes; that conversation prompted her to model generalizing by showing how a broader statement can sometimes be made when we notice a common feature of particular examples. As Participant C described, she was making a general connection for her students.

> *Participant B:* … Um, but it's sort of interesting because I, one other thing that sort of popped in my head was something a little bit different, um, where you can today, like today in [discrete mathematics], I, I, I was talking about equivalence classes, and I, I wrote down an example of an equivalence relation and then I, I s- sort of picked out some elements from the set and said, "Let's write down the equivalence class for this guy, and the equivalence class for this guy, and I went through a couple of them. Um, and then I said, you know, "Notice that if two of the elements of the set are equivalent to one another, then they have the equivalence classes." Um, and then I said, **"Oh, it turns out that this is always true," and I wrote down the statement of the theorem** and I, I sort of s- we didn't really manage to get through the proof, but we started talking about the proof a little bit**. Um, so like I mean, I guess that in some sense is generalization, right, because you're starting with sort of a specific example where a certain thing is true, but then you're immediately jumping to, obviously, this broad class of equivalence relations**. Um, so, so, in that case, the generalization kind of happened in an instant.

Example 2: student generated. As an example of student generated generalization, we share a case where Participant D described intentionally giving students chances to generalize. In this example, he discussed scaffolding tasks in which students might relate (in the sense of Ellis et al.'s (2017) framework) other proofs of irrationality to their initial experience with the square root of 2, ultimately extending ideas to more general cases. Here, we interpret that the instructor was attempting, through a choice of tasks, to provide opportunities for students to engage in generalizing activity and to formulate generalizations for themselves.

> *Participant D:* So for, for another example, in my eCampus course, um, I have the students do discussion board posts with each other where they're kind of required to work with each other on some exercises. And I think like in one of my videos, I, I go through the classic proof that the square root of two is irrational by contradiction. And some of the exercises I'll put on the activity are like, okay, can you prove the square root of three is irrational? Can you prove the square root of five is rational? **Can you, can you generalize this to proving other things are irrational?** This is as a, as something where, okay, I've, I've done the exercise for square root of two. Can you now see that this, this, this, these ideas generally apply to other, other proofs of irrationality of square roots?

Example 3: instructor and student generated. Finally, we offer an example in which both instructors and students together contribute to a generalization in the classroom. Participant A described his idea of "ungeneralizing" a math concept to a more concrete scenario that students could then use to help rebuild the more general concept themselves. In this case, we see the instructor's generalization served to support and direct student generated generalizations. In particular, he worked to find relevant examples and analogies, with the hope that students could then generalize from them. He also commented that up until the interviews with us, he had been thinking of these as analogies as opposed to instances of generalization, but during the interview he realized that generalization more accurately described his teaching intentions.

> *Participant A:* Well, it's, i-it, this is like hard 'cause I feel like what my, **part of my role as the teacher is, is, is to, is to take the abstract idea and provide specific examples to, that I choose carefully so that the students can then generalize from them**, to like to gain that deeper, intuitive understanding of the abstract concept through my ungeneralizing it.
> *Int. 1:* Mmm. So like, you're doing the ungeneralizing for them, is what you're saying?
> *Participant A:* So I think about – so that they can regeneralize themselves.
> *Int. 1:* That's interesting.
> *Participant A:* So thinking about like some multivariable calculus, I think is a great subject for this conversation because there's almost no new math to learn. There's just those concepts you've already known, but now applied in 3D.
> *Int. 1:* Mmm.
> *Participant A:* And so, so much of the class is like, remember how it used to work? Now it works this way and here are the differences. And so there's great ways to, (laughs) at least I think there's great ways of ungeneralizing the idea of what is a parameter in a vector valued function, and how does, what does it mean for a parameter to represent arc length versus represent time. And ungeneralizing that into different ways that different types of vehicles keep track of how, of their age. So if you look at like a car like you and I own, it has an odometer and its age is how many miles it's driven. But if you go to a construction site and you look at like an excavator, it doesn't have an odometer, it has an hours. How many hours has this machine been operated? And so there's these anti-generalizations, these analogies that I try to use so that the students can understand the abstract concepts so that they can begin to generalize.

While these are just brief examples of different ways that generalization might arise in postsecondary classrooms, we think that they highlight potential ways in which students may gain access to ideas related to generalization, even if instructors are making or describing generalizations. This distinction highlights that generalization might arise in a variety of ways – it does not only have to involve students themselves generalizing, but it might also involve instructors making clear attempts to explain general concepts, make connections, or generalize ideas for students. This is particularly noteworthy for some of the specific mathematical content that these instructors teach, which involves abstract concepts.

We want to emphasize that when instructors (and not students) generalized, this did not always suggest that instructors did not think it was worthwhile for students to generalize. Rather, they attempted to do some of the work with generalizing for interesting and nuanced reasons. Participant A's use of analogies involved him making connections, but it was for the purpose of setting up students to be able to engage in generalizing activity. As another example, Participant C gave an example of wanting their students to not overgeneralize in geometry, and he explained that focusing on 3 dimensions was a worthwhile use of their class time (and not, say, 17 dimensions, which would be a valid generalization but not one he wanted students to explore given the scope of his class). Thus, while he was supportive of the idea of generalizing and saw its value for students, in teaching that course, he was more inclined to direct students' generalizing so they would not get too far afield in their explorations.

As a final note, two of the instructors had not thought about whether or not they or their students had opportunities to generalize in class, but they were reflective about it once we raised the idea in the interviews. For example, Instructor F was able to reimagine the way she taught optimization problems to her calculus students as engaging in generalization of a procedure from a set of examples, and she noted that thinking of this activity in terms of generalization gave her new insight into how to explain optimization to her students.

Discussion and Limitations

We believe our initial instructor interviews raise questions for subsequent research about the effectiveness of student versus instructor generated generalizations in the classroom. That is, while it is clearly desirable to have students engage with generalizing activity, this distinction suggests opportunity for explorations into what (if anything) students gain from instructor's attention to generalizing. Based on prior work about students' experiences in lectures (e.g., Lew et al., 2016), it is likely that students may not necessarily pick up on points that instructors are trying to make about generalization. However, it is worth investigating whether there might be any benefit to instructors making explicit comments about generalization in their postsecondary classrooms, particularly as students encounter advanced, abstract concepts. Further, from a professional development perspective, it is worth investigating productive ways to engage instructors with exploring ways to incorporate generalization in their classrooms.

One limitation of this work is our reliance on instructor reports on their teaching, rather than observing their teaching (this is due in part to the pandemic and remote instruction); it would be good to observe teachers to see how generalization occurs in their classes. However, as we have seen, we contend that their self-reported data still offers useful insights in thinking about the nature of generalization and how it might be incorporated into postsecondary classrooms. We also acknowledge that we only interviewed six instructors, and more distinctions and nuances may arise by interviewing or observing more university instructors.

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Sameness Connections across Mathematics

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Helping students see conceptual connections between content areas is important for the development of flexible understanding, yet research on ways in which notions of sameness throughout math can be connected is limited. This study examines survey responses from mathematicians on the relevance of sameness to topics in abstract algebra and connections between sameness in algebra and other courses. Common connections are highlighted as are themes in the types of connections provided.

Keywords: Isomorphism, Homomorphism, Abstract Algebra, Instructional Practice, Sameness

A common motivation behind initiatives in math education is to see how content can be made relevant to students, especially pre-service teachers. Recent initiatives in undergraduate mathematics have largely approached this task by connecting K-12 content to advanced courses like analysis (e.g., Wasserman et al., 2018) and abstract algebra (e.g., Suominen, 2018) or by examining defined mathematical topics such as functions (e.g., Melhuish & Fagan, 2018; Melhuish et al., 2020) and binary operations (e.g., Melhuish et al., 2020) at various levels. In this paper, we take a slightly different approach by considering how an undefined concept, sameness, appears across math. Specifically, we address the following research question: What connections do algebraists see between sameness in abstract algebra courses and other math content?

Literature Review and Theoretical Perspectives

From the beginning of students' mathematical experiences, they are introduced to the notion of mathematical equality. Students are directed to attend to equal counts of objects already in Kindergarten (National Governors Association Center for Best Practices, 2010, p. 11) and continue utilizing notions of equality throughout their mathematical careers. Despite the centrality of equality to mathematics, students do not always have a conceptual understanding of equality. Multiple studies have examined students' conceptions of equality and have connected students' modes of understanding to their ability to do algebra (e.g., Alibali et al., 2007; Kieran, 1981). Specifically, students' ability to manipulate expressions while maintaining equality is aided by understanding equations as two expressions that relate to each other in a specific way instead of viewing the equal sign as an indication to compute something (Alibali et al., 2007). As students mature mathematically, notions of sameness are expanded from equality to congruence in geometry and verifying equivalence in trigonometric expressions (e.g., $\sin^2\theta + \cos^2\theta = 1$).

As students reach college mathematics, more complex ideas of equivalence arise such as isomorphism and homomorphism in abstract algebra. While some research has examined students' methods of verifying isomorphism (e.g., Dubinsky et al., 1994; Leron et al., 1995; Melhuish, 2018), recent work has examined students' understanding of isomorphism and homomorphism in other ways. For example, Hausberger (2017) considered how students understand "structure-preservation" of homomorphisms and other authors have examined isomorphism and homomorphism through the lens of conceptual metaphors (e.g., Melhuish et al., 2020; Rupnow, 2017; 2019).

However, work on mathematicians' views of isomorphism and homomorphism remains limited. Weber and Alcock (2004) highlighted algebraists' notions of isomorphism as meaning groups were “essentially the same” (p. 218) or that groups being isomorphic meant “one group was simply a re-labelling of the other group” (p. 218). When speaking with algebra instructors, we have found four major categories of metaphors used to describe isomorphism or homomorphism: mappings, sameness, sameness/mapping, and formal definition (Rupnow, 2021). Nevertheless, the prevalence and extent to which algebraists connect isomorphism or homomorphism to other notions of sameness in math and how such connections are made in instruction are unknown.

A pair of theoretical lenses for examining how knowledge in different contexts might be connected comes from Lobato (2012). In her paper, Lobato contrasts the mainstream *cognitive perspective on transfer* with *actor-oriented transfer* (AOT) and highlights places in which each can be effective. The mainstream perspective characterizes transfer as “how knowledge acquired from one task or situation can be applied to a different one” (Nokes, 2009, p. 2) and generally takes an observer’s perspective: an outside party determines whether or not transfer of some “desired” knowledge occurred. In contrast, AOT characterizes transfer as “the generalization of learning, which can be understood as the influence of a learner’s prior activities on her activity in novel situations” (Lobato, 2012, p. 233). Thus, AOT instead takes an actor’s perspective, where any expansion in the use of knowledge is viewed as transfer, even if the “desired” knowledge was not what transferred. Both of these perspectives can be useful in examining which concepts of sameness mathematicians transfer from lower-level classes through abstract algebra.

Methods

Data was collected via a survey that was sent to every 4-year college or university math department that offers abstract algebra in the United States. This survey addressed how algebraists think about sameness in general and in specific mathematical contexts. Participants were 197 mathematicians from 173 institutions who had taught at least one abstract algebra or category theory course in the last five years. For this paper, only the following three open-ended questions regarding sameness connections will be discussed:

1. Which abstract algebra topics lend themselves to deepening students’ understanding of mathematical sameness?
2. What connections do you see between sameness in abstract algebra and in prior math courses?
3. What sameness connections between abstract algebra and other courses do you (or could you) help students make when teaching?

These were the last non-demographic questions asked in the survey. Prior questions had asked about participants’ broad views of sameness in math as well as their specific views about isomorphism and homomorphism. We noted that many participants answered questions 2 and 3 similarly or referenced their answer to question 2 in their answer to question 3. For this reason, these two questions were grouped together for analysis.

Two independent coders used thematic analysis (Braun & Clarke, 2006) which included multiple iterations of coding (Anfara et al., 2002). First, responses were open-coded using descriptive coding (Saldaña, 2016) for words indicating any connection to sameness. For questions 2 and 3, an effort was made to also code the particular course in which the sameness connection was made. Responses could receive multiple codes. Any discrepancies between the two coders were discussed before coming to an agreement. Second, codes were organized according to theme. Third, responses coded within a given theme were examined as a group for consistency within the theme and to determine trends in general ideas captured by particular codes. Finally, the above steps were repeated to refine the codes and assure consistency.

Results

Connections to Sameness in Abstract Algebra

This section involves codes related to the following survey question: "Which abstract algebra topics lend themselves to deepening students' understanding of mathematical sameness?" The list of codes and number of participants who were coded in such a way are presented in Table 1.

Table 1: Connections to Sameness in Abstract Algebra

Category	Code	Participants
Equivalence Relations	Equivalence relation	29
	Bijection	7
	Isomorphism	113
	Automorphism	3
	Homomorphism	47
	Quotients	37
	Modular Arithmetic	13
	Group Actions	5
	Fundamental Theorems	23
Organization/Schema	Category Theory	9
	Classification Theorems	20
Broad Statements	No connections	5
	Everything	13

Equivalence relations. The majority of participants answered this question by referencing an equivalence relation or sets of equivalence classes. Several mentioned *equivalence relations* explicitly, e.g., "Equivalence classes. They are the essence of elemental sameness. Once [students] understand that different elements can be the same (under the right lens), then it comes naturally to view structures as being the same." Codes related to examples of equivalence relations were broken into three subcategories: renaming, structure loss, and relating the two.

Renaming. This subcategory involves codes related to a simple renaming of a structure: *bijection* (renaming sets), *isomorphism*, and *automorphism* (renaming algebraic structures). Although there is some debate in the literature about the level of equivalence implied by a renaming (Mirin, 2019), renaming (and its cousin relabeling) have previously been documented as ways mathematicians describe isomorphism (Rupnow, 2021; Weber & Alcock, 2004).

The most common code to appear in answers to this question was *isomorphism*, being mentioned by 113 participants (57%). Some suggested that isomorphism and sameness were linguistically interchangeable in the context of abstract algebra, e.g. "I want my students to think of 'the same' as 'isomorphic.'" Indeed, many highlighted its centrality to the study of algebra:

> Arguably, the concept of isomorphism is central to everything in the course. The whole point of teaching it to undergraduates is to help them see that the numbers and operations they grew up with are actually instantiations of more universal properties of a wide range of mathematical objects...that some of their intuition about the way numbers work translates into theorems about other cool objects.

Structure loss. This subcategory involves codes related to sameness that may involve some structure loss: *homomorphism*, *quotients*, *modular arithmetic*, and *group actions*. For example, many participants mentioned *homomorphisms* to refer to sameness of parts of structures. Often, participants noted that the sameness here lies in the relationship to quotients: "Homomorphism is useful in that it says something about being structure-preserving and you can find sameness in the fibers." In fact, one participant discussed the idea of *quotients* as an opportunity to explore and make precise various notions of sameness:

> In examining the fundamental set theoretic construction of passing from a set with an equivalence relation to the quotient set (set of equivalence classes) we have occasion to pay attention to different but related concepts: "two elements of X are the same", "two elements of X are equivalent", "two equivalence classes in X are equal." This should help the students to understand that there are different concepts related to "sameness" that need to be understood precisely.

***Relating the two*.** This subcategory includes the code *fundamental theorems*, which refers to what are commonly called "the isomorphism theorems" and/or "the fundamental homomorphism theorem." This family of theorems provides a bridge between two types of sameness discussed above: isomorphism and homomorphism. The theorems involve using a homomorphism between groups to create an isomorphism between related groups and using isomorphisms to reinterpret quotients, thereby permitting both loss of some structure and a renaming of subparts to occur. For example, one participant mentioned "the idea of the first isomorphism theorem, that [homomorphisms] are the same concept as quotient groups."

Organization/schema. Codes in this category give a way of classifying ideas about sameness in abstract algebra. For example, *category theory* provides general language with which to talk about structures and morphisms beyond just abstract algebra and can be used to define sameness more generally: "I introduce a little bit of category theory… to emphasize the 'context independent' way to say when two objects are the same." The code *classification theorems* is also categorized here because these types of theorems give a way to organize algebraic structures according to isomorphism classes or other broader classes.

Broad statements. A small number of participants were coded as *no connections* when they indicated that the idea of sameness was not well-defined enough for them to identify it with concepts from abstract algebra, e.g., "There is no such thing as mathematical sameness outside of specific context." In contrast, others were given the code *everything* when they indicated that all of the concepts in abstract algebra are related to sameness in some way, e.g., "All of them, in a sense. In all of algebra, the point is to distinguish between features that matter and features that don't." This idea of sameness being dependent on context was something that was frequently mentioned in answers to the questions featured in the next section.

Connections to Sameness in Other Courses

This section involves codes based on answers to the following survey questions: (1) "What connections do you see between sameness in abstract algebra and in prior math courses?" (2) "What sameness connections between abstract algebra and other courses do you (or could you) help students make when teaching?" The list of codes and number of participants who were coded in such a way are presented in Table 2.

Equivalence relation. The explicit mention of *equivalence relations* appeared more often in reference to other math courses than in answers to the abstract algebra specific question. This is likely because, as some participants pointed out, equivalence relations are often students' first experience with sameness beyond literal equality:

> The best reference point I can think of that most students will have had is the idea of an equivalence relation - we sometimes use an equivalence relation to regard two objects as the "same" (or related) even if they are not precisely the same object in the domain of the

relation. But abstract algebra takes that view of objects and applies it to entire structures, which themselves are composed of many objects.

Codes for specific types of equivalence relations are split up into the two subcategories below, which are analogous to the renaming and structure loss subcategories from the previous section.

Table 2: Connections to Sameness in Other Courses

Category	Code	Participants
Equivalence Relations	Equivalence relation	49
	Bijection	26
	Identities	4
	Equality	14
	Arithmetic—representations of numbers	21
	Geometry—congruence/isometry	31
	High school algebra—algebraic manipulation	5
	High school algebra—log/exponential function	10
	Calculus—limit	7
	Calculus—change of variables	3
	Calculus—coordinates	2
	Linear algebra—change of basis	7
	Linear algebra—isomorphism	26
	Graph theory—isomorphism	9
	Differential geometry—diffeomorphism	6
	Topology—homeomorphism	20
	Functions--one-to-one/onto	6
	Modular arithmetic	23
	Geometry—similarity	17
	High school algebra—graph transformations	3
	Calculus—derivatives and antiderivatives	18
	Differential equations—solving differential equations	4
	Linear algebra—transformations/homomorphisms	25
	Linear algebra—similar matrices	7
	Linear algebra—RREF	4
	Topology—continuous maps	3
	Analysis—almost everywhere	3
Language/ Organization/Schema	Category theory	13
	Proof techniques	7
Broad Statements	No connections	10
	Deepen understanding of sameness	53

Renaming/change of perspective. This subcategory involves codes related to a renaming of an object or a change of perspective that does not change the object itself. Some codes included here are direct analogs to abstract algebra's isomorphism that appear in the undergraduate curriculum, such as *linear algebra—isomorphism*, *graph theory—isomorphism*, or *topology—homeomorphism*. These are all fairly obvious connections to make, though some pointed out that the notions of isomorphism that students tend to see prior to abstract algebra are much more straightforward. For example, one participant noted: “Sometimes there is a graph theory unit in a discrete mathematics course, but somehow that’s still too intuitive. (Isomorphic graphs don’t really look like different objects...someone just drew them a little differently.)”

Others reached further back in the curriculum and included examples from high school (e.g., *geometry—congruence/isometry*) or elementary school (e.g., *equality, identities*). An interesting connection that many made was to *arithmetic—representations of numbers,* where participants pointed out that equivalent fractions can be reinterpreted as sets of equivalence classes. Still others made connections to renamings that provide a clear change of perspective or additional insight, such as *linear algebra—change of basis*: “For example, most students are familiar with change of basis before taking abstract algebra, and understand that using different coordinates (in calculus also) makes the problem look different, but is mathematically equivalent.”

Structure loss/change of object. This subcategory involves codes related to sameness that may involve some structure loss. For example, many made the connection to *linear algebra—transformations/homomorphisms*, which are a direct analog of homomorphisms in abstract algebra. Others mentioned *modular arithmetic* as one of the first places where students see more abstract notions of sameness and suggested it as a starting point for equivalence classes and quotients: “Modular arithmetic is really ‘the same’ as arithmetic on quotients (groups, rings, etc), so whenever we have a quotient, we should be able to define a modular arithmetic.” One interesting connection was to *calculus—derivatives and antiderivatives,* where participants noted an antiderivative is actually an equivalence class of functions that all have the same derivative.

Language/organization/schema. Codes in this category give a way of organizing and thinking about ideas of sameness within and across disciplines, such as *category theory*. One participant noted that “most (possibly not all) [connections to sameness] are captured by morphism-related notions in a suitable category.” The code *proof techniques* is also categorized here because the language and techniques used to write mathematical proofs provide ways to think about mathematical objects across disciplines.

Broad statements. A large number of participants (53 participants, 27%) indicated that abstract algebra is an opportunity for deepening understanding of sameness, although many did not explicitly state that they make attempts to deepen this understanding with their students. Some participants coded here said that the ideas of sameness in abstract algebra generalized or made more precise earlier concepts of sameness, e.g., “I think that students have to take a somewhat more liberal view of ‘sameness’ in abstract algebra than in most prior math courses.” Others stated that the idea of sameness is a thread that shows up throughout math, only with changes to what count as relevant characteristics:

> Emphasizing that “sameness” is context-dependent: what is and what is not “the same” depends on what we consider to be important for our purposes. I encourage [students] to keep an eye out for this in other courses and contexts, and to try to figure out what [are] the “essential characteristics” that are important in any given subject.

Again, some participants indicated that they do not focus on sameness or are not sure what connections they can make to sameness. Others indicated that they believe that abstract algebra is students’ first encounter with the idea of sameness, saying things like “I don’t think we have previously covered sameness in a way that would shed light on the topic.”

Discussion

Participants tended to equate sameness with concepts based on equivalence relations. Isomorphism was the most frequently mentioned abstract algebra concept that was linked to sameness, and homomorphism was a distant second. While this may be partially explained by prior questions in the survey directly asking about isomorphism and homomorphism, the fact that so many participants voluntarily cited them again and cited related concepts in other subjects suggests some importance placed on isomorphism and, to a lesser extent, homomorphism.

Codes classified as renaming, including *geometry—congruence/isometry*, *linear algebra—isomorphism*, *bijection*, and *arithmetic—representations of numbers*, were the most common connections to other courses. While it is not surprising that individuals would highlight analogues of isomorphism, especially in another form of algebra, the variety of specific types of renamings and the number of connections to K-12 topics were surprising.

Codes in the structure loss subcategory were mentioned less often than those in the renaming subcategory. This is likely because codes there refer to similar objects that may have some properties or partial structure in common, but do not share as many properties or structure as the renaming examples. These included connections to transformations in linear algebra, modular arithmetic (number theory/discrete math) or antiderivatives (calculus).

A few mathematicians resisted making sameness connections, suggesting that isomorphism was a “richer” type of sameness than students experienced previously. However, this opinion was expressed by a clear minority of respondents. In contrast, roughly a quarter of respondents viewed abstract algebra as deepening students’ understanding of sameness and made connections to a variety of prior math content, suggesting a view that sameness permeates mathematics.

While some participants only highlighted “obvious” examples of sameness, many made connections across multiple branches and/or levels of mathematics, suggesting sameness is relevant to much of the math curriculum. This highlights the importance of students transferring prior knowledge and experiences with sameness to the new contexts of isomorphism and homomorphism in abstract algebra (e.g., Lobato, 2012). However, the extent to which students transfer this knowledge independently or based on explicit classroom experiences remains an open question. Prior research in analysis (e.g., Wasserman et al., 2018) suggests that pre-service teachers are more likely to connect fully formed explanations to future teaching than general concepts. However, they also acknowledge that their pre-service teachers may not have “developed the web of connections to appreciate the conceptual insights that the explanations in our study (and perhaps real analysis in general) may provide” (p. 87). We suggest that taking a larger grain size, such as examining sameness across mathematics, may provide a way to create the “web of connections” necessary to support students’ transfer of big ideas across courses.

Conclusions and Future Work

In this study, we used actor-oriented transfer (AOT) to examine what mathematicians transferred across contexts with respect to sameness (Lobato, 2012). In accordance with mainstream transfer (Lobato, 2012), we also laid a foundation for future research by soliciting mathematicians’ views of what “should” transfer across contexts. Specifically, future research can examine what notions of sameness students are already transferring independently or based on instruction. Furthermore, some mathematicians did not claim to explicitly connect notions of sameness in abstract algebra while others did. Future research could examine how connections are made in class and see if some methods are more effective for encouraging transfer than others. In so doing, we hope to help math majors make meaningful connections across their studies and better prepare students to apply their knowledge flexibly in their future careers.

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Students' Examining and Making Sense of Proofs by Mathematical Induction

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Students have demonstrated difficulties in adopting the various techniques of proof in their upper-level mathematics coursework. One of these techniques of proof which students struggle with is mathematical induction. In this paper, we present an analysis of three groups of students in an Introduction to Proofs course as they made sense of and analyzed two sample induction arguments. To aid in our analysis, we utilized Stylianides' (2007) definition of proof with three components (accepted statements, modes of argumentation, modes of argument representation) to describe the components of proof students focus upon when they encounter induction arguments for the first time in an Introduction to Proof course.

Keywords: Mathematical Induction, Intro to Proof, Proof, Proving

While the transition to formal proof is difficult for many (Moore, 1994; Harel & Sowder, 2007; Stylianides et al., 2017), students have shown that there are unique challenges to adopting individual techniques of proof (e.g., direct proof, contradiction). One of these techniques of proof which can be particularly troublesome for students is mathematical induction. There are a few potential reasons for students' trouble. First, induction relies on the *well-ordering principle*, which has puzzled mathematicians into the 20th century (Hellman, 2010). Second, and potentially more importantly to novices of proof writing, induction uses the conditional statement in unique ways relative to other techniques of proof (See Table 1), with an embedded conditional as part of the proof structure (i.e., the induction step) and with the notion that proof by induction is utilized when proving an infinite sequence of conditional statements. This, combined with induction's necessary reliance on a particular example (i.e., the base case), has the potential to leave students confused about both the form and function (Stylianides et al., 2007; 2016) of proofs by mathematical induction.

Table 1
Conditional Use Per Proof Technique

Proof Technique	Conditional Use
Direct Proof	$P \rightarrow Q$
Proof by Contraposition	$\sim Q \rightarrow \sim P$
Proof by Contradiction	$(P \text{ and } \sim Q) \rightarrow F$
Mathematical Induction	$[P(1)]$ and $[P(n) \rightarrow P(n+1)]$
Disproof by Counterexample	$T \rightarrow F$

Studies on students' use and understanding of mathematical induction are limited. Students have expressed that proofs by mathematical induction lack explanatory power (Stylianides et al., 2007), although Stylianides and colleagues (2016) found that students' example use promoted such understanding when proving conjectures with mathematical induction. In regards to the components of mathematical induction, students have demonstrated a lack of understanding about the necessity of the base case (Dubinsky, 1986; 1990; Stylianides et al., 2007), and use circular reasoning when describing the inductive hypothesis (Harel, 2001; Palla et al., 2012). Stylianides and colleagues (2007) surveyed over 100 preservice elementary and secondary mathematics teachers and analyzed their response to two sample induction arguments as to whether or not the proofs were mathematically valid. In their conclusion, the authors suggest that the two tasks that they used in their survey could also potentially be used within classroom contexts and that future research would benefit from investigating such in-class experiences with the tasks. We continue their line of research by investigating small-groups of students in an Introduction to Proof course as they work together to evaluate two sample induction arguments. Our goal is to understand, within a natural classroom setting, how students make decisions about mathematical induction. In particular, we seek answers to the following research question: When students are asked to evaluate sample proofs by mathematical induction, what do they base their decisions upon with respect to whether the induction argument is valid?

Theoretical & Analytic Framework

In order to frame the ways in which students evaluate and make sense of induction proofs, we utilized Stylianides' (2007) description of mathematical proof, which is parsed into three categories: (i) Set of Accepted Statements; (ii) Modes of Argumentation; and (iii) Modes of Argument Representation (p. 292). Our adaptation is summarized in Table 2 below.

Table 2

Proof Component Framework and Description (Stylianides, 2007, p. 292)

	Stylianides Definition	Example Quotes: Group Proving
Set of Accepted Statements (S)	● Definitions, axioms, theorems ● Prior proven results	Billy: "I think it's important to look at the base case. So we should check to make sure it works for N equals one."
Modes of Argumentation (A)	● Modus ponens/ Modus tollens ● Listing of all possible cases (if finite), or methods like contradiction, contrapositive, induction ● Logical derivations of statements	Zoren: "I'm saying you move to K plus one, the left side is multiplied by something, K plus one has to be greater than five. I mean, that's the left side, the right side. You're just multiplying it by two."
Modes of Argument Representation (R)	● Language to describe arguments ● Physical representation of arguments (i.e. pictorial or symbolic representation)	Billy: "It looks like a good proof. It does look like it works out due to the algebra. But they didn't check the very starting case. "

We used this framework as a way to categorize the statements made by students as they worked in small groups to evaluate sample induction arguments. Our coding allowed us to determine if students were basing their decisions upon accepted statements, modes of argumentation, and/or modes of argument representation. Through categorizing student responses in this way, we hoped to better understand what students find most relevant in their process of evaluating sample induction arguments.

Methodology

Task & Participants

The data for this study came from the 1st Author's (in progress) dissertation data on undergraduate students in an Introduction to Proofs course as they completed small-group proving tasks. The study took place at a large, public university, in the southeast United States. Of the 11 students in the study, 2 were women and the remaining 9 were men, and all students were either majoring or minoring in mathematics. The small-group proving task used in this study was modified from Stylianides et al. (2007) study on preservice teachers' understanding of proofs by mathematical induction. The task (replicated in Figure 1) prompted students to discuss two sample induction arguments and to explain why or why not the given arguments prove the stated claims. The task differs from Stylianides et al. (2007) in that the students were not asked to choose from a list of predetermined options on the proof's validity.

Figure 1

Sample Induction Arguments

For every $n \in \mathbb{N}$ the following is true: $1+3+5+\ldots+(2n-1)=n^2+3$

Proof:

First, I assume that the result is true for n=k.

This means that, $1+3+5+\ldots+(2k-1)=k^2+3$.

I check whether the result is true for n=k+1:

$1+3+5+\ldots+(2k-1)+(2k+1)=(k^2+3)+2k+1=(k^2+2k+1)+3=(k+1)^2+3$

True.

Therefore, $1+3+5+\ldots+(2n-1)=n^2+3$ for all $n \in \mathbb{N}$ by mathematical induction.

For every natural number $n \geq 5$ the following result is true:

$$1 \bullet 2 \bullet 3 \bullet \ldots \bullet (n-1) \bullet n > 2^n$$

Proof:

I check whether the result is true for $n=5$.

$$1 \bullet 2 \bullet 3 \bullet 4 \bullet 5 = 120 > 2^5 = 32$$

So, I assume the result is true for n=k : $1 \bullet 2 \bullet 3 \bullet \ldots \bullet (k-1) \bullet k > 2^k$

Now, I check whether the result is true for n=k+1.

$$1 \bullet 2 \bullet 3 \bullet \ldots \bullet (k-1) \bullet k \bullet (k+1) > 2^k \bullet (k+1)$$

$$> 2^k \bullet 2 \qquad (\textit{because } k+1>2)$$

$$= 2^{k+1}$$

Therefore, $1 \bullet 2 \bullet 3 \bullet \ldots \bullet (n-1) \bullet n > 2^n$ for all n ≥ 5.

Students in this course had already been introduced to direct and indirect modes of proof, but were yet to be introduced to mathematical induction. Before class, students were asked to examine the two sample arguments and to be ready to discuss their initial thoughts. Prior to launching this task, the instructor (3rd Author) introduced students to the definition of mathematical induction shown in Figure 2.

Figure 2

Definition of Mathematical Induction Presented to Students

- To prove an infinite sequence of statements $P(1), P(2), P(3)$, … [that is, $P(n)$ for all $n \in \mathbb{N}$], we will need to prove the following:
- **Base case:** Prove $P(1)$ is true.
- **Induction step:** Prove $P(k) \Rightarrow P(k+1)$ for $k \in \mathbb{N}$.
- Then $P(n)$ is true for all $n \in \mathbb{N}$. Note: The statement $P(k)$ in the induction step is called the **induction hypothesis.**

Students then connected this definition to the domino analogy and graphic of S(1) → S(2) → … S(K) → S(K+1) →.... Thus, we were able to analyze students' initial conceptions and understandings of mathematical induction, whether they recognized the components of mathematical induction and could connect them to the definition, and whether they found this argument form convincing.

Data Analysis

Three videos, 19 minutes each, were transcribed by talk turn where small groups of students (3-4 students each) analyzed the induction arguments outlined in Figure 1. The three authors then independently applied Stylianides' (2007) framework (described above) to code each talk turn, where applicable. Each line could be coded with any combination of the three modes described in Stylianides' frame, and we decided to code a line if students appeared to be making a comment, asking a question, or making a decision related to the set of accepted statements (S), the modes of argumentation (A) or the modes of argument representation (R). After independent coding, the three authors met to come to unanimous agreements on each talk turn's code.

Limitations

This introduction to proof course was conducted entirely online due to Covid-19. As such, students worked in groups through their personal computers in Zoom breakout groups. We feel that this may have had an impact on their discussion when analyzing the sample arguments, as the difficulties of online learning and writing of proof are unknown. Further, no students were interviewed regarding their discussions and understanding of induction. We may have interpreted some student interactions in a manner different than they intended.

Results

We present the results for this study in two sections. First, we show how the various groups responded to Argument 1, in particular highlighting how they made sense of the base case (or lack thereof). Next, we present the various groups' responses to Argument 2, in particular highlighting how the groups of students made sense of the inductive step. Students in this study discussed the base case (for argument 1) in two main ways: (a) as something that needed to be confirmed as part of a checklist for performing mathematical induction, and (b) as a specific example that can aid in determining the truth value of the given claim. Likewise, students in this study focused on certain aspects of the inductive step (in argument 2): (a) algebra and the inductive hypothesis, and (b) why they switch variables between "N" and "K".

Base Case Theme 1: Confirming the Base Case: A Checklist

The first sample argument which students were asked to analyze is one for which the claim is not true, and the proof is also missing the base case. A student (or group of students) in all three groups recognized both of these inaccuracies in the proof and conjecture. Table 3 shows the percentage of the talk turns related to the base case in Argument 1 that were coded as S, A, or R.

Table 3

Percentage of talk turns related to the base case in Argument 1 coded as S, A, or R

	S	A	R	# Talk Turns in Arg 1 Related to Base Case
Group 1	43	57	0	7
Group 2	36	93	0	14
Group 3	39	77	31	13

In our analysis, we found that students' discussions around the first sample argument relied heavily on sets of accepted statements (S), seeing the base case as a necessary step to check from the given definition of induction as opposed to the first true statement in an infinite series of implications (i.e., S(1) → … → S(K) → S(K+1) → …). Example quotes from the various groups with this checklist-type language are shown in Table 4.

Table 4

Student Checklist-Type Language Quotes

	Group 1 - Zoran	Group 1- Eliza	Group 2 - Taylor	Group 3 - Billy
Student Quote	For induction proof, you have to start with proving like the N equals one case and then the N equals one case is false. So you can't even go on from there.	I think the first one is false because they didn't check for a base case.	Well, I don't know.... Aren't you supposed to prove the first step is true and then prove K. And then..	I think it's important to look at the base case. So we should check to make sure it works for N equals one

Base Case Theme 2: Disproving by Counterexample

While discussing the base case in Argument 1 some students also noted that the statement is not true in general. These discussions are shown in Table 5. Here their talk was different than when discussing the lack of a stated base case, as their attention turned to the validity of the claim given an initial value (e.g., n=1; n=4). In these instances, students argued that the given argument could not be a valid proof because the claim itself could be shown to be false through a single counterexample. Note, this rationale for why the proof is invalid does not require any understanding of proof by mathematical induction; however, the need that students felt above to check the base case within a proof by induction allowed students to quickly discover that the given claim was indeed false for that case (and hence in general).

Table 5

Students Disproving by Counterexample

	Group 1 - Zoran	Group 2 - Alex	Group 3- Billy and Tim
Student Quote	So I looked up the first one and it's not even true when you plug in values...Like I mean, if you plug in n=4, it will be 1+3+5+7, and that 1+3+5+7 is equal to 16, but if you look at the right side, n^2+3 is equal to 19. So without even thinking if it counts as induction, the statement is not even true. It's not true for n=1 either.	Yeah, and I don't really think there's a way to modify it to make it a proof because the statement is not true.	Tim: Does it work at [one] though? Billy: No, because 1would have to equal $1^2 + 3$. Tim: Yeah, so is the whole thing just false then? Billy: Yes.

Inductive Step Theme 1: Algebra and the Inductive Hypothesis

The second proof, a valid proof by mathematical induction, caused students to have some unique trouble. In particular, two of the groups had a detailed discussion about their struggle to understand the logical steps of the inductive hypothesis, similar to the results of Stylianides et al., (2007). Table 6 shows the percentage of the talk turns related to the inductive step in Argument 2 that were coded as S, A, or R.

Table 6

Percentage of talk turns related to the inductive step in Argument 2 coded as S,. A, or R

	S	A	R	# Talk Turns

Group 1	50	75	16.67	12
Group 2	85.71	71.43	14.23	7
Group 3	25	100	100	4

In this table we see the heavy reliance on both "S" and "A" for groups 1 and 2. In both of these groups, there seemed to be a tension between the logical and algebraic steps (A) and whether the argument satisfied the criteria for induction (S) with respect to the inductive step. For example, consider the following exchange between members of Group 1 as they worked together to determine if the inductive step was satisfied:

> **Eliza**: I think they want you to assume that you're going to be using some additional power of two. I think is the assumption that they're making here and why it's OK for them to say that K+1 is greater than two, that you can replace K+1 with two, because the assumption is on the right hand side, we're operating with powers of two.
> **Zoren**: I mean, say, like the left hand side is like four and the right hand side is three. If you replace that three with something smaller, it doesn't change whether the equality is true or false? I guess that is the idea.
> **Eliza**: Yeah.. That's I think that's part of it.
> **Zoren**: My question was like, so what if it's like three? Three is greater than four? And then you replace four with like two or something smaller. That would definitely change it from false to true. I guess. So, like switching something on the right hand side does. It could change the truth value.

In this exchange we see the group trying to make sense of whether or not these algebraic manipulations are logically valid (i.e., focusing on "A"). To these students, there seems to be a tension between the logical and algebraic steps (A) and whether this satisfies the accepted criteria for induction and the inductive step (S). To Zoren, it is not clear that each algebraic manipulation is its own implication statement. We observed this issue with other students in the algebraic manipulations and assumptions of the inductive step.

Inductive Step Theme 2: Why Switch Variables?

Table 6 demonstrates that when Group 3 was discussing the inductive step, they were primarily focused upon "A" and "R." The following conversation depicts the team's dilemma, considering why a switch from "N" to "K" is useful and/or warranted for a valid argument by mathematical induction:

> **Joe:** Oh, these two. OK. Over here. I still don't like that, what do you call it, let's say the problem is written in N. And then the first thing you do is put it a K, then K is equal to N, I never really understood that like. Why why would you substitute, like wouldn't working with N be just
> **Billy**: So what's the question.
> **Joe**: I'm just trying to figure out. So. I guess they would. What is it, they had N equals K?
> **Billy**: Yeah
> **Joe**: But I guess that's just like a framing thing, so you can get N=K+1
> **Billy**: Yeah. I mean, I guess it's just so you separate the original argument which had N's with the ones you're modifying.

In this exchange, we see a tension between the mode of argumentation (A) and mode of argument representation (R) in the necessity of replacing N with K. We see that the student, Joe, is focusing on the logic of the substitution (A), while Billy points out that this is a useful way to make a distinction between the statement of the conjecture and the representation of the inductive step (R).

Discussion & Conclusion

We pursued this research in an effort to understand, within a natural classroom setting, how students make decisions about mathematical induction. By studying small-groups of students as they worked together to evaluate sample induction arguments, we were able to gain insight about the students' decision-making and sense-making in real time, and through peer-to-peer discourse. Methodologically, we found this approach quite revealing in terms of the understandings exposed through peer discussions.

When analyzing students' small-group evaluations of Argument 1, we were able to identify two ways that students discussed the base case: (a) as something that needed to be confirmed as part of a checklist for performing mathematical induction, and (b) as a specific example that can aid in determining the truth value of the given claim. Our coding scheme brought these two themes to light, as (a) students focused on their perception of the two-part definition of induction (i.e., check off the base case), which we captured through our S (set of accepted statements) codes, and (b) students utilized specific numeric examples, including n=1 from the base case, in order to justify why the given conjecture must be false, which we captured through our A (modes of argumentation) codes.

When analyzing students' small-group evaluations of Argument 2, we identified two themes in terms of the students' focus on the inductive step: (a) algebra and the inductive hypothesis, and (b) why they switch variables between "N" and "K". Our coding scheme was likewise useful in supporting the identification of these themes as (a) students wrestled with how the algebraic and logical underpinnings of an argument (e.g., debating the transitivity of inequalities in Group 1), which was captured with A codes, aligned with the accepted process of the inductive step (i.e., P(k) implies P(k+1)), which was captured with S codes, and (b) students pondered the utility of substituting "K" for "N" in terms of the benefit to the argument (A codes) and in terms of the benefit to the communication and representation of the argument (R codes).

Note that in Stylianides et al. (2007), the authors posit that "by breaking the boundaries of students' normal experience, the[se] two tasks set up situations where procedural knowledge of the induction method is not enough for successful performance" (p. 164). Our findings suggest through discussing these tasks with their peers, students persisted in relying on their understood ritual (procedure) of induction. However, these tasks also allowed students to make connections beyond their initial procedural understanding of the definition of induction (S codes) and to the important aspects of the argument (A) and the representation of the argument (R). We believe that using such tasks in classroom discourse, as depicted in this study, could be beneficial as a starting point for introducing mathematical induction. We suggest that future research investigate the possible use of these tasks in classroom settings without first introducing the formal definition of mathematical induction. In this way, students may be less prone to rely upon the accepted ritual, and the environment could potentially further students' non-procedural understanding of mathematical induction.

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Predicting mathematics exam-related self-efficacy as a function of prior achievement, gender, stress mindset, and achievement emotions

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Addressing student affect around assessment is vital, given it is tightly interwoven with cognition. This study seeks to describe the relations between exam-specific affect and stress mindset in a university mathematics course. Participants (N = 356) completed a survey assessing their exam-related self-efficacy, achievement emotions, and stress mindset. The study demonstrated significant correlations between a stress-is-enhancing mindset with positive affect and a stress-is-debilitating mindset with negative affect. When controlling for prior achievement and gender, stress mindset was significant, and student exam-related emotions were dominant in explaining exam-related self-efficacy. The results are discussed with opportunities to adapt learners' stress mindset and the development of exam-related self-efficacy.

Keywords: self-efficacy, achievement emotions, stress mindset, affect, assessment

It is now well established that student performance on assessment is not exclusively a product of their understanding but, additionally, a function of their beliefs and emotions (Zan et al., 2006). Therefore, research should address and unpack the relationships between affect and cognition, to design mathematics assessments or interventions that promote an equitable experience for all students under exam conditions.

Literature Review

Self-Efficacy

Bandura (1997) defines self-efficacy as "the beliefs in one's capabilities to organize and execute the courses of action required to produce given attainments" (p. 3). Self-efficacy has been demonstrated to be a predictor of academic achievement (Pajares & Graham, 1999; Zimmerman, 2000). Reciprocally, in educational contexts, experiences of success positively influence the development of self-efficacy while experiences of failure impair it (Usher & Pajares, 2009). Since most postsecondary institutions measure students' progress through summative assessment, understanding and operationalizing student assessment-related self-efficacy is important for researchers and educators. Most research to date focuses on content-specific self-efficacy when considering assessment but not students' self-efficacy pertaining to context (such as beliefs around their ability to emotionally regulate). University mathematics educators do not generally have control over the prior achievement or past assessment experiences of students entering the course, so other potential factors that may contribute to explaining student self-efficacy to succeed in an exam environment must be investigated.

Achievement Emotions

Achievement emotions are emotions experienced by learners, which are related to achievement activities or outcomes (Pekrun, 2006). Research has demonstrated that positive emotions, like enjoyment, correlate positively with engagement, motivation, and performance, and negative emotions, like hopelessness, demonstrate the inverse relationship (Mega et al., 2014; Peixoto et al., 2017; Pekrun et al., 2017; Pekrun et al., 2019; Schukajlow & Rakoczy, 2016). Self-efficacy has been shown to associate positively with positive emotions and negatively with negative emotions (Pekrun et al., 2011; Luo et al., 2016). Further, it has been shown that high anxiety can undermine self-efficacy (Usher & Pajares, 2009). Pekrun et al. (2004) argue that self-efficacy relates to positive test-related emotions. Research is needed to understand how greatly students' emotions in and around mathematics assessments contribute to explaining their assessment-related self-efficacy.

Stress Mindset

Crum et al. (2013) challenge the view that stress is inherently negative and argue for the new construct of *stress mindset*, which is defined to be the extent of one's beliefs that stress has enhancing or debilitating consequences for stress-related outcomes. There is evidence to suggest that stress mindset may correlate with performance and the amount of stress psychologically experienced (Crum et al., 2013). However, there is limited research that investigates stress mindset in educational contexts, and particularly its relationship with affect. Keech et al. (2018) report stress mindset directly predicted perceived stress and indirectly predicted academic performance. Kilby and Sherman (2016) report a significant negative correlation between positive stress mindset with both perceived stress and trait anxiety. They did not find significant relationship between stress mindset and mathematics self-efficacy or mathematics anxiety. Research is needed to deduce whether assessment-specific stress mindset has relationships with achievement emotions and assessment-specific self-efficacy.

Procedure

Participants

This study is a cross-sectional analysis of the first time point in a longitudinal study over the course of a semester. The study was conducted during the second semester, 2020 at a major New Zealand university in a standard second year service mathematics course, designed to support other majors such as computer science, finance, physics, and other sciences. There were 410 students enrolled in the course at the start of the semester. Out of these, 379 provided their consent to the use of their data from the course and 364 consenting students completed the first survey. Participants were removed from the analysis if the survey was less than 50% completed or demonstrated sufficient evidence of straight-lining (N = 356). Included in the analysis were 193 students who reported their gender as male, 157 as female, three as gender diverse, and one who declined to answer. Missing data was inserted using EM-imputation.

Measures

The instruments employed in this study were first subject to pilot testing through a cross-sectional research design involving the completion of an anonymous survey by a sample of university students (N=301) procured through *Prolific* (a crowdsourcing platform based in the UK that pays individuals a nominal fee for participating in research projects).

Achievement Emotions Questionnaire (AEQ). The AEQ (Pekrun et al., 2011) contains five-point Likert items ranging from *Strongly Disagree* to *Strongly Agree* designed through the lens of the control-value theory to measure achievement emotions. This study focused on four emotions – enjoyment, hope, anxiety, and hopelessness. During pilot testing, confirmatory factor analysis on the four-factor adapted exam-related AEQ scale suggested the need to incorporate the temporal component of exam-related emotions. We proceeded with exam-related *before* emotions and *during* emotions as separate measurements. Several items were removed and four new items to measure hope and enjoyment during the assessment were introduced, based on the multi-component structure of achievement emotions as theorized in the control-value theory (Pekrun et al., 2011). Confirmatory factor analysis in this study demonstrated our *before* model offered an acceptable fit (χ/df = 2.627, CFI = .907, RMSEA = .068). The *during* model was acceptable with the inclusion of our four new items (χ/df = 3.067, CFI = .907, RMSEA = .076).

Measure of Assessment Self-Efficacy (MASE). The development and validation of this scale was reported in Riegel et al. (2020). The MASE items were developed following Bandura's (2006) recommendations and were designed to assess the participant's beliefs in their ability to understand, perform, and emotionally regulate while studying for and during an assessment. Responses to statements were measured using a slider scale from 1 to 100 (where 1 = Cannot do at all, 50 = Moderately sure can do, and 100 = Highly certain can do). Pilot testing supported a two-factor model of eight items, with latent factors "performance and comprehension abilities" and "emotional regulation". The model offered an acceptable fit in our study (χ/df = 3.015, CFI = .980, RMSEA = .075). Participants in the study responded to the scale under the following assessment scenario.

Mathematics exam scenario
Imagine that you've enrolled in a mathematics course like Maths 2XX that has a final EXAM worth 50% of your final grade. The exam contains short and long answer questions. The exam is invigilated and is two hours long.

Stress and Stress Mindset Measure (SMM). The stress mindset measure (Crum et al., 2013) measures the extent of participants beliefs that stress is enhancing or debilitating. Participants responded on a five-point scale from *Strongly Disagree* to *Strongly Agree*. Students were prompted with the exam scenario described previously. As this was hypothetical, the verb "are" in each statement was rephrased to "would be," for example, *the effects of this stress would be negative and should be avoided.* Pilot testing suggested that exam-related stress-is-enhancing and stress-is-debilitating mindsets were separate latent factors, and that one item should be removed from the scale. This two-factor model offered an acceptable fit in our study for exam-related stress (χ/df = 3.750, CFI = .957, RMSEA = .088). Additionally, we measured stress amount through asking participants *How stressful do you perceive this* ***mathematics EXAM*** *to be?* Responses were indicated on a nine-point scale ranging from *Not stressful at all* to *Extremely stressful*.

Academic achievement. Self-reported prerequisite grades for the course were collected.

Research Questions

- How does stress mindset associate with exam self-efficacy and achievement emotions?
- Do gender or stress mindset make independent contributions to predicting exam self-efficacy when controlling for prior achievement and emotions?
- How greatly do achievement emotions contribute to explaining exam self-efficacy when controlling for prior achievement?

Results

Table 1 presents a summary of the descriptive statistics and correlations of all latent factors.

