Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education

Editors:
Tim Fukawa-Connelly
Nicole Engelke Infante
Karen Keene
Michelle Zandieh

Pittsburgh, Pennsylvania
February 19-21, 2015

Presented by
The Special Interest Group of the Mathematics Association of America (SIGMAA) for Research in Undergraduate Mathematics Education

CITATION: In (Eds.) T. Fukawa-Connelly, N. Infante, K. Keene, and M. Zandieh, Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education, Pittsburgh, Pennsylvania.

Foreword

As part of its on-going activities to foster research in undergraduate mathematics education and the dissemination of such research, the Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education (SIGMAA on RUME) held its eighteenth annual Conference on Research in Undergraduate Mathematics Education in Pittsburgh, Pennsylvania from February 19 - 21, 2015. The conference is a forum for researchers in collegiate mathematics education to share results of research addressing issues pertinent to the learning and teaching of undergraduate mathematics. The conference is organized around the following themes: results of current research, contemporary theoretical perspectives and research paradigms, and innovative methodologies and analytic approaches as they pertain to the study of undergraduate mathematics education. The program included plenary addresses by Dr. Charles Henderson, Dr. Nicole McNeil, and Dr. Matthew Inglis and the presentation of over 160 contributed, preliminary, and theoretical research reports and posters.

The Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education are our record of the presentations given and it is our hope that they will serve both as a resource for future research, as our field continues to expand in its areas of interest, methodological approaches, theoretical frameworks, and analytical paradigms, and as a resource for faculty in mathematics departments, who wish to use research to inform mathematics instruction in the university classroom. RUME Conference Papers, includes conference papers that underwent a rigorous review by two or more reviewers. These papers represent current work in the field of undergraduate mathematics education and are elaborations of selected RUME Conference Reports.

The proceedings begin with the winner of the best paper award and the papers receiving honorable mention. These awards are bestowed upon papers that make a substantial contribution to the field in terms of raising new questions or providing significant or unique insights into existing research programs. RUME Conference Reports, includes the Poster Abstracts and the Contributed, Preliminary and Theoretical Research Reports that were presented at the conference and that underwent a rigorous review by at least three reviewers prior to the conference. Contributed Research Reports discuss completed research studies on undergraduate mathematics education and address findings from these studies, contemporary theoretical perspectives, and research paradigms. Preliminary Research Reports discuss ongoing and exploratory research studies of undergraduate mathematics education. Theoretical Research Reports describe new theoretical perspectives and frameworks for research on undergraduate mathematics education. Poster Reports were quite varied and described curriculum, research and theoretical contributions.

The conference was hosted by West Virginia Univerisity. Their faculty and student provided many hours of volunteer work that made the conference possible and pleasurable, we greatly thank the faculty, students and institution for their support.

Last but not least, we wish to acknowledge the conference program committee and reviewers for their substantial contributions to RUME and our institutions, for their support.

Sincerely,
Tim Fukawa-Connelly, RUME Conference Chairperson
Nicole Engelke Infante, RUME Conference Local Organizer
Karen Keene, RUME Program Chair
Michelle Zandieh, RUME Coordinator

Program Committee

Chair	
Tim Fukawa-Connelly	Drexel University
Local Organizer	
Nicole Engelke Infante	West Virginia University
Working Group Coordinator	
Jason Belnap	University of Wisconsin Oshkosh
Stacy Brown	California State Polytechnic University, Pomona
Kyeong Hah Roh	Arizona State University
Estrella Johnson	Virginia Tech
Karen Allen Keene	North Carolina State University, Raliegh
Elise Lockwood	Oregon State University
Ami Mamolo	University of Ontario Institute of Technology
Jason Martin	University of Central Arkansas
Michael Oehrtman	Oklahoma State University
Vicki Sealey	West Virginia University
Craig Swinyard	University of Portland
Allison Toney	University of North Carolina–Wilmington
Megan Wawro	Virginia Tech
Keith Weber	Rutgers University
Aaron Weinberg	Ithaca College

Additional Reviewers

Adiredja, Aditya
Alcock, Lara
Bagley, Spencer
Bleiler, Sarah
Boyce, Steven
Brown, Stacy
Bubp, Kelly
Byrne, Martha
Cadwalladerolsker, Todd
Capaldi, Mindy
Cawley, Anne
Champney, Danielle
Conner, Annamarie
Czocher, Jennifer
Davis, Owen
Dawkins, Paul Christian
Demeke, Eyob
Dibbs, Rebecca
Dominy, Morgan
Dorko, Allison
Ellis, Jessica
Eubanks-Turner, Christina
Firouzian, Shawn
Fisher, Brian
Fukawa-Connelly, Tim
Galle, Gillian
Glover, Erin
Goss, Melissa
Grundmeier, Todd
Hanusch, Sarah
Hauk, Shandy
Inglis, Matthew
Johnson, Estrella
Jones, Steven
Keene, Karen
Laforest, Kevin
Lai, Yvonne
Lande, Elaine
Larsen, Sean
Larson, Christine
Laursen, Sandra
Leckrone, Linda
Lew, Kristen
Lockwood, Elise
Mamolo, Ami
Martin, Jason
Matthews, Asia
Mejia-Ramos, Juan Pablo
Melhuish, Kathleen
Miller, David
Mills, Melissa
Moore, Kevin
Moore, Robert
Nepal, Kedar
Noll, Jennifer
Norton, Anderson
Oehrtman, Michael
Paoletti, Teo
Plaxco, David
Rasmussen, Chris
Roh, Kyeong Hah
Savic, Milos
Sealey, Vicki
Selden, Annie
Selden, John
Speer, Natasha
Stewart, Sepideh
Strand, Steve
Suominen, Ashley
Swinyard, Craig
Tallman, Michael
Thanheiser, Eva
Thomas, Kevin
Thompson, Pat
Toney, Allison
Tyne, Jen
Wagner, Joseph
Wasserman, Nicholas
Wawro, Megan
Weber, Eric
Weber, Keith
Weinberg, Aaron
Wescoatt, Benjamin
Zandieh, Michelle
Zazkis, Dov
Zazkis, Rina

Participant list

Reda Abu-Elwan
Aditya Adiredja
Solomon Adu
Valeria Aguirre Holgun
Husain AlAttas
Lara Alcock
Lily An
Christine Andrews-Larson
Naneh Apkarian
Spencer Bagley
Samantha Bailey
Donna Bassett
Nermin Bayazit
Mary Beisiegel
Jason Belnap
Ashley Berger
Annie Bergman
Sonalee Bhattacharyya
Sarah Bleiler
Tim Boester
Heather Bolles
William Bond
Linda Braddy
Kenneth Bradfield
Suzanne Brahmia
Carly Briggs
Stacy Brown
Kelly Bubp
Cameron Byerley
Martha Byrne
Mindy Capaldi
Marilyn Carlson
Drew Carpenter
Joshua Case
May Chaar
Danielle Champney
Daniel Cheshire
Morgan Chittum
Warren Christensen
Emily Cilli-Turner
Renee Clark
Nathan Clements
AnnaMarie Conner
Jeff Connor
John Paul Cook
Paul Dawkins
Eyob Demeke
Jessica Deshler
Jacqueline Dewar
Morgan Dominy
Allison Dorko
Sarah Dufour
Ashley Duncan
Irene Duranczyk
Antony Edwards
Houssein El Turkey
Amy Ellis
Jessica Ellis
Iwan Elstak
Nicole Engelke Infante
Sarah Enoch
James Epperson
Sarah Erickson
Jessica Brooke Ernest
Shahram Firouzan
Brian Fisher
Nicholas Fortune
Tim Fukawa-Connelly
Edgar Fuller
Khaled Furati
Gillian Galle
Carla Gerberry
sayonita ghosh hajra
Jennifer Glaspell
Jim Gleason
Erin Glover
Duane Graysay
Todd Grundmeier
Beste Gucler
Michael Gundlach
Caroline Hagen
William Hall
Sarah Hanusch
Nadia Hardy
Neil Hatfield
Shandy Hauk
Charles Henderson
Natalie Hobson
Charles Hohensee
Matthew Inglis
Billy Jackson
Gaya Jayakody
Erica Johnson
Elgin Johnston
Durrell Jones
Steven Jones
Surani Joshua
Shiv Karunakaran
Valerie Kasper
David Kato
Brian Katz
Jennifer Kearns
Karen Keene
Rachael Kenney
Sarah Kerrigan
Jalynn Keyser
Dana Kirin
Mile Krajcevski
Andrew Krause
Dave Kung
George Kuster
Kevin LaForest
Yvonne Lai
Sean Larsen
Renee LaRue
Maurice LeBlanc
Linda Leckrone
Younhee Lee
Kristen Lew
Ekaterina Lioutikova
Elise Lockwood
Guadalupe Lozano
Alison Lynch
Martha Makowski
Ami Mamolo
Jason Martin
Sarah Mathieu-Soucy
John Mayer
Nicole McNeil
Juan Pablo Mejia Ramos
Kathleen Melhuish
Yuliya Melnikova
Ashley Menillo
Hayley Miles-Leighton
David Miller
Melissa Mills
Thembinkosi Mkhatshwa
Kevin Moore
Robert C. Moore
Thomas Morgan
Kristy Murphy

Participant List cont.

Stacy Musgrave
Kedar Nepal
Xuan Hien Nguyen
Kristin Noblet
Jennifer Noll
Anderson Norton
Michael Oehrtman
Lori Ogden
Jeffrey Pair
Teo Paoletti
Hyejin Park
Spencer Payton
Matthew Petersen
Valerie Peterson
Tuyet Pham
Nathan Phillips
Amie Pierone
Alon Pinto
David Plaxco
Jeffrey Rabin
Chris Rasmussen
Beverly Reed
Zackery Reed
Paul Regier
Dylan Retsek
Bernard Ricca
Lisa Rice
Douglas Riley
Kitty Roach
Kimberly Rogers
Kyeong Hah Roh
David Roundy
Megan Ryals
Silvia Saccon
Jason Samuels
Luciane Santos
Milos Savic
Vicki Sealey
Annie Selden
John Selden
Soo Yeon Shin
Eric Simring
Ivanete Siple
Ann Sitomer
Kelli Slaten
John Smith
Osvaldo Soto
Natasha Speer
Doug Squire
Maria Stadnik
Harrison Stalvey
Christina Starkey
Irma Stevens
Sepideh Stewart
Steve Strand
April Strom
Heather Summers
Ashley Suominen
Judy Sutor
Julie Sutton
Rebecca Swanson
Craig Swinyard
Michael Tallman
Gabriel Tarr
Eva Thanheiser
Matt Thomas
Clarissa Thompson
John Thompson
Patrick Thompson
Anna Titova
Allison Toney
Andrew Tonge
Krista Toth
Heather Trahan-Martins
Melissa Troudt
Jeffrey Truman
Samuel Tunstall
Jennifer Tyne
Matthew Villanueva
Brittany Vincent
Joseph Wagner
Nathan Wakefield
Rebecca Walker
Nick Wasserman
Kevin Watson
Megan Wawro
Eric Weber
Keith Weber
Kelsea Weber
Matthew Weber
Ben Wescoatt
Diana White
Derek Williams
Erik Winarski
Ming Wu
Tetsuya Yamamoto
Nissa Yestness
Hyunkyoung Yoon
Siobahn Young
Balarabe Yushau
Michelle Zandieh
Matthew Zawodniak
Dov Zazkis
Rina Zazkis

Table of Contents

Conference Papers

Working Group Reports

Conference Reports

Guiding Reinvention of Conventional Tools of Mathematical Logic: Students' Reasoning About Mathematical Disjunctions

Paul Christian Dawkins
Northern Illinois University

John Paul Cook
University of Science and Arts of Oklahoma

Motivated by the observation that formal logic answers questions students have not yet asked, we conducted an exploratory teaching experiment with undergraduate students intended to guide their reinvention of truth-functional definitions for basic logical connectives. We intend to bridge the gap between reasoning and logic by inviting students to ask and answer questions that motivate logic as an objective science. We present categories of student strategies for assessing truth-values for mathematical disjunctions. Students' reasoning heavily reflected content-specific and pragmatic factors in ways inconsistent with the norms and conventions of mathematical, formalized logic. Despite this, all student groups reinvented the standard truth-functional definition for non-quantified disjunctions once they began reasoning about logic *by attending to logical connectives and by comparing their interpretations across various disjunctions. Students struggled to develop generalizable tools for assessing quantified disjunctions because they explored sets of examples in context-dependent ways.*

Key words: truth-functional logic, guided reinvention, disjunctions, reasoning about logic, quantification

In Toulmin's (1958) critique of formal, mathematical logic as a model for everyday argumentation, he points out that philosophers use at least four implicit models for the meaning of *logic*. The main conflict arises because logic ostensibly relates to ideas or language suggesting some connection to 1) psychology or 2) sociology (a *descriptive* function). In contrast, scholars view logic as an objective field not beholden to how people actually reason. So, other implicit models treat logic as 3) a technology for argumentation or 4) an objective science within mathematics (a *prescriptive* function). Despite this ambiguity, psychologists persistently investigate "deductive reasoning" as judged against formal logical structures (Stenning, 2002), which conflates the prescriptive and descriptive models. Psychologists do so in accordance with the classical philosophical view that "logical laws constitute the very fabric of thought" (Stenning & van Lambalgen, 2008, p. 9). However, throughout the 20th Century philosophers have largely abandoned early, formalized systems of logic as descriptions of everyday reasoning and language (Stenning, 2002), as have more and more psychologists (Evans & Feeny, 2004). Nevertheless, many other psychologists persistently try to apply tenets from mathematical, formal logic to human reasoning (Stenning & van Lambalgen, 2004b).

This debate reveals researchers' tendency to assume sophisticated, abstract, and technical models of language and argumentation are inherently correct or "rational," despite the frequency with which untrained adults speak and argue in alternative ways (e.g. Stanovich, 1999). We do not so assume that mathematical logic is in any way natural or inherent to "right thinking" (especially preconscious, untrained reasoning), but we do acknowledge that for students to be apprenticed into proof-oriented mathematical practice, they must learn to consciously conform their reasoning to the prescriptions of mathematical logic (Dawkins, 2014). Mathematical topics should afford the mathematical model of logic much more naturally than everyday topics (Dubinsky & Yiparaki, 1999). However, students' untrained mathematical reasoning frequently violates the norms and conventions of mathematical logic (e.g. Epp, 2003) regarding both linguistic interpretation and argumentation. So, we propose that proof-oriented instruction may need to surface and address the differences between mathematical and everyday discourse to help students consciously conform their language to mathematical norms and conventions. We thus proffer the notion of *reasoning about logic,* which we use to refer to students' conscious understandings (psychological) of the problems addressed by and solutions embedded in formal systems of mathematical logic (objective science).

Logic and Reasoning

As Toulmin's (1958) analysis suggests, there exists a gulf between formalized, mathematical logic and (even expert) mathematical reasoning (Rav, 2007). Though one may use the language of formal logic to describe the outputs of student reasoning, such models' fidelity to the causes or mechanisms of that reasoning can be highly tenuous (e.g. Dawkins, 2012). *Logic* as a formal system tends to entail assumptions of *abstraction* (Dawkins, 2014) such that the interpretation of a sentence remains constant regardless of the particular mathematical (semantic) content being discussed. However, some have thereby claimed that for students to reason logically, they should similarly abstract the form of an argument being analyzed (e.g. Stylianides, Stylianides, & Phillipou, 2004). It seems likely that such characterizations of students' reasoning processes or desired reasoning processes conflate the descriptive and prescriptive meanings of *logic*. When researchers compare the outputs of student reasoning to logical prescriptions or use the language of logic to describe student reasoning, they must be careful to clarify what is being modeled and where any logical "structure" is understood to reside. Many psychologists search for structure embedded in students' reasoning process (whether students are aware of it or not), and try to control for or at least explain the role of semantic content in such reasoning (Evans & Feeny, 2004). In contrast, we concur with Stenning (2002) who argued that "logic teaching has to be aimed at teaching how to [help students] find form in content" (p. 190), meaning logical structure emerges within students' conscious reasoning about mathematical content. Such structure is not embedded in language or the world, but rather in the interpretive processes by which we reason about them. Traditional methods of teaching logic appear likely to fall short of helping students impose logical structure within semantic reasoning, precisely because logic is generally taught as an independent subject. Teaching pre-abstracted logical tools independent of the mathematics it is intended to formalize runs the risk of isolating such learning from students' mathematical reasoning. As Stenning (2002) suggested, "formal teaching can be effective as long as it concentrates on the relation between formalisms and what it formalizes" (p. 187).

Gaps between everyday reasoning and logical prescriptions

It remains an attractive hypothesis to many, though, that there is a unique "deductive" domain of reasoning that is formalized by logic, but has natural psychological correlates in untrained reasoning. The preface to a recent introduction to proof textbook claimed, "The whole of mathematics... is merely a refinement of everyday thinking. Proving theorems [is] not a different way of thinking—it is merely a refinement of clear thinking" (Katz & Starbird, 2013, p. 2). Also, the psychologist Rips (1994) discussed "intuitions about deductive correctness by people with no formal training in logic" (p. 34). Research tasks that ask participants to assess the conclusions of arguments "logically" or "based only on the hypotheses" are assumed to tap into such intuitions. Such views of proof-oriented mathematics or deductive reasoning assume that people have an untrained awareness of the hypothetical truth of a claim (logical entailment) as distinguished from the empirical or absolute truth of a claim (personal belief). Along these lines, Rips (1994) postulated that the mental processes underlying such deductive reasoning reflect domain-independent rules of inference, approximate to formal systems of logic. In addition to imposing such abstract logical models based on analogies with computer programs, researchers expect subjects, based on brief task instructions, to accept the hypotheses of arguments without question (even if experientially false) and avoid unstated inferences common to everyday dialogue (such as Grice's, 1975, implicature that "some apples are red" implies "not all apples are red"). Any subject behavior incompatible with the researchers' logical competence model is then deemed irrational (e.g. Stanovich, 1999). In this way, researchers have applied highly sophisticated tools of meta-reasoning to the outputs or processes of students' untrained and pre-conscious reasoning (Evans, 2007) when the students are provided with sentences and arguments that are relatively alien to everyday discourse.

We argue rather that logic, in the sense of a formal field of study, answers questions that many students are unlikely to have fully comprehended or asked. To attribute meaning to a sentence frame ("… or…" or "if…, then…") independent of semantic content in order to generalize across all everyday use and experience is a highly esoteric task, were it even possible. To investigate truth using only limited epistemic resources (hypotheses and axioms) rather than all relevant knowledge is a highly technical practice, though mathematically indispensible. Is it reasonable to assume that everyday discourse will lead adults to consider whether an argument is formally acceptable by abstracting the form of the argument across all possible subjects of the argument (i.e. in all possible interpretations)? Is there evidence that untrained students can assess the form of an argument distinct from the subject of the argument? On the contrary, there is strong evidence that people commonly reason about sentences of the same logical form in very different ways when the semantic content changes (e.g. Barnard, 1995; Dubinsky & Yiparaki, 2000; Evans, 2005). As a result, some psychologists have argued that reasoning is in many ways determined by the semantic content the sentences being analyzed because people form mental representations of that content (e.g. Johnson-Laird & Byrne, 2002), which minimizes logic's relevance to reasoning altogether.

Logic learning for proof-oriented mathematics

So, there is evidence to reject the assumption that the norms and conventions of formalized, mathematical logic are implicitly embedded in untrained adults' reasoning processes. Mathematical logic, though, serves as the normative model of mathematical language and argumentation to which students in proof-oriented courses must conform their reasoning (Dawkins, 2014). Thus, proof-oriented instruction requires a means of helping students learn and abide by these norms and conventions, possibly by understanding their purpose and value. For this reason, we find Stenning and van Lambalgen's (2004a) approach to the relation between reasoning and logic quite helpful. They argued that researchers must distinguish two phases of student reasoning about "logical" tasks: *reasoning toward an interpretation* and *reasoning from an interpretation*. Students must first (intentionally or preconsciously) decide how to interpret given claims, what conventions of communication to adopt, and a representation system in which to approach the relevant semantic content. Once these choices are made, some system of logic should model their reasoning within that representation system. Since reasoning toward an interpretation is largely preconscious, helping students problematize their interpretive processes may help them consciously control those processes in some manner approximating logical structure.

In the context of proof-oriented mathematics education, one may expect greater accord between semantic reasoning and formalized, mathematical logic. First, mathematical language lacks some of the main pragmatic complexities that introduce variation into everyday interpretations of conditional statements (Johnson-Laird & Byrne, 2002). Also, inasmuch as the semantic content of a sentence strongly determines the emergent pattern of people's reasoning about it, mathematical content could implicitly imbue reasoning with structure more compatible with formal logical norms. After comparing students' interpretations of everyday and mathematical claims of the same logical form, Dubinsky and Yiparaki (2000) suggested:

> "As teachers, instead of trying to make everyday life analogies between ordinary English statements and mathematical statements, perhaps we should remain in the mathematical contexts and concentrate our efforts directly on helping students understand mathematical statements in their natural mathematical habitats" (p. 1).

Historical analyses suggest that mathematicians usually upheld formal logical conventions before such conventions were codified, as attempts to formalize mathematicians' proofs written prior to the emergence of modern formal logic rarely find significant logical gaps (MacKenzie, 2001). However, any links between students' reasoning about mathematical sentences and mathematical logic competence models requires empirical investigation.

In summary, mathematics educators with goals of modeling student reasoning or eliciting particular forms of reasoning must be careful in imposing formal logical structures onto the processes or outputs of student reasoning. If students are bringing their everyday linguistic tools to bear on mathematical sentences, there is evidence that their reasoning will be incompatible with mathematical logic (e.g. Epp, 2003) and generally not systematized (Johnson-Laird & Byrne, 2002; Stenning, 2002). So, we set forth in this study to understand the interpretive tools students bring to bear in assessing the truth of mathematical disjunctions, and how they reflect on and systematize those tools in a manner that approximates logical structure within their mathematical reasoning.

Study and Methods

To better understand the possible connections between students' emergent reasoning patterns and the norms of formal logic, we conducted an exploratory teaching experiment (Steffe & Thompson, 2000) using guided reinvention (Gravemeijer, 1994) heuristics to see whether and how students come to *reason about logic*. By *reason about logic*, we refer to students 1) consciously attending to the meaning of logical connectives and 2) systematizing their interpretation of statements of the same logical form so as to develop generalizable heuristics for assessing the truth-values of mathematical statements. Our goals were for students to reinvent 1) notions approximating truth function for disjunctions and conditionals and 2) means of evaluating truth functions for quantified statements containing predicates (propositions whose truth-values vary over the set of examples). The classical tools for achieving these two goals are truth tables and Venn diagrams as portrayed in Figure 1. According to this view, disjunctions entail three truth-values: [A], [B], and [A] or [B] (which is a function of the first two). In quantified disjunctions, [A] represents a predicate P(x) whose truth-value may vary such that the space of examples (imagined as points in a region) can be partitioned according to P(x)'s truth-value. Conventionally, the quantified disjunction is true only if every element of the set satisfies at least one of the predicates.

For our teaching experiment, we recruited pairs of undergraduate Calculus 3 students from a mid-sized university in the Midwestern United States. We chose this population because we desired participants who were mathematically proficient, but had taken no proof-oriented mathematics courses. The data in this paper reflects two such pairs' reasoning about disjunctions. One pair had both taken a university philosophy course in logic (Drew and Ron) while a second had no such formal training in logic (Eric and Ovid). Each pair attended 5-6 one-hour teaching

Disjunction ("[A] or [B]") Truth Table

[A] truth-value	[B] truth-value	[A] or [B] truth-value
T	T	T
T	F	T
F	T	T
F	F	F

Venn Diagram

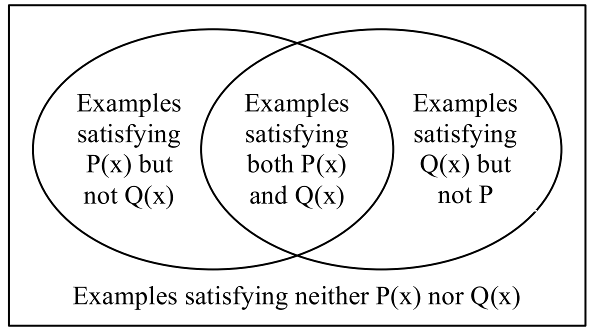


Figure 1: Classical tools for assessing the truth-values of disjunctions.

sessions outside of their normal class time. To prompt participants to reinvent truth-functions, we provided them with lists of mathematical disjunctions (Table 1) that they should determine as true or false. We only provided mathematical disjunctions because we wanted students to learn to structure their semantic reasoning, such that it might influence their later proof-oriented activity. On the three days spent on disjunctions, we directed them to 1) find patterns regarding why the statements were true or false, 2) write a "how-to guide" for determining whether disjunctions were true or false, and 3) develop a method for writing the negation of a disjunction, respectively. We privileged reasoning about logic by pushing students to abstract their strategies. The teaching experiment paradigm (especially using guided reinvention heuristics) allowed us to observe 1) whether and when students reasoned about logic, 2) whether students' reasoning showed any immediate or emergent patterns (that would constitute a pre-existing *logic* of disjunctions), and 3) observe any patterns of student reasoning that constituted barriers to adopting the normative mathematical interpretations of disjunctions. The interviewer (the first author) attempted to minimize any leading toward normative interpretations until the participants appeared to recognize or impose some structure upon their own reasoning about the provided statements (though mathematical information was readily provided upon request). He regularly asked students to articulate or elaborate upon their reasoning, restated their arguments back to them, and asked interview partners to respond to one another's claims.

The second author observed all interviews as an outside observer (Steffe & Thompson, 2000). Between each teaching session, the researchers discussed students' responses, formed hypotheses about student learning, viewed video of the teaching session, and designed activities for the next session intended to test and extend hypotheses about student learning. The hypotheses about patterns of students' *reasoning about logic* that emerged during the teaching experiment then formed the initial categories for the process of grounded theory-type (Strauss & Corbin, 1998) open and axial coding of the data. In retrospective analysis, we endeavored to code every major student action relevant to assessing and negating the given statements. Codes related to 1) truth-value assessment strategies (e.g. one condition false makes the disjunction false), 2) paraphrases of provided statements (e.g. introducing "either…or" language), 3) modes of reasoning about logic (e.g. attending to the meaning of *or*), 4) clarification of semantic information (e.g. identifying warrants such as "all squares are rectangles"), and 5) negating actions (e.g. negating [A or B] with [not A or not B]). We report here on persistent trends in students' interpretive behavior and emergent relationships between their strategies, interpretations, and particular disjunctions we provided.

Day 1 Disjunctions	Day 2 Disjunctions: How-to Guide
A1. Given an integer number x, x is even or x is odd. A2. The integer 15 is even or 15 is odd. A3. Given any two real numbers x and y, $x < y$ or $y < x$. A4. Given any two real numbers x and y, $x \leq y$ or $y \leq x$. A5. Given any real number y, y has a reciprocal $\frac{1}{y}$ such that $y * \frac{1}{y} = 1$ or y=0. A6. The real number π has a reciprocal $\frac{1}{\pi}$ such that $\pi * \frac{1}{\pi} = 1$ or $\pi = 0$. A7. The real number 0 has a reciprocal $\frac{1}{0}$ such that $0 * \frac{1}{0} = 1$ or 0=0. A8. Given any real number x, x is even or x is odd. A9. Given any even number z, z is divisible by 2 or z is divisible by 3. A10. Given any even number z, z is divisible by 4 or z is divisible by 3. A11. Given any even number z, z is divisible by 2 or z is divisible by 4. A12. Given any even number z, z is divisible by 4 or z+2 is divisible by 4.	B1. Given an integer x, x is an even number or x+1 is an even number. B2. 10 is an even number or 20 is an even number. B3. 13 is an even number or 6 is an even number. B4. 5 is an even number or 7 is an even number. B5. 8 is an even number or 37 is an even number. B6. Given any triangle, it is equilateral or it is not acute. B7. Given any triangle, it is acute, or it is not equilateral. B8. Given any triangle, the sum of the measures of the interior angles is 185.7° or the sum of the measure of the interior angles is 180°. B9. Given any quadrilateral, it is a square or it is not a rectangle. B10. Given any quadrilateral, it is not a square or it is a rectangle. B11. Given any rectangle, the interior angles are all right angles or the interior angles are all obtuse. B12. Given any two integer numbers x and y with x<y, there is an integer between x and y or x+1=y. B13. Given any two real numbers x and y with x<y, there is a real number between x and y or x+1=y. B14. Given any two natural numbers x and y with x<y, there is a natural number between x and y or x+1=y.

Table 1: Sample disjunctions from the first two instructional sessions.

Patterns of Student Assessment of Disjunctions

The disjunctions we provided fell into two forms: disjunctions of two statements with fixed truth-values (e.g. A2 or A7) and disjunctions of two predicates quantified over some set (e.g. A1, A5, or B9). Students generally exhibited different patterns of disjunction assessment regarding quantified and non-quantified statements. The following sections each describe a major pattern of disjunction assessment from prior literature or that emerged during the study.

Part False-All False Decision Heuristic

At times, students declared a non-quantified disjunction false because it contained a false component statement, which we shall refer to as the *False-False Heuristic*. For instance, several students initially rejected A2 because 15 is not even or A7 because 0 has no reciprocal. The frequency of this heuristic decreased over the course of the interviews, but some students struggled to avoid this interpretation when they considered the statement mathematically absurd (such as “0 has a reciprocal” or “$\pi = 0$” in A6 and A7 respectively). Once students in the study began to specifically attend to the “or” connective, these students mostly abandoned the strategy.

Part True-All True Decision Heuristic

Especially after students began to attend to the “or” connective, they declared non-quantified disjunctions true whenever they contained at least one true component statement, which we shall refer to as the *True-True Heuristic*. For instance, on the second day students all affirmed B5 because 8 is an even number. They also usually affirmed quantified disjunctions when all cases satisfied a single predicate, as in the case of A9.

During the first session with Eric and Ovid, the students used the False-False Heuristic until they reached A9. Initially they also declared it false because there were even numbers not divisible by 3. However, Eric then said, "It does say *or*, it doesn't say *and*. So if it's not divisible by 3 it is divisible by 2." This was the first time the pair attended directly to the connective *or* and its role in the sentence, which was the first observed pattern of *reasoning about logic*. The pair later reconsidered A2 (which they had declared false by the False-False Heuristic). Ovid noted, "If we're doing the *or* similar to down there [pointing to A9], we would have to go to true because, even though we know it's odd… it's either even or it's odd. That's what I'm getting out of it, it's one or the other." Though this is not tantamount to defining the connective *or*, Ovid's comparison of their interpretations of the two statements represents a conscious effort to systematize their interpretations of the disjunctions. Such statement comparison was the second pattern of *reasoning about logic* that emerged during the study. Eric also helped clarify the meaning of *or* by comparing it to the meaning of *and.* In each case, these comparisons directed the students' attention to their own interpretations of the language of the statements.

The next day, while discussing B10, Ovid stated more directly that, "Cause *or* for me means… either it could be one or the other or both." This articulation reflects the students' awareness of the repeated structure that each statement is made of two components, whose truth-values determine the truth-value of the statement overall. In this way, students distinguished the statement's three truth-values and related them via a truth-function. By this second day, their interpretation of *or* closely approximated the normative definition of the logical connective.

Fails Both Decision Heuristic

Students reliably declared non-quantified disjunctions false when both components were false and declared quantified disjunctions false when they found one example that failed both predicates, which we call the *Fails Both Heuristic*. However, students did not readily see how to segue this heuristic into a general strategy for finding counterexamples or negating disjunctions. We hypothesize they did not make this abstraction because they did not clearly relate the falsehood of one predicate with the truth of its negation, meaning that students did not spontaneously connect the observation that "it is a rectangle" is false to the claim that "it is not a rectangle" is true. Study participants interpreted properties as descriptions of individual cases in ways that did not lead them to link properties to alternative properties. This is not to say they would not have affirmed the negation claim "it is not a rectangle" if asked. Rather, study participants did not formulate their identification of a counterexample or search for a counterexample in terms of the properties that would be true of it (i.e. trying to falsify B9 by finding a non-square that is a rectangle). So, while study participants readily recognized sufficient conditions for a disjunction being false, they failed to abstract this approach in a manner approximating the standard logical negation (not[A] and not[B]).

Minimal Disjunction

On occasion, students rejected a disjunction because it contained extraneous conditions that were not realized, which we call the *Minimal Disjunction* interpretation. For example, Eric rejected A4 and called it false in comparison to A3 (which they declared true under the assumption that $x \neq y$). He preferred A3 because (under his assumption) it covered all possible cases rendering the equality conditions unnecessary. So, for these students A4 was not false because it made a false claim but was false because it included extraneous claims. Such pragmatic reasoning may explain why students rejected A2: they implicitly compared it to the more natural claim that "15 is odd." In this case, students used the *false* option to express "I wouldn't say this," inconsistent with the mathematical interpretations of truth and falsehood. In later studies, other students have more explicitly articulated that they do not interpret A5 in the same way as A6 or A7 because A5 requires the *or* to cover various cases, but the statements about 0 and π each contain a needless condition. This is another instance in which students appear to call upon implicit, pragmatic rules of everyday language (Grice, 1975) to critique the provided statements as unnatural rather than untrue.

Semantic Affirmation

Students enacted *Semantic Affirmation* of a disjunction whenever they attended only to the statement as a whole rather than to the component statements/predicates. For instance, students generally accepted claims such as A1 or A3 as true *prima facie*. Students showed no evidence of attending to particular numbers or the two predicates independently, and gave affirmation quickly. For example, when prompted to group the statements according to the reasons they were true, Eric contrasted statements like the first (A1) against others because the former were "rules" that they were taught in school. Eric struggled to generalize this category more precisely, but his sense that numbers being even or odd is a mathematical "rule" supports the view that he affirmed the statement as a single unit rather than as the coordination of two independent predicates.

Semantic Substitution

Semantic Substitution refers to instances when study participants affirmed statements like B6 and B9 because they used the warrant "equilateral triangles are acute" to paraphrase the statement with the tautology "Given any triangle, it is acute or it is not acute." Within the standard interpretation, this is an error because the class of acute triangles is larger than the class of equilateral triangles. However, study participants generally reasoned about particular cases rather than classes. Thus, if they imagined an equilateral triangle, it could also be called acute, which likely supported the linguistic substitution. We also hypothesize that students were attracted to the strategy because the paraphrases were much easier to assess.

Exclusive or

Prior to the study, we expected students to express non-conventional interpretations by employing the *Exclusive or* (the two component propositions cannot both be true) meaning that many professors take as paradigmatic of the divergence of everyday and mathematical language (Epp, 2003). However, only Eric adopted this interpretation, and did so for two of the given disjunctions. Furthermore, in the second case it was not his initial interpretation of the statement, but rather he adopted it 9 minutes into the pair's discussion of B10. In subsequent experiments, a few students more aggressively adopted an Exclusive or interpretation, which we take to imply that the findings from the two pairs in this paper should not be overgeneralized. Ironically, it is often the most astute study participants who have adopted the *exclusive or* interpretation, despite the mathematical community's adoption of an *inclusive or* convention. An explanation for this is that such students show greater awareness of their own interpretive processes and attend to the *exclusive or* interpretation in its everyday instantiations.

As with many cases in formalized logic, when there are two possible interpretations available for a single linguistic pattern, one interpretation is assigned to an alternative linguistic form. In the case of *or*, the exclusive interpretation is assigned to the *either… or…* linguistic form. Students in our study occasionally introduced *either… or…* language, generally when it was appropriate because no cases satisfied both conditions. However, no study participants seemed aware of this paraphrase or its correspondence to the alternative meaning of *or*.

Case-based Sentential Testing

Regarding quantified disjunctions, students frequently selected example cases and assessed whether they satisfied either proposition in the disjunction, usually passing from left to right, which we call *Case-based Sentential Testing*. For instance, Ron and Drew evaluated B1 by substituting various values for x (in contrast to their semantic affirmation of A1). Drew used their sequential test to describe how a statement could be false for a given case saying, “Well, you can say an *or* statement [disjunction] is false when both the original statement and then the backup statement are both, didn’t back up each other so both the statements are false, cause *or* [second predicate] did not back up the first statement which was already false.” Drew thus alternated between using the connective *or* to refer to the disjunction itself and to the latter predicate as the “backup” to the first condition. This mode of assessment entailed an asymmetry between the predicates because it was *statement-centered*.

The benefit of Case-based Sentential Testing was that, by serially selecting examples, it reduced quantified disjunctions to a sequence of non-quantified ones, affording the True-True Heuristic or the Fails Both Heuristic. This is not to say that students were aware of this relationship. In many cases, students did not perceive statement A2 as a “case of” statement A1 or statements A6 and A7 as instantiations of statement A5 (as the logician Copi’s principle of *Universal Instantiation* would suggest, Durand-Guerrier, 2008, p. 383). Students often brought different interpretive strategies to bear on quantified and non-quantified statements until students reflected on their interpretations and began to systematize them.

The sentential testing strategy did not provide study participants with a means of structuring the set of examples, and they employed very different strategies for testing examples depending upon the context. For statements like A9 and B1, students proceeded through the integers or even numbers and were quickly convinced whether the statement was true or false. For B12-14, Ovid let x=3 and then considering various values of y. In geometric contexts, though, students relied almost exclusively on familiar categories (obtuse, right, isosceles, trapezoid, parallelogram) such that a sequential test of examples did not guarantee they had considered all relevant cases. This contextual method of producing examples provided no viable abstraction similar to the Venn diagram, in which examples are partitioned by the predicates in the disjunction.

Categorical Partitioning

Unlike the case-based reasoning that students displayed on the majority of quantified disjunctions, students occasionally reasoned about properties without representative examples based on some implicit partitioning of possibilities such as even/odd or </>/=. Students using such *Categorical Partitioning* rejected statements like A3 because they left out one of the three possible order relations between two numbers and affirmed A4 along similar lines. This category of strategies does not include Semantic Affirmation where students treated the statement as obviously true based on prior knowledge or comparable Case-Based Sentential Testing when students chose to substitute particular values for the variables in the statement. We separate this strategy because students were able to implicitly use some warrant such as the trichotomy of order relations to reason about properties themselves rather than shifting their focus to cases that had those properties. This strategy was relatively uncommon in the data, but represents an alternative to their more frequent case-based approaches. Participants did not use this strategy in contexts where there was no clear partition by familiar categories, such as A9-12 or B6-11.