We conducted two hierarchical multiple regression analyses to explain the two factors comprising exam self-efficacy (performance/comprehension and emotional regulation),

Table 1. Descriptive statistics and correlations of latent factors

Variables	M	SD	α	1	2	3	4	5	6	7	8	9	10	11	12	13
1. SE - performance	65.19	17.21	.89													
2. SE - emotional	64.45	19.52	.86	.77**												
3. Anxiety (B)	3.34	0.75	.66	-.42**	-.45**											
4. Anxiety (D)	2.87	0.88	.86	-.37**	-.43**	.67**										
5. Hope (B)	3.27	0.67	.79	.55**	.52**	-.42**	-.28**									
6. Hope (D)	3.26	0.67	.73	.55**	.50**	-.43**	-.41**	.69**								
8. Hopeless (B)	2.41	0.79	.85	-.51**	-.54**	.53**	.53**	-.50**	-.52**							
8. Hopeless (D)	2.42	0.70	.85	-.53**	-.53**	.48**	.64**	-.41**	-.52**	.73**						
9. Enjoyment (B)	2.93	0.73	.73	.45**	.44**	-.43**	-.26**	.69**	.54**	-.37**	-.31**					
10. Enjoyment (D)	2.96	0.71	.73	.40**	.41**	-.37**	-.33**	.56**	.57**	-.35**	-.31**	.70**				
11. SMS enhancing	3.11	0.80	.83	.26**	.35**	-.27**	-.20**	.33**	.29**	-.23**	-.23**	.37**	.31**			
12. SMS debilitating	3.11	0.69	.70	-.31**	-.41**	.42**	.38**	-.23**	-.23**	.35**	.34**	-.31**	-.28**	-.59**		
13. Stress amount	6.42	1.76	-	-.29**	-.27**	.58**	.49**	-.30**	-.34**	.32**	.35**	-.30**	-.29**	-.23**	.38**	
14. Prior achievement	6.48	2.21	-	.38**	.29**	-.16**	-.11*	.29**	.31**	-.28**	-.24**	.22**	.18**	.16**	-.20**	-.12*

Note. **Correlation is significant at the 0.01 level; *Correlation is significant at the 0.05 level; SE = self-efficacy, (B) = before, (D) = during, SMS = stress mindset – specific (exam); Prior achievement (9 = A+, 1 = C-); N=356.

summarized in Table 2. The full models were statistically significant for both performance and comprehension self-efficacy R^2_{adj} = .45, F(13, 338) = 23.21, p < .0005, and emotional regulation self-efficacy R^2_{adj} = .44, F(13, 338) = 22.29, p < .0005. Prior achievement, which is known to have reciprocal effects with self-efficacy, significantly explained students' beliefs in their performance and comprehension abilities around an exam, R^2_{adj} = .14, F(1, 350) = 58.10, p < .0005, though not as greatly for their beliefs they could emotionally regulate during an exam, R^2_{adj} = .09, F(1, 350) = 33.88, p < .0005. We next included the demographic variable gender (the four students who identified as gender diverse or declined to answer were removed from the analysis). This variable was not significant for student performance/comprehension self-efficacy but made a small contribution of R^2_{adj} = .01, F(1, 349) = 4.40, p = .037 to explaining emotional regulation self-efficacy, with females feeling slightly less efficacious.

The third step added stress mindset as variable. A stress-is-debilitating mindset made a significant contribution to explaining both factors, whereas stress-is-enhancing was only significant for self-efficacy around emotionally regulating. Overall, stress mindset led to a significant increase for performance/comprehension self-efficacy in R^2_{adj} of .06, F(2, 347) = 14.84, p < .0005, and to emotional regulation self-efficacy in R^2_{adj} of .13, F(2, 347) = 32.22, p < .0005. After this we included stress amount during an exam, which significantly increased performance/comprehension self-efficacy in R^2_{adj} by .03, F(1, 346) = 12.25, p = .001, and emotional regulation self-efficacy in R^2_{adj} by .01, F(1, 346) = 5.73, p = .017.

The next four steps added the four different emotions (anxiety, enjoyment, hope, and hopelessness). The anxiety experienced before and during an exam were both significant, and resulted in increases of R^2_{adj} by .06, F(2, 344) = 15.25, p < .0005 for performance/comprehension self-efficacy and R^2_{adj} of .08, F(2, 344) = 19.26, p < .0005 for emotional regulation self-efficacy. Enjoyment made a significant contribution to both performance/comprehension self-efficacy of R^2_{adj} of .06, F(2, 342) = 16.01, p < .0005 and emotional regulation self-efficacy by R^2_{adj} of .04, F(2, 342) = 14.23, p < .0005. However, only enjoyment before an exam was significant. Hope increased self-efficacy for both performance/comprehension by R^2_{adj} of .07, F(2, 340) = 22.19, p < .0005 and emotional regulation by R^2_{adj} of .05, F(2, 340) = 14.49, p < .0005. Only hope experienced before an exam was significant for explaining emotional regulation self-efficacy. Finally, hopelessness also significantly contributed to the model for performance/comprehension self-efficacy by R^2_{adj} of .03, F(2, 338) = 32.22, p < .0005 and for emotional regulation self-efficacy by R^2_{adj} of .03, F(2, 347) = 10.36, p < .0005. For both variables, only hopelessness experienced during the exam was significant.

In the final models, prior achievement, hope before an exam, and hopelessness during an exam are significant predictors of exam self-efficacy. The final model of emotional regulation self-efficacy is also significantly explained by a stress-is-debilitating mindset.

Discussion

This study first sought to understand the relationship between exam self-efficacy, stress mindset, and emotions. The correlations between variables were in line with existing literature and theory. Self-efficacy correlated positively with positive emotions and achievement, and negatively with negative emotions and stress amount. As expected, a stress-is-debilitating mindset correlated positively with the reported amount of stress experienced, while a stress-is-enhancing mindset correlated negatively (Crum et al., 2013). Additionally, our results show that a stress-is-

Table 2. Hierarchical regression coefficients for exam self-efficacy

	β for exam performance and comprehension self-efficacy								β for exam emotional regulation self-efficacy							
Variable	Step 1	Step 2	Step 3	Step 4	Step 5	Step 6	Step 7	Step 8	Step 1	Step 2	Step 3	Step 4	Step 5	Step 6	Step 7	Step 8
Prior achievement	.38***	.38***	.33***	.32***	.30***	.27***	.19***	.17***	.30**	.31***	.23***	.22***	.21***	.18***	.12*	.09*
Gender		-.06	-.06	-.05	-.01	-.02	.02	.01		-.12*	-.11*	-.11*	-.06	-.07	-.05	-.06
SMS enhancing			.11	.11	.10	.03	-.03	-.02			.18**	.18**	.17**	.10	.06	.07
SMS debilitating			-.18**	-.12	-.04	-.05	-.11*	-.08			-.25***	-.21**	-.12*	-.13*	-.18**	-.14*
Stress amount				-.18**	-.02	.00	.03	.02				-.12*	.05	.07	.09	.08
Anxiety (B)					-.22**	-.12	-.09	-.09					-.19**	-.11	-.08	-.06
Anxiety (D)					-.14**	-.14*	-.10	.04					-.20**	-.19**	-.17**	-.05
Enjoyment (B)						.20**	.03	.04						.17*	.02	.03
Enjoyment (D)						.11	.01	.03						.12	.05	.07
Hope (B)							.27***	.23**							.26***	.21**
Hope (D)							.19**	.12							.11	.04
Hopeless (B)								-.04								-.12
Hopeless (D)								-.25***								-.17*
R^2_{adj}	.14***	.14***	.20***	.23***	.29***	.35***	.42***	.45***	.09***	.10***	.23***	.24***	.32***	.36***	.41***	.44***
ΔR^2_{adj}		.00	.06***	.03**	.06***	.06***	.07***	.03***		.01*	.13***	.01*	.08***	.04***	.05***	.03***

Note. ****p* < .0005; ***p* < .005; **p* < .05; (B) = before, (D) = during, SMS = stress mindset – specific (exam), N = 352.

enhancing mindset correlated positively with exam self-efficacy, positive achievement emotions, and prior achievement, and negatively with negative emotions, while stress-is-debilitating did the opposite. These correlations provide motivation to further investigate if stress mindset contributes to shaping exam-related beliefs and emotions, which in turn influence students' performance and experience of mathematics.

We also sought to understand the role of gender and stress mindset for exam self-efficacy when controlling for prior achievement. Nearly 50% of all exam self-efficacy variance is explained by prior achievement, stress mindset, and student emotions. Prior achievement is a known predictor of academic self-efficacy, however, our study indicates that it is not as great a predictor of students' beliefs they can emotionally regulate in exam conditions as it is for the belief in their ability to understand and perform. This could mean that repeated experiences of success do not as greatly develop beliefs they can remain calm or optimistic in the face of assessment, suggesting it may be important to separately address students' affect on assessments.

Gender accounted for a small but significant decrease in self-efficacy for women around emotional regulation. This distinction between performance/comprehension self-efficacy and emotional regulation self-efficacy may contribute to explaining why there have been mixed findings with this relationship historically (Hackett & Betz, 1989).

Stress mindset made a meaningful contribution to explaining both self-efficacy factors when controlling for prior achievement and gender, but more greatly for the emotional regulation factor of self-efficacy. A stress-is-debilitating mindset remained significant in the final model for explaining this factor. Self-efficacy has been shown to have reciprocal effects with academic achievement but can be slow to change through methods such as the accumulating mastery experiences. On the other hand, stress mindset has been shown to be malleable through short video interventions (Crum et al., 2013; Crum et al., 2017). Manipulating students away from a stress-is-debilitating mindset may provide a realistic opportunity to develop exam-related self-efficacy more rapidly. This would need to be investigated through a longitudinal study to determine if there is a causal relationship.

Finally, we were interested in the contribution of emotions to predicting exam self-efficacy. On both factors of exam self-efficacy, the addition of the four emotions accounts for around 20% of the variance, emphasizing the importance of considering student emotions when addressing self-efficacy. Hope before and hopelessness during an exam stayed significant in the final models for exam self-efficacy. Optimism before an exam aligns with the belief that they can succeed, but this perhaps also suggests that what influences the development of self-efficacy is how hopeful students have historically walked into exams and how despairing they have felt while taking an exam. This highlights how greatly a single assessment can damage the development of a learner's self-efficacy.

Limitations and Future Research

The MASE will be further validated using the data from the longitudinal study, specifically testing for measurement invariance. Our regression analysis has not yet considered possible interactions between variables, which could reveal nuanced relationships in the data. Finally, the data was collected during the Covid-19 pandemic and the previous semester had a lockdown, possibly influencing students' academic affect. However, we think it is unlikely it dramatically altered the relationship between the variables in this study. Stress mindset may have played a role in how students coped with these events, which we aim to understand in future research. Analysis of the longitudinal data will inform potential causal effects between these variables and relationship with exam performance in that semester.

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The Effectiveness of a Professional Development Video-Reflection Intervention: Mathematical Meanings for Teaching Angle and Angle Measure

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Abstract: In response to calls for more investigations of teachers' mathematical meanings for teaching (MMT) and the need to improve US mathematics teachers' and students' mathematical meanings, I investigated mechanisms for advancing post-secondary instructors' mathematical meanings and teaching practices in the context of their teaching precalculus mathematics using a research-based curriculum. This report presents results from clinical interviews with graduate teaching instructors (GTIs) to illustrate the impact of a video-reflection intervention on teachers' MMT and image of effective teaching practices. More specifically, I present two GTIs' expressed meanings for angle measure and image of effective teaching before and after attending a professional development seminar where a trigonometry video-reflection intervention was used. I conclude by hypothesizing that video-reflection interventions may orient teachers toward reflecting on the degree to which they are impacting student thinking by providing examples of high-quality teaching (Musgrave & Carlson, 2017) interactions.

Keywords: mathematical meanings for teaching, video-reflection intervention, quantitative reasoning

Introduction and Review of Literature

Although many math educators have investigated teachers' *mathematical knowledge for teaching* (MKT) (Ball, 1990; Ball & Bass, 2003; Ball, Hill, & Bass, 2005), and how it relates to students' learning (Ball, Thames, & Phelps, 2008; Hill & Ball, 2004; Hill, Ball, & Schilling, 2008; Hill, Schilling, & Ball, 2004; Hill et al., 2008), few researchers have investigated teachers' *mathematical meanings for teaching* (MMT). More concerning is the results of the few studies that have. As one example, Byerley and Thompson (2017) reported that most secondary mathematics teachers they studied provided formulaic or chunky descriptions of slope in contrast to thinking of slope as a rate of change that involves a multiplicative comparison between changes in two quantities. Other researchers who have investigated teachers' MMT have reported similar findings highlighting the impoverished nature of US teachers' mathematical meanings (Musgrave & Carlson, 2017; Yoon & Thompson, 2020; Thompson & Milner, 2018).

A teacher's meanings constitute her image of the mathematics she teaches (Thompson, 2013), her pedagogical decisions, and the language she uses to cultivate similar images in students' thinking (Thompson & Thompson, 1996), ultimately influencing the meanings her students are able to develop (Thompson, 2016). Prior research has shown that teachers' MMT develop in concert with their development of coherent mathematical meanings and reflection on the effectiveness of their instructional practices in supporting students' development of coherent meanings (Silverman &Thompson, 2008). Others have shown that teachers' meanings for foundational mathematical ideas can become more productive if they are engaged in interventions designed to support their development of coherent mathematical meanings (Musgrave & Carlson, 2017). However, little is known about the mechanisms for advancing instructors' teaching practices. Some researchers' have proposed implementing video-based professional development seminars (video clubs) (Sherin & Han, 2004; Seidel et al., 2009; Sherin & van Es, 2009; van Es, 2012; van Es et al., 2014). However, this research has been focused on the K-12 level and it has been shown that simply bringing teachers together does not ensure community development of desirable practices and norms (van Es, 2012). As such, investigating interventions for developing teachers' MMT and mechanisms that result in teachers' engagement in reflection on practice is of vital importance. This report addresses the following question: what is the impact of video-reflection intervention on teachers' MMT and image of teaching?

Theoretical Perspective

The problematic nature of trigonometry teaching and learning has been documented by many (Moore et al., 2016; Tallman, 2015; Thompson, 2008; Thompson et al., 2007; Weber, 2005). Researchers (Hertel and Cullen, 2011; Moore, 2012, 2014; Tallman, 2015; Thompson, 2008) have responded by leveraging quantitative reasoning (Thompson, 1990, 2011), to support students in conceptualizing angle measure. Within the theory of quantitative reasoning, a *quantity* is a quality of an object or situation that one conceives of as admitting a measurement process. *Quantification* is the process by which one assigns numerical values to qualities (Thompson, 1990). A quantity's *value* is the numerical result of a quantification process. It is crucial to distinguish between *numerical operations* and *quantitative operations.* Numerical operations are used to calculate a quantity's value, whereas a quantitative operation is the conception of two quantities taken to produce a new quantity (Thompson, 1990). Although quantitative reasoning has been shown to support students in conceptualizing angle measure, students and teachers often develop meanings that do not involve this way of thinking (Thompson, 2008).

Tallman and Frank (2018) described a productive meaning for angle measure grounded in quantitative reasoning. Namely, "a quantitative understanding of angle measure involves identifying an attribute of a geometric object to measure and conceptualizing a unit with which—and process by which—to measure it" (Tallman & Frank, 2018, p. 5). Many students and teachers recognize that measuring an angle involves quantifying the openness between two rays that meet at a common vertex. "However, the "openness" of an angle is a quantity only for those who can specify a particular attribute of a geometric object to measure, identify an appropriate unit of measure, and envision a measurement process" (e.g., iteration, multiplicative comparison) (Tallman and Frank, 2018, p.5). Thus, one may conceptualize measuring the openness of an angle by measuring the length of the arc of a circle subtended by the angle whose vertex lies at the center of the circle in a unit proportional to the circle's circumference that contains the subtended arc.

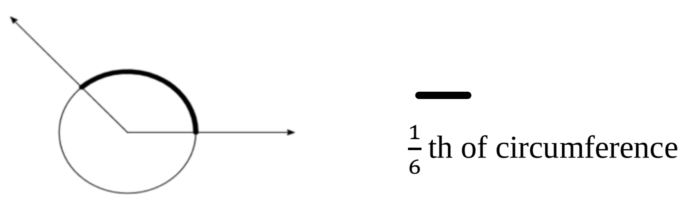

Figure 1. Angle Measure

For example, we could measure the angle in *Figure 1* by multiplicatively comparing the subtended arc length to the length of the circle's circumference. As such, the angle would have a measure of 1/3 in units of the circle's circumference. Namely, the subtended arc length is 1/3 times as large as the length of the circle's circumference. One could also imagine measuring the angle's openness in *Figure 1* by multiplicatively comparing the length of the subtended arc to the length of 1/6th of the circle's circumference. The angle would have a measure of 2 in units of 1/6th of the circle's circumference since the length of the subtended arc is two times as large as 1/6th of the circle's circumference. Thus, a productive understanding of angle measure involves conceptualizing angle measure as a measurement process that defines a multiplicative relationship between the subtended arc length and some unit proportional to the circle's circumference (Moore, 2014).

Methods, Subjects, and Data Collection

This study aimed to investigate the degree to which teachers' MMT and image of effective teaching shifted as a result of engaging in video-reflection tasks at a weekly professional development seminar. To accomplish this goal, I conducted two rounds of clinical interviews (Clement, 2000) with three graduate teaching instructors (GTIs) and recorded their engagement in a video-reflection task during a professional development seminar. At the time of the study, the GTIs were in their first semester of teaching using research-based *Pathways to Pre-Calculus* materials. The first round of clinical interviews took place a week before the instructors taught a lesson on angle and angle measure and attended a PD seminar where they discussed the video-reflection tasks. The second round of clinical interviews took place within three days of the instructors' lesson and the PD seminar. The analysis of the clinical interviews occurred in three phases as described in Simon (2019). The first phase involved my listening to the interviews to generate hypotheses about the teachers' ways of thinking. The second phase involved a more in-depth, line-by-line conceptual analysis aimed at describing aspects of the teachers' MMT for ideas of angle and angle measure. The third and final phase of this analysis involved identifying themes in the conceptual analysis to characterize the teachers' MMT.

The Intervention

The GTIs involved in this study participated in a professional development seminar that met once per week for one semester. During the 11th week of the semester, the GTIs were engaged in a video-reflection intervention designed to support instructors in implementing the *Pathways to Pre-calculus* curriculum with fidelity. The angle measure video-reflection intervention contains six video clips of an instructor teaching a lesson on angle and angle measure. The selected clips include segments of teaching in which the instructor demonstrated strong MMT, decentering abilities (Steffe & Thompson, 2008), teaching practices that involve speaking with meaning (Clark et al., 2008), mathematical care (Hackenburg, 2010), attending to student thinking, and conceptually-oriented explanations (Thompson & Thompson, 1996). The video-intervention also includes a set of questions that correspond to each of the video clips. We designed these questions to promote teachers' reflection on their mathematical meanings for angle and angle measure and practices they might engage in to support students' construction of coherent meanings for the ideas central to the lesson.

As one example, the trigonometry video-reflection intervention includes a clip of an interactive class discussion surrounding a task that asks students to determine which of three given angles has the largest measure. In this clip, the instructor poses multiple questions to students to elicit their meaning for angle measure and what they consider to be an appropriate unit for measuring angles. The video reflection intervention includes the following reflection questions for instructors to discuss after watching this clip.

1. In the video clip, the teacher asked, "what are you guys looking at when you're talking about angle measure?" Discuss with your partner (1) what you think the teachers' purpose for this question may have been? (2) What might be a more productive way the

teacher could have posed the question “what are you guys looking at when you’re talking about angle measure”?

2. At the end of clip two, the instructor draws three circles such that the angle’s vertex lies on the center of each circle. What might have been the teachers’ purpose for doing this? Discuss with a partner, what you might do next in your own class to help students develop the conception that an appropriate unit of angle measure is proportional to the circumference of a circle.

We hypothesize that the video-reflection interventions may orient teachers toward reflecting on the degree to which they are impacting student thinking by providing imagery of the ways of thinking the teachers need to develop in their students, and by providing examples of teachers listening to students’ thinking and asking questions to gain insight into students’ thinking.

Results

This section includes results from the pre/post interviews that highlight the GTI’s expressed meanings for angle measure and image of effective teaching. The participants were asked to complete multiple tasks designed to elicit their meanings for angle and angle measure in each of the clinical interviews. This section shares two instructors, Emani and Maliya’s, responses to the angle measure task in *Figure 2.*

In class, a student tells you that the measure of a given angle is 1.2 inches. Is this an appropriate measure for an angle? Why or why not?

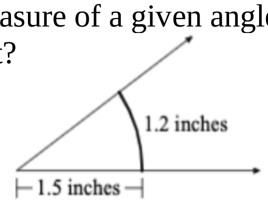

Figure 2. Angle Measure Task- Interview 1

GTIs’ Pre-Intervention Meanings for Angle and Angle Measure

Angle Measure as parts of a whole- the story of Emani

Emani’s expressed meaning for angle measure before engaging in the video intervention is best exemplified by her response (see Excerpt 1) to the task in *Figure 2.* It is noteworthy that Emani was the only instructor who consistently expressed that angle measure is a measure of openness between the two rays formed by the angle. Emani also expressed that she could determine the measure of an angle by counting how many units of measure are cut off by the angle’s rays. Thus, it appears as though Emani conceived of angle measure as fitting parts (some unit) into a whole (subtended arc).

Excerpt 1: *Emani’s expressed meaning of angle measure prior to intervention.*

Emani: Angle measure should be measuring how much openness we can measure between the rays. And measuring a length is not the same as measuring an amount of openness. So, the ways that jump out at me right now for measuring this angle, I could make up other ones, I guess, but are degrees or radians. Measuring in degrees would be like finding out how many 1/360ths [highlights portion of subtended arc] are cut off by the angle marked with 1.2 inches [traces over subtended arc].

While Emani was able to identify the attribute (length of the subtended arc) and a process by which to measure it (fitting pieces into a whole), she struggled to describe an appropriate unit for measuring the angle and why it was appropriate. In Excerpt 2, we can see that Emani correctly identified degrees and radians as appropriate units of measure, and she was also able to create her own unit of measure *diametans.* However, Emani was unable to provide a meaningful explanation for why degrees, radians, and *diametans* were appropriate units of measure.

Excerpt 2: *Emani's discussion of an appropriate unit for measuring angles*.

Interviewer: So earlier, I am going to take you back to the beginning when you first read this. You immediately read this task and you said, "oh like it could be degrees, or it could be radians, or I could make something up." So, what did you mean by "I could make something up"?

Emani: Hmm, umm, so, radians are measured by looking at the radius of the circle. Degrees are measured by looking at some portion of the circumference of the circle centered where the rays meet. So, I think we can measure any angle, or come up with any kind of angle measure as long as we're measuring in terms of some portion of the… like anything constant with the circle? Like we could come up with an angle measure instead of radians that is like *diametians.* So, you measure with the diameter of the circle. So, radians look at how many radius lengths are being subtended, are being cut off [indicates with hands that she is laying lengths next to each other] by the angle. So, we can do the same thing but with the diameter.

Angle Measure- the result of a quantitative operation: the story of Maliya

Maliya consistently expressed a meaning for angle measure as a comparison of the "distance you walk along that portion of the circumference that you marked with the two rays to the entire circumference of the circle" (see Excerpt 3).

Excerpt 3: *Maliya's expressed meaning for angle measure prior to intervention.*

Maliya: Ok this is harder than I thought it would be. An inch is not a unit that I would use to measure an angle because an angle would be umm, so an angle is going to measured in some unit relative to the circumference of the circle. So, to be more specific, if I have a circle and I want to know the angle between two rays, then I would say that you can measure an angle by comparing how far I have walked from the horizontal ray to this slanty ray (traces over the terminal ray) compared to the size of the circumference of the circle.

Thus, it appears as though Maliya conceived of angle measure as a multiplicative comparison of the subtended arc length and the circumference of the circle for which the angle is centered. I interpret Maliya's meaning for angle measure to be that of a quantitative operation as she consistently identified (see Excerpt 4) the result of the multiplicative comparison as the measure of a specific attribute (length of the subtended arc) in a particular unit (the circumference of the circle).

Excerpt 4: *Maliya's discussion of the angle measure task.*

Maliya: So, the endpoint of the angle (points to the angle's vertex) is going to be the center of the circle. So, I can say that 1.5 inches is the radius of the circle. So, if I want to know the circumference of the circle that's going to be 2 pi times as large as the length of the radius, so 1.5 times 2 pi. So, the 1.2 inches, I would say that is a measure of the length of that portion of the circle's circumference that is taken up by the arc between those two rays. But the measure of the angle would be a comparison of the length of the arc, 1.2 inches, divided by 1.5 times 2 pi inches.

Interviewer: So, what does 1.2 divided by 1.5(2pi) represent?

Maliya: To me it represents the measure of the length of the arc between the two rays given in the diagram in units of the circumference of the circle with radius 1.5 inches.

While Maliya expressed a quantitative meaning for angle measure, her descriptions of angle measure were limited to using the circumference as a unit of measure. Maliya did not recognize that she could measure the length of the subtended arc in radius lengths or any other unit proportional to the circumference of the circle.

GTIs' Post-Intervention Meanings for Angle and Angle Measure

Overall, the instructors' expressed meaning for angle measure remained consistent with their expressed meaning before engaging in the video-reflection intervention and teaching a lesson on angle and angle measure. Maliya continued to describe angle measure as a multiplicative comparison of the subtended arc and the length of the circumference. Emani's meaning for angle measure also remained consistent with the meaning she expressed before engaging in the video-reflection intervention. However, Emani was more closely able to provide a quantitative description of angle measure that involved identifying the attribute (length of the subtended arc) of the angle, and a process by which to measure it (fitting pieces into a whole), and a unit by which to measure the length of the subtended arc (fractional portions of the circumference). Although Emani was able to identify a unit of measure, she struggled to express what a correct unit of measure is meaningfully and was aware of her difficulty doing so. As one example, Emani expressed in the second interview, "I need to work on my own understandings and a better way to say this."

GTIs' Image of Effective Teaching Before the Intervention

The instructors were asked to describe their image of good/effective teaching before and after engaging in the video-reflection intervention. The instructors' image of good teaching expressed in the first interview is shown in *Table 1.* It is noteworthy that both Emani and Maliya expressed that good teaching involved attending to student thinking, awareness of how students are feeling in relation to their learning, and ensuring students are comfortable contributing to class discussions.

Table 1. Teachers' response to prompt: describe your image of good/effective teaching.

Teacher	Sample Excerpts from Clinical Interview 1
Emani	Good teaching is trying to understand where…how the students are understanding what is going on. I try to ask questions of my students like as we work through problems to try to find out how they're thinking so I can try to address how they're thinking about the problem.
Maliya	I would say, asking students questions really trying to get students input. It's a good practice that I tried to implement that exemplifies good teaching and I think really trying to help students to think for themselves and articulate their thinking as much as possible.

GTIs' Image of Effective Teaching After the Intervention

During the post interview, Maliya and Emani described asking questions to elicit student thinking as an essential quality of a good teacher. However, Maliya's response is particularly interesting as she described multiple practices that were exemplified in the video clips and reflective discussions included in the video-reflection intervention (see Excerpt 5). One goal of the video-reflection intervention is to promote a reflective discussion about the instructors' goals for students' learning. After participating in the video-reflection intervention, Maliya described "having clear expectations for what you want the students to learn" as an essential quality of good teaching. Maliya also expressed the importance of having a discussion-oriented classroom and an iterative conversation with students. Maliya's expression of the importance of having an iterative conversation with students is also interesting as the video-reflection intervention provided examples of a teacher eliciting, attending to, and responding productively to student thinking. Moreover, Maliya also mentioned, "really trying to speak with meaning in the class." It is noteworthy that *speaking with meaning* emerged as a practice that all three instructors identified as important throughout their second clinical interview.

Excerpt 5: *Maliya's response to the prompt "describe your image of good/effective teaching"*

Maliya: I do think having clear expectations for what you want the students to learn what you want the students to be able to master in terms of content and having clear expectations for assignments and assessments is important. But I do really like having the discussion-oriented classroom…and I'm really digging into their thinking when class is discussion oriented. I also do like having the visualization, especially because this course is so focused on quantitative reasoning that can be one of the most powerful ways to try to convey meaning to students and help students kind of convey their meaning…and just kind of like having an iterative conversation of, like, revising, okay, so this is what I think you said and then the student can correct you if that's not an accurate representation of what they said. So sometimes doing that can be useful to you like reiterating what you think the students said that can also help with students feeling comfortable revealing their thinking and really making clear and really trying to speak with meaning in the class.

Conclusions and Discussion

Overall, both of the instructors' meanings for angle measure remained relatively consistent with their meanings before engaging in the video-reflection intervention. Following their teaching of the section and their participation in a PD seminar, the instructors were more likely to distinguish between and angle and the measure of the angle. However, Maliya and Emani did not advance in their ability to describe a more general way of thinking for measuring an angle using a unit (such as a circle's radius length) that is proportional to a circle's circumference.

While the instructors' meanings for angle and angle measure did not shift after engaging in a video-reflection intervention, their image of effective teaching did. The instructors' participation in the video-reflection intervention appeared to have an impact on their image of teaching, including their recognition of the importance of *speaking with meaning* and posing questions to reveal student thinking. It is noteworthy that the one-hour video-reflection intervention had little impact on the instructors' conceptions of angle and angle measure. However, this result is not surprising as it supports others' findings (Thompson and Thompson, 1996; Thompson, 2016; Musgrave & Carlson, 2017; Tallman and Frank, 2018) that shifting teachers' conceptions is a gradual process that requires sustained and repeated opportunities to reason in new ways. Although we have developed video-reflection interventions for many units in the *Pathways to Pre-calculus* curriculum, this was the first time they were used with GTIs. In future semesters we plan to engage the GTIs more video-reflection interventions to investigate how teachers' MMT and image of effective teaching shift over the course of a full semester. As we continue to develop and refine the video-reflection interventions, we plan to include more video clips that show teachers' promoting students' engagement in quantitative reasoning since prior research has shown that teachers' attention to/focus on referencing quantities, including the starting point and directionality of the measurement, is useful for advancing student learning (Moore & Carlson, 2012; Joshua, 2019).

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Mathematical Sameness: An Ill-Defined but Important Concept

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Sameness is a notion that pervades mathematics through concepts like equality and isomorphism, but limited research has examined how mathematicians perceive sameness in mathematics. This study examines survey responses from mathematicians on the nature of mathematical sameness. Themes highlighted from responses include philosophical, proof-based, and informal notions of mathematical sameness.

Keywords: Isomorphism, Homomorphism, Advanced Mathematics, Sameness

Background and Literature Review

Math educators have been studying students' understanding of equality for many years (e.g., Renwick, 1932; Kieran, 1981). Research on students' understandings of higher-level notions of sameness, such as isomorphism, began more recently (e.g., Dubinsky, Dautermann, Leron, & Zazkis, 1994) and are still being studied (e.g., Melhuish, 2018; Rupnow, 2017). Nevertheless, research on types sameness, their utility, and problems for students' thinking have only recently been examined (Melhuish & Czocher, 2020). In this paper, we solicit experts' perspectives as we address the research question: How do algebraists describe mathematical sameness?

Prior research on equality, isomorphism, and homomorphism have largely focused on students' conceptions. Extensive research on students' understanding of equality in numerical contexts has connected students' understanding of the meaning of the equal sign to their later ability to understand and manipulate algebraic equations (e.g., Alibali, Knuth, Hattikudur, McNeil, & Stephens, 2007; Kieran, 1981). Research on isomorphism has examined how students determine if groups are isomorphic (e.g., Dubinsky et al., 1994; Leron, Hazzan, & Zazkis, 1995; Rupnow, 2017) as well as the use of properties of functions to draw conclusions about isomorphism and homomorphism (e.g., Melhuish, Lew, Hicks, & Kandasamy, 2020).

However, limited research has examined experts' conceptions of types of sameness. Weber and Alcock (2004) noted mathematicians' descriptions of group isomorphism as "re-labelings" or meaning groups were "essentially the same". Rupnow (2021) expanded on these notions with other instructor metaphors for isomorphism or homomorphism, such as "matching" and "structure-preservation". However, these descriptions related only to isomorphism and homomorphism, not to general descriptions of sameness in mathematics.

Cross-concept examinations of sameness have barely been examined. Melhuish and Czocher (2020) noted the context-dependent nature of sameness in mathematics that is not always fully elaborated when students are asked to respond to prompts related to sameness. For example, though Z_3 is isomorphic to a subgroup of S_3, we may not want students to suggest that Z_3 is a subgroup of S_3 because we want to attend to the elements or operations defined in the groups, not just structural similarity. However, the concepts that mathematicians think should be aligned has not been examined nor the ways in which they elaborate on the nature of sameness.

Conceptual Framework

Our theoretical lens for analysis is conceptual metaphors (e.g., Lakoff & Núñez, 1997), in which a source domain is used to structure understanding of a target domain. For example, "Happiness is an equation" and "Happiness is a warm summer's day" are metaphors that might be used to describe the target domain (happiness) in terms of two different source domains (an equation and a warm summer's day). The two metaphors provide different ways of understanding happiness: happiness as a concept that can be broken into constituent pieces and combined in standard ways, or happiness as a concept that is focused in-the-moment.

Conceptual metaphors have previously been used in math education to examine functions in high school and linear algebra (Zandieh, Ellis, & Rasmussen, 2017) as well as to examine isomorphism and homomorphism in abstract algebra (Rupnow, 2021). Here we examine the nature of sameness (target domain) through a variety of metaphors addressing philosophical notions of sameness, proof-based notions of sameness, and informal notions of sameness. For example, "Sameness is a context-dependent concept" is a philosophical description of sameness that emphasizes sameness has many interpretations.

Methods

Data were collected from a survey sent to every 4-year college or university math department that offers abstract algebra in the United States. This survey addressed how algebraists think about sameness in general and in specific mathematical contexts. Participants were 197 mathematicians from 173 institutions who had taught at least one abstract algebra or category theory course in the last five years. For this paper, only the first three questions from the survey, which broadly regard the nature of sameness in math, are discussed:

1. What does it mean to be the same in a math context? (Q1)
2. How do you know two things are the same in abstract algebra? (Q2)
3. How is "sameness" in abstract algebra similar or different from "sameness" in other branches of math? (Q3)

Responses to each question were coded independently, but each response could receive multiple codes. Descriptive coding (Saldaña, 2016) was used to formulate conceptual metaphor-based codes for use in thematic analysis (Braun & Clarke, 2006). In keeping with this approach, we used multiple rounds of coding (Anfara, Brown, & Mangione, 2002). Both authors coded independently, then came to consensus for each participant's response. Next, codes were organized according to theme. In this process, we concluded some codes should be split or combined. We then revised codes, regrouped codes, and recoded responses as necessary. Finally, we discussed revised codes to ensure consensus and consistency.

Results

We highlight three categories of response that we observed in our coding: philosophical, proof-focused, and informal notions of sameness. We characterize types of responses in each of these broad categories. Frequencies of codes are provided in Table 1.

Philosophical Notions of Sameness

In this section, we highlight participants' descriptions of sameness that relate to philosophical notions about the nature of sameness: sameness as context-dependent and sameness as context-bridging. Because all but 5 participants (97%) referred to sameness as context-dependent and 67% highlighted context-bridging, we showcase multiple responses characterizing these stances.

Context-dependence. We highlight five main flavors of context-dependent answer. These include general context-dependent responses, a focus on specific concepts that transcend disciplines, comparisons of sameness in different disciplines, and a focus on different strengths of sameness. Some responses combine these ideas.

A number of respondents wrote generally about sameness being context-dependent. A few believed that mathematical sameness was too vague of a concept to engage with in the survey. Many of these responses were also given the code *vague/don't engage*. For example, Participant 20 wrote, "I don't know because it's entirely unclear what "same" means here as you are using it in this survey or what it means in abstract algebra." Others elaborated; for instance: "First you need to decide which kind of "the same" you mean. Typically there is a list of features that need to be compared, either directly or by constructing the appropriate function" (Participant 92). Both of these responses highlight the variety of meanings that could be associated with sameness and emphasize the importance of ascertaining the correct type of "sameness" to respond.

Table 1. Code Counts for Each Question and Across Questions

Group	Codes	Q1 n(%)	Q2 n(%)	Q3 n(%)	Distinct n(%)
	Vague/don't engage	12(6%)	5(3%)	4(2%)	16(8%)
Philosophical	Context-dependent	107(54%)	47(24%)	146(74%)	192(97%)
	Context-bridging	19(10%)	3(2%)	123(62%)	131(67%)
Proof Based	Bijective	17(9%)	30(15%)	18(9%)	50(25%)
	Cardinality	3(2%)	9(5%)	6(3%)	16(8%)
	Same properties	8(4%)	3(12%)	10(5%)	19(10%)
	Logically equivalent	19(10%)	30(15%)	10(5%)	46(23%)
	Computation	4(2%)	28(14%)	7(4%)	37(19%)
	Invertibility	2(1%)	7(4%)	1(1%)	9(5%)
	Mutual subsets	2(1%)	2(1%)	1(1%)	5(3%)
Informal	Same behavior	52(26%)	24(12%)	29(15%)	77(39%)
	Indistinguishable	8(4%)	1(1%)	3(2%)	11(6%)
	Mutual replaceability	2(1%)	1(1%)	0(0%)	2(1%)
	Equivalence	22(11%)	5(3%)	8(4%)	28(14%)
	Structure-preservation	18(9%)	29(15%)	22(11%)	50(25%)
	Operation-preservation	1(1%)	15(8%)	8(4%)	19(10%)
	Matching	5(3%)	15(8%)	8(4%)	21(11%)
	Relabeling	16(8%)	13(7%)	6(3%)	27(14%)

Other participants highlighted differences between general terms used for sameness in math. For example, Participant 10 attended to which differences were important: "Same can mean many things—equal, isomorphic, equivalent. It's important to know what differences matter and what don't to determine what 'same' means." Here they listed three concepts that relate to sameness, while noting that "what differences matter" is central to knowing sameness.

Many participants highlighted specific types of sameness in different disciplines to illustrate context-dependence. For example, when responding to Q1, Participant 149 wrote:

> Many branches of mathematics concern themselves with classifying objects up to some notion of sameness. These notions vary by mathematical context. Topology, for example uses such notions as homeomorphic, homotopy equivalent and the like. Abstract algebra generally uses isomorphic….In general, when the objects are "sets with structure," sameness means that the sets map bijectively and the structures are respected by this mapping.

A key aspect here is examples of defined concepts to show that the contextual meaning of sameness is given by well-defined notions in various disciplines.

A number of participants focused on distinguishing between stronger and weaker types of sameness. Participants did this in two main ways: by discussing the strength of one or more notions of sameness from a particular part of mathematics or by contrasting the relative strength of notions of sameness in different disciplines. For example, Participant 15 noted:

> The main difference lies in what properties we want the sameness to capture, and whether we want a strong or weak notion. In category theory, for example, a weak notion of "sameness" (categorical equivalence, or isomorphism of skeletons) turns out to be much more fruitful/useful than the stronger version requiring a complete correspondence between them.

In addition to discussing the relative strength of two notions of sameness within the discipline of category theory, this participant is also discussing the utility of different strengths of sameness for gaining insight. An example contrasting strengths in different disciplines comes from Participant 42's response to Q3: "Compared to more classical subjects, isomorphism is similar to "congruence" in classical geometry but more flexible than "equality" in classical algebra." This participant seems to imply that equality is a stronger notion of sameness than isomorphism and congruence, but that isomorphism and congruence signify similar levels of sameness.

Other participants combined multiple aspects of context-dependence. For example:

> As in my earlier answers, "sameness" in abstract algebra is often taken to be the existence of an isomorphism. Sometimes, and especially in category theory, it is taken as a canonical isomorphism given by a universal property. Depending on the particular category used in another subject (e.g., homotopies in topology), we might get what appears to be a more flexible notion of sameness. In set theory, equality gives a much more rigid notion of sameness based on equality: having precisely the same elements. Having a bijection is typically not considered enough for sameness. Similarly, in geometry sameness is often treated more rigidly: points may look alike but are not the same. In computational problems, such as in public key cryptography, one only has sameness if the isomorphism can actually be computed in acceptable time. (Participant 14)

Here, the respondent listed six types of sameness and compared their relative strengths.

Context-bridging. Most explicit references to sameness as a context-bridging concept occurred in response to Q3, which prompted participants to compare and contrast sameness in abstract algebra with other math sameness. Some of these responses were brief, such as Participant 7's response: "Not different at all," suggesting that sameness has a universal meaning of some sort. Others elaborated on their view of sameness: "At the undergraduate level, I think it's not that much different from other classes like topology. Sameness usually means preservation of structure, and what abstract algebra is studying [is] structure" (Participant 3). Here the participant focused on structure-preservation as a cross-disciplinary idea.

A number of other responses, especially for the first question on sameness in math, connected to category theory: "If you have two objects in a category, they are the same if there is an isomorphism between the objects" (Participant 22). Although one might say that category theory is its own subject, we viewed this type of reference to category theory as a way to view math holistically. Thus, abstract references to objects being linked by an isomorphism within a 1-category were viewed as a way to view sameness in math as a unified concept.

Both. Many participants invoked ideas that combined context-bridging and context-dependent perspectives. Some invoked category theory to provide a framework for comparing sameness in different disciplines, such as Participant 21:

> If we're talking about isomorphism in a given category, then the similarity is through the definition of isomorphism, and the differences will be encoded in the properties of the category. Algebraic isomorphisms must preserve algebraic structures, while this is not a valid notion for homeomorphisms of space….

The participant starts by writing about sameness as a consistent concept interpretable across various places in mathematics via category theory, which implies a context-bridging aspect of sameness, but they also focus on sameness notions in different disciplines, highlighting their differences and therefore the contextual dependence on discipline for these notions of sameness.

Others wove context-bridging and context-dependent notions together through highlighting the specific shared and distinct aspects of types of sameness.

> As noted above, this notion of structural equivalence appears in many areas of mathematics. Isomorphism in abstract algebra is, for instance, analogous to homeomorphism in topology, similarity in geometry, equivalent statements in formal logic. It is distinct from these it that it is defined based on how binary structures behave under their operations.

Here, Participant 132 equates sameness with “structural equivalence”, which is a topic relevant across mathematics. However, they acknowledge that the specific objects differ in each discipline (e.g., binary structures with operations in algebra).

Proof-focused Notions of Sameness

Some participants used language associated with uniqueness proofs or sameness of object including same properties, establishing a bijection or same cardinality, deduction of logically equivalent statements, verification via computation, finding inverse mapping pairs, and finding mutual subsets. Although these codes were rare (each in under 15% of responses to each question), as a group they provide another way of viewing mathematical sameness.

We first address the single property metaphors: bijection, cardinality, and same properties. These relate to determining a specific well-defined characteristic is shared by two objects: numerosity (for bijection and cardinality) or other properties like being abelian (for same properties). For example, finding a bijection was a common aspect of describing isomorphism in abstract algebra for question two: “For most purposes in abstract algebra, two objects are the same if they are isomorphic, i.e., if there exists a structure-preserving bijection between the two objects” (Participant 46). These notions of sameness generally relate to a given definition of sameness (e.g., isomorphism) or can apply to simple contexts like sets to provide something to consider when determining sameness of objects. Lack of alignment in such cases would be useful in a proof context to verify objects are not the same.

The broad verification metaphors observed include deducing logically equivalent statements and verification via computation. These can be viewed as two ways of proving equivalence of objects: being able to use theorems and logic to show uniqueness or equivalence and exhibiting a mapping or invariant that satisfies a definition. For example, consider Participant 21, who highlighted both logical equivalence and verification by computing an invariant:

> …to be exactly the same, they would have to be identical, which would require proving something that equates the definitions of the two objects.…Or alternatively, you can invoke classification theorems, for example all groups with two elements are isomorphic. Or you may be able to classify an object up to isomorphism by computing its invariants, for example by finding the dimension of a vector space.

Notice, logical equivalence appears through “equating the definitions of the two objects” and “classification theorems” while computation appears through computing invariants. Both techniques can provide full proofs of sameness.

Two narrow verification metaphors also arose: finding inverse pairs and finding mutual subsets. While not prevalent in the data (5% and 3% of respondents), these metaphors can be viewed as specific techniques for demonstrating equivalence. For example, Participant 19 noted, “To prove two sets are the same, then we show that each is a subset of the other,” which can be viewed as a technique for demonstrating sameness of sets. Similarly, for isomorphism, mutual inverses were described as a technique for demonstrating sameness of structures: “By constructing the two morphisms that are inverses. Widely used constructions are formulated as theorems, like the First Isomorphism Theorem” (Participant 108). While these techniques can lead to proofs of sameness of objects, they only apply in certain disciplines. Mutual inverses are only sensible in a function context whereas mutual subsets are only sensible in a set context.

Informal Notions of Sameness

Participants used a number of informal terms to highlight sameness: same behavior, indistinguishable, mutual replaceability, equivalence, operation-preservation, structure-preservation, relabeling, and matching. These notions appeared in over 40% of responses.

Same behavior was the most common of the informal notions, occurring in 39% of responses. Same behavior was coded when participants spoke generically about shared properties or behavior without specifying what was meant by these terms. For example, Participant 34 wrote: “I tell my students that two mathematical objects are “the same” if they have the exact same properties. That is, if you know how one of them behaves, then you know how the other behaves.” While this explanation highlights what should be the same, attributes and behavior of two objects, it is not clear what those attributes or behavior are. Similarly, indistinguishable, mutual replaceability, and equivalence were used almost as synonyms for sameness, largely when addressing mathematical sameness in general. For example, Participant 72 wrote: “It depends on the situation! In a given situation, it means that for all purposes related to that situation, the two entities are indistinguishable.” This response seems to suggest mathematical sameness is about being similar enough to put objects in the same class for a given situation.

Other participants highlighted preservation of operation or structure when describing mathematical sameness. References to operation-preservation and structure-preservation largely seemed to refer to the homomorphism property (or its analogue in another branch of math). For example, Participant 157 described sameness in algebra in a way that seems to refer to isomorphism, with operation-preservation describing the homomorphism property: “For groups and rings, there must be a one-to-one onto map (i.e., a bijection) that preserves the operation(s).” Participant 16 used structure-preservation in a similar way when describing mathematical sameness: “We identify structures we think are important or that somehow characterize the object, and declare that things are the same if they share those structures. This is usually formalized by the concept of a structure-preserving map.” This example seems to imply that “structure” is the homomorphism property or an analogous idea in another discipline. Note, even though “structure” is often used to refer to objects like groups, rings, and other algebraic objects, this did not seem to be the structure intended in “structure-preservation.”

Relabeling and matching were used to describe sameness in a way that aligned with isomorphism in algebra contexts. For example, Participant 196 described sameness in abstract algebra as “Effectively you can relabel one using the labels from the other and the structures will be maintained (i.e., we can define an isomorphism between them).” Essentially, we can interpret this “sameness” as being able to interchange the names of elements without fundamentally changing the underlying object. Although similar to relabeling, we described responses as “matching” when more emphasis was placed on the act of pairing elements in different objects instead of looking at structural relationships as a whole. For example, Participant 172 wrote:

> You know that they are the same if you can prove that an isomorphism between them exists. There are many ways to do this: for instance, for small enough groups, finding a way to organize the elements in two groups so that their group tables correspond suffices.

Note that the act of matching elements in Cayley tables was highlighted as a verification method, which has a qualitatively different sense from changing names to create a new, equivalent group.

Discussion and Future Work

Mathematicians talk about sameness in a variety of ways. Notions of sameness are present throughout math, have many names, and have many subtle distinctions. Sameness connects to much of what we do in math, whether verifying objects all satisfy a shared definition or verifying objects are the "same" up to a sufficient level, which permits reasoning about the object in a new way. Mathematicians' responses highlighted this balance through the general views of sameness as context-bridging and context-dependent. Mathematicians emphasized that "sameness" needs to be clearly defined. However, this emphasis on qualifying sameness leads to questions of motivation: is emphasis placed on precise definitions because this is a mathematical norm, or do mathematicians view students as too inexperienced to determine what needs to be the same in a given context? Potentially, both factors are important. Certainly, evidence suggests students can attend to unintended aspects when considering sameness (Melhuish & Czocher, 2020).

While not as common as the philosophical codes, a number of ways of thinking of sameness applicable to proof settings were highlighted. Though students often consider specific properties that hold for isomorphic groups (Dubinsky et al., 1994; Leron et al., 1995), it is less clear how much they examine context-limited notions like mutual inverses in algebra and mutual subsets in analysis. This aligns with the need to consider content-based differences when examining students' understanding of proof (Dawkins & Karunakaran, 2016).

Mathematicians used many types of informal language to discuss sameness, including same behavior, structure-preservation, and relabeling. While these types of language have been used for isomorphism and homomorphism previously, seeing them employed in a broader participant pool is encouraging (Rupnow, 2021). Furthermore, the use of informal language sets up an interesting contrast with the emphasis on the context-dependence of sameness. On one hand, a majority of respondents highlighted the importance of context to making sense of sameness and emphasized context clarity in discussing sameness. However, informal language, by definition, introduces ambiguity into statements. Future research should examine how mathematicians balance precise language with their informal language, both for themselves and when teaching.

Furthermore, what students take away from informal expressions is unclear. Hausberger (2017) specifically notes that "structures" are often left undefined, leaving students to determine for themselves what precisely is preserved in a given context. Although mathematicians in this study seemed to refer to relationships between elements when referring to structure-preservation, their uses of the word structure were not always clear. Recall Participant 3's statement "Sameness usually means preservation of structure, and what abstract algebra is studying [is] structure". The "preservation of structure" seems to refer to relationships between elements, but "what algebra is studying is structure" could be interpreted in an object (e.g., groups) sense.

This study suggests mathematicians have a variety of philosophical approaches to sameness, including math as both context-dependent and context-bridging. Furthermore, mathematical sameness can refer to discipline specific notions like isomorphism or general notions related to proof, such as logical equivalence. Finally, mathematicians invoked a number of informal ideas to describe sameness, despite emphasizing the importance of precision in mathematics.