“If Not, Then” Reasoning

Some students adapted the case-based sentential approach by anticipating that cases satisfying one proposition were automatically “covered.” They thus began to focus only on those cases not described by a chosen predicate. For instance, Ron evaluated B7 by focusing only on triangles that were not acute. He inferred that this entailed obtuse triangles and noted that an obtuse triangle cannot be equilateral. Thus the cases excluded by the first condition must be captured by the second (notice this line of reasoning applies to right triangles, which he ignored). We call Ron's heuristic *"If Not, Then" Reasoning* because his approach might be paraphrased, "If a triangle is not acute, then it will not be equilateral." When Drew wanted to consider equilateral triangles, Ron contested based on the criteria in their case-based sentential approach: "We not talking (sic) about that though. If it's acute, that's it. We don't have to worry about the not equilateral." What distinguishes this approach from case-based sentential testing is how it allowed Ron to ignore some cases ("we not talking about that") according to the propositions in the statement. In this way, such reasoning began to structure the examples according to the propositions in the statement, and in this way is more *case-centered*. It allowed Ron to focus on salient cases rather than employing a random or exhaustive case search.

"If not, then" reasoning also helped some study participants identify counterexamples because it focused their attention on half of the conditions for negating a disjunction ([not A] and [not B]). For instance, regarding B9 Eric said, "yeah like the *or* statement is like, 'If it's not a square, it can't be a rectangle either,' but it could be a rectangle if it's not a square." Eric only focused on non-squares and used a deontic paraphrase (Cheng, Holyoak, Nisbett, & Oliver, 1986) of the second condition ("can't be" instead of "is not"). This focused his attention on the possibility that a non-square could be a rectangle, helping him identify a counterexample to the disjunction. Though some students applied this heuristic several times, they continued to vacillate to other strategies and none of the participants abstracted it into a working definition for the *or* connectives quantified over a set. They also did not segue this strategy into a general method for finding counterexamples or negating disjunctions.

Reasoning toward an interpretation and reasoning about logic

The mere categorization of student strategies does not adequately capture a striking aspect of students' reasoning processes during the teaching experiment: how their interpretations vacillated even as students considered a single mathematical disjunction. The following episode from the second meeting with Eric and Ovid demonstrates this trend. The students were attempting to assign truth-values to B9 and B10.

E (1)[1]: [Considering B9] If it's a square, it's not a rectangle. Well, squares are rectangles, but…

O (2): "Is not a rectangle," that could mean it's a parallelogram or anything like that too, right so I would say it's true.

E (3): There's a square. There's not a rectangle. It could be the rectangle. I don't think a rectangle is considered a square. A square is, they're all even sides. They're all equal sides… But a square is a rectangle.

O (4): With equal sides.

E (5): But it's a specific rectangle, yeah… So I'd say it's false.

O (6): Umm, but for a quadrilateral it doesn't mean they all have to be right angles. You could have a parallelogram that is also not a rectangle.

I (7): So [Eric], what was your reasoning for saying it was false?

E (8): Well it could be a square, or it could be a rectangle that isn't a square. So.

I (9): So you have a rectangle that isn't a square

E (10): So it can be a square or it can be a rectangle or it can be anything else.

I (11): So that makes it false because

E (12): It's saying, "If it's not a square it can't be a rectangle." But it could really be anything.

I (13): What do you think [Ovid], do you see his line of reasoning?

[1] Each turn taken by a dialogue participant is numbered for reference.

E (14): A quadrilateral could be a parallelogram and it's not a square and it's not a rectangle. But it could be a rectangle that isn't a square.

O (15): Yeah cause "not a rectangle" that's just a parallelogram then, or a square. So I would say that it's true.

E (16): Or it could be angled [holding forearms up as parallel diagonal lines] or it could be 90 degrees [rotating arms to vertical orientation], it could be anything. It could be a rectangle if it's not a square. So, like, if it's not a square, it could still be a rectangle. This is saying, "it's either a square or it's not a rectangle." It could be a square, it could be a rectangle, it could be, like, an angled quadrilateral. So it's giving you, yeah like the *or* statement is like, "If it's not a square, it can't be a rectangle either," but it could be a rectangle if it's not a square.

O (17): So it's either a square or a parallelogram, which is not a rectangle. So the only, so actually the only way that this is false if the "any quadrilateral" is a rectangle.

E (18): Yeah.

O (19): So, based on that I would say false then. Cause it's "not a rectangle," and you're given a rectangle, then that doesn't satisfy either.

E (20): Right…

O (21): [Now considering B10] Okay, so then I would say that's false too.

E (22): Yeah. Cause it's basically like a third thing it could be that doesn't satisfy those two.

I (23): So explain the "third thing."

E (24): Like a parallelogram that is at angles isn't considered a rectangle, is it?

I (25): No. It is not a rectangle.

E (26): Right. Cause a rectangle is 90 degrees all around. It would be like, is a square, or it's a rectangle, or is a parallelogram.

I (27): So you are kind of forming three groups, there's that stuff that's like squares, rectangles, and… other stuff like parallelograms.

E (28): If they included all possible quadrilaterals, but this is pretty much saying there is only two types of quadrilaterals, when there could be a third… I think there's an instance where neither of those would be satisfied, but it would still be a quadrilateral.

This episode exemplifies several notable trends in the students' reasoning processes. Even by this task on the second day, students' reasoning was still highly focused on the mathematical subject matter rather than on abstracting to some generalized syntax (p's and q's or truth tables). This was a consequence and intention of our task design, but it highlights how unnatural classical logical abstraction is for students, even when an expert would estimate it valuable for solving the tasks provided. Study participants spent much more time and displayed more diverse strategies when reasoning about the geometric items as compared to many others. This was not because they were not aware of the relevant warrants such as "all squares are rectangles," though the students had to elaborate how such claims interacted with the given disjunctions. In other cases, students had difficulty with geometric items because they lacked a clear means of enumerating the examples for their Case-Based Sentential Testing.

In this episode, however, Eric quickly divided all quadrilaterals into three relevant cases: squares, rectangles, and all others. Ovid seemed to need to find some representative for the third category, which is why he paraphrased "not a rectangle" with "is a parallelogram" in turns 2 and 15, but did not challenge the three-case partition. Ovid took some time to understand Eric's argument that one of the three cases was not covered by the given conditions, so Eric had to state and restate his thinking. To do so, he alternated between an If Not… Then… Paraphrase (turns 12 and 16) and an Either… Or… Paraphrase (turn 16). In this way, participants in the study often alternated between the various strategies either for the purpose of convincing themselves or their partner of the verity or falsity of a given statement. Sometimes their grasp on a single interpretation was tenuous enough that their partner's alternative explanation shifted their interpretation and they would abandon the previous. As Dubinsky and Yiparaki (2000) noted, "It was as though the statement was a window from which they were looking out. The students described what they saw looking out the window, but they did not see the window itself" (p. 23). The goal of the guided reinvention experiment was to make students more aware of the window of interpretation, but often students shifted or maintained various interpretive stances without being able to control or compare the various viewpoints. Such shifting of viewpoints helped the students reach equilibrium in their understanding of each statement, but was not always sufficient for systematizing their linguistic interpretation. It is these varying patterns of interpretation that we equate with Stenning and van Lambalgen's (2004a) notion of *reasoning toward an interpretation* before *reasoning from an interpretation*. Eric came to a stable understanding of B9 and its relationship with the set of quadrilaterals. Eric then reasoned from that interpretation regarding B10 in a way that was less productive.

B10 is true because each of the two conditions cover two of the three categories in Eric's partition of cases (squares, rectangles, and everything else). Instead of seeing this new referential structure, Eric anticipated that B10 would maintain the same one-to-one relationship between conditions and categories (turn 28) and that B10 would have a counterexample (Figure 2). Eric was so convinced of this that he spent about 10 minutes trying to identify which of his three cases would serve as a counterexample, often shifting his interpretation of the disjunction to suit his desired outcome that it be false. Eric's anticipation represents a form of *reasoning about logic* because Eric recognized a referential relationship in a disjunction and abstracted that relationship beyond the particular conditions in the statement. Unfortunately, the structure that Eric abstracted was not generalizable because conditions such as "is a rectangle" and "is not a square" can entail multiple categories in his partition of the set of quadrilaterals. As we stated before, study participants did not generate a structure for reasoning about quantified disjunctions that approximated the Venn diagram, but this is not to say that they did not attempt to abstract certain structures such as Eric's Two Out of Three pattern he observed in B9.

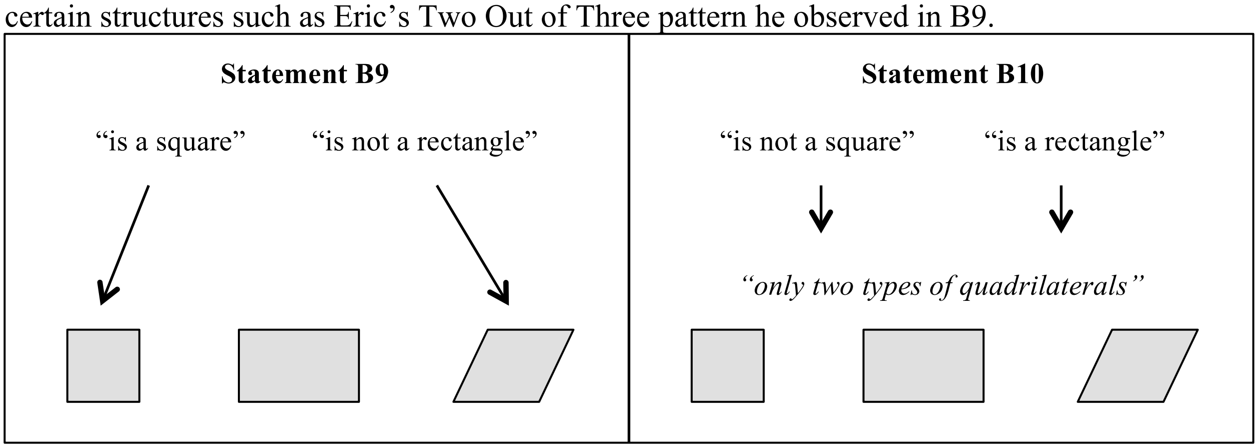


Figure 2: Eric's abstraction of the referential structure of B9.

When asked to explain why there were only three relevant categories, Eric and Ovid were unable to justify this choice. They instead began citing other categories of quadrilaterals such as trapezoids and rhombi. The space of quadrilaterals was for them already partitioned by familiar categories rather than by the novel partition induced by the conditions in the statement (as shown in Figure 1). Only after the interviewer prompted the students to classify each of the cases according to the two conditions in the statement and group those with the same truth pattern did the students recognize why there were only three relevant cases (since there are, for instance, no non-rectangle squares). They were soon able to abstract the identification of rhombi and parallelograms into non-square non-rectangles, but this connection had not occurred to them prior to interviewer prompting. Thereafter, Eric and Ovid became relatively fluent with the four possible types of examples (TT, TF, FT, and FF), but they consistently represented each possibility by an exemplar rather than reasoning about the abstract truth-value pattern alone.

Conclusions and Implications

Consistent with prior literature, students did not begin the experiment with fixed, content-independent meanings for *or*, as revealed by the diverse behaviors they exhibited in interpreting mathematical disjunctions. Table 2 organizes these behaviors into five categories: non-quantified strategies, pragmatic strategies, property-based strategies, case-based strategies, and instances of reasoning about logic. While the non-quantified strategies show a clearer progression from non-normative to normative modes of interpretation, the progression is less clear for the various strategies students used for quantified disjunctions. This reflects the fact that study participants were able to reinvent the standard truth-functional definition for *or* when the two propositions had fixed truth-values (as in the non-quantified case or statements like A9), but did not reinvent any strategy for assessing non-quantified disjunctions that approximated the Venn diagram without explicit guidance. That is to say, the standard truth function emerged rather naturally from students' own strategies once they began to attend to and reflect on those strategies. While students used several recurrent strategies for quantified disjunctions that we as researchers could identify, students struggled to reflect on and generalize them. Furthermore, we hypothesize that the case-based nature of most of their strategies inhibited their development of a Venn diagram type strategy for assessing disjunctions: namely that the union of the sets entailed by the two conditions is the universal set. As such, we expect that instructional activities that help students connect properties with sets rather than single cases might better facilitate the emergence of more conventional and generalizable modes of interpretation for quantified disjunctions.

Non-quantified strategies	Pragmatic strategies	Property-based strategies	Case-based strategies	Reasoning about logic
False-False Heuristic	Semantic affirmation	Semantic substitution	Case-based sentential testing	Comparing connectives
True-True Heuristic	Minimal disjunction	Categorical Partition	"If not… then…" reasoning	Comparing interpretations
Fails Both Heuristic	Deontic paraphrase			Abstracting referential structure
Exclusive *or*				Defining *or*

Table 2: Categories of student interpretive behaviors regarding mathematical disjunctions.

The pragmatic strategies especially demonstrate how students initially framed some statements in ways incompatible with the standard logical form of quantified disjunctions ($\forall x \in S, P(x) \vee Q(x)$). Researchers have tended to assume that all of these statements are of the same logical form, independent of any reader's interpretation, but we concur with Stenning's (2002) caution against such assumptions because, "Talking of… finding form in content could be misleading if it gave the impression that there is a unique form waiting to be found. The skill can as well be thought of as imposing form on content which more adequately captures its active nature and the range of outcomes" (p. 195). The goal of our teaching experiment was to help students identify generalizable interpretations they could apply to all of the given disjunctions, which would entail them framing each statement in a uniform way. Students certainly began to recognize each statement as being composed of two components that have independent truth-values and that the overall truth-value depended upon the component truth-values. However, their methods of identifying and testing cases largely depended upon the semantic context. Their use of property-based strategies relied on the availability of an exhaustive partition of familiar categories such as <, >, and = or some problematic paraphrase to a tautology.

We highlight instances of students *reasoning about logic* by which we mean conscious recognition the problems that formal logic solves as well as students' solutions posed for these problems. In our study, this generally entailed systematizing their interpretations and strategies for assessing mathematical disjunctions. We identified at least four types of reasoning about logic in these teaching experiment, as listed in Table 2. Initially, the *or* connective remained ostensibly invisible to study participants. Once students attended to the *or*, they used several strategies to find appropriate meanings for it. They compared it to the connective *and* as well as comparing their interpretation of *or* across provided disjunctions. Eric's anticipated solution to B10 also represented reasoning about referential structure, though it did not prove generalizable. In several such cases, students spontaneously adopted some generalized language for defining *or* or discussing conditions for declaring disjunctions true or false.

We propose this construct as a viable way to define and elicit "logical structure" within students' own mathematical activity (as opposed to in language or meaning itself). While this does not preclude research trying to describe the implicit logic of students' preconscious and untrained reasoning, we think it constitutes a necessary direction for mathematics education research on proof-oriented mathematics instruction. This is because students must be trained to consciously impose normative logical structure in their reasoning about mathematical content. Such structure does not a priori reside in students' thinking or in mathematical language, as our data demonstrate. Rather, this structure is comprised of a useful set of conventions of linguistic interpretation that solve a set of problems that students may need to recognize as problematic before they will adopt them as useful and later intuitive. Certainly for mathematicians, these conventions codify "what the statements say," but students may need to look at the language more consciously before it will "speak to them" in the same way.

References

Barnard, T (1995) The Impact of Meaning on Students' Ability to Negate Statements, in *Proceedings of th 19th PME Conference, Recife, Brazil* vol. 2 p. 3-10.

Cheng, P.W., Holyoak, K.J., Nisbett, R.E., & Oliver, L.M. (1986). Pragmatic versus syntactic approaches to training deductive reasoning. *Cognitive Psychology, 18*, 293–328.

Dawkins, P.C. (2012). Extensions of the semantic/syntactic reasoning framework. *For the Learning of Mathematics, 32*(3), 39-45.

Dawkins, P.C. (2014). Disambiguating research on *logic* as it pertains to advanced mathematical practice. In (Eds.) T. Fukawa-Connelly, G. Karakok, K. Keene, and M. Zandieh, *Proceedings of the 17th Annual Conference on Research in Undergraduate Mathematics Education*, Denver, Colorado.

Durand-Guerrier, V. (2008). Truth versus validity in mathematical proof. *ZDM*, *40*(3), 373-384.

Epp, S. (2003). The role of logic in teaching proof. *The American Mathematical Monthly, 110*, 886-899.

Evans, J. (2005). Deductive reasoning. In Holyoak, K.J. & Morrison, R.G. (Eds.) *Cambridge Handbook of Thinking and Reasoning*. (pp. 169-184) Cambridge, NY: Cambridge University Press.

Evans, J. (2007). *Hypothetical thinking: Dual processes in reasoning and judgement*. Hove, UK: Psychology Press.

Evans, J. & Feeny, A. (2004). “The Role of Prior Belief in Reasoning” In J. P. Leighton & R. J. Sternberg (Eds.), *The nature of reasoning* (pp. 78-102). Cambridge, U.K.; New York: Cambridge University Press.

Gravemeijer, K. (1994). *Developing Realistic Mathematics Education*. Utrecht: CD-β Press.

Grice, H.P. (1975). Logic and conversation. In Cole, P. and Morgan, J. (Eds.) *Syntax and Semantics: Speech Acts, volume 3*. London, UK: Academic Press.

Johnson-Laird, P. N., & Byrne, R. M. J. (2002). Conditionals: A theory of meaning, pragmatics, and inference. *Psychological Review, 109,* 646– 678.

Katz, B. & Starbird, M. (2013). *Distilling Ideas: An Introduction to Mathematical Thinking*. Washington, D.C.: Mathematical Association of America.

MacKenzie, D. (2001). *Mechanizing proof: Computing, risk, and trust*. Cambridge, MA: The MIT Press.

Rav, Y. (2007). A Critique of the formalist-mechanist version of the justification of arguments in mathematicians' proof practices. *Philosophia mathematica. 15*(3), 291-320.

Rips, L. J. (1994). *The psychology of proof: Deductive reasoning in human thinking*. Cambridge, MA: MIT Press.

Simon, M., Saldanha, L., McClintock, E., Akar, G.K., Watanabe, T., & Zembat, I.O. (2013). A Developing approach to studying students' learning through their mathematical activity. *Cognition and Instruction, 28*(1), 70-112.

Stanovich, K. (1999). *Who is Rational? Studies of individual differences in reasoning*. Mahwah, NJ: Lawrence Erlbaum Associates.

Steffe, L.P., & Thompson, P.W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267-307). Hillsdale, NJ: Erlbaum.

Stenning, K. (2002). *Seeing reason: Image and language in learning to think.* New York, NY: Oxford University Press.

Stenning, K. & van Lambalgen, M. (2004a). A little logic goes a long way: basing experiment on semantic theory in the cognitive science of conditional reasoning. *Cognitive Science, 28*, 481-529.

Stenning, K. & van Lambalgen, M. (2004b). The natural history of hypotheses about the selection task: Towards a philosophy of science for investigating human reasoning. In Manketelow, K. & Chung, M.C. (Eds.) *Psychology of Reasoning: Theoretical and historical perspectives* (pp. 127-156). New York, NY: Psychology Press.

Stenning, K. & van Lambalgen, M. (2008). *Human Reasoning and Cognitive Science*. Cambridge, MA: The MIT Press.

Strauss, A. & Corbin, J. (1998). *Basics of Qualitative Research: Techniques and Procedures for Developing Grounded Theory (2nd edition).* Thousand Oaks, CA: Sage Publications.

Stylianides, A.J., Stylianides, G.J., & Philippou, G.N. (2004). Undergraduate students' understanding of the contraposition equivalence rule in symbolic and verbal contexts. *Educational Studies in Mathematics, 55*, 133-162.

Toulmin, S. (1958). *The uses of argument*. Cambridge: Cambridge University Press.

How do mathematics majors translate informal arguments into formal proofs?

Dov Zazkis
Arizona State University

Keith Weber
Rutgers University

Juan Pablo Mejía-Ramos
Rutgers University

In this paper we examine a commonly suggested proof construction strategy from the mathematics education literature—that students first produce an informal argument and then work to construct a formal proof based on that informal argument. The work of students who produce such informal arguments when solving proof construction tasks was analyzed to distill three activities that contribute to students' successful translation of informal arguments into formal proofs. These activities are elaborating, syntactifying, and rewarranting. We analyze how engaging in these activities relates to students success in proof construction tasks. Additionally, we discuss how each individual activity contributes to the translation of an informal argument into a formal proof.

Key words: [proof; argumentation; Toulmin scheme; formalization]

Proving is central to mathematical practice. Consequently, a primary goal of advanced mathematics courses at the university level is to have mathematics majors become proficient at writing proofs. Unfortunately, research has demonstrated that this goal is rarely met. Numerous studies have documented that mathematics majors perform poorly when presented with proof construction tasks (e.g., Alcock & Weber, 2010; Hart, 1994; Iannone & Inglis, 2010; Moore, 1994; Weber, 2001; Weber & Alcock, 2004). Researchers have documented many causes for mathematics majors' difficulties in writing proofs, including a poor conceptual understanding (Hart, 1994; Moore, 1994), a lack of proving strategies (Weber, 2001), and not knowing where to begin when given a proving task (Moore, 1994). However, research on how mathematics majors can or should successfully write proofs has been comparatively sparse. In this paper, we examine one suggestion from the literature—that students can first produce informal arguments for why a statement is true and then base their proofs on these informal arguments (e.g., Garuti, Boero, & Lemut, 1998; Raman, 2003; Weber & Alcock, 2004).

Theoretical perspective

Basing proofs on informal arguments

Boero (1999) observed that a proof—the product of one's mathematical reasoning—must satisfy certain formal constraints, but the reasoning used to generate this proof need not. In particular, when proving a statement, one can first construct an informal argument that convinces oneself that the statement is true and then use this informal argument as a basis to construct a proof (e.g., Garuti, Boero, & Lemut, 1998; Raman, 2003; Weber & Alcock, 2004). There is a difference between the informal arguments and proofs that one may generate. An informal argument may be viewed as a form of personal persuasion (Douek, 2009) where one convinces oneself that a mathematical assertion is true. A proof is a form of validation where one convinces oneself that the assertion is a necessary logical consequence of things one knows to be true (Douek, 2009; Duval, 2007).

To distinguish between an informal argument and a proof in an advanced mathematical context, we follow Stylianides (2007) who proposed assessing whether an argument is a proof along three criteria: (i) the facts that are taken as the starting points of the proof, (ii) the representation system that is used, and (iii) the validity of the methods of inference used in the proof.

(i) In a proof, each assertion must be an acceptable assumption (e.g., an axiom, a definition), a statement accepted by one's mathematical community as true within this context (often because it had been proved previously), or inferred from previous assertions. In contrast, in an informal argument, assertions merely need to be statements that the individual believes are true.

(ii) Mathematical proofs are written in a unique representation system using a combination of natural language, algebraic notation, and logical symbols (cf., Weber & Alcock, 2009). Each term in the proof has a precise meaning. Informal arguments may express mathematical concepts in other ways, such as using graphs and diagrams. They may also express ideas less precisely. For instance, in an informal argument, one might say, "$f(x)$ will eventually overtake $g(x)$" rather than "there exists a real number c such that if $a>c, f(a)>g(a)$".

(iii) In a proof, new statements need to be deduced from previous statement via a valid warrant—that is, a method of deduction that is accepted as valid by the mathematical community in that context. In an argument, new statements may be inferred from previous ones by a personal warrant—a method of inference that the individual believes is likely (or perhaps guaranteed) to yield true statements. As opposed to a proof, an informal argument may, for instance, involve generalizing from a particular example, making a perceptual inference about a function from its graph, or making an abductive inference.[1]

Benefits of informal arguments in proving

In the past two decades, a number of researchers have advocated that students base their proofs on informal arguments. This is a driving force behind the research program of the Italian school, whose proponents endorse proofs having a cognitive unity where, under particular circumstances, there is a continuum between a student's production of a conjecture and how the student proves it (e.g., Bartolini Bussi, et al., 2007; Garuti, Boero, & Lemut, 1998; Pedemonte, 2007). Raman (2003) contended that it is desirable for students to base their proofs off a key idea, where a key idea connects students' informal private ways of knowing why a statement is true with the formal proof that students produce for public consumption.

Support for these recommendations typically comes from the analysis of episodes of students successfully basing proofs off of informal arguments (e.g., Alcock & Weber, 2010; Douek, 2007; Garuti, Boero, & Lemut, 1998). In these episodes, students often gain insights by studying informal representations of mathematical concepts (e.g., graphs, diagrams, prototypical examples) and/or using non-deductive methods of inference (e.g., generalizing from a specific example). These insights appeared to be easier to discern via non-deductive reasoning than if one worked exclusively at a formal level. For instance, it is often easier to see that a function is increasing or strictly positive by studying its graph than working with its formula.

[1] To avoid misinterpretation, we deliberately avoid making both psychological or normative judgments on whether personal warrants or valid warrants provide complete conviction. For instance, philosophers argue that some perceptual inferences can and should provide complete conviction under some circumstances even though most mathematicians would not consider these to be valid (e.g., Azzouni, 2013). Likewise, a string of valid inferences might not provide complete conviction. For an extended discussion of these issues, see Weber, Inglis, and Mejia-Ramos (2014).

Researchers have touted a number of benefits for having students base proofs on informal arguments, including improved success on proof writing tasks (Douek, 2007), greater learning opportunities (Weber, 2005), a better understanding of the proving enterprise in mathematics (Raman, 2003), a better appreciation of proof as a problem solving tool (Schoenfeld, 1991), and greater conviction in the propositions that are proven (Weber & Alcock, 2009).

Limitations to basing proofs on informal arguments

Duval (2007) cautioned that there is often a large gap between arguments and proofs; bridging this gap can be a difficult and cognitively complex task. To highlight this difficulty, Samkoff, Weber, and Lai (2012) asked eight research mathematicians at prestigious universities to prove the following claim: "Prove that $f(x)=sin(x)$ is not injective on any interval of length greater than π". All eight mathematicians drew a graph of the sine function and used this graph to convince themselves that the statement was true, yet producing a proof of this statement was deceptively difficult. Participants spent an average of 18 minutes completing the proof and only four of the proofs were fully valid. If translating an intuitive argument into a proof in calculus is challenging for research mathematicians, we can expect that this activity may be daunting for undergraduates. While the research literature contains numerous examples of students successfully basing proofs of informal arguments, there are also instances where students were unable to make this translation (e.g., Alcock & Weber, 2010; Pedemonte, 2001, 2002). Reflecting on her own teaching with diagrams, a common representation on which to base an informal argument, Alcock (2010) wrote:

> *Diagrams can provide insight, but it is not always easy for students to make detailed links between what is in the diagram and what is in a formal proof. This means that the step between seeing that a result must be true and proving it can seem insurmountable. (p. 232).*

Based on these findings, it seems overly optimistic to hope that most students can base their proofs off of informal arguments without greater instructional support. One goal of this paper is to describe the activities that successful mathematics majors engage in to write proofs based on informal arguments. Distilling the specific activities that students participate in to write proofs in this way may be a fruitful starting point for researchers hoping to design instruction that improves students' ability to produce proofs based on informal arguments.

Research on bridging the gap between argumentation and proof

In recent years, researchers concerned about the gap between informal arguments and proofs have begun to look at how this distance can be traversed. Much of the research can be divided into two categories: analyzing the types of arguments that are easier to translate into proofs and designing classroom environments that help bridge this gap.

In the first category, Pedemonte and her colleagues have conceptualized the distance between the informal arguments that students construct and the formal proofs that could result from those arguments (Pedemonte, 2007, 2008; Pedemonte & Reid, 2011). For instance, Pedemonte observed if the structural distance is too great—that is, if the methods of inference used in the informal argument are significantly different from the deductive inferences required in the corresponding formal proof—students will have trouble producing this proof (e.g., Pedemonte, 2007). Similarly, if the content distance is too great—that is, if the mathematical ideas in an informal argument and its corresponding proof differ—students will face similar difficulties in writing a proof (Pedemonte, 2001). This analysis has yielded useful insights into why students have difficulty basing their proofs off of informal arguments as well as what types of informal arguments are likely to serve as a good basis for a proof.

The second category of studies conceptualizes the role of the instructor in helping students build proofs of informal arguments, and includes creating instructional environments that encourage this behavior (e.g., Bartolini Bussi et al., 2007) and teacher moves that may facilitate students with this transition (e.g., Stylianides, 2007).

In this paper, we will explore how mathematics majors can and do bridge the gap between informal arguments in mathematical proofs. To investigate this broad issue, we will address the following questions:

(i) What activities do mathematics majors engage in when they successfully write a proof based on an informal argument?

(ii) To what extent can these activities account for their success?

The answer to these questions can inform instruction by highlighting what skills and practices students may need to learn to write proofs based on informal arguments.

Methods

Corpus of data used in this study

The data from this paper came from a large-scale study in which 73 mathematics majors were observed constructing 14 proofs. For the sake of brevity, we only report the details of the study germane to this paper. We recruited 73 mathematics majors from a large public university in the United States who had recently completed a second linear algebra course. Most of these participants were seniors. Participants met individually with an interviewer for two sessions that lasted approximately 90 minutes each. In one session, the participants worked on linear algebra proving tasks; in the other, they worked on proving tasks in calculus. In each session, participants were presented with a proving task that could be approached either syntactically or semantically (in the sense of Weber & Alcock, 2004). Participants were asked to "think aloud" as they completed this task and were told to write up a proof as if they were submitting it for credit on a course exam. Participants were given up to 15 minutes to complete each task. At any point during their work, the participants had access to a computer with a graphing calculator application that enabled participants to make basic calculations and view the graph of any function that they wished. This process was repeated until the participant had completed all seven linear algebra or calculus tasks.

This corpus yielded a total of 1022 proof attempts across the 73 participants. However, in this paper, we focus on the 37 proof attempts where participants provided an informal argument for why the statement was true. We used the following coding scheme to determine when an informal argument occurred. We flagged for each time a participant represented a concept or a situation. We coded the representation as syntactic if it involved stating the formal definition of a concept or expressing a situation algebraically. We coded the representation as semantic if the participant represented the concept in some other way, such as using a diagram, graph, or prototypical example. We flagged for each time a participant drew an inference (i.e., produced a new piece of information that they believed to be true). Whenever possible, we attributed each inference that a participant drew to a specific representation. If the participant drew an inference from a syntactic representation, we coded this as a syntactic inference. If the participant drew an inference from a semantic representation, we coded this as a semantic inference. (e.g., if a participant showed $f(x)$ was increasing by using algebraic manipulation to show $f'(x)>0$, that would be a syntactic inference. If the participant inferred this from looking at the graph of $f(x)$, that would be a semantic inference). We operationalized the notion of informal argument as any multistep argument concluding with the statement to be proven that contained at least one semantic inference. There were 37 such arguments in our dataset. There was a high level of inter-rater reliability (greater than .7) for each step in the coding process across the large dataset.

Analysis

Each final proof that the participants produced was coded as being valid or invalid. Two research assistants, who are not authors of this paper, coded each proof as valid or invalid. There was 96% agreement on their coding across the data set. Among the 37 proof attempts considered in this study, 14 were coded as valid and 23 were coded as invalid.

Following Pedemonte (2007), for each of the 37 proof attempts, we used the basic Toulmin (2003) scheme to analyze each inference that the participant drew in his or her informal argument and final proof. According to the basic Toulmin scheme, each inference (or mini-argument) contains three parts, the claim (C) being advanced, the data (D) used to support the claim, and the warrant (W) that necessitates how the claim follows from the data. In many cases, a warrant was not explicitly stated by the participant. In these cases, if possible, we would infer the warrant that the participant was using. (e.g., if the participant said, "a and b are negative so ab is positive", we could infer the warrant that connected the data, "a and b are negative", to the claim, "ab is positive", is the deductive warrant that "the product of two negative numbers is a positive number"). Comparing Toulmin schemes allowed us to notice differences between the participant's initial informal argument and their final proof.

For the 14 successful proof attempts, we used an open coding scheme in the style of Strauss and Corbin (1990) to categorize the ways that the mathematics majors attempted to transform their informal argument into a proof. This process yielded three categories of activity: syntactifying, rewarranting, and elaborating.[2] Once these categories were created and defined, we went through each of the 37 proof attempts, seeking out any evidence that participants attempted to engage in these activities.

4. Results

General observations

The 37 informal arguments analyzed for this report were produced by 22 participants. Twelve participants produced an informal argument on a single task and collectively produced two valid proofs on their 12 attempts. Ten participants produced informal arguments on multiple tasks in this study (collectively 25 across the ten participants) and produced proofs on 12 of these attempts. It is interesting to note that those who produced multiple informal arguments had a much larger success rate on writing formal proofs than those who produced only a single informal argument (48% vs. 17%). An informal argument was produced for nine of the fourteen tasks. The specific tasks for which students produced informal arguments are available in Table 1. This table details the number of informal arguments produced for each task and the number of these informal arguments that were accompanied by correct proofs.

[2] Two other categories of activities, termed *rearranging* and *consolidation*, emerged in the coding process. These activities involved reordering inferences and removing duplicate inferences from arguments, respectively. They are not discussed here because they occurred infrequently and had little bearing on student success with proofs.

Task	C1	C2	C3	C4	C5	C6	C7	L1	L2	L3	L4	L5	L6	L7
# of informal arguments	7	4	8	1	4	8	0	0	0	0	2	0	1	2
# of correct proofs	2	1	2	1	1	5	0	0	0	0	2	0	1	0

Table 1: Informal arguments by task

Categories of activity

Elaborating

We regarded elaborating as occurring if participants attempted to add more detail to the proofs that were being constructed. This occurred in two different ways: Participants would justify statements that they took for granted (D) in their informal arguments by adding supporting data and warrants (D_0 and W_0), or make explicit warrants that were initially implicit (W_i) in their informal arguments. We illustrate this in Figure 1.

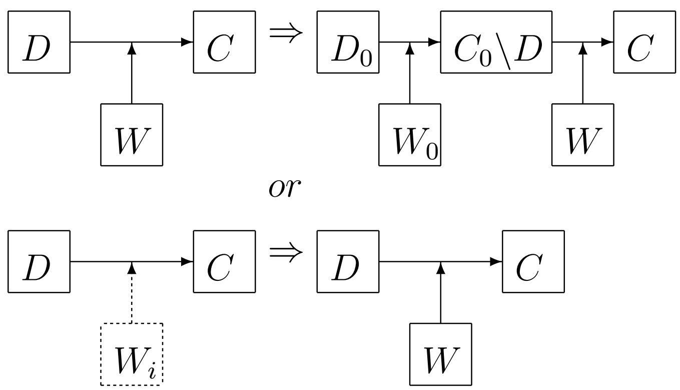


Figure 1. Elaborating

The following example illustrates the first type of elaboration: justifying claims initially taken for granted. It occurred during a participants work on problem C6 (Prove that $\int_{-a}^{a} \sin^3(x)\,dx = 0$ for any real number a)

Student A: It [*sin*^{3}*(x)*] must be an odd function. […] Right it'll be symmetrical across the identity line, which would mean that the integral from negative a to zero should be the negation of zero to a. And so it would be zero.

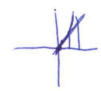
In this excerpt the participant has an informal argument that $\int_{-a}^{a} \sin^3(x)\,dx = 0$. Notice that within this argument the assertion that *sin*^{3}*(x)* is odd is treated as a known fact (data). Immediately following this informal explanation the participant begins to elaborate this statement by providing a justification for this assertion.

Student A: I'm trying to think how to show that sin of x cubed is odd. So basically I'd have to show that f of negative x has to equal negative f of x. Is that right... yes. So sin cubed of negative x... sine by definition is an odd function [writes *sin(-x)=-sin(x)*]. Uh Yeah. So sin cubed negative is equal to sin negative x times sin negative x which is equal to sin of x times sine of x times sin of x. Which is sin of x cubed. Quantity cubed. [writes:

$sin^3(-x)=sin(-x)sin(-x)sin(-x)=(-sin(x))(-sin(x))(-sin(x)) =-sin^3(x)$] So it's odd.

In the above excerpt the participant uses $sin(x)$ being odd (D) to justify that $sin^3(x)$ is odd (C) given a simple algebraic manipulation (W). In doing so he provides additional information regarding why the statement $sin^3(x)$ is odd is true. Student A shifts the starting point for the proof from $sin^3(x)$ is odd to $sin(x)$ is odd, which he believed to be more mathematically appropriate.

A student may also elaborate by replacing an implicit warrant in their informal argument with an explicit one in their formal proof. The following excerpt is taken from student B's work on problem C1 (Suppose $f(0)= f'(0))=1$. Suppose $f'(x)>0$ for all positive x. Prove that $f(2)>2$).

Student B: If the second derivative is greater than zero then f prime of x is increasing. So we know that f prime of zero equals one[draws: ]. So the derivative at zero equals one and the derivative is always increasing then the slope is greater than one after zero. Which means f of 1 is greater than one and f of 2 is greater than two. Well it makes sense.

In the above student B produced an informal argument that relied on a graph. Notice that he, among other things, argues that $f'(x)$ being increasing and $f'(0)=1$ (D) implies that $f'(x)>1$ for x>0 (C). The implicit warrant here is the definition of increasing. Later when he writes a formal proof this warrant is no longer implicit:

Student B: [saying what he writes] If f double prime of x greater than zero, then f prime x is increasing for all positive x. Thus for any x sub 1 comma x sub two in the interval zero to infinity such that x sub 2 is greater than x sub 1 f prime of x sub 2 is greater than f prime of x sub 1. f prime of zero equals one. Thus f prime of x sub 2 is greater than f prime of x sub 1 is greater than one. The derivative at any point greater than zero is greater than 1…

Notice that in his proof he uses the formal definition of increasing ($x_2>x_1 \Leftrightarrow f(x_2)>f(x_1)$). Since this was an implicit warrant in the informal argument and is now used explicitly as a sub-step in the proof, elaboration has occurred. Notice, however, that the underlying reasoning has not changed. So even though the proof involves taking smaller steps than the informal argument the path the reasoning follows is similar.

Syntactifying

A participant was coded as engaging in syntactifying when he or she attempted to take a statement in the informal argument that was given in what he or she perceived to be non-rigorous terms and translate it into what he or she considered to be a more appropriate representation system for proofs. Such actions included removing references to a diagram used in the informal argument and replacing them with more conventional mathematical terminology, or introducing algebraic or logical notation. In terms of Toulmin's scheme, we can regard syntactifying as translating the data (D), claim (C) and/or warrant (W) of an informal argument into corresponding data (D'), claim (C') and/or warrant (W') in another representation system, without intending to change the original meaning of D, C or W. We illustrate this with Figure 2.

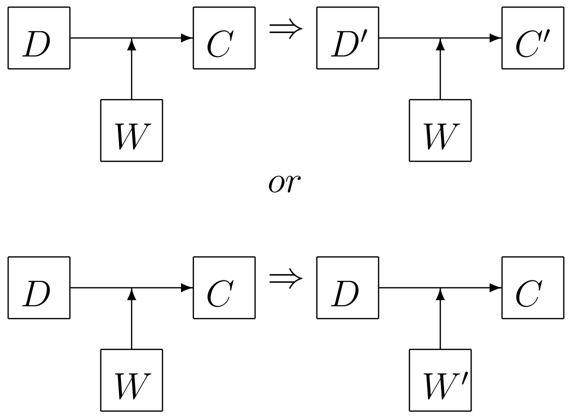


Figure 2. Syntactifying

The following informal argument occurred in student C's work on problem C3 (Prove the derivative of an even function is odd.)