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Student Mathematical Reasoning and Construction in a Digital Environment with Regard to Conceptions of Circles

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This study addresses what students' conceptions of circles are and how students' interaction with a digital environment influences their development of those conceptions. Two undergraduates from calculus 1 courses participated in exploratory teaching interviews for two one-hour sessions in fall 2019. In this study, we focus on moments in which the students' mathematical reasoning in regard to circles was on display. The two students demonstrated similar conceptions of and construction of circles on a physical medium at the start of Day 1. However, they interacted differently with the digital environment in the rest of the teaching interviews.. Consequently, their conceptions at the end of Day 2 were markedly different. We will discuss the progression of each student and how their interactions with the digital environment influenced their conceptions of circles, paying particular attention to the students' reasoning throughout.

Keywords: Geometry, digital environment, Euclidean construction, circle conception, teaching geometry

Visualization plays an important role in the teaching and learning of geometry as geometry pertains to the study of shapes and figures. On the other hand, research has reported that many students struggle with visualizing geometric shapes and figures that properly convey specific mathematical concepts (Walcott et al., 2009). As guiding students from a general conception (e.g. "This is an equilateral triangle because it looks like one") to a very specific conception of shapes (e.g. "This is an equilateral triangle because all sides of the triangle are equal") is a key element of geometry education (Jones, 2000), digital software and environments are considered devices to combat students' conceptions of geometric shapes and challenge students to rely on specific properties of shapes instead (Walcot et al., 2009). In particular, mathematics education research has shown that the use of digital manipulatives developed from dynamic geometry software such as GeoGebra could be helpful to improve their visualization skill (Baki et al, 2011; Dogan & Içel, 2011; Laborde, 2006). In line with this standpoint,this study focuses on students' conceptions of circles and how digital environments may influence students' development of conceptions of circles. To be specific, we developed GeoGebra applets and implemented them to see how digital environments may influence students' development and conception of circles. By analyzing two undergraduate students' construction and use of circles in the GeoGebra applets during two days of exploratory teaching interviews (Castillo-Garsow, 2010; Moore, 2010; Sellers, 2020) we address the following research questions:

1. What are students' conceptions of circles as they are evoked in geometry construction problems?
2. How are students' conceptions of circles evoked as they interact with the digital environment?

Theoretical Framework

When we analyzed students' conception of circles in the digital environment, we took for granted that different students may conceive of circles in different ways. Indeed, we noticed that students mainly used two conceptions. In some moments, students' reasoning and construction were based on circles as a geometric shape and in other moments, their reasoning and construction were based on circles as a geometric object (Sakauye, 2020).

Geometric shape. Students who think of circles as a geometric shape do not appear to consider the properties of circles in their reasonings or constructions. That is, they describe and construct circles simply as a shape and do not take into account any properties of the circle in doing so. We can categorize students' conceptions of circles as a *geometric shape* if their explanations and constructions do not include the use of the properties of a circle.

Geometric object. Students who think of circles as a geometric object include the properties of circles in their reasonings and constructions. We can categorize students' conceptions of circles as a *geometric object* if their explanations and constructions include the use of the properties of a circle, such as that all points on the circle are a fixed length from the center.

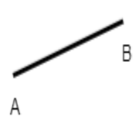
Our categorization of students' conceptions of circles is in line with the van Hiele levels of geometric understanding (Mason, 2009). In particular, the conception of circles as a geometric shape corresponds to van Hiele level 1 because students at van Hiele level 1 recognise shapes by their appearance, not by their properties (Mason, 2009). The conception of circles as a geometric object also corresponds to van Hiele level 2 or higher since students at that level recognise shapes by their properties (Mason, 2009).

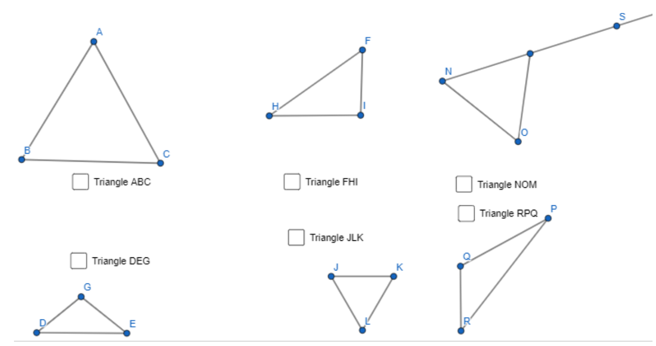
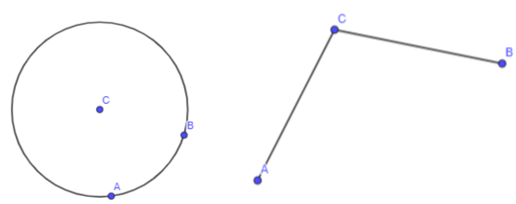
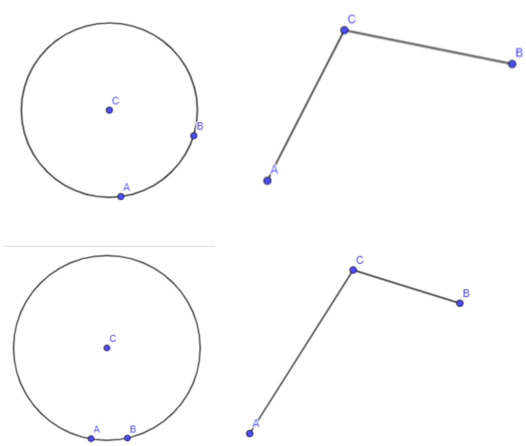
Methodology

Two undergraduates from calculus 1 courses individually and voluntarily participated in exploratory teaching interviews (ETIs) (Castillo-Garsow, 2010; Moore, 2010; Sellers, 2020) for two days in fall 2019. We selected students from calculus 1 classes because they were already introduced to geometry concepts like circles and equilateral triangles, but would not have learned the formal constructions yet - if they had already learned the constructions, their responses in this study would be reports of what they learned in class, not authentic reactions to the digital environment. The first author of this paper acted as the teacher-researcher, while the second author and another researcher served as witnesses of the ETIs. Data included videotapes of sessions, audio recordings, copies of students' written work, as well as screen recordings on a tablet used to access the digital software (a GeoGebra web page designed by Fischer (2019)). In this study we focus on moments in which the students' reasoning in regard to circles was on display, both in the digital software and on a physical medium.

Structure of the ETIs

The experiment took place over two days. A list of the tasks and the accompanying diagrams are included in the table below.

Task name	Diagram	Task instructions
Pretest Day 1		Given a segment AB, construct an equilateral triangle using a ruler and compass on paper.

True or false statement Day 1	☐ A line is a one-dimensional figure with no thickness that extends indefinitely. ☐ A line segment is a bounded or finite line ☐ A circle is a set of points in a plane, equidistant from a center point. ☐ A triangle is a three-sided polygon.	Answer whether the question about circles is true or false.
Triangle exercise Day 1		Check if each triangle is an equilateral triangle using the circle tool.
Circle exercise (original) Day 1		For the diagram with the curve of the circle drawn: Are the points on the circle equidistant from the center of the circle? For the diagram without the curve of the circle drawn: Are the two segments equidistant?
Circle exercise (revised) Day 2		For each of the diagrams with the curve of the circle drawn: Are the points on the circle equidistant from the center of the circle? For each of the diagrams without the curve of the circle drawn: Are the two segments equidistant?
Construction of equilateral triangle in GeoGebra Day 2		Given a segment, construct an equilateral triangle using the circle tool and the segment tool in GeoGebra.

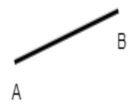
Posttest Day 2		Given a segment, construct an equilateral triangle using a ruler and compass on paper.

Table 1. A list of the tasks used in this study, including the instructions given. The original circle exercise can be accessed here https://www.geogebra.org/m/kmwyaua8 *and the rest of the tasks in GeoGebra can be accessed here* https://www.geogebra.org/m/hfqxjr4k

Data Analysis

The data collected was part of a larger study. In this paper, we report only our analysis related to student conceptions of circles. Drawing from recordings of the ETIs, we analyzed the students' words, gestures, and diagrams they produced by moments, to capture changes in students' reasoning.

Ongoing analysis was done spontaneously in the ETIs, described by Steffe and Thompson (2000, p. 276) as "creat[ing] new hypotheses and situations on-the-spot" in exploratory teaching. Thus, we formed hypotheses about the students' reasoning, and tested those hypotheses immediately in the form of questions. Our ongoing analysis also informed our decision to include additional exercises for Day 2, as described in the methodology.

After the ETIs were completed, we used Corbin and Strauss's (2014) grounded theory in the sense that two categories of students' conception of circles emerged from the data analysis: circle as a geometric shape and circle as a geometric object (see theoretical framework). We relied on these two categories to analyze our data and arrive at our results. Then, we examined what the students' conception of circles were and how they were evoked in each stage of the ETIs. Finally, we compared the two students in terms of their conception of circles and how their interactions differed in the digital environment.

Results

We found that the two students demonstrated similar conceptions of circles in the pretest as they constructed circles in the physical medium. During Day 1, they also displayed similar conceptions of circles in the digital environment. However, on Day 2, they displayed different trajectories of conceptual development in regard to circles. In this results section, we report our analysis on the different progression of each student and how their interactions with the digital environment influenced the development of their conceptions of circles differently.

Pretest Results

The main task in the pretest was to construct an equilateral triangle using a ruler and compass. A typical way to construct an equilateral triangle is constructing the intersection between two circles, each centered at one endpoint of the segment and with the segment as the radius. Neither student constructed the triangle in this way.

When constructing an equilateral triangle during the pretest, both students first checked the tick marks on the ruler to measure the length of the given segment, then attempted to construct two other sides of a triangle from each endpoint of the given segment. Both also checked the length of each segment with the use of a ruler as a measuring tool. Athena explained that what she had constructed was an equilateral triangle because all of the sides were "the same length AB", which was the length of the given segment she had measured. Faith had similar reasoning, that she only needed to measure segment AB in order to replicate the same measured length on the sides of the triangle. Since they both relied on measurement as their method of construction, neither seemed to consider the compass (or circles) as a necessary tool for construction of an equilateral triangle, and thus did not use it.

Day 1 Results

At the beginning of Day 1, the interviewer asked the students if a circle was a set of points on a plane equidistant from a center point. Both students verbally accepted the statement as presented to them in the digital environment. While completing the triangle exercises, both students had a specific methodology for constructing circles (shown in Figure 1). They created a point, T, in the middle of the triangle, intended to be the center of triangle ABC. Then, they dragged their pens back and forth from the point T, (which enlarges and shrinks the radius of the electronically drawn circle). When it appeared the shape of the circle included points A, B, and C, they lifted their pens off the screen, which constructed the circle centered at T. When they lifted their pens, they had not chosen a radius, so the pens were not directed to any point in particular, and the GeoGebra software automatically created a point where the pens were lifted off the screen. They both used a similar method to construct circles in the circle exercises.

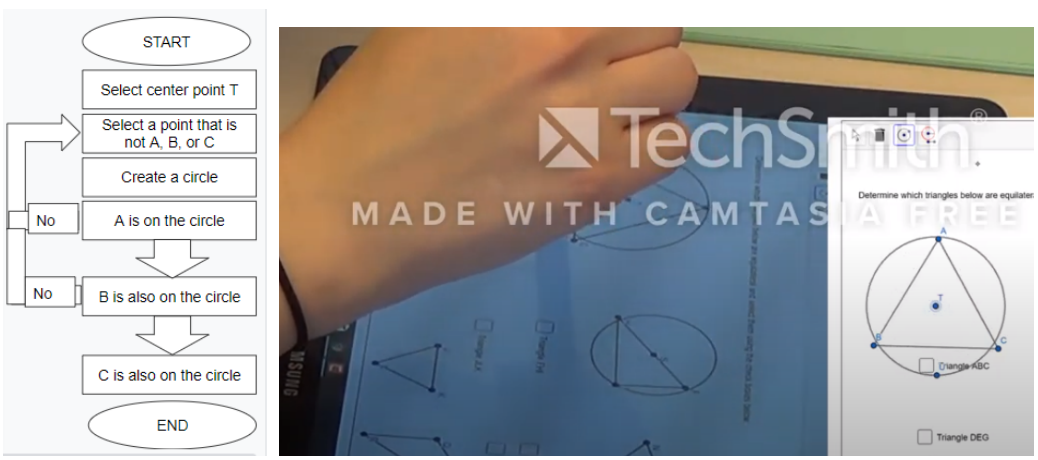


Figure 1. A flowchart of the method both Athena and Faith used to construct circles on Day 1 (left) and a screenshot of Athena using the method to construct a circle centered at T (right).

Once the circles were constructed, both students included the property that all points on a circle are equidistant from the center in their reasoning.

Findings from Day 1. The students' conceptions of circles remained similar on Day 1. Athena and Faith's verbal explanations demonstrated their conception of circles as geometric objects; however, they did not consider the properties of a circle in their constructions. Neither used the property that radii are equal in length to construct a circle, because neither selected a point on the circle as the endpoint of a radius. They instead dragged their pens across the screen, which enlarged the electronically drawn circle, so they could determine which points "look like" they are on the circle. Both students viewed construction as drawing the geometric shape based on visual stimulus (GeoGebra) because neither Athena nor Faith included radii in their construction of circles. Since they consider the properties of a circle (including equidistant radii) in their reasoning, but not in their construction of circles, we categorize their reasoning about circles as conceptions of geometric objects and their conceptions of circle construction as conceptions of geometric shapes.

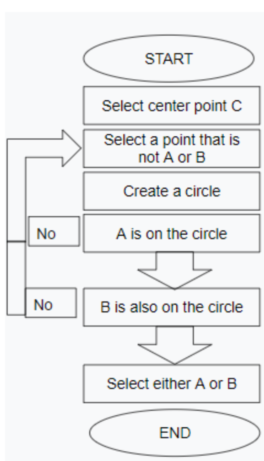
Results of Day 2

In response to the students' consistent strategies in constructing circles based on the visual shape of the circle demonstrated in Day 1, the research team developed and implemented a new requirement for construction in GeoGebra: the instruction "don't create new points". As GeoGebra will construct a point if the point selected by the user was not previously defined, this seemed like an effective intervention for the students' reliance on visual stimulus in construction. In this section, we describe individually each student's response to the new requirement in regard to her reasoning and construction of circles and analyze the major differences between the two students' conceptions on Day 2.

Findings from Day 2. We found that during Day 2, two students' construction of circles changed differently in regard to their integration of the new requirement.

Athena's reasoning was similar to her explanations during Day 1. Although the interviewer intervened with a new requirement for GeoGebra use, such instructional intervention did not have an impact on her conception of circles. Rather, her verbal reasoning continued to demonstrate a conception of circles as geometric objects. In regard to construction of circles, Athena did not seem to change her conception of construction of circles as geometric shapes. The change in her method (i.e. tracing a circle before selecting a point on the outside of the circle) seems to indicate that her accommodation of the intervention was to add a step to her previous method (checking that the point she chose would not create a new point). Instead of changing her conception of circles to a geometric object (e.g. including the property that a circle has a radius), she continued to drag her pen back and forth, relying on the shape of the constructed circle to determine the size of the circle.

On the other hand, Faith's verbal reasoning and construction of circles demonstrated a conception of circles as geometric objects by the end of Day 2. Following the new requirement, to not construct another point when constructing a circle in GeoGebra, she selected a point as the center of the circle, then dragged her pen across a segment she had chosen as a radius to select the endpoint of the radius on the circle. As she had chosen a radius in the construction of circles, we considered her conception of circles as geometric objects.



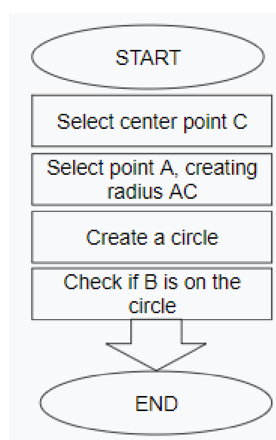


Figure 2. A flowchart of the method Athena (left) and Faith (right) used to construct circles in Day 2.

Using the digital environment to adapt their reasoning, the students fulfilled the new construction requirement. Athena simply added to her method of constructing circles - her method is different only in that instead of creating a new point on blank space, she traced a curve from the blank space to a previously defined point. The way she constructed circles in the digital environment did not include her reasoning about the properties of circles. Faith, on the other hand, seemed to change her method in order to integrate the new requirement. Instead of relying on the visual shape of the circle in construction, Faith used the properties of a circle (that all the radii were equal in length) to construct her circles.

Posttest Results

The research team administered the posttest on paper in order to observe what impact, if any, the transition back to physical materials would have on the students' reasoning. We observed that students' reasoning in the posttest had changed significantly from the pretest.

Findings from the Posttest. Both students' conceptions of circles in construction and reasoning did not change in the posttest. Athena's verbal reasoning about the properties of a circle demonstrated her conception of circles as geometric objects; however, her use of the compass without selecting radii of the circle in her construction demonstrates her conception of construction of circles as geometric shapes. Faith, on the other hand, selected radii in her construction of circles, and also reasoned using the property that radii of a circle are equal, demonstrating her conception of circles as geometric objects both in construction and verbal reasoning.

Conclusion

The GeoGebra environment created by Fischer (2019) was helpful to the students in this study as a concrete method of visualization for both their reasoning about and of circles. Athena commented on how interesting GeoGebra is and Faith said that GeoGebra "seemed like a cool program". In addition to the students' positive feedback about the application itself, GeoGebra seemed to help the students progress in their construction of an equilateral triangle and influenced their conception of circles.

Although both Athena and Faith began the ETIs with similar conceptions (demonstrating conceptions of circles as geometric objects verbally, but constructing circles with a conception of circles as geometric shapes), their conceptions at the end of the ETI were different. While Athena accommodated the teaching intervention without changing her conceptions of circle construction as a geometric shape, Faith incorporated the conception of circles as geometric objects in her constructions, with the inherent property that all radii of a circle are equal.

In regard to how the students' conceptions of circles were evoked as they interacted with the digital environment, our results indicate that the students demonstrated different conceptions of circles in their reasoning and construction. While further study is needed, the results of the students' changes in reasoning and construction suggest that GeoGebra has the potential to influence students' conception of geometric objects, such as circles, which would make it an invaluable tool in geometry classrooms, both in-person and online.

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How do Mathematics Instructors at Hispanic Serving Institutions Justify their Instructional Decisions?

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An underutilized strategy for addressing the underrepresentation of Black, Indigenous, and people of color (BIPOC) in the U.S. STEM workforce is directing more attention to Minority Serving Institutions (MSIs). Unlike some MSIs, Hispanic Serving Institutions (HSIs) are defined by the current demographics of their student population rather than the intention of the institution to serve a specific population. As instructors are the mediators between students that attend the HSIs and mathematics content, we use the framework of professional obligations (Herbst & Chazan, 2012; individual, interpersonal, disciplinary, and institutional) to investigate if or how they are being intentional about the ways their practice can impact BIPOC students. We found from preliminary analysis of nine interviews that instructors are recognizing the four professional obligations in some common ways. A tension exists between the disciplinary obligation to portray mathematics as culture-free and attending to students as individuals while fostering an inclusive classroom environment.

Keywords: Hispanic-Serving Institutions, Professional obligations

Introduction

The underrepresentation of Black, Indigenous, and people of color (BIPOC) in the U.S. STEM workforce is a widely recognized issue (NAS, NAE, and IOM, 2011; NSF, 2017; Palmer et al., 2015). An underutilized strategy to address this problem is to direct more attention to Minority Serving Institutions (MSIs) that serve large percentages of these students (NASEM, 2019). MSIs provide the primary pathways to STEM careers for minority students and are thus best positioned to address the urgent STEM workforce supply issue (NASEM, 2019).

There is a large variation in the types of MSIs that exist, so the best ways to bolster support for the faculty and students are likely not uniform. Postsecondary institutions with an enrollment of at least 25% or more Latinx undergraduate full-time students can apply for Hispanic Serving Institution (HSI) status (U.S. Department of Education, 2021). Unlike Historically Black Colleges and Universities (HBCUs) that are created with the intention of serving a specific demographic, institutions become HSIs based on the current demographics of their student population.

In the NASEM (2019) report on MSIs, a literature review of what practices and strategies best create the most success for students of color revealed the importance of intentionality. This included the creation of initiatives, policies, and practices that were intended to create a culturally mindful environment. While the report described several exemplary program-level initiatives, the review of the literature revealed that little is known about the intentionality of the faculty and lecturers working directly with students (NASEM, 2019). For instructors, we view intentionality as the making of instructional decisions with a conscious justification. We suspect, due to the designation of HSIs by their student demographics rather than their historical mission, that instructors at HSIs might not be well prepared to purposefully serve their BIPOC students. This is consistent with research by Hubbard and Stage (2009) that found few differences between HSI faculty and their faculty counterparts at predominantly white institutions in terms of perceptions and attitudes towards teaching and students. Recent and continuing shifts in demographics (Laden, 2004) have left many institutions to grapple with the implications of being an HSI.

Instructors are a crucial aspect of students' experiences and success in mathematics disciplines because they are the mediators between the content and the students. Mathematics instructors are used to a sense of autonomy over the content and pedagogy in their classrooms, even given the constraints of course syllabi and curriculum (Hora & Ferrare, 2012), and while they have extensive training in their disciplines, they often have little pedagogical training when they begin teaching (Ellis et. al., 2016). Instructors at HSIs are well-positioned to address the underrepresentation of BIPOC students in the STEM workforce, but can also replicate the existing inequities if they rely on their disciplinary enculturation to shape their pedagogical choices. Thus, we gathered information on the intentionality of faculty at HSIs in making their instructional decisions. This can inform the types of policies, professional development, and other strategies that can best support BIPOC students at HSIs.

Theoretical Framing

Given the importance of intentionality, we investigated how instructors made decisions that explicitly serve the student populations in their institutions. Cohen et al. (2003) posed that instruction is not merely about what teachers do, but also about how instructors shape the relationships between the teacher, students, and content, within the environment that surrounds all three. Very little research exists on instruction in the particular contexts of STEM at HSIs. Yet we know how important this environment is, in particular, for retaining students in STEM fields (Hubbard & Stage, 2009; NASEM, 2019).

We draw from the theory of *practical rationality* (Chazan, Herbst & Clark, 2016; Herbst & Chazan, 2012), which poses that instructors not only draw from individual resources like content and pedagogical content knowledge (Shulman, 1986) and individual beliefs for making teaching decisions, but also from social resources. One such social resource that instructors can use to justify their actions is *professional obligations*, which are shared by members of the profession and are tied to the environments in which they work.

Herbst and Chazan (2012) identified four professional obligations that serve as justifications for instructional actions (Herbst & Chazan, 2012). The four obligations include: 1) upholding the policies and practices of the department, institution, or state (*the institutional obligation*); 2) attending to the unique psychological, emotional, and other needs of students (*the individual obligation*); 3) representing the particular discipline being taught accurately and authentically (*the disciplinary obligation*); and 4) maintaining a classroom environment that is socially and culturally appropriate (*the interpersonal obligation*). For example, instructors could justify the decision to use more active learning in their classroom using the obligations: the individual obligation could motivate them to meet the needs of students who could learn more from instant feedback on their thinking; the disciplinary obligation could motivate them to give students an experience practicing science or mathematics in a more authentic way.

With this framing, we ask the following research question: How do faculty recognize their professional obligations to justify instructional decisions in the HSI context? We hypothesize that the way instructors recognize their professional obligations might show areas for more instructor intentionality in the HSI context. This could inform the ways administrators and researchers support faculty growth and development.

Methods

We used a random stratified sampling technique to identify the institutions we asked to participate in this study. This paper is based on nine interviews with undergraduate mathematics instructors from eight different HSIs. Participants represent a range of institution categories (associate's, master's, doctoral) and locations (West, Southwest, and Southeast). They come from a larger multidisciplinary (including physics, biology and chemistry) dataset of 40 interviews still being conducted at the time of this paper's submission. See Table 1 for participant details.

We coded the transcripts for each of the four professional obligations and looked across instances of each obligation to see emerging themes. The goal of this process was to create an organized framework of teacher decision-making in HSI environments.

We report evidence of the various ways STEM faculty at HSIs recognize the four professional obligations. Our data cannot, however, support causal claims about these themes being a product of the HSI institution status.

Results

We found that all four professional obligations contributed to the ways that participants in our study justified their instructional decisions. We highlight ways each professional obligation manifested in the context of HSI undergraduate mathematics environments.

Institutional obligation: Rare unless mandated by law

Brittany and Jessica noted that part of the institutional obligation that instructors now must recognize in the state of California is new legislation that prohibits mathematics coursework that lengthens the time of graduation (Assembly Bill No. 705, 2017). Instead of using placement exams that sometimes lead students through a lengthy string of remedial mathematics courses, math departments now progress students through any non-credit remedial mathematics courses in one year. Jessica explained that she supports the bill because, previously, the timeline to get through the mathematics courses "weeds out a lot of people".

The new legislation in California was the only explicit recognition of an institutional obligation related to instructing at an HSI by any of the nine instructors. Matthew began the interview by apologizing and asking what an HSI was, indicating that he had likely not made any instructional decisions based on departmental initiatives or policies related to teaching at an HSI. Most instructors denied much involvement from the department or institution in shaping their courses. For example, Francesca said the department reviewed her class, "But after a semester, they pretty much left me alone, because I know my standards are very high." Thus, any recommendations or requirements by the department or institution for policies that would directly support their BIPOC students went mostly unnoticed by instructors as they made pedagogical decisions.

Individual obligation: Different identities and assessment

Three instructors recognized the obligation towards meeting the needs of individual students by acknowledging the different identities that students brought to the classroom. Brittany and Samuel explained that, in comparison with the students they taught at a non-HSI, the students they teach at their current institutions more often carry identities beyond their student identity. Brittany said, "Student is a third title. I mean, we have mom as the first title, or dad, or caretaker or, a business owner, construction worker." Samuel recounted his initial shock when students in his class talked about teaching the mathematics content to their own children.

Table 1. Demographics, position, institution Carnegie classification, location, and field of each participant

Pseudonym	Self-identified demographics	Position	Carnegie classification	State	Field
Jessica	Hispanic or Latina female	Tenure-track Instructor	Associate's Colleges: Mixed Transfer/Career & Technical-High Traditional	CA	Mathematics
Brittany	White female	Assistant Professor	Associate's Colleges: High Transfer-Mixed Traditional/Nontraditional	CA	Mathematics
Steven	Caucasian male	Professor	Master's Colleges & Universities: Larger Programs	CA	Mathematics
Matthew	White male	Associate Professor	Associate's Colleges: Mixed Transfer/Career & Technical-High Nontraditional	TX	Mathematics Education
Samuel	White male	Assistant Professor	Doctoral University: High Research Activity	TX	Mathematics Education
Theresa	White female	Associate Professor	Doctoral University: High Research Activity	TX	Mathematics Education
Francesca	White female	Adjunct Professor	Associate's Colleges: High Transfer-Mixed Traditional/Nontraditional	CA	Mathematics
Niya	Indian female	Lecturer	Doctoral University: High Research Activity	TX	Mathematics
Tomas	White, Latino male	Visiting Professor	Doctoral University: Highest Research Activity	FL	Mathematics

Samuel and Brittany both expressed that this changed their view of assessment. Samuel said that, since coming to an HSI, he stopped taking attendance and instead reached out to students individually if he saw they missed class. Brittany has a nontraditional grading scheme for homework, so students have opportunities to practice and are not penalized for having other priorities that make meeting deadlines difficult. We interpreted Samuel and Brittany as not viewing things like late work as a product of laziness; instead, the students' complex identities caused these instructors to view late work as a product of their students' competing priorities. Consequently, they were inclined to work with their students to help them fit mathematics in with the rest of their lives.

Disciplinary obligation: Utility of content and representing a culture-free topic

Three instructors justified their instructional decisions as helping their students to realize that mathematics is useful and applicable to other courses and real-life situations. For example, Brittany explained that she wanted students to be able to critically analyze the statistics they encounter outside the classroom. Steven worked with undergraduate learning assistants (Otero, 2015) to implement student-centered instruction in his course. One of the learning assistants pointed out content that was useful in his later coursework. Steven used this knowledge to stress to students the utility of the content they were learning.

How the instructors represent the discipline depended on how they viewed the discipline. Two instructors expressed views that their mathematics departments considered mathematics to be a culture-free subject. Francesca praised her department for this: “We don't see a color or sex or anything. We see a raw being that you can teach mathematics to. Mathematics has no color.” Theresa, a math educator, explained that “We have a lot of faculty who are very aware of issues of identity and professional perception of self in the discipline, but by no means the majority. … Overwhelmingly, the department leans towards math is math. Math is neutral. Math doesn't discriminate.”

Two instructors, including Francesca, expressed that they taught and treated all students similarly due to this understanding of the discipline. Francesca, however, continued that “mathematics is this infinite number of like sides to it, you know, like a diamond, like more than a diamond. And the way people understand is different.” So although she felt that mathematics is an objective subject, she also found herself teaching it differently to different individuals. The stance that mathematics is culture-free is closely related to instructors’ (lack of) concerns for the interpersonal environment of their classroom.

Interpersonal obligation: Inclusivity and Student engagement

Three of the mathematics instructors explicitly stated that concerns for the inclusivity of their classroom did not play a role in their teaching. For example, Jessica stated that the demographics of her students did not play a role in her teaching. Instead, she relied more on the questions students asked or the correctness of their answers to adjust her teaching. For those that did have concerns about inclusivity, one way instructors used this obligation to justify their actions was in terms of making the course content more inclusive. Matthew and Samuel sought to adjust examples so they were more relevant to the populations in their class. Theresa tried to be inclusive by being aware of the experiences that the students who were African American or Hispanic were having in the class. She tried to give extra attention to whether they were comfortable sharing their ideas and whether they were feeling heard by herself and the groups they were working in on a day-to-day basis.

The interview protocol did not directly ask about student engagement, but many instructors expressed that they aimed to foster an environment that encouraged students to engage with each other. In response to a question about maintaining an inclusive environment, Jessica said she facilitated activities on the first day of class to give students a sense of community and connections to other students within the classroom. For example, she has students discuss in groups what fears they might have after reading the syllabus. Steven tried to get people working together inside the class so that they would be more likely to reach out to other students for help outside the class as well.

Scholarly Significance

As the primary mediators between the Latinx students that attend HSIs and the STEM content, instructors have the opportunity to impact students' experiences as they prepare to enter the workforce. We asked how faculty recognized their professional obligations to justify instructional decisions in the HSI context. Instructors at HSIs gave justifications for their instructional decisions that recognized the four professional obligations (summary in Table 2). This study highlights where potential might exist for instructors to better serve their BIPOC student populations. Information about how they are recognizing the obligations revealed tensions between the belief that mathematics is above culture and wanting to foster an inclusive community that respects the unique identity of individuals. Without having opportunities to address this tension, it is likely to remain unresolved.

Table 2. Themes from the ways HSI mathematics faculty recognized their professional obligations in their instructional decision-making

Professional Obligation	Ways the professional obligations were recognized
Institutional	● Adapting classes to the requirements around remedial courses
Individual	● Respecting the varied identities of the students ● Changing view of assessment to better accommodate students' lives
Disciplinary	● Aiming to show students the utility of the discipline ● Viewing mathematics as a culture-free discipline
Interpersonal	● Using or creating inclusive materials ● Fostering student engagement

These accounts inform three ways to support instructors. First, this work offers clarity about an unresolved tension that may exist among mathematics instructors' professional obligations. Instructors may struggle to treat the discipline as culture-free while recognizing their individual and interpersonal obligations, and never encountered the opportunity to address it directly. Second, this work shows what facets of professional obligations instructors are not taking into account. Policy-makers or departments could use this information to motivate how they try to direct instructors' efforts. For example, instructors did not mention practices associated with culturally relevant pedagogy, either from their own volition or incentivized by the institutions where they work. To the contrary, some instructors reported that they make decisions to treat all students similarly. Departments or policymakers could use these results to legitimize professional development on such a topic. The results from this study show that instructors may be open to attending to the different ways individual students learn, even if they view mathematics as culture-free.

Third, if instructors value certain facets of professional obligations, but there are better actions that could serve those intentions, this could inform the types of resources made available to those instructors. For example, we found that instructors prioritize student engagement as part of their interpersonal obligation. If instructors are unaware of practices (or the need for practices) that make BIPOC students feel a greater sense of belonging so they can engage with their peers, professional development can be tailored to those needs by appealing to instructors' interpersonal obligation.

As we continue this work we will ask if these themes appear in a larger, representative national sample of STEM instructors at HSIs. We intend to look for patterns of these themes coupled with the use of evidence-based practices and similarities and differences across disciplines. For example, in the natural sciences more than mathematics, more instructors describe giving students authentic disciplinary experiences through activities that model what scientists do. Involving students in laboratory research to develop their disciplinary identity is mentioned more often than involving students in theoretical mathematics research. As these decisions are mediated by content, we expect that even if the professional obligations remain the same, how instructors instantiate them will vary by discipline.

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Sign Language Variation and Iconicity in Undergraduate Mathematics

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Two American Sign Language interpreters and two Deaf instructors were asked to sign various undergraduate mathematical terms and definitions. Various types of iconicity were identified in the signs and each term had more than one sign demonstrated for it. The participant's preference and choice for signs reflected their mathematical beliefs and prior experience. While the motivation for all participants was to accurately translate the meaning of the mathematical concepts, not all participants shared the same type of preference for sign type. One out of the four participants preferred to use signs that represent the notation used for the concept, while the other three participants preferred to use signs that showed some conceptual meaning.

Keywords: Math Education, Sign Language, Deaf Education, Linguistics, ASL, Iconicity

Background Literature

Deaf students drop of out of college at alarmingly high rates (Walter, Foster & Elliot, 1987; Walter, 2010). Most deaf undergraduates attend a university where a mainstream classroom is the only environment (Stinson, Liu, Saur & Long 1996), and thus might need to utilize interpretation services for American Sign Language. American Sign Language (ASL) is the primary language of the Deaf Community in the United States. Not all deaf people use ASL to communicate, but many students choose to utilize ASL interpreters in the classroom.

Savic and I conducted an initial qualitative study with the aim of developing an understanding for what Deaf and Hard of Hearing (DHH) students' experience is like in undergraduate mathematics (Simmons & Savic, 2018). The students acknowledged struggles related to instructor communication and struggles related to incorrect or confusing sings used by interpreters with words like "sign" and "function" (which have multiple meanings in English). If there is confusion about which signs to use for certain mathematical ideas, the investigation of ASL use in higher level mathematics may provide insight into the difficulties DHH students face in undergraduate mathematics courses.

Several researchers have used embodied cognition to analyze how students' mathematical conceptions are grounded in their bodily experiences (Núñez, 2003; Smith 2009). Núñez et al. (1999) explain that embodied cognition grounds the situatedness of knowing and learning in bodily experience and bodily actions "realized through basic embodied cognitive processes and conceptual systems" (p. 46). Embodied cognition also takes the position that mathematics does not exist independently from human understanding. "Reality is constructed by the observer, based on non-arbitrary culturally determined forms of sense-making which are ultimately grounded in bodily experience" (p. 49).

Theoretical Perspective

Meir and Tkachman define iconicity as "a relationship of resemblance or similarity between the two aspects of a sign: its form and its meaning" (2014). In essence, the opposite of iconicity is arbitrariness.

Embodied cognition has been used to study gesture in mathematical talk (Edwards 2008; 2009) and sign language use in elementary mathematics (Krause, 2017, 2018, 2019). Krause (2019) adopted theoretical constructs from Edwards work on categorizing gestures (Edwards, 2009). *Iconic-physical* reference represents physical objects or actions. For example, the sign used for the German word for simplifying uses the dominant index finger to "cross out" numbers from what looks like a numerator and denominator in the air. *Iconic-symbolic* reference represents symbolic notation or graphical representations of mathematical ideas. Krause (2019) noted that the sign for simplifying had iconic-symbolic elements because the sign represented fraction notation spatially: the index finger would be placed horizontally to represent the fraction bar. She used her own theoretical construct, *innerlanguage iconicity* to refer to signs that reference other mathematical ideas or concepts by similarity in performance of the sign. For example, the German sign for denominator "borrows" the sign for number within the performance of the sign for denominator.

The study provides evidence that sign language influences conceptualizations through what the student perceives as iconicity of signs. All three types of iconicity reference were seen to influence conceptualization of corresponding mathematical ideas. The author points out that the categorization is not "exhaustive" but can "sensitize practitioners to the choice and design of signs they use in the mathematics classroom" (p. 15).

Research Question

What types of variation and iconicity are there in the way ASL interpreters and Deaf instructors sign undergraduate mathematical terms and definitions (and what reasons do they express for these choices)?

Methods

Because of the lack of information about ASL use in undergraduate mathematics, exploratory interview methodology lends itself to provide a basis of information to conduct future studies in this area of research. Interviews held through Zoom in a variety of ways helped gather information about ASL use by interpreters, DHH students, and a Deaf instructor in undergraduate mathematics courses. The participants were able to choose their preferred mode of communication: spoken English, text-chat interview, and the use of an ASL interpreter were types of communication the participants chose to utilize. They also consented to have pictures and videos shown in reports including their faces for the purpose of retaining the meaning of the signs (as facial expression is an important element of ASL).

The four participants of this study (all residing in different states and given pseudonyms) consist of two ASL interpreters and two Deaf mathematics students/instructors. Martha and James are the two ASL-interpreter participants. James has been interpreting ASL for 27 years and is hired through an agency to interpret at a large Southwestern university. Martha has been interpreting for 35 years and is currently hired on as staff at a community college known for having a large population of deaf students.

The other two participants of the study are Deaf and have had experience learning and teaching mathematics in college: Andrew and Thomas. Andrew has a BS in mathematics from Gallaudet (a university for deaf students) where he was able to take mathematics from Deaf instructors and all his classes were taught in ASL. He is currently a PhD student and has taught 45 credits worth of mathematics in ASL. Thomas majored in four STEM fields and chose to have some of his mathematics and physics courses interpreted in ASL. He taught an undergraduate applied mathematics course in ASL and had an interpreter for his all-hearing class.

The interviews were one-hour in length and consisted of two phases: general experience discussion and term card signing. During the first half of the interview, participants were asked questions about their general experience with ASL and mathematics, sign creation for non-standardized terms, and techniques used when signing about mathematics in the classroom. During the second phase of the interview, participants were presented with term cards that contained a mathematical term they have probably experienced (based on their response about which math courses they are familiar with) along with a definition for the term. The definitions were pulled from a variety of sources to simulate encountering a new topic in the classroom and account for course variety across institutions. All four participants were given the same instructions: to show the signs they use and have seen for the term and to sign the definition written below it.

The analysis of the data consisted of identifying each sign given for the term cards and categorizing those signs by iconicity categories: innerlanguage iconicity and iconic-symbolic iconicity. Innerlanguage iconicity has three subcategories: conceptually-linked signs, English-linked signs, and initialized signs. Iconic-symbolic reference has two subcategories: notational signs and graphical signs. While the main categories are drawn from Edwards (2009) and Krause (2019), the subcategories were created as themes found during data analysis and are defined in the results section.

Results

I have categorized the variation of sign use leaning on Krause's iconicity construct, innerlanguage iconicity (2019) and Edwards gesture iconicity construct, iconic-symbolic reference (2009). I first define the category, give an example of a sign in the category, and provide transcript of a participant's explanation or reasoning for their sign choice.

Innerlanguage Iconicity

Leaning on Krause's definition of innerlanguage iconicity, an innerlanguage iconic sign is a sign that shows reference to another sign. This type of reference can show a linkage of the referenced concepts to the concept that the sign represents. This could be in the form of borrowing the same handshape, movement, or position of the reference sign.

Conceptually-linked sign: A sign that has iconicity to another sign with the intention of showing conceptual relation between the two ideas, I refer to as a conceptual sign. Out of the nine term cards, conceptually-linked signs were given for only one of the main terms, *Derivative*. However, four of the eight variations of signs given for *Derivative* were conceptually-linked signs.

Figure 1 illustrates how Martha uses the same motion to the sign for "*change*" but uses the handshape for the letter "*d*". The use of the same motion as "*change*" is intentional in that it links the related concept of change to the concept of derivative.

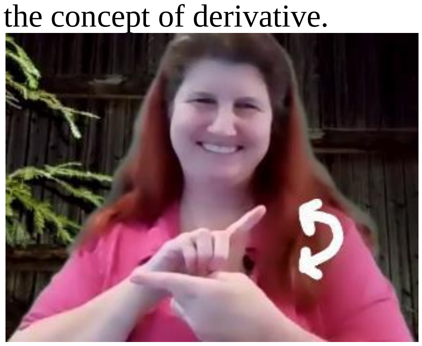

Fig. 1 Martha (interpreter) is shown demonstrating the sign they use for Derivative.

In Figure 2, Thomas uses the sign "derive-from" for Derivative. This is an intentional linking of the concept to-derive-from and the concept of Derivative.

Fig. 2 Thomas (Deaf student/instructor) is shown demonstrating one of the signs he uses for derivative. He uses the sign "derive-from" for Derivative.

English-linked sign: I use the term English-linked sign to refer to signs that incorporate signs from another concept that uses the same English word but can be considered a separate concept. Figure 3 shows Andrew using the sign "*line*" for *Linear*, and Figure 4 shows him using the sign "*independent/independence*" for *Independence*.

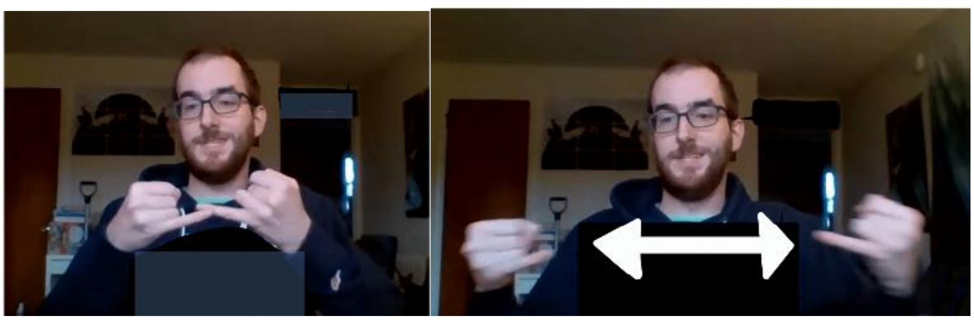

Fig. 3 Andrew (Deaf student/instructor) is shown demonstrating the first half of the sign for Linear Independence, also "line".

Fig. 4 Andrew is shown demonstrating the second half of the sign for Linear Independence, also "independence/independent".

Andrew explained that he doesn't think the sign shown above is a very good sign because the sign for line doesn't exactly mean "linear", but it is the sign that is used for the term "*linear*". It is the sign typically used for *Linear Independence* and was given by all the participants who were shown the card.

Initialized sign: The discussion of initialization came up during Martha's interview, but is seen throughout the language of ASL. If a sign is initialized, it uses the fingerspelled version of the first letter of the word. For example, the sign for "mathematics" uses the handshape for the letter "*m*" on both hands facing in as they brush across each other with the dominant hand on top. Different mathematics topics will use the same motion but with the handshape of the first letter of the topic: algebra uses the "*a*" handshape and calculus uses the "*c*" handshape. Out of the nine term cards, an initialized sign was given by at least one participant for four of the main terms (this does not include completely fingerspelled signs which lie outside the iconicity categories).

While signing the term card for function, Martha discussed different signs she uses for *Domain* as seen in Figure 5.

Fig. 5 Martha (interpreter) is shown demonstrating two different signs she uses for Domain.

The first sign for *Domain* uses an open dominant hand moving side to side across the other arm, and the second sign uses the handshape for "*d*" with the same motion. She demonstrated similar signs for *Range* using the handshape for "*r*" and moving her hand up and down (versus side to side).

Martha described her use of initialization in the classroom for the purpose of helping the student learn the vocabulary term:

> I sometimes include more initialization if it has to do with terms that are going to come up on the test… As a technical interpreting tool, I will sometimes bring that into my interpreting to help students.

Iconic-Symbolic Reference

A sign with iconic-symbolic reference shows reference to symbolic notation or graphical representation of the signed concept. It can use motion, handshape, and position to mimic how the notation or graph looks.

Notational Sign: A notational sign has iconic reference to the symbolic notation used to reference the concept.

As illustrated in Figure 7, James (interpreter) uses the handshape for "*d*" and motions a "*z*" shape in the air. While interpreting for a physics student, he collaborated with other ASL users to create a unique sign for *Derivative*. He mentioned that the sign "is definitely not standard. [It's] something that the physics student, a linguist and [himself] worked on and came up with." It was influenced by the notation of . Andrew, during his interview, described a sign used by many people at RIT that resembles this sign.

Fig. 7 James (interpreter) is shown demonstrating his primary sign for Derivative.

Graphical sign: A graphical sign has iconic reference to how a mathematical idea can be represented graphically.

Fig. 8 James is shown demonstrating a sign for slope.

Both versions of the sign (seen in Figure 8) might be used for slope generally, however, the left picture, would be used to represent positive slope; the right picture would be used to represent negative slope. Notice, this is signed from the signer's perspective (rather than the audience's perspective).

Category Summary

Table 1 illustrates whether or not a sign in the given iconicity categories was presented by at least one participant for the term.

Table 1. Category breakdown of given signs for term cards.

	Innerlanguage Iconicity		Iconic-Symbolic Reference		
Terms	Conceptually-Linked	English-linked	Initialized	Notational	Graphical
Function	No	No	Yes	No	No
Limit	No	Yes	No	Yes	No
Derivative	Yes	No	Yes	Yes	No
Rate of Change	No	No	Yes	No	No
Slope	No	No	No	Yes	Yes
Span	Yes	No	No	No	No
Linear Independence	No	Yes	Yes	No	No
Concavity	No	No	No	No	Yes
Continuous Function	No	Yes	No	No	No

Discussion

There were not only differences in the way each participant signed the terms and definitions presented to them in the study, there were also different ways the signers translated the cards, often offering various ways to sign each term and definition. As ASL is not standardized at the undergraduate mathematics level, sign variation documentation can help bring awareness to alternative ways of signing mathematical ideas. Students may struggle with unfamiliar signs if they attend college in a different region than where they learned mathematics in high school.

Krause found that iconicity can influence how a student can conceptualize corresponding mathematical ideas (2019). The data from this study suggests that an interpreter, or Deaf instructor might choose or prefer a sign as a result of their mathematical experience, not only with exposure to certain signs, but their conceptions of the mathematics as well. This inevitably will impact the sign usage and mathematical conceptions of DHH students (it may not be clear in exactly what way). This is an important result, as there is no mathematics education requirement for interpreters to interpret at the university level. It is up to the interpreter to be prepared to accurately translate the material, which possibly shouldn't be expected from interpreters who are paid by the hour of interpreting (rather than additionally being paid for time spent preparing).

While there were two categories of innerlanguage iconicity and iconic-symbolic reference, there were two major themes that seemed to underly sign choice or sign preference among the participants. Martha, Andrew, and Thomas tended to use or prefer signs that referenced the concept directly or a related mathematical concept in the performance of the sign. Typically, these signs would lie in the Conceptually-Linked and Graphical Sign categories. James, on the other hand, stated and demonstrated that he preferred signs that represented the notation of the concept (Notational Sign category). While his sign preference was different, his motivation was the same as the other three participants. They all emphasized the importance of accurately translating the mathematical concepts' meaning. James believes that a sign which bears resemblance to a related concept's sign (innerlanguage iconcity) won't always be conceptually accurate across multiple contexts. He stated that by avoiding using more conceptual signs, you can avoid attributing misconceptions to mathematical ideas. Martha believed initialization could help students with the translation of concepts from ASL back to English for test preparation. While the interpreters had more inclination of including more notation and English-related signs, their motivations were grounded in how certain signs could offer more support for the students' conceptual understanding in the classroom.

Conclusion

The ability to express mathematics in a language that can represent three-dimensional space has a great opportunity for future study. Understanding the different ways signers choose to express mathematics in ASL and why they choose to express it in a certain way can provide insight into how language plays a role in conceptualizing certain mathematical ideas. Sign Language is different to spoken language in that it provides a way to fluently discuss graphical spaces in a way that is unavailable to spoken language. Iconicity plays a big role in sign languages, and its role in undergraduate mathematics might help us understand how DHH students think about mathematics. This, in turn, could lead us to a new or deeper understanding of how all students might think about mathematics.

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Two Students' Conceptions of Solutions to a System of Linear Equations

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Systems of linear equations (SLE) comprise a fundamental concept in linear algebra, but there is little research regarding the teaching and learning of SLE, especially students' conceptions of solutions. In this study, we examine students' understanding of solutions to SLE in the context of an experientially real task sequence. We interviewed two undergraduate mathematics majors, who were also preservice teachers, to see how they thought about solutions to SLE in $\mathbb{R}^3$, especially linear systems with multiple solutions. We found participants used their knowledge of SLE in $\mathbb{R}^2$ to think about systems in higher dimensions, sometimes ran into algebraic complications, and initially did not find the third dimension intuitive to think about geometrically. Our findings highlight students' ways of reasoning with infinite solution sets, such as moving toward the notion of parametrization.

Keywords: linear algebra, systems of linear equations, student thinking, solution

Systems of linear equations (SLE) are a core concept in linear algebra. Having a deep understanding of this concept helps in understanding many applications, as well as other topics in linear algebra. Instruction often foregrounds solving methods that focus first on unique solution cases, before moving to the much more common cases of no solution or infinitely many solutions. There is evidence that such procedural instructional approaches to SLE are often inadequate for helping students make sense of SLE in these more common cases (Litke, 2020; Vaiyavutjamai & Clements, 2006). Procedural instruction links to memorization of rules for knowing when a system has no solution or infinitely many solutions and what that means for the geometry of the SLE (Huntley et al., 2007).

In the current study, we examine two students' understanding of solutions to SLE, from both algebraic and geometric interpretations. We used two research questions to guide our analysis: *(1) How did students reason about solutions in the context of the system of equations task sequence? (2) What did students wonder about in this same context?* We discuss the ways participants consider solutions to SLE and questions they voice as they work through a task sequence designed as part of a research-based curriculum design project in linear algebra.

Literature

Research regarding SLE is limited but growing. In this small pool of research, some regard solutions to SLE, whether through task development (e.g., Possani et al., 2010; Sandoval & Possani, 2016) or examining students' understanding of solutions algebraically and geometrically (e.g., Harel, 2017; Huntley et al., 2007; Oktaç, 2018; Zandieh & Andrews-Larson, 2019). For SLE instruction, Possani et al. (2010) developed a task about a set of streets, asking students which streets could be closed for road work without disrupting the flow of traffic. Major parts of this task were modeling the context, using variables to represent important quantities, and then writing a system based on those understandings. Possani and colleagues argue that their task helped students create a solid foundation for understanding SLE and their solutions, and that students were able to transfer what they learned in this task to other problems regarding SLE. Sandoval and Possani (2016) developed a task to see when moving between representations in $\mathbb{R}^3$ (e.g., graphs versus equations of vectors or planes) makes sense for students. They found students could naturally interpret geometric representations of solutions to SLE but that connecting geometric representations to algebraic representations to be less natural.

In regard to student thinking about SLE, several authors have found single solutions to SLE to be more intuitive for students to interpret than systems with no or infinitely many solutions (Harel, 2017; Huntley et al., 2007; Oktaç, 2018). Oktaç (2018) and Huntley et al. (2007) noted that when students worked to interpret results of algebraic manipulation like $8x = 8x$ or $2 = 4$, students often rely on memorized rules to decide which had infinitely many solutions or no solution. Huntley and colleagues also noted these rules were unhelpful to students in deciding whether parallel lines or the same line in a graph meant no solution or infinitely many solutions. Zandieh and Andrews-Larson (2019) found that, when solving SLE, students tended to row reduce on matrices or manipulate the given systems. Students who used the row reduced echelon form were generally successful in solving, but connections between row reduced forms and their geometric representations were not straightforward for them to interpret. Thus, the research points to a need for more tasks that provide opportunities for students to algebraically and geometrically examine SLE with infinitely many solutions, especially those in $\mathbb{R}^3$.

Theoretical Framework

We drew from Realistic Mathematics Education (RME) when designing the task sequence. RME is a theory used to design mathematics tasks and involves Freudenthal's (1973, 1991) framing of mathematics as a human activity. Three core heuristics of RME are guided reinvention, emergent models, and didactical phenomenology (Larsen, 2018). The first refers to which aspects of a concept students are to reinvent along with an instructor's guidance. The second refers to models students develop to organize their mathematical activity. The third refers to the selection and use of phenomena in a task sequence to build toward a mathematics concept. We wanted students to develop reasoning about what the *collection* of solutions to an SLE looks like and how those solutions relate to one another (both numerically and geometrically). We hoped students would be able to build imagery for that collection of things (solutions) by working through our task sequence. Our goal is to understand our participants' reasoning related to this reinvention of infinite solution sets.

Another important heuristic in RME is the use of experientially real starting points (Gravemeijer, 1999), an aspect of didactical phenomenology. These starting points are intended to help students engage in the development of a mathematical idea, rather than have them simply apply prescribed procedures to a situation. In our task context, students find valid combinations of meals in a meal plan. We assume most students are either familiar with having a meal plan that requires a limited number of meals, or, at least, familiar with experiences less specific, like using a limited amount of money to make purchases. Historically, SLE have been used in agricultural and economic situations (Andrews-Larson, 2015), which present contexts for reasoning about SLE (i.e., didactical phenomenology). In terms of the mathematics in the tasks, we assume students have prior experience writing equations that represent a situation, listing solutions (e.g., to at least one or two equation equations), graphing (more in 2d than in 3d), and manipulating equations using substitution and elimination.

Methods

This study is part of a broader NSF-funded project (1914793, 1914841, 1915156) focused on the development of curricular materials in inquiry-oriented linear algebra. This particular task sequence was designed to help students create or reinvent ways of thinking about solutions to SLE, especially infinite solutions sets. Participants were two undergraduate students at a large public university in the Southeastern United States. Both were mathematics majors and preservice secondary mathematics teachers. Both participants were white; one was a woman ("R"), and one was a man ("L"). The students were recruited by asking faculty in a local secondary teacher preparation program to identify two math majors preparing to be secondary math teachers who had taken Calculus I but not linear algebra, were neither atypically "strong" nor "weak" in terms of their mathematical preparation and would be willing to explain their reasoning. Two of the authors interviewed the participants across four consecutive days on Zoom, working through as much of the designed task sequence as possible. We used a paired teaching experiment (PTE) to see how students reasoned with the tasks (Steffe &Thompson, 2000) and to allow for discussion between the participants and the interviewer. When working through the task sequence, participants briefly worked individually before discussing their reasoning. The interviewer prompted participants to think about a problem and explain their ideas, asking questions to further their thinking at some points.

Task Sequence

The tasks posed to participants in the PTE begins with an experientially real situation focused on a university meal plan (see Figure 1). During the first session, both participants worked to identify numbers of breakfasts, lunches, and dinners satisfying the given constraints, as well as estimate the number of options. During the second session, the pair of participants worked to represent the set of solutions they identified in a three-dimensional space using the corner of a cardboard box. During the third session, they explored the geometry of linear systems with three unknowns using Geogebra. During the fourth and final session, they generated examples of systems of equations with specified numbers of solutions and developed generalizations based on this work. In this proposal, we focus on participants' work during days 1 and 2 of the PTE.

PTE Day 1. Participants were first prompted to consider a constraint regarding the number of meals that can be purchased in the context of the meal plans (Figure 1, Part 1). They were asked to list a few different choices for the 210-meal plan and estimate the total number of possible 210-meal plans. The goal for this part of the task was to have the participants reason about and organize a large solution set (not to treat it as a combinatorics problem).

Participants were then asked to consider an additional constraint related to the cost of meals (Figure 1, Part 2). Participants were asked to write equations corresponding to each constraint, and to identify a solution that satisfied the constraints for the number of meals but not the cost (and vice versa) as well as a solution that simultaneously satisfied both constraints. This pressed participants to consider solutions that would simultaneously satisfy two different equations.

PTE Day 2. Participants were first asked to predict what it would look like if all of the solutions to the number of meals constraint ($b+l+d=210$) were graphed. (Note: here b, l, and d are the number of breakfasts, lunches, and dinners, respectively.) They were then asked to find a way to represent (algebraically and geometrically) all of the "no dinner" meal plans (ignoring the 105-meal maximum from day 1), and to make similar predictions corresponding to the cost constraint equation ($5b+7l+10d=1500$). Finally, they were asked to consider the graph of both sets of solutions simultaneously using the corner of a cardboard box to represent $\mathbb{R}^3$.

Part 1: Number of Meals. A university meal plan called the "210-meal plan" requires that a student purchase exactly 210 meals for a 15–week semester (i.e., on average 2 per day for 15 x 7=105 days).
Part 2: Costs of Meals. You just made an estimate of how many different choices would fit the requirements of the 210-meal plan. As you read the brochure more carefully, you noticed that the cost of your 210 meals must add up to exactly $1500. Breakfasts cost $5 each, lunches cost $7 each, and dinners cost $10 each.

Figure 1: The Meal Plans Context.

Data Sources, Analysis, and Limitations

Data sources consisted of video recordings of the interviews, field notes, and participant work. The four authors watched the video recordings of the interviews together, taking notes regarding what we each noticed regarding participants' reasoning about solutions to linear systems as we watched the participants work through the task sequence. Two of the authors used an emergent coding method (Glaser & Strauss, 2017) along with researcher notes to locate moments that offered insight into participants' conceptions of solutions. These instances were selectively transcribed and recordings were reviewed for additional details. Due to space constraints, we limit our analysis to the first days of the PTE. Though our interpretations were triangulated across data sources and study authors, we do not make broad claims about the generalizability of these findings. Rather, our findings document possible reasoning paths and productive starting points of students engaging in the described task sequence.

Findings

In the first day, participants first with the SLE algebraically, one equation at a time. In connecting finding solutions to counting solutions of the first equation, participants wondered about how solutions might be ordered. Participants then considered solutions to the entire SLE and started down a path toward parametrization. On day 2, participants began connecting their algebraic work to the graph of the SLE.

"Are we thinking about (105, 105, 0) being different from (105, 0, 105)?"

In this part of the task, participants looked for solutions to a linear equation ($b + l + d = 210$) and estimated the number of solutions to that equation in the context of the given "real world" constraints. In trying to estimate the number of solutions, Student R asked, *"Are we thinking about (105, 105, 0) being different from (105, 0, 105)?"* Student L replied, *"Does it matter?"*

Here, we saw participants reasoning with and wondering about order in solutions. When adding three numbers, order does not matter, but when counting solutions, order does matter. We saw both participants wonder how solutions with the same three values might count as the same or different solutions. This question prompted the development of a shared notation, that breakfast is the first value, lunch the second, and dinner the third.

"Get all the variables in terms of one variable"

In the second part of the task, an additional constraint was added., and participants began to use algebraic manipulation they learned in secondary school, primarily substitution. We found that, because the system had many solutions, both participants ended up in some circulation of algebra, seemingly looking for a single value (as evidenced by Student L's comment that the result was *“probably going to be a positive number”*). Both ended up plugging different things into equations but found *“the variables will cancel”* (Student L) or *“everything zeroed out”* (Student R), as shown in Figures 2 and 3. Student L articulated a goal for plugging things in: *“I’m just trying to get all the variables in terms of one variable like* [R] *was doing, but every time I try it I guess they’ll either cancel or like I’ll run into another issue where I just can’t. Pretty much what I’m trying to do is try to relate all the quantities to like one specific variable.”*

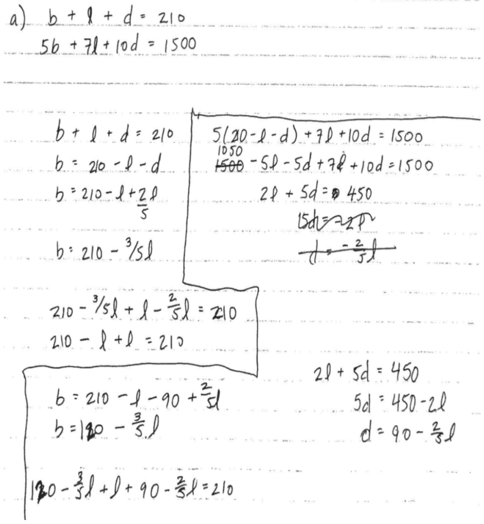

Figure 2. Student R’s algebra.

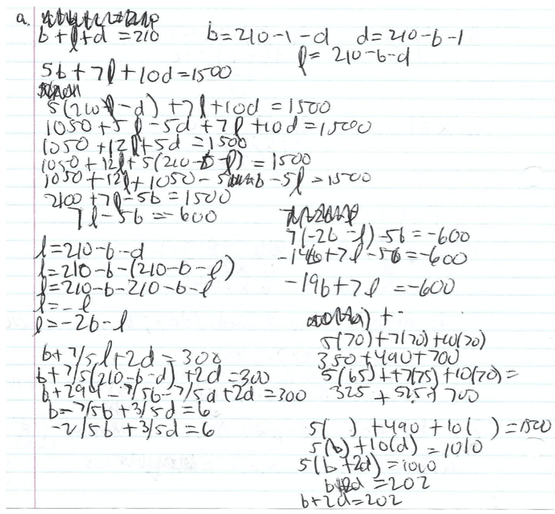

Figure 3. Student L’s algebra.

After both participants used specific solutions they had found to the first equation to see if they work for the second equation, the interviewer prompted Student R to try using an equation she found in conjunction with her previously identified solutions to the first equation. Student R had found dinner in terms of lunch, d = 90 – 2/5 l, shown at the bottom right of Figure 2.

I: Okay, so talk me through. How did you put those two ideas together?

R: [Shows work] Okay, so I have the d = 90 – 2/5l and I plugged in 70 and I got d = 62. So then I plugged that into the price equation with my 70 times 7 and I got 5b + 1110 = 1500. And then I got b = 78, which is how many breakfasts there are. So then I added 78 + 62+70 and that equals 210.

I: And then did it satisfy the 1500? Or I guess you plugged into the 1500 one, didn’t you?

R: Right.

I: Okay, so you found one that works.

R: Yes. The ordered triple is (78, 70, 62). Which is a very random number. So I’m like how are we supposed to find more. So does it work like that, well obviously every number doesn’t work. So do you have to like get lucky with the number you choose to plug in?

I: What do you mean every number doesn’t work?

R: Because we know that some, like (70, 70, 70) doesn’t make both equations true but then when lunch is 70, there is a solution. So like, if I just like plugged in 75 for lunches like, how are we supposed to know which number works or is it just guess and check?

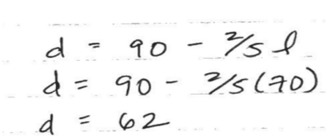

Figure 4. Student R finds d in terms of l.

Student R used information from her solution to the first equation, (70, 70, 70), that she found earlier in the interview to find a solution to the SLE. By plugging 70 in for lunch into an equation for d in terms of l (Figure 4) and using the second equation to find b, she found that (78, 70, 62) was a solution for both equations. Student R had also previously stated that *"because* [the equation] *has to equal exactly 1500, and lunch is $7, we have to make sure that whatever number, however many lunches we have, is a multiple of 5."* She planned to plug 75 in for l in her next iteration because she knew all values of l in the solutions needed to be a multiple of 5. Through this process, Student R was able to find many solutions to the SLE but said she did not *"know why they worked."* Here, both participants were heading toward the notion of parameter and using that parameter to find solutions. Student R wondered why plugging in values for one variable led to finding multiple solutions to the SLE.

"They would intersect because we found… numbers that will satisfy both equations"

On day two, participants were asked to think about the graph of each equation of the SLE, especially considering each line corresponding to skipping a certain meal throughout the semester. In thinking about how to graph a 'no dinners' line, both participants drew on their prior knowledge of graphing a line in the xy-plane:

R: I first started listing out some of the triples with d equaling 0 but then I was like wait, I think there is an easier way to do this… b+l is 210 … and then I just graphed that which was a line with the y and x intercept both being 210 and connected them.

L: I kind of did the same thing where I just plugged in 0 in the equation and I got b+l=210 but then from there I wasn't really sure what to do. Obviously, b or l can be any number between 0 and 210 including 0 and 210. ... But ... I'm not really sure how to even display that geometrically either.

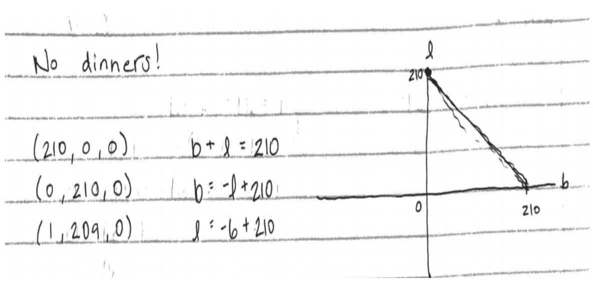

Figure 5. Student R's work in the plane.

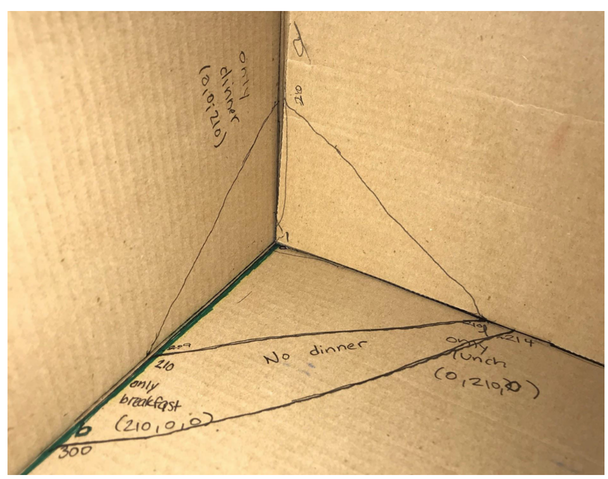

Figure 6. Student R's work on the box.

Student R showed L her graph (see Figure 5) to which he replied, *"Ok, so just a line."* To this point neither participant had tried to illustrate the "no dinners" meal plans or any other solutions in the box. After some help from the interviewer in setting up a coordinate system in the corner of a box, the participants started using their knowledge of lines in 2-dimensions, to create the "no dinners," "no lunches," and "no breakfasts" lines in the box (see Figure 6). The participants then wondered what all the solutions to the b+l+d=210 equation would be. They considered a pyramid or a triangle and whether it would be hollow. L suggested that *"the d would get higher, but the b and the l would get closer in"* and R recognized that if we considered all real numbers (not just integers) then *"it would be flat and all the points would be connected."*

The interviewer next prompted the participants to consider the second equation starting with the "no dinner" line for that equation. Student R added this to her box (see Figure 6) and noted that the triangle is no longer equilateral. The interviewer asked, *"What do you think it means that the lines don't intersect?"* Student R initially noted *"the first equation is inside the graph of the second equation. So, in that way they do intersect, or they overlap."* When the interviewer asked for clarification, Student L stated a different perspective, then analogized to equations of lines.

L: I would think that eventually they would intersect because we found that there are numbers that will satisfy both equations. [Student R: Hmm. That's true.]

I: Why would that tell you that they would meet?

L: Because if there's two lines that intersect at a certain point, then that point is a solution … then that x value that they hit at will be able to be plugged in and you get the same y for both of them.

The participants made substantial progress in building their geometric understanding of 3D solution sets using the box, but still wondered about the exact shape of the intersection of these two equations. That question would not be answered until Day 3.

Discussion

We focus our discussion on the potential of our task sequence to support participants' reinvention of infinite solution sets. Our task setting functioned as an accessible entry point for participants to anchor their algebraic and graphical approaches. Students' prior knowledge of SLE acted as productive beginnings of students' reasoning toward our goal of guided reinvention of infinite solution sets. Asking participants to count possible solutions pressed them to organize their reasoning and notation, identifying the meaning of the *order* in listing solutions as ordered triples. Participants leveraged their prior knowledge about systems, particularly substitution, when the second equation was introduced. Students began by looking for a single unknown value. When participants linked solutions they had identified to the first equation to their algebraic manipulation working with the pair of equations, Student R identified a type of parameterized solution. Finding values for b and d in terms of l seemed to broaden Student R's prior algebraic reasoning to allow her to generate solutions more easily by varying values of l. Initially, the parameterized solution was not as intuitive for students in understanding *why* it was generating solutions, but this was alleviated through a discussion. Finally, using a box and no-dinner lines, participants extended their understanding of lines in a plane to a three-dimensional context. This linked their prior conceptions of solutions as numbers that satisfy the given equations to points of intersection. They also algebraically reasoned with points of intersection as x-values that will get the same y-values in both equations. Here, we found that students' conceptions of solution and what they wondered about largely related to the context of the task and drew from students' prior knowledge. The task allowed students to extend their reasoning about solutions to less familiar situations (i.e., from a single solution to a large solution set), leading to their reinvention of infinite solution sets.

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Student Understandings of Properties of Linear Transformations

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Student understanding of linear algebra concepts is a growing research area. This study explores students' conceptualization of linear transformations and the various ways in which they use such conceptualization to reason through linear transformation problems after they have completed a linear algebra course. Three students participated in a task-based interview. Through analyzing interview data using a grounded theory approach, emerging themes were found indicating that students' exposure to linear transformations in other courses and the nature of these experiences impacts how they further conceptualize linear transformations. Notably, the way that the participants were engaged with linear transformations within their other courses for their major seemed to influence their use of geometry, algebra, and proofs in determining whether a given transformation is linear. One implication of this study is a need to engage students with more real-life applications of linear transformations in linear algebra courses.

Keywords: Linear algebra, linear transformations, representation, reasoning.

Introduction

Linear algebra is a course that many students are required to take as a prerequisite for other mathematics courses as well as courses in other disciplines. It is important to consider with what conceptions students leave their linear algebra courses to determine how they will view and interact with linear algebra concepts in future courses. Knowing students' potential conceptions can inform instructors of both linear algebra and other courses that utilize linear algebra topics on how to develop research-based learning opportunities on these topics for improved understanding.

There has been a large amount of research investigating student reasoning in the areas of span, linear independence, eigenvectors, and eigenvalues. (See Stewart, Andrews-Larson, & Zandieh, 2019 for a recent overview.) However, these studies focus on mostly students' geometric reasoning of these topics (Stewart, Andrews-Larson, & Zandieh, 2019). To have a robust conceptualization of linear transformations, a student needs to perceive them as more than just geometric movements. "Linear transformations are functions from one vector space to another, often $\mathbb{R}^n$ to $\mathbb{R}^m$, with particular linearity properties (they preserve addition and scalar multiplication)" (Bagley et al., 2015, p.36). In other words, a robust conception that connects the geometric with these algebraic and abstract descriptions of linear transformations is important. Other studies have shown that students need more opportunities to make connections between their geometric descriptions of linear transformations and the algebraic representations of transformations (Bagley et al., 2015; Andrews-Larson et al., 2017). Thus, there is still a need for more research to gain deeper knowledge on student understanding of properties linear transformations: what they represent (for students) and how they can be used (Stewart, Andrews-Larson, & Zandieh, 2019). By determining more about this understanding, the Mathematics education community can further the design practices of linear algebra topics to incorporate existing student conceptions and challenges that would move these conceptions forward. The purpose of this research is to further investigate the conceptions and reasoning used by students who have taken a linear algebra course to utilize linear transformations. My research questions are: What conceptualizations do undergraduate students have of linear transformations? In particular, what mathematical reasoning do students implement in problem-solving situation related to linear transformations?

Literature Review

How to implement geometric ideas about linear transformations in the classroom has been a focus of research, and studies have shown benefits for student understanding by having geometric representations and application problems in related lessons. Bagley, Rasmussen and Zandieh (2015) considered students conceptions of linear transformations as functions. They specifically explored ways in which students made connections between the concept of function they learned in previous algebra courses with the concept of transformations they learned in their linear algebra course. Bagley et al. (2015) described student conceptions that were not necessarily productive as pseudostructural conceptions (Sfard, 1991). The authors stated this type of conception seemed to cause students to struggle to identify the need for vector notation or how it was involved in linear transformations. Andrews-Larson, Wawro, and Zandieh (2017) also emphasized that students' conceptions of linear transformations needed improved flexibility, and the authors offered a hypothetical learning trajectory to help build this flexibility.

Stewart, Andrews-Larson, and Zandieh (2019) provided a summary of existing research in Mathematics education pertaining to linear algebra and gave directions for future research. They found that in the research regarding student reasoning and understanding, there was significant work considering geometric representations in linear algebra, proofs, and eigen theory. However, the authors pointed out that there was less research regarding students' algebraic reasoning with linear transformations and other properties of linear transformations. They called for research in the teaching of linear transformations and claimed that further research needs to be conducted to determine what claims from these studies can be further generalized. In knowing this, research in the teaching of linear algebra can grow and change.

Turgut (2019) does specifically study student understanding of linear transformations and explores student reasoning regarding matrix representations using geogebra, and the implications for teaching linear transformations. This study focused on different matrix and geometric representations of linear transformations and investigated how students interacted with technology modelling linear transformations. Turgut (2019) found that students had a difficult time transitioning to matrix representations of linear transformations from other forms of representations. This, the most current research on linear transformations available to my knowledge, indicates a need for further exploration of student understanding that encompasses contexts outside of the use of technology. The study I present in this paper aims to consider student understanding of linear transformations in a different way from Turgut (2019) by exploring students' reasoning involving different representations of linear transformations, as matrices, as functions, geometrically, and so forth. Furthermore, this study differs from the existing research by incorporating grounded theory methodology to explore not only students' differing conceptions and use of representations but also to investigate their ways of reasoning with representations after completing a linear algebra course.

Methodology

Due to the limited nature of existing studies that capture students' conceptions of linear transformations and students' uses of properties of linear transformations after students complete a linear algebra course, this study is designed to develop a substantive theory (as opposed to "grand" theory) to expand on the existing research. The grounded theory methodology (Merriam & Tisdell, 2015) guided this qualitative inquiry to address the research questions: What conceptualizations do undergraduate students have of linear transformations? And, what mathematical reasoning do students implement in problem-solving situation related to linear transformations? The interview data from three students were analyzed using themes and a constant comparative method and a core category emerged in the data, building to a substantive theory.

Theoretical Framing

I utilized the theoretical framework of constructivism that assumes students construct their knowledge through their experiences in various settings and with others (Crotty, 2015). Centered in this perspective a conception is defined as "a theoretical construct within 'the formal universe of ideal knowledge'; the whole cluster of internal representations and associations evoked by the concepts" (Sfard, 1991, p. 3). This theoretical construct together with Sfard's (1991) duality principal of conception shaped my overall design of the study. Students' conceptions at the time of the interview were assumed to be displayed through their utterances including written outputs and gestures. During the constant comparative stage of data analysis, I operationalized Sfard's (1991) operational and structural conceptions descriptions to understand the ways in which participants performed "processes, algorithms, actions" (p. 4) on the notion of linear transformations as well as their (flexible) use of representations of linear transformations.

Methods

In this grounded theory study, semi-structured in-depth interviews were conducted to explore participants' current conceptions of linear transformations by asking them to identify whether given transformations were linear or not. Participants were asked to solve linear transformation problems and provide their reasoning. Interviews were 30-90 minutes in length and were audio and video recorded and transcribed after the interview. The interviews were conducted during Spring 2020 semester at a mid-sized university in the Rocky Mountain Region of the United States. At this university physics and mathematics majors are required to take the linear algebra course offered by the Mathematics Department. The department offers only one linear algebra course every semester and the course is matrix-oriented that covers the core topics including eigenvalues and eigenvectors with limited focus on proofs. The linear algebra course serves as a pre-requisite for several undergraduate courses including abstract algebra, numerical analysis, and modeling courses.

Students from mathematics courses for which linear algebra is a prerequisite were invited to participate in the study. To select participants, students were given a survey with questions about their year in school, when they took linear algebra, their mathematics course history, their majors, available times for interview, and their preferred pseudonyms and pronouns. Three participants, Perry, Jed, and Summer (preferred pseudonyms) were selected for the study according to their availability for an interview, their year in school, mathematics course history and major to include a varying range of experiences. All three students participated individually in one interview. Perry (pronouns:they/them) was a junior secondary education mathematics major and took linear algebra in Fall 2019 semester. Jed (pronouns: he/him) was a senior physics major and took linear algebra in Fall 2018 semester. The third participant, Summer (pronouns: she/her), was a senior mathematics major with an emphasis in applied math and applied statistics and took linear algebra in Spring 2017 semester.

Data Collection and Analysis

Each participant was given a series of transformations to consider in matrix, function, graphical, and descriptive formats. These included 2x2 matrices, linear functions, and geometric depictions of images transformed on a set of axes. The interview included a variety of the types of interview questions listed in Zazkis and Hazzan (1998) to address the research questions. For example, I used "give an example and non-example of a linear transformation" tasks.

Within a grounded theory approach, open and axial inductive coding processes are fundamental to develop a substantive theory. This study followed these processes to analyze the interview transcripts. In the first cycle of coding, transcripts were categorized according to their relevance to each research questions. There was some overlap where participant responses applied to both research questions. Open descriptive coding was then used to assign words and phrases to these transcript "chunks". In the second cycle of coding, axial coding was used to relate these chunks of transcript based on their open codes. Through this axial coding, themes emerged. Some of these themes were about students' conceptions that were reported in existing research studies such as geometric and algebraic interpretations. New themes that emerged were related to the students majors and other courses they had taken.

Results

The analysis of the interview data indicates that three participants had three different conceptions of linear transformations and approached the problems in different ways. There were common themes in some of their approaches to individual problems, and the type of notations they used including geometric examples, algebraic examples, and informal algebraic proofs. In addition, there were commonalities among participants' reasoning and the ways in which they made connections to their coursework and majors in such reasoning instances.

Insight into participants conceptualizations of linear transformations

Perry seemed to have an evolving conception of linear transformations. Jed's conception seemed to be more robust, and Summer's conception seemed to be process oriented (Sfard, 1991). Perry's interview began with them struggling to recall what a linear transformation was. They relied on their concept of lines and transformations of lines specifically. As the conversation continued, Perry began to recall some of the types and properties of linear transformations:

Perry: Like vectors always start from the origin. Or they act like they always start from the origin so reflecting them over a line doesn't really do anything to the vector.

Interviewer: Can you draw what you are talking about?

Perry: Yeah I guess I could also just flip it over a line like this and have it be there (draws picture reflecting a vector over line y=x). Oh it's like we can alter the magnitude of the vectors…so we can multiply this vector by two and get wow ok, bigger vector.

Recalling what you can do with a vector geometrically and the realization of vectors "starting at the origin" helped Perry to reconstruct their understanding of linear transformations and properties such as the zero (vector) has to be mapped to zero (vector). They often updated their definition along the way. The following exchange occurred when they were unable to come up with a matrix representation when asked whether $T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}|x|\\|y|\end{bmatrix}$ is a linear transformation:

Perry: Because I am getting the idea that I need to write a matrix to have a linear transformation, which breaks my definition, in this case at least.

Interviewer: What's giving you that idea?

Perry: I could do it for all the other ones. Why can't I do it to this one? I don't like it.

Jed was able to recall almost immediately what a linear transformation was and used a formal mathematical definition of linear transformations. He said, "I mean algebraically I would say that it is a, any function t that satisfies this [pointing at the last line in Figure 1 is linear. And then generally if we're acting on an n dimensional space, these are represented by n by n matrices."

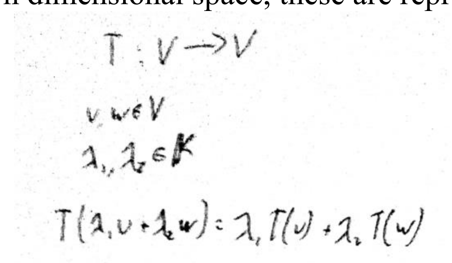

Figure 1. Jed's definition for linear transformations

Jed also gave a correct example and non-example of linear transformations and was able to articulate why these did and did not meet the definition for linear transformations. Overall, Jed seemed to have a more connected conception of linear transformations.

Summer had difficulty recalling linear transformations. She referred to her statistical knowledge and she said, "Coming from my stats background…linear is usually what I think of with like, linear regression. So relating things to one another in just like, one degree, not necessarily multiple, that's where my mind goes." Without further prompting she was unable to produce a working definition for linear transformation, so without indicating that it was the definition for a linear transformation (to determine if she would recognize it as such) I provided her with a copy of "rules" and asked her to determine if the transformations aligned with these rules. She did not say anything at this point about whether she recognized this to be a linear transformation, although towards the end of the interview she said she had wondered if that was the case. Overall, Summer's conception of linear transformation was limited to processes, rather than being abstract.

Insight into student reasoning in problem solving

As Perry refined their definition of linear transformations, they relied less on geometric interpretations and the more they used algebraic and matrix notations and their arguments became mathematically formal with resemblance of a mathematical proof. When presented with matrix representations of linear transformations they seemed to be hesitant. However, they conjectured that multiplying that matrix by a vector would transform the vector. They initially did not recall how to multiply matrices, but after giving it some thought, they correctly multiplied the given matrix by a generic vector to determine how the matrix transformed the vector both algebraically and geometrically. In this process their work resembled the notation and reasoning used in formal mathematical proofs.

Jed gave examples and non-examples of linear transformations as discussed in the previous section. He was able to use his formal definition to prove using algebraic methods whether a transformation was linear or not and give a matrix representations and determine the type of transformation geometrically without using specific examples of vectors and was able to give geometric interpretations when prompted. Jed also noted that he felt his knowledge of linear algebra was more from his physics (major) courses rather than the linear algebra course he took:

Jed: [M]ost of my knowledge comes from the functional analysis that is used in quantum mechanics. So we use a lot of stuff with basically finding the eigenfunctions and eigenvalues of differential operators. And then another decent chunk of it comes from knowledge from Tensor analysis and differential geometry that I did with an independent study.

Summer used a combination of geometric ideas and algebraic ideas to determine whether the transformations were linear or not but did not use formal algebraic notation. She also did not view matrices as representations of linear transformations. When presented with the matrix representation $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ of a linear transformation she attempted to construct her own definition for that notation, referring to the first column as the "input vector" and the second column of the matrix as the "output vector".

Summer: Oh, okay. So if it's doing the transforming, what I would assume is that your first column is your input vector essentially, and then your second column, the on the right hand side is your output vector. So we'd have a linear transformation of multiplying that vector by two would be my assumption, and that's also coming from thinking of matrices from a neural network standpoint. Because we've been doing that with our project in Math 437. And that's how I've kind of been remembering matrices is lots of like input-output kind of stuff.

With significant prompting from the interviewer, she was able to recognize that if the matrix itself was the transformation, then matrix multiplication would be how it transformed something a vector or any matrix with exactly two rows. Additionally, the excerpt from Summer's interview above provides another example of experience in other, more recent coursework having an influence on a participant reasoning with linear transformations.

Discussion

Both Summer and Perry appear overall to have pseudostructural conceptions (Sfard, 1991) of linear transformations. Perry demonstrating a pseudostructural conception of matrix representations of transformations aligns with the results of Bagley et al. (2015) This seemed to impact their ability to use matrix representations to determine what the transformation does and whether it is a linear transformation. These preliminary results also show a tendency of these students to rely on concepts from their most recent coursework rather than recall their linear algebra course specifically and for Jed and Summer this was specifically courses related to their major and interest areas. Summer often drew on her statistical knowledge to reason through linear transformations until she was given a different definition of transformation. Jed specifically described his experience in physics courses as having a large impact on his knowledge of linear algebra and how he was able to reason through these linear transformations.

Both Perry and Summer are math majors, and both utilized geometric interpretations more frequently than Jed. Summer however was more likely to use examples rather than proof techniques which seems to align with her applied math major. She has not taken proofs-based courses that Jed and Perry have, and considered examples enough to support her arguments. Perry would use generic examples as can be found in formal mathematical proofs and used formal algebraic notation. Jed was able to answer these questions about linear transformations seemingly without much difficulty. He had a significant amount of recent coursework that required ideas from linear algebra specifically. He also had coursework involving formal proofs of linear algebra related concepts. These experiences seemed to lead to his default approach of formal algebraic proofs. He demonstrated flexible switch between geometric and algebraic explanations when asked. Overall, Jed seemed to have more than a pseudostructural conception. This theme aligns with results of Karakok (2019) in that students' conceptions continue to evolve particularly in their future courses. Karakok (2019) noted that physics students' conception of eigen-values and -vectors showed changes from completion of a linear algebra course to completion of a series of quantum mechanics courses.

Conclusion

The purpose of this research study was to explore students' conceptions of linear transformations and their reasoning during problem-solving situations that involved properties and representations of linear transformations. The themes observed in this study lead to the substantive theory that at this university the conceptions of students whose main exposure to linear transformations has been the introductory linear algebra course are likely to remain at most at the pseudostructural level without further coursework. This prevents them from drawing connections between the matrix representations of linear transformations and their geometric implications. It seems that the ways in which students are exposed to linear algebra concepts beyond the introductory linear algebra course impact their conceptions of linear transformations and the ways that they reason in problem-solving situations with linear transformations.
This study is limited in scope due to the small sample size of three students. However, the findings have many implications specific to this university and for future research. The results of this study could be used to inform instruction not only of linear algebra but also of other mathematics courses. As prior research studies suggest that pseudostructural conception of linear transformations is problematic for students of all majors, more research should be done in alterations that can be made to help students restructure their conceptions of linear transformations. An alteration similar to that presented in Andrews-Larson et al. (2017) might be implemented to improve the flexibility of the students' conceptions. Because this course is required for many students of different backgrounds and majors, attention should be placed on making sure all representations of linear transformations are being addressed and that connections between these representations are made clear to students. In follow up courses within certain majors, adjustments may need to be made in understanding what type of conceptions of linear transformations those students may have when entering the course. When a robust, coherent conception of linear transformations is the goal for students in a mathematics program, it is important that students are given opportunities to make connections between and among many representations and contexts of linear transformations frequently in linear algebra courses and future courses. Linear algebra is viewed as essential for many students, and we need to equip them to be prepared to use these concepts in future courses and careers.

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Discourse Patterns in a Small Group "Collaboration": The Case of Uma and Sean

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In this study we present the results of a discourse analysis of the interactions between two partners, Uma and Sean, through a lens of positional theory. During nearly five hours of small group work in a teaching experiment, the way in which each partner used language to position each other's thinking as mathematically significant and establish a collaborative environment varied dramatically. Specifically, Uma shouldered the burden of continuously working to maintain collaboration, oftentimes at the expense of having her thinking positioned as mathematically significant. On the other hand, Sean regularly offered little opportunity for Uma to engage openly with his thinking, while simultaneously positioning his own thinking as mathematically significant. Language enacts and constructs identification with social groups and positions of privilege; thus, we also describe the role of Uma and Sean's identities, particularly gender-roles, in potentially explaining the nature of their interactions.

Keywords: discourse analysis; gender-identities; group work; attitude, affect, and beliefs

Collaborative group work in undergraduate mathematics has become an essential pedagogical feature of active learning (CBMS 2016), was deemed an element of ambitious teaching in college calculus (Sonnert, Sadler, Sadler, & Bressoud, 2014), and has been associated with undergraduate students' positive attitudes (Sonnert et al., 2014) and interest (Williams, 2016) towards learning mathematics. Research examining what students say during class discussions with a lens focused on mathematical ideas and reasoning is essential for understanding collective and individual development. Such research has demonstrated the importance of interacting with peers' thinking for students' own learning (Rasmussen, Apkarian, Tabach, & Dreyfus, n.d.), and aligned mathematical development with the nature of collective mathematical argumentation during class discussions (e.g., Rasmussen, Wawro, & Zandieh, 2015). Additionally, language does much more than establish normative ways of reasoning about mathematics. In fact, discourse allows conversants to enact and construct their identities—consciously and subconsciously (Gee, 2014; Langer-Osuna & Esmonde, 2017). Therefore, discourse occurring within small groups simultaneously offers students opportunities to learn mathematics and enact or construct their identities. The purpose of this study is to re-examine the collaboration between two students—Uma and Sean—attending to how their use of language positioned each other's mathematical thinking as significant and established a collaborative working environment.

Theoretical Framework

This study is guided by positioning theory (Davies & Harré, 1999) which posits "individuals construct subject positions in relationship with others during interactions, in particular through talk, that make claims about who people are within particular social contexts" (Langer-Osuna & Esmonde, 2017, p. 639). Within the positioning theory framework, we draw on Gee's (2014) discourse analysis, which asserts that language is used for seven building tasks. Specifically, this study attends to the language used for the tasks of building significance and communal practices. In the context of small groups working on a mathematical task, discourse analysis together with positioning theory offers a lens to examine how language is used by individuals to position members' mathematical thinking in relation to each other and how—or perhaps whose—language is responsible for maintaining communal practices. Positioning theory also considers the role of broader societal norms and expectations that may manifest in local contexts, discerning "how power and privilege are distributed, enacted, and taken up" (Davies & Harré, 1999, as cited in Langer-Osuna & Esmonde, 2017, p. 639).

Methods

The purpose of this study is to investigate discourse within a small group partnership while problem-solving to explore how collaborations take place. Specifically, we address the following research questions: How do partners working on a mathematical task use language and actions to: (1) position others' thinking as significant, and (2) establish a collaborative environment?

Setting and Participants

Data for this study come from a larger study investigating the relationships between student engagement and emerging mathematical development. The larger study had six preservice teachers participating in a teaching experiment focusing on concepts of logarithms. Participants were recruited from the elementary and secondary mathematics education majors taking mathematics courses during the semester of data collection. Participants attended 5 one-hour sessions, held weekly, which were not affiliated with any course. For this study, we focus on the interactions of one small group: Uma and Sean.

All participants were in their junior year of a secondary mathematics education program. Sean identified as a White man and was a "non-traditional" student with previous military experience. Uma identified as a White woman and was a "traditional" undergraduate student, who had recently changed majors from elementary education. Of the six participants, Uma was the only participant who was not enrolled in a course taught by the teacher-researcher.

Each of the student groups formed a heterogendered pair, and Uma and Sean self-selected to work together. During the study it came to our attention that across all groups, it appeared that the ideas of the men were held in higher regard than the ideas of the women, and that the men had authority to decide which—if any—of the women's ideas were pursued. The rationale for selecting Sean and Uma was twofold. First, the themes identified in the group interactions were the most salient with this pair. Second, the contrasting identities of Uma and Sean speak to the role that student identity plays in group interactions.

Background from Previous Study

Additional aspects of the larger study also informed this secondary analysis and discussion of our results. The larger study included semi-structured post interviews with participants in conjunction with the teaching experiment sessions. Individual interviews with Uma and Sean shed light on their experiences. Specifically, Uma expressed the important role Sean played in her ability to fully engage with the mathematical tasks. Uma explained, "without [Sean], I probably would have just sat there like staring… I would have picked some points and put them on there, but I wouldn't be very confident… I didn't necessarily know what to put on my own papers, I was like before I do anything I wanted to see where [Sean] was going with it." On the other hand, Sean described the nature of their partnership as, "I tend to work alone very well… and I don't think we were necessarily working together, we may have agreed on some things, but I don't think we were truly, you know, working in unison." In the context of this study, we take this aspect of their individual experiences to suggest that Uma and Sean may have had contrasting desires for establishing a collaborative working relationship.

Data Collection & Analysis

The teaching experiment sessions were recorded using a Swivl. This technology allowed for each small group of students and the teacher-researcher to have designated microphones, one main camera to capture the entire room, and one hand-held camera carried by the teacher-researcher. Cloud-based software allows viewers to specify microphones and cameras on playback, allowing us to listen to group conversations, and toggle between two cameras to view participant actions. We transcribed all of the small group interactions involving Uma and Sean.

Data analysis of the video and audio data took place in four stages. First, we identified every meaningful interaction between Uma and Sean, which became our unit of analysis. We define a "meaningful interaction" as an exchange of one or many talk-turns around a common on-task topic, idea, or subject. Interactions may contain one or many brief intervals of silence as long as subsequent talk-turns continue to concern the same subject and no changes to interlocutors occur. We considered instances when the teacher-researcher entered or left the conversation to constitute new interactions. Next, we transcribed every interaction and coded talk-turns using a codebook of verbs. Due to space constraints, our codebook can be viewed as an external document.

Depending on whether we were considering positioning mathematical thinking as significant or establishing collaborative environments, we categorized verbs based on the level of participation or whether the interaction was open or closed. For example, one student *agreeing* with another's mathematical thinking without offering an explanation for their agreement, indicates a low participation in positioning the other's thinking as significant. Moreover, *stating* claims without data or warrant is a closed form of collaborating because it does not offer opportunity for continued interaction. On the other hand, *revoicing* a partner's thinking indicates a higher level of participation in positioning the other's thinking as significant, and offers the opportunity for a continuation of the conversation. We discuss our codes further as we unpack transcripts presented in the results.

Then, for each interaction we created summaries surrounding how each student used language and actions to position other's mathematical thinking as significant and establish a collaborative environment. We analyzed these summaries to identify patterns of language that Uma and Sean tended to employ for each of the two discourse tasks (Gee, 2014).

Results & Discussion

We will discuss how each partner, Uma and Sean, used language to (1) position the other's thinking as significant, and (2) establish a collaborative environment during the study.

Positioning Other's Thinking as Significant

As the group worked through the mathematical tasks in the five sessions, each member made mathematical contributions which moved the group forward. However, how each member positioned the other's thinking as significant differed substantially. Generally, Uma's strategy for elevating Sean's thinking as mathematically significant was to agree with his ideas, ask clarifying questions, and revoice Sean's ideas in her own terms. In contrast, Sean seldom outwardly positioned Uma's thinking as mathematically significant. Instead, if Sean viewed Uma's ideas as significant, he would either immediately begin implementing Uma's idea (without agreement or acknowledgement), promptly make a mathematical statement building off of her idea (again, without explicit agreement), or provide a simple acknowledgement of Uma's speaking.

Interactions 1 and 2 below highlight how each student used these strategies. In Interaction 1, Uma contributes an idea to the group, in the form of a question posed to Sean. Her question is met by Sean asking a clarifying question, so he can immediately begin implementing her idea, punching numbers into his calculator. In Interaction 2 we see an example of how Uma responds to Sean's ideas by agreeing and revoicing his ideas in her own language. Meanwhile, Uma's ideas are met by Sean either solely acknowledging that he was aware of Uma speaking, or with a mathematical statement, neither of which suggest Sean engaged with any of Uma's arguments.

Uma: Would it help to change these...so that they are all in the same [notation]?
Sean: So, write it like, 5 times 10--squared [concurrent with Uma below]
Uma: --to the second. Yeah.
Sean: 1 times 10 to the 3 [head down, saying what he is writing]. 1 point 8 [trailing off, punching numbers into calculator]

Interaction 1: Uma presents a mathematical idea in the form of a question, which Sean then implements

Uma: Um, so, like the concerns for too much paper? Is that what the concerns that we were, like the space?
Sean: Yeah. So, either the two concerns were either the clustering or having to extrapolate that line out so far. And, you know, our model and the student's model are really similar, right?
Uma: Mmmhmmm.
Sean: [looks at cell phone while Uma is talking]
Uma: It's just the different interval, kind of yeah. I mean, it makes it so that you can get an idea of the whole thing on one sheet of paper, so it doesn't feel like too much. [Uma looks at Sean and realizes he is looking at his phone]
Sean: I think, um…
Uma: I think if it were, if it was linear and was on a whole thing, it might look a little bit more overwhelming with how long it would have to be.
Sean: I think it, it's kind of important…the thing that drives you to use our model is that there are so many dates that are recent, right? So, that allows us to spread some things out, and towards the end they are kind of compressed.
Uma: Yeah, it kind of puts them back.

Interaction 2: Uma presents an open mathematical argument, which is met with a (closed) mathematical statement.

As seen above, the interactions between Uma and Sean during the group collaboration have very different tones. Uma presents mathematical arguments as a way of inviting Sean to interact with her reasoning, whereas Sean makes statements about his mathematical thinking, which are closed for further conversation. Uma accepting and revoicing Sean's ideas demonstrates a wish to reach a consensus on another's ideas, reiterating her desire to build a collaborative community.

Building a Collaborative Environment

Analogous to the strategies by Uma and Sean when positioning the other's ideas as significant, each student had vastly different approaches to building a collaborative community. In general, the ways Uma used language to build a collaborative partnership with Sean were open, whereas Sean's language was relatively closed. Each student relied on two primary strategies for building a community. Uma posed direct invitations to Sean for him to explain his reasoning, or phrased her ideas as questions, as an invitation for collaboration. Sean, on the other hand, frequently made statements or simply answered Uma's questions without offering further explanation unless asked. Sean invited Uma to share her mathematical thinking in noticeably few interactions. Yet, when he did, he used the same strategy, but its intention depended on whether he was confident in his thinking or not.

Interaction 2 provides an example of how these two used one of their strategies. Uma initiates this interaction by inviting Sean to engage with her idea of what concerns their solution took into account—which was actually revoicing Sean's ideas from a prior interaction. This invitation is then met by Sean asserting his ideas in closed statements. In Interaction 2, we see an example of how Sean poses questions that outwardly appear to be invitations but are not open for Uma to voice her thoughts. Instead, Sean's statement, "Our model and the student's model are really similar, right?", only allows for Uma to agree or disagree with his idea—with a clear expectation of agreement.

Alternatively, in the moments where Sean was unsure of his mathematical thinking, he posed genuine invitations for Uma to engage with his thinking. In Interaction 3, we see both Sean and Uma working through a task which has them both stumped. Again, Uma begins the conversation by inviting Sean to explain his thinking. Sean then provides a closed statement, which Uma uses as another starting point for an invitation. During their conversation, it becomes clear that Sean is unsure of how to approach the problem. However, Sean's actions change after he voiced his confusion, as he began to pose genuine invitations for Uma to engage with his thinking.

Uma: So, now the question is, when you take in the overlapped time, does this still preserve the same amount?
Sean: As that 10 to 1? That is the question. I don't think that it is.
Uma: Mmmhmm. Do you have any thoughts leaning you more towards why it's not...
[15 seconds of silence]
Sean: If we graphed them longer, we would end up with something like that.
Uma: Mmmhmm.
Sean: We're saying then that is equal with that, or this is 10 times that, but I think this.
Uma: That overlap takes away from it being able to be 10 times that?
Sean: Yeah, I just don't know how to…
Uma: Yeah, how to explain…
[Sean states what he is thinking, Uma agrees]
Sean: You know what I mean? If I keep overlapping more and more, then they become the same time period. So, I don't think any overlapping is…
Uma: It conflicts. I guess, yeah, that is a good way to…
Sean: Does that make sense?

Interaction 3: Uma invites Sean to explain his reasoning, Sean and Uma voice their confusion, Sean invites Uma to engage with his idea.

Discussion & Conclusion

This study outlines discourse patterns emerging within small group partnerships while solving mathematical problems. The themes identified illustrate how interactions within a group can elevate or diminish other's thinking, as well as build or demolish collaborative environments. Furthermore, these results illuminate the role a student's identity plays in these interactions.

Throughout the study, Uma actively sought out a collaborative community, while Sean appeared ambivalent in their collaboration. Uma used language as a tool when attempting to create the collaboration she desired, consistently inviting Sean to share his ideas, posing questions about the tasks, and clarifying his ideas. Meanwhile, Sean voiced that he viewed their collaboration as simply "working in the same space," and thus appears to have paid little attention to the environment his actions produced. Rarely does Sean provide Uma with a genuine invitation to share her thinking about mathematics. Instead, the "invitations" Sean provides Uma are followed by a statement of his thinking, disregarding any argument or question Uma made. The few instances that Sean acknowledges Uma's thinking, he does not probe her thinking any further, whereas Uma repeatedly asks Sean questions about his thinking. Notably, how Uma cultivated collaboration simultaneously positioned Sean's mathematical thinking as significant.