Student C: Okay, Like okay, since it's symmetric about the y-axis, so it's like a mirror and then all the tangent lines, all the derivatives would be like some values [pointing at the left side of the graph of an even function] and then this would just, since it's a mirror would be the negative of them [pointing at the right side of the graph]. So it would be odd.

In the above excerpt student C draws a semantic inference to justify the result. She argues that since even functions are symmetric about the y-axis (D) the y-axis acts like a mirror (C). This mirror property is then used as data to justify that $-f'(a) = f'(-a)$ for all a, which is in turn used to conclude that the function would be odd. The warrants used are implicit and perceived from the graph of x^2, which is used as an example of a generic even function. In the continuation of the script while constructing a formal proof she engages in syntactifying when she shifts away from discussing tangents in terms of the graph.

Student C: How do I put that into words? [...] This is what we want f prime of negative x equals negative f prime of x. [writes $f'(-x)=-f'(x)$]. Okay, so if we take the derivative at negative [pointing at a tangent of the graph of an even function left of the origin], this would be the negative of f of x's derivative [pointing at a tangent of the graph right of the origin], which makes sense. So how do we get from f of negative x equals f of x [writes $f(-x)=f(x)$]? Use the definition? Okay lets try that. So, let's see, f prime of x equals. So by the definition of derivative, its like, as this approaches this point [drawing a sequence of points of the graph approaching a point in that graph] then the tan line of that [drawing the corresponding secant lines]. This is the limit at a. Either way, f of x minus f of a. over x minus a. [writes $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$] So, on once side [pointing at the graph with one hand and the algebraic expression with the other] it would be positive, and one side it would be negative. And that's what we want.

Notice that Student C first syntactifies both the end point and the starting point of the informal argument. She begins with the end point, stating that she is trying to show that $f'(-x)=-f'(x)$. This syntactifies her claim that the derivatives on one side are the negation of the other. She then syntactifies the initial data when she writes that $f(-x)=f(x)$. This analytic definition of even function replaces the graphical definition she used in her informal argument. Although the chain rule can be used to warrant going directly from the data to the claim, which would ignore her previous informal argument regarding mirror tangents, she instead begins to build a proof based on her informal argument. Student C uses her graph to express the slope of a tangent at a point algebraically as the limit of the slopes of secants. As such, she syntactifies the claim that tangents to the graph at points reflected over the y-axis would have slopes that had the same magnitude but different signs. By syntactifying these statements of her informal argument she moved from working with semantic (graphical) representations to syntactic (analytic) representations; and in doing so she shifted to a more appropriate representation system for presenting proofs.

Rewarranting

Many informal arguments employ warrants that are not permissible in a proof. Such warrants include perceptual reasoning and generalizing from examples. A participant was engaged in rewarranting if the participant tried to find a new, more appropriate reason for a claim than the one used in their informal argument. In terms of Toulmin's scheme, we can regard rewarranting as replacing a *personal warrant* (W) from the informal argument (i.e., a warrant that the participant believes is likely to yield or guaranteed to yield truth) with a *valid warrant* ($W^{\oplus}$) (i.e., a warrant that the participant believes is considered valid by the mathematical community). This is illustrated in Figure 3. Essentially, the participant attempts to write a sub-proof that shows the claim is a valid deductive consequence of the data[3].

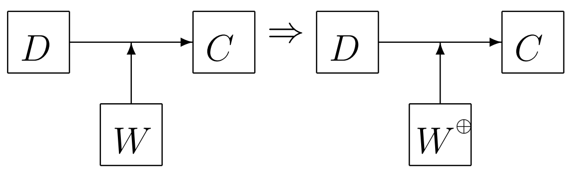


Figure 3. Rewarranting

Student D's work on problem C3 (Prove the derivative of an even function is odd) illustrates this activity.

Student D: f is even. So let me just draw what an even function might look like. So across the origin. [Draws an arbitrary function with reflectional symmetry about the x-axis]. […] So basically, if I look at a slope on this function [draws a tangent on the negative side of the previously drawn function], then I look on the other side [draws a mirror tangent on the positive side], it's the same slope but negative. So that's going to show that f prime is odd.

In the above excerpt student D draws an arbitrary even function and produces an informal argument similar to the one student C produced in the syntactifying section. In this argument he uses the arbitrary even graph to argue that corresponding tangents of even functions have slopes that are negatives of each other. This inference regarding tangents and their slopes is then used as data to support the claim that the derivative of an even function is odd. The

[3] Syntactifying a warrant differs from rewarranting since syntactifying does not change the underlying meaning of the warrant. For instance, replacing the semantic warrant "because the function $y=f(x)$ lies above the line $y=x$ in the Cartesian plane" with the syntactic warrant "because $f(x)>x$" is an example of syntactifying. Here the underlying meaning and usage has not changed, only the representation system used to communicate it. There may be cases where it is difficult to disambiguate syntactifying of a warrant and rewarranting. However, we did not encounter such cases in our data. The research team agreed regarding which translation activity occurred within particular excerpts.

implicit warrant used to support this claim is that slopes correspond to derivatives. In summary, the informal argument in the above excerpt uses a sequence of semantic inferences to link the data that *f(x)* is even to the claim that *f '(x)* is odd. Immediately following his informal argument student D begins to construct a plan for turning his informal argument into a proof:

Student D: So if I look at the definition of derivative as a slope and then I find the derivative on the negative side using the fact that it's even I should get the negative of the derivative, showing that f prime is odd.

In the above excerpt, student D expressed a plan for turning his informal argument regarding corresponding tangents into a formal proof. Student D explicitly makes this connection when he says "if I look at the definition of derivative as a slope."

Also, notice that he has constructed a plan for linking the newly syntactified data to the claim that the function is odd. This plan involves using substitution of the analytic definition of even to show that *f '(-x)=-f '(x)*. The plan involves linking the data that *f(x)* is even to the claim *f '(x)* is odd using a warrant that is different from the link used in his informal argument. If he had stopped at this point he would have been coded as attempting to rewarrant, since he has not yet implemented his plan. The subsequent excerpt details his execution of the plan.

Student D: So I'll say that f prime of negative x equals f of negative x minus f of negative x plus h over h equals the limit of this [writes: $f'(-x)=\lim_{h\to 0}\dfrac{f(-x)-f(-x+h)}{h}$] and it exists. And now because f is even I'm going to replace f of negative x with f of x. and I'm going to replace f of negative x plus h with f of x minus h [writes: $=\lim_{h\to 0}\dfrac{f(x)-f(x-h)}{h}$]. But now because I have x minus h this is the same thing as the definition of derivative, of the negative of the derivative [writes: $=-\left[\lim_{h\to 0}\dfrac{f(x)-f(x+(-h))}{-h}\right]$]. So I can just replace h with negative h. So the limit as h approaches zero is the same thing as the limit as negative h approaches zero [writes: $=-\left[\lim_{(-h)\to 0}\dfrac{f(x)-f(x+(-h))}{-h}\right]$]. And that by definition is negative f prime of x [writes: $=-f'(x)$]. And that means that it's odd.

In the above excerpt student D algebraically manipulates the limit definition of derivative to show that *f '(-x)=-f '(x)*. This changes the nature of the warrant that links the data that *f(-x)=f(x)* to the claim that *f '(-x)=-f '(x)*. The warrant in the original informal argument was semantic and based on the link between slope and derivative. It linked a graphical observation regarding mirror slopes (D) to the claim that f '(x) is odd (C). This warrant is replaced here by a string of algebraic manipulations. The new warrant is not simply a translation of the previous warrant into a new representation system that leaves the meaning of the warrant unchanged; it is a different route to linking the data and claim, one which is conducive to the analytic representation being used.

As illustrated above student D does not view these arguments as wholly separate. He views the submitted analytic proof detailed in the above excerpt as a translation of the informal argument. In moving from a graphical (semantic) representation to an analytic (syntactic) representation the warrants used need to shift in order to reflect the representation. This process of warrant replacement is what we term rewarranting.

In general, we noticed that rewarranting usually occurred after the data and claim had been syntactified. This is likely because statements represented syntactically invite valid deductive reasoning.

How all three activities contribute to translation

Students who attempted to participate in all three activities had a much greater rate of success with producing correct proofs (this is discussed in detail in the prevalence of activities section below). Thus, it is useful to illustrate how all three activities work in tandem to contribute to the translation of an informal argument into a proof. Below is student D's work on problem C6 (Prove that $\int_{-a}^{a} \sin^3(x)\,dx = 0$ for any real number a)

> Student D: Yeah, ok, so we are just going to use the fact that sin x is an even function [he means odd]. So if you look at this integral [shades the part of the *sin*^{3}*(x)* graph left of the origin] and you look at that [shades the part of the *sin*^{3}*(x)* graph right of the origin] they are the same area because sin is the same function. It's just negative. Yeah okay.

In the above excerpt Student D has an informal argument that $\int_{-a}^{a} \sin^3(x)\,dx = 0$, which relies on intuitions regarding the symmetry of the graph of *sin*^{3}*(x)*, which he sketched as he read the problem statement. Although he incorrectly identified the function as even instead of odd, he is still relying on the notion of odd symmetry when he says "they are the same area because sin is the same function. It's just negative." This is an indication that he recognizes the role symmetry plays in the problem, but has mislabeled the type of symmetry. In the continuation of the excerpt he corrects this mistake and uses the three activities to translate his informal argument into a proof.

> Student D: Lets just attempt to do the proof right now. We know sin is an even function, so sin of x equals... I'm sorry, sin is an odd function. Yeah, when I said even I meant odd. I'm just trying to confuse you.[…] It's an odd function, so negative sin of x equals sin of negative x [writes *-sin(x)=sin(-x)*]. Cube both sides and you get negative sin cubed equals sin cubed of negative x [writes *-sin*^{3}*(x)=sin*^{3}*(-x)*].

At this stage he has corrected his earlier mistake regarding labeling *sin*^{3}*(x)* as even instead of odd. He has also elaborated in a similar way to what Student A did in the elaborating section. That is, he is no longer treating *sin*^{3}*(x)* being odd as initial data but instead uses *sin(x)* being odd as initial data and cubing both sides as a warrant to justify that *sin*^{3}*(x)* is odd. Notice also that he is no longer referring to the symmetry in terms of the graph. He has instead expressed the oddness property in terms of algebraic notation and has therefore syntactified. In the continuation of the excerpt he begins to rewarrant the argument.

> Student D: And then I'm going to use this fact [points at previously shaded graph]. So the integral from negative a to a of sin cubed of x dx equals the integral from negative a to zero of sin cubed of x dx plus the integral from zero to a of sin cubed of x dx [writes $\int_{-a}^{a} \sin^3(x)\,dx = \int_{-a}^{0} \sin^3(x)\,dx + \int_{0}^{a} \sin^3(x)\,dx$]. And then for the one on the left I'm going to… flip it.

At this stage student D has used the "areas cancel" intuition from his informal argument to create a strategy for connecting the data that *sin*^{3}*(x)* is odd to the claim that

$\int_{-a}^{a} \sin^3(x)\,dx = 0$. In other words he has concluded from his informal argument that the right side and left side will cancel and has broken up the integral into the two pieces which he intends to show cancel each other. This can be thought of as syntactifying his implied both sides cancel inference. He also begins to consider rewarranting strategies when he states "And then for the one on the left I'm going to flip it." At this stage he has attempted to rewarrant by verbalizing an algebraic strategy but has not yet rewarranted.

Notice that when working with the graph the fact that the sides cancel can be inferred via a visual warrant. However, in an algebraic context there is no simple one step procedure that affirms this cancelation. So justifying the cancelation in this algebraic context requires a different warrant to the one used in the informal argument. In the continuation of the excerpt Student D works to generate this alternate justification.

Student D: I'm going to do a change of variables because I want to get sin cubed of negative x. So that it will cancel out with this one. So I'll say u equals negative x [writes: *u=-x*]. d u equals negative d x [writes du=-dx]. So then I'll get integral from negative-negative a to negative zero sin cubed of negative x. I mean negative u. replace dx with negative du.[writes $= \int_{-(-a)}^{-0} \sin^3(-u)(-du) + \int_0^a \sin^3(x)\,dx$] Okay. And re-write it once more. So Integral from a to zero of sin cubed of negative u. I'm just making this more complicated than it needs to be. Negative sin cubed of u... negative du plus the same thing [writes $= \int_a^0 -\sin^3(u)(-du) + \int$ "]. Yeah. And then when you flip a and b [we presume by a and b, student D was referring to the upper and lower bounds of integration] on the left side it just makes the whole thing negative [writes $= -\int_0^a \sin^3(u)(du) + \int_0^a \sin^3(x)\,dx$].[…] And remember x is a dummy variable. So they're the same thing. Yeah okay. [Writes *"=0"* and hands back paper.]

In the above excerpt student D rewarrants the cancelation inference from his informal argument by replacing it with a series of algebraic steps. Specifically u substitution and the oddness of $sin^3(x)$ were used to manipulate the left integral into one equivalent to the negation of the right integral. Student D then uses "*x* is a dummy variable" to warrant the fact that the two integrals actually cancel in spite of them having two different variables.

As illustrated by the above excerpts, each of the three translation activities played an important role in facilitating Student D's production of a correct final proof based on his informal argument. This occurred in spite of their being an error in his initial informal argument. It is likely that the process of translating the informal argument brought this error to the surface. So it may be the case that the translation activities, under certain circumstances, may function as mechanisms for error detection as well as mechanisms for translation of informal arguments.

Prevalence of these activities in successful proofs

In Table 2, we present the frequency with which a participant attempted to engage in these activities as a function of whether they were able to successfully produce a proof. As Table 2 illustrates, participants who successfully produced proofs were significantly more likely to engage in elaborating, syntactifying, and rewarranting, with the most pronounced difference occurring for rewarranting. Those who were successful in writing a proof usually engaged in all three activities, while those who were not successful rarely engaged in all three.

	Total Number	Elaborating	Syntactifying	Rewarranting	All three activities
Successful	14	11 (79%)	12 (85%)	12 (85%)	11 (79%)
Unsuccessful	23	12 (52%)	15 (65%)	9 (39%)	4 (17%)

Table 2: Relating the three activities to successful proving.

Slicing the data another way, there were 15 instances in which a participant engaged in all three activities, and they succeeded in writing a proof 11 times (73% of the time). Here it is interesting to note that in three of the four instances in which engaging in all three activities did not yield a proof, the participant produced other informal explanations in this study. Collectively, these three participants produced six other informal explanations. In all six of these cases, they were successful in producing proofs. Among the 22 instances in which a participant did not engage in all three activities, the participants only succeeded in writing a proof three times (14% of the time); in two of those successful instances, the proof that was produced did not appear to be based on the informal argument.

It is important to note that Table 2 examined whether a participant attempted to engage in the activity, not if they engaged in the activity successfully. Consequently, we believe a key factor in determining success in proof writing for these participants was their willingness to try to elaborate, syntactify, and rewarrant.

Discussion

The data in this paper contributes to the literature on bridging the gap between informal arguments and proofs. We highlighted three activities—elaborating, syntactifying, and rewarranting—that are used by students attempting to write a proof based on an informal argument. We used examples of student work to illustrate how each of the three activities contributed to the creation of a valid proof based on an informal argument. Elaboration adds additional details to an argument in part by justifying why assumed facts are true. This coincides with part (i) of Stylianides' (2007) framework, when elaboration shifts what initial data is used as a starting point for a proof. Syntactifying is used to translate data, claims and/or warrants stated in terms of informal representations and the participants natural language to the representation system of proof. If successful, this results in an argument that uses the appropriate representation system (part (ii) of Stylianides framework). Finally, rewarranting which corresponds to part (iii) of Stylianides framework, is used to replace personal warrants with valid ones.

We observed that there was a relative scarcity of informal arguments produced across this large data set (37 instances across 1022 proof attempts). We also noted that participants who produced multiple informal arguments were more likely to successfully produce proofs. From this observation, we conjecture that one reason that students have trouble bridging the gap between informal arguments and proofs is that they lack the experience of producing informal arguments. In this respect, we support research into the design of instructional environments that encourages students to do so (e.g., Bartolini Bussi et al., 2007).

We also observed that participants who engaged in syntactifying, rewarranting, and elaborating once their informal arguments were produced enjoyed far greater success in proof-writing than those who did not. Consequently we hypothesize that some of students' difficulties with bridging the gap between informal arguments and proofs is that students do not appreciate the importance of and are not able to successfully engage in elaborating, syntactifying, and rewarranting. Designing instruction where these activities are specifically targeted has the potential to improve mathematics majors' abilities to write proofs and would be a useful direction for future research.

Acknowledgments

This material is based upon the work supported by the National Science Foundation under grant DRL-1008641. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

References

Alcock, L. (2010). Interactions between teaching and research: Developing pedagogical content knowledge for real analysis. In R. Leikin & R. Zazkis (Eds.) *Learning through teaching mathematics*. Springer: Dordrecht.

Alcock, L., & Weber, K. (2010). Undergraduates' example use in proof production: Purposes and effectiveness. *Investigations in Mathematical Learning*, *3*(1), 1-22.

Azzouni, J. (2013). That we see some diagrammatic proofs are perfectly rigorous. *Philosophia Mathematica, 21*, 247-254.

Bartolini Bussi, M., Boero, P., Ferri, F., Garuti, R., & Mariotti, M. (2007). Approaching and developing the culture of geometry theorems in school. In P. Boero (Ed.) *Theorems in school: From history, epistemology, and cognition to classroom practice.* (pp. 211-217). Rotterdam: Sense Publishing.

Boero, P. (1999, July/August). Argumentation and mathematical proof: A complex, productive, unavoidable relationship in mathematics and mathematics education. *International Newsletter on the Teaching and Learning of Mathematical Proof.*

Douek, N. (2007). Some remarks on argumentation and proof. In P. Boero (Ed). *Theorems in School: From historic, epistemology, and cognition to classroom practice.* (pp. 137-162). Sense Publishers: Rotterdam.

Douek, N. (2009). Approaching proof in school: from guided conjecturing and proving to a story of proof construction. *Proceedings of the 19th ICMI Study Conference* (Vol, 1, pp. 142-147). Taipei, Taiwan.

Duval, R. (2007). Cognitive functioning and the understanding of mathematical processes of proof. In P. Boero (Ed). *Theorems in School: From historic, epistemology, and cognition to classroom practice.* (pp. 163-181). Sense Publishers: Rotterdam.

Garuti, R., Boero, P. & Lemut, E. (1998). Cognitive unity of theorems and difficulty of proof. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd PME Conference* (Vol. 2, pp. 345–352). Stellenbosh, South Africa.

Hart, E. (1994). A conceptual analysis of the proof writing performance of expert and novice students in elementary group theory. In J. Kaput, and Dubinsky, E. (Ed.), *Research issues in mathematics learning: Preliminary analyses and results* (pp. 49-62). Washington: Mathematical Association of America.

Iannone, P. & Inglis, M. (2010). Self-efficacy and mathematical proof: Are undergraduates good at assessing their own proof production ability? In *Proceedings of the 13th Conference for Research in Undergraduate Mathematics Education*. Raleigh, North Carolina.

Moore, R.C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics*, 27, 249-266.

Pedemonte, B. (2001). Some cognitive aspects of the relationship between argumentation and proof in mathematics. In M. van den Heuvel-Panhuizen (Ed.), PME-25, vol. 4, (pp. 33–40). Utrecht, Netherlands.

Pedemonte, B. (2002). Relation between argumentation and proof in mathematics: cognitive unity or break. In J. Novotná (Ed.) *Proceedings of the 2nd Conference of the European Society for Research in Mathematics Education* (Vol. 2, pp. 70-80). Czech Republic.

Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics, 66*(1), 23-41.

Pedemonte, B. (2008). Argumentation and algebraic proof. *ZDM*, *40*, 385-400.

Pedemonte, B., & Reid, D. (2011). The role of abduction in proving processes. *Educational Studies in Mathematics*, 76, 281-303.

Raman, M. (2003). Key ideas: What are they and how can they help us understand how people view proof? *Educational Studies in Mathematics, 52*, 319-325.

Samkoff, A., Lai, Y., and Weber, K. (2012). Mathematicians' use of diagrams in proof construction. *Research in Mathematics Education*, 14, 49-67.

Schoenfeld, A. (1991). On Mathematics as sense-making: An informal attack on the unfortunate divorce of formal and informal mathematics. In J.F. Voss, D.N. Perkins, and J.W. Segal, eds. *Informal Reasoning and Education*, (pp. 311-344). Lawrence Erlbaum: Hillsdale, NJ.

Strauss, A. & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques.* London: SAGE.

Stylianides, A. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education, 38*, 289-321.

Toulmin, S. (2003). *The uses of argument.* Cambridge: Cambridge University Press.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, *48*, 101-119.

Weber, K. (2003). Students' difficulties with proof. In A. Selden and J. Selden (Eds.) Research Sampler, 8. Available from the following MAA website on the teaching and learning of mathematics: http://www.maa.org/t_and_l/sampler/research_sampler.html. Last downloaded January 14, 2014.

Weber (2005). Problem-solving, proving, and learning: The relationship between problem-solving processes and learning opportunities in proof construction. *Journal of Mathematical Behavior*, 24(3/4), 351-360.

Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics, 56*, 209-234.

Weber, K., & Alcock, L. (2009). Semantic and syntactic reasoning and proving in advanced mathematics classrooms. Invited chapter on research on proof at the undergraduate level for M. Blanton, D. Stylinaiou, and E. Knuth (eds.) *The Teaching and Learning of Proof across the K-16 Curriculum*.

Weber, K. Inglis, M., & Mejia-Ramos, J.P. (2014). How mathematicians obtain conviction: Implications for mathematics instruction and epistemic cognition. Educational Psychologist, 49, 36-58.

Conceptualizing and reasoning with frames of reference

Surani Joshua
Arizona State University

Stacy Musgrave
Arizona State University

Neil Hatfield
Arizona State University

Patrick Thompson
Arizona State University

Though commonly used in math and physics, the concept of frames of reference is not described cognitively in any literature. The lack of a careful description of the mental actions involved in thinking within a frame of reference inhibits our ability to account for issues related to frames of reference in students' reasoning. In this paper we offer a theoretical model of mental actions involved in conceptualizing a frame of reference. Additionally, we posit mental actions that are necessary for a student to reason with multiple frames of reference. This theoretical model provides an additional lens through which researchers can examine students' quantitative reasoning.

Keywords: Frames of Reference, Quantitative Reasoning, Theoretical Perspective

Consider the following problems that students encounter routinely in high school:

- Bobby is 3 years older than Lucy. When Bobby is *x* years old, how old will Lucy be?
- A particular engine can propel a boat at a maximum of 32 miles per hour. The boat travels 30 miles upstream from Port Adele to Port Chimney and then back, at maximum speed. The captain dropped a branch in the water before starting, estimating the downstream current as 6 mph. Considering just travel time, how long will the round-trip take?
- Yolanda and Sydney ran in the same marathon. Sydney ran 5/3 times as fast as Yolanda. If Sydney finished the 26.2-mile race in 4 hours, what was Yolanda's average speed?

Students often struggle to manage the dual perspectives required in each task (Bowden et al., 1992; Panse, Ramadas, & Kumar, 1994; Monaghan & Clement, 1999); for instance, the first scenario provides a comparison of Bobby and Lucy's age relative to Lucy's age, then switches to describing Bobby's age from Bobby's perspective, and finally asks for Lucy's age relative to Bobby's. A student must similarly tease apart the ways in which the framing of information about quantities in a scenario switches between two frames in the other two examples. In our own work investigating teachers' meanings on similar tasks, we identified a need to isolate the type of reasoning involved in answering the above tasks within quantitative reasoning.

Our search of the literature provided just a few references, all in physics education, that deal with tasks of this nature (Bowden et al., 1992; Panse et al., 1994; Monaghan & Clement, 1999). In line with the physics terminology, we choose to describe the extra layer of complexity in the above problems as issues of "frames of reference". In this report, we introduce what we mean by *a conceptualized frame of reference* and *reasoning with frames of reference*, and explain why this is an area that deserves attention by the math education community.

A definition of the noun phrase "frame of reference" would suggest that a frame of reference is an object external to the person reasoning with it. Such a perspective does not align with our goal of describing what it might mean for an individual to conceptualize a frame of reference.

Therefore, we articulate the mental activity involved in conceptualizing and reasoning with frames of reference. While the *products* of the mental activity we describe align with the classical definition for frame of reference as a coordinate system or a system of measures, our emphasis is on the mental actions a student must employ to conceptualize a frame of reference. In particular, we use the phrase "frame of reference" to refer to a set of mental actions through which an individual might organize processes and products of quantitative reasoning (Thompson, 2011). As such, conceptualizing frames of reference and quantitative reasoning are interrelated, with frames of reference providing an additional lens with which to look at quantitative reasoning.

Conceptualizing a Frame of Reference

An individual can think of a measure as merely reflecting the size of an object relative to a unit or he can think of a measure within a system of potential measures and comparisons of measures. An individual conceives of measures as existing within a *frame of reference* if the act of measuring entails: 1) committing to a unit so that all measures are multiplicative comparisons to it, 2) committing to a reference point that gives meaning to a zero measure and all non-zero measures, and 3) committing to a directionality of measure comparison additively, multiplicatively, or both.

Committing to a Unit

As an example, a student can think about the measure "4.5 feet" in different ways. If the student focuses only on the value "4.5" and sees the unit as of secondary (or perhaps no) importance, there is no meaningful connection between the unit and the value for this student. In contrast, if the student sees a multiplicative relationship between the unit and the value, this provides a meaning for the measure. In this second case, "4.5 feet" is a length that is 4.5 times as long as the length of an object that is taken as a standard foot. A student who sees this relationship and the importance of unit in establishing meaning for each measure has taken the first crucial step towards conceptualizing a frame of reference.

Committing to a Reference Point

As a demonstration, consider the phrases "distance Ben walked" and "distance Ben walked from his house today". Both phrases describe quantities. The first phrase is vague and leaves a reader wondering if the quantity described is Ben's distance walked today, Ben's distance walked in his room, or the distance Ben walked since his birth. As such, the ambiguity in the phrase "distance Ben walked" creates ambiguity in the meaning of a measure. Saying the measure of "distance Ben walked" is *m* units fails to provide usable information for an individual trying to reason about the situation. Moreover, the vagueness of "distance Ben walked" would make it possible for an individual to inadvertently change his meaning for "distance Ben walked" while reasoning within a complex situation. He might define formulas or expressions to model the situation without understanding that his inconsistent meanings for the quantity make his model incoherent. Another possibility is that two individuals can read a situation and internally ascribe different meanings to the quantity "distance Ben walked" (by assigning different reference points) without realizing that they have done so. They might then discuss a problem and never realize that they are talking past one another because they are operating and speaking within two different frames of reference.

The specificity of "the distance Ben walked from his house today" makes it a more useful description of a quantity. In particular, we can confidently say that if the measure of the quantity "distance Ben walked from his house today" is zero, then Ben hasn't left his house today. Similarly, if the measure of that quantity is b units, $b > 0$, then Ben walked b units outside of his house. The commitment to a reference point attributes a meaning to every measure of the quantity and avoids the problems associated with ambiguity described above.

Committing to a Directionality of Measure Comparison

Consider a student designing a study to investigate the relationship between people's weight and Vitamin C consumption. The student plans to weigh each participant at the start and at the end of a two-month period, during which the participants will consume various amounts of Vitamin C daily. The student plans to examine the changes in the participants' weights. This student could imagine these comparisons in two different ways. If the student is oriented to think always of positive changes, then the student would make the following kinds of statements: "Josh is 6 pounds heavier at the end of the study" and "Wanda is 6 pounds lighter at the end of the study". In this case, the student has not thought of the comparison of measures within a frame of reference. Rather, the student adjusted his description so that a comparison always results in a positive number. Such adjustments constantly alter the directionality of comparison in order to think of the larger measure relative to the smaller. Should the student be asked what a participant's change of 1.5 pounds means, he could not say definitively whether the participant gained or lost weight.

Alternatively, suppose that the student commits to a comparison of "pounds heavier at the end than at the beginning". The additive comparison that the student has in mind is the post-weight minus the pre-weight. Here, the student would make statements like: "Josh is 6 pounds heavier" and "Wanda is –6 pounds heavier." In these statements, the student made use of the same direction in comparing the measures. Unlike the other case, the student now definitely interprets a change of 1.5 pounds as the individual weighed 1.5 pounds more at the end of the experiment than at the beginning.

We note that this commitment to the directionality is crucial when making multiple comparisons. For instance, most students can mentally shift between "heavier than" and "lighter than" when comparing two people's weights. However, the activity of comparing three or more people's weights proves much more difficult without committing to a directionality within a frame of reference.

An analogous commitment to a directionality when comparing measures holds for multiplicative comparisons. A student thinking within a frame of reference will be able to say "x is 3 times as large as y" and "y is one-third as large as x." A student who avoids committing to a directionality of comparison will only be able to make the first statement, possibly because of a discomfort with non-integers.

As a final note, we emphasize that we are *not* suggesting people should commit to a single reference point or a single directionality of comparison for their entire engagement in a task. In fact, it is often the case that while solving problems, an individual must conceptualize more than one frame of reference. The commitments we refer to only occur *within* the act of conceptualizing one frame of reference; a student can choose to work with a different frame of reference for the same quantity within one context, but while working within one frame, he works consistently with the choices of reference point and directionality of comparison he made in order to conceptualize that frame of reference. The conceptualization of multiple frames of reference then requires further mental actions to bring information from multiple frames together, an activity we call *reasoning with multiple frames of reference*.

Reasoning with Multiple Frames of Reference

We identify two types of reasoning that a student might employ when engaging in a task that necessitates conceiving of multiple frames of reference. The first type is that a student *coordinates multiple frames of reference* when he finds the relationship between one or more quantities' measures in two frames, such that he can determine a measure given in one frame from a measure given in the other. A student who has coordinated two frames of reference could, given an event's representation in one frame, represent that event in another frame in order to compare similar quantities. The second type of reasoning is that of a student *combining multiple frames of reference* when he considers two different quantities simultaneously within their respective frames of reference. Below we discuss the mental actions that are associated with each type of reasoning.

Coordinating Multiple Frames of Reference

A student coordinates multiple frames of reference by carrying out three sets of mental actions. She must first recognize the need to transform the measures of quantities measured in different frames of reference into measures measured in the same frame of reference. Second, a student must coordinate known *measures* of quantities in different frames in order to answer her question. Third, she must use those known measures to coordinate the *frames*.

We illustrate these mental actions in the context of the task presented in Figure **1**.

Two children, Alice and Bob, walk together from school to home. Alice starts measuring the distance they have traveled by counting the sidewalk squares they have crossed since passing the tree. Bob starts counting the sidewalk squares they have crossed since passing the stop sign and noticed that there were 3 squares between the tree and the sign. Let u be the number of sidewalk squares Alice has counted. Write an expression that gives Bob's count of sidewalk squares.

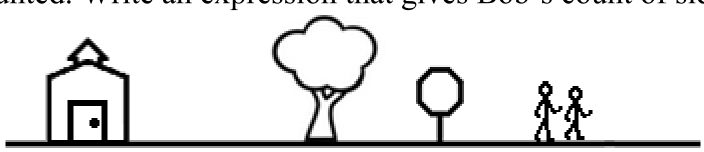


Figure 1. The Alice and Bob task.

Before beginning to coordinate multiple frames of reference, the student must first recognize that Alice and Bob each conceived of a comparable quantity within separate frames of reference. The student's recognition of this fact coincides with her envisioning what a distance of zero squares means to both Alice and Bob. The student must recognize that for Alice, "zero squares" means that the children are at the tree; likewise the student understands that "zero squares" to Bob means that the children are at the stop sign.

While the student could answer the prompt with a statement such as "Let v represent the number of squares that Bob has counted", she may feel the need to make use of the given definition for u. However, in attempting to use u, she imagines shifting from Alice's measurements (and frame of reference) to Bob's measurements (and frame of reference). The student anticipates that for the shift to work, she needs to find a commonality between the two frames of reference. The stem of the task in Figure **1** provides the student with a useful point of commonality between the frames. The student knows that Alice and Bob walk along the same path, counting the same sidewalk squares, with Alice starting to count at a tree and, three squares later, Bob starts counting at the stop sign. The stop sign serves as a point of commonality between the two frames of reference. The student knows that for Alice the stop sign is 3 squares from the tree. Likewise, she knows that Bob views the stop sign as 0 squares from itself. Thus, a measure of 3 squares for Alice, 3_{Alice}, is the same point along the path as 0 squares for Bob, 0_{Bob}. In establishing the link $3_{\text{Alice}} \equiv 0_{\text{Bob}}$, the student has coordinated known measures of comparable quantities from two different frames of reference. To fully coordinate the two frames of reference, the student must establish the relationship between the measure of a quantity in one frame of reference and the measure of the comparable quantity in other frame of reference. The student imagines that if Alice and Bob are at the stop sign and move forward one square, then both of Alice's and Bob's counts will increase by one; thus $4_{\text{Alice}} \equiv 1_{\text{Bob}}$. She anticipates that as they keep moving forward *any amount*, both Alice and Bob will increase their counts (e.g. they move forward another 0.5 squares, $4.5_{\text{Alice}} \equiv 1.5_{\text{Bob}}$). Likewise, she imagines that if Alice and Bob moved backward one square, their counts would increase by -1; thus $2_{\text{Alice}} \equiv -1_{\text{Bob}}$. In examining these connections based from the point of commonality, the student anticipates that Bob's count will always be 3 squares less than Alice's count. This supports the student in expressing Bob's count as $u - 3$ using Alice's frame of reference.

Coordinating multiple frames of reference is cognitively demanding. It requires that a student conceive each frame as a valid frame, be aware of the need to coordinate quantities' measures within them, and carry out the mental process of finding a relation between the frames while keeping all relative quantities and information in mind.

Combining Multiple Frames of Reference

A student *combines* frames of reference when she considers multiple quantities that exist within separate frames of reference simultaneously. Combining frames of reference is a separate act from coordinating frames of reference. When combining frames of reference, the student does not have a goal of expressing measures of one or more quantities in terms of different frames. Rather, the student's goal is simply to hold quantities from multiple frames of reference in mind concurrently. In the above section, the student would have combined Alice's frame of reference with Bob's frame of reference had she stated "Alice and Bob's home is both u squares from the tree *and* $u - 3$ squares from the stop sign". As a further example, coordinate systems allow us (mathematicians, teachers, and students) to represent the measures of different quantities simultaneously when those measures stem from potentially different frames of reference. Figure **2** shows two examples of this; a coordinate system combining Alice's and Bob's frames of reference as well as a coordinate system for air temperature in Fahrenheit and Celsius. Students' acts of joining two or more number lines that represent measures of (one or more) quantities in different frames of reference, and anticipating that ordered pairs (or n-tuples) give information about the measures in relation to each other, is the heart of combining multiple frames of reference.

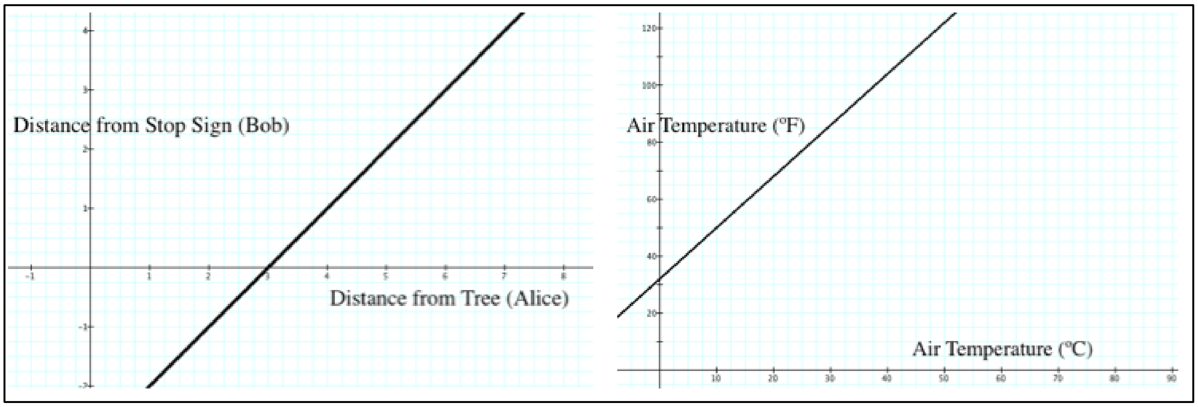


Figure 2. Examples of coordinate systems as combining multiple frames of reference.

Coordinating and Combining Multiple Frames of Reference

We note that when the student imagines a point (an ordered pair) along either line in Figure 2 as representing the measures of quantities in different frames of reference, she has combined the frames. If, however, she sees the line not just as representing a set of coordinated measures of quantities, but as a transformational relation between values of the quantities, she sees the graph as representing a functional relationship between the quantities.

Placing Our Theoretical Perspective amongst Others

Our interest in frames of reference and reasoning with frames of reference came about in an unexpected way. While analyzing teachers' responses to two items intended to target proportional thinking and rate of change, we found that teachers' responses to both items revealed struggles with coordinating quantities measured in what we came to realize were different frames of reference. Bowden et al. (1992) looked at the different approaches students used to analyze problems that involved an object moving inside another moving object (such as vector addition or proportional reasoning) and concluded that few students focused on "distinguishing frames of reference" (p.263-264). Bowden et al. noted that they attempted to characterize students' meanings based on their entire transcripts; however, Bowden et al. did not explain what they meant by "frames of reference". Rather, they used "frame of reference" as the possession of some object, e.g. "the frame of reference of the boat," Likewise they did not explain what they meant by "students' meanings." Monaghan and Clement (1999) wrote that computer simulations helped students develop mental imagery and ability to switch between frames of reference (e.g., as in a scenario involving a moving car and a plane flying overhead). However, they did not define or explain what they meant by frames of reference other than using pointers as Bowden et al. did. In further work they continued to use the construct of frames of reference without explicating what they meant by it (Monaghan & Clement 2000). Panse et al. (1994) investigated and identified "alternative [unproductive] conceptions" that students had about frames of reference, such as the idea that a frame of reference was a concrete object with boundaries or that a frame of reference is defined by the existence of a concrete object. While they did valuable work in describing alternative conceptions that hindered students' ability to reason about physical situations, they did not describe their normative conception of frames of reference. In all literature focusing on the idea of frames of reference or student thinking thereof, the authors presume that they and their readers share a common understanding of what "frames of reference" entails.