While Uma repeatedly invited Sean to share his thinking, she also constantly agreed with the statements Sean would make. Hand in hand, these actions positioned Sean's thinking as significant. Moreover, Uma would often pose questions to Sean to which she appeared to know the answer, implying that Uma positioned Sean's thinking above hers. Yet, Sean never elevated Uma's thinking in the same way. Rather, if Sean agreed with Uma's argument, he would respond with another statement. It is important to note that some of the questions Sean posed to Uma position his own thinking as significant. As seen in Interaction 2, Sean often posed questions to Uma seeking agreement with his ideas. Additionally, these questions would frequently occur after Uma had a discussion with the teacher-researcher about her own mathematical ideas. Sean would undercut the ideas Uma had voiced to position his own thinking as significant. For example, after the teacher-researcher checked in on the pair's progress in the first session and Uma explained her reasoning for their work, Sean asked Uma, "But we do agree that this block has more time in it than this block, right?"—repositioning his own thinking over the ideas she had voiced.

In these interactions we see inequities for both how students build a collaborative space and how they position another's thinking as significant. While Sean spent little to no effort building a collaborative space, Uma used a myriad of different methods to do so. Impressively, Uma managed to position Sean's thinking as significant even though he provided very few openings for her. Uma created and maintained a collaborative environment by posing direct invitations to Sean and regularly applauding Sean's mathematical contributions—actions which were never reciprocated. We discourage readers from ascribing a lower mathematical ability to Uma based on her actions. Instead, we believe Uma is an inspiration—single handedly building a collaborative environment, while also demonstrating her effort in thinking about the mathematical tasks.

Inequities in a group collaboration could arise from a variety of factors, not least of which are the identities of the students involved. Uma and Sean's identities differed in three notable ways: (1) their age, (2) major, and (3) gender. Uma was a "traditionally" aged undergraduate student and Sean was an older, "non-traditional" student. The differences in their age could have led Uma to view Sean as more of an authority figure than a peer, and thus defaulted to his thinking over her own. Second, Uma had just changed her major from elementary education, while Sean had been in the secondary program since he started. Because of this, Uma and Sean had not interacted previously. Additionally, Uma was the only student in the study who had not taken a course with the teacher-researcher. Each of these factors could have left Uma feeling like an outsider, not having established an identity with the group prior to the study. Finally, Uma identified as a woman and Sean as a man. In addition to the other factors, this difference in gender holds the potential to explain a large part of Uma's interactions, as they align with societal expectations for women and the identities women in mathematics are known to inhabit.

Although the members of the other groups did not share as substantial of differences in backgrounds as Uma and Sean, the impetus for this investigation came from interactions of *all* groups participating in the study. Every group shared one common characteristic: heterogendered pairs. It is well documented that, compared to the men in their major, women in STEM experience a lower sense of belonging (e.g., Good, Rattan, & Dweck, 2012; Holmegaard, Madsen, & Ulriksen, 2014), confidence (e.g., Ellis, Fosdick, & Rasmussen, 2016; Fennema & Sherman, 1978; Steel & Ambady, 2005), and self-efficacy (Dweck, 2008). However, society also dictates that women should be polite, accommodating, and reassuring. These "desirable" characteristics substantially impact the way women engage in discourse in the mathematics classroom. As evidenced by Uma's hesitancy to share her own ideas, these societal expectations suggest that women should not be overly confident, as that can be seen as assertive. Instead, Uma sought reassurance of her ideas by posing them to Sean as questions, inviting him to engage, and more frequently positioning his thinking above her own.

A student's ability to learn in a [mathematics] classroom should not be tied to any aspect of their identity. Our research has demonstrated the inequities that can arise from group collaborations, both in whose ideas are acknowledged as significant and the effort that is placed on building a collaborative environment. Additionally, we have explored the role a student's identity plays in these collaborations, and the expectations placed on the ways students interact.

These findings hold broad implications for the mathematics education community. First, the inequity in Uma and Sean's collaboration, and the remainder of the groups in the study, necessitate the thoughtful inclusion of group work by mathematics teachers. As teachers, we cannot blindly assume that group work will increase student learning, when the cognitive lifting required of some students in these spaces may inhibit their ability to learn. While Rowe's (1988) study saw a dramatic increase in student confidence with same-sex pairs, we do not believe there is a quick "fix" to this issue. Students' identities are multifaceted and each component impacts the way they walk through the world and through a mathematics classroom.

This lack of understanding of how to build equitable group collaborations in an undergraduate mathematics classroom demands attention from the mathematics education community. The findings of this study came from a larger study which was not designed to investigate discourse patterns in group collaborations. Yet, our insight into how language was used in these collaborative spaces is poignant. Research on building group norms for collaborations that value *all* mathematical contributions as significant and create spaces for *every* student to share their thinking is of utmost importance to the field. Moreover, equitable group collaborations at the undergraduate level have yet to be explored and are of critical importance to increasing diversity and inclusivity in mathematics.

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Low Engagement Explained by the Satisfaction/Frustration of the Basic Psychological Needs: The Case of Uma

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Student engagement is one of the most robust predictors of student achievement and behavior (Klem & Connell, 2004). Therefore, a larger study was conducted to investigate undergraduate students' engagement while learning mathematics. A pre-service mathematics teacher, Uma, reported drastically lower levels of engagement in the larger study. Given that the satisfaction of the basic psychological needs for autonomy, competence, and relatedness facilitates the internalization of motivation and, consequently, students' engagement (Ryan & Deci, 2009), a case study was conducted to examine Uma's low engagement through the Basic Psychological Needs Theory (BPNT) in a 5-week teaching experiment focused on developing notions of logarithmic change. Nine themes emerged from the analyses of the videos and an interview, which explain the low levels of engagement through the BPNT. The case of Uma provides rich information about how the satisfaction/frustration of the BPNs is related to engagement levels.

Keywords: Basic Psychological Needs Theory, engagement, case study

Engagement in mathematical activity and learning mathematics are inseparable (Middleton, Jansen, & Goldin, 2017). Moreover, engagement is inseparable from the context in which learning takes place, which includes the mathematical context, the social context, and the individuals' emotions, beliefs, and values associated with mathematical activity (Middleton et al., 2017). As such, student engagement is an essential factor in students' mathematics learning experiences.

We conceptualize engagement from the perspective of flow theory (Csikszentmihalyi, 1975, 1990). This theory considers engagement as the collection of interest, enjoyment, and concentration (Shernoff, Csikszentmihalyi, Schneider, & Shernoff, 2003). Student engagement is deeply intertwined with motivation and some of its theories. One of those theories is Ryan and Deci's (2017) Self-Determination Theory, in specific, the sub-theory named the Basic Psychological Needs Theory (BPNT). According to Ryan and Deci (2007), the BPNT explains that autonomy, competence, and relatedness are universal basic psychological needs in humans that drive motivation and whether we are motivated by intrinsic or extrinsic factors. The BPNT's main assumption is that optimal motivational function is achieved with the satisfaction of all three needs. Therefore, in mathematics education, the satisfaction of the needs for autonomy, competence, and relatedness will facilitate the internalization of motivation and, consequently, students' engagement, achievement, and adjustment (Ryan & Deci, 2009). Conversely, it has been found that the frustration of these needs contributes to amotivation and disaffection, which could be described as a lack of engagement (Ryan & Deci, 2009).

Considering the importance of engagement and its strong relation to the basic psychological needs, the purpose of this case study is to examine Uma's low engagement in a 5-week teaching experiment focused on developing notions of logarithmic change through the lenses that the BPNT provides. Therefore, the research question that we address is: How does the satisfaction and/or frustration of Uma's basic psychological needs of autonomy, competence, and relatedness explain Uma's engagement?

Background

This case study focuses on Uma, a junior in secondary mathematics teaching who had recently changed majors from elementary education. Uma, along with five other pre-service secondary mathematics teachers (PSMTs), participated in a 5-week teaching experiment focused on developing notions of logarithmic change for one hour each week. The purpose of the larger study was to investigate undergraduate students' engagement while learning mathematics (Williams, López Torres, & Barton Odro, under review). The PSMTs worked in self-selected heterogender pairs on a problem-solving task previously used to investigate students' thinking about multiplicative change (Confrey, 1991). Uma's partner was Sean; however, the pair was joined by Frank for the final session to work as a group of three.

Sean was a non-traditional secondary mathematics teaching candidate also in his junior year. Sean had returned to university after a successful career in the military. All participants except Uma had multiple courses together as a cohort of PSMTs, including a course taught by the second author of this paper. The teaching experiment was not associated with any course.

Uma regularly reported lower levels of engagement than her peers throughout the teaching experiment and was the only student to report many instances of exceptionally low engagement - operationalized as one standard deviation below her mean. The whole group's average level of engagement was 10.8 (out of 15). Uma's average was 5.9. Uma's drastically lower levels of engagement prompted us to delve into her experience using the BPNT.

Theoretical Framework

The three BPN of autonomy, competence, and relatedness are defined as follows:

> [*Autonomy*] refers to feeling willingness and volition with respect to one's behaviors…The need for autonomy describes the needs for individuals to experience self-endorsement and ownership of their actions – to be self-regulating in the technical sense of that term…*Competence* refers to feeling effective in one's interactions with social environment – that is, experiencing opportunities and supports for the exercise, expansion, and expression of one's capacities and talents…*Relatedness* refers to both experiencing others as responsive and sensitive and being able to be responsive and sensitive to them – that is, feeling connected and involved with others and having a sense of belonging. (Ryan & Deci, 2017, p. 86).

The satisfaction and frustration of each of these BPNs look very different at the functional level. For example, students who satisfied their need for autonomy in a teaching experiment will very likely feel that they have been doing what interests them in the experiment, whereas a student whose need for autonomy is frustrated will very likely feel that the activities in the teaching experiment are just things they have to do. Students who satisfied their need for competence in a teaching experiment will very likely feel confident in their contributions to the mathematical task, especially within their small group; whereas a student whose need for competence is frustrated will very likely have a lack of confidence in their contributions. Finally, students who satisfied their need for relatedness in a teaching experiment will very likely feel a sense of belonging to that group of students, whereas students whose need for relatedness is frustrated will very likely feel that they do not belong to that group.

Methods

This case study intends to develop an in-depth description and analysis of Uma's low engagement through the BPNT (Creswell & Poth, 2017). The data for this study consists of a stimulated recall interview and video recordings of the teaching sessions, which were transcribed. The interviews began with a discussion of students' overall engagement experiences during the experiment and then focused on three clips – one depicting an instance of personally high or low engagement, a second showing an instance when the interviewee made a meaningful mathematical contribution, and a third (the same third clip was used for each pair) of an instance when the woman partner made a *potentially meaningful* mathematical contribution that was not pursued seemingly at the man's discretion.

Transcripts from stimulated recall interviews were analyzed using Ryan and Deci's BPNT. Significant statements were identified, which were coded according to which psychological need they referenced and whether it suggested that the need was satisfied or frustrated. Then, codes were condensed to themes which emphasized how satisfied or frustrated psychological needs were related to students' interest, enjoyment, and/or concentration (i.e. engagement). The authors then collaborated to review the coding and settle any discrepancies.

Results

A total of nine themes emerged related to the satisfaction/frustration of Uma's basic psychological needs. The themes follow a chronological pattern to tell a coherent story where we can see a progression of her needs rather than being divided into the different BPN.

Uma's Shock by the Frustration of her Need for Relatedness

One of the first things Uma realized once she joined the first session, as she put it, was the following: "It became obvious pretty quickly that they all had a relationship and past experiences, and they had been in a lot of classes together. So, I initially felt out of place and was like 'Why am I a part of this?'" This realization came as a shock to her. She described feeling "apprehensive" and "fearful". Uma further explained that she started asking herself why she volunteered for the teaching experiment and described that initial moment in the first session as "nerve-wracking". In addition, she expressed that it did not help that the mathematical tasks were about logarithms because it had been a long time since she has worked with them. Therefore, not only did she feel socially disconnected and with a low sense of belonging, but she also felt misplaced when it came to the mathematical content. Thus, the frustration of the need for relatedness (both socially and mathematically) may explain her self-reported low levels of engagement during the first session, which placed her levels of engagement significantly lower than her peers.

Uma's need for relatedness was clearly frustrated at the beginning of the teaching experiment. As she mentioned: "It did take a while to get comfortable with the group." In fact, she explained that her enjoyment levels could have been higher had she not "felt so out of place with the group." Therefore, it seems that Uma's frustration of the need for relatedness obstructed feeling higher levels of enjoyment and, consequently, overall engagement in the mathematical task. Unfortunately, fitting in the group and trying to feel more comfortable around the rest of the participants seemed to be an additional struggle that she would have to overcome in order to become more engaged in the mathematical activity.

Uma's Lack of Familiarity with the Concept of Logarithms and Uma's Frustration for the Need of Competence

As stated before, Uma's need for relatedness was frustrated at the beginning of the teaching experiment. Additionally, she mentioned: "It's been a long time since I have done log work. So, it's kind of like pushing the envelope of what I remember and what I've done. So, it was a little difficult on that aspect of things." Uma also expressed: "I think like there were times where I was like 'I don't know what's going on. I can't keep up. I don't know what's happening.'" Therefore, it seems that her need for competence was frustrated. Moreover, she expressed: "The interest was like losing a little bit because it wasn't an immediate like 'Oh, I know what the answer is' kind of thing." This suggests that her interest in the mathematical activity seems to be mostly based on whether she can immediately succeed in it or have a clear solving strategy. In summary, initially, she had a low level of familiarity with logarithms, which made the task more challenging. In turn, this frustrated her need for competence and reduced her levels of interest and enjoyment. Thus, the initial frustration of her need for competence also seems to explain her low levels of engagement.

Low Autonomy due to lack of Familiarity with both the Social and Mathematical Contexts

The need for autonomy is also crucial when it comes to motivation and engagement (Ryan & Deci, 2009). Regarding this, when Uma felt out of place both socially and mathematically, she mentioned: "I felt like I was outside of it or something. That made me question like what's the purpose, like what I was doing there. The interest kind of wasn't there to begin with because I was like I don't know if this is like the right, the best place for me." Notice that the hallmark of autonomy is that one's behaviors are self-endorsed and/or congruent with one's authentic interests and values. However, Uma was experiencing incongruence and conflict and was behaving contrary to her volition. As such, her need for autonomy was frustrated.

Initially, when she felt like she did not belong, her interest in the activity was depleted, and so her autonomy withered. Although the fact that she joined the group voluntarily was an autonomous decision, she was no longer behaving autonomously. Again, it feels like her sense of belonging explained this change in autonomy, which emphasizes how important feeling related can be. Moreover, the relationship between Uma's autonomy and interest appears to have been interdependent - one does not influence the other "first", instead Uma's interest and sense of autonomy were intertwined.

Considering the first three themes, we can conclude that Uma's basic psychological needs for autonomy, competence, relatedness were all frustrated at the beginning of the teaching experiment. These needs especially influenced her interest and enjoyment during the teaching experiment. Therefore, it seems that all these needs explain, at least partially, her low levels of engagement at the start of the experiment.

The Mediating Effect of the Satisfaction of the Need for Relatedness on the Relationship Between Uma's Enjoyment and Engagement

As Uma got to interact with Sean, her partner, she realized that "It was nice to have somebody to bounce ideas off them and go 'Oh, yeah! That makes sense.'" This was especially true for her because of her lack of recent experiences working with logarithms. By that time, Uma had opportunities to interact with Sean as a team, and they showed sensibility toward each other. Thus, Uma started to consider Sean as a responsive partner, which increased her sense of belonging at least within the pair.

Being part of a smaller group may have helped her navigate her way out from feeling like an outsider to feeling a greater sense of belonging and enjoyment. Particularly, she felt like she could ask for help and discuss ideas with her partner. This was still not the case between Uma and the rest of the group. Therefore, her need for relatedness seemed to have been partially satisfied (or frustrated) at this time as she felt more comfortable with Sean but not yet with the rest of the group.

Adjusting to a new group often requires time and interactions. This seems to have been Uma's case. For example, she explained: "through each session, [enjoyment] went up because every time we came in and we were looking at the same problem, and we were talking about previous sessions and I was like 'Okay. Yeah.' So, like I am a part of what's going on here." Therefore, with time, there was an increased familiarity with the group and with the mathematical activity. As she explained, that helped her to slightly satisfy her need for relatedness and increase her levels of enjoyment.

Therefore, for Uma, the association between enjoyment and engagement seems to be mediated by the need for relatedness in such a way that, when the need for relatedness is frustrated, her enjoyment in the mathematical activity is low. Consequently, she reports low overall levels of engagement. On the other hand, when the need for relatedness is satisfied then she can feel higher levels of enjoyment and allow for greater engagement with mathematics.

The Initial Effects of Having a Partner on Uma' Needs for Competence

As described earlier, Sean partially satisfied Uma's need for relatedness but having a partner influenced her other BPNs as well. Uma considered Sean a good resource to seek scaffolding from whenever she felt stuck. When talking about her partner, she described that it was nice to have that "assurance" that she was "going down the right path". She often talked to him for "guidance" and to "set an example of where his thoughts were and where he was planning on going." Therefore, two things seemed clear. First, she was not very confident in her own ideas. Second, she considered Sean as the most knowledgeable one of the pair. Although the second does not necessarily contribute to the frustration of her need for competence, the first one does. It was this lack of confidence that often impeded her to continue making progress in the task without Sean's help during the first few sessions. In turn, this interruption in the flow due to what seemed to be a frustration of the need for competence may have decreased her concentration, which could also explain her low level of engagement.

A Phenomenon of Autonomously Motivated Dependence

As it was explained earlier, Uma often relied on Sean's understanding of the problem and help when she began to struggle. Like she mentioned:

> So, I was trying to look for his guidance and see what my thought was before I would be like 'Okay. That makes sense.' Cuz I know I was just kind of like watching as he like put things down on the paper because I think I didn't necessarily know what to put on my own papers. I was kind of like, before I did anything, I wanted to see where he was going with it.

Up to this point, it seems like she could not self-regulate her behavior concerning the task. Also, she is not confident in her competence and in her ability to make much progress in the task by herself, making her seem to depend on Sean. However, as a team, Uma wanted their thinking to converge and she consistently checked on Sean's progress and thinking. In fact, she was the one who initiated most of the conversations intended to share their understanding. As expressed before, she was interested in those interactions and she also valued them. She decided to rely on him, which she considered being very beneficial for their progress on the task. Therefore, her dependence on him up to this point seems to be autonomous. That is, it was an action congruent to her authentic interest in group work and with her values of what having a partner means. After all, autonomy does not refer to independence (Ryan & Deci, 2017). Thus, this seems to be an interesting phenomenon of an autonomously motivated individual who is dependent on others.

The Effect of Uma's Higher Familiarity with the Mathematical Task in Uma's Satisfaction for the Need of Competence, and its Consequences

Familiarity or relatedness is again another important factor in Uma's levels of competence. It got to a point when she became more confident. As she stated: "It was starting to seem a little bit familiar and I could like pick out things that I had previously done. So, I could kind of see where to go and how to like get somewhere knowing what I knew previously." She also mentioned that her highest level of interest was at this moment when she became more familiar with the mathematical activity which Uma attributed to increased interest, enjoyment, and concentration. Therefore, higher familiarity with the mathematical tasks led to higher levels of competence which led to higher levels of engagement.

Greater familiarity also led her to greater understanding and enjoyment, and perhaps independence. Like she mentioned, higher familiarity "built up that confidence a little bit I think because I felt like I actually knew what was going on and so I could feel that like interest level was higher." Therefore, as she became more familiar with the logarithmic tasks, she felt more confident, allowing the need for competence to be satisfied as she felt that she could be successful given that, as she stated, "it actually made it to make sense to me."

Satisfying the Basic Psychological Needs

Around the third session, Uma explained that she started to feel part of the group as the sessions progressed; thus, slightly satisfying her need for relatedness. Additionally, she mentioned: "I was at my highest level of interest because it was starting to seem a little bit familiar and I could like pick out things that I had previously done...So I was like 'Oh. I actually kind of want to know what's going on because it actually started to make sense to me." Understanding what was going on mathematically, made her interest increase. Therefore, her behavior was more autonomously regulated in that her actions were not only self-endorsed but were also driven by her interest. Hence, her need for autonomy seemed to also be satisfied. Afterward, she expressed: "I actually knew what was going on." This suggests that she also felt competent. In sum, it seems that Uma's basic psychological needs were at least slightly satisfied by the third session, the session in which she reported the highest level of engagement.

The Fragility of Uma's Basic Psychological Needs

The satisfaction of a BPN does not guarantee that it will remain satisfied. One can feel competent around some people and incompetent around others; one can feel related to someone but later feel out of place, and one can behave autonomously and then lose interest. The state of having the BPNs satisfied can be fragile, especially when the context changes. This is the case of Uma in the last session when Frank joined her and Sean.

Based on video recordings and transcripts of the final session, the nature and amount of contributions Uma made to the small group diverted from the progress she had made through the experiment to that point. As mentioned before, Sean and Frank were in multiple courses together without Uma and the two seemed to have a comfortable working relationship. Throughout the final session, Sean and Frank routinely discussed with each other while problem-solving and would check in with Uma to see "what [did] you get for problem 2?" (Frank, session 5). In this regard, Uma was displaced socially and mathematically, where her involvement was limited and her mathematical contributions were not sought after or included during problem-solving. Therefore, Uma may have not felt as comfortable working with Sean as before and may have not had the opportunity to create a connection with Frank. Moreover, Uma may have also felt less competent as before given that her contributions were not recognized by her partners. Thus, although she seemed to have at least slightly satisfied her BPNs by session 3, it seems that adding Frank to the pair may have slightly frustrated her needs for relatedness and competence again. In turn, this may explain her low levels of engagement.

Conclusion and Implications

Uma's reported levels of engagement in an undergraduate mathematics teaching experiment were drastically lower than her peer's levels of engagement. As stated in these nine emergent themes, Uma's lower levels of engagement may be explained by the satisfaction/frustration of the basic psychological needs of autonomy, competence, and relatedness. Specifically, Uma's BPNs were all frustrated at the beginning of the teaching experiment, and this formed the starting point of her motivational and engagement process which placed her at a drastically lower level than her peers.

The need for social and mathematical relatedness was the first to be frustrated, which also seemed to mediate her enjoyment and engagement. Later on, she developed autonomously motivated dependence on her partner. Then, familiarity with her partner and logarithms helped to slightly satisfy her needs for relatedness and competence, particularly when Uma realized she belonged in the sessions and could contribute in mathematically meaningful ways. It was then when she finally engaged slightly autonomously and independently. However, none of these needs were satisfied strongly as the introduction of another student to the small group made her needs to be frustrated again, demonstrating how fragile the satisfaction of her needs was.

In the end, we see that Uma could not enjoy the tasks when she did not feel related to her partners, her concentration declined when she did not feel competent, and she would lose interest when she perceived a lack of competence which affected her autonomy negatively. Since engagement is defined as the collection of interest, enjoyment, and concentration, we can say that the satisfaction/frustration of the BPNs explains, at least partially, her engagement levels.

The case of Uma provides rich information about how engagement works in undergraduate students. As it can be seen, the need for relatedness is crucial for engagement in the classroom. As such, instructors should create opportunities for students to create connections among themselves, with the instructor, and with the content. Also, it is important to identify and integrate students who might feel alienated. This is especially important now in the context of COVID-19 as students tend to feel like they are not creating social connections. On the other hand, it should always be a goal to satisfy students' need for competence as it allows students to behave more wholeheartedly and autonomously. However, students may feel competent but not autonomous. Therefore, it is important to integrate students' interests in the mathematical tasks and provide opportunities for them to make choices. That is, students should not feel that everything that is done in the classroom has been imposed on them as it frustrates their need for autonomy. Finally, the frustration of any of these needs will risk withering students' engagement, and ultimately their opportunities to learn. After all, an optimal motivational function is achieved with the satisfaction of all three needs.

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Virtual Reality Applications of Math Education in Calculus: Visualizing Triple Integrals

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This study details the embodied, symbolic, and formal reasoning of an undergraduate student when determining proper bounds for a triple integral problem within a virtual reality program called Calcflow. While the participant initially had difficulty inputting valid bounds, once he learned the appropriate constraints, he gradually connected various aspects of his symbolic reasoning with his embodied reasoning through progressive experimentation in Calcflow. Results suggest a virtual reality environment can help ease students' difficulties in visualizing three-dimensional shapes and thereby provide them the opportunity to develop deeper reasoning about these shapes and their corresponding equations.

Keywords: Bounds, Triple Integral, Virtual Reality, Embodied, Symbolic

Introduction and Literature Review

Research on difficulties with students' understanding of single variable calculus is extensive (e.g., Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996; Oehrtman, 2009; Tall & Vinner, 1981), and some of these difficulties might generalize to multivariable calculus (Dorko & Weber, 2014). Similar studies (e.g., Orton, 1983; Mahir, 2009; Nguyen & Rebellow, 2011) have indicated that visualizing a 3D object and setting up an integral may be the most difficult part of the solution (Sheikh & Fray, 2017). Students appear able to evaluate given integrals analytically but struggle to make connections between the integral bounds and the given 2D or 3D object. Thus, Sheikh and Fray (2017) asked students to use MATLAB to generate and view 3D solids, and to sketch projections, contours, and cross-sections of the solid, hoping these tasks would help students set up triple integrals to find volume for a given object. They found not many students mastered these tasks, particularly for cases where split integrals were required, or multiple bounding surfaces were involved. By utilizing Zazkis, Dubinsky, and Dautermann's (1996) Visualization-Analysis model, they reported that students' conversions between representations were problematic. Similarly, Kashefi, Ismail, Yusof, and Rahman (2012) wrote "for most students, imaging and sketching in 3-dimensions were the greatest difficulties that they encountered when doing non-routine problems in multivariable calculus" (p. 5537).

Habre (2001) reported that while computer programs assisted visualization, they did not develop students' ability to visualize without assistance. In contrast, Purnomo, Winaryati, Hidayah, Utami, Ifadah, and Prasetyo (2020) concluded a sequence of tasks completed in Maple helped students improve in multivariable calculus. Perhaps this distinction can be partially explained by the nature of the tool utilized. Zbiek, Heid, and Blume (2007) noted that the usefulness "of a cognitive tool becomes most apparent in the area of multiple representations when it is used to establish a 'hot link', that is, a dynamic connection between two representations such that an action taken in one representation is automatically reflected in the other" (p. 1174). The tool should also have a high degree of mathematical fidelity: the ability for the technological rendering to correctly match the underlying mathematical principle it is representing. They note it is important to encourage students both to play with the tool and reflect on the actions they took with it, to discover and develop mathematical meaning.

Kang, Kushnarey, Pin, Ortiz, and Shihang (2020) tested a program designed to help students visualize multivariable calculus concepts and found participants in the experimental group felt using a virtual reality (VR) program was more helpful to them than the control group's slides. However, the control group outperformed the experimental group. The authors suggested this may have occurred because the VR group learned without instructor guidance, while the slide group was paced. This hypothesis implies VR "cannot be a panacea for understanding advanced mathematics, and some form of intervention (whether via better implemented technology, or via in-person diagnosis of the student's ability) is required" (p. 58945). In a study on spatial visualization via VR, Herrera, Abalo, and Ordóñez (2019) claimed that VR helped students visualize mathematical concepts, especially for visualizing intersections between surfaces, noting increased average class score and reduced failure rate. Thus, usage of VR may help remove one of the primary obstacles to multivariable integrals-visualization of 3D objects. The current study thus attempts to establish whether a similar result holds in another mathematical context.

This study investigated whether utilizing a VR program as a visual aid in a series of directed tasks can help resolve difficulties in visualizing 3D graphs and finding bounds for a double or triple integral. This is an exploratory study designed to investigate some of the qualitative phenomena that learning math in VR is likely to elicit. Participants were asked to complete tasks involving double and triple integrals while utilizing *Calcflow*, a VR program. The research question is: What kinds of reasoning do students use as they attempt to find bounds of integration for a triple integral while working in a VR environment using multiple representations?

Theoretical Framework

The main principle of embodied cognition is that reasoning is inextricably linked to the physical environment. There are a variety of interpretations of this statement (see Alibali & Nathan, 2012; Lakoff and Nuñez, 2000; Nemirovsky, Rasmussen, Sweeney, & Wawro, 2012; Wilson, 2002), but for the author, this assumption carries with it the corollary that "learning is doing." Thus, speech, gesture, or technological actions all serve as evidence for the types of reasoning the participants might utilize. Thought processes are invisible, though we might speculate on potential thought processes based on physical action. Notably, in a VR environment, the lines between gesture and technological actions are significantly blurred.

Tall (2010) details a world of embodied mathematical thinking, which shares some overlap with the embodied cognition perspective. The embodied world is described by Tall (2010) as consisting of "our operation as biological creatures, with gestures that convey meaning, perception of objects that recognize properties and patterns…and other forms of figures and diagrams" (p. 22). The author thus considers the embodied world to consist of reasoning about geometric shapes, graphical phenomena, and connections of abstract mathematical phenomena to real-world contexts. Thus, the embodied world bears resemblance to both embodied cognition and *NCTM*'s (2009) definition of geometric reasoning. The symbolic world, as described by Tall, consists of "practicing sequences of actions until we can perform them accurately with little conscious effort. It develops beyond the learning of procedures to carry out a given process (such as counting) to the concept created by that process (such as number)" (p. 22) So, the author characterizes the symbolic world as consisting of reasoning about algebraic or algorithmic processes, similar to *NCTM*'s definition of algebraic reasoning (2009). Finally, Tall writes that the formal world "builds from lists of axioms expressed formally through sequences of theorems proved deductively with the intention of building a coherent formal knowledge structure" (p. 22) Thus, the author considers the formal world to consist of structural relationships between mathematical concepts, and so includes proofs, theorems, conjectures, and formal logic. This study investigates how participants engaged with each of these worlds.

Method

Participants consisted of two undergraduate students who had previously completed the entire calculus sequence (Calculus I-IV) at a southwestern university and are referred to as Mike and Connor in this paper. Connor had completed the calculus immediately before the interview took place, while Mike was a senior who had completed the calculus sequence two years before the interview. Each participant was interviewed separately. This paper focuses on Connor. A one-hour task-based interview was conducted, immediately after the spring semester had concluded. In this interview, participants were to use *Calcflow* to determine proper bounds for a given shape. *Calcflow* allowed participants to enter bounds via a virtual keyboard, and generated the surface corresponding to those bounds. Thus, *Calcflow* requires symbolic input and displays corresponding embodied output. The left side of Figure 1 shows the bounds already entered when the program starts, and the corresponding surface as a blue graph within a transparent cube. This cube can be picked up, moved, dilated, and rotated. Participants could also summon virtual chalk to draw freely in the virtual environment in full 3D, though this action generates no other embodied output. The right side of Figure 1 shows such equations drawn with the chalk.

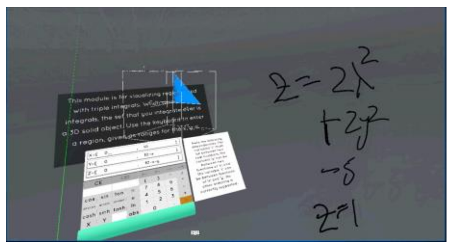

Figure 1: A snapshot of Calcflow

Interviews were recorded with screen-capture software to record participants' interactions within the virtual environment, and an external camera to record their gesture and other movements within the real physical world. These videos were analyzed, with particular attention paid to which of the three worlds participants were utilizing at any given point. All recorded speech, gesture, and technological actions were transcribed to an Excel spreadsheet, and each line coded according to which world or worlds the participant appeared to be utilizing. Summaries were then written detailing the participants' progression through each task. Finally, referencing the Excel spreadsheet and raw data as needed, I performed a cross-case and within-case (Merriam, 2009; Patton, 2001) analysis on these written summaries.

Results and Discussion

The interviewer started the interview by helping Connor fit the headset and introducing the VR controls. Then, the interviewer read the following question: Find the bounds for $\iiint_E xydV$, where E is the solid bounded by $z = 2x^2 + 2y^2 - 5$ and $z = 1$.

Connor made the connection between the symbolic expression $z = 1$ and its corresponding embodied planar shape immediately and recognized $z = 1$ as the upper bound. He observed:

> And, so I'm writing out the $x, y,$ and z plane, and I know that it's bounded at least (Creates dotted lines in VR at $z = 1$ above the y-axis and above the x-axis) by z equals 1. So I made a plane (Moves hand over the dotted lines, back and forth a few times) at z equals 1 as the upper part of it.

When the interviewer asked Connor what the resulting shape should be, he said "paraboloid." Connor began sketching a graph by hand by plotting individual points, then instead wrote several symbolic algebraic equations, including what appears to be $2 - 5 = 2x^2 - 2y^2$. Connor divided this equation by 5 and then erased it. When the interviewer asked him about his algebra, he said, “I tried moving the 5 to the other side, but that’s for an ellipse.” This attempt at symbolic reasoning suggests Connor had some trouble finding the intersection of the two bounding equations. The interviewer asked about the cross-section of the bounds, and after a short pause, Connor answered “$2x^2 + 2y^2$? That’d be like a circle.” Connor wrote $2x^2 + 2y^2$ and claimed after a longer pause, “so that is a circle with radius $\sqrt{2}$, or something like that”. Connor may have thought $2x^2 + 2y^2 - 5 = 1$ would become $2x^2 + 2y^2 = 5 - 1 = 4$ when combining constants, so this mistake may have been just a sign error. Connor drew a circle of radius $\sqrt{2}$ on the $xy-$ plane, suggesting he was making connections between his symbolic and embodied reasoning regarding the shape of the cross-section, but not its positioning. So, at the beginning of the interview Connor appeared to have difficulty coordinating aspects of his symbolic and embodied reasoning when multiple bounding surfaces were involved, as in Shiekh and Fray (2017). Notably, he was sketching by hand at this stage, so he did not have the benefit of immediate computer-generated output, which might explain some of his difficulty. Similar partial correspondences are evidenced by his completed sketch, as he identified $z = 1$ as the upper bound but drew an inverted paraboloid and reflected it across the xy −plane. (Figure 2).

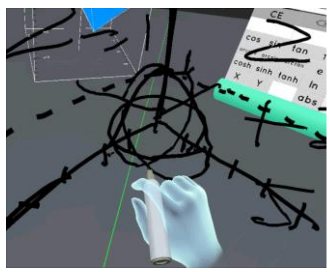

Figure 2: Virtually hand-drawn paraboloid and its reflection across the $\boldsymbol{xy}$ −plane

The interviewer noted *Calcflow* fixes the order of integration as $dzdydx$, so the x bounds must be constant, the y bounds can only involve the variable x, and the z bounds can involve x and y. Connor still input many bounds that violated these constraints, perhaps because *Calcflow* rendered output even for some of these expressions. For instance, the bounds $x^2 \leq x \leq 2, 0 \leq y \leq 2$, and $1 \leq z \leq -1$, generated output that Connor called a “cube” (see Figure 3), and $x^2 \leq x \leq 2$, $2x^2 \leq y \leq 2$ created a blue graph that showed a sequence of vertical blue lines stretching back and to the right (see Figure 4).

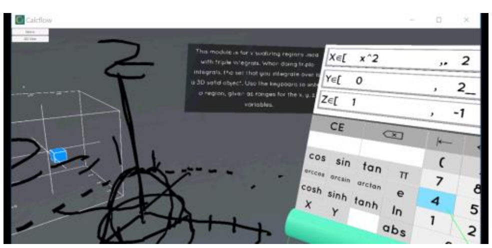

Figure 3: Self-referencing bounds

Figure 4: Infinite blue graph

This behavior reduces *Calcflow*’s mathematical fidelity (Zbiek et al., 2007), which can cause a learner to have difficulty distinguishing which aspects arise from the mathematics and which arise from the tool. Connor avoided further self-referential bounds only after a second warning from the interviewer. He entered $2 \leq x \leq -2, 2x^2 \leq y \leq -2$, and $1 \leq z \leq -1$. This input generated a rectangular prism with a parabolic cylinder cut out (see Figure 5). Connor observed:

2 is less than x is less than negative, I got a parabola. That's pretty cool. Um, so that's $2x$. Okay then I have, understood, okay. So the y part, so this, this part right here (Waves cursor over trough shape along the axis stretching to the right) must be the y-axis.

Connor thus connected his symbolic input to the embodied output. By changing the lower y bound to $2x^2$ and observing the resulting graph, Connor inferred the parabola must lie along the y −axis. He extended this connection between his embodied and symbolic reasoning, saying, "to make it a circle, I'd have to put the circle equation into y maybe." Connor thus entered the bounds $2 \le x \le -2$, $(2 - x^2)^{\frac{1}{2}} \le y \le -2$, and $1 \le z \le -1$, producing a blue graph that Connor called "something kind of circular" (Figure 6).

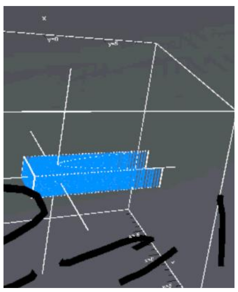

Figure 5: Connor's "parabola"

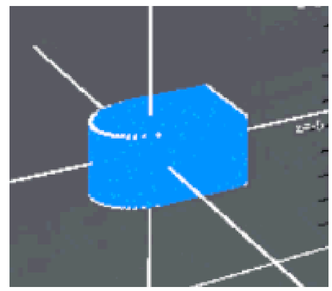

Figure 6: "Kind of circular"

Note he continued this pattern of making embodied observations and changing the symbolic bounds to address that observation. He said, "The back half is not circular though, so I need to figure out how to fix that. The back half is straight." Connor typed $(2 - x)^{\frac{1}{2}}$ into the lower y bound and $-(2 - x^2)^{\frac{1}{2}}$ into the upper y bound, which resulted in a full cylinder output.

Hey there we go. (Looks at blue graph which now displays a full cylinder instead of a half cylinder. And then you need to shift it. (Changes lower y bound to $(2 - x^2)^{\frac{1}{2}} - 5$. Looks at blue graph, which now displays a rectangular prism with cylinder shapes cut out of either end. Highlights upper y bound and upper z bound.) Okay. (Changes upper y bound to $-(2 - x^2)^{\frac{1}{2}} - 5$, looks back at graph which is now a cylinder centered on the y axis (not the origin), but pointing in the z direction.)

Thus, Connor connected the symbolic -5 to a graphical embodied "shift". When he changed the upper z −bound from 0 to $-(2x^2 + 2y^2 - 5)$, he noticed that "when I put in the negative it just reflected over the axis," as seen in Figure 7.

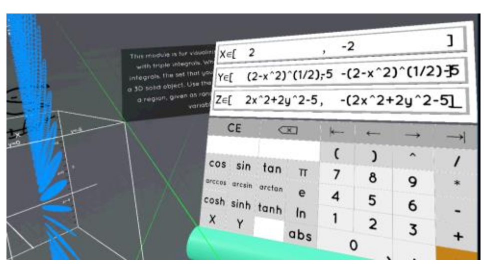

Figure 7: Reflection over axis

Afterwards, Connor observed that he had added a -5 onto the y bounds. He removed these and seemed excited by the result (see Figure 8). The bounds at this stage were $2 \le x \le -2$,

$(2-x^2)^{\frac{1}{2}} \leq y \leq -(2-x^2)^{\frac{1}{2}}$, and $0 \leq z \leq 2x^2+2y^2-5$.

Oh, and then I put a minus 5 and a plus 5. Let me delete this see what happens to (Backspaces -5 out of lower y bound) Ooh okay. (Looks at blue graph which looks like a paraboloid on one side but fanning upward infinitely on the other side. Backspaces -5 out of upper bound. Looks back at blue graph, which was an upward facing paraboloid with a cylinder on top, as shown in Figure 9.) Now we're getting somewhere.

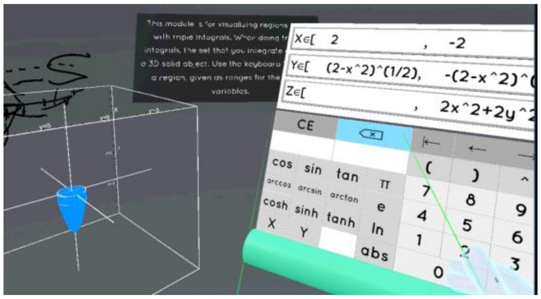

Figure 8: Cylinder on paraboloid

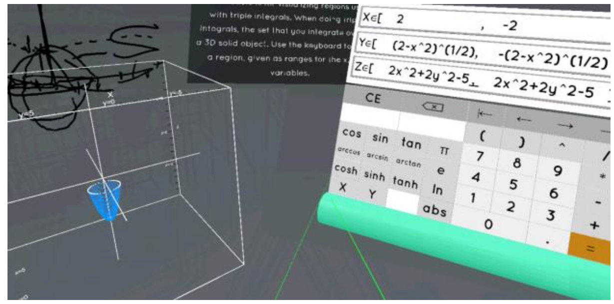

Figure 9: The very thin line

Connor deleted the -5 out of the upper z bound as well, which resulted in a cylindrical output. He then changed the lower z bound to the same as the upper z bound, and again connected his "thin line" embodied output to the identical expressions for z in his symbolic input:

I got a very thin, Oh! I understand why. It's doing a very thin line (Looks over at blue graph.) because I made the z, and it's the same thing. (Mouses cursor over lower z bound. Moves cursor back and forth between upper and lower z bounds)

The interviewer asked whether this shape matched his sketch, and Connor laughed and said no.

Interviewer: What's wrong with it?

Connor: So, I thought it was more spherical in shape, but this is definitely like, this makes more sense that it's paraboloid shaped.

At the end of the task, the bounds read $2 \leq x \leq -2, \sqrt{2-x^2} \leq y \leq -\sqrt{2-x^2}$, and $2x^2+2y^2-5 \leq z \leq 0$. Although at this stage of the interview the upper z bound was not correct, recall Connor recognized the plane $z=1$ as an upper bound at the beginning of the interview. Circular equations bound x and y; they just use an incorrect radius. The $x-$ and $y-$ bounds are backward, likely because *Calcflow* makes no observable distinction in its rendering based on this ordering. So, Connor came close to correct bounds through experimentation, observation, and coordination of symbolic and embodied expressions in *Calcflow*.

Qualitative Post-Interview

Connor independently described the experience overall in a way that reads like a natural summary of the embodied cognition theoretical perspective, saying:

[*Calcflow*] allow[s] you to, um, play around, and see the shapes you're getting towards as you get to your final function that you want to see. And it gives you a sense of how each one of the different variables and their placement in x, y, or z is helpful in, you know, making the graph the way it is.

Similarly, Connor contrasted the difficulty of making sense of 3D figures on paper with the comparative ease of making sense of these same figures in a VR environment. He explained:

When it's depicted in the calculus book that we use, it uses, like, shading in lines, but it's difficult to find the difference between, um, let's say a plane is coming toward you and

> sloping down. It's hard to see whether they're saying that that is an edge or whether the lines are coming toward you so close that they're not an edge, but it's slightly sloping up, and so this, this allows me to grab it and actually, you know, manipulate it and move it around.

Connor remarked it was easier to draw 3D shapes in *Calcflow* than on paper, probably because to do so in *Calcflow* one need only move their hand through the air in the same way they would as if they were tracing an actual 3D object of the same shape. On the other hand, Connor seemed discouraged from experimenting by the input mechanism, asking the interviewer on one occasion to verify he had the correct bounds before typing them.

Concluding Remarks

The results of this investigation highlighted that Connor developed connections between his symbolic and embodied reasoning with the help of *Calcflow*, that he was misled at times by some breaches in mathematical fidelity, and that he felt he could draw, manipulate, visualize, and comprehend 3D objects more easily in *Calcflow*'s VR environment than on paper. By gradually connecting aspects of embodied and symbolic reasoning, Connor was able to come close to finding correct bounds. *Calcflow* likely helped him connect these two forms of reasoning by requiring symbolic input and producing corresponding embodied output, and aided by this "hot link", Connor eventually appeared to attend to both representations in tandem. This dynamic enabled Connor to experiment and determine how the graph changed in response to the input bounds, and it further enabled him to test the validity of his inferences and generalizations immediately. Like the coordination between algebra and geometry seen in Zazkis et al. (1996), the coordination between embodied and symbolic reasoning itself may have helped drive Connor's progress. Early on, Connor appeared to have difficulty attending to multiple representations, while later he discovered and leveraged connections between them. In the qualitative post-interview, Connor claimed it was easier to visualize the 3D shapes with VR than with other 2D representations, even interactive ones, suggesting both that such visualization might be one of the primarily obstacles in determining these bounds and that *Calcflow* can help alleviate this burden. There were no identified instances of Connor utilizing formal reasoning as described by Tall (2010). Given Connor's repeated struggles avoiding self-referencing bounds, the author agrees with Zbiek et al. (2007) that high mathematical fidelity in a program is desirable, and Kang et al. (2020) that "technology itself cannot be a panacea for understanding advanced mathematics, and some form of intervention (whether via better implemented technology, or via in-person diagnosis of the student's ability) is required" (p. 58945).

Most of Connor's praise seemed to revolve around the embodied graphical output of *Calcflow*, and most of his criticism around the symbolic input mechanism, which proved difficult to use. Thus, future research could focus on whether a similar VR program that improves on this aspect would help further. For example, *Geometer's Sketchpad* allows direct manipulation of the shapes, as well as their corresponding equations. Troup (2018) suggests that moving from embodied to symbolic reasoning can offer different affordances than moving from symbolic to embodied reasoning. Thus, it may be beneficial to allow for similar manipulation of these shapes in *Calcflow*, so that students can change the embodied representation to see changes in the corresponding representation, to allow for a bi-directional "hot link". One could also find and investigate tools that further leverage the affordances of VR, such as those that would allow investigation of cross-sections obtained from slicing a solid, or various projections.

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Exploring Affective Dimensions of Women's Experiences in the Secondary-Tertiary Transition in Mathematics

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This report describes a pilot study conducted in Spring 2019. We set out to accomplish two tasks. First, through semi-structured interviews, we sought to understand characteristics of women and underrepresented minority (WURM) undergraduate mathematics majors' experiences pertaining to the secondary-tertiary transition (STT) in mathematics. Second, we designed and implemented a supportive intervention to address aspects of the STT that emerged from the outcome of the interviews. We draw upon Di Martino and Zan's (2010) three-dimensional model of attitude, which reifies the role of affect in the STT, and focuses on students' vision of mathematics, perceived competence in mathematics, and emotional disposition towards mathematics. In this report, we present data from three WURMs' perspectives to document three related dimensions in the STT and discuss whether and how our intervention supported these students' transition.

Keywords: affective dimensions, secondary-tertiary transition, women in mathematics

The secondary-tertiary transition (STT) in mathematics learning has attracted many researchers' attention from a variety of contexts (e.g., Clark & Lovric, 2008; De Guzman et al., 1998; Gueudet, 2008; Tall, 2008; Thomas et al., 2015). The STT has been commonly attributed to the knowledge gap in mathematics that students experience when introduced to advanced mathematics. These difficulties, cited as the "transition problem", have been predominantly connected to cognitive difficulties relevant to conceptual-epistemological change (Witzke et al., 2016) and/or under-preparedness in school mathematics (Wade et al., 2016). De Guzman et al. (1998) identified potential difficulties that impact students' success during the STT in mathematics education from the perspectives of (a) cognitive and epistemological, (b) social and cultural, and (c) didactical, all of which helped to scrutinize numerous challenges that undergraduate mathematics majors face. Further, Di Martino and Gregorio (2019) investigated the STT from affective dimensions, which added a new perspective to explore the phenomenon. Considering affective aspects of learning, the intersectionality in one's identity increases the complexity of the STT phenomenon. Investigating affective factors can shed light on how gender, race, ethnicity, and educational backgrounds can play a role during the STT. Recognizing mathematics as a historically masculine-dominated academic community (Solomon, 2012), the current report aims to explore the salient elements of women and underrepresented students' (WURMs) lived experiences by addressing the affective aspects.

Literature Review

After more than two decades, the STT in mathematics has been problematized as discontinuities in institutional levels between secondary school and university mathematics related to changes in the nature of mathematics (Gueudet et al., 2016), advanced and formal mathematical thinking (Tall, 2008), mathematical discourse (Sfard, 2007), and learning and teaching practices between the two institutions (Bosch, 2014). Alcock and Simpson (2002) expressed the complexity of difficulties in the STT as an amalgam of multiple transitions. Kajander and Lovric (2005) referred to the STT as a complex phenomenon involving multiple issues that need to be addressed. Gueudet (2008) articulated the STT almost as a rupture.

Recently, Di Martino and Gregorio (2019) reflected on it as a crisis that students undergo from a perspective of affective domains by taking beliefs, attitudes, and emotions toward mathematics into account. Various underlying factors during the STT may lead students to feel alienated (Hernandez-Martinez, 2016; Solomon & Croft, 2016) and even drop out (Andrà et al., 2011; Di Martino & Gregorio, 2019; Rach & Heinze, 2017). Rarely, the challenges in the STT compelled students to develop diverse strategies for sense of belonging and attainment to be able to persist in the major (Holmegaard et al., 2014; Solomon, 2012).

Institutional inequity, gender gap, and stereotype threat constitutes barriers for WURMs to persist in STEM majors (Master & Meltzoff, 2016). Recognizing the multi-layered nature of identity, several studies recommended looking at experiences of WURMs in mathematics with an emphasis on the intersectionality (Eddy & Brownell, 2016; Starr, 2018). In this paper, we discuss various experiences of WURMs who have been traditionally positioned as underrepresented participants of mathematics communities in institutions of higher education in the US. Yet, the body of literature which examines how gender and intersectionality play a role during the STT is limited. To address this issue, our research question was: In what ways do WURM students articulate their vision of mathematics related to their perceived competence and emotional response to the STT in mathematics?

Theoretical Perspectives

Many scholars have recognized the importance of affective factors that play a part in learning mathematics (e.g., DeBellis & Goldin, 2006; Di Martino & Zan, 2010; McLeod, 1992). Previous studies in mathematics education attempted to include affective factors in examination of cognitive processes (e.g., Mandler, 1989), yet a few of them focused on the interplay between these affective and cognitive aspects. Even though several scholars have taken the affective domains into consideration, the majority of research is primarily concerned with the role of beliefs during problem-solving activities (e.g., Schoenfeld, 1985) rather than investigating the interaction among beliefs, attitudes, and emotions. Di Martino and Zan (2011) conceptualized attitudes as a bridge between emotions and beliefs in a three-dimensional model of attitude (TMA). The TMA views attitudes as an outcome of interconnections among and between the subjective components of vision of mathematics (i.e., beliefs on the nature of mathematics), perceived competence (i.e., beliefs about self), and emotional disposition toward mathematics (Di Martino & Zan, 2010). Di Martino and Zan (2010) described the vision of mathematics based on whether students refer to the role of reasoning (i.e., relational understanding) or sets of rules for memorization (i.e., instrumental understanding) in reflecting on their mathematics engagement (Skemp, 1978). Next, the perceived competence in mathematics is related to students' beliefs about their success, which depends on different factors such as grades, teachers' appreciation, and/or understanding. Finally, the emotional disposition toward mathematics captures positive or negative feelings when students reflect on their relationship to mathematics (Di Martino & Gregorio, 2019). Students describe their experiences with mathematics by using expressions such as like, dislike, apprehension, or joy when they tell their story (Di Martino & Zan, 2010). Investigating the underlying reasons for these emotional expressions is a useful method in understanding students' attitudes. Di Martino and Zan (2010) claimed that students often refer to one or more of these dimensions when elaborating on their story of mathematics. Students tend to connect their cognitive difficulties to affective factors such as interest, enjoyment, and perceived competence (Di Martino & Gregorio, 2019; Furinghetti et al., 2013; Liston & O'Donoghue, 2010). Beliefs about mathematical competence and feelings about mathematics become even more critical when looking at the experiences of women (Sax et al., 2015; Solomon et al., 2011) in gendered mathematical environments.

Method

Settings and Participants

We conducted a small pilot study in 2019 Spring with 12 participants who were either pure mathematics (PM) majors or double majors in PM and secondary mathematics teaching (SMT). In this report, we present findings from three female students (Table 1), whose experiences highlight the key aspects of our findings. Our aim was to gain insight into the salient features of the STT for students enrolled in a southeastern university in the US, particularly from WURMs' standpoint. First, we interviewed students to capture their STT experiences along three dimensions of interest (emotional disposition towards mathematics, vision of mathematics, and perceived competence in mathematics). Next, an analysis of the interviews aided our design of a supportive intervention, which confronted aspects of the STT via an examination of the (historical) changes in the nature of mathematics (in this case, related to the concept of derivative). Data collected during the four-session seminar (each session was three hours in duration) include survey responses, video recordings of seminar sessions, and student reflection diaries. In this study, we focused on a small portion of data from the interview and seminar (reflection diaries from days 1 and 4) to report our initial observations related to the three aforementioned dimensions of the STT. In our analysis, we adopted a grounded theory approach by following open, axial, and selective coding to group students' responses according to three dimensions of our theoretical framework (Charmaz, 2006). We reduced the raw data into smaller units to take analytical descriptive notes for "explicating implicit actions and meaning" (Charmaz, 2006, p. 50). Next, we recorded useful quotes from the participants to illustrate their affect-related experiences. In the next section, we highlight our initial findings on how WURM students articulated their STT experiences with university mathematics, as well as how our intervention contributed to their perception of the STT.

Table 1. Selected participants and data sources (WURM pilot study, 2019).

Students	Major/Year	Data Sources
Karine	PM/Sophomore	Interview, Reflections, Survey
Malya	PM & SMT/Junior	Interview, Reflections, Survey
Evelyn	PM & SMT/Senior	Interview, Reflections, Survey

Preliminary Findings

In this report, we present episodes from Karine, Malya, and Evelyn to address the three dimensions from the perspectives of WURMs. We note that Karine identified herself as Latina, Malya as Black, and Evelyn as White. We first discuss the vision of mathematics to capture how changes in the nature of mathematics impacted students' emotions. Second, we provide excerpts pertaining to students' perceived competence and highlight their accompanying emotional responses. Finally, we discuss the seminar intervention and highlight students' emotional disposition toward advanced mathematics courses and whether and how the seminar contributed to their transition experiences.

Vision of Mathematics

In the interview, we asked students to discuss their STT experiences. We observed that students expressed certain challenges by emphasizing changes in the nature of mathematics by contrasting school and university mathematics. Initially, the students were not aware of the STT. After discussing the phenomenon with them, students were able to recognize its impact on their experiences. In the following excerpts, Evelyn and Malya similarly articulated that the transition had begun when they were introduced to more advanced courses (e.g., after Calculus 3), while they perceived lower division courses as an extension of secondary school mathematics.

> The real transition didn't really happen for me until I started to take the major level of math classes. Because I feel for the most part like, the [Calculus sequence was] pretty much formula-based and the things were still tangible. … we're still using numbers, whereas now I feel it's a completely different strain of math; it is all theoretical; it is all proof-based and it is just something very different. (Evelyn, Interview 2019)

Malya also stated that the STT was rough for her, especially after Calculus 3, despite her good preparation in high school. Also, she explicitly reflected her emotional dispositions in coping with difficulties during the STT. Later, she found some resources for support. Also, it is worth noting that Malya attributed her struggle in the STT to her inner mathematics abilities, which also captures her lower perceived competence in mathematics.

> I think it [the transition] was rough, but I also think that it wasn't as rough as it could have potentially been. … entering Calc 3 right after [the] first semester here…was bad. Like, I know that was bad, but I also think that I had resources here on campus. At least the resources that I was able to gain access to and find easily here that made me feel like I wasn't doing as terribly as I could have… I feel like a lot of the transition that was bad for me was kind of like a mind thing. It wasn't that I couldn't do it. It was just that I was thinking I couldn't do it and thinking that made me feel more reluctant to do my homework and more reluctant to ask for help and basically just talk about the things that were going on and knowing that I was struggling, but it didn't feel comfortable enough to bring up the struggle to anyone. (Malya, Interview 2019)

We also captured emotionally charged utterances from Evelyn's interview, which were similar to how Malya articulated her experience. Surprisingly, we detected that the professor privileged a passing grade rather than understanding of content. The following excerpt illustrated Evelyn's inner unsettled feeling, which also reflected that she valued understanding in mathematics for success as opposed to solely receiving a passing grade.

> I find that like, every semester, I panic because I think I am failing the class. I get stressed out because I am not doing well on these tests. I am getting like 60s. But then, the professors are always like: Yeah, that's average. Don't worry about it. We are going to just curve the grade. So then, I end with these B-plusses, a couple of Cs, and [in] so many harder [classes]. And, I feel like, I struggle in the next class because I didn't do well on the first class and then everything builds, and it's just like a snowball effect. I feel like the transition won't end until I graduate because I am still struggling to really handle this new strain of mathematics that I am dealing with. (Evelyn, Interview 2019)

Karine's reflections on the STT in mathematics seemed less emotionally laden compared to Evelyn's and Malya's. One can expect students to reflect on their emotions more often when confronted with the changes in the nature of mathematics, which began after the Calculus 3 sequence for Evelyn and Malya. As a sophomore, Karine might not have spent as much time in the major to make sense of how courses will look as she advances in course taking. In the following excerpt, Karine highlighted some changes in her vision of mathematics related to university mathematics courses.

> I guess this stuff at the beginning of the course isn't that big of a jump. It's like functions. It's set theory, which we've been using set notation for a long time. The concept of functions and mapping and stuff that comes up in Linear Algebra. It's more theoretical, but it's not like this huge jump. It's just kind of phrasing it in a different way and looking at it from a different angle, [like] the metric spaces… At the beginning of the semester, the proofs class was like, 'Oh my gosh' But then… I've gotten the groove of things. I was like, okay, I'm going to start having a schedule for when I'm going to work on this homework. … That's when I started talking with this math friend. … We spend like three hours going through this stuff for this class. And then, once we leave that initial kind of like, 'oh gosh, what are we doing?' sort of thing then, I was like, okay, this isn't really too bad. (Karine, Interview 2019)

Karine's experiences in dealing with the abstract nature of mathematics showed a necessary change in vision of mathematics, as well as learning and studying strategies. It seems that Karine was fully cognizant that she needed to put forth more effort to overcome difficulties that arose in the STT related to the nature of mathematics. Additionally, it is worth noting that she sought to engage in mathematics conversations with one of her peers which might be an indication of her need for academic and social support. Overall, considering these three students' reflections, it is reasonable to argue that mathematics majors need social and academic support. They might need to share experiences and notice common challenges about mathematics in their program of study. These ideas led us to organize a seminar in 2019 to create a small academic space for these women to talk about their mathematical experiences and dispositions.

Perceived Competence in Mathematics

We noticed that students expressed conflicting beliefs about their perceived competence, which seems to be linked to increasing abstractness in the nature of mathematics. Specifically, Evelyn and Malya reflected on changes in the nature of mathematics after Calculus 3 and how their perceived competence was influenced by this change in upper division mathematics courses. Karine also mentioned how this change in the nature of mathematics "[shook] her confidence" and led her to seek additional help to cope with the STT. Accordingly, we detected that students reflected on their emotions at varying levels. Depending on variances in lived experiences of these three women, their emotional reaction levels were also notably different.

> In Calc 3, ODE and Linear Algebra, yes, I did feel like that [confident]. Because there were clear steps that could help me. And, we were still using numbers. But then in the major-related classes, I do not feel competent or confident with my work. I find that I am … just writing down stuff that we went over in class. And hoping that somehow it makes sense and is a good enough proof to prove something. (Evelyn, Interview 2019)

Malya expressed similar sentiments regarding her perceived competence while taking Calculus courses. Interestingly, we noticed that Malya pointed out her lower confidence before Calculus 3, which seems different from Evelyn's experience. Considering opportunities and the intersectionality, Malya's high school preparation might not have sufficiently prepared her for lower division mathematics courses (Wade et al., 2017).

> I don't know if it's [my confidence that has] changed because… even though I am confident about … anything less than Calc 3 now, I do remember it in the moment while taking them, I was not so confident about it. And so, I think, my confidence levels have been about the same while I'm taking the courses at least. (Malya, Interview 2019)

We identified similar patterns in Karine's reflection on perceived competence in her mathematics experiences. Even though Karine stated that being surrounded by other good students is intimidating, she viewed them as a resource for improving her knowledge.

> Sometimes I do feel like, oh my gosh! I made a really dumb mistake on an integral, or I had to look up something that I knew where that should have been really obvious. So then… that just shakes my confidence. But … I don't really let it get to me because just why would I do that? That's just a waste of my time. …and then seeing other people… I guess better or more advanced in math, sometimes it's like, oh geez, that's intimidating. But it's also why am I worried about that? They can help me. (Karine, Interview 2019)

As seen from the excerpts, each of the three women mentioned that at some point there were times when they did not feel as confident as they had previously. Interestingly, we observed connections between increasing abstractness in mathematics and students' inclination to question their mathematical competence (Di Martino & Gregorio, 2019). This tendency often prompted their unproductive emotions.

The Intervention Seminar: Students' Emotional Reflections on the STT in Mathematics

During the first seminar session, we asked students to reflect on their STT experiences in the major. We captured evidence of affective dimensions in the STT that influenced our participants' mathematical experiences and emotional dispositions. We observed that the seminar positively contributed to their experiences in the STT regarding emotional dispositions when dealing with more advanced mathematics. Creating a safe, collegial space for mathematics majors seemed to encourage them to engage in mathematical activities such as thinking, feeling, and talking about mathematics. Notably, Evelyn expressed loneliness considering the lack of representation of women in various mathematical spaces and different hardships in her transition.

> Knowing that there is a name to this phenomenon is comforting. When dealing with "the transition problem," I often felt alone in this transition, especially being surrounded by mostly men who did not seem to have this problem. (Evelyn, Seminar 2019, Day 1)

Similarly, Malya's reflection on the first day provides evidence that even though she was not aware of a transition phenomenon in mathematics, it resonated with her. Malya touched upon the prerequisite knowledge related to secondary school mathematics that did not warrant success to proceed to advanced university mathematics (Rach & Heinze, 2017). Malya's reflections highlighted her views on the different nature of mathematics in the two institutions as well as a necessary adjustment of learning strategies (Clark & Lovric, 2008).

> I feel like this transition is similar to a well-known undercover secret. A lot of folks acknowledge that there is this period of transition between school and university, yet they are not doing much about it. There seems to be a lack of effort in attempting to shorten or lessen the "transition." My teachers warned me about how much more studying I would need to do and how schoolwork is nothing like university-course work. I understand that going to college is not for everyone, but for those who are going to college, I think it is important for them to have that exposure (Malya, Seminar 2019, Day 1)

On the fourth day, we asked students how the seminar was useful in dealing with challenges in their mathematics program when the changes in the nature of mathematics were considered. Karine stated that the seminar was helpful for her to develop more productive attitudes toward dealing with advanced mathematics. She also commented on how the academic content of the seminar was useful for her to gain a better understanding of an important mathematical concept that will benefit her in course work. Karine's reflection was an exemplar of mathematical confidence, which seemed to be uplifted by conceptual support provided in this seminar.

> I feel more confident and excited with regards to my 4000-level math classes next semester, including Analysis. I definitely feel like I have a better grasp of the derivative and differential as a mathematical object, which is a huge relief since deep down there was still that last bit of confusion of what a derivative actually was and how it came to be. It was amazing! I loved every part of it. (Karine, Seminar 2019, Day 4)

Moreover, Evelyn pointed out her positive attitudes toward advanced mathematics after the seminar, which previously did not seem to be so. From a pre-service teacher perspective, Evelyn commented on in-class practices that did not benefit her conceptual understanding, especially for upper-level mathematics courses. Group discussions and interactions were important for her to become more open to abstract mathematics.

> This seminar improved my attitudes toward transition. When dealing with formal mathematics, I was miserable and struggled a lot. After the seminar, I am a little more open to dealing with abstract mathematics. I think if there was a little more discussion, explanation, or collaboration, like the seminar, then I think I could have gotten more out of the class. I think this seminar was really useful. (Evelyn, Seminar 2019, Day 4)

Implications

This pilot study suggests that STT experiences might differ across country contexts. Also, we conjecture that WURMs' STT experiences vary based on high school preparation, year in the mathematics major, and gender. Using affective aspects of the STT is promising for capturing students' vision of mathematics and perceived competence. Our research provides valuable insights into perseverance strategies and support mechanisms that WURMs need in coping with challenges and how they configure their new mathematical space. We posit that future studies should examine more deeply how academic and social support interventions can contribute to WURMs' mathematical experiences in STT, while considering the notion of identity.

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Implementation of IBL in Undergraduate Calculus Classes

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We report the results of a survey of calculus instructors from colleges and universities across the US related to their reported awareness and usage of Inquiry Based Learning, instructional practices, and beliefs about teaching and learning. Cluster analysis of the data revealed three distinct types of self-identified IBL-users based on the proportion of in-class time students spent in different activities, including a large number of instructors who lecture for the majority of class time. The teaching practices of that sub-group did not differ significantly from those of the group of instructors who reported no knowledge of IBL. Variation in level of agreement with positive statements about lecture shows some association with in-class instructional practice; all four groups indicate strong agreement with statements that inquiry practices support learning.

Keywords: Instructional practices, Calculus, STEM, Individual Characteristics

Introduction

Inquiry forms of instruction have been part of broader educational reforms going back at least to the work of John Dewey (Artigue & Blomhøj, 2013) and one manifestation of this in mathematics education has been inquiry-based learning (IBL). Broadly, the pedagogical approach called IBL has been practically defined as having students engage with each other and in actively reinventing mathematics through a carefully constructed sequence of high-level tasks while an instructor acts as a 'guide on the side' to facilitate students' activity and discussions towards the curricular content agenda (Artigue & Blomhøj, 2013; Ernst et al., 2017; Hayward et al., 2016; Laursen & Rasmussen, 2019; Rasmussen & Kwon, 2007). The IBL community typically offers guidelines and principles to instructors; with less emphasis prescribed, specific pedagogical choices for implementing IBL. While not narrowly defined, IBL is usually associated with student-centred instruction and activities such as working in small groups, presentations, and whole-class discussions and decreased use of instructor-activities such as lecturing and solving problems (Hayward et al., 2016; Kogan & Laursen 2014; Laursen et al. 2014). The variety in implementations of IBL lends to be conceived of a spectrum of related activities, as opposed to a ridged instructional format.

Given how broadly IBL is defined and how similar the tenets of IBL are to other active learning approaches more generally (Ernst et al. 2017; Rasmussen & Kwon, 2007; Rasmussen, Marrongelle, Kwon, & Hodge, 2017; Walczyk & Ramsey, 2003), the way in which IBL is implemented by individual instructors may be largely dependent on widely varying instructor beliefs. While IBL is fairly well-known and promoted among mathematicians (see results in Apkarian et al., 2019), there are open questions about whether IBL is being implemented in ways that are consistent with the tenets of IBL, something other researchers have questioned as well (Stains & Vickery, 2016). We do not know if IBL implementation translates into a clearly distinct set of instructional practices within active learning, or the extent to which IBL implementations differ from teacher-centred instructional approaches. Thus, the goal of our paper is to examine whether self-identified IBL implementors have a clearly definable set of instructional practices and attitudes, and the extent to which they differ from those who have never heard of IBL. In this paper we address the following research questions:

1. What is the range of instructional practices among instructors of undergraduate calculus courses who report using IBL in their classes, and how does this compare to the practices of those who have no knowledge of IBL?
2. What attitudes toward teaching and learning are associated with different instructional profiles among self-identified IBL users, and how does that compare to those who have no knowledge of IBL?

Literature Review

Prior research has tracked the evolution of IBL as it has become a more formalized, yet broadly conceptualized, pedagogical approach (Ernst et al. 2017; Haberler et al., 2018; Laursen & Rasmussen, 2019; Rasmussen & Kwon, 2007; Rasmussen et al., 2017). Earlier work identified one, then two, then three, and now four guiding principles which are outlined on the *Academy of Inquiry Based Learning* website (inquirybasedlearning.org). Those principles, reviewed by Lauren and Rasmussen (2019) as Inquiry Based Mathematics Education (IBME) were synthesized as 1) deep student engagement in rich mathematics, 2) student collaboration with peers and instructors, 3) instructor inquiry into student reasoning, and 4) instructor attention to equity. In terms of the principle of deep student engagement in rich mathematics, students are to make significant contributions to the curriculum progression and math concept development. Students may not be provided answers or methods to follow ahead of time but instead have to wrestle with ideas before making conclusions; the goal of the instructor is to get the students work on activities that are novel and challenging to them. The second principle, collaboration, involves students working in groups, as a class, and or individually in order to learn how to effectively communicate mathematics and deepen their understanding (e.g. individuals going to present a proof for the class to evaluate); the goal of an instructor is to help students engage with other students' thinking and facilitate cross-talk among students). The third principle of IBME, instructor inquiry in students' reasoning, has been operationalized in various ways in prior research but generally involves instructors eliciting student ideas publicly and building on them (Laursen & Rasmussen, 2019; Rasmussen & Kwon, 2007; Rasmussen et al., 2017); the goal of the instructor is to build on and extend student work. Earlier work by Rasmussen and Kwon (2007) noted that this instructor inquiry helps instructors construct models of student thinking and learning, provides instructors opportunities to gain deeper insight into mathematical ideas by reflecting on student activity, and positions instructors to build off student thinking by posing follow up questions or tasks. The fourth, most recent addition, calls on instructors to attend to classroom dynamics which arise in interactive settings and consciously support an equitable and respectful experience for each student. This principle has been put forward as an aspirational characteristic of IBME (and IBL), not an inherent one (Laursen & Rasmussen, 2019).

Given the broad conceptualization of IBL and how innovative instructional practices are then taken up (Scanlon et al., 2019; Stains & Vickery, 2016), implementation of IBL is likely to vary widely between instructors. One determining factor in how variations in teaching practices arise is instructors' beliefs (Leatham, 2006). Prior research has identified how beliefs, and specifically beliefs about teaching and learning, impact instructional practice (Hoyles, 1992; Leatham, 2007; Philipp, 2007; Speer, 2008; Sztajn, 2003). We conjecture that while IBL is diffused as an innovative instructional practice, the broad nature of IBL will allow instructors to implement IBL in various ways, which will be partly determined by instructors' individual beliefs and goals.

Methods

Survey Overview and Key Items

This work is part of a larger study about factors impacting the uptake of research-based instructional practices in postsecondary chemistry, mathematics, and physics, which included the development and administration of a web-based survey informed by related research in undergraduate STEM education (e.g., Apkarian et al., 2019; Johnson et al., 2018; Gibbons et al., 2018; Henderson & Dancy, 2009; Lund & Stains, 2015; Walter et al., 2016). The survey is an amalgamation and adaptation of items from validated instruments which gathered information about instructors': general instructional practices (Landrum et al., 2017, Walter et al., 2016); beliefs about teaching, learning, and students (Aragón et al., 2018; Chan & Elliott, 2004; Dweck et al., 1995; Johnson et al., 2018; Meyers et al., 2006; Tollerud, 1990); and departmental climate and culture (Walter et al., 2014). The survey also included a section on demographic questions related to both professional and personal identities and roles (see Apkarian et al., 2020 for additional detail). Survey participants were instructed to consider a particular, recently taught course (in mathematics, a single-variable calculus course) when responding.

In this report we focus on data from a subset of the items related to instructional practice and instructor's beliefs about teaching, learning, and students. Our analysis incorporated two items about instructional practice. The first (Table 1A) asked instructors to report the percentage of in-class time (in a typical week) students spent in four kinds of activity: working individually, working in small groups, participating in whole-class discussions, and listening to the instructor lecture or solve problems. The second (Table 1B) asked instructors to rate their familiarity with IBL[1] on a five-point scale from "I have never heard of this" to "I currently use it in this course to some extent." Seven belief items (Table 1C) were queried using a six-point Likert scale from strongly disagree to strongly agree, with no neutral option.

The survey was distributed electronically in Spring 2019, and 3769 instructors of introductory STEM courses at 851 postsecondary institutions across the U.S responded. Of these, 1349 were instructors of undergraduate single-variable calculus courses; here we report only on responses from the 967 who responded to all of our target items. We further reduced the dataset to include only 366 respondents: those we refer to as IBL users (289 instructors who report currently using IBL in their calculus course) and non-IBL (nIBL) instructors (77 instructors who report never having heard of IBL).

Table 1. Survey question, in which participants: (A) selected the percentage of class time spent in each of four activities from drop-down menus of 0, 5, 10, …, 95, 100; (B) selected one of five options describing their level of awareness or experience with IBL; and (C) indicated their level of agreement with seven Likert items. Statements 1-4 are positive statements about lecture; statements 5-7 are statements aligned with an inquiry perspective on learning. These statements were presented to participants as part of a larger set and in a scrambled order.

[**A**] During a typical week, what proportion of time during regular class meetings (i.e., lecture sections) do students spend doing the following? (Answers must total 100.)	[ ] Working individually [ ] Working in small groups [ ] Participating in whole-class discussions [ ] Listening to the instructor lecture or solve problems
[**B**] Please indicate your awareness and (if applicable) usage of … <u>Inquiry-Based Learning (IBL):</u> A broad range of empirically validated teaching methods which emphasize	1. I have never heard of this 2. I know the name, but not much more 3. I know about this, but have never used it in this course

[1] The full survey asked mathematics instructors about 12 distinct RBIS, but for this paper we focus on IBL only.

(a) deeply engaging students and (b) providing students with opportunities to authentically learn by collaborating with their peers	4. I have tried it in this course, but no longer use it 5. I currently use it in this course to some extent
[**C**] To what extent do you agree/disagree with the following statements? *Strongly disagree, disagree, slightly disagree, slightly agree, agree, strongly agree* 1. I think lecture is the best way to teach. 2. Students learn best from lectures, provided they are clear and well- organized. 3. I think lecture is the only way to teach that allows me to cover the necessary content. 4. The major role of a teacher is to transmit knowledge to students. 5. I think students learn better when they struggle with the ideas prior to me explaining the material to them. 6. Making unsuccessful attempts is a natural part of problem-solving. 7. Learning means students have ample opportunities to explore, discuss, and express their ideas.	

Data Analysis

Our goal was to investigate how different or similar were the instructional practices of calculus instructors implementing IBL in their classes. Specifically, we wanted to classify the 289 self-reported IBL-users based on their reported breakdown of how class time is spent (Fig.1). To accomplish this task, cluster analysis was conducted to identify homogeneous groups of participants and the four clustering variables were the instructional practices: working individually (WI); working in small groups (WSG); participating in whole class discussion (WCD); and listening to the instructor lecture or solve problems (LL).

To determine the optimal number of clusters, a combination of hierarchical and non-hierarchical methods was used. Hierarchical cluster analysis using the Centroid Clustering method with Euclidean Distance as a measure of distance revealed that the respondents could be clustered into three distinct groups. To test for robustness, a *k*-means cluster analysis was conducted. The *k*-means cluster profiles were in agreement with those obtained from hierarchical cluster analysis, thus supporting the three-cluster solution. A one-way analysis of variance within the clusters was then performed to examine the difference among the four instructional practices across the three clusters. Tukey's post hoc HSD testing was used to assess pairwise differences in the amount of class time spent in each activity between instructor groups.

The final stage of analysis involved examining individual beliefs about teaching and learning associated with instructors across the three clusters of self-reported IBL-users and those who reported to have no knowledge of IBL (nIBL). Within each cluster, we examined the participants' beliefs around instruction. We compared the means of the participants' responses to seven Likert items (Table 1C) using Tukey's HSD test. Participants responded to the items on a measuring scale of 1 (strongly disagree) to 6 (strongly agree).

Results

Instructional Profiles

The *k*-means cluster analysis of the instructional practices of self-declared IBL users identified three distinct groups of practitioners. Based on their instructional profiles (Fig. 2), we refer to the three clusters of IBL-users as MSG (mostly small group work), MW (mixed ways), and ML (mostly lecture). The MSG group report the highest average of class time spent in small-group work (53%); the MW group do not have a dominant instructional activity; ML is the largest group and report the highest average of class time spent in didactic lecture (62%). For context, we compare these instructional profiles to of those who report never having heard of IBL (nIBL). As shown in Figure 1 and confirmed in Table 2, the nIBL group reports a class time distribution very like that of the ML group (average 65% class time spent in lecture). For each of the four instructional activities, group membership has a statistically significant main effect on the amount of time spent in that activity (see Figure 2 caption).

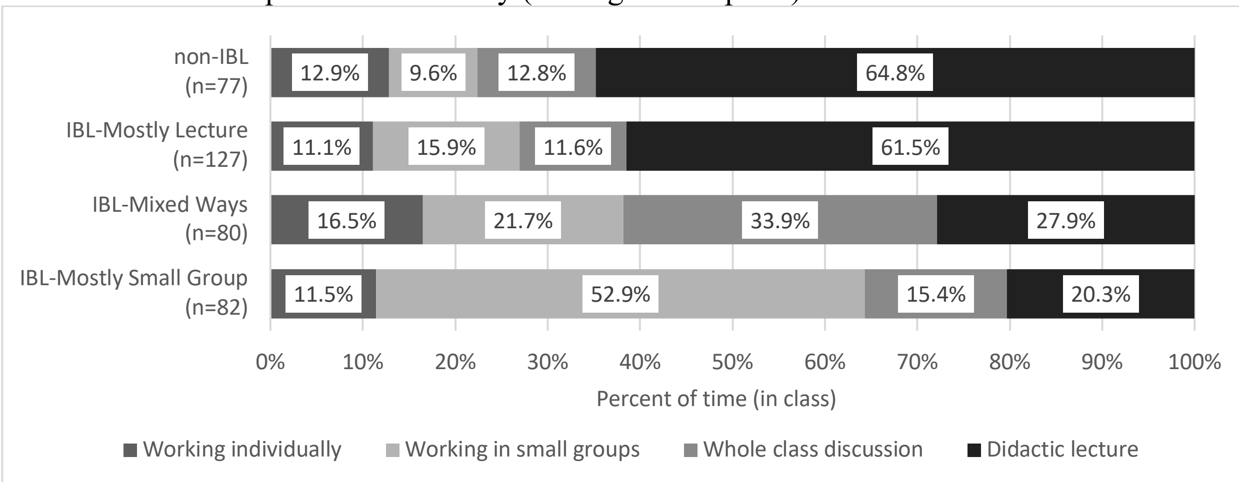

Figure 1. Stacked bar charts of the percentage of class time spent in each of four pedagogical activities. ANOVA indicate group impacts the amount of time students spend in each activity: working individually $F(3,362)=4.07$, $p<0.01$, $\eta^2=0.03$; working in small groups $F(3,362)=231$, $p<0.001$, $\eta^2=0.66$; whole class discussion $F(3,362)=73.32$, $p<0.001$, $\eta^2=0.38$; and didactic lecture $F(3,362)=196.9$, $p<0.001$, $\eta^2=0.62$.

We then compare the instructional practices across these three groups and the non-IBL users using (Table 2). The three types of IBL users varied in their instructional practices and there were significant differences between the groups. ML and nIBL were distinguished only by a medium-sized effect of time spent working in small groups; there are large differences in the amount of time these groups spend in lecture compared to MW and MSG (who differed slightly). Small group work varied the most across groups, and individual work time the least.

Table 2. Results of Tukey HSD post hoc testing at 95% family-wise confidence intervals between instructor groups for each of the four pedagogical activities. Cells show difference of group means.

	MSG-MW	MSG-ML	MSG-nIBL	MW-ML	MW-nIBL	ML-nIBL
Working individually (WI)	-5.0* d=0.5[Med]	*ns*	*ns*	5.4** d=0.6[Med]	*ns*	*ns*
Working in small groups (WSG)	31.2*** d=2.9[Lrg]	37.0*** d=3.2[Lrg]	43.3*** d=3.4[Lrg]	5.8** d=0.5[Med]	12.1*** d=1.0[Lrg]	-6.3** d=0.5[Med]
Participating in whole class discussion (WCD)	-18.6*** d=1.5[Lrg]	*ns*	*ns*	22.4*** d=2.0[Lrg]	21.1*** d=1.5[Lrg]	*ns*
Listening to instructor lecture (LL)	-7.6** d=0.7[Med]	-41.2*** d=3.3[Lrg]	-44.5*** d=2.4[Lrg]	-33.6*** d=2.7[Lrg]	-36.9*** d=2.0[Lrg]	*ns*

*p<0.05, **p<0.01, ***p<0.001; differences which were not statistically significant (ns) are omitted.
Effect sizes calculated for significant differences using Cohen's d: 0.2 < Small < 0.5< Medium < 0.8 < Large.

Attitudes to Teaching and Learning

In terms of individual beliefs about teaching and learning, we calculated the group mean scores for seven items (Figure 2) and conducted a Tukey HSD test (Table 3).

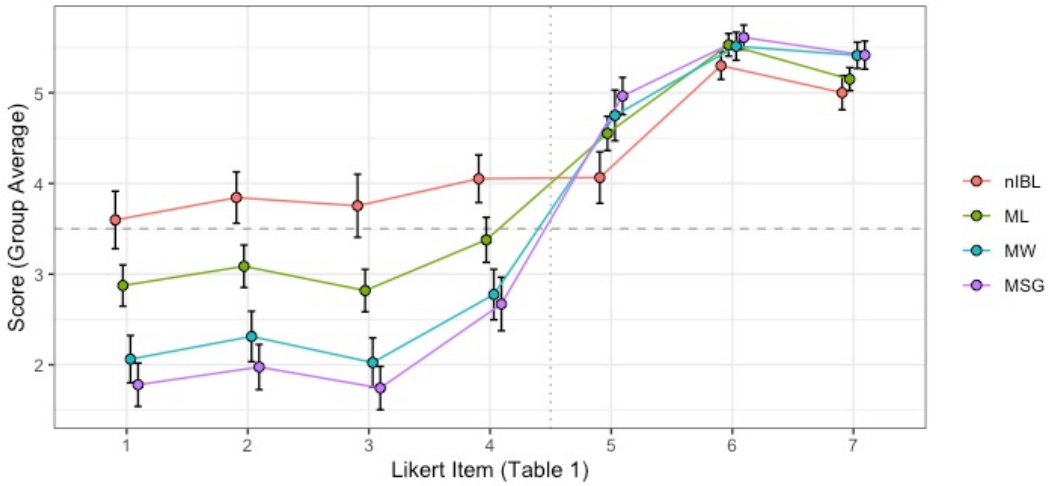

Figure 2. Mean and 95% CI of the mean of each instructional group's responses to each of the seven selected Likert items. A vertical line separates the statements about lecture (1-4) and statements about learning through inquiry.

The MSG and MW groups were indistinguishable in their responses to these Likert items. All four groups indicated agreement with statements that inquiry supports learning, with some variation. The groups differed in their responses to the positive statements about lecture. MSG and MW indicated the strongest disagreement, followed by IBL-users who mostly lecture, and the nIBL group reported mild agreement.

Table 3. Results of Tukey HSD testing (differences of group means) at 95% family-wise confidence intervals.

	MSG – MW	MSG – ML	MSG – nIBL	MW – ML	MW – nIBL	ML – nIBL
1. I think lecture is the best way to teach	*ns*	-1.1*** *d*=0.9[Lrg]	-1.8*** *d*=1.5[Lrg]	-0.8*** *d*=0.7[Med]	-1.5*** *d*=1.2[Lrg]	-0.7*** *d*=0.5[Med]
2. Students learn best from lectures, provided they are clear and well-organized	*ns*	-1.1*** *d*=0.9[LRg]	-1.9*** *d*=1.6[Lrg]	-0.8*** *d*=0.6[Med]	-1.5*** *d*=1.2[Lrg]	-0.8*** *d*=0.6[Med]
3. I think lecture is the only way to teach that allows me to cover the necessary content	*ns*	-1.1*** *d*=0.9[Lrg]	-2.0*** *d*=1.5[Lrg]	-0.8*** *d*=0.6[Med]	-1.7*** *d*=1.3[Lrg]	-0.9*** *d*=0.6[Med]
4. The major role of a teacher is to transmit knowledge to students	*ns*	-0.7*** *d*=0.5[Med]	-1.4*** *d*=1.1[Lrg]	-0.6** *d*=0.5[Sm]	-1.3*** *d*=1.1[Lrg]	-0.7** *d*=0.5[Med]
5. I think students learn better when they struggle with the ideas prior to me explaining the material to them	*ns*	0.4* *d*=0.4[Sm]	0.9*** *d*=0.8[Lrg]	*ns*	0.7*** *d*=0.6[Med]	0.5* *d*=0.4[Sm]
6. Making unsuccessful attempts is a natural part of problem-solving	*ns*	*ns*	0.3* *d*=0.5[Sm]	*ns*	*ns*	*ns*
7. Learning means students have ample opportunities to explore, discuss, and express their ideas	*ns*	*ns*	0.4** *d*=0.5[Med]	*ns*	0.4** *d*=0.6[Med]	*ns*

Adjusted *p<0.05, **p<0.01, ***p<0.001; differences which were not statistically significant (ns) are omitted. Effect sizes calculated using Cohen's d (rounded to one decimal place) 0.2 < Small < 0.5< Medium < 0.8 < Large

Discussion

The primary objective of this report was to characterise self-identified calculus IBL implementors based on proportion of in-class time students spent on four different activities alongside their beliefs about teaching and learning. First, we hoped to gain insights into how different or similar IBL classrooms would be and examine the differences, which we conjectured may exist, between IBL and non-IBL classrooms. Second, we sought to examine to what extent instructors' beliefs about lecture and non-lecture align with their instructional practices.

Addressing our first research question, we found the instructional practices of IBL instructors varied remarkably. Our cluster analysis identified three distinct groups – with significant difference in instructional practice. In one cluster small group work dominated the instructional time; in one cluster class time was pretty evenly distributed between four different instructional practices; and, in one cluster students spent the majority of in-class time listening to the instructor lecture. If we take into account the number of IBL instructors in each of the three clusters we see that a significant portion of these self-identified IBL users, about 43% of them, were sorted into the "mostly lecture" group, where students spent on average 62% of in-class time listening to the instructor lecture. In terms of instructional practice, there was limited variability reported by this "mostly lecture" cluster of IBL implementors and instructors who reportedly have never heard of IBL. Thus, our findings imply that it would be a mistake to assume a consistency between IBL courses or to assume that in IBL classrooms the majority of class time is spent on student-centred activities. Further, we concur with researchers (e.g., Hayward et al., 2016) that future efforts to spread IBL should examine factors that influence implementing IBL and support instructors to incorporate IBL practices in their instruction.

Second, we found statistical alignment across all the IBL clusters (despite instructional differences) with regards to their attitudes toward inquiry learning, but these attitudes were also shared by those with no knowledge of IBL. Thus, acknowledging inquiry as supportive of learning does not appear associated with stated IBL usage or instructional practice. However, attitudes toward lecture do appear to have this association. Self-declared IBL users indicated more disagreement with positive statements about lecture than the nIBL group, and there is a tentative association between agreement with positive statements about lecture and the percentage of class time spent lecturing. The responses of self-identified IBL-users whose classes are dominated by lecture are intriguing, and further analysis is needed to understand how this group describe their understanding of IBL as a pedagogical and/or teaching philosophy, their instructional practices, and how their attitudes and beliefs about teaching and learning drive their practical pedagogical decisions within and outside the classroom. Our findings suggest that beliefs about lecture are more related to pedagogical choices than beliefs about inquiry, and that this is an avenue for further investigation by those seeking to shift instructional practice.

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What is Instantaneous Rate of Change?

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The purpose of this study is to examine students' meanings for the derivative at a given input value. While students may verbally state that the derivative represents instantaneous rate of change, that does not imply that they have a coherent meaning for the difference quotient as an average rate of change. This study explores students' responses to a typical calculus 1 problem that requires the use of derivatives to determine a linear approximation. This study analyzes student responses about instantaneous rate of change and the differences in meanings that students may hold about it.

Keywords: Derivative, Instantaneous Rate of Change, Calculus, Student Meanings

The concept of derivative is foundational to the learning of higher topics in the STEM fields. However, researchers report that the learning of derivative is difficult (Tall, 1981; Park, 2013; Zandieh, 2000; Oehrtman, 2002; Monk, 1994; Ubuz, 2007; Yu, 2020). Therefore research on how students reason about the idea of derivative will be useful for informing efforts to improve the teaching of derivative. The seemingly paradoxical issue that students have to resolve is interpreting the derivative at a given input value ($f'(3) = 6$) as regarding a single instance (at the input of 7), yet derivatives are about change, so how can you talk about change if there is only one instance involved? Perhaps due to this issue, students hold multiple disconnected meanings about derivative (Zandieh & Knapp, 2006) as a coping mechanism to get through their calculus courses. Since derivatives represent something that we call "instantaneous rate of change", then students' meanings for rate of change are pertinent to their understanding of derivative. Yet, researchers (Byerley et al., 2012; Simon & Blume, 1994; Castillo-Garsow, 2010) have shown that students have differing ideas about what a rate of change is. Due to these issues related to learning the idea of derivative and rate of change, the purpose of this study is to explore students' meanings about the idea of derivative as instantaneous rate of change. The research question this study examines is:

How do students interpret the derivative in a context as instantaneous rate of change?

Literature Review

The literature on derivatives has been extensive thus far, ranging from the limiting process (Oehrtman, 2002; Roh, 2008, Ferrini-Mundy & Graham, 1991; Weber et al., 2012), conceptions of rates of change (Byerley & Thompson, 2017; Hackworth, 1995; Thompson, 1994, Confrey & Smith, 1994), and student understandings of the derivative as a function (Park, 2013; Baker et al., 2000; Aspinwall et al., 1997; Asiala et al., 1997; Borgi et al., 2018; Zandieh, 2000).

Derivatives are about quantities changing; thus, how students reason about quantities changing is central to how students construct a meaning for instantaneous rate of change. Castillo-Garsow (2010,2012) investigated how students may be reasoning about change and identified two distinct student images of change, *chunky* and *smooth*. A student engaging in chunky reasoning imagines the value of a quantity changing discretely from one point to another but with no imagery of motion between those two points. A student engaging in smooth reasoning imagines a change in a quantity as a change in progress by conceptualizing a quantity as taking on values as time flows continuously and smoothly. These different ways of imagining changes in quantities would likely influence how a student interprets a rate of change.

According to Zandieh (2000), there are four representations of derivatives that students tend to recall: graphically as the slope of the tangent line, verbally as instantaneous rate of change, physically as speed, and symbolically via the limit definition of derivative. Zandieh's derivative framework has been central in calculus education research (e.g., Petersen et al., 2014; Roundy et al., 2015; Zandieh & Knapp, 2006) and has led other researchers to hypothesize the desired understandings we hope students will hold (Jones & Watson, 2017). While Zandieh's (2000) framework is a useful tool in comparing students' meanings with a desired meaning, one gap in the literature is how students conceptualize the value of an instantaneous rate of change. In order to make changes in the standard Calculus curriculum, we need to investigate the meanings that students are constructing about derivative and analyze how and why students build these meanings. This paper explores how some students explain their meaning for instantaneous rate of change and characterizes the ways of thinking that these students employed.

While a student may verbally explain a derivative as a rate of change, research has shown that students have various understandings about what a rate of change is. Byerley et al. (2012) investigated calculus students' understanding of division and found that many students used additive reasoning when interpreting a rate's value. These students considered the rate's value as how much they needed to add to the function's output. Simon & Blume (1994) reported that additive reasoning does not lead to students' conceptualizing division quantitatively; instead, students use addition to replace thinking multiplicatively. This additive reasoning is also what Castilo-Garsow (2010) described when his student explained an interest rate as how much money to add to a bank account every year. If students use additive reasoning about in contexts that require them to use the idea of rate, this will affect how they approach learning and using the idea of derivative and what they imagine instantaneous rate of change at a point represents.

Theoretical Perspective

The Mathematics of Students

This study employs Radical Constructivist theories of learning (Thompson, 2000), taking the perspective that it is impossible to know another's thinking. Therefore, investigating student thinking aims to build models of students' mathematics (Steffe & Thompson, 2000) that may be used as an explanatory model for why students produce specific responses. The use of this approach to build an explanatory model of students' thinking can then be useful for describing the process of developing a productive meaning for instantaneous rate of change. I use 'meaning' the same way that Thompson (2013) uses it to describe mathematical meaning. It is the organization of an individual's experiences with an idea that determines how the individual will act. Meanings are personal, and they might be incoherent, procedural, robust, or productive, but individuals use these meanings to respond to mathematics tasks and make sense of and access mathematical ideas. For example, a person's meaning for derivative might only be associated with calculating limit of the difference quotient, while another's meaning for derivative involves the slope of a tangent line. Since meanings are personal, if one student writes a response similar to another student, it cannot be assumed that they both have the same meaning.

Conceptual Analysis

In a typical Calculus 1 course, instantaneous rate of change is introduced to students via the limit definition of derivative, $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{(x+\Delta x)-(x)}$ (Stewart, 2013; Larson et al., 2006). One productive interpretation of the limit is the multiplicative relationship (the value of $f'(x)$) between a variation in a function's output ($f(x+\Delta x)-f(x)$) and a variation in the input ($(x+\Delta x)-(x)$) so long as the variation from the input value x is arbitrarily small ($\lim_{\Delta x \to 0}$). The convergence of this limit is what we call "instantaneous rate of change," which represents the relationship between two varying quantities with respect to one another's relative size of variation. To use a derivative value as a rate of change having some value m, one must imagine the input quantity varying while simultaneously imagining the output quantity varying m times as much as the input quantity's variation. This is the same meaning we might attribute to an average rate of change over a small interval, in that someone is imagining the necessary constant rate of change to achieve the same accrual in one quantity with respect to the accrual size of the other quantity. Additionally, a student also has to recognize that the output variation will be *essentially equal* to the actual variation since the quantity is not changing at a constant rate of change. In this case, what it means to be *essentially equal* is that the approximation of the variation in the output by assuming a constant rate of change will be so close to the actual variation that the difference between the two is imperceptible.

Methodology and Data Analysis

This study employed clinical interview methodology (Clement, 2000). Clinical interview methodology consists of an interviewer, a single student, and a camera to record each interview. During these interviews, the interviewer asks the student to engage in a mathematical task. These tasks intend to explore how students are reasoning about them. As the interview progresses, the interviewer creates models of the student's understandings and tests their hypotheses by asking questions and probing the student's thinking. Additionally, the interview makes no teaching moves since the purpose is to understand how the student is thinking about a particular topic.

Clinical interviews were conducted with 25 students at a southwestern university who were enrolled in a Calculus 2 course or were at the end of their Calculus 1 course. This study was part of a larger study that involved tasks designed to help investigate students' understandings of the derivative at a point and their meaning for rate of change. Only the first task is presented here.

This study was analyzed using Open and Axial Coding for moment-by-moment coding of students' responses and interpretations (Stauss & Corbin, 1998). Using the codes from each student, I conducted a thematic analysis (Clarke & Braun, 2013) across moments within each students' moments and across different students' moments. This thematic analysis aimed to identify and analyze the patterns of student responses to model the types of thinking that students were engaging in.

Results

In this study, each student was presented with a task about interpreting the derivative at a point and then performing a linear approximation [Figure 1].

Given that $P(t)$ represents the weight (in ounces) of a fish when it is t months old,
a.) Explain the meaning of $P'(3)=6$
b.) If $P\ (3)\ =\ 15$ and $P'(3)\ =\ 6$ estimate the value of $P\ (3.05)$ and say what this value represents.

Figure 1: The Fish Task

18 of the 25 students wrote an interpretation of $P'(3) = 6$ as representing instantaneous rate of change. By examining the written responses, two types of answers emerged from these 18 students; one stated the units as ounces per month, and the other stated units as ounces [Figure 2]. The other seven students did not include rate of change in their explanation. Some of these students only explained that it was the slope of the tangent line or the result of differentiation. However, they did not express another type of understanding that indicated they were associating the derivative at a point with instantaneous rate of change.

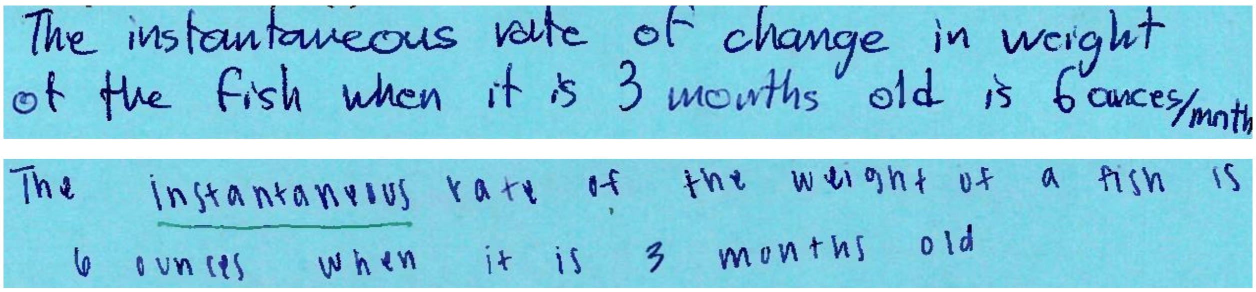

Figure 2: Students' Written Interpretation of $P'(3) = 6$

After students wrote their responses, each student was asked to explain their written statement. Even within the eight students who chose "ounces per month" as their units, different meanings emerged as students attempted to articulate their thinking. Similarly, the other ten students who chose "ounces" as their unit also expressed a variety of meanings despite having written similar responses. Figure 3 categorizes the meanings that students articulated when asked about what instantaneous rate of change meant to them in this situation. About one-third of the students chose ounces as their unit and this suggests that these students have not conceptualized a rate of change as involving two varying quantities. Some of these students conveyed that they interpreted the value of a rate as an additive difference in the weight of the fish (as opposed to a ratio between the two quantities of weight and time elapsed), which aligns with other researcher's findings that students confuse rate quantities with amount quantities (Byerley et al., 2012, Castillo-Garsow 2010).

Instantaneous Rate of Change with units as "Ounces per Month" (8 Students)

- 2 students described how weight and time would change in some interval of time
- 3 students stated "if the rate does not change then the fish will gain 6 ounces in 1 month"
- 1 explained that 6 ounces per month was found by finding the average rates of change over arbitrary intervals involving the time value of 3 months
- 1 described it as the average rate of change over the entire fish's lifetime
- 1 explained it as speed at a moment like "the reading of a speedometer"

Instantaneous Rate of Change with units as "Ounces" (10 Students)

- 3 students conveyed that the fish would grow 6 ounces in the entire 3rd month
- 5 said "if the rate does not change then the fish will gain 6 ounces in 1 month"
- 1 explained that the fish will grow 6 ounces every 3 months
- 1 described the average rate of change over the 3rd month.

Figure 3: Summary of Students' Explanations for Instantaneous Rate of Change

It is noteworthy that even if cases where students provided the same answer, they did not necessarily share the same meanings. Among this pool of 18 students, I identified eight different meanings for instantaneous rate of change. The following excerpts exemplify this finding.

Andrew was a 2nd-year student enrolled in a Calculus 2 course. He explained instantaneous rate of change as an amount of change in a 1-unit change in the time [Table 1].

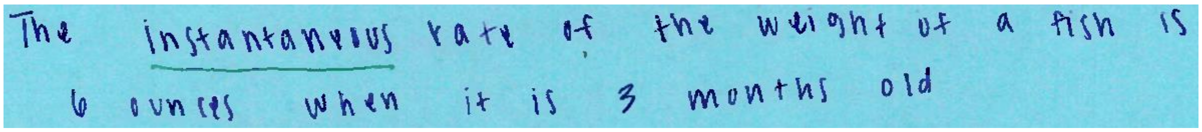

Table 1: Andrew's Explanation for the Derivative at a Point

1	And:	So it's at three months and it's growing by six ounces… it's weight increased by six
2		ounces… that's how much it grew that month so for the third month it grew six
3		ounces.
4	Int:	Like in that entire month?
5	And:	It gained 6 ounces.
6	Int:	So you're saying that if I have like a calendar month **Draws a calendar**, over the
7		course of all this you're saying that the total change in weight is…
8	And:	6 ounces yeah
9	Int:	Okay, so what are the units for 6?
10	And:	Change in ounces.