Expanding the Theory of Quantitative Reasoning

The few times an author (usually of a textbook) did explicitly describe what he or she meant by a frame of reference, the description focused on a frame of reference as an object or objects. Typical definitions range from "a coordinate system with a clock" (Young and Freedman 2011) to "a rigid system of 3 orthogonal rods welded together" (Carroll 2004) to "a set of observers at rest relative to each other" (de Hosson et al. 2010), with no further discussion about how students must conceptualize a frame of reference in order to reason with them. Such definitions support a student in focusing on the object of a frame of reference itself. In contrast, a key moment in developing our theory was when we began framing the question as "How does a student think about measures within a frame of reference?" As we said earlier in this manuscript, we defined a fully conceptualized frame of reference by stating that "An individual conceives of *measures as existing within a frame of reference* if the act of measuring entails [three commitments]." In other words, the mental actions, behaviors, and skills that we traditionally associate with someone "understanding frames of reference" (whatever that means) have nothing to do with how one thinks about frames of reference and everything to do with how one thinks about quantities.

In 1993 in his first article about quantitative reasoning, Thompson defined a quantity by saying that a "person constitutes a quantity by conceiving of a quality of an object in such a way that he or she understands the possibility of measuring it" (Thompson, 1993). He also added in an unpublished 1990 paper that this includes implicitly or explicitly thinking of appropriate units (Thompson, 1990). We find this to be a useful definition that provides a place to start thinking and talking about quantities, especially with younger children. However, curricula that seek to emphasize quantitative reasoning have highlighted further aspects of quantities, such as measuring a quantity in relation to a reference point (Carlson et al., 2013).

Therefore, we define the idea of a *framed quantity*, which refers to when a person thinks of a quantity with commitments to unit, reference point, and directionality of comparison. As an example, consider a person who thinks about measuring how far Yolie has traveled as she walks her dog, understanding that appropriate units would be linear units such as feet, meters, and miles. This person is thinking about a quantity. In contrast, a person thinking about measuring Yolie's displacement to the east from her front door in meters is conceiving of a framed quantity. Not only does this person's mental construction have all the aspects of a conceptualized quantity, but it also shows a commitment to a unit (meters), reference point (front door) and directionality of comparison (displacement to the east yields positive measures). In other words, the quantity is so well defined that any measure value contains all the necessary information to understand its meaning. If x = Yolie's displacement to the east from her front door (meters), then $x = 3$ means that Yolie is 3 meters to the east of her front door and $x = -5$ means that Yolie is -5 meters to the east of her front door (which could be interpreted as being 5 meters west of her front door if wanted, but also provides the same specific meaning without this reframing). No extra qualifiers are needed to make sense of the value, and there is a clear directionality of comparison: the value always says how much further in the eastern direction Yolie is than her front door.

In Thompson's 2011 paper he identified a number of dispositions that would aid students' construction of algebraic thinking from quantitative thinking, including a disposition to represent calculations in open form, propagate information, think with abstract units, and reason with magnitudes. To this list we can now add that a disposition to think about measures within a frame of reference, and specifically with a direction of comparison, aids students in algebraic thinking. In constructing formulas students are often perplexed as to how to choose between $a -$

b and $b - a$, or a/b and b/a. This confusion can now be explained by thinking about how students do or do not commit to a directionality of comparison. Let us think about a student that is comparing the heights of husbands and wives in a study of couples. If the student sometimes frames the results of the comparison as "the husband is 6 inches taller than the wife" and other times "the husband is 2 inches shorter than the wife" then he is internally switching between two quantitative operations, which have corresponding formulas of $h - w$ and $w - h$, where h represents the husband's height and w represents the wife's height, both in inches. Naturally such a student would have difficulty in developing a formula to compare heights. In contrast, another student may commit to a directionality of comparison by deciding the value of his measure will always describe 'how much taller the husband is than the wife'. Since such a commitment entails always using the same quantitative operation, such a student will have far less obstacles to describing his process in symbolic form as $h - w$.

Applications of the Frame of Reference Construct

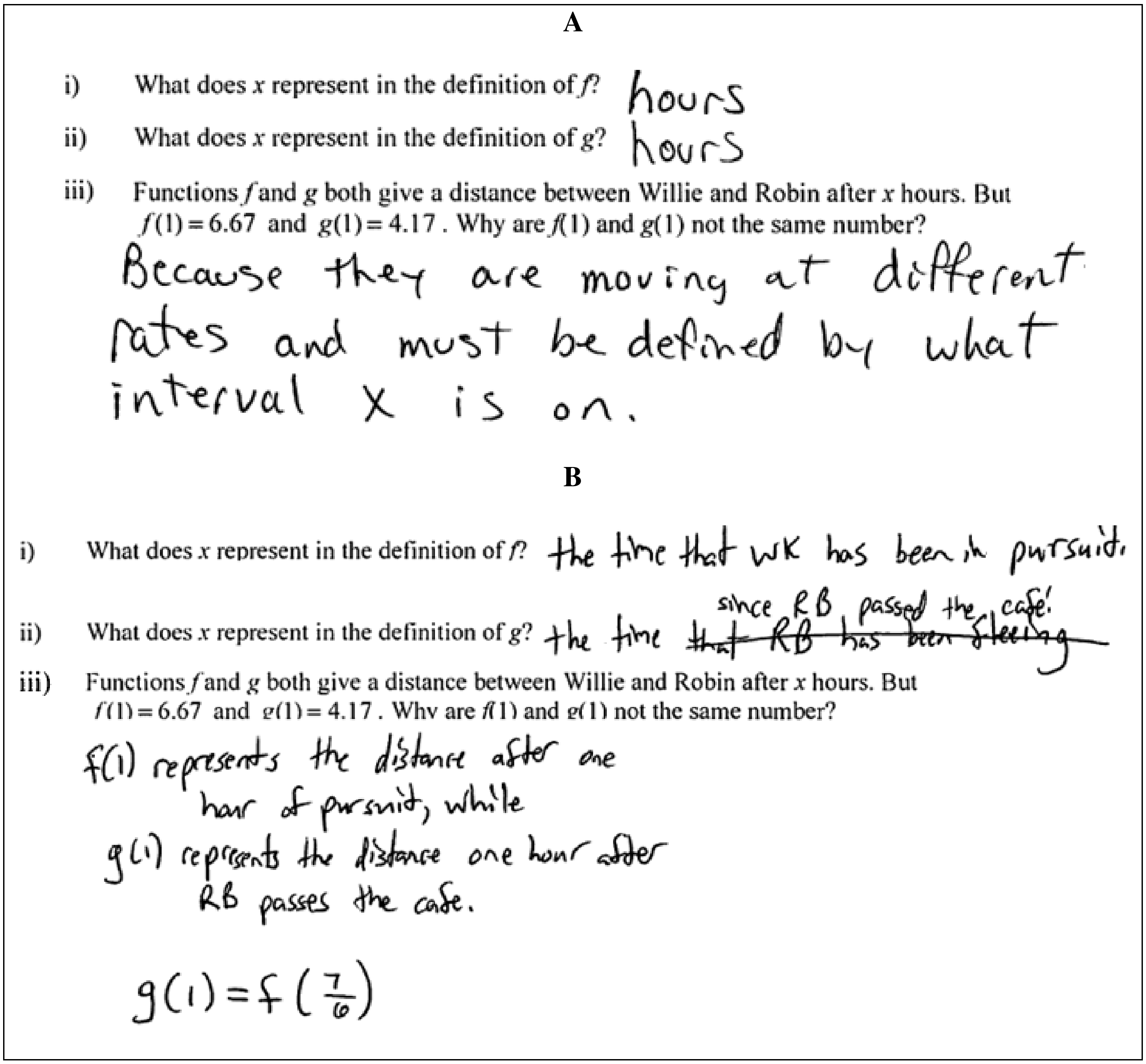
In our description of a conceptualized frame of reference and reasoning with multiple frames of reference, we deliberately used simplified tasks to illustrate the mental actions a person would have to take. However, we feel that the power of these constructs lie in their explanatory power in far more complex tasks. Below we illustrate two such tasks in detail, as well as sample responses from high school math teachers.

The task in Figure **3** presents two functions with non-equivalent rules (i.e., $f(x) = 15x\text{-}50/3$ and $g(x) = 15x\text{-}65/3$) to represent the same quantity (i.e., the distance between the two men). The fact that these two different functions can both represent the same quantity as a function of time creates difficulties for students (and teachers) trying to understand the scenario.

> Robin Banks ran out of a bank and jumped into his car, speeding away at a constant speed of 50 mi/hr. He passed a café in which officer Willie Katchim was eating a donut. Officer Katchim got an alert that Robin had robbed the bank, jumped into his patrol car, and chased Robin at a constant speed of 65 mi/hr. Willie started 10 minutes after Robin passed the café.
>
> Here are two functions. They each represent distances between Willie and Robin.
>
> $f(x) = 65x - 50\left(x + \frac{1}{6}\right),\ x \geq 0.$ i) What does x represent in the definition of f?
>
> $g(x) = 65\left(x - \frac{1}{6}\right) - 50x,\ x \geq 1/6.$ ii) What does x represent in the definition of g?
>
> iii) Functions f and g both give a distance between Willie and Robin after x hours.
> But $f(1) = 6.67$ and $g(1) = 4.17$. Why are $f(1)$ and $g(1)$ not the same number?

Figure 3. Robin Banks Task. Adapted from Foerster, (2006). © 2014 Arizona Board of Regents. Used with permission.

A person who can both conceptualize and coordinate frames of references, however, can see that this seeming paradox is resolved when one acknowledges that all measurements are taken from some reference point. Willie's distance from the café is $65x$ miles where x is Willie's travel time in hours, and Robin's distance from the café is $50x$ miles where x is Robin's travel time in hours. However, the x's in these expressions have different meanings because they are measured from different reference points: the moment when Willie left the café and the moment when Robin left the café. To make a comparison of these two distances requires coordinating the two frames and re-expressing either measure in the other's frame. The distance between the men as described by $f(x)$ is the result of re-expressing Robin's distance from the café using Willie's "stopwatch", or frame, because at every point in time Robin has driven 1/6 hours more than Willie. Likewise, the distance between the men as described by $g(x)$ stems from re-expressing Willie's time using Robin's "stopwatch", or frame, because at every point in time Willie has driven 1/6 hours less than Robin.



A

i) What does x represent in the definition of f? hours

ii) What does x represent in the definition of g? hours

iii) Functions f and g both give a distance between Willie and Robin after x hours. But $f(1) = 6.67$ and $g(1) = 4.17$. Why are $f(1)$ and $g(1)$ not the same number?

Because they are moving at different rates and must be defined by what interval x is on.

B

i) What does x represent in the definition of f? the time that WK has been in pursuit.

ii) What does x represent in the definition of g? the time ~~that RB has been fleeing~~ since RB passed the cafe!

iii) Functions f and g both give a distance between Willie and Robin after x hours. But $f(1) = 6.67$ and $g(1) = 4.17$. Why are $f(1)$ and $g(1)$ not the same number?

f(1) represents the distance after one hour of pursuit, while g(1) represents the distance one hour after RB passes the cafe.

$g(1) = f\left(\frac{7}{6}\right)$

Figure 4. Sample responses to Robin Banks task

Figure 4 displays two sample responses to the Robin Banks task given by high school math teachers. In Figure 4A parts i) and ii) the respondent did not think about the quantities with respect to a reference point, and so had no way to answer part iii) meaningfully. In contrast, in Figure 4B we see in parts i) and ii) that the respondent conceptualized the quantities with respect to specific reference points, and was also able to correctly coordinate the two frames in part iii).

Our frames of reference construct is also useful for examining individuals' struggles with situations devoid of motion. As an example, consider the task in Figure 5 that asks the reader to compare consecutive changes in the interval [1, 2].

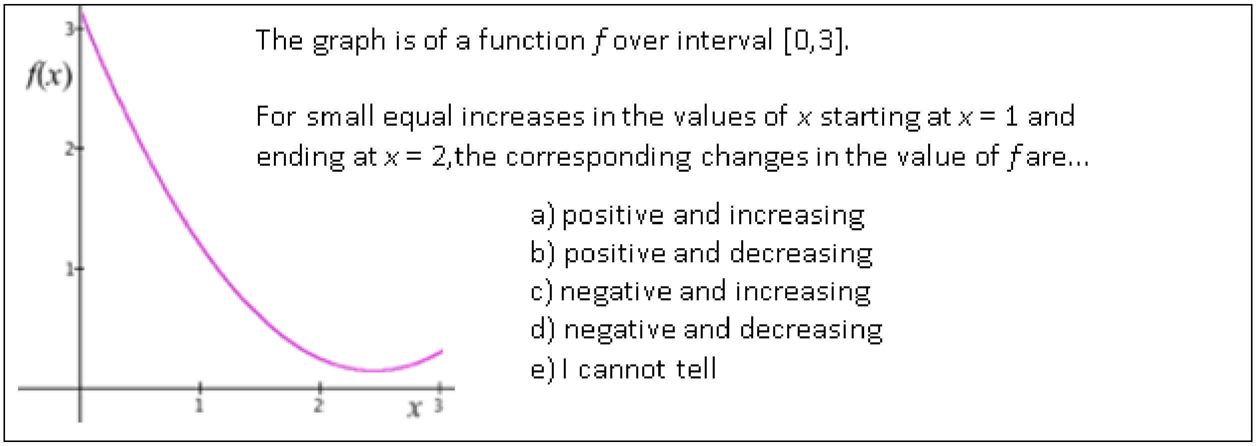


Figure 5. Comparing Changes Task. © 2014 Arizona Board of Regents. Used with permission.

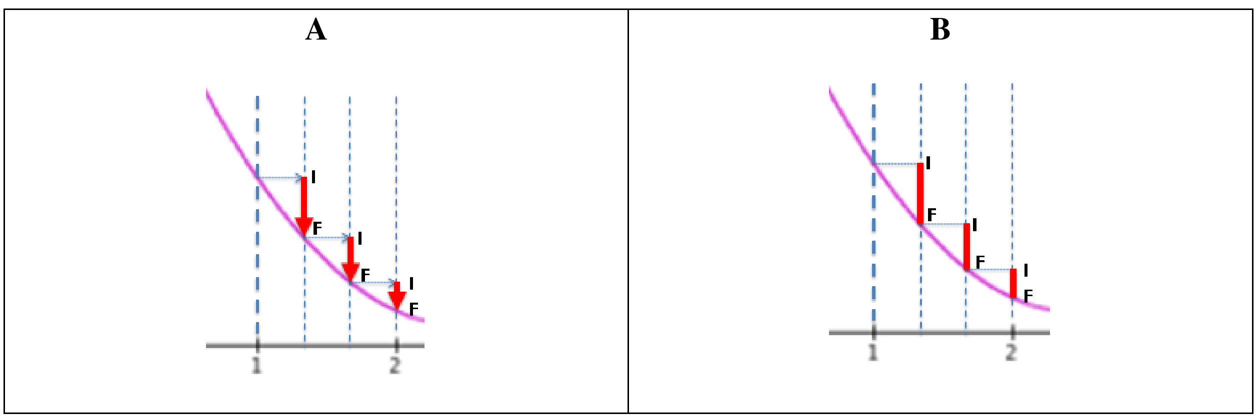


Figure 6. Different Visualizations of the Comparing Changes Task

This task proves challenging for people who do not think about changes within a frame of reference – specifically, people who do not maintain a directionality of comparison. Consider two hypothetical students: Dean who chooses option d) and Cathy who chooses option c). Assume both students understand the directionality of changes well enough to visualize changes as in Figure 6A.

Dean says that the changes are negative and decreasing because he has inadvertently switched the direction of his comparison between deciding "the changes are negative" and "the changes are decreasing." To determine that the changes are negative, he is engaging in a quantitative operation that we can formulize as [final *y*-value] – [initial *y*-value] and obtains a negative value for each. However, in deciding that the changes are decreasing, he is really only considering the magnitude of those changes, essentially switching his mental image to that shown in Figure 6B and engaging in a quantitative operation that we can formulize as [initial y-value] – [final y-value]. In comparison, Cathy says the changes are negative and increasing because she has maintained her directionality of comparison. For both her “changes are negative” and “changes are increasing” decisions, she engages in a quantitative operation that can be formulized as [final y-value] – [initial y-value]. We gain insight into individuals’ difficulties with this task by noticing a lack of commitment to directionality of comparisons.

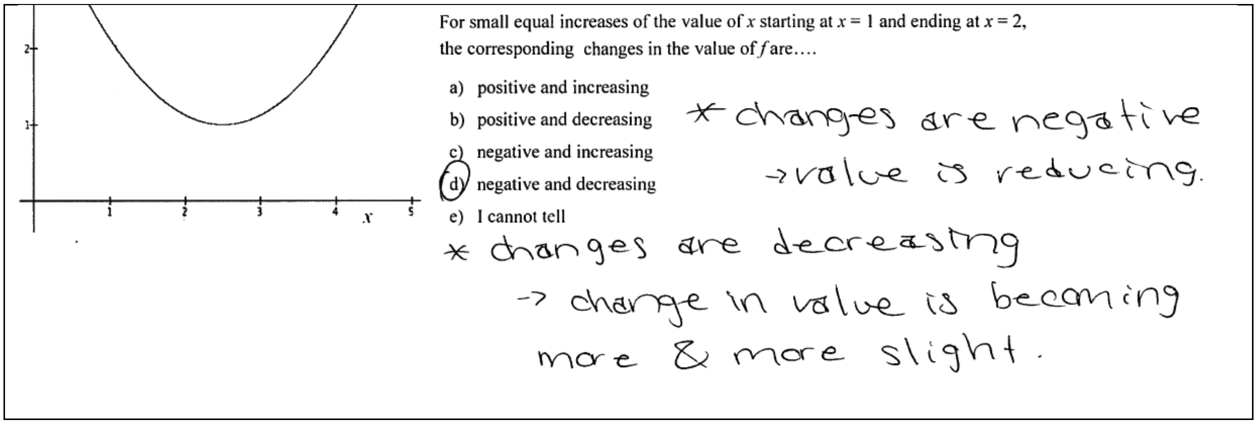


Figure 7. Sample response to Comparing Changes task

Figure 7 displays a sample response to the Comparing Changes task given by a high school math teacher. Note that the teacher’s justification for his comment “changes are negative” refers to a directionality: “value is reducing.” However, his comment “changes are decreasing” uses the language “changes in value are becoming more and more slight”, which we see as a strong indication that the teacher suddenly switched to looking at magnitudes.

Discussion

The above are two examples where the constructs of a conceptualized frame of reference and reasoning with multiple frames of reference have explanatory power and potential for improving instruction. As we developed our descriptions of these constructs, we started to see applications in a variety of other domains. Below we give brief descriptions of some of these domains and where we see potential for future research and teaching.

Personal experiences in teaching pre-calculus and calculus had shown us that students frequently conflate the value of a quantity and a change in that quantity, which leads to difficulties in understanding the ideas of change, slope, constant rate of change, and rate (derivative) functions. This confusion may be explained by a lack of attention to reference point for each measure; if a student does not commit to a reference point when measuring a quantity, there is little meaningful difference between the measure of the total quantity and a change in that quantity over a given interval. On the other hand, developing the idea that the total quantity is really a change from (a reference point of) zero provides parallel ideas with which to distinguish the two. Highlighting reference point commitment in teaching and discussion may help to alleviate this confusion.

Students frequently categorize all motion within a false dichotomy of “real motion” vs. “imagined motion”, where an object is only “really moving” if it is moving with respect to the surface of the Earth, and the measure of its speed or velocity is only “real” if measured with respect to the surface of the Earth (Panse et al. 1994). This hinders their ability to deal with relative motion tasks and has been a focus of study in physics education (Monaghan and Clement 1999). For example, students cannot accept that a bike moving 15mph towards a sign is also moving 5mph with respect to a walker and moving backwards with respect to a car. While Monaghan and Clement worked on developing their students' visual imagery, we believe that teaching students about conceptualizing all quantities as measured with specific reference points, and comparing quantities with specific directionalities of comparison, may prove beneficial.

This common student struggle with "real" versus "imagined" motion stems from a lack of understanding of the fundamental physics principle of relativity (Bandyopadhyay 2009) that states that there can be no way of verifying that any reference frame (or object) is at absolute rest, and therefore the entire notion of absolute rest should be abandoned. We believe emphasizing that a reference point is mandatory for any measure to be meaningful can provide a backdrop for students to also accept that what we talk about as motion measure in the real world always comes with its implicit assumption of a reference point (the surface of the Earth), and that if all reference points are arbitrary then the surface of the Earth is as well.

One of the most common struggles students have in physics is in understanding the concepts of velocity and acceleration. For example, researchers have found it extremely difficult to change the student perception that a positive acceleration means an object must be speeding up (when in fact it may be going from -5mph to -2mph, meaning it is slowing down but increasing in velocity). We have found in personal conversations that even professors who are known for their work in physics education have been teaching students that an object going from -10mph to -20mph means that "the velocity is increasing in the negative direction" probably to deal with these types of misunderstandings. But not only are such descriptions physically and mathematically inaccurate, they result in descriptions that are incompatible with observations about change and rate of change that can be derived from calculus. We believe that teaching students about a commitment to directionality of comparison is far more consistent and fruitful way to approach these concerns.

Panse et al. wrote a detailed description of seven alternative conceptions that students have about reference frames (Panse et al. 1994). Alternative conceptions 1, 2, 3, 4, and 6 are the consequences of seeing a reference frame as a physical object, while alternative conceptions 5 and 7 are the consequences of not fully understanding the principle of relativity. As we developed our constructs we identified an eighth alternative conception: the idea that a frame of reference is useful primarily (or only) for an observer that remains at the origin of the frame's coordinate system. We see the potential to reduce the number of students that develop all eight alternative conceptions in discussing frames of reference with students only in terms of three commitments on the part of the observer.

We are grateful to an audience member at our presentation of this paper at the RUME 18 conference, who offered the idea of electric potential as another concept we can reconceive through our constructs for frames of reference. It is true that students struggle with the idea of electric potential, and our minds immediately went to the struggles that physics and engineering students have with Kirchoff's second laws for circuits. Briefly stated, Kirchoff's second law states that the sum of the changes in electric potential around any loop in a closed circuit must be zero. Students often struggle with how to apply the rule because they feel a need to know where in the circuit the potential is "really zero" so that they can start their calculations there, not understanding that (like absolute rest) there is no such thing as absolute zero electric potential. These student difficulties may be alleviated by the same measures that help students to understand the principle of relativity in motion.

We believe that research on frames of reference and student thinking about frames of reference is warranted by the difficulties that students have with "typical" frames of reference problems. We think that the framework *conceptualized frame of reference* that we proposed offers new insight on student difficulties and contributes to a foundation for further research.

References

Bandyopadhyay, A. (2009). Students' ideas of the meaning of the relativity principle. European Journal of Physics, 30, 1239.

Bowden, J., Dall'Alba, G., Martin, E., Laurillard, D., Marton, F., Masters, G., Walsh, E. (1992). Displacement, velocity, and frames of reference: Phenomenographic studies of students' understanding and some implications for teaching and assessment. *American Journal of Physics 60*, 262-268.

Carlson, Marilyn, Oehrtman, Michael & Moore, Kevin. (2013). Precalculus: Pathways to Calculus, A Problem Solving Approach (4th Ed.). Rational Reasoning.

Carroll, S. M., & Traschen, J. (2005). Spacetime and geometry: An introduction to general relativity. *Physics Today*, *58(1)*, 52.

De Hosson, C., Kermen, I., & Parizot, E. (2010). Exploring students' understanding of reference frames and time in galilean and special relativity. *European Journal of Physics, 31,* 1527.

Foerster, P.A. (2006). *Precalculus with trigonometry: Concepts and applications* (2nd ed.). Emeryville, CA: Key Curriculum Press.

Thompson, P. W. (1990). *A theoretical model of quantity-based reasoning in arithmetic and algebraic*. San Diego State University, Center for Research in Mathematics & Science Education.

Ji Shen, Ji & Confrey, Jere (2010) Justifying alternative models in learning astronomy: A study of K-8 science teachers' understanding of frames of reference. (2010). *International Journal of Science Education, 32(1),* 1-29.

Monaghan, J., & Clement, J. (1999). Use of a computer simulation to develop mental simulations for understanding relative motion concepts. *International Journal of Science Education, 21(9)*, 921-944.

Monaghan, J., & Clement, J. (2000). Algorithms, Visualization and Mental Models: High School Students' Interactions with a Relative Motion Simulation. Journal of Science Education and Technology, 9(4), 311-325.

Panse, S., Ramadas, J., & Kumar, A. (1994). Alternative conceptions in Galilean relativity: frames of reference. *International Journal of Science Education, 16(1)*, 63-82.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education*. WISDOMe Mongraphs (Vol. 1, pp. 33-57). Laramie, WY: University of Wyoming.

Thompson, P. W. (1990). A theoretical model of quantity-based reasoning in arithmetic and algebraic. Unpublished manuscript, Center for Research in Mathematics & Science

Education. Available at http://pat-thompson.net/PDFversions/ATheoryofQuant.pdf

Young, H., Freedman, Roger A., & Ford, A. Lewis (2011) University Physics with Modern Physics. Addison-Wesley.

A Perspective for University Students' Proof Construction

John Selden
New Mexico State University

Annie Selden
New Mexico State University

This theoretical paper suggests a perspective for understanding university students' proof construction. It is based on the ideas of conceptual and procedural knowledge, explicit and implicit learning, cognitive feelings and beliefs, behavioral schemas, automaticity, working memory, consciousness, and System 1 and System 2 cognition. In particular, we discuss proving actions, such as the construction of proof frameworks that could be automated, thereby reducing the burden on working memory and enabling university students to devote more resources to the truly hard parts of proofs.

Key words: proof construction, behavioral schemas, automaticity, consciousness, System 1 and System 2 cognition

Introduction

In this theoretical paper we suggest a perspective for understanding university mathematics students' proof constructions and for improving and facilitating their abilities and skills for constructing proofs We are interested in how various types of knowledge (e.g., implicit, explicit, procedural, conceptual) are used during proof construction, in how such knowledge can be constructed, and in how one can control and direct one's own thinking. If that were better understood, then it might be possible to better facilitate university students' learning through doing, that is, through proof construction experiences. Although one can learn some things from lectures, this is almost certainly not the most effective, or efficient, way to learn proof construction, which is a kind of activity. Indeed, inquiry-based transition-to-proof courses seem to be more effective than lecture-based courses (e.g., Smith, 2006). In this paper, we are referring just to inquiry into proof construction, not into theorem or definition generation, although these are also interesting areas of study. These ideas emerged from an ongoing sequence of design experiment courses meant to teach proof construction in a medium-sized US PhD-granting university.

The Courses

There were two kinds of design experiment courses. One kind was for mid-level undergraduate mathematics students and was similar (in purpose) to transition-to-proof courses found in many U.S. university mathematics departments (Moore, 1994). In the U.S., such courses are often prerequisite for 3rd and 4th year university mathematics courses in abstract algebra and real analysis. The other, somewhat unusual, kind of course was for beginning mathematics graduate students who felt that they still needed help with writing proofs. The undergraduate course had from about 15 to about 30 students and the graduate course had between 4 and 10 students. Both kinds of course were taught from notes and devoted entirely to students attempting to construct proofs and to receiving feedback and advice on their work. In order to include the kinds of proofs found in typical subsequent courses and to provide students with as many different kinds of proving experiences as possible, both courses included a little sets, functions, real analysis, and algebra. The graduate course also included some topology. More information on the graduate course can be found in Selden, McKee, and Selden (2010, p. 207).

Introductory Psychological Considerations

Much has been written in the psychological, neuropsychological, and neuroscience literature about ideas of conceptual and procedural knowledge, explicit and implicit learning, automaticity, working memory, consciousness, beliefs and feelings such as self-efficacy, and System 1 (S1) and System 2 (S2) cognition (e.g., Bargh & Chartrand, 2000; Bargh & Morsella, 2008; Bor, 2012; Cleeremans, 1993; Hassin, Bargh, Engell, & McCulloch, 2009; Stanovich, 2009; Stanovich & West, 2000).

In trying to relate these ideas to proof construction, we have discussed procedural knowledge, situation-action links, and behavioral schemas (Selden, McKee, & Selden, 2010; Selden & Selden, 2011). However, more remains to be done in order to weave these ideas into a coherent perspective.

In doing this, two key ideas are working memory and the roles that S1 and S2 cognition can play in proof construction. *Working memory* includes the central executive, the phonological loop, the visuospatial sketchpad, and an episodic buffer (Baddeley, 2000) and makes cognition possible. It is related to learning and attention and has a limited capacity which varies across individuals. When working memory capacity is exceeded, errors and oversights are likely to occur. The idea behind S1 and S2 cognition is that there are two kinds of cognition that operate in parallel. *S1 cognition* is fast, unconscious, automatic, effortless, evolutionarily ancient, and places little burden on working memory. In contrast, *S2 cognition* is slow, conscious, effortful, evolutionarily recent, and puts considerable call on working memory (Stanovich & West, 2000). Also, System 2 cognition is thought to monitor System 1 cognition and to sometimes take over cognition when System 1 appears to be going astray. Of the several kinds of consciousness, we are referring to phenomenal consciousness—approximately, awareness of experience.

In both proving and learning to prove, it appears to be important for both teachers and students to understand the various kinds of progress to be made and the kinds of tasks to be performed. We have divided these into two parts, one more directly related to producing a proof text, and the other more related to the psychological influences on cognition and the mind when producing that text. We turn now to the first of the two parts of the proposed perspective.

Part 1: Producing the Proof Text

The Genre of Proofs

Just as there are distinctive features of poems and news articles that constitute those specific genres, there are a number of distinctive features that seem to commonly occur in published proofs. These features tend to reduce unnecessary distractions to validation (reading/reflecting to judge correctness) and thus increase the probability that any errors will be found, thereby improving the reliability of the corresponding theorems. Proofs are not reports of the proving process, contain little redundancy, and contain minimal explanations of inferences. They contain only very short overviews or advance organizers and do not quote entire statements of previous theorems or definitions that are available outside of the proof. Symbols are generally introduced in one-to-one correspondence with mathematical objects. For example, one does not say, as students sometimes do, "Let $x \in R$. Now let $y = x$." Finally, proofs are "logically concrete" in the sense that, where possible, they avoid quantifiers, especially universal quantifiers, and their validity is often seen to be independent of time, place, and author. Details can be found in Selden and Selden (2013).

Structure in Proofs

A proof can be divided into a formal-rhetorical part and a problem-centered part. The *formal-rhetorical* part is the part of a proof that depends only on unpacking and using the logical structure of the statement of the theorem, associated definitions, and earlier results. In general, this part does not depend on a deep understanding of, or intuition about, the concepts involved or on genuine problem solving in the sense of Schoenfeld (1985, p. 74). Instead it depends on a kind of "technical skill". We call the remaining part of a proof the *problem-centered* part. It is the part that *does* depend on genuine problem solving, intuition, heuristics, and a deeper understanding of the concepts involved (Selden & Selden, 2011).

Perhaps the most common feature of writing the formal-rhetorical part of a proof or subproof is what to do when the theorem statement starts with a universal quantifier, such as, "For all $x \in X$ …". One normally starts its proof with "Let $x \in X$", meaning that the variable x in the statement will be regarded in the proof as "fixed, but arbitrary", that is, as a single unspecified constant. This facilitates constructing proofs about infinite sets and changes most logic required from predicate to propositional calculus, which we think is closer to common sense reasoning.

We have noticed informally that a considerable number of our beginning transition-to-proof course students tend not to carry out the above action (of considering x as fixed, but arbitrary). Also, we have reported on an interview with a returning graduate student, Mary, who said that because of her real analysis teacher's instructions, she (successfully) carried out the action of considering a fixed, but arbitrary $\varepsilon>0$ in her proofs and even appended a reason why it was needed. That is, she would write, at the end of her proofs, "Because ε was arbitrary, the theorem has been proved for all ε." She reported to us that she did not feel that doing so was appropriate for about half a semester (Selden, McKee, & Selden, 2010, p. 209). Mary's difficulty suggests that some ideas in beginning proof construction may be adopted only slowly by some students even when they carry out the associated actions and provide appropriate warrants.

Proof Frameworks

A major feature that can help one write the formal-rhetorical part of a proof is what we have called a *proof framework*,[1] of which there are several kinds, and in most cases, both a first-level and a second-level framework. For example, given a theorem of the form "For all real numbers x, if $P(x)$ then $Q(x)$", a first-level proof framework would be "Let x be a real number. Suppose $P(x)$. … Therefore $Q(x)$," with the remainder of the proof ultimately replacing the ellipsis. A second-level framework can often be obtained by "unpacking" the meaning of $Q(x)$ and putting the second-level framework between the lines already written for the first-level framework. Thus, the proof would "grow" from both ends toward the middle, instead of being written from the top down. In case there are subproofs, these can be handled in a similar way. A more detailed explanation with examples can be found in Selden, Benkhalti, and Selden (2014). A proof need not show evidence of a proof framework to be correct. However, we have noticed that use of proof frameworks tends to help novice university mathematics students write correct, well-organized, and easy-to-read proofs (McKee, Savic, Selden, & Selden, 2010).

[1] We first introduced the idea of proof framework in Selden and Selden (1995), but have expanded considerably on this idea since then.

Operable Interpretations

Another feature that can help one write the formal-rhetorical part of a proof is converting definitions and previously proved results into *operable interpretations*. These interpretations are similar to Bills and Tall's (1998) idea of operable definitions. For example, in the courses described above, given a function f: $X \rightarrow Y$ and $A \subseteq Y$, we define $f^{-1}(A) = \{ x \in X \mid f(x) \in A\}$. An operable interpretation would say, "If you have $b \in f^{-1}(A)$, then you can write $f(b) \in A$ and vice versa." One might think that this sort of translation into an operable form would be unnecessary or easy, especially because each symbol in $\{ x \in X \mid f(x) \in A\}$ can be translated into a word in a one-to-one way. However, we have found that for some students this is not easy, even when the definition can be consulted. We have also noted instances in which students have had available both a definition and an operable interpretation, but still did not act appropriately. Thus, *actually* implementing an operable interpretation is separate from *knowing* that one *can* implement it.

One can also have operable interpretations for situations in a partly completed proof. For example, when a conclusion is negatively phrased (e.g., a set is empty or a number is irrational), one might early in the proving process attempt a proof by contradiction. Also when the conclusion asserts the equivalence of two statements, or that two sets are equal, often the proof should be divided into two parts, in which there are two implications to prove. Finally, if in a partly completed proof, one has arrived at a statement of the form p or q, the proof can be divided into two cases, one assuming p and the other assuming q.

We suggest that students, or small groups of students, can and should develop some operable interpretations independently of a teacher. However, if or when this should be done in a particular course is a design problem.

Part 2: Psychological Features of Proof Construction

In this second, psychological part of the perspective, we view proof construction as a sequence of actions which can be physical (e.g., writing a line of the proof or drawing a sketch) or mental (e.g., changing one's focus from the hypothesis to the conclusion or trying to recall a relevant theorem). The sequence of all of the actions that eventually lead to a proof is usually considerably longer than the final proof text itself. This fine-grained action approach appears to facilitate noticing which actions should be taken to write various parts of a proof correctly, which beneficial student proving actions to encourage, and which detrimental student proving actions to discourage.

Situations and Actions

What matters from the point of view of a student, who is learning to construct proofs, is the perceived situation. By a perceived, or *inner, situation* in proving, we mean a portion of a partly completed proof construction, perhaps including an interpretation drawn from long-term memory that can suggest a further action. The interpretation is likely to depend on recognition of the situation, which is easier than recall, perhaps because fewer brain areas are involved (Cabeza, et al., 1997). An inner situation is unobservable. However, a teacher can often infer an inner situation from the corresponding outer situation, that is, from the, usually written, portion of a partly completed proof.

Here we are using the term, *action*, broadly, as a response to a situation. We include not only physical actions (e.g., writing a line of a proof), but also mental actions. The latter can include trying to recall something or bringing up a feeling, such as a feeling of caution or of self-efficacy

(Selden & Selden, 2014). We also include "meta-actions" meant to alter one's own thinking, such as focusing on another part of a developing proof construction.

Examples of Inner Situations

Norton and D'Ambrosio (2008) have described what amounts to an illustration of the distinction between an inner situation and an outer situation for two middle school students, Will and Hillary, who viewed the same external situation involving a fraction such as 2/3. Hillary had (in her knowledge base) a partitive fractional scheme, as well as a part-whole fractional scheme, while Will had only the second scheme. This caused Will and Hillary to "see" the external situation differently, that is, to have differing inner situations, and hence, to act differently. In particular, Hillary was able to solve a problem that Will could not. Will could only solve the problem after he had developed a partitive fractional scheme, and then presumably experienced a richer inner situation.

In the above illustration, Will's internal view of the external situation could not be enriched by a concept of fraction that was not yet available in his knowledge base. But that is not the only way for two persons to have significantly differing inner situations for the same external situation. Selden, Selden, Hauk and Mason (2000) reported on mid-level undergraduates in a first course on differential equations attempting to solve moderately non-routine beginning calculus problems. Many students did not solve the problems, even though the solutions called on familiar calculus facts, as ascertained by a subsequent routine test. For example, one problem asked: Does $x^{21} + x^{19} - x^{-1} + 2 = 0$ have any roots between –1 and 0? Why or why not? This problem could not be solved by simple algebraic techniques, but could be solved by noticing that $f(-1) > 0$ and $f'(x) > 0$ on [-1,0). Selden, Selden, Hauk, and Mason (2000) were able to show that a number of the students could not solve moderately non-routine problems for which they had adequate information in their knowledge bases. Apparently, the students were unable to bring this information to mind, that is, bring it into consciousness, because their knowledge bases lacked certain links between "kinds" of problems and information that might be useful in solving them. Thus, the students were unable to enrich their views of the external situations to create inner situations, including connections to resources such as a function is increasing where its derivative is positive, that might have stimulated the enactment of appropriate problem-solving behaviors.

Situation-Action Links, Automaticity, and Behavioral Schemas

If, during several proof constructions in the past, similar situations have corresponded to similar reasoning leading to similar actions, then, just as in classical associative learning (Machamer, 2009), a link may be learned between them, so that another similar situation evokes the corresponding action in future proof constructions without the need for the earlier intermediate reasoning. Using such *situation-action links* strengthens them, and after sufficient practice/experience, they can become overlearned, and thus, automated. We call automated situation-action links *behavioral schemas*.