Andrew interpreted instantaneous rate of change as "it's weight increased by 6 ounces" and "that's how much it grew that month" [Lines 1-2]. He also conveyed that he thought that the fish "gained 6 ounces" in the entire third month [Line 5]. This suggests that Andrew interpreted $P'(3) = 6$ as the difference in weight between the 4th month and the 3rd month, i.e., $P'(3) = P(4) - P(3)$. When asked about the units on 6, Andrew said it was a "change in ounces" [Line 10]. Andrew described the change as the difference in weight between two points in time. For him, this value of 6 did not describe a multiplicative relationship between variations in weight and time. Instead, it was how much weight to add to the fish's weight at the 3rd month to get the fish's weight at the 4th month. Andrew's description is akin to the kind of additive reasoning that Byerley et al. (2012) described. If we consider the verb tenses that Andrew employed, we can see that Andrew mostly used past tense verbs to describe the variation in the fish's weight. Andrew stated, "it's weight increased" [Line 1], "that's how much it grew… it grew 6 ounces" [Lines 2-3], and "it gained 6 ounces" [Line 5]. This suggests that Andrew might have thought that the values of the fish's weight were already there, and that $P'(3)$ was describing the variation in the fish's weight between the 3rd and 4th month. Andrew did not seem to be thinking about the value of this rate as describing the multiplicative relationship between the two varying quantities; instead, it appeared that Andrew was using chunky reasoning by thinking of an entire discrete chunk of change in the entire 3rd month. Additionally, Andrew never mentioned that time was passing; in fact, the only instance where he described an interval of time was at the beginning where he stated: "that month so for that month" [Line 2]. Andrew's explanations provide evidence that he was thinking about a chunk of time instead of time varying continuously.

Jerry was a 3rd-year Computer Science major at the end of his Calculus 1 course. Jerry explained instantaneous rate of change as an average rate of change [Table 2].

The instantaneous rate ~~the~~ the fish is gaining weight in ounces at specifically 3 months is 6 ounces/months

Table 2: Jerry's Explanation for the Derivative at a Point

1	Int:	So can you tell me what you wrote means to you?
2	Jer:	At 3 months, the rate that the fish is gaining weight is 6 ounces… so let's say the fish
3		is gaining weight then at 3 months… so when you… it is like the average
4	Int:	Average?
5	Jer:	Yeah like the 3 is the amount of 3 months if it were to ummm like the average like a
6		straight line graph **uses his finger to motion a straight line going up and to the right**
7		at 3 months would be 6 ounces per… would be the rate like the average rate over
8		those months
9	Int:	So instantaneous rate of change means average rate then?
10	Jer:	Like it is an average for those 3 months, and we are measuring it at that specific time
11		of 3 months, like instantaneous.

Initially, Jerry struggled to articulate what he meant by instantaneous rate of change and ends by saying that it is an average [Lines 2-3]. Next, Jerry explained that he interpreted the value of the derivative of 6 as the average rate of change of the fish's weight over the entire fish's age up to the input value of 3 months [Lines 5-8]. Jerry's responses convey that he was thinking about a slope or a constant rate of change as he moved his hand to motion a straight-line graph when describing an average [Lines 5-6]. Afterward, Jerry confirmed that he interpreted instantaneous rate of change as referring to an average [Line 10]. Additionally, it appeared that to Jerry, the word '*instantaneous*' referred to an instant of time when he states "that specific time of 3 months, like instantaneous" [Lines 10-11]. One possibility for Jerry's explanation is that he may be recalling that the derivative involves the slope between two points ($\frac{f(b)-f(a)}{b-a}$) while forgetting the limiting process involved. Later on in the interview, Jerry continued to explain rate of change only in the context of a graph. This suggests that Jerry's image of change was primarily about the physical shape of a function's graph.

Randy was a 2nd-year in a Calculus 2 course explained instantaneous rate of change as a rate of change over a small interval of time [Table 3]

A. At 3 months old, the fish is growing at a rate of 6 oz per month

Table 3: Randy's Explanation for the Derivative at a Point

1	Ran:	Like the instantaneous rate of change, so like… that's how much it's changing by
2		over a process of time. *Slides his hands to motion*
3	Int:	So like over the first three months it gained 6 ounces?
4	Ran:	No like…let's say like from…. like 2.9 to 3.1, you can assume like the average rate
5		of change is 6, like from there to there it would keep changing by like 6 ounces per
6		month.
7	Int:	So the average rate of change between 2.9 and 3.1 is always going to be 6?
8	Ran:	No, but I mean I'm just like trying to explain it cause it's not changing if time isn't
9		changing

Randy did not initially write "instantaneous rate of change," but when explaining what he wrote, Randy immediately said that it was "like the instantaneous rate of change" [Line 1]. As Randy explained what instantaneous rate of change meant to him, he articulated that he was thinking about "how much it's changing over a process of time," and as he said this, he slid his hands to express the imagery of this thinking [Lines 1-2]. As he continued explaining, Randy indicated that he was thinking about a variation in the input since he said "like from… 2.9 to 3.1…like from there to there" [Lines 4-5]. Randy clarified that he was not thinking of 2.9 and 3.1 as fixed values. Instead, he chose them because he was "trying to explain it cause it's not changing if time isn't changing" [Lines 8-9]. Here Randy articulated that he thought about quantities varying and associated the rate of change with these varying quantities. However, he did not explicitly say how the value of the rate of change corresponded to these quantities changing. It is important to note that Randy seemed to be thinking about quantities changing smoothly through intervals due to his choice in verb tense when he described a quantity changing. Randy repeatedly used "changing" when asked about what he meant; "changing by over a process of time [Lines 1-2], "keep changing by like 6 ounces per month" [Lines 5-6], and "it's not changing if time isn't changing" [Lines 8-9]. His repeated use of the word "changing" as well as his hand sliding motion, evidence that Randy was engaging in smooth reasoning by thinking about a change in progress.

Discussion

From the results of this study, we can see that students hold differing meanings about instantaneous rate of change. With these different meanings, one would expect that students would solve part b of the fish task (a linear approximation problem) in various ways as well. All but one student produced a correct solution, however, did not engage in the same ways of reasoning to do so (Yu, 2020). If a teacher did not inquire into how students produced their solutions, they would likely believe that their students understand instantaneous rate of change the same way they do. This also indicates that our current assessments on linear approximation are insufficient in evaluating students' meanings about instantaneous rate of change.

The results of this study indicate that students do associate derivative with instantaneous rate of change, yet they may have different meanings about what instantaneous rate of change is. This study supports past findings (Zandieh, 2006; Park 2013) that students in a traditional calculus course might have unconventional meanings about derivatives and instantaneous rate of change. Therefore, teachers should consider what meaning they desire for their students to hold for derivative and instantaneous rate of change, and how students may be supported in building coherent and productive meanings. In particular, the results of this study suggest that Calculus teachers should attend to the meanings that students bring into their classroom about rate of change since many students consider rates as an amount to add instead of describing the multiplicative relationship between two varying quantities. Typically, most Calculus textbooks (Stewart, 2013; Larson et al., 2006) do not explicate a meaning for rate of change and likely assume that students understand what a rate of change is. I argue that teachers should intentionally spend time to aid students in developing a robust meaning for rate of change so that they can better understand a derivative as representing instantaneous rate of change.

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Adapting a Single Local Instructional Theory into Multiple Instructional Sequences

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Local instructional theories (LITs) are often described as a generalized sequence of steps for students' guided reinvention of some mathematical concept. They consist of a series of tasks and the accompanying rationale for the tasks, where the rationale is described in student's mathematical activity. Literature suggests that the rationale can be used to adapt LIT's into different instructional sequences however it fails to provide specifics as to how these adaptations are achieved. This report details how the design heuristic of didactical phenomenology can be used to create multiple instructional sequences for students with different mathematical backgrounds, thus contributing to our understanding of the theory of realistic mathematics education.

Keywords: realistic mathematics education, didactical phenomenology, group theory

Abstract algebra is a required course for almost every undergraduate mathematics student (Blair, Kirkman, & Maxwell, 2013) and it is notoriously difficult for them (Larsen, 2010; Leron, Hazzan, & Zazkis, 1995; Weber & Larsen, 2008). In fact much of the literature on student understanding in abstract algebra highlights student struggles (Weber & Larsen, 2008). One of the many contributing factors to student difficulties in the subject is the abstract nature of the content of the course (Hazzan, 1999). In an attempt to help students navigate the abstract nature of the content, I recently engaged a variety of students in a real-world application of abstract algebra by investigating molecular structures. The findings reported here are from a larger design experiment aimed at developing a local instructional theory (Gravemeijer, 1998) for the guided reinvention of a classification system (i.e. flowchart) for chemically important symmetry groups (Bergman & French, 2019; Bergman, 2020).

As students begin to develop a system for identifying various symmetry groups, they need to decide on a group representation that is encapsulated, or compact, enough to be used as the outputs for their classification system. While there are a number of ways to represent the group concept (Bergman et al., 2015), one such as Cayley tables would be rather impractical to serve as the terminating points of a flowchart. Therefore, students were ultimately encouraged and supported in using something like a group name, i.e. $\mathbb{Z}_3$, D_8, $\mathbb{Z}_2$ x $\mathbb{Z}_2$. This report gives evidence and explanation to how I used the design heuristic of didactical phenomenology to support students with different mathematical backgrounds in their use and reinvention of group names, thus attending to the research question: *What kinds of mathematical activity can promote the evolution of the students' informal knowledge and strategies into more powerful ways of thinking, symbolizing, and reasoning?*

Theoretical Framing

The design study reported here utilized the instructional design theory of realistic mathematics education, RME, in order to develop a local instructional theory, LIT (Gravemeijer, 1998). RME provided both an underlying theoretical perspective and three accompanying design heuristics. The notion that mathematics is a human activity instead of a collection of facts and procedures is the theoretical foundation of RME (Freudenthal, 1971). The first design heuristic of RME, the reinvention principle, suggests that by engaging students in the guided reinvention of a particular mathematical concept, we are creating opportunities for them to meaningfully engage in this activity of mathematizing. When used as a design heuristic, the reinvention principle promotes the idea of starting the reinvention process in a context that is experientially real to students (Gravemeijer & Doorman, 1999). The reinvention/evolution from an informally situated student strategy to conventionally recognizable, formal mathematics is evoked using the second design heuristic, emergent models. Emergent models help to describe the overall evolution of the students' mathematical activity. Since the formal mathematics that is eventually developed is rooted in the students' approaches, this particular development process helps students have a greater sense of ownership over the formal mathematics created. The students' use of the encapsulated group representation described in this report can be seen as the final version of the students' model for their unique symmetry group, which was eventually incorporated into their classification system.

The third and final design heuristic of RME is didactical phenomenology. Larsen (2018) states that, "Simply put, didactical phenomenology tells the designer that an instructional sequence meant to support the learning of a piece of mathematics should be situated in a context that can be productively organized by students using that piece of mathematics." (p.25) As a heuristic it is helpful to the design researcher in two distinct ways. First, didactical phenomenology is used to help determine an appropriate context for the students to engage with. For example, since the goal of the study was to support students in reinventing a classification system for the various symmetry groups of molecules, I began by having them first investigate the symmetry groups of specific molecules with a variety of group structures. The second way in which didactical phenomenology is useful to the design researcher is in helping the evolution of the student's mathematics from informal to more formal mathematics through progressive mathematizing. The specifics of how didactical phenomenology was used to promote this evolution in students' use of group representations are presented as the main results of this report.

Methods

The overarching design study consisted of three teaching experiments which were conducted with pairs of students with varying mathematical backgrounds. The data reported here is from the second and third teaching experiments exclusively. The second teaching experiment, TE2, involved a pair of senior level undergraduates, Arthur and Stu, who had recently completed a traditional proof-based introductory group theory course. The third teaching experiment, TE3, was conducted with Ada and Sophie, sophomores majoring in mathematics and engineering, respectively. Ada and Sophie had recently completed an introductory linear algebra course and neither had any previous experience with group theory or advanced mathematics.

The corpus of data included video recordings of the teaching experiment, all the written work which was scanned into pdf documents, and researcher memos created during ongoing analysis. TE2 consisted of 12 sessions lasting 60-90 minutes each, and TE3 was 11 sessions also lasting 60-90 minutes per session. Between sessions, memos (Maxwell, 2013) were used for recording: the goals for each session, a short recap of the students' mathematical activity after each session, and a description of how well the goals were met. During retrospective analysis content logs were used to record a description of the students' activity, highlight places the students moved forward in the LIT, and helped organized my understanding of the student's activity overall.

Results

The results are organized in two cases, each detailing a different instructional sequence created in response to the students' unique mathematical activity. Each case contains a description of the student's initial model of their symmetry groups and an account of the transition of this informal model into a model for more formal mathematical activity using the notions of didactical phenomenology.

Case 1: Students with prior group theory experience

For the students in TE2 who had previous exposure to conventional group theory concepts, their goal for representing their symmetry groups was to use "standard group names." By "standard name" I am referring to the paradigmatic representatives of the isomorphism classes to which each symmetry group belongs. For example, there is only one algebraic structure that satisfies all of the group axioms with three elements. This isomorphism class is often referred to as $\mathbb{Z}_3$ even though the elements in the set are often referring to something other than $\{0,1,2\}$ under modular addition. It is common practice in abstract algebra, and also in chemistry, to use a single name to describe an entire isomorphism class. In chemistry, they do it quite explicitly by saying ammonia belongs to D_6.

The mathematics offered by the students as a starting point. When asked to identify the symmetry groups of water, ammonia, and ethane (in an eclipsed configuration) students in TE2 seemed to have some access to standard group names. Evidence of student's mathematical activity naming the symmetry group of ethane is given below.

Stu: I think we could rep… like one way I might wanna think about starting is um, we know that we could compare it to this (ammonia) because this half of ethane looks an awful lot like ammonia, right? (see Figure 1 below)

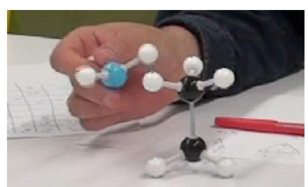

Figure1. Stu's visual comparison of ammonia and ethane

Arthur: yeah

Stu: So I feel like it might be something like $\mathbb{Z}_2$ cross ammonia. If that makes any sense?

How the student's progressive mathematization was supported using didactical phenomenology. Referring to the symmetry group of ethane as "$\mathbb{Z}_2$ cross ammonia" is mathematically inaccurate for a number of reasons; ammonia isn't technically a group, none of the elements in the symmetries of ethane are integers under modulo addition, and there is no external direct product between two groups. However, Stu's suggestion was also quite promising for a number of reasons as well. The symmetry group of ethane, which would conventionally be considered an internal direct product, is isomorphic to the external direct product D_6 x $\mathbb{Z}_2$ and the students were already comfortable with the symmetries of ammonia as they had recognized it as equivalent to a more familiar group from their introductory group theory course, the group of symmetries of a triangle. This posed the instructional design question: How can students be supported in connecting their intuitive notions of different group structures with conventional representations?

Where's the D_6 and where's the $\mathbb{Z}_2$? To better understand the students' use of a direct product, I asked them to explain how they were seeing the groups D_6 and $\mathbb{Z}_2$ in their group of symmetries of ethane. Note that after Stu's previous suggestion of "$\mathbb{Z}_2$ cross ammonia" we discussed that the symmetry group of ammonia could also be referred to by the more conventional name of D_6.

Researcher: Last time you said that the symmetry group of ethane was D_6 x $\mathbb{Z}_2$ can you tell me, what's the D_6 and what's the $\mathbb{Z}_2$? Like where is D_6 and where is $\mathbb{Z}_2$?

Arthur: D_6 was the configuration of the three kind of top hydrogen as we were looking at them and $\mathbb{Z}_2$ encoded which face was up. Or in the case of these colorings, just cause we needed a way to tell the two triangular faces apart. So it was a case of the six combinations and then like a binary dimension to that, which is what crossing $\mathbb{Z}_2$ encoded.

This description of their symmetry group is encouraging. Their portioning of the set into two subsets is powerful since the group D_6 x $\mathbb{Z}_2$ can easily be partitioned into two copies of D_6 where one has a 0 in all the second components and the other has a 1 in all of them. By describing all the symmetries as the product of something in D_6 and "either color-swapped or not" they seem to be capturing the notion that all the elements in the group can be represented as some hk for two subgroups H and K, which is a necessary condition of internal semi-direct products.

Prove that your symmetry group for ethane is isomorphic to a more conventional representation of D_6 x $\mathbb{Z}_2$. In order for Arthur and Stu to meaningfully connect their intuitive notions of group structures to more conventional representations, they spent a lot of time in TE2 trying to prove that their symmetries of ethane were isomorphic to D_6 x $\mathbb{Z}_2$. A more detailed discussion of how the students' mathematical activity related to isomorphisms was supported using didactical phenomenology is beyond the scope of this conference report.

How do you know you have a D-something x $\mathbb{Z}_2$? What would you look for? After having spent a lot of time focused on various representations of D_6 x $\mathbb{Z}_2$ and making arguments about it's group structure, the students were asked to consider a similar group structure in a more generic context.

Reseracher: If I just handed you another molecule could you tell if it was going to be a *D*-something cross $\mathbb{Z}_2$? What would you look for? Describe something that you would expect to have a *D*-something cross $\mathbb{Z}_2$ symmetry group.

Arthur: I'd look for that regular polygon. I would look to be able to see it, how do I say this… looking for regular polygon, I feel good about that part of the process, and then I guess saying like how many nested symmetries now, because I can look at this square and there's another square that I can look at that's indistinguishable from the first. So that's kind of our $\mathbb{Z}$-category, our number of color options.

Researcher: …and then the $\mathbb{Z}_2$, sorry one more time…

Stu: Is coming from this plane of reflection, between the two polygons. And it's not really the molecules it's the plane of reflection.

The goal for this task was to test the students' familiarity with identifying groups that are isomorphic to external direct products and to see if they had begun to gain more ownership over the use of conventional group names. Stu's understanding that a plane of reflection is necessary to pair with a regular polygon is particularly encouraging. Stu's attention to an independent symmetry can be interpreted as evidence of him being at least implicitly aware of the importance of the commutativity of r_h (the horizontal reflection associated with their "color swap") with the other symmetries. This symmetry is independent as in it doesn't have an impact on the orientation, which again is a function of commutativity. What Stu is seeing is important because it is the consequence of this commutativity, and commutativity establishes the equivalence between their internal direct product and the external direct product they used to describe their symmetries. While a strong argument can be made that Stu is observing the necessary conditions for an internal direct product to be met, it is also clear that he faces an obstacle in using the formal language necessary to describe why what he is observing is the necessary condition for this to have the proper group structure.

Summary of Case 1. Arthur and Stu start with a representation of a symmetry group that was based on their situated activity with ammonia and their previous exposure to external direct products. This informal activity was the motivation for a didactical phenomenological analysis aimed at supporting students in the use of more formal group names in a new more general mathematical reality. In other words, the students had gone from an informal notion of what a direct product represents to being able to describe what kinds of molecules will have this particular kind of group structure and why.

Case 2: Students without prior group theory experience

For the students in TE3 who had no previous exposure to group theory concepts, my goal for them was to reinvent a compact representation *like* group names to use in their flowchart since conventional group names were unavailable to them. Early in the teaching while establishing geometric symmetries as an experientially real starting point, Ada and Sophie were guided in reinviting the group concept by investigating the symmetries of square. This activity led them to describe their groups using what are called group presentations. Group presentations are comprised of a set of generators, and the relations between them.

The mathematics offered by the students as a starting point. When asked to identify the symmetry groups of water, ammonia, and ethane (in an eclipsed configuration) students in TE3 created a group presentation for each as seen in Table 1 below.

Table 1. Student's group presentations for the symmetry groups of water, ammonia, and ethane.

Water	Ammonia	Ethane (eclipsed)
$\mathbb{Z}_2 \times \mathbb{Z}_2$	D_6	$D_6 \times \mathbb{Z}_2$
0F=F 0R=R 2R=0 2F=0 Associative FR=RF Inverses	$R^3=0$ $F^2=0$ 0R=R 0F=F $RF=FR^3$	RF=FR $T^2=0$ $R^3=0$ $F^2=0$ FR=RF $RT=TR^2$ TF=FT

How the student's progressive mathematization was supported using didactical phenomenology. While this representation of a group is useful for capturing the structure of the group, the students needed a more concise representation to serve as the output for their classification system. In this case the student's mathematical activity posed the instructional design question: How can students be supported in transitioning from the use of group presentation to a more compact representation like a group name? Since conventional group names were unavailable to their students due to their lack of previous exposure to group theory, the question became: How can students be supported in the reinvention of group names? Fortunately, the students' mathematical activity of creating group presentations provided a context in which to seek solutions to these questions by conducting phenomenological analyses.

Test out your system on a new set of molecules. Ada and Sophie were given a new set of molecules and asked to determine the symmetry group for each, as seen in Table 2 below. This new set of molecules were specifically chosen to give the students an opportunity to work with each of the most common group structures found in chemically important symmetry groups: cyclic, dihedral, cyclic x $\mathbb{Z}_2$, and dihedral x $\mathbb{Z}_2$. It was necessary to expose the students to each of the group structures so that they would have sufficient names for each group in their flowchart. When given the new molecules, the students quickly and efficiently produced group presentations, seen in Table 2.

Table 2. Student's group presentations for the symmetries of a new and expanded set of molecules.

Tetra-aza copper II	Hydrogen peroxide	Boric acid	BrF_5	cyclobutene
$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_3$ x $\mathbb{Z}_2$	D_8	D_8 x $\mathbb{Z}_2$
TAC $R^4=0$ 4 symmetries No Fs No Ts	Hydrogen Peroxide $R^2=0$	Boric Acid $R^3=0$ $T^2=0$ 6 symmetries $TR=RT$	Brain Slug 4 rotations $R^4=0$ $F^2=0$ 8 symmetries $RF=FR^3$	Cyclobutane $R^4=0$ $F^2=0$ $T^2=0$ 16 symmetries $RF=FR^3$ $TR=RT$ $TF=FT$

Make a 'score card' of written rules for each molecule. Once the students had exposure to each of the different flavors of symmetry groups, Ada and Sophie were asked to make "score cards" for each of the molecules they had encountered so far containing each of their group presentations. Creating the score cards involved going though previous work to find each of the group presentations; doing so gave Ada and Sophie an opportunity to both standardize their notation and streamline their presentations. This standardization can also be interpreted as a formalization of their mathematics and therefore evidence of progressive mathematization.

Sort the score cards into collections/flavors/kinds of symmetry groups. Which seem more similar? Which hangout together? When asked to sort the score cards into collections of similar groups types Ada and Sophies initially offered two different strategies. Ada started by gathering the pure rotational groups together, whereas Sophie gathered the groups with order 3 rotations. Sophie speaks first suggesting that they put all the groups with the same order rotation together, but then quickly stops saying:

Sophie: Oh wait, what about the F's and the T's?

Ada: I kinda inherently wanna be like, oh there are ones that have R's and F's , R's and T's, R's, T's, and F's, or just R's.

Sophie: Yeah, it seems, it's totally grouping them.

Ada: I think that would be like my gut instinct.

They quickly work together to arrange the groups into four collections based on the existence of symmetries within the group (seen in Table 3). This categorization was extremely exciting because not only were the students one step closer to using a more useful group representation for their flowchart, this is precisely how group structures are conventionally categorized in the context of chemistry.

Table 3. Student's sorting of group presentation by the existence of symmetries.

Contain's *R* and *F*		
Water 4 symmetries $R^2=0$ $F^2=0$ $RF=FR$	Ammonia $R^3=0$ $F^2=0$ ~~$RF=FR^2$~~ $RF=FR^{n-1}$	BRF5 (Brain Slug) 8 symmetries $R^4=0$ $F^2=0$ $RF=FR^3$

<table>
<tr><th>Contains R and T</th><th>Contains R, T, F</th><th>Contains only R’s</th></tr>
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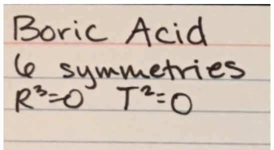
<td>
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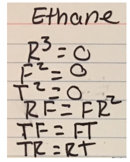
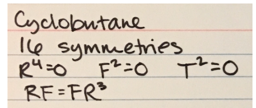
<td>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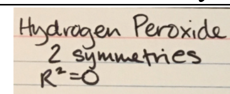
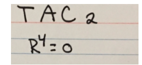
<td>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Come up with a name for each of the collections. Once Ada and Sophie were comfortable with their grouping of the groups into four categories, they were asked to come up with a “name” each of their categories of groups. The students quickly agreed on naming each collection of groups based on the symmetries that exist within the groups. Therefore, they ultimately created 4 symmetry group types; *RT group*, *RF group*, *R group*, and *RFT group*.

Summary of Case 2. In order to support Ada and Sophie in describing symmetry groups with concise group representations, the students were provided with supplemental tasks and prompts to support their progressive mathematizing. The sequence of tasks used to help the students move from group presentations to group names gives evidence to how the students’ informal mathematical activity was leveraged into more formal mathematics.

Discussion and Conclusion

My goal has been to show how didactical phenomenology can be used to create multiple instructional sequences for students with different mathematical backgrounds while engaging in the same LIT. The pairs of students in the two teaching experiments had access to two very different sets of mathematical experiences. For students in TE2 their previous exposure to group theory led them to focus on the use of conventional group names. However, the students had only an informal notion of the group structures needed to describe molecular structures. For students in TE3 their lack of access to conventional groups granted them an opportunity for reinvention. Didactical phenomenology was used in both cases to create tasks that could promote the transition of the student’s informal ideas into models-for more formal mathematical activity. By attending to aspects of the students’ mathematical activity that begged to be further mathematized by the use of a group name, individualized instructional sequences were created to support students in their reinventing and use of group names.

Implications

The implications from these results are twofold. First, this work adds to our collective understanding of the theory of realistic mathematics education. Literature describing LIT’s states that the sequence of tasks included in a LIT is meant to be general in the sense that it can be adapted to work in different instructional sequences (Larsen, 2018). However, when LIT’s are presented they are often described in terms of the mathematical activity of students with similar mathematical backgrounds (Larsen, 2009; Larsen & Lockwood, 2013; Swinyard, 2011; Wawro, Zandieh, Rasmussen, & Andrews-Larson, 2013). This work begins to shed light on how exactly these adaptations can be made. Second, this work starts to push on the notion of pre-requisites and access in advanced mathematics. Access to advanced math nearly universally comes after a long path of calculus and differential equations with no breaks. This is a sequence of opportunities to get kicked out with no alternative ways to get back, where we use surviving as a gatekeeper. The instructional sequence resulting from TE3 explicitly provides a way for students to engage in meaningful advanced math without much previous experience as a pre-requisite where they can be supported in building formal mathematics off their own informal activity.

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Making Connections: Advanced Perspectives on School Algebra

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Fostering conceptual understanding is a main goal of mathematics education. Yet, students do not often see connections between topics. The recognition of connections can be fostered by making them explicit in instruction and by encouraging students to struggle with the relevant mathematics. Instances in which knowledge from one domain is brought to bear in another domain are examples of transfer; in particular, backward transfer describes the ways in which learning new information influences prior knowledge. We solicited narratives from individuals with a strong mathematics background describing instances during which they recognized a connection between two mathematics topics. We present two of these narratives which contain instances of backward transfer connecting advanced mathematics to the binomial theorem. These narratives suggest that backward transfer is also facilitated by explicit instruction and productive struggle. We discuss implications for education of backward transfer that is induced by learning advanced mathematics, particularly for pre-service teachers.

Keywords: backward transfer, conceptual understanding, binomial theorem, connections

Introduction

Broadly construed, mathematics teaching focuses on two central aspects: conceptual understanding and procedural knowledge. Procedural knowledge fixates on students' abilities to accurately apply mechanical techniques, including algorithms and formulas as well as algebraic manipulations and routine proof techniques. Conceptual understanding, however, links procedures with one another and with deeper meanings. Our research focuses on conceptual learning: specifically, we explored the influence of learning advanced mathematics on conceptual understanding of school algebra.

Recent research has explored the formation of cognitive connections between advanced mathematics and school mathematics (e.g., Murray et al., 2017; Wasserman, 2018; Weber et al., 2020). In particular, questions have been raised about the benefits of taking advanced mathematics courses such as combinatorics and linear algebra for pre-service teachers who will likely never teach the content of these courses. In this paper, we share personal narratives from two individuals who recognized connections between advanced mathematics and the binomial theorem, a common topic in high school algebra courses. We then discuss factors that may have precipitated the formation of these connections and conclude with a discussion of the potential impacts of advanced mathematics study on the understanding of fundamental mathematics concepts.

Theoretical Framing and Research Question

Conceptual Understanding

We adopt Hiebert and Grouws' (2007) definition of *conceptual understanding* as "mental connections among mathematical facts, procedures, and ideas" (p. 380). This is distinguished from *skill efficiency*, or "the accurate, smooth, and rapid execution of mathematical procedures" (ibid., p. 380). Combined, conceptual understanding and skill efficiency form the backbone of modern mathematics instruction, yet Brophy (1997) notes that we have much more knowledge of how learners develop skills than we do of how they build concepts. Although it has been shown that different kinds of teaching differentially support the learning of these two aspects of mathematics, there is not a simple correspondence between a single method of teaching and a single outcome; for example, there is not a causal relationship between expository teaching and rote (procedural) learning, nor between discovery teaching and meaningful (conceptual) learning (Ausubel, 1963; Hiebert and Grouws, 2007).

Nevertheless, from their meta-analysis of empirical studies on measures of student learning, Hiebert and Grouws (2007) identified two key features of teaching that promote conceptual understanding. First, teachers and students should attend *explicitly* to concepts by "treating mathematical connections in an explicit and public way" (p. 383). Examples of these kinds of activities include "discussing the mathematical meaning underlying procedures, asking questions about how different solution strategies are similar to and different from each other, considering the ways in which mathematical problems build on each other or are special (or general) cases of each other, attending to the relationships among mathematical ideas, and reminding students about the main point of the lesson and how this point fits within the current sequence of lessons and ideas" (p. 383). Second, students should *struggle* with important mathematics. Hiebert and Grouws use the term *struggle* to mean that "students expend effort to make sense of mathematics, to figure something out that is not immediately apparent... The struggle we have in mind comes from solving problems that are within reach and grappling with key mathematical ideas that are comprehendible but not yet well formed" (p. 387).

Backward Transfer

Connections among content may be established by means of *transfer*, the process by which "learning in one context or with one set of materials impacts on performance in another context or with other related materials" (Perkins & Salomon, 1992, p. 3). In experimentation, transfer is a rare phenomenon, especially *far transfer*, which occurs between contexts that are only distantly related in the mind of the learner (Barnett & Ceci, 2002; Perkins & Salomon, 1992; Sala & Gobet, 2017). Hohensee (2011) and Hohensee et al. (2019) extended the notion of transfer to include *backward transfer*, defined by Hohensee (2011) as "the influence on prior knowledge by the acquisition and subsequent generalization of new knowledge" (p. 13). For example, Hohensee et al. (2019) studied the influence of participating in an instructional unit on quadratic functions on high school algebra students' views of linear functions.

Although transfer is considered rare in experimental settings, two mechanisms for its occurrence have been identified (Perkins & Salomon, 1992). *Reflexive*, or *low road*, transfer, occurs when a well-practiced routine is triggered by familiar stimulus conditions; *mindful*, or *high road*, transfer, occurs as a consequence of a deliberate search for connections.

Research Question

In light of the research on conceptual understanding and transfer, we investigated the following research question: How does the learning of advanced mathematics influence conceptual understanding of school mathematics? In the coming sections, we discuss the personal narratives of two individuals who made conceptual connections between school algebra and more advanced areas of mathematics. These connections led to deep mathematical insights and new perspectives on elementary mathematics, which we conceptualize as instances of backward transfer. We discuss these connections in terms of Hiebert and Grouws' (2007) key features for promoting conceptual understanding as well as Perkins and Salomon's (1992) mechanisms for transfer.

Making Connections

We collected narratives from a total of 14 individuals, all of whom hold at least a master's degree in mathematics or statistics. We asked these individuals to tell us a story about an experience in which they have "suddenly recognized a connection between two mathematical concepts [they] hadn't noticed before, particularly an experience that helped [them] understand some aspect of mathematics better or in a different way." Some individuals told of connections formed between different parts of the standard K-12 curriculum (e.g., the Pythagorean theorem and the Euclidean distance formula) or between different topics from advanced mathematics (e.g., Taylor polynomials and the comparison test for series convergence). We took particular interest in those narratives that told of connections between advanced mathematics and the basic mathematics taught in school. In this section, we share two stories of learners making connections between advanced mathematics and basic algebra. Because these stories are unique to those who experienced them, we have endeavored to present the following narratives as they were presented to us so as to preserve the storytellers' voices. We withhold our analyses of these narratives until the subsequent section, Theoretical Connections in the Literature.

Robin – Connections to Combinatorics

At the time when she shared her story, Robin was a graduate student working toward a PhD in mathematics education, having already completed a master's degree in mathematics. Her story is about a connection she made between the binomial theorem, which she learned in high school, and combinatorics, which she learned later at university.

The binomial theorem relates combinations $\binom{n}{k}$, often read as "n-choose-k," with the expansion of binomials via exponentiation: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$, where $n \geq 0$. Robin learned the binomial formula in her high school algebra course, but she described feeling unsure about what role choice played in this equation. She agreed that these combinations produced the correct coefficients when she worked out examples, and she could even see that these combinations yielded the entries in Pascal's triangle, but when she thought about $\binom{n}{k}$, she wondered: "What are we choosing here? Why does something have to be 'chosen'?" And was it a coincidence that the coefficients in the binomial expansion just happened to correspond to the values of $\binom{n}{k}$?

When Robin was later enrolled in a course in combinatorics at university, she connected the "choice" with the "jumps" she made while expanding a binomial expression. She explained this by writing $(x+y)^5$ as $(x+y)(x+y)(x+y)(x+y)(x+y)$, and drawing arcs connecting the x in the first binomial with x's and y's from the later binomials, as in Figure 1. It became clear that the combination was, in fact, counting the number of ways one could "choose" exactly k x's from the n possible ones. She noted the "nice symmetry" that this creates in the resulting polynomial.

$$(x+y)(x+y)(x+y)(x+y)(x+y)$$

Figure 1: "Jumps" indicating one way of choosing three x's when expanding a polynomial, as described by Robin.

This imagery also spurred Robin to notice why it must be true that $\binom{n}{k} = \binom{n}{n-k}$ – that one can choose k x's or *not* choose *n*-*k* x's in the same number of ways. Given five copies of $(x +$

y), one can choose exactly 3 x's in the same number of ways that one can *not* choose exactly 5-3=2 x's. Furthermore, Robin already knew the fact that the coefficients generated using the binomial theorem corresponded with the entries in Pascal's triangle, but this realization helped her to see the connection between the rule for generating the terms of Pascal's triangle and the combinatorial statement of Pascal's formula: $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$.

Martin – Connections to Linear Algebra

Martin holds a PhD in mathematics and is a university professor of mathematics and statistics. Martin, like Robin, saw the binomial theorem from a new perspective: linear algebra. At the time of his epiphany, Martin was a graduate student, preparing a lesson on the binomial theorem for his college algebra students. Martin's recent research had involved working with families of orthogonal polynomials, and he had spent a lot of time recently switching between bases for the vector space of univariate polynomials, $\mathbb{R}[x]$.

While he was preparing this lesson, Martin "had the familiar feeling of doing something [he] had been doing recently elsewhere," and he suddenly realized the binomial theorem could be thought of as "telling us how to expand a vector written in terms of the second basis as a linear [combination] of the vectors in the first basis." In other words, the binomial theorem could be thought of as a way to switch between bases of the vector space $\mathbb{R}[x]$, just as he had been doing in his research.

Martin gave the following example. The nonnegative integer powers of x (i.e., $1 = x^0, x, x^2, x^3, \ldots$) are a basis for the vector space $\mathbb{R}[x]$ of polynomials with coefficients in $\mathbb{R}$. Moreover, the nonnegative integer powers of $1+x$ (i.e., $(1+x)^0 = 1, (1+x), (1+x)^2, (1+x)^3, \ldots$) constitute a different basis for the same vector space, $\mathbb{R}[x]$. This means that an equation like $(1+x)^3 = \sum_{k=0}^{3} \binom{3}{k} x^k = 1 + 3x + 3x^2 + x^3$ can be thought of as taking the vector $(1+x)^3$ written in terms of the second basis and rewriting it as the linear combination $1 + 3x + 3x^2 + x^3$ of vectors from the first basis. Martin reflected favorably on the experience: "Both things got simpler by virtue of their relation to each other. [I was] happy to have seen it, a bit surprised that I hadn't heard it said that way before." Moreover, because of the connection between Pascal's triangle and the coefficients of the binomial theorem, Martin realized that Pascal's triangle can be thought of as a change of basis matrix by simply writing the entries of Pascal's triangle in a lower triangular matrix (Figure 2).

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & \cdots \\ 1 & 3 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Figure 2: Pascal's triangle written as a change of basis matrix.

Theoretical Connections in the Literature

In our participants' narratives, we see that the factors identified by Hiebert and Grouws (2007) for the development of conceptual understanding may also promote backward transfer. We extend Perkins and Salomon's (1992) nomenclature of *mindful transfer* and *reflexive transfer* to the case of backward transfer. Furthermore, we argue that making connections explicit during instruction may promote mindful backward transfer.

We presented two narratives of instances in which individuals made connections between topics in advanced mathematics and the binomial theorem, a topic frequently taught in high school algebra classes. Neither of these individuals described any procedural change in the ways they operationalized the binomial theorem when solving problems, but these connections led to conceptual shifts in the ways they understood the binomial theorem. As such, these connections are examples of changes in what Hiebert and Grouws (2007) called *conceptual understanding*, as Robin and Martin developed new "mental connections among mathematical facts, procedures, and ideas." (Hiebert & Grouws, 2007, p. 380) as a result of their interactions with advanced mathematics – combinatorics and linear algebra, respectively.

In these two narratives, we also see examples of backward transfer: mindful backward transfer in Robin's case, and reflexive backward transfer in Martin's. Robin's connection between combinatorics and the binomial theorem originated in her desire to understand the connection and her own effort exerted while using the theorem in her combinatorics class. She had encountered the binomial theorem in high school and had reason to believe that such a connection existed due to the presence of combinations in the formula for binomial expansion and direct instruction on the connection to Pascal's triangle. However, her desire to understand *why* combinations were related to the idea of binomial expansion had caused her to consciously seek resolution. Martin's connection, however, was not tied to his effort exerted toward understanding the binomial theorem; rather, it came as a consequence of exerting effort on related tasks in his research. Even though he did not realize the tasks were related to the binomial theorem while he worked on them, his need to revisit the binomial theorem in his teaching forced him to interact with it, and the recency of his work on (unbeknownst to him) related mathematics left him primed to make the connection. While Martin was not actively searching for a connection, he was surprised to find one as he experienced "the familiar feeling of doing something [he] had been doing recently elsewhere."

These stories are indicative of the two key features Hiebert and Grouws (2007) identified for the promotion of conceptual understanding, namely, the need for teachers to attend explicitly to connections in instruction and the need for learners to struggle with the related mathematics. In the cases of Robin and Martin, their connections did not take place during direct instruction on the binomial theorem, but rather at a later time: either during a later mathematics course or while working on research and teaching. Yet, in Robin's case, explicit instruction had given her an awareness that she should search for a connection she had not yet made, thus making mindful backward transfer possible. And in each of these stories, we see that the person was actively engaged in learning advanced mathematics, yet the connections they realized were to mathematics they had encountered much earlier in their careers. Their struggles to understand *advanced* topics induced them to develop new understandings of more basic topics that aligned with these advanced ideas. Robin's struggle was directly related to the binomial theorem and her exploration of the role combinations played. In contrast, Martin's struggle was tangential: he did not describe a struggle with the binomial theorem, but rather a struggle with the content he related to the theorem, that of changing bases for the vector space of polynomials. Both instances of transfer resulted from focused engagement with the mathematics involved in the connections that were eventually formed.

These narratives point to two important implications for education and the training of future teachers. The first is that the formation of important mathematical ideas and connections between ideas often results from engagement with advanced mathematics. Each of these narratives tells a story about an individual who learned something new about a concept long after they were taught the concept. While neither of these individuals was prevented from performing the requisite tasks assigned in their coursework, and indeed may have already possessed some conceptual understanding of the binomial theorem, each of them developed a deeper understanding of how the binomial theorem related to other concepts through participation in advanced mathematics. If one of our goals as educators is that our students should possess rich conceptual understanding, these stories suggest that, even in cases where connections are made explicit, students may benefit from engagement with advanced mathematics.

Questions are often raised about why prospective schoolteachers are required to take such advanced mathematics courses as combinatorics and linear algebra if the reality is that they will never teach the advanced topics contained in those courses. The second implication of our analysis is that these advanced courses provide learners with an environment in which they must see basic mathematical ideas from a new perspective through engagement in productive struggle. Each of these stories tells us about how moving on to more advanced mathematics allowed these individuals to see how the basic mathematical procedures they had learned in school fit into the larger context of mathematics. The benefit of advanced mathematics courses for teachers may not lie in the advanced mathematics, itself, but rather in the advanced perspectives that teachers develop about the fundamental mathematics they will one day teach to others.

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Examining Instructor Decision-Making Using Two Frameworks in the Context of Inquiry-Based Learning

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In this paper, I use two frameworks—Schoenfeld's theory of teaching in context (2010) and Herbst and Chazan's practical rationality theory (2012)—to make sense of an undergraduate mathematics instructor's decision-making when teaching with inquiry-based learning (IBL) methods. After providing a brief review of the relevant literature and the frameworks, I illustrate the analysis of one decision-making situation. I then discuss the affordances and challenges of the theories. The goal is to better understand what each framework offers in relation to the context and the nature of the analyzed data, and how findings from each framework can contribute to research on decision-making in the context of undergraduate teaching and IBL.

Keywords: teacher decision-making, university mathematics teaching, inquiry-based learning

Explaining why teachers do what they do has been a fundamental research problem on teaching, built on the central assumption that teachers are professionals who make reasonable decisions (Herbst & Chazan, 2017; Shulman, 1986a). Investigating teacher decision-making has mainly taken place in K-12 mathematics education settings. In a review of 60 articles on mathematics teachers' decision-making, perception, and interpretation, Stahnke, Schueler, and Roesken-Winter (2016) found only two articles assessing teacher decision-making in higher education. Given that postsecondary settings are distinctly different (e.g., faculty in general have more autonomy), I conjecture that faculty decision-making is different from that of K-12 teachers, thus, an important area of research in its own right.

In this study, I deliberately focus on teacher decision-making when implementing inquiry-based learning (IBL) methods in university mathematics courses[1]. While IBL has been appraised as a promising alternative to lecturing (Kogan & Laursen, 2014; Yoshinobu & Jones, 2012), the studies on the effects of IBL have not been entirely consistent in showing positive student outcomes (e.g., Freeman et al., 2014; Johnson, Keller, & Fukawa-Connelly, 2018; Laursen Hassi, Kogan, & Weston, 2014; Sonnert, Sadler, Sadler, & Bressoud, 2015). Moreover, because IBL has emerged from mathematicians' teaching practices rather than research findings, conceptualizations of IBL are still under development (e.g., Laursen & Rasmussen, 2019; Mesa, Shultz, & Jackson, 2020). Although these conceptualizations provide useful building blocks for practitioners, it is not obvious how instructors operationalize them. I propose that one way to explain the inconsistent student outcomes associated with IBL is to understand the source of differences in IBL implementation: teacher decision-making. This paper is a step in that direction.

I used two theoretical frameworks—Schoenfeld's theory of teaching in context (2010) and Herbst and Chazan's practical rationality theory (2012)—to illustrate the process of analyzing decision-making situations (for more, see Gerami, 2020). My goal through these preliminary analyses is to gain insights regarding how studying undergraduate mathematics instructors' decision-making can be approached theoretically in the context of IBL.

[1] See Ernst, Hodge, & Yoshinobu (2017), Laursen & Rasmussen (2019), and Yoshinobu & Jones (2013) for descriptions of IBL methods in undergraduate mathematics education within the U.S. context.

Background on Research on Teacher Decision-Making

Research on teacher decision-making has two main strands: cognitive and sociocultural. The cognitive strand started during the cognitive evolution in psychology in the 1960s and gained more traction as a result of Shulman's (1986b) call for using cognitive psychology in research with teachers. This approach assumes that individuals make decisions based on personal and idiosyncratic schemas (e.g., what has worked in the past) and as such, focuses on individual teachers rather than on teachers as a group (e.g., Bishop, 1976; Borko & Shavelson, 1990; Shulman & Elstein, 1975). Because of "theoretical and methodological limitations that keep the research at the scale of case studies of individual teachers" (Herbst & Chazan, 2017, p. 110), this strand has continued without acceleration in its growth. To address these shortcomings, a sociocultural orientation, that assumes teaching as a social activity, has emerged. This strand situates decision-making within ecological considerations, allowing researchers to investigate teacher decision-making in specific situations, and across teachers, as members of professional communities.

I chose two teacher decision-making frameworks—Schoenfeld's (2010) theory of teaching in context and Herbst and Chazan's (2012) practical rationality theory—for three reasons: (1) both frameworks, while situated within K-12 mathematics education research, have been empirically tested and found fitting in investigating mathematics instruction; (2) these theories have been increasingly appearing in empirical research in postsecondary settings (e.g., Lande & Mesa, 2016; Mesa & Herbst, 2011; Paterson, Thomas, & Taylor, 2011; Herbst & Shultz, 2019); and (3) each represents a different strand: Schoenfeld's theory is an individual-centered perspective of teacher decision-making, whereas practical rationality takes an ecological perspective and has been used to study decision-making across teachers. This allowed me to investigate IBL instructor decision-making from two complementary perspectives and reflect on the benefits of each framework in the given context.

Two Theoretical Frameworks to Study Teacher Decision-Making

Theory of Teaching in Context

The central claim of the theory of teaching in context is that people's in-the-moment decision-making in well-practiced, goal-oriented and knowledge-intensive domains, such as teaching, "can be fully characterized as a function of their orientations, resources, and goals" (Schoenfeld, 2010, p. 182). Three constructs—goals, orientations, and resources—are used to *model*, and even *predict*, teachers' in-the-moment decision-making in the classroom. By modeling, Schoenfeld means that the theory provides explanations of how and why teachers make decisions as they teach, with decision defined as the selection of goals and actions that are consistent with one's orientations and resources. The modeling is done with fine-grained analyses paired with subjective calculations for each teacher (see Schoenfeld, 2010).

In Schoenfeld's (2010) theory, teachers' decisions are always consistent with a set of goals of varying degrees. These goals are not always conscious; the teacher may make decisions that serve unconscious goals. When a goal is achieved, the goal with the next highest priority takes its place. Orientations, as an inclusive term that "encompass[es] beliefs, dispositions, values, tastes, and preferences" (p. 15), shape how teachers perceive different situations, establish their goals, and use their resources to achieve those goals. Similar to goals, orientations can be unconscious. As the third construct of the theory, resources are brought into a decision-making situation and are subjective to the specific teacher. Knowledge is a significant resource; once a piece of knowledge is activated, related knowledge and actions are also accessed and activated, resulting in established patterns of behavior such as routines and scripts. Resources can also be non-intellectual, such as material (e.g., curriculum, learning manipulatives), institutional (e.g., computer labs, online teaching platforms), or social (e.g., colleagues' support) resources. The theory assumes that in a contextualized situation, the teacher as an individual with a pre-existing set of goals, orientations, and resources, reevaluates their goals and makes decisions that are consistent with them. When a situation is familiar, existing routines and scripts are accessed, making the process more automated to reduce cognitive load. In unfamiliar situations, the subjective expected value of each available option is calculated and the option with maximum expected value is picked. Next, the teacher enacts the plan and monitors its effectiveness. Decision-making starts again when a goal is achieved or a plan is interrupted.

Theory of Practical Rationality

The theory of practical rationality of mathematics teaching uses two main constructs—norms and obligations—to unpack mathematics teacher's justifications behind their actions (Herbst & Chazan, 2012.). When teachers are asked to explain the rationality behind their instructional actions, norms and obligations are two sources they use. Norms are tacit expectations in teachers' and students' patterns of behaviors and responsibilities (Chazan, Herbst, & Clark., 2016); as cultural behaviors, they can go unnoticed, without the need to justify them, because they are implicit and typical (Moore-Russo & Weiss, 2011; Webel & Platt, 2015). Nonetheless, teachers may make different decisions under the same set of norms. Professional obligations, as types of social norms, are a set of resources that teachers, as professionals belonging to professional communities, use to justify deviating from the norms. The environments (e.g., department, institution) that an individual in the position of a mathematics teacher is placed in impose these social norms and values that "constrain and condition the position of mathematics teacher" (Chazan et al., 2016, p. 1066). Herbst and Chazan identify four stakeholders that a mathematics teacher has obligations to: the discipline of mathematics, including the knowledge and its applications, the valid representations of the knowledge, and the disciplinary practices (*disciplinary obligation*); the students and their individual needs (*individual obligation*); the society, including the appropriateness of patterns of interpersonal and social interactions, and acceptable societal dynamics (*interpersonal obligation*); and the institution and its regimes, such as policies and scheduling (*institutional or schooling obligation*). The four sets of obligations are, however, not exhaustive, as obligation to other professional groups are possible. The framework is considered situative rather than individualistic because it sees "teaching as an activity system involving positions, roles, and relationships" (Herbst & Chazan, 2012, p. 601). In other words, while teachers can and do make individual choices, these choices have some costs, because teaching is a cultural act with environmental constraints on their position as teachers.

Methods

This analysis is part of a larger study exploring 20 mathematics instructors' decision-making in IBL lower division courses via a survey and a follow-up interview. The decision-making situations section of the survey (the main data source for this paper) asked instructors to choose a lower-division course that they have taught with IBL methods and to describe five situations where they made a decision: about the content, about the course materials, about the assignments and assessments, about methods of teaching, and while teaching. For each situation, participants were asked to (1) list all options available to them when making the decision, (2) the desirable and undesirable outcomes of each option, (3) explain what they chose and why, and (4) identify the use and frequency of various teaching methods (e.g., lecturing, whole class discussion) and their learning objectives for their students. The survey also inquired about instructors' background and demographic information, their definition of IBL, reasons for using IBL, and personal gains and concerns about IBL. I followed up their survey's responses via a semi-structured hour-long interview.