Features of Automaticity

In general, it is known that a person executing an automated action tends to: (1) be unaware of any needed mental process; (2) be unaware of intentionally initiating the action; (3) executes the action while putting little load on working memory; and (4) finds it difficult to stop or alter the action (Bargh, 1994). However, not necessarily all four occur in every situation. Morsella (2009) has pointed out

> Regarding skill learning and automaticity, it is known that the neural correlates of novel actions are distinct from those of actions that are overlearned, such as driving or tying one's shoes. Regions [of the brain] primarily responsible for the control of movements during the early stages of skill acquisition are different from the regions that are activated by overlearned actions. In essence, when an action becomes automatized, there is a 'gradual shift from cortical to subcortical involvement …' (p. 13).

Because cognition often involves inner speech, which in turn is connected with the physical control of speech production, the above information on the brain regions involved in physical skill acquisition is at least a hint that forming behavioral schemas not only converts S2 cognition into S1 cognition, but also suggests that different parts of the brain are involved in access and retrieval.

In particular, there may be a shift from cortical to subcortical involvement. Neural activity associated with doing mathematics is generally located in the frontal and parietal lobes (Norton, 2015). Also, more resources (in both the frontal and parietal lobes) have to work in concert when a person is doing tasks with higher cognitive demand because those tasks require greater use of working memory and executive function (Sauseng, Klimensch, Schabus, & Doppelmayr, 2005). Thus, it is important to conserve those resources for working on high cognitive demand tasks such as the truly hard parts of problems or proofs.

Behavioral Schemas as a Kind of Knowledge

We view behavioral schemas as belonging to a person's knowledge base. They can be considered as partly conceptual knowledge (recognizing and interpreting the situation) and partly procedural knowledge (the action), and as related to Mason and Spence's (1999) idea of "knowing-to-act in the moment". We suggest that, in using a situation-action link or a behavioral schema, almost always both the situation and the action (or its result) will be at least partly conscious.

Here is an example of one such possible behavioral schema that can conserve resources. One might be starting to prove a statement having a conclusion of the form p or q. This would be the situation at the beginning of the proof construction. If one had encountered this situation a number of times before, one might readily take an appropriate action, namely, in the written proof assume not p and prove q or vice versa. While this action can be warranted by logic (if not p then q, is equivalent to, p or q), there would no longer be a need to bring the warrant to mind.

It is our contention that large parts of proof construction skill can be automated, that is, that one can facilitate mid-level university students in turning parts of S2 cognition into S1 cognition, and that doing so would make more resources, such as working memory, available for such high cognitive demand tasks as the truly hard problems that need to be solved to complete many proofs.

The idea that much of the deductive reasoning that occurs during proof construction could become automated may be counterintuitive because many psychologists (e.g., Schechter, 2012), and (given the terminology) probably many mathematicians, assume that deductive reasoning is largely S2.

The Genesis and Enactment of Behavioral Schemas

The action produced by the enactment of a behavioral schema might be *simple*. It might also be *compound*, such as a procedure consisting of several smaller actions, each produced by the enactment of its own behavioral schema that was "triggered" by the action of the preceding schema in the procedure. Multi-digit subtraction of natural numbers is an example of such a compound behavioral schema. When viewed in a large grain-size (mainly compound actions), behavioral schemas might also be regarded as habits of mind (Margolis, 1993). Habits of mind are similar to physical habits, and people are similarly often unaware of, or do not remember, them (as habits).

It appears that consciousness plays an essential role in triggering the enactment of behavioral schemas for doing mathematics. This is reminiscent of the role consciousness plays in reflection. It is hard to see how reflection, treated as selectively re-presenting past experiences, could be possible without first having had the experiences and without each experience triggering the next. We have developed the following six-point theoretical sketch of the genesis and enactment of behavioral schemas (Selden, McKee, & Selden, 2010, pp. 205-206).

1) Within very broad contextual considerations, behavioral schemas are immediately available. They do not normally have to be remembered, that is, searched for and brought to mind before their application. This distinguishes them from most conceptual knowledge and episodic and declarative memory, which generally do have to be recalled or brought to mind before their application.

2) Simple behavioral schemas operate outside of consciousness. One is not aware of doing anything immediately prior to the resulting action – one just does it. Thus, the enactment of a simple behavioral schema that leads to an error is not under conscious control, and we should not expect that merely understanding the origin of the error, or being shown a counterexample, would prevent future reoccurrences. Compound behavioral schemas are also largely not under conscious control.

3) Behavioral schemas tend to produce immediate action, which may lead to subsequent action. One becomes conscious of the action resulting from a behavioral schema as it occurs or immediately after it occurs.

4) Behavioral schemas were once actions arising from situations through warrants that no longer need to be brought to mind. So one might reasonably ask, can several behavioral schemas be "chained together" and act outside of consciousness, as if they were one schema? For most persons, this seems not to be possible. If it were so, one would expect that a person familiar with solving linear equations could start with $3x + 5 = 14$, and without bringing anything else to mind, immediately say $x = 3$. We expect that very few (or no) people can do this, that is, consciousness of the results of enacting the individual schemas is required

5) An action due to a behavioral schema depends on conscious input, at least in large part. In general, a stimulus need not become conscious to *influence* a person's actions, but such influence is normally not precise enough for doing mathematics. For example, in many psychological experiments a stimulus-response connection is considered established when its occurrence *departs from chance* over multiple trials.

6) Behavioral schemas are acquired (learned) through (possibly tacit) practice. That is, to acquire a beneficial schema a person should actually carry out the appropriate action correctly a number of times – not just understand its appropriateness. Changing a detrimental behavioral schema requires similar, perhaps longer, practice.

Implicit Learning of Behavioral Schemas

It appears that the *process* of learning a behavioral schema can be implicit, although the situation and the action are in part conscious. That is, a person can acquire a behavioral schema without being aware that it is happening. Indeed, such unintentional, or implicit, learning happens frequently and has been studied by psychologists and neuroscientists (e.g., Cleeremans, 1993; Cleeremans & Jiménez, 2001). In the case of proof construction, we suggest that with the experience of proving a considerable number of theorems in which similar situations occur, an individual might implicitly acquire a number of relevant beneficial behavioral schemas. As a result he or she might simply not have to think quite so deeply as before about certain portions of the proving process, and might, as a consequence of having more working memory available, take fewer "wrong turns".

Something similar has been described in the psychology literature regarding the automated actions of everyday life. For example, an experienced driver can reliably stop at a traffic light while carrying on a conversation. But not all implicitly learned automated actions are positive. For example, a person can develop stereotypical behavior without being aware of the acquisition process and can even be unaware of its triggering situations (Chen & Bargh, 1997). This suggests that we should consider the possibility of mathematics students implicitly developing similarly unintended negative situation-action links, and the corresponding detrimental behavioral schemas, during mathematics learning, and in particular, during proof construction.

Detrimental Behavioral Schemas

We begin with a simple and perhaps familiar algebraic error. Many teachers can recall having a student write $\sqrt{}(a^2 + b^2) = a + b$, giving a counterexample to the student, and then having the student make the same error somewhat later. Rather than being a misconception (i.e., believing something that is false), this may well be the result of an implicitly learned detrimental behavioral schema. If so, the student would not have been thinking very deeply about this calculation when writing it. Furthermore, having previously understood the counterexample would also have little effect in the moment. It seems that to weaken/remove this particular detrimental schema, the triggering situation of the form $\sqrt{}(a^2 + b^2)$ should occur a number of times when the student can be prevented from automatically writing "$= a + b$" in response. However, this might require working with the student individually on a number of examples mixed with nonexamples.

For an example of an apparently implicitly learned detrimental behavioral schema for proving, we turn to Sofia, a first-year graduate student in one of the above mentioned graduate courses. Sofia was a diligent student, but as the course progressed what we came to call an "unreflective guess" schema emerged (Selden, McKee, & Selden, 2010, pp. 211-212). After completing just the formal-rhetorical part of a proof (essentially a proof framework) and realizing there was more to do, Sofia often offered a suggestion that we could not see as being remotely helpful. At first we thought she might be panicking, but on reviewing the videos there was no evidence of that. A first unreflective guess tended to lead to another, and another, and after a while, the proof would not be completed.

In tutoring sessions, instead of trying to comprehend, and work with, Sofia's unreflective guesses, we tried to prevent them. At what appeared to be the appropriate time, we offered an alternative suggestion, such as looking up a definition or reviewing the notes. Such positive suggestions eventually stopped her unreflective guesses, and Sofia was observed to have considerably improved in her proving ability by the end of the course (Selden, McKee, & Selden, 2010, p. 212).

Feelings and Proof Construction

The word "feeling" is used in a variety of ways in the literature so we will first indicate how we will use it. Often feelings and emotions are used more or less interchangeably--both appear to be conscious reports of unconscious mental states, and each can, but need not, engender the other. However, we will follow Damasio (2003) in separating feelings from emotions because emotions are expressed by observable physical characteristics, such as temperature, facial expression, blood pressure, pulse rate, perspiration, and so forth, while feelings are not. Indeed, Damasio has described a brain operation during which the patient was awake and could report on her state of mind. She experienced both feelings and emotions, but clearly at different times (Damasio, 2003, pp. 67-70).

Feelings such as a feeling of knowing can play a considerable role in proof construction (Selden, McKee, & Selden, 2010). For example, one might experience a feeling of knowing that one has seen a theorem useful for constructing a proof, but not be able to bring it to mind at the moment. Such feelings of knowing can guide cognitive actions because they can influence whether one continues a search or aborts it (Clore, 1992, p. 151). We call such feelings that can influence cognition *cognitive feelings*. When we speak of feelings here, we mean non-emotional cognitive feelings.

For the nature of feelings, we will follow Mangan (2001), who has drawn somewhat on William James (1890). Feelings seem to be summative in nature and to pervade one's whole field of consciousness at any particular moment. For example, to illustrate what it might mean for a feeling to pervade one's whole field of consciousness, consider a hypothetical student taking a test with several other students in a room with a window. If, at a particular time, the student looks at his test, then towards the other students, and finally out of the window, at each of the three moments he or she perceives information from only that moment. But if the student feels confident (i.e., has a feeling of knowing) that he or she will do well on the test during one of these moments, then he or she will also feel confident during the other two. This suggests that feelings are especially available to be focused on and can directly influence action.

Additional (nonemotional cognitive) feelings, different from a feeling of knowing, are a feeling of familiarity and a feeling of rightness. Mangan (2001) has distinguished these. Of the former, he wrote that the "intensity with which we feel familiarity indicates how often a content now in consciousness has been encountered before", and this feeling is different from a feeling of rightness. It is rightness, not familiarity, that is "the feeling-of-knowing in implicit cognition". Rightness is "the core feeling of positive evaluation, of coherence, of meaningfulness, of knowledge". In regard to a feeling of rightness, Mangan has written that "people are often unable to identify the precise phenomenological basis for their judgments, even though they can make these judgments with consistency and, often, with conviction. To explain this capacity, people talk about 'gut feelings', 'just knowing', hunches, [and] intuitions". Often such quick judgments (i.e., the results of S1 cognition) can be correct, but they sometimes need to be checked, that is, S2 cognition needs to "kick in" and override such incorrect quick judgments.

Finally, we conjecture that feelings may eventually be found to play a larger role in proof construction than indicated above, because they provide a direct link between the conscious mind and the structures and possible actions of the unconscious mind.

Self-Efficacy

In order to prove harder theorems, ones with a substantial problem-centered part, students need to persist in their efforts, and such persistence is facilitated by a sense of self-efficacy. According to Bandura (1995), self-efficacy is "a person's belief in his or her ability to succeed in a particular situation" (Bandura, 1995). Of developing a sense of self-efficacy, Bandura (1994) stated that "The most effective way of developing a strong sense of self-efficacy is through mastery experiences," that performing a task successfully strengthens one's sense of self-efficacy. Also, according to Bandura, "Seeing people similar to oneself succeed by sustained effort raises observers' beliefs that they too possess the capabilities to master comparable activities to succeed."

According to Bandura (1994), individuals with a strong sense of self-efficacy: (1) view challenging problems as tasks to be mastered; (2) develop deeper interest in the activities in which they participate; (3) form a stronger sense of commitment to their interests and activities; and (4) recover quickly from setbacks and disappointments. In contrast, people with a weak sense of self-efficacy: (1) avoid challenging tasks; (2) believe that difficult tasks and situations are beyond their capabilities; (3) focus on personal failings and negative outcomes; and (4) quickly lose confidence in personal abilities.

Bandura's ideas "ring true" with our past experiences as mathematicians teaching courses by the classical Moore Method (Mahavier, 1999). Classical Moore Method advanced undergraduate or graduate courses are taught from a brief set of notes[2] consisting of definitions, a few requests for examples, statements of major results, and those lesser results needed to prove the major ones. Exercises of the sort found in most textbooks are largely omitted. In class meetings, the professor invites individual students to present their original proofs and then only very briefly comments on errors.[3] Once students are able to successfully prove their first few theorems, they often progress very rapidly in their proving ability, even without any apparent explicit teaching, and persist even when subsequent proofs are more complex or require creating new mathematical objects or lemmas.

Why should this be? We conjectured then, and also conjecture now, that students obtained a sense of self-efficacy from having proved their first few theorems successfully, and that this sense of self-efficacy grew over time and helped them persist in explorations, re-examinations, and validations when these were needed in proving subsequent difficult theorems.

Seeing Similarities, Searching, and Exploring

How does one recognize situations as similar? Different people see situations as similar depending both upon their past experiences and upon what they choose to, or happen to, focus on. While similarities can sometimes be extracted implicitly (Markman & Gentner, 2005), teachers may occasionally need to direct students' attention to relevant proving similarities. On the other hand, such direction should probably be as little as possible because the ability to *autonomously* see similarities can, and should, be learned.

For example, it would be good to have general suggestions for helping students "see", without being told, that the situations of a set being empty (i.e., having no elements), of a number being irrational (i.e., not rational), and of the primes being infinite (i.e., not finite) are similar.

[2] A brief set of notes for the first semester of an undergraduate abstract algebra course taught by the classical Moore Method is provided in the Appendix of Selden and Selden (1978).
[3] We emphasize that this description of classical Moore Method courses, which we have sometimes taught in the past, differs significantly from our teaching of the two design experiment courses described earlier in this paper.

That is, while the three situations—empty, irrational, and infinite—may not seem similar on the surface, they can be rephrased to expose the existence of a negative definition. And, unless students rephrase these situations, it seems unlikely that they would see this similarity and link these situations (when they occur as conclusions to theorems to prove) to the action of beginning a proof by contradiction.

In addition to automating small portions of the proving process, such as writing proof frameworks, we would also like to enhance students' searching skills, that is, their tendency to look for helpful previously proved results. We would also like to enhance students' tendency to "explore" various possibilities when they don't know what to do next. In a previous paper (Selden & Selden, 2014, p. 250), we discussed the kind of exploring entailed in proving the rather difficult (for students) *Theorem: If S is a commutative semigroup with no proper ideals, then S is a group*. Well before such a theorem appears in a set of course notes like ours, one might provide students with advice, or better yet, experiences showing the value of exploring what is not obviously useful. For example, one could discuss the usefulness of starting with $abba = e$, for arbitrary $a, b \in S$, when attempting to show commutativity of a semigroup with identity e, having $s^2 = e$, for all $s \in S$ (as discussed in Selden, Benkhalti, and Selden, 2014).

Using this Perspective to Analyze Students' Proof Attempts

We hope the following analysis that highlights actions in proofs and proving will provide insights into what might be emphasized when teaching particular groups of students.[4]

In examining students' proof attempts, we are not just looking for mistakes or misconceptions, but rather we are looking for possible detrimental actions, possible beneficial actions, and for potential beneficial actions *not* taken. Below we give two examples of how we have analyzed students' (incorrect) proof attempts (Selden, Benkhalti, & Selden, 2014).

Example 1. The student had attempted to prove the following on an examination. *Theorem: Let S be a semigroup with identity e. If, for all s in S, ss = e, then S is commutative*. Here we are examining, and analyzing, the student's written work using our theoretical perspective of actions. The lines are numbered for convenient reference. The student's accompanying scratch work consisted of the definitions of identity and commutative. The proof went as follows:

1. Let S be a semigroup with an identity element, e.
2. Let $s \in S$ such that $ss = e$.
3. Because e is an identity element, $es = se = s$.
4. Now, $s = se = s(ss)$.
5. Since S is a semigroup, $(ss)s = es = s$.
6. Thus $es = se$.
7. Therefore, S is commutative. QED.

Analysis. Line 2 only hypothesizes a single s and should have been, "Suppose for all $s \in$ S, $ss = e$." With this change, Lines 1, 2, and 7 are the correct first-level framework.[5] There is no second-level framework between Lines 2 and 7. This was a beneficial action *not* taken and should have been: "Let $a \in S$ and $b \in S$. ... Then $ab = ba$." inserted between Lines 2 and 7.

[4] We are not suggesting that this kind of proof analysis be used as a way of grading, or marking, students' work. Rather we are suggesting that this kind of proof analysis might be helpful for teachers and course designers.

[5] Proof frameworks are not the only way to start a proof, but these students had participated in a course in which they had been encouraged to, and had often found it useful to, begin their proofs by writing a proof framework.

Line 3 violates the genre of proof by including a definition easily available outside of the proof. Lines 3, 4, 5, and 6 are not wrong, but do not move the proof forward. Writing these lines may have been detrimental actions that subconsciously primed the student's feeling that something useful had been accomplished, and thus, may have brought the proving process to a premature close.

Example 2. Next we consider another student's proof attempt of the following theorem on an examination. *Theorem. Let S and T be semigroups and $f{:}S\to T$ be a homomorphism. If G is a subset of S and G is a group with identity e, then f(G) is a group*. Here, again, we are examining, and analyzing, the student's written work using our theoretical perspective of actions. The lines of the student's proof attempt are numbered for convenient reference.

1. Let S and T be semigroups and $f{:}S\to T$ be a homomorphism.
2. Suppose $G \subseteq S$ and G is a group with identity e.
3. Since G is a group and it has identity e, then for each element g in G there is an element g' in G such that $gg' = g'g = e$.
4. Since f is a homomorphism, then for each element $x \in S$ and $y \in S$, $f(xy)=f(x)f(y)$.
5. Since $G \subseteq S$, then $f(gg')=f(g)f(g')$. So $f(gg') = f(g'g) = f(e)$.
6. So $f(G)$ has an element $f(e)$ since f is a function.
7. Therefore, $f(G)$ is a group. QED.

Analysis. The student has written the first-level framework, namely Lines 1, 2, and 7, correctly, assuming that Line 7 was written immediately after writing the first two lines. To complete the proof framework, the student should have unpacked the last line and written the second-level framework. That is, the student should have considered $f(G)$ and noted that there are three parts to prove, namely, that $f(G)$ is a subsemigroup, that there is an identity in $f(G)$, and that each element in $f(G)$ has an inverse in $f(G)$. This unpacking of the conclusion is a beneficial action *not* taken.

Instead, in Line 3, the student wrote into the proof the definition of G being a group, and in Line 4, stated what it means for f to be a homomorphism. These actions are not wrong, but they do *not* move the proof forward and are detrimental because they can give the student a feeling that something useful has been done. Perhaps, in Lines 5 and 6, the student was trying to show the existence of an identity and inverses in $f(G)$ and was unsuccessful, but one cannot know this. If the second-level proof framework had been written, the proof would have been reduced to three easier parts, each of which also has a proof framework, and this might have been helpful to the student.

Teaching and Research Considerations

The above considerations can lead to many possible teaching interventions. This then brings up the question of priorities. Which proving actions, of the kinds discussed above, are most useful for mid-level university mathematics students to automate, when they are learning how to construct proofs? Since such students are often asked to prove relatively easy theorems—ones that follow directly from definitions and theorems recently provided—it would seem that noting the kinds of structures that occur most often might be a place to start. Indeed, since every proof can be constructed using a proof framework, we consider constructing proof frameworks as a reasonable place to start. Furthermore, some students do not write the second-level proof framework, perhaps because they have difficulty unpacking the meaning of the conclusion. So it would also be good to work on that.

Also, helping students interpret formal mathematical definitions so that these become operable might be another place to start. This would be helpful because one often needs to convert a definition into an operable interpretation in order to use it to construct a second-level proof framework. However, eventually students should learn to make such interpretations themselves.

Finally, we believe this particular perspective on proving, using situation-action links and behavioral schemas, together with information from psychology and neuroscience, is mostly new to the field and is likely to lead to additional insights.

References

Baddeley, A. (2000). The episodic buffer: A new component of working memory? *Trends in Cognitive Science*, *4*(11), 417-423.

Bandura, A. (1994). Self-efficacy. In V. S. Ramachaudran (Ed.), *Encyclopedia of human behaviour* (Vol. 4, pp. 71-81). New York: Academic Press.

Bandura, A. (1995). *Self-efficacy in changing societies*. Cambridge: Cambridge University Press.

Bargh, J. A. (1994). The four horseman of automaticity: Awareness, intention, efficiency and control in social cognition. In R. Wyer & T. Srull (Eds.), *Handbook of social cognition*, Second Edition, Vol. 1 (pp. 1-40). Mahwah, NJ: Lawrence Erlbaum Associates.

Bargh, J. A., & Chartrand, T. L. (2000). Studying the mind in the middle: A practical guide to priming and automaticity research. In H. T. Reid & C. M. Judd (Eds.), *Handbook of research methods in social psychology* (pp. 253-285). New York: Cambridge University Press.

Bargh, J. A., & Morsella, E. (2008). The unconscious mind. *Perspectives on Psychological Science*, *3*(1), 73-79.

Bills, L., & Tall, D. (1998). Operable definitions in advanced mathematics: The case of the least upper bound. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2 (pp. 104-111). Stellenbosch, South Africa: University of Stellenbosch.

Bor, D. (2012). *The ravenous brain*. New York: Basic Books.

Cabeza, R., Kapur, S., Craik, F. I. M., McIntosh, A. R., Houle, S., & Tulving, E. (1997). Functional neuroanatomy of recall and recognition: A PET study of episodic memory. *Journal of Cognitive Neuroscience, 9*, 254-265.

Chen, M., & Bargh, J. A. (1997). Nonconscious behavioral confirmation processes: The self-fulfilling consequences of automatic stereotype activation. *Journal of Experimental Social Psychology*, *33*, 541-560.

Cleeremans, A. (1993). *Mechanisms of implicit learning*: *Connectionist models of sequence processing*. Cambridge, MA: MIT Press.

Cleeremans, A., & Jiménez, L. (2001). Implicit learning and consciousness: A graded, dynamic perspective.Retrieved December 1, 2014, from http://srsc.ulb.ac.be/axcWWW/papers/pdf/01-AXCLJ.pdf.

Clore, G. L. (1992). Cognitive phenomenology: Feelings and the construction of judgment. In L. L. Martin & A. Tesser (Eds.), *The construction of social judgments* (pp. 133-162). Hillsdale, NJ: Lawrence Erlbaum Associates.

Damasio, W. (2003*). Looking for Spinoza: Joy, sorrow, and the feeling brain*. Orlando, FL: Harcourt.

Hassin, R. R., Bargh, J. A., Engell, A. D., & McCulloch, K. C. (2009). Implicit working memory. *Conscious Cognition*, 18(3), 665-678.

James, W. (1890). *The Principles of Psychology*. New York: Holt.

Machamer, P. (2009).Learning, neuroscience, and the return to behaviorism. In J. Bickle (Ed.), *The Oxford handbook of philosophy and neurosciences* (pp. 166-176). Oxford: Oxford University Press.

Mahavier, W. S. (1999). What is the Moore Method? *PRIMUS*, *9*, 339-354.

Mangan, B. (2001). Sensation's ghost: The non-sensory 'fringe' of consciousness. *Psyche*, *7*(18). Retrieved September 29, 2009, from http://psyche.cs.monash.edu.au/v7/psyche-7-18-mangan.html.

Margolis, H. (1993). *Paradigms and barriers: How habits of mind govern scientific beliefs*. Chicago: University of Chicago Press.

Markman, A. B., & Gentner, D. (2005). Nonintentional similarity processing. In T. Hassin, J. Bargh, & J. Uleman, *The new unconscious* (pp. 107-137). New York: Oxford University Press.

Mason, J., & Spence, M. (1999). Beyond mere knowledge of mathematics: The importance of knowing-to-act in the moment. *Educational Studies in Mathematics*, *28*(1-3), 135-161.

McKee, K., Savic, M., Selden, J., and Selden, A. (2010). Making actions in the proving process explicit, visible, and "reflectable". In *Proceedings of the 13th Annual Conference on Research in Undergraduate Mathematics Education*. Retrieved Feb. 10, 2011, from http://sigmaa.maa.org/rume/crume2010/Archive/McKee.pdf .

Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics*, *27*(3), 249-266.

Morsella, E. (2009). The mechanisms of human action: Introduction and background. In E. Morsella, J. A. Bargh, & P. M. Goldwitzer (Eds.), *Oxford handbook of human action* (pp. 1-34). Oxford: Oxford University Press.

Norton, A. (2015). Neural correlates for action-object theories. *Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education*. Retrieved February 27, 2015,from http://timsdataserver.goodwin.drexel.edu/RUME-2015/rume18_submission2.pdf.

Norton, A., & D'Ambrosio, B. S. (2008). ZPC and ZPD: Zones of teaching and learning. *Journal for Research in Mathematics Education*, *39*(3), 220-246.

Sauseng, P., Klimensch, W., Schabus, M., & Doppelmayer, M. (2005) Fronto-parietal EEG coherence in theta and upper alpha reflect central executive functions of working memory. *International Journal of Psychophysiology*, *57*, 97-103.

Schechter, J., (2013). Deductive reasoning. In H. Pashler (Ed.), *Encyclopedia of the mind*. Los Angeles, CA: SAGE Publications.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

Selden, A. A., & Selden, J. (1978). Errors students make in mathematical reasoning. *Bosphorus University Journal*, *6*, 67-87. Available online through www.academia.edu.

Selden, A., McKee, K., & Selden, J. (2010). Affect, behavioural schemas, and the proving process. *International Journal of Mathematical Education in Science and Technology*, *41*(2), 199-215.

Selden, A., & Selden, J. (2013). The genre of proof. In M. N. Fried & T. Dreyfus (Eds.), *Mathematics and mathematics education: Searching for common ground* (pp. 248-251). New York: Springer.

Selden, A., & Selden, J. (2014). The roles of behavioral schemas, persistence, and self-efficacy in proof construction. In B. Ubuz, C. Hasar, & M. A. Mariotti (Eds.). *Proceedings of the Eighth Congress of the European Society for Research in Mathematics Education* [CERME8] (pp. 246-255). Ankara, Turkey: Middle East Technical University.

Selden, A., Selden, J., Hauk, S., & Mason, A. (2000). Why can't calculus students access their knowledge to solve non-routine problems? In A. H. Schoenfeld., J. Kaput, & E. Dubinsky, (Eds.), *Research in collegiate mathematics education. IV. Issues in mathematics education: Vol. 8.* (pp. 128-153). Providence, RI: American Mathematical Society.

Selden, J., Benkhalti, A., & Selden, A. (2014). An analysis of transition-to-proof course students' proof constructions with a view towards course redesign. In T. Fukawa-Connolly, G. Karakok, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 17th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 246-259). Denver, Colorado. Retrieved March 1, 2015 from http://sigmaa.maa.org/rume/RUME17.pdf .

Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*, *29*, 123-151.

Selden, J., & Selden, A. (2011). The role of procedural knowledge in mathematical reasoning. In B. Ubuz (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4 (pp. 124-152). Ankara, Turkey: Middle East Technical University.

Smith, J. C. (2006). A sense-making approach to proof: Strategies of students in traditional and problem-based number theory courses. *Journal of Mathematical Behavior*, *25*, 73-90.

Stanovich, K. E. (2009). Distinguishing the reflective, algorithmic, and autonomous minds: Is it time for a tri-process theory? In J. Evans & K. Frankish (Eds.), *In two minds: Dual processes and beyond* (pp.55-88). Oxford: Oxford University Press.

Stanovich, K. E., & West, R. F. (2000). Individual differences in reasoning: Implications for the rationality debate? *Behavioral and Brain Sciences, 23,* 645-726.

Conceptualizing equity in undergraduate mathematics education: Lessons from K-12 research

Aditya Adiredja, Oregon State University
Nathan Alexander, University of San Francisco
Christine Andrews-Larson, Florida State University

Research on equity in mathematics education has been one of the primary foci among K-12 researchers. However, in research on undergraduate mathematics education, equity research has yet to establish and maintain the presence and consistency that aligns with issues of inequity related to fairness, access and opportunity. In K-12 research, the focus has shifted from individual context to socio-cultural context and now to understanding the social and political aspects of power and identity in mathematics education research. In RUME, the use of theories related to socio-cultural contexts has been increasing but there has not been a major shift in the focus of the research toward addressing issues of equity. In this paper, Gutiérrez' (2009) four dimensions of equity: access, achievement, power and identity are used as a conceptual framework to interpret insights from K-12 equity research and apply them to existing studies in RUME. The report ends with open questions and directions for equity research at the postsecondary level.

Keywords: equity, socio-political perspective, identity, power

People of color, women, individuals with disabilities, and people who live in their intersections, continue to be underrepresented in postsecondary mathematics. For example, as of 2012, African American/Black, Hispanic/Latin@[1] and Native Alaskan and American Indian *combined* comprised 20% of Bachelor's, 18% of Master's and 8% of Doctorate degrees in mathematics (NSF Science and Engineering Indicators, 2014). At the faculty level, the latest Conference Board of the Mathematical Sciences (CBMS) survey of mathematics departments in the United States shows people from the same group comprised only 6% of the total mathematics faculty (Blair, Kirkman and Maxwell, 2013). The report captures the ongoing dearth of faculty of color in mathematics (Walker, 2014). The reality of the underrepresentation is situated in the broader context of increasing the number of science, technology, engineering and mathematics (STEM) graduates (Harper, 2010; Executive Office of the President, PCAST, 2012), and the increasing awareness about racial injustice in today's America (e.g., #BlackLivesMatter movement on social media). The broader societal context drives much of our work around equity.

In the Research in Undergraduate Mathematics Education (RUME) community, studies that specifically focus on the learning experiences and opportunities of historically marginalized groups of students have also been underrepresented. A review of the conference proceedings from the past four years shows that these type of studies comprised 5-10% of the total presentations. From our broader literature search, we also found a limited number of mathematics education studies at the postsecondary level that specifically focuses on issues relevant to marginalized groups. As Gutiérrez (2013) noted, a decade ago, Lubienski and Bowen (2000) found a similar representation of equity related articles in K-12 mathematics education journals, though more recently there has been an increase in representation at national

[1] Gutiérrez (2013) uses "@" to indicate both an "a" and "o" ending (Latina and Latino) to be more gender inclusive.

conferences. The undergraduate mathematics education community has yet to fully initiate and maintain the presence and consistency of equity studies observed in the broader mathematics education community. Altogether, these contexts present a need to rethink theoretical perspectives and research-based practices related to equity in undergraduate mathematics education.

Gutiérrez (2002) defines equity research as research that explicitly focuses on efforts to understand and mitigate systematic differences in opportunities and experiences in education for different groups of students, particularly on ways that these differences privilege some groups of students over others. Embedded in this definition is the effort to depart from the fixation on differences in achievements between groups of students, but instead to focus on ways that different students experience education and educational opportunities differently. This definition also fosters consideration of issues of privilege and power that might have contributed to systematic differences in students' experiences.

One common response to these systematic differences is what many argue to be an equal and fair approach to education: treat all students the same way regardless of their background. This *colorblind* approach to equality fails to take into consideration students' identities and the important ways they influence students' experience in education (Martin, 2003). This approach also privileges the identities and practices of the dominant group of students while continuing to perpetuate systematic marginalization of other groups (Bonilla-Silva, 2003). The ideas embedded in Gutiérrez's (2002) definition of equity differentiate the goal of striving for equity from the notion of equality in education.

This paper aims to stimulate a discussion about a need for more equity-focused research at the postsecondary level by considering insights gleaned from equity research at the K-12 level. In addition to the aforementioned issues of underrepresentation, in the next section we use the historical and theoretical development of equity research at the K-12 level to further motivate the need to advance equity research at the postsecondary level. We then use a conceptual framework from Gutiérrez (2009) to help us organize and interpret existing research at the K-12 level. Specifically, we consider the four dimensions of equity: *access, achievement, identity* and *power* in discussing existing equity related research in RUME, and ways that we can use the findings from the K-12 level to advance our work at the postsecondary level.

Development of Equity Research Framework at the K-12 Level

In this section, we offer a brief overview of existing frameworks for equity research, highlighting cognitive, socio-cultural, and socio-political perspectives. In particular, we examine the origins of these frames and discuss their relationship to educational research. We aim to put these lenses on equity in education in conversation with lenses on learning commonly used in RUME research.

Variations in the approaches to understanding student learning have generated frameworks upon which new knowledge in the field of mathematics education has developed, including knowledge around issues of equity. In general, perspectives have come from an array of disciplines but can be structured into broad categories that help to summarize the literature on equity scholarship in mathematics education. These parent domains, the *cognitive*, *social-cultural*, and *socio-political*, while not mutually exclusive or exhaustive, help to establish an obtuse view of the trajectory of equity research in the field of mathematics education. Moreover, in the last three decades, the field has witnessed an increase of critical theory at the intersection of these domains as a means to highlight issues of identity and power (Skovsmose & Greer, 2012).

Historically, theories and research on teaching and learning have engaged cognition and cognitive development as a means to better develop education research into a science. In the early development of the field of mathematics education, for example, empirical studies were used to understand cognitive development patterns among school children (Kilpatrick, 1992). Resultantly, early research in the field tended to take on psychological approaches to research in which teaching was situated as a *treatment* and learning as an *effect*. In this particular frame, equity research was not central in the then-developing field of mathematics education. Moreover, at the time the field primarily functioned under the belief that student learning was based in large on individual cognition, void of effects outside of focused teaching and learning contexts.

Studies linking cognition with other non-cognitive factors did not emerge until what Kilpatrick (1992) identified as a period in mathematics education during the 1950s where more interdisciplinary work began. Saxe and de Kirby's (2014) discussions of methodological approaches to studies examining, for example, cognition and culture, followed the traditional *dichotomous* and *intrinsic relation* approaches. Early research in the field tended toward a *dichotomous* lens in which culture is viewed as having an effect on student cognition, or an *intrinsic relation* lens in which culture and cognition are viewed as mutually situated in daily activities (Saxe & de Kirby, 2014). Elsewhere, linkages made between social and cultural contexts led to the development of research on social interactions in learning contexts (Vygotsky, 1978) and identity development (Erikson, 1950; Lave & Wenger, 1991), among others.

Saxe and de Kirby's (2014) framework and review is useful for better understanding the trajectory of early mathematics education research from cognitive, individually-focused contexts to more sociological, culturally-centered aspects of learning and the issues that have come as a result of working to intersect these domains. Specifically, the authors and others (e.g., Nasir, 2005) identify portions of this trajectory and note that researchers generally situate discipline-specific contexts separately, which seemingly imply a one-way directional influence versus an integrated more nuanced structured relation between culture and cognition (Nasir, 2005).

In some intermediary views on teaching and learning, student knowledge was framed as structuring of cognition in sociocultural contexts (Cobb & Yackel, 1996; Nasir, 2005), which provided nuanced frames to better understand equity issues in education contexts. Specifically, these and similar references situated earlier contexts on teaching and learning in both individual and social contexts. As a result, research on identity and identity development in the field sprouted, as we see through seminal studies on developing identities in learning contexts (e.g., Lave & Wenger, 1991). Further development on views about teaching and learning were prompted in part by two theoretical contributions: on one end, the research developing to help situate student learning and equity contexts in cognitive, social, and cultural contexts and, on the other end, a much more apparent issue with the political aspects and politics of research in the field.

Apple (1992) argued that knowledge is highly political. The author argued that the way that society treated certain kinds of knowledge to be more legitimate than others spoke to the distribution of power in society. Apple's work raises the question: whose knowledge do we leverage in education and in creating curricular reforms based on that knowledge, and who truly benefits from these reforms? Connectedly, in mathematics education, Martin (2000, 2013) has contended with the ideas of existing narratives that support the marginalization of groups of people. One such narrative is the idea that knowledge production is "neutral and impartial, unconnected to power relations" (p. 323). This is in direct line with the research on the politics of education research (Apple, 1992).

Gutiérrez (2013), along with Martin (2000, 2013), brings these challenges to the field of mathematics education, and pushes the field to take the "sociopolitical turn." The sociopolitical perspective considers knowledge, power and identity to be interrelated and "arising from (and constituted within) social discourses" (Gutiérrez, 2013, p. 40). Adopting such a perspective involves "uncovering the taken-for-granted rules and ways to operating that privilege some individuals and exclude others" (*ibid,* p. 40). As the field witnesses the development of theories related more broadly to the social-cognitive and socio-cultural domains, researchers have argued for a need to examine the sociopolitical domain (Gutiérrez, 2013).

Situating the current state of RUME in the development of the K-12 research suggests some productive research directions for equity research at the postsecondary level. One direction is for the RUME community to consider other theoretical frameworks that prioritize non-cognitive factors and socio-cultural context in learning. In recent years, we have begun to see the uptake of socio-cultural perspective and the situative perspective on learning, as reflected in the work presented in the RUME community (e.g., Katz, Post, Savic & Cook, 2015). This suggests that parts of the RUME community are equipped with theories to consider issues of equity more directly in their work. Perhaps in the near future, we as a community can begin exploring questions about the role of identity and identity development in mathematics education at the postsecondary level.

RUME research continues to focus heavily on students' and teachers' individual cognition and practices. While one direction in research is to consider theoretical frameworks that prioritize non-cognitive factors, another direction is to consider the social and political contexts in which cognition occurs with individual cognition studies. As Gutiérrez (2013) and Apple (1992) suggest, we need to begin to explore questions about the kind of the knowledge and practices we privilege in teaching and learning in postsecondary mathematics. At the K-12 level, some researchers have responded to this question by trying to leverage and build on cultural aspects of the students' communities in designing curriculum (Civil, 2006). At the postsecondary level, we are constrained by the more abstract nature of mathematics and the fact that it becomes increasingly reliant on previous formal mathematical knowledge. Recognizing those constraints, what other non-dominant knowledge can we leverage in instruction? What is the role of students' more informal knowledge in learning formal mathematics? We discuss some ways that some researchers have answered this question in a later section. Thus, in addition to considering the social and political contexts in which cognition occurs, we need to explore the extent to which the nature of mathematics at the postsecondary level is similar and/or different from the mathematics at the K-12 level, and the corresponding implications for equity research in the teaching and learning of postsecondary mathematics.

In the next several sections, we discuss more specific progress and findings in equity research at the K-12 level, and attempt to put them in conversation with some studies at the postsecondary level. In order to frame this discussion, we introduce Gutiérrez' (2009) framework for equity as a lens through which we interpret lessons from existing research at the K-12 level, and we also use this framework to discuss ways that these lessons can inform our work at the postsecondary level. Finally, we elaborate on the subtleties and meaning of the constructs from Gutiérrez (2009) in the context of existing studies at the postsecondary level.

Four Dimensions of Equity

Gutiérrez (2009) introduces four dimensions of equity: *access, achievement, identity* and *power* to problematize standard conceptions of equity research and practices in K-12 mathematics education (See Figure 1). She places the four dimensions on two axes of equity: the access and achievement axis, and the power and identity axis. Access, as a precursor to achievement, refers to learning resources related to students' opportunity to learn and participate in mathematics, e.g., good instructors, rigorous curriculum, and classroom structures that invite participation. Achievement focuses on student learning outcomes. This ranges from students' learning outcomes on a particular topic to students' ability to productively use mathematics to participate in society.

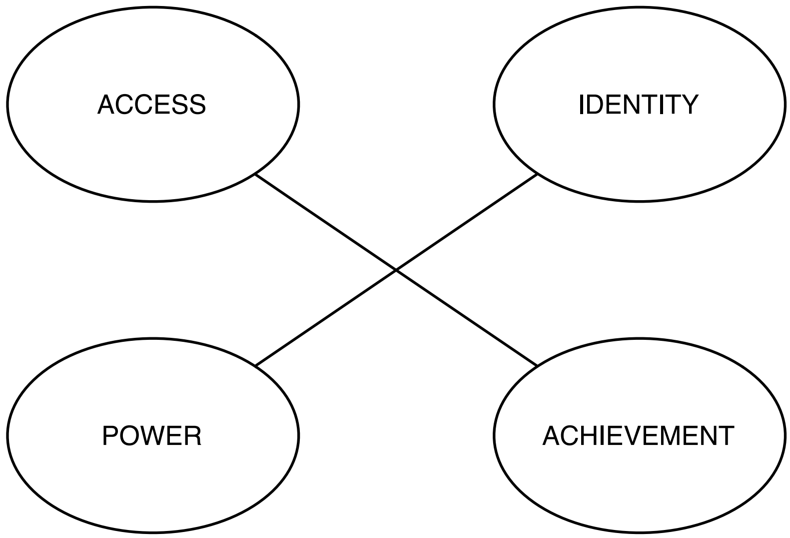


Figure 1. The two axes of equity. Diagram adapted from Gutiérrez (2009).

On the other axis lies identity and power. To consider identity is to recognize students' relationship with the broader world. This consideration involves understanding students' pasts, the contribution of their culture and heritage, and also the ways that they are "racialized (Martin, 2007), gendered and classed (Walkerdine, 1988)" in different social contexts (as cited in Gutiérrez, 2009). Power accounts for the role of learning in "social transformation" at different levels (Gutiérrez, 2009, p. 6). To consider power is to explore the degree that learning challenges or disrupts existing power distribution and dynamics in society, which are often based on race, gender and social class. This can be achieved by helping students use mathematics to critique social issues, examining who speaks and makes decisions during class time, and considering what counts as productive mathematical knowledge. The purpose of conceptualizing equity in this way is to highlight the relationships and tensions among these dimensions, which allows us as a community to acknowledge and combat longstanding inequities.

Problems and Progress in K-12 Equity Research

We organize problems and progress in K-12 equity research using Gutiérrez' (2009) four dimensions of equity. We first discuss critiques ofthe fixation with closing achievement gaps at the K-12 level as it pertains to issues of access and achievement. Then we consider the recommendation for considering socio-political nature of mathematics education, specifically as it pertains to issues of power and identity. We bring these lessons into post-secondary contexts by using the four dimensions of equity to reinterpret existing studies in RUME.

Achievement Gap Studies and the Access and Achievement Axis

Issues surrounding equity in K-12 rarely move beyond the static goal of closing the achievement gap, and often fail to account for issues related to access into mathematics. At the elementary and secondary levels, the question of educational equity has been most often centered on closing what many have framed as the mathematics achievement gap (Gutiérrez, 2013). In general, researchers have referred to the mathematics achievement gap as the difference in achievement outcomes between two or more groups of students. However, race has been a dominant focus in discussing the achievement gap, with researchers oft noting the differences between White, and often Asian students' mathematics outcomes and those of other racial-ethnic groups, most often other students of color (Martin, 2000, 2013).

Some researchers have cautioned against this way of framing equity as it supports deficit thinking and negative narratives about marginalized groups, and relies on one-time cross sections of data on achievement (Gutiérrez and Dixon-Roman, 2011; Martin, 2013). Gutiérrez (2008) points out that this fixation with "gap-gazing" also fails to acknowledge differences in access to learning opportunities experienced by members of marginalized groups. The nature and scale of these differences in learning opportunities vary widely: from the level of expertise of the students' mathematics teachers, to students' access to rigorous curricula and instruction, to classroom participation structure, to students' opportunities to take higher-level mathematics courses (Gutiérrez, 2009). Elsewhere, researchers have discussed the detrimental effects on pedagogy and practice in light of these largely racialized references (Martin, 2000, 2013).

Access and Achievement in the Postsecondary Context

Uri Treisman's work with the Emerging Scholars Program (Fullilove & Treisman, 1990; Treisman, 1992; Treisman, 1985) serves as a productive context to discuss the tension between access and achievement in the postsecondary mathematics context. The study is one of the earliest studies around equity specifically in undergraduate mathematics. The Emerging Scholars program was a response to an achievement gap in calculus between Chinese students and the African American students at UC Berkeley. On the one hand, the study challenges many preconceptions about African American students, and it problematizes the way in which institutions can fail to acknowledge differences in access to learning opportunities experienced by marginalized groups of students. On the other hand, this work suffers from a common criticism of achievement gap studies, in that the study positioned the African American students' practices in a deficit way while it privileged the practices of the Chinese students.

Treisman (1992) documented and challenged commonly held assumptions among faculty members about reasons for low performance of African American students in calculus at UC Berkeley. Through interviews with students and their families, the study dispelled the commonly held faculty explanations for low performance: lack of motivation, preparation, family support, and economic resources. Having ruled out those factors, Uri Treisman set out to understand differences in study practices of minority students, comparing those of the Chinese and the African American students. The rationale for this choice was that both were minority groups, but the Chinese students were performing well in calculus whereas the African American students were not.

While the study started with an aim to investigate an achievement gap, the analysis revealed differences in opportunities to participate in doing mathematics between the two groups. That is, instead of focusing solely on performance, the study looked at achievement gap as differences in opportunities to learn (Flores, 2007). Fullilove and Treisman (1990) found that the Chinese students had access to different resources as a result of studying in groups, learning from upperclassmen, and critiquing one another's work. They found that the African American students on the other hand were studying in isolation. In order to mitigate this difference in learning opportunities, Treisman created recitation sections that incorporated opportunities for students to work together on challenging problems. When the access to opportunities for learning was leveled, differences in achievement between the groups seemingly disappeared.

Closing the achievement gap between the African American students and students from dominant groups came at some cost. While learning from the practices of the Chinese students was productive, it positioned the African American students as deficient. Like many achievement gap studies, it suffered from the criticism that it was focusing on how to make the African American students be more like the Chinese students (Gutiérrez, 2013), thereby setting a particular racial hierarchy. This perpetuation of the narrative of Asians as good with mathematics has also been shown to have implications in the way that African American students see themselves as doers of mathematics. Shah (2013) found that high school students would attribute their African American classmate's success in mathematics to having "some Asian in them" (p. 1). It perpetuates the narrative that success in mathematics is not for Black students.

The African American students in Treisman's study were high achieving (Treisman, 1992). They were admitted to UC Berkeley with comparable academic background and test scores to other students who were admitted, but their strengths in mathematical practices were underexplored in the study. This contrasts with Walker's (2014) work with Black mathematicians and McGee and Martin's (2011) work with successful Black mathematics and engineering students. Their work emphasizes the practices and identities of the Black students and mathematicians, and the nature of their success, independent from any comparison to other racial groups.

Walker (2014) reconceptualized mathematical achievement using practices within and outside of school that involve the mathematicians' kinships and networks. By exploring the lived experiences of Black mathematicians, she uncovered more than just the personal achievement of the individual mathematicians, but also how achievement was a product of support of community from the person's home, neighborhood and schools, and personal network. McGee and Martin (2011) explored practices of successful Black mathematics and engineering students in managing *stereotypes threat* (Steele, 1997) in their field of study. The authors were able to document ways that *stereotype management* developed for many of these students. Being critical about perceptions and attributions of Black behavior, mastering cultural code-switching and always attempting to be "on top of things" were some of the ways through which students in the study developed ways to challenge stereotypes against them. Walker (2014) and McGee and Martin (2011) were able to explore the ways that the person's racial and academic identity influence their trajectories to and through mathematics. Along with Treisman's work, this work challenges existing narratives about Black/African American people and mathematics. Their work also highlights and brings us to issues of identity and power in mathematics education.

Sociopolitical Turn and the Identity and Power Axis

Research done from the sociopolitical perspective pays careful attention to ways that identities influence participation in mathematics (Gutiérrez, 2013). Challenging the status quo with respect to what counts as acceptable knowledge and practices in mathematics is one of the ways that the sociopolitical perspective strive to "*transform* mathematics education in ways that privilege more socially just practices" (p. 40). As we mentioned earlier, the sociopolitical perspective treat knowledge, identity and power as interrelated and as a product of social discourse. This stands in contrast with the treatment of identity as static cultural marker, as it is sometimes conceived in existing equity studies.

In addition to race, equity studies have also consistently focused on the effect of other demographic factors and differences in fairness, access, and opportunity. These factors encompass socioeconomic status, gender, language status, and they have been studied both singularly and in tandem. For example, Reardon (2013) argued that socioeconomic status plays a more significant role in differential outcomes than race. However, Gutiérrez (2013) noted that research that focused on achievement gaps often treated identity as "a fixed, overarching metanarrative, owned by the individual" that simply serve as "a classification system"(p. 45).

The sociopolitical perspective instead views identity as "dynamic" and "multivocal" (Gutiérrez, 2013, p. 46). Identity is dynamic in that it is not owned but rather *performed* depending on the context the person is in. Identity is multivocal in the way that it is negotiated by the person and the environment. Gutiérrez (2013) summarized, "The self, therefore, is a collection of interconnected identities constituted in *practices* such that any given practice positions an individual through and in race, class, ethnicity, sexuality, gender, religion, language, and so forth" (emphasis added, p. 46). Others have argued that one cannot examine issues of access and opportunity without first contending with the impact of the intersection of these identities, i.e., issues of *intersectionality* (Collins, 2000; Crenshaw, 1991).

We know that particularly in mathematics education, gender plays a prominent role in conversations about to equity and achievement. Yet, researchers still tend to adopt a static view of gender in equity research (Alexander, 2013; Oakes, 1990). Collins (2000) focused on ways that gender, sexuality, race, class and nationality serve as mutually constructing systems of oppression. This is to say, that a person does not experience oppression solely as a result of the color of their skin, but rather that gender, sexual orientation, social class and disability status all contribute to a person's lived experience. Such treatment of intersectionality is largely absent in equity research. In addition to lack of consideration of intersectionality and dynamic treatment of identity, other factors, such as language status, sexual orientation and disability status have been deemphasized in the literature. Thus, a consideration of the political contexts of undergraduate mathematics education research associated with social, cultural, and cognitive issues is not only warranted, but also further underlined by its absence in the literature (Gutiérrez, 2013; Martin, 2013).

Identity and Power in Postsecondary Context

In this section we consider two studies: Saundra Laursen's work around Inquiry-Based Learning (IBL) and Keith Weber's work around mathematicians' proof practices. We use them to discuss ways that issues of power and identity have been discussed in the context of postsecondary mathematics, albeit without necessarily using those particular terms. In this way, we also suggest ways in which this work might be productively reinterpreted through an equity lens.

Laursen, Hassi, Kogan and Weston (2014) documented ways in which access to different learning opportunities can mitigate differences in achievement and attitudes among students. IBL approaches to teaching undergraduate mathematics focus on ways to provide students opportunities to engage with problem solving and problem posing, develop and test conjectures and solution paths in collaboration with peers, as well as collaboratively justify and critique arguments. This approach stands in contrast to more traditional lecture-based instructional approaches. In addition to supporting previous research that showed the effectiveness of IBL for positively impacting previously low-achieving students in sizeable and persistent ways (Kogan & Laursen, 2013), the study also documented ways in which IBL had the potential to support women in developing positive self perceptions as capable doers of mathematics.

Laursen et al. (2014) drew on data from over 100 courses offered across 4 campuses to compare students' interest and confidence before and after taking IBL and non-IBL courses, as well as their self-reported learning gains. The authors identified two important differences by gender. First, women's self-reported learning gains in non-IBL sections were lower than those of men, while this gender gap was not present in IBL sections. However, comparable performance in the subsequent coursework between these men and women suggest that the differences in learning gains were perceived rather than actual differences. Second, interest and confidence in doing mathematics decreased more for women in non-IBL sections compared to men, whereas they increased for women in IBL sections. Taken together, the authors argued that IBL instruction "leveled the playing field by offering learning experiences of equal benefit to men and women" (Laursen et al., 2014, p. 412).

One might critique this study for some of the reasons discussed earlier in that the study focused on eliminating gaps (differences) between men and women. The study also (treated gender as a static marker, with little consideration of other aspects of students' identities. On the other hand, the study focused on challenging the participation structure in mathematics through IBL. The simple triangulation with students' performance in the subsequent class problematized the narrative that female students are less confident and less capable as compared to male students. Not only did it show that the performances of the male and female students were comparable, but it also attributed the lower confidence to the influence of traditional participation structure on female students' self-perception. While the study still challenged existing narratives about female students and considered the implications of power redistributions in the classroom through IBL, consideration of the intersectionality of the students' identities were deemphasized. For example, how would controlling for race in addition to gender in these IBL classes influence the results? How many of the students were of a different race, and how many were international students? We explore this issue of intersectionality and power through Keith Weber's work around mathematical proof practices.

To further discuss issues of power in undergraduate mathematics education, we discuss two particular pieces of work: Weber, Inglis and Mejia-Ramos (2014), and Weber and Alcock (2004). Together, the authors of these pieces challenged common misperceptions of students' "unproductive" proof practices. For example, in Weber and Alcock (2004), the authors argued against work that uses Tall and Vinner's (1981) notion of *concept image* and *concept definition*—a common theoretical framework in the RUME community—that positioned students' prior knowledge as a hindrance in learning formal mathematics. They instead documented and illustrated ways that students' intuition and prior knowledge can play an important role in producing correct proofs. Similarly, Weber et al. (2014) challenged the perception that expert mathematicians simply do not do what novice mathematics students do in gaining conviction about proofs. For example, the study found that some mathematicians in the study used examples to gain conviction about a mathematical proof, a practice that is often associated with novice mathematics students. Returning to the political nature of knowledge and practice, these works nicely illustrate ways that studies about individual cognition and practice can challenge ways that we as a community have been privileging certain practices over others.

However, similar to many studies around individual knowledge and practices, these studies de-emphasized issues related to participants' identities. One particular finding in Weber et al. (2014) can assist in illustrating this point. In the paper, the authors reported on a study in which they interviewed highly successful mathematicians and their practice in reading proofs. They had one mathematician who refused to believe a theorem simply because it was written by another mathematician, while most mathematicians in the study stressed that they did not check published proofs for correctness. When asked why this mathematician still read proofs in journals, the mathematician responded,

> "I would like to find out whether their asserted result is true, or whether I should believe that it's true. And that might help me, if it's something I'd like to use, then knowing it's true frees me up to use it. If I don't follow their proof then I would be psychologically disabled from using it. Even if somebody that I respect immensely believes that it's true" (p. 44).

This lone mathematician happened to be a female mathematician. Was it a coincidence that it was a female mathematician that spoke differently? We are not claiming that her refusal to use a proof that she did not follow was solely a function of her gender. At the same time, to assume that her gender had nothing to do with that statement might be shortsighted.

Herzig (2004) summarized existing research on attrition and persistence of doctoral students and identified particular obstacles for women and students of color. Herzig (2004) reported that according to the 2003 Committee on the Participation of Women of the Mathematical Association of America in, female graduate students experience sexist behaviors from their faculty, which include professors who openly express the opinion that women are not as "smart, dedicated, or talented as men" (p.192). The underrepresentation of female faculty also contributed to the isolation of women in mathematics. For example, Blair, Kirkman and Maxwell (2013) reported females comprised only 29% of full-time faculty.

Those two facts alone provide warrant to further explore the mathematician's comment about verifying the proof. To what extent was the necessity to verify another mathematician's proof a result of positioning of this mathematician as a woman in mathematics? More broadly, what additional pressures do female mathematicians experience in their practice, and how might these pressures influence their proof practices? We believe that considering these questions, and more broadly, issues of identity and its relationship to power can provide important nuances and insights into Weber and colleagues' work.

Summary

Lessons from K-12 equity research have the potential to transform the research that is being done at the postsecondary level. Moving away from achievement gap research, and towards research that focuses on uncovering privilege and challenging the status quo in mathematics education is one important lesson that can be applied to postsecondary mathematics education. Careful consideration of power and identity are necessary for understanding equity issues in the postsecondary context. At the same time, we hope that our discussion also illustrates the interconnectedness and the tensions among the four dimensions of equity (Gutiérrez, 2009). For example, as we illustrated, a discussion about Black students' *access* and *achievement* in Fullilove & Treisman (1990) and Treisman's (1985) study could not be separated from a discussion about the intersections of students' *identities*. By challenging existing narratives about Black students through an investigation of these students' lived experiences, these authors disturbed and challenged the *power* distribution in mathematics education. In particular, it challenged the narrative of who can be successful in mathematics, and how such success can be achieved. Thus, as part of the summary, we offer a revision to Figure 1 that we adapted from Gutiérrez (2009) to reflect the interconnectedness of these four dimensions in Figure 2.

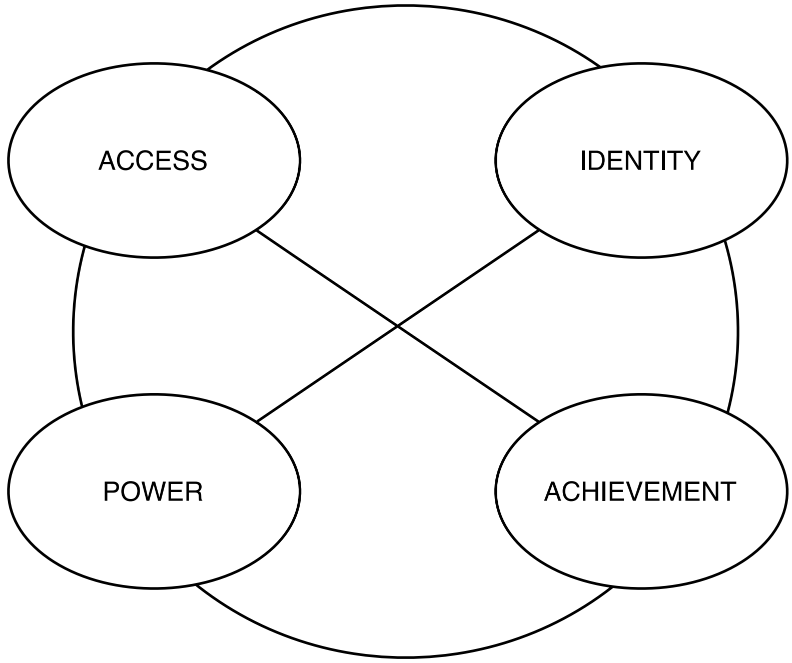


Figure 2. The Interconnectedness of the Four Dimensions of Equity

Discussion and Implications

While we argue in this report that there is a paucity of equity-focused studies in RUME, we hope that we have been successful in illustrating ways that lessons from equity research at the K-12 level can help conceptualize and advance equity research at the postsecondary level. Conceptualizations of equity for research should lead to focused conversations and generate shifts in pedagogical practices. Thus, we close with potential research foci and open questions that resulted from our discussion.

The first question is about studies of individual cognition and practices. How can we be more mindful of equity issues while studying individual student cognition or practices? One helpful thing to do is to provide a better representation of research participants by documenting and reporting on participants' backgrounds. As we discussed with the example of Keith Weber's work, cognitive researchers wield a lot of power. As researchers and practitioners of mathematics at the postsecondary level, we decide what type of practices and thinking we deem to be valuable, which is relevant to issues of equity. Thus, being mindful of the claims we make about what type of knowledge or practices we value is a very important consideration. At the same time, this presents a challenge to existing theoretical frameworks around cognition and student thinking and sense making. To what extent do existing frameworks privilege a particular type of thinking while marginalizing others? This is an open question that we present to the community.

The second question deals with issues of intersectionality of identity. As we discussed earlier, consideration of intersectionality in K-12 equity research is still minimal. While we can agree that a person's identity matters in the way that they participate in mathematics, empirically we have yet to think of ways to consider the intersections of a person's identities and to see how that intersection influences the way that the person participates in mathematics at the postsecondary level. For example, we can begin exploring the lived experiences of female mathematicians of color. Of the 29% female faculty members we mentioned earlier, 60% of them were White (Blair, Kirkman and Maxwell, 2013). What might be some of the factors that contribute to the underrepresentation of women of color in mathematics? Returning to study by Weber et al., perhaps we can explore the extent to which the proof practices of an Eastern European female mathematician similar or different from that of an Asian female mathematician. The purpose of asking these questions is not to *essentialize* a particular group of individuals, i.e., claiming that a particular group of people behave or think a certain way. Instead these questions aim to explore the intersection of people's identities and ways that mathematics as an institution privileges parts of people's' identities but not others, which brings us to the last open question.

The third question deals with institutional factors that might have contributed to the marginalization of underrepresented groups. Postsecondary mathematics is an institution, and from the data we presented about the ethnic breakdown of faculty members in mathematics, it is a *white institutional space* (Martin, 2013, p. 323). Part of the work for equity at the postsecondary level is to explore, as suggested by the sociocultural perspective, some of the taken-for-granted rules of this institution that might have privileged certain groups while marginalizing others. For example, a researcher found that instructors of [teaching] methods courses in mathematics departments reported equity as one of the lowest areas of need of support, whereas instructors of methods courses in college/school of education reported it as one of the highest areas of need (Sean Lee, personal communication, February 11, 2015). Do these differences point to issues about how equity is understood and valued in different institutional contexts?

We are excited by the prospect of using this theoretical report to generate conversations in the RUME community about equity research and equity-focused lenses on learning. It is our belief that these conversations have the potential to advance the ways in which our community understands and values issues of equity, while offering new tools and insights for conceptualizing the current work of those in our community.

References

Alexander, N. N. (2013). Gender Inequality: Mathematics. In *Sociology of Education: An A-to-Z Guide*, J. Ainsworth (Ed.) (pp. 308–309). Thousand Oaks, CA: Sage.

Apple, M. W. (1992). Do the standards go far enough? Power, policy, and practice in mathematics education. *Journal for Research in Mathematics Education*, 412-431.

Blair, R., Kirkman, E.E., & Maxwell, J.W. (2013) Statistical abstract of undergraduate programs in the mathematical sciences in the United States Fall 2010 CBMS Survey.

Bonilla-Silva, E. (2003). Racism without racists: Color-blind racism and the persistence of racial inequality in the United States. Lanham, MD: Roman & Littlefield.

Civil, M. (2006). Building on community knowledge: An avenue to equity in mathematics education. In N. Nasir & P. Cobb (Eds.), *Improving access to mathematics: Diversity and equity in the classroom* (pp. 105–117). New York: Teachers College Press.

Cobb, P., & Yackel, E . (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. Educational Psychologist, 31, 175-190.

Collins, P. H. (2000). Gender, black feminism, and black political economy. *The Annals of the American Academy of Political and Social Science*, *568*(1), 41-53.

Crenshaw, K. W. (1991). Mapping the margins: Intersectionality, identity politics, and violence against Women of Color. *Stanford Law Review*, 43, 1241–1299.

Blair, R., Kirkman, E.E., & Maxwell, J.W. (2013) Statistical abstract of undergraduate programs in the mathematical sciences in the United States Fall 2010 CBMS Survey.

Executive Office of the President. President's Council of Advisors on Science and Technology. *Report to the President. Engage to Excel: Producing one million additional college graduates with degrees in science, technology, engineering, and mathematics.* Washington, DC: Executive Office of the President. President's Council of Advisors on Science and Technology, 2012. www.whitehouse.gov/administration/eop/ostp/pcast/docsreports

Flores, A. (2007). Examining disparities in mathematics education: Achievement gap or opportunity gap? *The High School Journal, 91*(1), 29-42.

Fullilove, R. E., & Treisman, P. U. (1990). Mathematics achievement among African American undergraduates at the University of California, Berkeley: An evaluation of the mathematics workshop program. *Journal of Negro Education, 59*, 463–478.

Gutiérrez, R. (2013). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education*, *44*(1), 37-68.

Gutiérrez, R. (2009). Framing equity: Helping students "play the game" and "change the game." *Teaching for Excellence and Equity in Mathematics*, *1*(1), 5-7.

Gutiérrez, R. (2008). A "gap gazing" fetish in mathematics education? Problematizing research on the achievement gap. *Journal for Research in Mathematics Education, 39*(4), 357–364.

Gutiérrez, R. (2002). Enabling the practice of mathematics teachers in context: Towards a new equity research agenda. *Mathematical Thinking and Learning, 4*(2 & 3), 145–187.

Gutiérrez, R. & Dixon-Román, E. (2011). Beyond Gap Gazing: How Can Thinking About Education Comprehensively Help Us (Re)envision Mathematics Education?. In *Mapping equity and quality in mathematics education* (pp. 21-34). Springer Netherlands.

Harper, S. R. (2010). An anti-deficit achievement framework for research on students of color in STEM. In S. R. Harper & C. B. Newman (Eds.), *Students of color in STEM: Engineering a new research agenda. New Directions for Institutional Research* (pp. 63-74). San Francisco: Jossey-Bass.

Herzig, A.H. (2004). Becoming mathematicians: Women and students of color choosing and leaving doctoral mathematics. *Review of Educational Research, 74*(2), 171-214.

Katz, B., Post, R., Savic, M. & Cook, J.P. (2015). An analysis of sociomathematical norms of proof schemes. Paper presented at Special Interest Group of the Mathematical Association of America (SIGMAA) on Research in Undergraduate Mathematics Education (RUME) 2015 Conference: Pittsburgh, PA.

Kilpatrick, J. (1992). A history of research in mathematics education. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 3–38). New York: Macmillan.

Kogan, M., & Laursen, S.L. (2013). Assessing long-term effects of inquiry-based learning: a case study from college mathematics. *Innovative Higher Education*, 1–17.

Laursen, S. L., Hassi, M. L., Kogan, M., & Weston, T. J. (2014). Benefits for Women and Men of Inquiry-Based Learning in College Mathematics: A Multi-Institution Study. *Journal for Research in Mathematics Education*, *45*(4), 406-418.

Lave, J. & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge university press.

Lubienski, S. T., & Bowen, A. (2000). Who's counting? A survey of mathematics education research 1982-1998. *Journal for research in mathematics education*, 626-633.

McGee, E. & Martin, D. (2011). "You would not believe what I have to go through to prove my intellectual value!" Stereotype management among academically successful black

mathematics and engineering students. *American Educational Research Journal, 48*(6), 1347-1389.
Martin, D. B. (2013). Race, Racial Projects and Mathematics Education. *Journal for Research in Mathematics Education*, 44(1), 316–333.
Martin, D. B. (2000). *Mathematics success and failure among African-American youth: Theroles of sociohistorical context, community forces, school influence, and individual agency*. Mahwah, NJ: Erlbaum.
Nasir, N. (2005). Individual Cognitive Structuring and the Sociocultural Context: Strategy Shifts in the Game of Dominoes. The Journal of the Learning Sciences, 5 - 34.
National Science Foundation, *Science and Engineering Indicators 2014.* Arlington, VA.
Oakes, J. (1990). Opportunities, achievement, and choice: Women and minority students in science and mathematics. Review of Research in Education, 16, 153-222.
Reardon, S. F. (2013). The widening income achievement gap. *Educational Leadership*, *70*(8), 10-16.
Saxe, G. B., & de Kirby, K. (2014). Cultural context of cognitive development. *Wiley Interdisciplinary Reviews: Cognitive Science*, *5*(4), 447-461.
Shah, N. (2013). Racial Discourse in Mathematics and its Impact on Student Learning, Identity, and Participation. Dissertation, University of California, Berkeley.
Skovsmose, O., & Greer, B. (2012). *Opening the cage: critique and politics of mathematics education* (Vol. 23). Springer.
Steele, C. M. (1997). A threat in the air: How stereotypes shape the intellectual identity and performance, *American Psychologist*, *52*(6), 613-629.
Tall, D.O. & Vinner, S. (1981). Concept image and concept definition in mathematics, with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151-169.
Treisman, U. (1992). Studying students studying calculus: A look at the lives of minority mathematics students in college. *College Mathematics Journal*, *23*(5), 362-372.
Treisman, U. (1985). A study of the mathematics performance of black students at the University of California, Berkeley. Dissertation, University of California at Berkeley.
Vygotsky, L. S. (1978). Mind and society: The development of higher mental processes.
Walker E. N. (2014). Beyond Banneker: Black mathematicians and the paths to excellence. Albany: State University of New York Press.
Weber, K. & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics*, *56*, 209-234.
Weber, K., Inglis, M., & Mejia-Ramos, J. P. (2014). How mathematicians obtain conviction: Implications for mathematics instruction and research on epistemic cognition. *Educational Psychologist*, *49*(1), 36-58.

Variations in Implementation of Student-Centered Instructional Materials in Undergraduate Mathematics Education

Christine Andrews-Larson
Florida State University

Valerie Kasper
Florida State University

Pedagogical reforms in undergraduate STEM courses are garnering increasing attention in the literature and from national organizations in disciplines such as mathematics, physics, and chemistry. While there is significant evidence to support the effectiveness of classroom-based pedagogical reforms, the way in which these reforms are taken up by those not involved in their development varies widely. This comparative case study seeks to better understand the ways in which student-centered instructional reforms in undergraduate mathematics are implemented. Data is taken from videotaped instruction of three participating instructors at three different institutions as they work to implement a student-centered instructional unit focused on supporting students' understanding of span and linear independence in undergraduate linear algebra. This analysis examines the way in which these instructors structured class time when implementing the unit, and the nature of opportunities for students to explain their thinking in the context of whole class discussions.

Key words: instructional change, classroom research, discourse analysis

Background, Literature, and Research Questions

The number of students in the United States entering and completing undergraduate and graduate science, technology, engineering, and mathematics (STEM) programs is declining, and this decline has been connected to the nature and quality of instruction in undergraduate STEM courses (Fairweather, 2008; Seymour & Hewitt, 1997). Because of the social and economic implications posed by a reduction in STEM graduates, funding for research aimed at improving instruction in undergraduate STEM courses has increased (National Science Foundation, 1996; National Research Council, 1999, 2011). While there is significant evidence to support the effectiveness of classroom-based pedagogical reforms, these reforms have failed to spread, and the research on how to improve instruction at scale in undergraduate STEM is limited (Fairweather, 2008; Henderson, Beach & Finkelstein, 2011).

Classroom-based research has identified student-centered instructional methods that are related to greater conceptual learning gains in mathematics (Hiebert & Grouws, 2007). Such student-centered instruction is marked by two primary characteristics: (1) students have opportunities to engage in cognitively demanding tasks, and (2) students have opportunities to engage in mathematical argumentation with their peers (Hiebert & Grouws, 2007). The cognitive demand of a task distinguishes whether it is one in which students are asked to recall or reproduce terms/procedures (low cognitive demand), or one in which students are asked to develop, generalize, or justify a solution method (high cognitive demand) (Stein & Lane, 1996). Students' engagement in mathematical argumentation entails justifying and explaining their thinking and evaluating arguments developed by their peers. While Hiebert and Grouws's (2007) findings are drawn from a synthesis of K–12 mathematics literature, theirs are consistent with findings from a variety of undergraduate STEM courses including differential equations, physics, and chemistry (e.g., Deslauriers, Schelew, & Wieman, 2011; Kwon, Rasmussen, & Allen, 2005; Lewis & Lewis, 2005).

Classroom-based research on effective instructional methods is compelling, but there is evidence that a variety of both individual instructor and institutional factors can substantially influence the ways in which pedagogical reforms are taken up by those not involved in their development (Henderson et al., 2011). For instance, in the K–12 mathematics education literature, instructors' knowledge of both content and how students think about that content, as well their beliefs about instruction and students' capabilities, have been related to the likelihood that they will pose cognitively demanding tasks to their students, and to the likelihood that they will enact those tasks in a way that maintains the cognitive demand (Charalambous, 2010; Garrison, 2013; Son, 2008). Investigative studies have also identified challenges postsecondary mathematics instructors encounter when working to adopt more student-centered instructional methods; a central theme in this research is the need for instructors to develop an understanding of student thinking to plan for and lead discussions that effectively build on students' solution strategies (Johnson & Larsen, 2012; Speer & Wagner, 2009; Wagner, Speer, & Rossa, 2007). In addition to these individual level factors, institutional factors have been related to the success or failure of instructional change strategies. In a review of the literature on instructional change strategies in undergraduate STEM education, Henderson and colleagues (2011) find that top-down policies and designing and disseminating curricular materials are strategies that have proven unsuccessful in generating instructional change in STEM; effective change strategies consider teachers' conceptions of instruction and align with existing institutional contexts. Factors that have served as impediments to instructional change include student attitudes, faculty perceptions of departmental expectations regarding coverage, time demands on instructors, and departmental culture (Henderson & Dancy, 2007; Enderle, Southerland & Grooms, 2013).

Given the need for instructional change in undergraduate STEM, it is necessary to better understand the ways in which individual instructors engage in efforts to change instruction. More specifically, a better understanding of the ways in which student-centered instructional materials are interpreted and implemented has the potential to inform the design of the materials and instructional supports needed to generate instructional change strategies. Our research questions are: (1) What challenges do instructors report facing when working to change instruction by using student-centered instructional materials, and how do these challenges shift when instructors use the same materials in subsequent semesters? (2) In what ways do instructors structure class time and opportunities for students to explain their thinking when implementing student-centered instructional materials, and how do these structures and opportunities shift when instructors use these materials in subsequent semesters?

Data Sources and Methods of Analysis

This analysis involves conducting a comparative case study (Yin, 2003) of data that was collected during Fall 2013 and Fall 2014 from three participating instructors at three undergraduate institutions that implemented one unit of student-centered instructional materials in the context of an intact introductory linear algebra course. Participating instructors were recruited on a volunteer basis, but deliberately selected to represent a variety of institutional contexts (see Table 1 below). More specifically, instructors were selected to represent both public and private institutions of varying size and geographic location in the U.S.—one in the Midwest, one in the Northwest, and one in the Northeast. The instructional unit is intended to span approximately 4–5 one-hour class sessions. Student prerequisites for the course as well as instructor background and training vary by institution.

The participating instructors are "best case" implementers in many regards: All have an expressed interest in mathematics education, and all have some background knowledge of and interest in teaching with this set of linear algebra instructional materials (through conference presentations attended or interactions with colleagues familiar with the materials). However, the participating instructors also differ along a number of key dimensions that are likely to be important in shaping the way they interpret the materials and the way in which they actually implement the instructional materials in their classrooms. This case selection is appropriate both theoretically and pragmatically, as the study aims to explore the challenges encountered by individuals aiming to achieve instructional change. Indeed, it has been documented elsewhere that undergraduate STEM instructors commonly see a need for instructional change but do not attempt to implement significant changes to their teaching for a variety of reasons (Henderson & Dancy, 2007). This study aims to generate theory about what instructional change looks like by examining the work of those seeking to implement it.

Instructor	Academic Specialty	Institution Type	# of Tenure-earning math faculty in department	Duration of employment in department
A	PhD mathematics	Private 4-year college	10-15	5 years
B	PhD mathematics education	Public 4-year college	Less than 5	1 year
C	MA mathematics education	Public PhD-granting university	More than 40	15 years

Table 1: Profiles of instructor characteristics and institutional context

Data for this analysis is taken from video recordings of classroom instruction of implementation of the instructional materials in 2013 and 2014, as well as a series of audio-recorded interviews conducted with instructors before, during, and after they had completed implementation of the instructional unit in 2013 and 2014. In order to identify challenges instructors reported in implementing the instructional materials, as well as shifts in challenges from the first year to the second year of implementation, field notes from the interviews were summarized to identify themes in the challenges noted by instructors.

In order to understand how instructors structured class time and built on student thinking when implementing the instructional materials in 2013, we conducted a two-phase analysis. In the first phase of this implementation analysis, we generated content logs for each class session. These were generated in a table format, with columns for timestamp, description of classroom events (in which teacher contributions were distinguished from student contributions), discourse structure (small group work, whole class discussion, or lecture), notation and language introduced, and other notes. In order to generate a broad characterization of instructors' use of time, we coded instructors' use of time into three broad categories: small group work, whole class discussion, and lecture. Segments of the class when the whole class was focused on the instructor providing information to students were coded as lecture; segments of the class when the whole class was focused on a single conversation in which multiple students contributed ideas to the conversation (e.g. by explaining or justifying their solution strategy) were coded as whole class discussion. We then used these content logs to generate initial categories for how instructors structured opportunities for students to share their thinking, as well as to summarize their overall use of time.

Instructors indicated in 2014 interviews that they felt much more able to dig into and build on students' thinking. In order to explore this issue, we decided to closely examine the whole class discussions facilitated by instructors following students' work on task 1 only in 2014. This would allow us to examine more carefully the ways in which instructors structured opportunities for students to share their ideas and approaches in whole class discussion, and the ways in which instructors built on those ideas – and then to examine how these compared to 2013 implementation.

By looking across these two years of implementation, we identified two broad categories for ways in which instructors structured opportunities for students to share their thinking, and two broad categories for ways in which instructors built on student thinking. Finally, we compare shifts in use of these structures from 2013 to 2014 in the context of whole class discussions following students' work on task 1.

Instructor Support Materials

The instructional support materials used in this study were developed across a series of four classroom teaching experiments in linear algebra, and are described elsewhere (Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012). Following the model described by Lockwood, Johnson, and Larsen (2013), the instructor support materials aim to support instructors in implementing student-centered teaching. These materials include a sequence of student-centered, cognitively demanding tasks that build toward understanding of key mathematical concepts, a rationale for the instructional sequence, information regarding common student strategies, and suggestions for implementation (e.g., guiding questions for whole-class discussions after students have worked on the tasks). The rationale for the design of the instructor support materials is firmly rooted in the literature (Ball & Cohen, 1996; Collopy, 2003; Davis & Krajcik, 2005) and has been shown to be useful for instructors (Lockwood et al., 2013).