Here, I use data from Leah—a full-time untenured lecturer at a PhD-granting public university, teaching pre-calculus for the first time, with 13 years of teaching experience—who described herself as knowledgeable of IBL and comfortable with implementing it. When asked to explain the situation where Leah made a decision about the mathematics content covered in the course, she wrote: "[W]hen we were studying trig[onometry] functions, I had to basically skip tan[gent] all together so that we could better cover cos[ine] and sin[e]. I showed one single example of a graph of the tan[gent] function." Leah listed four available options she could choose from: (1) cover Tangent graphs with some book example in class via short lecture, (2) skip Tangent graphs altogether, (3) ask students to read about Tangent graphs on their own, and (4) cover Tangent graphs in class like any other topic covered in class. In reflecting on the desirable and undesirable outcomes of each option (see Table 1), Leah chose option (1): "I gave a short lecture/example and didn't include it in any of their assessments ... I did this so that we could spend the extra time on graphs of sin[e] and cos[ine] instead. We needed more time on this."

Table 1. Leah's options and the desirable and undesirable outcomes of each option

Options	Desirable Outcomes	Undesirable Outcomes
(1) Cover Tan graphs via mini lecture	students see some new examples, allows more time on Cos and Sin	short lecture likely results in the students not learning
(2) Skip Tan graphs altogether	more time to spend on Cos and Sin graphs	students do not see examples of the Tan graph, problematic for calculus series
(3) Ask students to learn Tan graphs on their own	more time to spend on Cos and Sin graphs	have to follow up (only a couple students would read the book), students not learning
(4) Cover Tan graphs in class	reinforce ideas from the Cos and Sin graphs	No more time to spend on to better learn Cos and Sin graphs

Illustration of Analyses

Using Theory of Teaching in Context

To use the theory of teaching in context to explain why an instructor chose a specific option in a situation, I attended to each participant as an individual instructor by: (1) going through a process of "getting to know the instructor" before analyzing the decision-making situations, (2) analyzing all the decision-making situations for each instructor at a time, and (3) triangulating the findings with other available data. To get to know the instructor and triangulate the findings, I listened to the interview, read responses to other parts of the survey, and took notes when I found the information relevant to the situation. Analyzing the decision-making situations consisted of: (a) deciding whether the situation was familiar or unfamiliar; (b) identifying the dimensions that the instructor used to evaluate the options for unfamiliar situations; (c) identifying goals, orientations, and resources essential in the situation at hand; and (d) triangulating them with other data, describing how the option was chosen in the light of the findings from Step (a)-(c).

Getting to know the instructor. To teach her pre-calculus course for the first time, Leah heavily depended on her colleagues for advice and course materials. As her priority was to make the course inquiry-oriented, she adjusted the curriculum, the examination, and the grading structure for the course, which she referred to as good learning opportunities for herself. While Leah oriented herself towards promoting inquiry into mathematics for her students, she also oriented herself towards inquiry into student thinking about mathematics. This is also manifested in her beliefs about lecturing; she equates lecturing with "feeding the materials to the students" without them "really learning" the materials or her learning about student thinking. Therefore, she avoids lecturing unless absolutely necessary. Based on three of the five decision-making situations, it became apparent that Leah prefers covering content in class with students because she believes that students learn better in class with her support.

Step (a). The decision-making situation was captured as: whether and how to cover graph of Tangent functions. The situation was coded as unfamiliar because it is Leah's first time making this decision as it is her first time teaching pre-calculus.

Step (b). The desirable and undesirable outcomes were used to determine two dimensions on which she assessed each option: (D1) students learning Sine and Cosine graphs deeply, with Leah's support; and (D2) students learning Tangent graphs deeply, with Leah's support.

Step (c). I identified three goals for Leah: (G1) covering content efficiently in terms of time, (G2) assuring that students learn the concepts deeply, and (G3) preparing students for the next course. I identified both material and intellectual resources. The course textbook, as a material resource, is discussed when Leah talks about students learning about Tangent graphs by themselves using their textbook, or showing them some examples from the textbook via a mini lecture. In terms of intellectual resources, Leah had access to a script for teaching graphs of Tangents in the form of a short lecture. Moreover, her decision-making was navigated by her knowledge of her students' level of understanding of the graphs of Sine and Cosine functions and her knowledge of the future courses. In terms of orientations, Leah believes that spending more time on graphs of Sine and Cosine functions in class would allow students to engage with the content more "deeply" and learn the materials better. She believes that covering graphs of Tangent functions could be an opportunity for students to better understand graphs of Sine and Cosine functions. She finds graphs of Sine and Cosine functions as more fundamental and necessary to learn. While shying away from lecturing, Leah orients towards covering new content in class; this is evident from option (3)'s desirable outcomes, when she says that if she asks students to learn Tangent graphs at home, then she would still need to follow up with them to make sure that they have learned them.

Step (d). Because the decision-making situation was unfamiliar, Leah's decision-making is modeled by calculating subjective expected value of each option, where the option with highest value is chosen. Instead of assigning numeric values to each dimension for each option like Schoenfeld (2010) did, I attributed a satisfaction score (none, low, medium, high) based on the degree to which Leah described the options satisfying the dimensions (see Figure 1). Given that Leah values both dimensions and her goals, we can predict that she would not choose option (2) because it would not satisfy D2. Next, because Leah finds graphs of Sine and Cosine function fundamental, we can also predict than she values D1 more than D2, meaning that she would choose an option that would satisfy D1 more than D2. This eliminates option (4). So Leah is left with options (1) and (3). At this point, I reached a bypass where I could not explain which option Leah would choose based on her orientations, resources, and goals. Yet, knowing that Leah chose option (1) with both dimensions scored for medium satisfaction instead of option (3), where D1 has a high satisfaction score and D2 has a low satisfaction score, I wondered if the "Medium/Medium" combination subjectively sounded better to Leah, than the "High/Low" combination. Looking across her other decision-making situations, I found that Leah also chose "Medium/Medium" combinations over others. However, because only five decision-making situations were elicited from Leah, I cannot confirm that she mostly orients towards options that satisfy more dimensions moderately, rather than options that satisfy some dimensions more than others.

	D1: students learning Sine and Cosine graphs deeply, with Leah's support	D2: students learning Tangent graphs deeply, with Leah's support
(1) Cover Tangent via mini lecture	Medium	Medium
(2) Skip Tangent graphs altogether	High	None
(3) Ask students to learn on their own	High	Low
(4) Cover Tangent graphs in class	Low	High

Figure 1. Satisfaction score attributed to each option on the dimensions. The satisfaction scores are associated with colors: None = red; Low = light pink; Medium = light green; High = dark green.

Takeaways. The main challenge I encountered using this theory was differentiating between goals, orientations, and dimensions, as they overlap or inform one another. Schoenfeld (2010) neither defines dimensions, nor describes how he draws them, nor explains the relationship between these constructs. To carry out my analysis, I conceptualized: 1) goals as what teachers want to be achieved; 2) dimensions as the criterion that teachers think about when weighing different options; and 3) orientations as how teachers perceive everything associated with teaching and learning, including the goals and dimensions. Although the framework has been used to model in-the-moment decision-making, I expanded the framework to include outside-class situations. It seems that to harness the modeling power of the theory for both types of situations, multiple instances of similar decision-making situations are needed; this would allow the researcher to unpack automatic teacher decision-making in familiar situations and find more accurate satisfaction scores in unfamiliar situations.

Using Theory of Practical Rationality

Analyzing the decision-making situations with this framework consisted of three main steps. First, I identified the situation's normative behavior(s) and whether the normative behavior was breached by the instructor choosing the option they chose. Second, I determined whether the instructor's choice breached the norms. Third, I identified all instructor's *considerations* (why the instructor would or would not choose an option) when asked to provide the desirable and undesirable outcomes of their available options and explain their reasoning for choosing one of the options. Although the theory has been used to examine teachers' justifications for their *actions*, analyzing instructor's justifications for considering other options added nuance to the findings by showing that instructors may consider a more diverse set of professional obligations. Fourth, I coded considerations for types of resources that instructors used to justify breaching or not breaching the normative behaviors (norm, the four professional obligations) and open coded the ones that did not fit the norms and the four professional obligations. Some considerations were coded for multiple justification resources because they did not solely align with one.

Analyzing the situation. I identified two norms governing the situation: Leah was obliged to teach Tangent graphs because of her position as the instructor (obligation to institution) and she was supposed to teach new content in class so that students learn the materials deeply (instructional norm and an obligation to students). I identified four considerations from Leah's descriptions of her options' desirable and undesirable outcomes: students learning graphs of Sine and Cosine better (individual obligation); more time should be spent on the graphs of Sine and Cosine because they are fundamental materials (disciplinary obligation), covering content that prepares students for the next course (disciplinary and individual obligation), and saving class time (institutional obligation). Choosing option (1), giving a mini lecture, breached the second norm. This was justified by all four considerations. In other words, Leah justified not covering Tangent graphs in depth using professional obligations to the students, to the institution, and to the discipline of mathematics.

Takeaways. A major challenge in using Herbst and Chazan's (2012) theory was identifying norms in teaching lower division IBL mathematics courses, as these have not yet been documented at the level of detail as in K-12 settings. The framework assumes that teachers who teach the same content in similar context (e.g., high school geometry) make decisions under the same set of norms within similar didactical situations. I identified the norms underlying each situation based on the instructor's responses, which made it salient that instructors make decisions under different sets of norms than those found in K-12 settings, and that these norms may not be similar even across the same lower-division undergraduate course (Gerami, 2020).

Discussion

Much was revealed about Leah as an individual decision-maker because of Schoenfeld's (2010) framework's exclusive attention to her goals, orientations, and resources. The theory's attention to individual instructors can be useful in this context because it may help explain the diverse implementation of IBL by showing the variety of goals, orientations, and resources. I hypothesize that this theory can be used across instructors by positioning them in the same situation and investigating their goals, resources, orientations, and dimensions. Nonetheless, the way the theory binds these constructs is idiosyncratic to each teacher, making the theory in its whole inadequate for studying decision-making across IBL instructors.

Using Herbst and Chazan's (2012) theory allowed me to identify the actions that Leah found to be normative and the justifications she relied on for diverging from such norms. My preliminary findings show that these obligations look different across instructors, possibly because of institutional differences or instructors' diverse perceptions and implementation of IBL, and that IBL instructors may be obligated to novel stakeholders, such as the IBL community (Gerami, 2020). More research is needed to understand these norms and the sources of their differences.

Given the importance of cultural and social forces in university mathematics teaching (McDuffie & Graeber, 2003), Schoenfeld's (2010) theory will not paint a comprehensive picture of college mathematics teacher's decision-making. On the other hand, Herbst and Chazan's (2012) theory highlights such societal influences. Paying attention to both cognitive and sociocultural influences via complementary frameworks appears to be the best way forward; as Sfard (1998) puts it, "it seems that the most powerful research is the one that stands on more than one metaphorical leg" (p. 10).

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Foundational Ideas for Understanding the Constant Rate of Change

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This report presents a conceptual analysis that describes the productive ways of thinking of a hypothetical student in learning the idea of constant rate of change. The paper characterizes a hypothetical learning trajectory with a hierarchy of ideas that we conjecture to be foundational for understanding constant rate of change. The study briefly presents an instructional sequence of tasks guided by the hypothetical learning trajectory to promote learning the foundational ideas for understanding the constant rate of change.

Keywords: constant rate of change, conceptual analysis, hypothetical learning trajectory, quantitative reasoning, proportional reasoning

Introduction

Across the country, most Precalculus curriculum in college repeats the procedural approaches that students learn in high school, and therefore, students are less successful in passing Calculus in college (Sonnert & Sadler, 2014). Byerly (2016) reported Calculus students' and teachers' weak meanings of measurement, fraction, and constant rate of change. Students start building their meanings for constant rate of change and proportionality in their early mathematics learning. In Precalculus and Calculus, students get the opportunity to build meanings for the idea of constant rate of change in a sophisticated way. Thompson (2008) discusses that the constant rate of change is foundational to understanding linear functions and supporting ideas like an average rate of change, proportionality, and slope. Studies have reported the disconnections students and teachers have as they treat these ideas as different sets of actions and as associated with unrelated contexts (Lobato, 2006; Lobato & Siebart, 2002; Lobato & Thanheiser, 2002; & Coe, 2007). The coherent meanings for ratio, rate, and proportionality are foundational to the idea of constant rate of change.

My goal is to study an instructional sequence that is designed to support students in understanding and connecting ideas of a quantity, change in a quantity, variable, formula, ratio, rate, linear function, slope, an average rate of change, etc. that are related to learning and understanding the idea of a constant rate of change. Reasoning abilities that support students in making connections between the ideas related to a constant rate of change involve quantitative reasoning (Thompson, 1994), covariational reasoning (Carlson, Jacobs, Coe, Larsen, Hsu, 2002; Thompson & Carlson, 2017), and proportional reasoning (Lamon, 2007; Lesh, Post & Behr, 1988; Thompson, 1994). If we say two quantities have a constant rate of change, then we mean that two quantities are varying together; any change in one quantity's variation remains proportional to the corresponding change in another quantity's variation. Standard precalculus curriculum approach to a constant rate of change is often limited to solving linear equations and finding the slope of the "equation of the line." It rarely focuses on the meanings of quantities, ratio, rate and constant rate of change. Therefore, it is important to discuss a conceptual analysis that discusses the productive ways of thinking to understand the idea of constant rate of change. Furthermore, it is relevant to design a hypothetical learning trajectory (Simon, 1995; Simon & Tzur, 2004) that presents an order of learning goals for students with a series of tasks that promote the specific learning goals to understand the idea of a constant rate of change. The primary research question I want to address is- what understandings are foundational to understand the idea of constant rate of change?

Theoretical Background

Mathematics educators and researchers have investigated students' understanding of proportionality, rate, and ratio at various mathematical grade levels, with findings consistently revealing student difficulties in applying proportional reasoning when interpreting the rate of change in a modeling context (Tourniaire & Pulos, 1985; Doerr & O'Neil, 2011; Orton, 1983 & Yoon, Byerley & Thompson, 2015). Learning the idea of constant rate of change associates to the concept of rate, ratio, proportionality, changes in quantities, linear functions, and rate of change functions. According to Lamon (2007), early definitions of "ratio was considered a comparison between like quantities (e.g. pounds: pounds) and a rate a comparison of unlike quantities (e.g. distance: time)" (p.634), and over the years researchers have developed multiple definitions and distinctions of ratio and rate (Lesh, Post & Behr, 1988; Kaput & West, 1994). After working with students and teachers for a substantial amount of time, Thompson (1994), Thompson & Thompson (1994 & 1996) suggested that the distinction between rate and ratio depends on an individual's mental operations on how she comprehends the given rate and ratio context. Thompson (1994) defined ratio as a result of comparing two quantities multiplicatively. The two quantities that are compared multiplicatively, can be measured in the same unit or they can be measured in different units. In either case, if an individual conceives a mental operation of comparing two quantities multiplicatively, she conceives the result of the comparison as a ratio. Thompson (1994) defined rate as a reflectively abstracted constant ratio. When one conceives the idea of rate, she thinks about the rate as a characteristic of how quantities are covarying. A person reconceives a ratio as a rate when she applies the ratio to a different situation and think about that ratio as a rate that characterizes covariation between quantities. In another words, a ratio is a multiplicative comparison of the measures of two non-varying quantities and a rate is the proportional relationship between two varying quantities' measures (Thompson & Thompson, 1994).

Proportional reasoning as a theory interacts with the theory of quantitative reasoning (Thompson, 1988, 1990, 1993, 1994, 2011) as one conceives quantity as a measurable attribute of an object, then she conceives a multiplicative comparison of two fixed quantities. She conceives ratio as a result of the multiplicative comparison of the quantities. She conceives rate as a proportional relationship between the measure of two varying quantities. If the value of a quantity varies (or changes) in a situation, we say the quantity is a varying quantity, and if the value of a quantity does not vary in a situation, we say the quantity is fixed. When two quantities vary in relation to each other, the mental operations that support that dynamic images in students' thinking is referenced as covariational reasoning (Carlson et al., 2002; Thompson & Carlson, 2017). When a student engages in proportional reasoning, she simultaneously engages in covariational and quantitative reasoning. A student engages in proportional reasoning when she conceives the invariant relationship of quantities in dynamic situations or applies her understanding of proportionality in mathematical modeling situations. Constructing the idea of rate involves envisioning two quantities in a situation vary smoothly and continuously, and the changes (increases or decreases) in one quantity or variable's value is a simultaneous result of changes in another quantity or variable's value; and as the two quantities covary, the multiplicative comparisons of their measures remain proportional.

Conceptual Analysis

I used radical constructivism (Glaserfeld, 1995) as the theoretical perspective of this report. Radical constructivism claims that an individual constructs his own knowledge and it is not directly accessible to others. Glaserfeld (1995) also made a case that two individuals can reach a certain level of mutual agreement by providing compatible interpretations about each of their actions in the conversation. This idea of intersubjectivity seems to support the constructs conceptual analysis (Glaserfeld, 1995), hypothetical learning trajectory (Simon, 1995), that are relevant to investigate students' development of the foundational ideas in learning the idea of constant rate of change.

Thompson (2008) considered conceptual analysis as a tool to explain the useful ways of thinking for learning an idea. As one example of a conceptual analysis, Thompson (2008) described what is involved in understanding and learning the idea of constant rate of change and its role in understanding the idea of an average rate of change, linear functions, proportionality, and slope. In his conceptual analysis, he discussed the important role of quantitative, covariational, and proportional reasoning for learning the idea of a constant rate of change. Thus, if a student develops the deep meaning of constant rate of change, she should conceptualize the relationship between two quantities as continuous variation in their values, and the relationship remains proportional as two values of two quantities vary together.

I use the candle burning context (Carlson, Oehrtman, & Moore; 2016) to illustrate the productive ways of thinking of the idea of constant rate of change. This conceptual analysis describes what we have observed and hypothesize to be productive ways of thinking for learning the idea constant rate of change.

An 18-inch candle is lit and burns at a constant rate of 2.25 inches per hour. What is the length of the candle when it's been burning for 4.1 hours?

After reading the problem statement, it is necessary to realize that as the candle is burning since it is lit, the remaining length of the candle (in inches) is decreasing since the number of hours the candle is lit. The length of the candle (in inches) does not change if it is not lit. The remaining length of the candle (in inches) depends on the number of hours since the candle is lit. Therefore, the two quantities covarying together are the remaining length of the candle (in inches) and the number of hours since the candle is lit. Let *r* represents the remaining length of the candle (in inches) and let *t* represent the number of hours that have elapsed since the candle was lit. *r* can be written as an output of a function *f*, where $f(t)$ represents the remaining length of the candle since it was lit (in inches) in terms of the number of elapsed hours *t* since the candle was lit. We can say, $r = f(t)$ and where r is the output of the function f and f is a function of t. Here, r and t are variables that represent the varying values of the two covarying quantities- the remaining length of the candle since it was lit (in inches) and the number of hours since the candle was lit.

Representing the remaining length r of the candle since it was lit (in inches) in terms of the numbers of hours t since the candle was lit involves conceptualizing how quantities' values change together. The candle was originally 18 inches tall and burns at a constant rate of 2.25 inches per hour. To elaborate the problem statement, we can say for any change in the number of hours since the candle was lit (since $t = 0$) the change in the remaining length of the candle in inches is decreasing at a rate of 2.25 inches per hour. After *t* hours since the candle was lit the change in the remaining length would be *t* times as large as 2.25 inches less than 18 inches. There is a continuous variation in the number of hours since the candle is lit and a continuous variation in the remaining length of the candle since the candle was lit. The remaining length of the candle (in inches) is always the number of hours since the candle was lit times as large as 2.25 inches shorter than the original length of the candle. The expression $-2.25t + 18$ represents the remaining length r of the candle in inches and we can write a formula $r = -2.25t + 18$ in terms of t.

We can also write the formula using function notation, $f(t) = -2.25t + 18$, where t is the input of the function and independent variable that represents the number of hours since the candle was lit and $f(t)$ represents the functional relationship between r and t in terms of t, where r is the output of the function and dependent variable, and f is the function name. The function formula provides a description of how t and $f(t)$ vary together.

Moving forward, a student needs to conceptualize what is the length of the candle after it burns for 4.1 hours. Here, she needs to imagine the remaining length of the candle (in inches) when 4.1 hours have elapsed since the candle was lit. At this point, it is essential to conceptualize 4.1 hours of burning as a change of 4.1 hours since the candle was lit. In this context, the reference point is $(t_1, r_1) = (0,18)$ where r_1 represents the length of the candle before it was lit and t_1 represents the number of hours since the candle was lit. The reference point is a point in a context from which we imagine the variation in variable's value when the quantities are covarying together. As the change in the value of the remaining length of the candle since it is lit (in inches) remains constant with the change in the value of the number of hours since the candle was lit, we can represent the change in the value of the remaining length of the candle (in inches) since it is lit with Δr and the change in the value of number of hours since the candle was lit with Δt and $\Delta r = -2.25\Delta t$. The change in value of the remaining length of the candle since it is lit is decreasing with respect to the change in the value of the number of hours since the candle was lit. Therefore, the constant rate of change is -2.25 inches per hour. For any value of t, the change is away from 0 is also the value of t, $\Delta t = t - 0 = t$. Any value of t can be substituted into the function formula and is interpreted as a value of Δt changing away from $t = 0$. Similarly, the change in the value of the candle's remaining length (in inches) is a change in value of r, away from 18, $\Delta r = r - r_1 = r - 18$. We can write, $r - r_1 = -2.25(t - t_1) \rightarrow r - 18 = -2.25(t - 0) \rightarrow r = -2.25t + 18$ which is an equivalent form of the function formula $f(t) = -2.25t + 18$, where $r = f(t)$.

Therefore, if we want to know the length of the candle when it's been burning for 4.1 hours then $t = 4.1$ and $r = -2.25 * (4.1) + 18 = -9.225 + 18 = 8.775$, where the expression $-9.225 + 18$ represents the candle's length changes by -9.225 inches when the time elapsed by 4.1 hours and the candle is 8.775 inches long 4.1 hours after being lit.

The function $f(t) = -2.25t + 18$ represents a linear functional relationship between the remaining length of the candle since it is lit (in inches) and the number of hours since the candle was lit. As we vary t, the value of t is a change away from 0 and the change in length (in inches) is always -2.25 times as large as the change in elapsed time (in hours). The function represents a proportional relationship between the change in values of the remaining length of the candle since it is lit (in inches) with respect to the change in the number of hours elapsed since the candle was lit. The change in both quantities' values remain proportional for any amount of change in one quantities value with respect to another. When changes in one quantity's value are proportional to the corresponding changes in the other quantity's value there is a constant rate of change. Representing the relationship between both quantities values in a graphical context is always a straight line. Dynamic images of variation in the value of remaining length of the candle (in inches) since it is lit with respect to the variation in the value of the number of hours since it is lit is useful to track the proportionality in both quantities' values along with the linear representation of change in their values.

Hypothetical Learning Trajectory

I present a hierarchy of ideas and describe them as I conjecture these ideas to be foundational for understanding constant rate of change. Simon (1995) and Simon & Tzur (2004) discusses hypothetical learning trajectory (HLT) as an order of learning goals for students with a series of tasks that promote the specific learning goals to understand any idea. Based on Simon's idea of HLT we present a hierarchy of the ideas that are foundational to learning the idea of constant rate of change (Table 1). Table 1 elaborates the hypothetical order of introducing the ideas, the ideas or sub ideas that are foundational for understanding constant rate of change and the required student understanding for each of the section.

Hypothetical order of ideas	The foundational ideas for understanding constant rate of change	Desired student understanding
1.	Quantity: i. Varying or fixed quantity ii. Covariation iii. Changes in quantity's varying values	Given a problem statement, a student starts thinking about identifying fixed and varying quantities in the situation. She thinks about which quantities are varying independently or with respect to another quantity; or value of one quantity that does not change in the context. She also thinks about whether the varying quantities in the quantity changing together or not; and if two or more quantities are changing with respect to one another she will think about them as covarying quantities. Then she will think about how changes in one quantity's value is related to changes in another quantity's value.
2.	Representation: i. Variables ii. Expressions iii. Formula iv. Variables (delta notation) to express change in quantity's values v. Graphical representation	The student thinks about using variables to represent the value of varying quantities. She thinks about using variables, numbers or operators to express the value of a quantity as an expression. She conceives a formula as a tool for expressing the value of one quantity in terms of another, when the quantities are covarying. She thinks about change in a quantity's value as it is always away from a reference location. She represents the changes in quantity's values with delta notation. She thinks about presenting the relationship between quantities in a graphical representation. She will think about a cartesian coordinate system as a system where she can represent the values of the independent quantity in a horizontal axis, and the values of the dependent quantity in a vertical axis. She then thinks about a graph as a representation of how the covarying quantities are varying together.
3.	Times as large as comparison i. Ratio ii. Proportionality	She thinks about two numbers A and B, as A/B is a measure of A in units of B. She thinks if A/B as a ratio then A is A/B times as large as B (provided $B \neq 0$) and B is B/A times as large as A (provided A$\neq$ 0).

	iii. Rate	She thinks two quantities are proportional if the measure of one quantity's value in units of the other's value is a constant as the two quantities vary together. Two quantities are proportional if the value of one quantity is always the same number of times as large as the value of the other quantity. She perceives rate as an instantaneous image of a constant ratio, when one of the two covarying quantity's value is measured in units of another quantity's value.
4.	Constant rate of change	She thinks about constant rate of change as relating two co-varying quantities, when changes in one quantity's value are proportional to the corresponding changes in the other quantity's value. She thinks about using variables x and y to represent the values of two quantities that change together, then if the quantities are related by a constant rate of change then $\Delta y = m.\Delta x$ where Δy represents the changes in y values and Δx represents the changes in x values. The changes in y's values are m times as large as changes in x's values.
5.	Linear graph	She perceives a graph representing constant rate of change is always linear with a slope of the constant rate. Any change in one quantity's value is proportional to the corresponding change in the other quantity's value when the quantities are changing at a constant rate. She thinks proportionality is a restriction on how the quantities are varying together.
6.	Function i. Function notation ii. Independent and dependent quantity iii. Functional representation	She thinks function f defines a rule of a function that expresses how a function's independent quantity and dependent quantities are related as their values change together. If there are two covarying quantities x and y that possess a functional relationship and f is a function, then $y = f(x)$ and $f(x)$ is represented in terms of x. Here, she thinks x represents the independent quantity and y is the dependent quantity. f is a function name, values for x are input values and values for y for corresponding x values are output values of the function f.
7.	Constant rate of change function or linear function	She thinks a function that represents relationship between changes in the values of two covarying quantities and changes in one quantities value remain constant with corresponding changes to another quantities value, the function is a constant rate of change function or linear function.

Table 1. The hypothetical order of ideas that are foundational for understanding a constant rate of change

Implications and Limitations

One of the implications of the proposed HLT (Simon & Tzur, 2004) is to develop lessons and homework problems in a way where a student will build her foundational understandings of the idea of constant rate of change by reflecting upon her actions and the effects of those actions when working on the problems. As a methodological implication the HLT is a sophisticated tool to design sequenced tasks to conduct teaching experiments (Steffe & Thompson, 2000). The teaching experiment will focus on characterizing students' ways of thinking as they complete tasks designed to support their understanding of the foundational ideas in learning the idea of constant rate of change. The episodes of teaching experiments will inform the researchers to revise the HLT and make adjustments on the tasks to support new hypotheses as the results will unfold. This paper is limited to the theory of conceptual analysis and hypothetical learning trajectory in discussing the foundational ideas for understanding constant rate of change. I have designed a teaching experiment to investigate students' thinking of the foundational ideas for understanding the constant rate of change. The results of the teaching experiment will be addressed in the future. The conceptual analysis and the HLT will be useful to other researchers and teachers to design sequenced lessons or tasks to promote learning the idea of constant rate of change. The following figure 1. presents an example of an instructional sequence of tasks that informs the foundational ideas for understanding constant rate of change-

Sarah started a trip with 15 gallons of gas in her car's tank. Her car uses gasoline at the constant rate of 0.05 gallons per mile. Sarah also drove at a constant speed of 60 miles per hour until the car's tank was empty.

Q. What quantities are varying in this situation?
Q. How these quantities change together?

Q. What does it mean to say that Sarah's car uses gasoline at the constant rate of 0.05 gallons per mile?
Q. If Sarah's car uses gasoline at the constant rate of 0.05 gallons per mile, then how far Sarah can drive with one gallon of gasoline?
Q. How much gasoline (in gallons) is consumed when Sarah drove a distance of 5 miles?
Q. How much gasoline (in gallons) is consumed when Sarah drove a distance of 0.08 miles?
Q. Write a formula that represents the amount of gasoline g (in gallons) remaining in the car's tank in terms of the distance s (in miles) Sarah has driven since she has started the trip.

Q. What does it mean to say Sarah drove at a constant speed of 60 miles per hour until the car's tank was empty?
Q. Write a formula that represents Sarah's distance s (in miles) in terms of the amount of time t (in hours) since she has started the trip.
Q. How much gasoline (in gallons) was consumed when Sarah drove for one hour maintaining a constant speed of 60 mph?
Q. How much gasoline (in gallons) was consumed when Sarah drove for 1/3 of an hour maintaining a constant speed of 60 mph?
Q. Write a formula that represents the amount of gasoline g (in gallons) remaining in the car's tank in terms of the amount of time t (in hours) since she has started the trip.

Q. How does the amount of gasoline g (in gallons) remaining in the car's tank varies with respect to the amount of time t (in hours) since she has started the trip?
Q. How far (in miles) from Sarah's starting point she would drive until the car's tank is empty?
Q. How long (in hours) Sarah would drive until the car's tank is empty?

Q. Construct graphs – (a) that represents the amount of gasoline g (in gallons) remaining in the car's tank in terms of the distance s (in miles) Sarah has driven since she has started the trip.
(b) that represents Sarah's distance s (in miles) in terms of the amount of time t (in hours) since she has started the trip.
(c) that represents the amount of gasoline g (in gallons) remaining in the car's tank in terms of the amount of time t (in hours) since she has started the trip.

Figure1. A sequence of tasks for learning the foundational ideas of constant rate of change.

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Relative Size Reasoning: A Cross-Cutting Way of Thinking for Learning Precalculus Ideas

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The goal of this paper is to describe a cross-cutting way of thinking which may be productive for learning precalculus ideas. I describe the construct, "relative size reasoning" which consists of thinking about the relative size of two quantities as they change together. This way of thinking entails quantitative and covariational reasoning. I describe the mental operations involved in relative size reasoning and how it can be used by providing three illustrations of relative size reasoning in three precalculus ideas and conclude with future research.

Keywords: relative size reasoning, relative size, way of thinking, quantitative reasoning

Introduction

The ideas of measurement and measurement comparisons (such as fractions or ratios) are mathematical concepts introduced to students in elementary school. However, students in university mathematics courses struggle with comparing two *quantities* in terms of their *relative size (e.g Byerley, 2019; Tallman, 2015)*. Thinking about the *relative size* of two quantities entails envisioning the measure of one quantity in units of the other, provided the two quantities are of the same unit of measurement (the magnitude of one measured in units of the magnitude of the other). If they are not of the same unit of measurement, it involves thinking of the measure of one quantity in multiples of the measure of the other quantity. For this paper, I introduce the idea of relative size reasoning to characterize the dynamic imagery involved in thinking about the relative size of two quantities as the two quantities' values change in tandem (Thompson, 1994; Carlson et al., 2002).

Traditional or conventional mathematics textbook designers create sections and chapters that are disconnected and developed without regard to NCTM standards for mathematics teaching and learning (Cady et al., 2015). Teachers rely heavily on these disconnected textbooks and curriculum to structure their classroom instruction (Valverde, Bianchi, Wolfe, Schmidt, & Houng, 2002). As a result, teachers are not equipped to support students in reasoning about these connections in the classroom (Thompson & Saldanha, 2003). Thus, students are not supported in developing the cross-cutting "way of thinking" that is productive for understanding foundational ideas (e.g., rate of change). For example, students who do not view a quotient as representing the relative size of the ratio's numerator and denominator will not be equipped to understand the idea of constant rate of change.

The foundational reasoning necessary for thinking about constant rate of change includes *quantitative reasoning* (Thompson, 1990, 2011), reasoning multiplicatively (Thompson 1994b; Thompson and Thompson 1994, 1996; Thompson & Carlson 2017), coordinating two covarying quantities and their changes (*covariational reasoning*) (Thompson & Saldanha, 1998; Carlson et al., 2002; Thompson & Carlson 2017), and thinking about the proportional correspondence between the changes in the two quantities. Relative size reasoning is useful for thinking about constant rate of change because it provides students with an opportunity to approach problems with a way of thinking that increases their ability to make meaning for the mathematical structures they create, and determine the appropriate operations needed to evaluate some quantity in the problem, which entails *quantitative* and *covariational reasoning*. Lobato and colleagues have reported that many students do not view the idea of proportionality, slope, rate of change, and average rate of change as being connected (Lobato, 2006; Lobato & Siebert, 2002; Lobato & Thanheiser, 2002). In fact, it has been documented that university students have difficulty modeling function relationships involving rates of change (Carlson, 1998; Thompson 1994a). Specifically, understanding ideas such as constant rate of change is central to understanding ideas in calculus such as (a) the difference quotient (Byerley & Thompson, 2014), (b) the concept of the derivative (Byerley et al, 2012; Zandieh, 2000) and (c) the fundamental theorem of calculus (Thompson 1994). These findings suggest that students may need to be supported in using relative size reasoning. Comparing two quantities multiplicatively has been documented to be important for understanding ideas in specific mathematical topics such as measurement (Thompson et al., 2014), constant rate of change (Thompson, 1994a, 1994b, 1995; Thompson & Saldanha, 1998; Carlson et al. 2002), rational functions (Pampel, 2017), trigonometry (Moore, 2014, 2010; Tallman & Frank, 2018; Thompson, 2008), exponential functions (O'Bryan, 2018), and difference quotient (Byerley, 2019). More research is needed to understand the nature of relative size reasoning, how it develops in students, and how it contributes to learning other ideas in mathematics. Since *relative size reasoning* has not been an explicit focus of research, this study investigated the question:

What are the mental operations involved in relative size reasoning and how is relative size reasoning used?

Review of the Literature

I initially discuss how relative size reasoning is used when learning and understanding precalculus level ideas. I follow by providing an overview of prior research that discussed the role of relative size reasoning when learning topics in elementary school through calculus.

Relative Size Reasoning in Rational Functions Literature

When one has reasoned about rational functions productively, she can "coordinate the covarying relationship of the polynomial in the numerator (the input and output quantity of the polynomial) and the polynomial in the denominator (the input and output quantity of the polynomial)" (Pampel, 2017). This relationship represents the relative size of the value of the numerator in terms of the value of the denominator, which also varies with the input quantity. Using relative size reasoning to think about rational functions leads to reasoning quantitatively about characteristics of rational functions like end behavior, asymptotes, and holes rather than relying on algebraic manipulations (Pampel, 2017).

Relative Size Reasoning in Trigonometry Literature

Thompson (2008) argued that students' difficulties in understanding trigonometry lie in the essential meanings they construct from grades 5 through 10, such as their meaning for an angle. Students are frequently introduced to the idea of angle measure as representing a location of an angle's rays (e.g., a right angle is 90 degrees, and a straight angle is 180 degrees). It is also common for students to have weak or incoherent meanings for what it means to measure an angle's openness in degrees or radians. Measuring an angle's openness in degrees consists of thinking about the angle's openness in units of $1/360^{th}$ of any circle's circumference. When measuring an angle in radians, we compare the size of the arc length relative to the circle's radius length (Moore, 2014, 2010; Tallman & Frank, 2018; Tallman 2015; Thompson, 2008). In other words, we are measuring the subtended arc of any circle cut off by the angle's rays in units of its respective radius length.

Relative Size Reasoning in Exponential functions Literature

O'Bryan (2018) claimed that understanding the idea of exponential functions relies on comparing two quantities multiplicatively. When one has reasoned about exponential functions coherently, she can "conceptualize an exponential function as a function with a constant growth factor or constant percent change over all equal sized intervals of the domain" (O'Bryan, 2018). In other words, students with a more developed understanding of exponential functions recognize that the growth factor for an n-unit change in the independent variable is a comparison of the relative size of two function values over that n-unit change.

Relative Size Reasoning in Division and Fractions Literature

Thompson and Saldanha (2003) characterized a mature understanding of fractions as involving multiplicative reasoning and strong meanings for measurement, multiplication, division and quotient. When students understand the idea of division (which entails measuring and partitioning) they understand that "any measure of a quantity induces a partition of it and that any partition of a quantity induces a measure of it" (Thompson & Saldanha, 2003). When fractions and division are thought of multiplicatively, "A is m/n times as large as B" means that A is m times as large as $1/n$ of B (Thompson & Saldanha, 2003; Byerley & Thompson, 2012; Thompson et al., 2014). Thompson and Saldanha (2003) emphasize a direct link to reciprocal relationships of relative size to conceptualize fractions. This is similar to a person who is thinking with a *Steffe Magnitude* which Thompson, Carlson, Byerley, & Hatfield (2014) defined as one who is reasoning about measurement of a quantity's size relative to the size of its unit and the reciprocal relationship between them. Thompson et al. (2014) described a conceptual analysis for developing a mature magnitude scheme that involves ideas such as multiplication, division, proportional correspondence, measure, fraction, and quotient as a measure of the relative size of two quantities. Conceptualizing division as relative size enables students' conceptualization of divisors that are not-integer values (Byerley, Hatfield, and Thompson, 2012). Understanding reciprocal relationships of relative size entails conceptualizing the connectedness between ideas of measure, multiplication, fractions, and division. Thompson and Carlson (2017) and Byerley, Hatfield, and Thompson (2012) also argued that to understand the concept of rate of change, it involves an understanding of quotient, ratio, accumulation, and proportionality, which consists of thinking about the comparison of two quantities.

Theoretical Perspective

The theoretical perspective used for this study is radical constructivism (Glasersfeld, 1995) which proposes that every individual has their own reality, knowledge is constructed based on previous personal experiences, and knowledge lies in the mind of the individual. It is important to note that knowledge is not transferable from one person to another. Only through models can we portray hypothetical models of students' *schemes* and *meanings*. A scheme is an "organization of actions, operations, images, or schemes which can have many entry points that trigger action – and anticipations of outcomes of the organization's activity" (Thompson et al., 2014). When I say meaning I am referring to the "ideas or ways of thinking that someone intends to convey to someone else and uses signs or symbols to do so" (Thompson, 2013). Meanings that a person has exist in the mind of that individual and the person that is interpreting them. Therefore, my intention is not to classify how all students engage in relative size reasoning, but rather provide a model of an epistemic student's thinking (Steffe & Thompson, 2000).

Relative size reasoning is a *way of thinking* that is needed for conceptualizing a wide range of mathematical ideas, while engaging in relative size reasoning relies on one having mature understandings of measurement, fractions, multiplication, division, ratio, proportionality, and magnitude. Relative size reasoning includes comparing the size of two quantities by envisioning the measure of one quantity in units of the other, provided the two quantities are of the same unit of measurement (the magnitude of one measured in units of the magnitude of the other). If they are not of the same unit of measurement, it involves thinking of the measure of one quantity in multiples of the measure of the other quantity. When I refer to a *way of thinking*, I am referring to a pattern that a person developed for utilizing specific meanings for reasoning about specific ideas (Thompson et al., 2014). Constructing productive meanings for these ideas involves *quantitative reasoning* (Thompson, 2011) and *covariational reasoning* (Carlson et al., 2002; Thompson and Carlson, 2017).

To engage in relative size reasoning, one must first engage in *quantitative reasoning* (Thompson, 1990, 1993, 1994, 2011) which begins with conceiving of a quantity. A *quantity* is a "quality of something that one has conceived as admitting some measurement process" (Thompson, 1990). *Quantification* is the process of assigning numerical values to quantities (Thompson, 1990, 2011). Thompson (1990, 1994b, 2011) described *quantitative reasoning* as "the analysis of a situation into a quantitative structure- a network of quantities and quantitative relationships." *Quantitative relationships* are 3 quantities where two quantities and a quantitative operation are used to make a third (Thompson, 1990). For instance, constant speed is a quantity that determines how many times as large the change in distance traveled is, in comparison to the corresponding change in time. In order to make this comparison, the student must also conceptualize the change in distance and change in time as quantities. Then the student must think about the changes in the quantities as the quantities vary in tandem which entails *Covariational Reasoning*, "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other." (Carlson et al, 2002). Thompson and Carlson (2017) argued that covariational reasoning entails developing the ability to imagine quantities' values varying smoothly and continuously.

Three Vignettes of Relative Size Reasoning

I propose that relative size reasoning is a cross-cutting *way of thinking* that is used when learning and using numerous ideas in mathematics. I provide three vignettes grounded in research previously presented above to illustrate a student's mental operations when engaging in relative size reasoning and why it is propitious for conceptualizing these situations.

Vignette 1

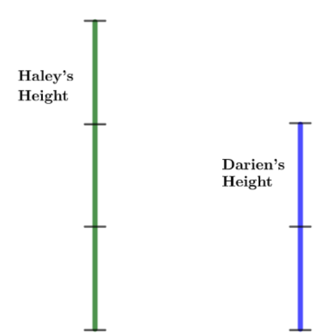

Figure 1. An image to represent the comparison of Haley and Darien's Heights

Suppose a student is first asked to compare Haley's height to Darien's height and then Darien's height to Haley's height (see Figure 1). The student must conceptualize the quantities in the situation: Darien and Haley's heights. When the student compares Haley's height to Darien's height, she must imagine measuring the magnitude (length) of Haley's height relative to the magnitude (length) of Darien's height. So the student must conceptualize Darien's height as the unit of measure. Therefore, using her measurement scheme, she imagines Darien's height as a measuring stick to determine Haley's height in units of Darien's height. Then using her scheme for fraction and division, she may think about how many times as tall Haley's height is relative to Darien's height. A student who exhibits these mental actions likely understands and can express that Haley is $\frac{3}{2}$ times as tall as Darien.

Then when comparing Darien's height to Haley's height, the student must imagine measuring the magnitude (length) of Darien's height in units of the magnitude (length) of Haley's height. Now she must reconceptualize the situation where Haley's height is the measuring stick. Then the student may think about the multiplicative comparison as Darien is $\frac{2}{3}$ times as tall as Haley. Engaging in relative size reasoning to compare Haley and Darien's heights entails thinking of each measured height as a measurement unit or measuring stick used to measure the other person's height.

An example of a student who reasons in a way other than using relative size reasoning to think about comparing Haley and Darien's heights may think about the fraction $\frac{2}{3}$ or $\frac{3}{2}$ as "2 over 3" or as a rule that tells you to divide 2 by 3. This may result in the student seeing a fraction as a command to divide, and less likely to conceptualize the result of division as determining the relative size of one quantity in terms of another.

Vignette 2

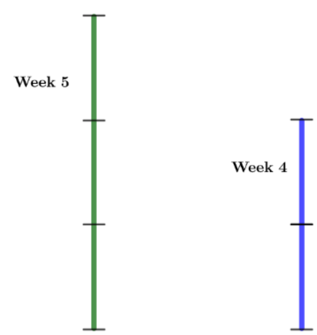

Figure 2. An image of the sunflower's height from week 4 to week 5. Task adapted from CCR exam (Carlson, Madison, West, 2015).

Suppose we consider a task adapted from a CCR exam (Carlson, Madison, & West (2015). A sunflower is planted and grows by 50% each week as pictured in Figure 2. Given the height of the sunflower after 4 weeks, a student must find how tall the plant will be in week 5. Engaging in relative size reasoning while responding to this problem entails imagining the quantities in the problem context: the time in weeks since the sunflower was planted and the height of the plant. The student must visualize both the time since the sunflower was planted and the height of the sunflower as measurable attributes of the plant and consider the height of the sunflower while imagining the amount of time elapsed since the sunflower was planted.

When thinking about this situation the student must have a meaning for percent which entails understanding that the size of Quantity A is n% of the size of Quantity B means that the size of Quantity A is n one one-hundredths of the size of Quantity B (O'Bryan, 2018). The student’s percent scheme includes the idea that the quotient $\frac{A}{B} = n \cdot 0.01$; the student interprets these symbols as expressing that the size of Quantity A is 0.01 times as large as Quantity B. Therefore, she sees that Quantity B is the measuring stick used to determine the size of Quantity A. In addition, the student must have a meaning for percent change that entails describing the change in the value of a quantity as a percentage of the original value.

Specifically, for this question, a student must imagine using the height of the plant in week 4 as the measuring unit. Given that the plant grows by 50% each week, one must conceptualize the sunflower growing by 50% and seeing an increase of 50% as the percent change as each week passes. So, when comparing the height of the sunflower in week 5 to week 4, she would see that the height of the sunflower in week 5 is 150% of the sunflower’s height in week 4. The student would see the multiplicative comparison of the height of the sunflower in week 5 as 1.5 $or\ \left(\frac{3}{2}\right)$ times as tall as it was in week 4. Determining the reciprocal relationship of the height of the sunflower in week 4 relative to week 5 requires conceptualizing the sunflower’s height in week 5 as the measuring unit. Understanding the reciprocal relationship in terms of percentage entails the idea that the height of the sunflower in week 5 is the reference quantity and the height in week 4 is about 66.67% of the plant’s height in week 5. Thus, the sunflower’s height in week 4 is $\frac{2}{3}$ times as tall as it is in week 5. Engaging in relative size reasoning to compare the heights of the sunflower for weeks 4 and 5 entails thinking of the sunflower’s height in a certain week as a measurement unit used to measure the sunflower’s height for another particular week.

Vignette 3

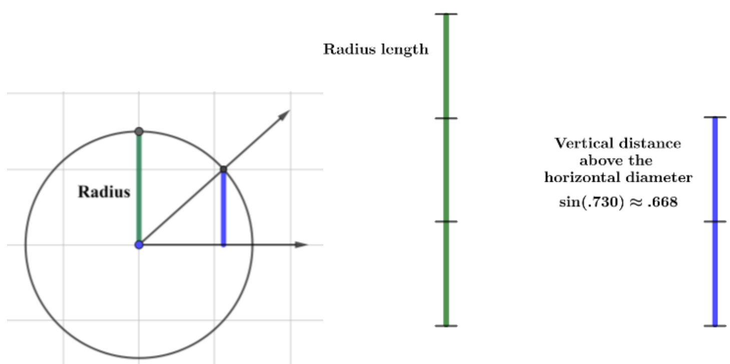

Figure 3. Left: An image of a circle with an angle centered at the circle’s center with a terminal point at $sin(0.73)$*. Right: An image of the height of the terminal point’s vertical distance above the circle’s center relative to the radius of the circle.*

Another example of using relative size reasoning to make sense of a situation would be to describe the meaning of the output value of the sine function given an angle measure of 0.73 radians as pictured in Figure 3. Using relative size reasoning to think about the sine function entails imagining that, when given an angle whose initial ray is in the 3 o’clock position and vertex is at the center of a circle, the sine function will take the angle's radian measure as an input and output the terminal point’s vertical distance above the circle's center measured in radius lengths. The student would have to conceptualize the radius length as the measuring stick. With this meaning the student may reason about the terminal point’s vertical distance above the circle’s center in radius lengths given an angle with a measure of 0.73 radians using the sine function: $sin(0.73) \approx 0.668 \approx \frac{2}{3}$. Thus, she must conceptualize the relative size of the terminal point's vertical distance above the circle's center measured in units of the radius lengths. The student must conceptualize that the terminal point's vertical distance above the circle's center measured in radius lengths is $\frac{2}{3}$ times as long as the radius length. Understanding the reciprocal relationship entails using the terminal point's vertical distance above the circle's center as the unit of measure for determining the relative size of the radius in comparison to the terminal point's vertical distance above the circle's center. This conceptualization includes seeing that the radius length is $\frac{3}{2}$ times as long as the terminal point's vertical distance above the circle's center.

An example of a student who views the meaning of the sine function in ways other than relative size reasoning may include thinking about "opposite over hypotenuse" or "SOH" in the acronym SOH-CAH-TOA. Students who learn trigonometric functions as SOH-CAH-TOA will be at a disadvantage when it comes to needing to think about the arguments of these functions and what they represent. Especially since recognizing that angle measure in radians represents this proportional relationship between the subtended arc of any circle cut off by the angle's rays and the corresponding circle's radius requires relative size reasoning. Engaging in relative size reasoning to compare the terminal point's vertical distance above the circle's center to the circle's radius entails thinking of each measured length (either the radius or vertical distance) as a measurement unit or measuring stick used to measure the other length.

Discussion, Limitations & Future Research

Since conjecturing *relative size reasoning* is foundational for understanding many ideas in mathematics, it is necessary to emphasize this as an important way of thinking for students to acquire. When engaging in relative size reasoning, thinking about comparing any two quantities (fixed or varying) in the previous situations, one must reason about the relative size of one quantity in units of the other and see the reciprocal relationship as well. Students who think about fractions and division as a command to divide may be limited to seeing the meaning of the result of division and the connections between their ways of thinking while approaching problems in different topics. Students who think about trigonometric functions as an acronym may be limited when it comes to thinking about measuring an angle in radians. Empirical studies are needed to further refine the conceptual analysis presented, to further investigate how students spontaneously use (or don't use) relative size reasoning, and to explore the role of relative size reasoning in learning and using other ideas in precalculus and calculus.

I hypothesize that supporting student's in engaging in relative size reasoning will strengthen their ability to characterize the covariation in two quantities as their values change together and understand key concepts (constant rate of change, exponential growth, derivative) in mathematics.

Future Research Questions:

1. What meanings do students display or exhibit about the concept of relative size across multiple precalculus topics: constant rate of change, proportionality, exponential functions, rational functions, and trigonometry?
2. What are ways that a mathematics curriculum and a teacher can support students in developing relative size reasoning?

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