The specific instructional unit that is the focus of this analysis is organized around the goal of supporting students in developing a conceptual understanding of span and linear independence of sets of vectors in R^n. The instructional sequence is organized around a sequence of four primary tasks that are designed to support students in developing intuition that coordinates geometric and symbolic representations of linear combinations of vectors and properties of sets of vectors. In the first task, students are given two "modes" of transportation whose motion corresponds to the vectors $\begin{vmatrix} 3 \\ 1 \end{vmatrix}$ (whose movement results in a displacement of 3 miles East and 1 mile North of its starting location each hour) and $\begin{vmatrix} 1 \\ 2 \end{vmatrix}$ (whose movement results in a displacement of 1 miles East and 2 mile North of its starting location each hour), and asked to find if they can reach Old Man Gauss, who lives 107 miles East and 64 miles North of their home. Task 2 asks if there is anywhere Old Man Gauss can hide (which leads to the defining of span as the set of all possible linear combinations of a set of vectors). Task 3 shifts to R^3 and asks whether "non-trivial" journeys can be taken that start and end at home (which leads to the defining of linear dependence and independence). Task 4 asks students to generate linearly dependent and independent sets of specified numbers of vectors in R^2 and R^3 or explain why it is not possible to do so, and to develop at least 2 generalizations they think are true based on this work.

Findings

In this section, we first offer a brief overview in of shifts in the challenges instructors reported in implementing the materials from year 1 to year 2. We then summarize the ways in which instructors made use of class time in year 1 implementation of the instructor materials. We then provide examples to illustrate each of our categories for how instructors structured opportunities for students to share their thinking and the ways in which instructors built on student thinking. Finally, we compare instructors use of these structures in whole class discussion following students' work on task 1 in 2013 to 2014. Our discussion considers implications for these findings, particularly with regard to shifting challenges faced by instructors and its implications for the implementation of student-centered instructional materials.

Shifts in challenges reported by instructors from 2013 to 2014

One of the most striking trends that emerged from our examination of data from interviews with instructors was the shift in demands experienced by instructors from the first to the second year of implementation. During the first year of implementation, instructors' talk about planning and use of the instructional materials tended to focus on issues of timing within class sessions, pacing of material across class sessions, aligning materials with the course textbook (both for mathematical coherence and for issues of assigning homework), and ways of grouping students and getting them to talk in class. This is in contrast with the second year of implementation, when instructors' talk in interviews focused much more heavily on students' mathematical thinking. This shift in instructors' talk about their instruction was reflected in their implementation of the materials themselves in a way that is consistent with one instructor's description of this shift:

> Last year, I introduced the tasks, but it was more like here you go. Let's see what happens with this. I was more willing this year to let discussions go… I knew where everything was going. I could see the forest instead of focusing on the individual trees. Last year I was very uncomfortable doing that, I think I cut in too soon or imposed my own strategy too soon.

While this comment does not highlight the challenges instructors experienced in their work outside of class time to plan for use of the materials, it highlights the way in which this instructors' in-class experience implementing the materials shifted as well as the way in which the knowledge instructors gained in the first year of implementation impacted the instructional choices they made in subsequent implementation.

Implementation: Use of class time in 2013

In 2013, all three instructors allocated a similar amount of class time to the instructional sequence; those with 50 minute classes used 6-7 days of instruction whereas the instructor with 90 minute classes used about 4 days of instruction. However, as shown in Figure 1, instructors differed in their allocation of time among small group work, whole class discussion, and lecture.

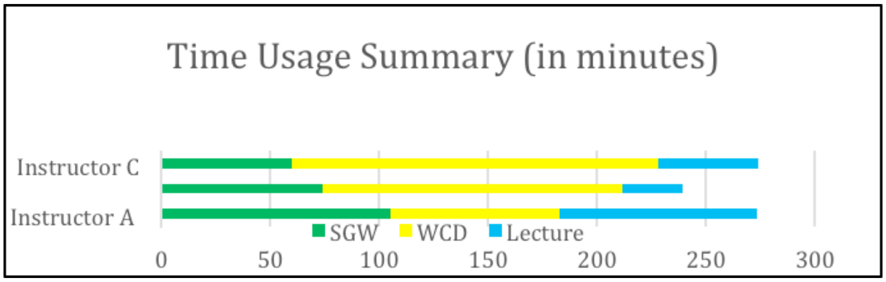


Figure 1: Net use of time by instructors

In order to better illustrate how instructors structured time on a day-to-day basis, we created a diagram to show the trajectory of time use for each instructor as shown in Figure 2. The discourse patterns in Instructor A's class were the most consistently structured: following an initial lecture on the first day that gave an overview of the course, there was always a chunk of time for small group work, followed by time for discussion of student approaches, followed by a short lecture. This stands in stark contrast with the discourse structure in instructor B's class, where there were frequent shifts between small group work and whole class discussion. The discourse structure in instructor C's class, on the other hand, was marked by longer periods of whole class discussion, and less frequent use of small group work.

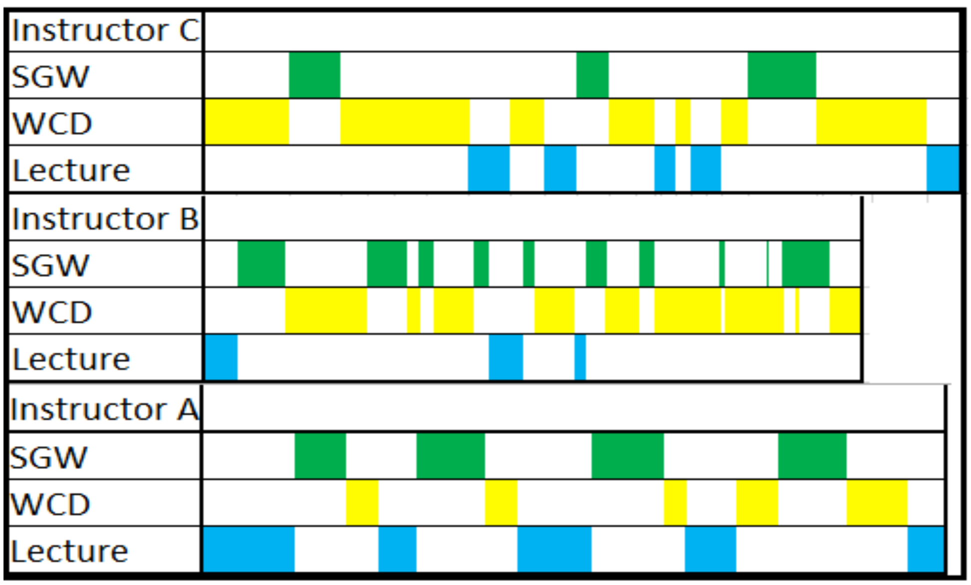


Figure 2: Trajectory of Time Use

While these differences in use of time are interesting, they offer little insight into the nature of the opportunities students had to share and explain their thinking, and how students' ideas were leveraged to advance the mathematical agenda of the class as a whole – thus we turn our focus to this issue in the following section by drawing on video data of classroom instruction from both 2013 and 2014. Importantly, we note that all instructors across both years did use cognitively demanding tasks by virtue of using the central tasks designated in the instructional materials, and that all instructors across both years tended to maintain the cognitive demand of the task by requiring students to develop their own approaches and methods to solving. Given the prevalence of use of tasks of low cognitive demand in mathematics instruction and the tendency of instructors in K-12 settings to lower the cognitive demand of mathematics tasks, we find this promising for future efforts toward achieving instructional change in the context of undergraduate mathematics instruction.

Structures for Student Sharing & Building on Student Thinking in 2013 and 2014

In our implementation analysis, we identified four ways in which instructors elicited and built on student contributions in whole class discussion following students' work in groups on a problem-solving task. For the purposes of this paper, we name these four structures as follows: (1) getting students to talk, (2) getting students to explain, (3) using student ideas to explain or formalize, and (4) using student ideas as the basis for a new mathematical question or task.

Structures 1 and 2 focus on the student contributions elicited, and are defined in terms of the things students actually said (rather than what the instructor may have been hoping students would say). We delineated these two structures because we noted some substantial differences across classes in the ways in which students were able to make contributions to whole class discussions. More specifically, in some instances students contributed explanations of what their group did in trying to solve the problem posed – and these explanations typically consisted of at least a couple of full sentences characterizing their approach. We characterize these as instances as structure 2: getting students to explain. In contrast, we observed instances in which students' contributions to whole class discussions were limited to sentence fragments that did not entail a complete thought (sentence), claim, or justification. We characterize whole class discussions in which student contributions are of this nature as instances of structure 1: getting students to talk. Subsequently, we will illustrate each of these structures with examples.

Structures 1 and 2 are relatively content neutral, whereas structures 3 and 4 are more complex in that they rely on the nature of student contributions made, as well as what the instructor did with those contributions mathematically. Further, structure 3 and 4 typically coincide with structure 2 as they depend on student explanations as a basis for what the instructor does next mathematically. More specifically, structures 3 and 4 characterize the way in which the instructor uses the content of student contributions to further the mathematical agenda of the class. What delineates structure 3 from structure 4, largely, is who assumes mathematical authority for advancing the mathematics beyond students approaches to the task presented. In structure 3, "using student ideas to explain or formalize," the instructor assumes mathematical authority by explaining mathematical connections (e.g. between groups' approaches) or formalizing language or notation, but by doing so in a way that the instructor explicitly relates to ideas brought forth by students. In structure 4, "using student ideas as the basis for a new mathematical question or task," the instructor pushes mathematical authority onto students by asking them to engage in an idea set forth by students in the context of a new question or task. Based on our data, it appears that it is difficult for instructors to implement structure 4 smoothly their first time implementing the materials. Further, we suspect that structure 4 is intertwined with a particular set of goals for how instructors aim to engage their students in the development of the mathematics.

In the sections below, we offer examples to illustrate each of the four structures and then discuss shifts we saw in the use of these structures from year 1 to year 2 across instructors in the context of whole class discussions following small group work on task 1 of the instructional sequence.

Structure 1: Getting Students to Talk
The example we share of an instructor "Getting students to talk" takes place in the context of an instructor working to facilitate a whole class discussion after the completion of the first task in the provided instructional materials. Students had worked on the task for about 20 minutes during the first day of class. Because there was not time for a whole class discussion on this first day, the instructor told us in a debriefing interview that she took pictures of examples of the work of several groups and put them into a powerpoint in order to have a whole class discussion around the variety of approaches used by different groups in the class. The instructor began the whole class discussion around student approaches to the previous day's task as follows:

I: I wanna look at how things were written and how they were solved, okay, so that I get a feel for what everybody can do in terms of solving systems. So, there was the first group. They wrote their equation as vectors, right, and set it equal. Then what did they do?
S: A system of equations.
I: Yeah, they got to a system of equations. Okay, and they solved it by- what does it look like? How'd they solve that?
S: Elimination.
I: What are they doing here? Right there. …Okay. Um, so if you look at how they're solving it, through the three x's. Why?
S: Elimination.
I: Elimination. How many people know how to do elimination? Yeah? One, two, three, everybody? Okay. So elimination involves doing what?
S: Crossing stuff out.
I: Crossing stuff out, but how?
S: Inverse.
I: Not an inverse. Well, maybe an inverse, so to get from here to here what did they do?
S: Multiply. [choral response]
I: Yeah, they multiplied the bottom one by...?
S: Three...

The discussion continued in this manner for 8 minutes, with students offering brief contributions (typically one word or phrase) about the approaches to representing and solving depicted in the powerpoint slides. The instructor then shifted into a powerpoint presentation focused on interpreting the situation from a vector equation perspective and a systems of equations perspective, raising the question of whether the solution is unique. The nature of student contributions (contributing at the level of a word or phrase) across this class period is consistent with the excerpt shown above.

Structure 2: Getting Students to Explain
In contrast to structure 1, wherein students talk in whole class discussions but do not provide an explanation of their thinking, structure 2 captures instances in which students do provide explanations of their thinking in whole class discussion. Our example of structure two takes place immediately following students work in small groups on task 1.

I: Okay so let's reconvene here. That is beautiful, I like that [gesturing to group's work]. All right so Laura why don't you kind of hold up your board and kind of explain to everybody the method that your group used.

S: So, something times the 3,1 plus something times the 1,2 vector equals 107, 64. So I just distributed the a and the b and then added them. So it makes a vector 3a plus b comma a plus 2 b, equals 107, 64. And then it equaled 3a plus b equals 107. And a plus 2b equals 64. And then I just did a little algebra. Multiplied by negative 2 to cancel the 2b right there. Then I just solved for a and b. And it equaled 30 and 17.

I: Okay so you have what we call a vector equation and then you used that to produce a system of equations a system of linear equations and you used some kind of substitution method to figure out what a was and what b was. Okay and can you guys, Mike can you explain reorient your white board and explain what's going on there in your diagram?

S: The first time we just went and rode the hover board the first time, rode it for 30 hours and then it ran out of gas or battery power so then we jumped on the magic carpet and rode it for 17 hours, then we went back home and we wanted to go again so then we rode the hover board then the magic carpet but then we had to recharge it a few times and it starts to suck so then we went halfway which would be 15 hours on our hover board then we rode the magic carpet for its duration of 17 hours then we switched again to stretch our legs for the last 15 hours.

I: Now you don't have to show us on the board because I think it's kind of small but can you tell me the vector equation that represents the stutter step method, the staircase method?

S: Yes it's uh 15 times the vector 3, 1 plus 17 times the vector 1, 2 plus fifteen times 3,1.

In this structure, students contributed explanations of their groups' approaches to the problem to the whole class discussion. This stands in contrast to structure 1, when students contributed in ways that didn't provide explanations of their thinking. We also note that in this instance, Laura's group's strategy highlights their work to solve a system of equations as an approach to task 1, whereas Mike's explanation is consistent with a vector equation interpretation and highlights the interpretation of the scalars weighting the vectors.

We conjecture that students were able to contribute in much more substantive way than in structure 1, at least in part, because they were being asked to explain what their own group did rather than speculate what another group did just based on the inscriptions other groups had created in a previous class period. We also think the specificity about who was being asked to share may have contributed to students' offering of more substantive explanations of their thinking in the discussion.

Structure 3: Using Student Ideas to Explain or Formalize

Our third structure, "Using student ideas to explain or formalize" captures instances in which instructors use student ideas, approaches, or explanations as a basis for formalizing mathematical terminology or notation, or for offering a mathematical explanation. This structure typically took place following structure two in which students explain their thinking, though we can imagine instances in which student work might be publicly displayed and the instructor might narrate aspects of students' written work or explanations to achieve this goal (though such a non-verbal structure likely entails a greater likelihood that the instructor might misinterpret students' thinking).

Our example of structure three takes place after students have worked in groups on task 1, and after a couple of groups' approaches have been explained by students. In the excerpt below, the instructor asks another group to explain their approach. The instructor then explains how this approach relates to the approaches described by previous groups, and uses this to highlight an important relationship between vectors equations and systems of equations.

I: How about group number 3, can, um. So, this group did the systems approach as well. Um they had a different, um, similar but different approach for how they came up with the equation. So you can you share that with us?

S: So what we figured out was that you've got a vector equation here and it's going to be some scalar multiple of the first vector, plus some scalar multiple of the second vector is going to be equal to the vector on the end. And from there we just distributed these in and basically everything that is on the top has got to equal it in some way and everything on the bottom has got to equal in some way so system of linear equations. I'll share you that part, so that's how we got it.

I: Ok, and so, let's… let me write that up here. Essentially what they did, and I'm going to call, just to be consistent with our notation so far in the class, I call it x1 and x2. So essentially, they took an approach that maybe goes one step further back from what these two groups did. Right, so this, I'd like to point out that this agrees exactly with the approach that these two groups did and even the group that did the, um, approximation and guess and check approach, ultimately. So, what they did was use how we define scalar multiplication or how we have been using scalar multiplication rather, and moved x1 into the vector. Multiplied by the scalar. Multiplied x2 in, and then said everything, and this is where what Calvin was saying comes into play. Everything in the horizontal competent must add up to 107 and everything in the vertical component must add up to 64. Ok so this is, essentially their first step was to us the column vector equation. Which then of course leads right into the system of linear equations. So there's a point that I want to make here. And the point is that systems of linear equations will enable us to solve vector equations pretty handily.

As the instructor spoke, he made reference to, and adjusted an inscription he had made on the board. Prior to this exchange, the instructor had written on the board the vector equation $30\begin{bmatrix}3\\1\end{bmatrix}+17\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}107\\64\end{bmatrix}$ as a way of recording the approach of the group who used the "guess and check" method. The instructor had used this to introduce the language of linear combinations by circling the left hand side of this equation and drawn an arrow below that pointed to the words "linear combination of $\begin{vmatrix}3\\1\end{vmatrix}$ and $\begin{vmatrix}1\\2\end{vmatrix}$." During the last turn of talk in the exchange shown above, the instructor erased the coefficients 30 and 17 in the vector equation that had been written and replaced them with x_1 and x_2 so that the vector equation now read "$x_1\begin{vmatrix}3\\1\end{vmatrix}+x_2\begin{vmatrix}1\\2\end{vmatrix}=\begin{vmatrix}107\\64\end{vmatrix}$." This shift in denoting the scalars of the vector equation helped the instructor link the approaches he had denoted with the vector equation to systems of equations approaches used by other groups.

Structure 4: Using student ideas as the basis for a new mathematical question or task

Our final structure, "Using student ideas as the basis for a new mathematical question or task" was the least frequently observed, and appeared to be the most challenging to implement productively -- but when it was implemented productively, it seemed to be particularly powerful. We observed this structure play out in two ways: one was when an instructor highlighted an approach of a particular group and sent all students back to their groups to re-solve the task using the approach of that group (and relate it to their own approach). For example, one instructor asked all groups to illustrate their solution using a tip-to-tail graphically interpretation (as one group had done) of a vector equations when most groups had used a systems of equations approach to a particular problem.

The other way in which we saw this structure play out was when an instructor used different (sometimes conflicting) responses or interpretations to a question to generate a point of discussion. In this case, an instructor first spent time asking students to explain their group's approach, established that there were two primary approaches (vector equation with tip-to-tail graphical interpretation in which the solution appeared as weights on the vectors), and system of equations in which the solution appeared as a point of intersection, and asked students to return to their groups and decide if their group thought the solution was unique. A polling of the class revealed that half the groups thought the solution was unique and half thought it was not. This created an opportunity to hear how various groups were interpreting what it meant for a solution to be unique, and to then clarify the intended meaning of a unique solution, thereby generating consensus about the uniqueness of the solution.

Shifts in implementation from 2013 to 2014:

In both 2013 and 2014, we observed all instructors making explicit efforts to build on students' ideas and approaches; this was done in a variety of ways. Overall, instructors' elicited and built on student explanations more in 2014, as depicted below in Table 2. This table indicates how many student groups there were in each class (note considerable variation in class size), how many of those groups' approaches were publicly represented (e.g. displayed on white boards around the room, on a powerpoint, document camera, etc.). We note that all instructors prioritized public display of students' work, and that there were not explanations provided by more than 4 student groups in any instance. The only instructor who exhibited a decrease in the number of student explanations in whole class discussion following task 1 in year 2 was instructor A, and one or more student explanations were part of whole class discussion in both years. We subsequently discuss shifts observed with regard to the way each instructor facilitated whole class discussion following students' work on task 1.

The most notable shifts were observed in instructor C, who in 2013 struggled to elicit student explanations and in 2014 not only elicited student explanations (structure 2), but effectively used them to move forward the mathematical agenda of the class (structures 3 and 4). We note that the examples provided for structures 1 and 4 both come from instructor C. We also note that instructor C collaborated closely with a colleague at the same institution who was teaching with the same set of instructional materials, and speculate this is likely related to these significant shifts.

Instructors B and C both exhibited an increase in structure 3 from 2013 to 2014. In particular, both instructors were observed explicitly using student approaches to introduce and formalize mathematical language and notation in 2014. Instructor B did use student ideas as the basis for new mathematical tasks in 2013, but these emergent tasks were less centrally related to the mathematical goals of the unit than those observed in 2014.

Instructor	Approx # of student groups	# of student approaches represented publicly	# of student approaches *explained* by students in WCD	Discussion structures observed
C- 2013	14	14	0	1
C- 2014	10	10	4	2,3,4
B-2013	2	2	2	2,4
B-2014	5	5	4	2,3,4
A-2013	8	8	4	2,3
A-2014	8	8	1	2,3

Table 2: Student approaches to task 1 explained in whole class discussion

Instructor A's implementation in 2013 was similar to that of instructors B and C in 2014 in eliciting of multiple student explanations and using these explanations to introduce and formalize mathematical language and notation. Importantly, instructor A had used a version of the materials at least once prior to the 2013 implementation, so instructor A's implementation in 2013 was contextually similar to that of instructors B and C in 2014. It is interesting to note that in 2014, instructor A elicited fewer student explanations, but similarly used student approaches to 'narrate' a particular mathematical storyline in both 2013 and 2014.

Overall, we highlight the increase in structures 2 and 3 as indicators that document important shifts in implementation of the materials – namely shifts that indicate in increase in student explanations as well as an increase in instructor formalization and explanation that built on those explanations. We conjecture that instructors' familiarity with student approaches and the mathematical storyline of the unit played an important role in supporting these shifts.

Discussion & Next Steps

This work set out to better understand the challenges experienced by instructors working to implement instructional change through the use of research-based, student-centered instructional materials. Based on instructor reports, the logistical challenges instructors experience (e.g. paciing, alignment with other curricular resources, homework selection, grouping students and getting them to talk) shift significantly from their first time implementing their materials to the second time implementing the materials. Concurrently, the classroom implementation of these materials was seen to shift in that instructors' eliciting and building on student thinking increased from the first time implementing the materials to the second time.

We conjecture that these two shifts are important to consider as part of the broader conversation about instructional change, and that these two shifts are related in important ways. We conjecture that the first time through the materials, instructors learn a lot about the kinds of strategies students use, how long the tasks take their students, what is challenging for their students – and this learning has the potential to help instructors develop a vision of how these elements might fit together to form a coherent mathematical narrative, as well as how to structure class time in ways that orchestrates the creation of this narrative together with students. The second time through the materials, instructors are positioned with this knowledge AND they have more cognitive resources available to devote to thinking about student strategies than they did when they were having to worry about issues of pacing, alignment, HW selection, getting students to talk, and figuring out and anticipating the kinds of things students might say and do and how to make use of those instructionally.

This work has the potential to inform the work of those working to support instructional change in undergraduate STEM fields as well as K-12 STEM fields. Specifically, it documents the shifting nature of challenges experienced by highly expert instructors working to teach in a new way. This has the potential to inform and contextualize the work of those who work with pre-service and in-service teachers, particularly those who support and/or examine the implementation of curricular innovations. Additionally, this work points to the potential value of identifying structures through which instructors can make incremental changes to their instructional practice. It is important to acknowledge the difficulties inherent to implementing student-centered instructional materials the first time through. We argue there is a significant amount of instructor learning that takes place in this context (and it deals with both the 'fitting' of new materials into their current practice or conceptions of practice, 'fitting' new materials into any current structures, and learning of the in-the-moment kinds of ideas and strategies students will bring and what to do with them.

In order to advance efforts to scale up student-centered instructional innovations, it is important for us to understand the variety of ways in which such innovations might be implemented, and challenges in implementation. This work contributes to the literature that documents the nature of challenges experienced by instructors working to implement innovative instruction, and also offers insight into how those challenges shift over time. Importantly, we relate these challenges and shifts to the ways in which instructors make use of instructional time in this context and identify instructors' ways of of structuring whole class discussions that allow students to contribute in meaningful ways.

Further work is needed to understand the institutional factors and pedagogical reasoning that informs decisions about how to structure the sharing of student explanations. More broadly, there is a need to articulate learning trajectories for instructors of undergraduate mathematics who want to engage in instructional change; such trajectories have the potential to inform the development of instructional supports and help instructors enact incremental changes as they learn to fit new instructional methods to their current institutional setting and instructional practice.

References

Ball, D. L. & Cohen, D. K. (1996). Reform by the book: What is—or might be—the role of curriculum materials in teacher learning and instruction reform? *Educational Researcher*, *25*(9), 6–8, 14.

Charalambous, C. Y. (2010). Mathematical Knowledge for Teaching and Task Unfolding: An Exploratory Study*. *The Elementary School Journal*, *110*(3), 247-278.

Collopy, R. (2003). Curriculum materials as a professional development tool: How a mathematics textbook affected two teachers' learning. *The Elementary School Journal*, 103(3), 287–311.

Davis, E. A. & Krajcik, J. S. (2005). Designing educative curriculum materials to promote teacher learning. *Educational Researcher*, *34*(3), 3–14.

Deslauriers, L., Schelew, E., & Wieman, C. (2011). Improved learning in large enrollment physics class. *Science*, *332*(6031), 862–864.

Enderle, P. J., Southerland, S. A., & Grooms, J. A. (2013). Exploring the context of change: Understanding the kinetics of a studio physics implementation effort. *Physical Review Special Topics-Physics Education Research*, *9*(1), 010114.

Fairweather, J. (2008). Linking evidence and promising practices in science, technology, engineering, and mathematics (STEM) undergraduate education. *Board of Science Education, National Research Council, The National Academies, Washington, DC*.

Garrison, A. L. (2013). *Understanding Teacher and Contextual Factors that Influence the Enactment of Cognitively Demanding Mathematics Tasks* (Doctoral dissertation, Vanderbilt University).

Henderson, C., Beach, A., & Finkelstein, N. (2011). Facilitating change in undergraduate STEM instructional practices: An analytic review of the literature. *Journal of Research in Science Teaching*, *48*(8), 952-984.

Henderson, C., & Dancy, M. H. (2007). Barriers to the use of research-based instructional strategies: The influence of both individual and situational characteristics. *Physical Review Special Topics-Physics Education Research*, *3*(2), 020102.

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. *Second handbook of research on mathematics teaching and learning*, *1*, 371-404.

Johnson, E., & Larsen, S. (2012). Teacher listening: The role of knowledge of content and students. *Journal of Mathematical Behavior*, *31*, 117–129.

Kwon, O. N., Rasmussen, C., & Allen, K. (2005). Students' retention of mathematical knowledge and skills in differential equations. *School Science and Mathematics*, *105*(5), 227–239.

Lewis, S. E., & Lewis, J. E. (2005). Departing from lectures: An evaluation of a peer-led guided inquiry alternative. *Journal of Chemistry Education*, *82*, 135–139.

Lockwood, E., Johnson, E., & Larsen, S. (2013). Developing instructor support materials for an inquiry-oriented curriculum. *The Journal of Mathematical Behavior*.

National Science Foundation. (1996). *Shaping the future: New expectations for undergraduate education in science, mathematics, engineering, and technology.* (NSF 96-139). Arlington, VA: National Science Foundation.

National Research Council. (1999). *Transforming undergraduate education in science, mathematics, engineering, and technology.* Washington, DC: National Academy Press.

National Research Council. (2011). *Promising Practices in Undergraduate Science, Technology, Engineering, and Mathematics Education: Summary of Two Workshops*. Natalie Nielsen, Rapporteur. Planning Committee on Evidence on Selected Innovations in Undergraduate STEM Education. Board on Science Education, Division of Behavioral and Social Sciences and Education. Washington, DC: The National Academies Press.

Seymour, E., & Hewitt, N. (1997). Talking about leaving: Why undergraduates leave the sciences. Boulder, CO: Westview.

Son, J. W. (2008). *Elementary teachers' mathematics textbook use in terms of cognitive demands and influential factors: A mixed method study*. ProQuest.

Speer, N. M., & Wagner, J. F. (2009). Knowledge needed by a teacher to provide analytic scaffolding during undergraduate mathematics classroom discussions. *Journal for Research in Mathematics Education*, *40*(5), 530–562.

Stein, M. K., & Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project. *Educational Research and Evaluation*, *2*, 50–80.

Wagner, J., Speer, N. M., Rossa, B. (2007). Beyond mathematical content knowledge: A mathematician's knowledge needed for teaching an inquiry-oriented differential equations course. *Journal of Mathematical Behavior*, *26*, 247–266.

Wawro, M., Rasmussen, C., Zandieh, M., Sweeney, G. F., & Larson, C. (2012). An inquiry-oriented approach to span and linear independence: The case of the magic carpet ride sequence. *PRIMUS*, *22*(8), 577–599.

Yin, R. K. (2003). *Case study research design and methods*. Thousand Oaks, CA: Sage Publications.

Best practices for the inverted (flipped) classroom

Spencer Bagley
University of Northern Colorado

The inverted, or flipped, classroom model is attracting the attention of many researchers, practitioners, and administrators in undergraduate mathematics programs as a way to navigate the tension between coverage and engagement, and to respond to the problem of increased class sizes and decreased budgets. The literature contains many reports on successful implementations, which vary widely in content delivery and student engagement. However, a core set of commonalities shared by these successful implementations forms the nucleus of a list of best practices for flipping a class. I discuss the theoretical underpinnings of the inverted model and the best practices suggested by the literature, and examine as a case study an inverted calculus class that did not follow these emerging best practices.

Key words: inverted class, flipped class, calculus, hybrid model, classroom research

Inverted (or flipped) classrooms are a revision of the traditional lecture-based classroom model. There are many different approaches to teaching an inverted class, but the common feature is that some lecture content is delivered outside of class time, often via internet videos. The class time thus freed up is typically spent in problem-solving activities with instructor assistance.

Viewed through the lens of sociocultural learning theories, the inverted model is a theoretically-grounded way to increase student understanding. Content delivery, which is less conceptually demanding and thus requires less expert help, is moved (wholly or partially) outside of the classroom; more demanding problem-solving tasks, wherein students can benefit more from expert assistance, replace content delivery during class time. Thus the utility of class time, where the more-knowledgeable other is physically present, is maximized (Vygotsky, 1978; Talbert, 2014).

In particular, many instructors use the inverted model to provide students with more opportunities for active engagement. Numerous studies have shown that student success increases when students are actively engaged (Freeman et al., 2014), and the inverted model frees up class time for active learning by moving content delivery outside of class.

Researchers have studied inverted classrooms in a variety of disciplines in undergraduate education, including physics (Deslauriers, Schelew, & Wieman, 2011), economics (Lage et al., 2000), computer science (Gannod, 2007; Gannod et al., 2008), mathematics (Talbert, 2014), and biology (Moravec, Williams, Aguilar-Roca, & O'Dowd, 2010). Many investigators have seen remarkable improvement in learning outcomes over traditional classrooms, as well as favorable reactions from their students. Examining these reports yields a list of important common features of successful inverted classes.

Success Reports in the Literature

First, I present a brief look at four representative reports on successful inverted classrooms in different disciplines, and point out important commonalities in their implementation. Deslauriers et al. (2011) compared student learning gains over one week of two large-enrollment introductory undergraduate physics classes, one taught by an experienced, highly-rated professor in traditional lectures, and the other taught using an inverted method by an instructor who was inexperienced but trained in physics education and pedagogy. The inverted class utilized pre-class readings paired with a brief online quiz. In class, the instructor emphasized deliberate practice "thinking scientifically," with a mix of clicker questions with student-student discussion, small-group tasks with a written response, and demonstrations. Both classes covered a common unit on electromagnetic waves and completed a common end-of-unit test jointly developed by the instructors involved. The mean score on the end-of-unit test in the experimental inverted section was 74 ± 1%, more than two standard deviations higher than the mean score of 41 ± 1% in the control section. To assess students' reception of the inverted method, the experimenters asked students to complete an online survey after the unit. 90% of students in the experimental section indicated that they enjoyed the inverted technique, and 77% felt they would have learned more if the whole course had been taught in this style.

Lage et al. (2000) studied students' perceptions of an introductory economics course taught using an inverted model. Lectures were available via videotape and PowerPoint with sound, and students were assigned to complete worksheets while watching the relevant lecture. These worksheets were collected and graded for completeness. Class time was spent in small groups, conducting economic experiments or labs; for example, an auction for a can of cola enabled students to plot a price-demand curve. On an end-of-term survey, students had favorable reactions to the course, generally agreeing with survey questions such as "I prefer this classroom format to a traditional lecture." The instructors also noted that students were more motivated, asked more questions, and enjoyed the group-work components of the course.

To free up class time for active learning exercises in an introductory biology class, Moravec et al. (2010) shifted some content into "learn before lecture" (LBL) activities. They moved four to five slides from PowerPoint lectures used the year before into either narrated PowerPoint videos or PDF worksheets, made available two days before class. Students were assigned to submit electronic copies of either their completed worksheet or the notes they took on PDF versions of the PowerPoint slides; each LBL activity was completed by over 90% of their students. The instructors then used in-class time freed up by shifting content into LBL activities to engage students in active-learning exercises. For instance, students answered clicker questions on transport through nuclear pores, or interacted with physical models demonstrating the transcription of mRNAs by ribosomes. On the final exam, students performed 21% better on the questions assessing content delivered through LBL activities, compared to <3% improvement on all other questions (typical of year-to-year variability in difficulty of exam questions). Additionally, students reported that the LBL activities were helpful in learning the course material and preparing for lectures, as well as reviewing material later in the term.

Talbert (2014) used inverted classroom design principles to structure a series of in-class workshops in linear algebra. He created highly-structured pre-class assignments called "guided practice," which included learning objectives, a collection of resources, a set of exercises, and requirements for submitting responses. An example in-class workshop asked students to work in pairs or threes to explain whether given numbers and vectors are eigenvalues and eigenvectors of a particular matrix, and then to explore the results of repeatedly applying a stochastic matrix to different initial vectors. Students enjoyed these workshops, and every student rated themselves as either "satisfied" or "very satisfied" with their learning in the workshops.

The results of these four studies, and others in the literature, are consistent. Most students report that they enjoy the inverted model and find it useful for their learning. The trend of performance data in those studies that report it is that students perform as well or better in inverted classrooms than in traditional classrooms. Despite the differences in content area, delivery mechanism, and use of class time, these reports also share commonalities in their implementation: pre-lecture activities were tailored to the particular class, often personally created by the instructors or researchers; time formerly occupied by lecture was replaced with active-learning exercises with the substantial involvement of the instructor; and students were held accountable for completing pre-lecture activities.

A Less Successful Example.

I now turn to a case study of an inverted calculus class that did not share these important commonalities. I examined this class as part of a broader study comparing student success in four calculus classes taught at the same university in the Fall 2012 semester using different pedagogical strategies. The other three classes were a traditional lecture-based class, a class based on "student-centered" lectures with time in class to work on problems similar to ones modeled on the board by the instructor, and a student-centered lecture-based class using Geometer's Sketchpad applets developed by the instructor to help students develop conceptual understanding. I refer to these three classes as the Lecture class, the Lecture with Discussion (LD) class, and the Lecture with Discussion and Technology (LDT) class, respectively.

After surveying the literature on the inverted model, I hypothesized that students in the inverted class would be more successful than those in the other classes, and in particular those in the Lecture class; however, this was not the case. For the purposes of this study, I operationalize "success" in three ways: persistence in STEM major tracks, expert-like attitudes and beliefs about mathematics, and performance on the common final exam. In each of these three measures, students in the inverted class were less successful than those in other classes.

To measure persistence in STEM major tracks, I examined the enrollment records of each section of Calculus I and II for the four semesters immediately following the Fall 2012 semester. I restricted my analysis to STEM-intending students, i.e., those who were declared in a STEM major track in Fall 2012. I classified STEM-intending students as *persisters* if they enrolled in Calculus II by Spring 2014 and remained declared in a STEM major throughout this period; otherwise, I classified them as *switchers*. The percentage of switchers in a given class is thus a measurement of the rate of non-persistence in STEM major tracks. Students from the inverted class switched out of STEM majors at a rate of 22.8%, significantly higher than the overall switching rate of 17.5%.

Students in each of the four classes completed start-of-term and end-of-term surveys examining demographics, preparation, and beliefs and attitudes about mathematics. Despite a lack of significant differences between the four classes at the beginning of the term, students in the inverted class scored significantly lower than those in the other classes on end-of-term measures of confidence in their mathematical abilities, enjoyment of mathematics, and interest in taking more mathematics classes.

To assess performance, I collected scores on the common final exam. ANOVA revealed significant differences between the classes: students in the inverted class scored on average approximately 7 points lower than students in the Lecture and LDT classes. When using ANCOVA to control for student preparation (as measured by the Calculus Concept Readiness test; Carlson, Madison, & West, 2010), these differences remained, though were no longer statistically significant. Additionally, in each of the three other classes, students who took a calculus course in high school scored significantly higher on the final exam than other students. However, in the inverted class, there was no significant difference between students with different levels of prior calculus experience.

The surprising difference between the quantitative results in my study and those reported in the literature drove me to seek explanations in qualitative data. I conducted focus group interviews from students in each of the four classes, asking students in each class to rate their satisfaction with the way their class was conducted. I also observed several class sessions in order to understand a typical day in each class. I found that the inverted class in my study departed in significant ways from those in the literature. I also identified three categories of student dissatisfaction with this implementation of the inverted model; remarkably, the things that students were concerned about were precisely the ways in which this inverted class departed from the reports in the literature.

To understand the concerns raised by students in the focus group interview, it is important to first draw a picture of a typical day in the inverted classroom in my study. The professor in the inverted class summarized his approach to the inverted model in general terms on his syllabus: "In the inverted model, students begin their learning at home via a variety of resources, then complete their learning in class by training on exercises." To complement this general description with a detailed account of the daily activity of this class, I examined the syllabus and course website, spoke with the three teaching assistants (TAs) assigned to the course, and asked students in the focus group to describe a typical day in class.

Several days before each class session, the professor posted links on his website to videos and other resources discussing the material that would be covered in class. The videos came from a variety of sources, including Hippocampus, Khan Academy, PatrickJMT, and MIT's OpenCourseWare collection. Additionally, the professor commonly provided some text-based resources from Wikipedia and online textbooks such as Strang (1991).

Students were expected to prepare for class by watching videos or reading text materials. Students were free to choose which resources to use; in general, in order to accommodate a wide variety of student learning styles, more resources were provided for a given day's lesson than any individual student would use. There was no mechanism to check whether or not students had watched videos or read materials before class.

As students entered the classroom, they signed in on an attendance sheet, and received a worksheet described in the syllabus as "a sequence of increasingly challenging exercises." There were usually between 10 and 20 exercises on a worksheet. The entirety of the 100-minute class time was spent working on the problems on the worksheet. Most students chose to work in self-assigned groups of four to six, while a few generally preferred to work by themselves.

Except for the first day and exam days, the professor did not attend class. Instead, three TAs were assigned to attend class and answer student questions. The TAs were not empowered to stop class to hold a brief mini-lecture, even if a substantial number of students all had the same question. Thus, the atmosphere of the classroom was more like a tutoring lab than a classroom with one central authority.

Near the end of each class session, the TAs would announce which of the problems on the worksheet would be collected. Students would recreate their work on that problem and turn it in before leaving. Their work was graded and returned in a later class session.

The five students in the focus group interview were uniformly and vociferously dissatisfied with the implementation of the inverted model. Their comments clustered into three intertwining categories: problems with the pre-class videos, problems with the in-class activities, and a feeling of disconnect from the professor. (All student names in the following are pseudonyms.)

Problems with pre-class videos

Early in the focus group interview, a student named Sarah said, "I feel like that's the biggest problem in this class, is the videos are not applicable to the work." Paige agreed: "The way this professor puts his problems doesn't correspond with the way we're taught on the videos [from] Hippocampus, or Wikipedia [articles] he puts on there, just like random stuff. It just isn't cohesive." The feeling that the pre-class videos did not prepare students for the in-class worksheets had adverse effects on students' confidence:

> Melissa: I watched the videos and I understand it going in, I feel very confident, and then I get that paper [the in-class worksheet] and I'm like, well, I give up already.
> Int: So you feel like you understand things after watching the videos, but then...
> Melissa: It just doesn't relate to the worksheets, yep.
> Paige: We're getting a good understanding of calculus watching the videos, but just not the way he wants it done. That's where it gets confusing.

Students in this class often gave up on in-class activities because they did not feel adequately prepared. This theme was expressed by several other students throughout the interview.

Because of the problems with the pre-class videos, the students felt that they were not useful. When I asked students what they did to prepare for tests, none of the students reported watching the videos again. This is in contrast to the findings reported by Moravec et al. (2010), whose students re-used the learn-before-lecture activities to review for tests (see also Lage et al., 2000). Some of the students in the inverted calculus class gave up on the videos entirely, preferring to rely on previous knowledge of calculus:

> Bob: To be honest, I haven't watched very many of the videos. … I've watched videos maybe two to three times out of the whole semester. I mean, most of what I remember is from high school.
> Paige: Yeah, what's keeping me going in this class is math classes in high school I took.

This corroborates the quantitative finding that in the inverted class, unlike in the other classes, students who took calculus in high school did not perform better than other students. It seems that these students were less likely to watch the videos and instead relied on their old knowledge. Recall also that there was no specific mechanism holding students accountable for watching the videos.

Bob had prior experience with inverted classroom design, having taken an inverted statistics class in high school. In his high school class, "it was nice because the teacher devoted a lot of time, because he created the videos himself." By contrast, the instructor of the calculus class did not make the videos himself. Bob continued, "If he were to make the time in terms of creating the videos himself and shaping the videos towards his class, I think it would be more beneficial than just pulling random [internet videos.]" When I asked the students what other resources they would have liked to have in their class, Bob replied, "I wish he had videos that he made himself, or that had more direct correlation to the class we were taking, ... covering all the concepts on the worksheets that he put."

It does not appear that the students felt it was required to have all the videos made by the professor. Paige observed that "at the beginning of the year, the videos corresponded well, because it was just a lot of simple stuff." Variation in presentation seems to be less of an issue for foundational early material in a course.

The general thread of students' comments about the pre-class videos is that because the pre-class videos were not made by the professor, they were not applicable to the in-class work the professor required. Students felt that the videos did not adequately prepare them to complete the in-class worksheets. This was a source of frustration, because the videos failed in their express purpose. Students commonly directed this frustration toward the professor, who they seemed to regard as having abrogated his responsibility to prepare them for the work in the class.

Brousseau's (1997; see also Herbst & Kilpatrick, 1999) construct of the didactical contract is a useful way to understand the students' frustration. The didactical contract is the usually-unspoken set of expectations and obligations to which the instructor and the students in a given class believe they are held. The didactical contract in a typical classroom might include the following: the instructor's responsibility to the students is to teach them the material, and the students' responsibility to the instructor is to attend classes and do the work assigned by the instructor. Students can expect that if they complete the assigned work, they will be well-prepared for the assessments that will follow. In this inverted class, students felt that they were ill-prepared for the worksheets despite having watched the assigned videos; they thus felt that the instructor had breached the didactical contract by failing to create or select adequate videos.

Problems with in-class activities

In addition to the failure of the in-class worksheets and videos to articulate well, students identified several other concerns with the in-class activities. One concern was the lack of structure in class time:

> Sarah: An hour and forty minutes straight of doing word problems is kind of a lot, at least for me. I don't know, I can't just...
> Paige: I get a headache.
> Bob: I get distracted all the time.

One of the affordances of the inverted model identified by Talbert (2014) is that students who have difficulty managing their time outside of class are at a disadvantage in the traditional classroom, since "higher-level cognitive tasks often require extensive periods of time for work and reflection; these segments of time are often mismanaged or are simply unavailable to many students." He argues that the inverted model, in which high-level tasks are done in class "where the instructor is present to guide students in efficient and effective work," removes this disadvantage (p. 362). In this Inverted class, students were presented with 100-minute blocks of time designated for working on problems; however, these blocks were not further structured by the instructor or TAs. Thus, while students in the Inverted class at least had time set aside for working on problems, the difficulties of managing that time effectively were still present.

Another concern was that the problems on the worksheet became too difficult too rapidly, jumping, as Sarah said, from "zero to a hundred":

> Ben: I feel like he has too many problems that go way too deep into the concept. I mean, he'll start out basic, like let's just say it was 2 + 2 = 4. And then by the end it'll be, if all you're trying to do is learn addition, he'll have 2 times this times this plus this, just so you can get the concept of adding. He'll have sine squared, squared, to the third, or something like that.
> Bob: I mean maybe that's the level of calculus for college that we need to be at, and that's completely understandable, but let's work to it rather than just going from, hey, simple sine is cos, and then jumping to what's sine cos sine to the fifth or something.

This concern is likely related to the lack of structure in class time (and to the absence of the instructor): with stronger scaffolding from the instructor or TAs, and thus more structure provided to class time, exercises increasing in difficulty would be less problematic. This was borne out in the next line of transcript:

> Melissa: And then when we get to those problems, we all need help, because all of us don't understand that level, but there's only three TAs. So we'll sit there and they'll [say,] "I'm gonna start the problem, but then I gotta walk away and start it for somebody else."

Giving TAs the authority to conduct mini-lectures, and thus impose more structure on class time, would likely have ameliorated this concern. If the TAs saw or expected that many students would have the same question about the same problem, allowing them to explain the question to the entire class at once would have been more efficient and likely more effective.

The lack of structure of class time in this inverted class is a large departure from the inverted models reported in the literature, which are much more structured by the involvement of the professor. For instance, Lage et al. (2000) engaged their students in highly-structured economic experiments, while Moravec et al. (2010) used clicker questions and class demonstrations to structure their class time. It is likely that these highly-structured activities contributed to students reporting that the inverted approaches were more enjoyable than traditional lecture formats (Lage et al., 2000). A large block of unstructured problem-working time, as found in this inverted class, is likely less interesting and less motivating for students.

Disconnect from the professor

The third category of concern was that the instructor did not come to class sessions. This theme arose first when I asked a follow-up question about the pre-class videos:

> Int: So the videos that you see, are they made by your instructor, or just chosen by him from other sources?
> Bob: No, he chooses all of them.
> Paige: He never makes them.
> Bob: He just references them out to different online sources.
> Paige: We've only seen him once.
> Sarah: Twice.
> Bob: Yeah, we've seen him on the first day, we've seen him on the test [days]. Three times he's showed up. I mean, I know he's busy, but -
> Sarah: Yeah, it's kind of ridiculous, to be honest. I mean, the TAs are there to help us, but it would be nice to talk to a professor. Like during class, if he was there.

Melissa also complained that there were questions about the professor's expectations that the TAs were unable to answer:

> Since he's not there, when we ask the TAs what does he want with this problem, they say 'I don't know,' because no one knows what he's thinking. And so we're like, is this type of problem -- how should we set this up, or do this? And they're just like, 'I don't know what to tell you.'

None of the students in the focus group attended their professor's office hours, because they felt disconnected from the professor. Sarah said that she wouldn't go to office hours because "I don't know my professor's name." Later in the interview, Paige added, "I feel like it would be weird if we went to them, because we don't know him." Melissa agreed, and said she felt "it would be really awkward."

The students in the focus group interview felt like the instructor's absence negatively impacted their understanding of calculus and decreased their confidence moving forward in future mathematics classes:

> Sarah: I could have done so much better in a different class.
> Paige: I have friends in another [calculus class] and they have a professor who teaches them and like...
> Ben: They say they're really good, too.
> Paige: Yeah, really good professors, really understand, and then there's me, and I'm like, I get it to an extent, but then I feel behind. I'm nervous for Calc 2.

Paige later said that the instructor's absence also decreased her enjoyment of mathematics:

Math is my favorite subject. Since I was in elementary school, I was like, I love math! But now like this semester, I sit there sometimes and I'm like, why don't I get this? Because math is my class where I get this, it's easy to me, but when we don't have a professor, it's -- I kinda sit there and [say], oh no. It's kind of discouraging, I guess.

Near the end of the focus group interview protocol, I asked several questions about the instructor's attitude toward students. The students' responses were generally negative:

Ben: I feel like we're kind of a nuisance to him. Just because of the fact that he only comes for tests, and when he does, he's really short.

Bob: I don't know. He didn't leave a very good impression the first day, like he was -- It almost seemed like he was being rude to people.

Sarah: I grew up around professors, and a lot of the time you know they're here for the research aspect. So I feel - I don't know if he does research, but I feel like that's quite possible, that he doesn't actually care about teaching.That he's not here for the students.

These views parallel those reported by Seymour (1997, 2006), who conducted exit interviews with students completing STEM degrees as well as those who had changed their major. Both categories of students Seymour interviewed reported taking classes from unavailable, disinterested faculty with an implicit or explicit dislike for teaching.

Again, Brousseau's (1997) construct of the didactical contract is a useful way to discuss students' frustration with their absent instructor. Since the instructor did not create the videos or attend class sessions, the perception of the students was that he did not do anything. In reality, however, he spent considerable time, energy, and effort to create the in-class worksheets. From the students' perspective, he did not hold up his end of the didactical contract. Several students in the focus group pointed out that they felt underprepared for Calculus II, or that they enjoyed mathematics less after having taken this class than they had before, and laid the blame for these feelings at the professor's feet. Most had attended class and watched videos as assigned, but did not feel adequately prepared since the professor was not involved.

The TAs, who assumed the instructor's role as authority figures in the classroom, were not given enough authority or enough training to fill that role effectively. Further, there are some questions that TAs are unable to answer, no matter how proficient in mathematics or how well-trained they are. In particular, TAs cannot answer questions about the instructor's expectations, as Melissa pointed out. For these questions, answers must come from the instructor; his absence, and the accompanying unanswerable questions, was thus a source of frustration for both the TAs and the students.

The absence of the professor led to a feeling of disconnect. Students did not attend the professor's office hours, because they felt that they did not know him well enough; "it would be really awkward," one student said. They doubted that he cared about their learning, viewed him as disinterested and unavailable, and felt that he saw them as "a nuisance." This led to a profusion of negative feelings about the instructor and the class in general.

The instructor's absence from the class, another large departure from studies in the literature, is a plausible explanation for the lack of positive quantitative results reported earlier. One of the main objectives of the inverted model is to get students in the same room as the more-knowledgeable other (Vygotsky, 1978) when working on the tasks with the greatest cognitive demand, so that assistance can be provided when it is most needed (Gannod, Burge, & Helmick, 2008). To achieve the full benefit of the inverted model, the instructor must be present and actively involved. Indeed, as Talbert (2014) writes, "open lines of communication between the instructor and the students are critical to the success of the inverted classroom" (p. 365). The failure of this iteration of the inverted model is an example of what can happen when the lines of communication are closed off.

Possible Best Practices.

The reports of successful inverted classes in the literature share several commonalities: the pre-lecture activities were tailored to the particular class, often personally created by the teachers or researchers; students were held accountable for completing the pre-lecture activities; and time formerly occupied by lecture was replaced with active-learning exercises led by the instructor of the class. The concerns reported by students in the inverted calculus class I studied implicate the failure of the class to replicate these important commonalities. I thus propose these features as the beginning of a list of best practices that should be adopted for an inverted class to be successful. Here, I discuss why each of these features is plausibly necessary for success.

The first feature is that pre-lecture activities are closely tailored to the class. The students in the inverted calculus class read the articulation failure between the pre-class videos and the in-class worksheets as a breach in the didactical contract: the videos were meant to prepare the students for the in-class work, but they did not, so they failed in their purpose. This caused some students to disengage entirely from watching the videos. While instructors in the literature typically create pre-lecture activities themselves, this does not seem strictly necessary, particularly for early foundational material. Perhaps this is because for this material, variations in presentation are less impactful; a lesson on the power rule, for instance, likely looks much the same no matter who delivers it. Ongoing work in this research program is surveying teachers who use the inverted model to determine, among other things, the balance they strike between creating videos and using pre-existing resources. Future work could attempt to find criteria for when pre-lecture activities can be borrowed from other sources and when it must be developed in-house.

The second feature is accountability for completing pre-lecture activities. This can take many forms, from handing in a filled-in worksheet, to completing a clicker quiz at the beginning of class, to using a content management system to ensure that students clicked the link to a video. It may even be as simple as making it clear to students that pre-lecture activities won't be reviewed in class. Accountability measures can be effective in motivating students to complete activities; Moravec et al. (2010) reported that each of their "learn-before-lecture" activities was completed by over 90% of their students. Without accountability, however, there is no guarantee that students will complete the activities; it is no surprise that students often do not do things they are not accountable for.

The third common feature is the use of active-learning activities in class, led by the instructor. One key motivation for the inverted model was to allow students the opportunity to engage with challenging material with the instructor physically present to provide scaffolding and support. The inverted model thus contrasts with traditional models, which assign students to complete challenging tasks at home without the instructor's help. Without engaging, well-structured activities, or without the instructor present, this affordance of the inverted model is lost.

References.

Brousseau, G. (1997). *Theory of didactical situations in mathematics.* Edited and translated by N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield. Dordrecht: Kluwer.

Carlson, M., Madison, B., & West, R. (2010). The Calculus Concept Readiness (CCR) instrument: Assessing student readiness for calculus. *arXiv preprint arXiv:1010.2719*. Retrieved from http://arxiv.org/abs/1010.2719

Deslauriers, L., Schelew, E., & Wieman, C. (2011). Improved learning in a large-enrollment physics class. *Science*, *332*(6031), 862–864. doi:10.1126/science.1201783

Gannod, G. C. (2007). WIP: Using podcasting in an inverted classroom. *Proceedings of the 37th IEEE Frontiers in Education Conference. IEEE*. Retrieved from http://www.colorado.edu/MCDB/MCDB5650/InvertedClassroom07.pdf

Gannod, G. C., Burge, J. E., & Helmick, M. T. (2008). Using the inverted classroom to teach software engineering. *Proceedings of the 30th international conference on Software engineering* (pp. 777–786). Retrieved from http://dl.acm.org/citation.cfm?id=1368198

Herbst, P., & Kilpatrick, J. (1999). Pour lire Brousseau. *For the Learning of Mathematics, 19*(1), 3–10.

Lage, M. J., Platt, G. J., & Treglia, M. (2000). Inverting the classroom: A gateway to creating an inclusive learning environment. *The Journal of Economic Education*, *31*(1), 30-43. doi:10.2307/1183338

Moravec, M., Williams, A., Aguilar-Roca, N., & O'Dowd, D. K. (2010). Learn before Lecture: A Strategy That Improves Learning Outcomes in a Large Introductory Biology Class. *Cell Biology Education*, *9*(4), 473–481. doi:10.1187/cbe.10-04-0063

Seymour, E. (1997). *Talking about leaving: why undergraduates leave the sciences*. Boulder, CO: Westview Press.

Seymour, E. (2006). *Testimony offered to the Research Subcommittee of the Committee on Science of the U.S. House of Representatives hearing on Undergraduate Science, Mathematics, and Engineering Education: What's Working?* Retrieved from http://commdocs.house.gov/committees/science/hsy26481.000/hsy26481_0f.htm

Strang, G. (1991). *Calculus*. Wellesley, MA: Wellesley-Cambridge Press.

Talbert, R. (2014). Inverting the linear algebra classroom. *PRIMUS*, *24*(5), 361-374. doi:10.1080/10511970.2014.883457

Vygotsky, L. S. (1978). *Mind in society.* Cambridge, MA: Harvard University Press.

Instantiation Practices During Conjecturing Activity: Implications from the use of technology

Jason K. Belnap & Amy Parrott
University of Wisconsin - Oshkosh
belnapj@uwosh.edu parrotta@uwosh.edu

Proof is a complex mathematical activity with which students struggle as they transition from K-14 to abstract mathematics. This transition may be eased by developing mathematical practices during more accessible mathematical activities, such as conjecturing. Recent studies in both proof and conjecturing suggest that instantiation practices, practices surrounding the generation and selection of examples during mathematical inquiry, are key to success in both activities. In our own study, most participants utilized GeoGebra, a dynamic geometry software, to facilitate their investigations; however, although this software (in theory) makes sophisticated instantiation practices possible, it did not appear to advance participants' instantiation practices. In this paper, we detail the participants' use of the GeoGebra and raise implications, questions, and cautions for teaching and research.

Key words: conjecturing, dynamic geometry software, instantiation practices, technology

Background

Providing students with what it takes to be successful in mathematics is one of the primary focuses of mathematics education research and a very complicated challenge to address. This is partly because becoming successful in mathematics entails more than simply memorizing facts or practicing procedures. It involves the acquisition of many cultural practices, many of which are subtle and not fully understood, overtly addressed, or intentionally encouraged. The Conference Board of the Mathematical Sciences in the Mathematical Education of Teachers II explained (2012):

> "A primary goal of a mathematics major program is the development of mathematical reasoning skills. This may seem like a truism to higher education mathematics faculty, to whom reasoning is second nature. But precisely because it is second nature, it is often not made explicit in undergraduate mathematics courses. A mathematician may use reasoning by continuity to come to a conjecture, or delay the numerical evaluation of a calculation in order to see its structure and create a general formula, but what college students see is often the end result of this thinking, with no idea about how it was conceived." (p.55-56)

Successfully traversing the entire timeline of mathematical experience (from kindergarten to professional mathematician) requires students to become proficient in inconsistent and incompatible "mathematical" cultures. As Lockhart (2002) observed, K-12 mathematics (and even early undergraduate college classes) values the acquisition of facts and rules, the mastery of procedural proficiency and algorithm application--usually provided through direct instruction and practiced by rote. Upper-division and graduate level coursework (and professional mathematical activity) on the other hand value the ability to understand and generate new mathematical ideas, connect mathematical concepts, and advance the theoretical knowledge of the field.

This gap creates a well-known challenge to the undergraduate student. Somewhere in the midst of the undergraduate experience, usually in an introductory linear algebra or analysis course or an introduction to proofs course, students experience this sudden cultural shift. They are rapidly introduced to classes requiring an understanding of axiomatics, deductive logic, argumentation, and precise (and specialized) language, all of which center around the mathematical practice *proof*.

We assert that this gap could be bridged through the study of mathematical practices, by which we mean the values and actions utilized by members of the professional mathematical community as *they* engage in authentic mathematical work. For example, the heavy attention that the mathematical education community has paid to those practices surrounding the mathematical activity of proof has helped us better understand proof. We have learned that proof is a highly complex cultural activity; mastering it requires the acquisition of many different cultural practices and skills (Alcock, 2010, Harel & Sowder, 1998). To begin, students must come to understand the need for, value of, and purposes of proof. They must also: a) develop discipline-specific language; b) acquire specialized linguistic skills; c) become proficient at deductive logic; d) understand conditional, equivalent, and nonequivalent statements; e) become aware of the roles and uses of cultural artifacts such as definitions, axioms, and theorems; f) become proficient at deriving meaning from written statements; and g) understand and become proficient at acceptable proof structures (e.g. direct proof, proof by contradiction, and proof by induction). In addition, when asked to ascertain the truthfulness of mathematical statements, they must engage in mathematical inquiry by generating examples that test the statement's limits and validity.

Closely related to proof, our own research focuses on understanding mathematical practices surrounding the mathematical activity of conjecturing (Belnap & Parrott, 2013, 2014, 2015). This includes all actions, strategies, and efforts contributing to the development and formulation of new (to the participant) mathematical ideas (i.e. formal mathematical conjectures). Our findings suggest that conjecturing is a form of mathematical inquiry that is highly accessible (even to novices) and draws upon many of the same mathematical practices that proof does--potentially serving as an accessible avenue for providing early mathematical enculturation experiences (Belnap & Parrott, 2013, 2014).

Conjecturing and Geometric Inquiry

In this paper, we present some interesting cases that we encountered during our broader qualitative research study--the full methodological and analytical details of this study are provided in (Belnap & Parrott, 2013, 2015). The goal of this research was to understand what conjecturing entailed by engaging a diverse set of participants (individually) in a conjecturing task and contrasting their conjecturing practices. We purposively selected a diverse set of eight participants at various levels of mathematical maturity. Three were expert (i.e. research) mathematicians with different specialties, three were apprentice mathematicians (i.e. graduate students), and two were novice mathematicians (i.e. undergraduates at the aforementioned cultural transition point). We selected student participants to provide a diverse range of mathematical abilities, as judged by their instructors.

Each participant individually took part in an *overt-conjecturing task*, defined as one in which conjecturing is the sole purpose of the task, unshrouded by other goals or purposes (Belnap & Parrott 2014). As detailed in Belnap & Parrott (2013), we gave each participant a copy of the task (shown in figure 1) and a variety of resources (including both traditional construction tools and a computer with GeoGebra); then provided ample time to investigate the definitions and generate conjectures. Following the task, we gave participants a short break (during which we discussed our observations in preparation for the interview). After the break, we then conducted an interview to clarify each participant's approach and activities. We video recorded the participants, their work, and computer use during both task and interview; for triangulation of data, we also kept observation notes and collected participants' written work.

Figure 1: Conjecturing task completed by each participant.

Consider a generic quadrilateral $ABCD$ and the three types of derived quadrilaterals: the angle bisector quadrilateral, the midpoint quadrilateral, and the perpendicular bisector quadrilateral. Your task is to explore the relationships between quadrilateral $ABCD$ and each of these different types of derived quadrilaterals; make conjectures based upon your observations.

Task 3: Formal Definitions of Derived Quadrilaterals

Angle Bisector Quadrilateral: The *angle bisector quadrilateral* of a quadrilateral $ABCD$ is a quadrilateral $A'B'C'D'$, where A', B', C', and D' are each the intersection of the angle bisectors of the corresponding and subsequent vertices (e.g. A' is the intersection of the bisectors of $\angle DAB$ and $\angle ABC$).

Midpoint Quadrilateral: The *midpoint quadrilateral* of a quadrilateral $ABCD$ is a quadrilateral $A'B'C'D'$, where A', B', C', and D' are the midpoints of $\overline{AB}, \overline{BC}, \overline{CD}$, and $\overline{DA}$ (respectively).

Perpendicular Bisector Quadrilateral: The *perpendicular bisector quadrilateral* of a quadrilateral $ABCD$ is a quadrilateral $A'B'C'D'$, where A', B', C', and D' are each the intersection points of the perpendicular bisectors of the two adjacent sides (e.g. A' is the intersection of the perpendicular bisectors of $\overline{DA}$ and $\overline{AB}$).

Using grounded theory techniques (Strauss & Corbin, 1998), we systematically analyzed the data (Belnap & Parrott, 2013, 2015). Starting with the least experienced novice and working up to the experts, we alternated between data collection and analysis, using initial findings to inform subsequent data collection. During the process, we annotated both observed and reported behaviors, clustered them by commonality, and defined these emerging themes which became categories. Each category described a critical aspect along which participants' conjecturing practices differed. These categories included: a) their overall process and problem-solving approach to the task, b) the objects that they created during and for their investigation, c) the nature of their observations (i.e. what they noticed, attended to, and looked for), d) the qualities of their written conjectures, and e) what was required for an idea to qualify as a conjecture for the participant. Using these categories as guides, we revisited the data. For each participant and category, we gathered detail and synthesized it into a vignette or a synopsis, describing the participant's practices in regard to that category.

Since this general analysis, we have begun examining each category in greater detail. Looking across participants, we have compared and contrasted the vignettes for each category, providing descriptions of ways in which participants' practices compared and differed. For example, a cross-comparison of category b (objects created for investigation) compared with related work and theory published by Lockwood et al. (2013, 2014) resulted in an understanding of differences in the strategic purposes and use of participants' examples during conjecturing (Belnap & Parrott 2015).

As detailed in Belnap & Parrott (2015), we found that more expert participants approached the selection and generation of examples in strategic ways. They were careful and intentional in these instantiation practices, using examples in systematic ways to generate, discover, and validate conjectures. Intermediate participants likewise created and chose their examples purposively, but without the organization and analysis characteristic of expert approaches. As an extreme and contrasting case, our most novice participant did not even consider multiple examples when generating conjectures, instead generalizing from a single example.

Building off of work completed by Lockwood et al. (2013, 2014) concerning the usage of examples in proof development, we identified specific instantiation strategies employed by our participants during the conjecturing process (Belnap and Parrott 2015). These strategies largely mirror those identified by Lockwood et al. (2013, 2014), with adaptations and omissions deemed necessary to accurately reflect the conjecturing context. Table 1 describes each of these strategies.

Almost all participants chose to utilize the software GeoGebra (a dynamic geometry environment or DGE) as part of their investigative process; even the one that didn't, opted to do so during the interview. Furthermore, most of the participants only generated examples within the DGE. Because of this, we became curious regarding these questions: What role did the DGE play in participants' investigations? What benefits did participants derive from its use?

We have not completed a thorough study or analysis of these questions. As we revisited our data with these questions in mind, we uncovered some unusual cases regarding the way

Table 1: Instantiation Strategies Used in Conjecturing

Strategies (and Subtypes)	Description
Multi-Stage Example Exploration	Exploration based on a systematic change in the selection of examples used.
Changing in Complexity	Progressing from simple examples to more complex or complicated ones (or visa versa)
Changing in Extremity	Progressing from simple, typical cases to more extreme or special cases (or visa versa)
Changing in Generality	Progressing from special cases to more general cases (or visa versa)
Exhaustive	Progressing in a sequence that would ultimately (i.e. potentially in the limit sense) exhaust all possibilities
Property Analysis	An exploration of the properties of examples and how these properties are related to the conjecture
Systematic Variation	Taking a known case and systematically changing it by making small adjustments

participants chose to utilize (or *not* utilize) the software available to them. In this paper, we present these cases, suggesting a boundary between what technology may and may not induce in individuals' mathematical activity.

Technology - Dynamic Geometry Environments (DGE)

All of the participants in our study except one chose to utilize GeoGebra (a DGE) as a central tool for facilitating their exploration of the task's definitions. DGEs are software environments (e.g. GeoGebra, Geometer's Sketchpad, Cabri) that allow the creation of primitives (such as points, lines, and circles) and dependency relationships among them (such as the midpoint of a given line segment). These environments update in real-time based on changes in the primitives (Laborde et al., 2006), allowing the user to manipulate objects and immediately observe the consequences of those changes.

Within a DGE, users are able to construct objects that retain their properties even when manipulated. For example, if we created a triangle's medians by *constructing* each side's midpoint and attaching a line to each midpoint and the vertex opposite it, then these lines will remain medians, even if the vertices are moved onscreen. This type of construction is called a *robust construction* (Laborde et al., 2006). On the other hand, if a user were to simply make three lines, choosing their placement by appearance (i.e. so they looked like medians), but without building the midpoints or vertices into their construction, this lines would *not* remain medians when the vertices were moved. Such a construction is known as a *soft construction* (Laborde et al., 2006), since it does *not* resist manipulation.

The investigative power of DGEs lies in robust constructions. These constructions enable the user to discover *invariants*, meaning observable properties of the figure. Similar to constructions, invariants can be classified as either soft or robust. *Soft invariants* are those that are affected by manipulation; *robust invariants* are resistant to manipulation. (Laborde et al., 2006). For example, if we construct a triangle's medians and manipulate it, we will always see the lines intersecting at the same point; their intersection is a robust invariant (because a triangle's medians are always concurrent). On the other hand, when carefully dragging the triangle's vertices we may notice that *sometimes* a median is perpendicular to the side that it bisects; this property would be a soft invariant (because it only happens when the triangle is isosceles).

A feature of DGEs that is critical to investigations, is the ability to drag objects and observe the results in real-time. The research literature reports three main types of dragging (Laborde et al., 2006). The first is *wandering dragging*, which is randomly dragging to discover a soft invariant. The second is *guided dragging*, which is done with the intent of obtaining a particular shape. Finally, we have *lieu muet dragging,* which is done to maintain a specific property (Laborde et al., 2006).

Baccaglini-Frank and Mariotti (2011) describe a strategic combination of these actions that is used in conjecturing activities called *maintaining dragging*. This consists of three parts. The user begins by wandering dragging to induce a soft invariant. Then he proceeds with lieu muet dragging in order to identify or conjecture conditions associated with the invariant, and finally will verify the conjecture through a dragging test.

Strategic use of DGEs in constructing and manipulating examples can reveal important mathematical properties. For example, a conjecturer's *hypothetical* investigation of our task's perpendicular bisector quadrilateral (PBQ) definition might proceed as follows. He begins by constructing a fully manipulable quadrilateral (parent object) and off of its sides constructs its PBQ. He then engages in wandering dragging by manipulating one of the parent's vertices.

During these manipulations, he induces a soft invariant, where the PBQ degenerates to a point--many common special cases of the parent quadrilateral yield this, including all squares, rectangles, and isosceles trapezoids. Wanting to find-out what causes this degeneracy, he carefully moves this vertex in order to maintain the invariant (lieu met dragging). He notices that the definition degenerates whenever this vertex is on the circle that passes through the other three vertices (i.e. as long as the quadrilateral is concylic). To check, he constructs a robust but fully manipulable concyclic quadrilateral along with its PBQ. For this construction, the degeneracy is a robust invariant, supporting his conjecture. In this way, strategic construction and manipulation of examples using the DGE can reveal important geometric properties and lead to strong mathematical conjectures.

Results

Almost all participants utilized the software during their conjecturing work with intent and purpose. They reported choosing to use the DGE because it facilitated the construction of accurate representations and afforded the examination (through manipulation) of a large number of examples in a short time. In fact, a commonly reported benefit of DGE use was the ability to examine both numerous and diverse examples, which enabled some advanced instantiation strategies, such as those discussed in Belnap and Parrott (2013).

Dr. Sam - a case of strategic software usage.

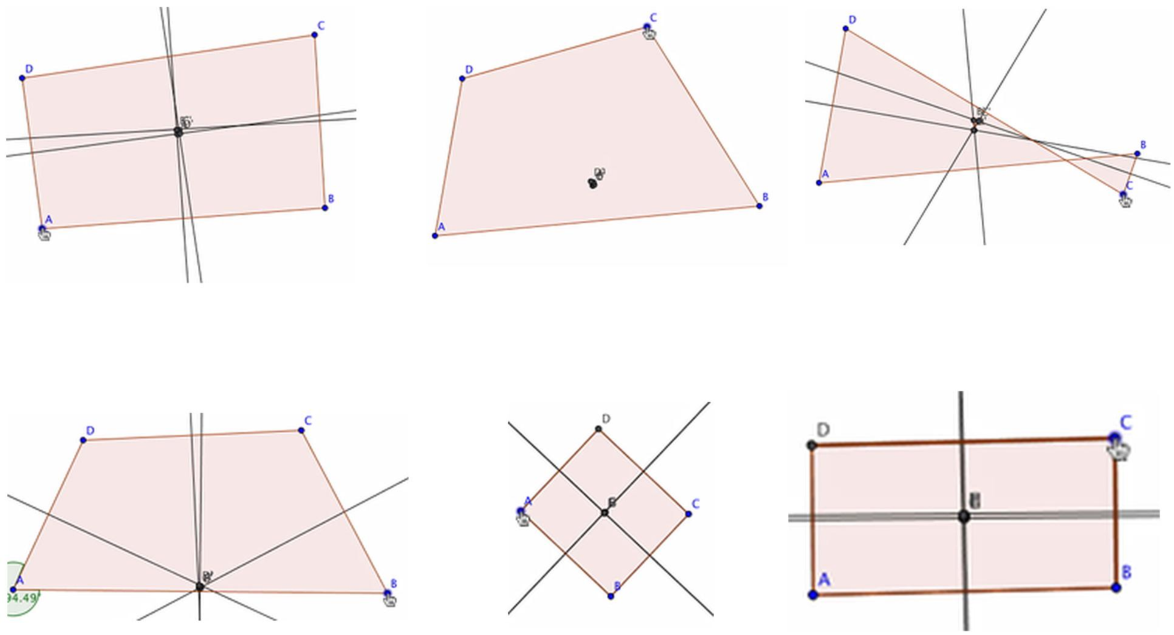
As a starting point and basis for comparison, consider one of our experts Dr. Sam. During his investigation, Dr. Sam engaged in various instantiation strategies, strategies afforded by DGE usage. Dr. Sam considered numerous, diverse examples through the manipulation of dynamic models.[1] At the same time, he was strategic and intentional in his instantiation practices, utilizing and switching between various types of multi-stage example explorations (namely increasing in extremity, increasing in complexity, and decreasing in generality) and engaging in property analysis (Belnap and Parrott 2015).

Dr. Sam conducted his initial investigations through *strategic* wandering dragging. He systematically moved the parent quadrilateral through the soft construction of several common types of conventionally-defined quadrilaterals (e.g. rhombi, parallelograms, and isosceles trapezoids), observing the consequences for the derived quadrilateral. He also sought out more extreme and diverse cases, including extreme changes in the shape and proportions of the parent quadrilaterals as well as situations when the parent quadrilateral was concave or not even a quadrilateral (due to intersecting sides). He explained, ``I tend to look at extreme cases. … You know, I think that's sort of a feature of, of mathematicians, to say, `Ooh, how can this go wrong?' … Yeah… I think of the convex quadrilaterals as being ugly ones... They're not nice quadrilaterals. I said, `Well let's see what happens when, when we look at one of those.' '' (Interview transcript) In this way, Dr. Sam systematically used examples that ***increased in complexity***.

During his explorations, he also periodically ***decreased the generality*** of his selected examples. When (during his wandering dragging) he encountered an interesting soft invariant, he would

[1] In each of Dr. Sam's models, the derived quadrilateral was created to be dependent on a general (parent) quadrilateral, allowing him to manipulate the parent quadrilateral and immediately see the effects on the derived quadrilateral. During most of his explorations, he used a model in which the parent quadrilateral was a general quadrilateral, whose vertices could all be freely moved.

Figure 2: Degenerate Perpendicular Bisector Quadrilaterals (PBQs) induced during Dr. Sam's investigation



Note: Snapshots arranged chronologically in rows from left to right (Video recordings). In the top three and first on the second row, the parent quadrilateral's vertices are fully manipulable. In the last two, the parent quadrilateral is a robust parallelogram.

stop to construct the specific case to make manipulation easier and ensure there were no errors in his observations.

During periods of his investigation, Dr. Sam utilized examples that ***increased in extremity***, that is sequences of examples that sought out the boundary between situations in which the construction yielded the defined quadrilateral and those where it did not (i.e. degenerate conditions) . For example, using wandering dragging, he hunted down situations that cause the PBQ to be degenerate. He explained, ``Then there were some… I think in the last case--looking at examples where the um, quadrilateral doesn't exist--and there--it seems like there are tons of them, and one of them was a certain kind of trapezoid, so I was trying to tweak my picture, to make it so." (Interview) His initial efforts utilized a PBQ with a parent quadrilateral whose vertices were fully manipulable. He manipulated these vertices, pausing near various degenerate cases, including when the parent is close to being a rectangle, isosceles trapezoid, and non-polygon. Later efforts involved the manipulation of PBQs constructed off of special parent quadrilaterals. In one of these, he used a robust isosceles trapezoid (with a 60 degree angle) as the parent quadrilateral, making the degeneracy a robust invariant. In another, he used a robust parallelogram as the parent quadrilateral, using which he encountered the degeneracy as a soft invariant—induced when the parent quadrilateral looked like a square or non-square rectangle. Figure 2 shows snapshots of these invariants.

Dr. Sam used the software to generate mathematical ideas and discover properties of the defined quadrilaterals. Through the systematic exploration described, he made several conjectures concerning when the PBQ is degenerate. He used the DGE to provide accurate representations and measurements that allowed him to generate a number of conjectures (shown in Table 2), some of which are visually evident and others that were not.

Dr. Sam was systematic and strategic in the examples he created and selected for his investigation. Drawing upon the DGE's ability to create accurate and manipulable representations, he systematically examined a large number of diverse examples. This empowered him to use strategy in his example usage; he considered sequences of conventionally defined shapes, looked for extreme and unusual cases, and examined examples that increased in specificity (decreasing in generality). Furthermore, the DGE's precision and measurement features facilitated an analysis of the objects' measurable properties.

Table 2: Dr. Sam's conjectures

Quadrilateral Type	Conjectures
ABQ [1] Conjectures:	- If ABCD isn't convex, the vertices A'B'C'D' must be reordered, else quad isn't well defined □A'B'C'D'
	- derived quad need not be inside □ABCD
	- If ABCD is a parallelogram, so is □A'B'C'D'
	- If ABCD is a rhombus, A'B'C'D' doesn't exist (angle bisector for opp angles are identical)
	- If ABCD is a parallelogram, then □A'B'C'D' is a rectangle
	- If ABCD is a rectangle, then A'B'C'D' is a square
MPQ [1] Conjectures:	Given □ABCD
	- □A'B'C'D' is always a parallelogram (even if □ABCD isn't convex)
	- Area of □A'B'C'D' is ½ Area of □ABCD
	- □A'B'C'D' is a square [iff] □ABCD is a square
	- □A'B'C'D' is a rectangle [iff] □ABCD is a rectangle
PBQ [1] Conjectures:	- □A'BC'D' is undefined in many circumstances:
	- If □ *ABCD* isn't convex
	- If □*ABCD* is a 60-60-120-120 trapezoid
	- there are other cases where *A'=B'=C'=D'*
	- If □*ABCD* is a rectangle
	- □A'B'C'D' is a parallelogram [iff] □ABCD is
	- □A'B'C'D' gets large as □ABCD approaches a line
	- □A'B'C'D' gets smaller as □ABCD approaches a rectangle
	- congruent when □ABCD is a 45-135 parallelogram
	- in that case the set theoretic intersection is a square
	- □A'B'C'D' is a rhombus [if] □ABCD is a rhombus
	- Side length of □A'B'C'D' is side length of □ABCD x tan *a* [diagram shows *a* to be the angle at A]

Note: [1]ABQ, MPQ, and PBQ stand for Angle Bisector Quadrilateral, Midpoint Quadrilateral, and Perpendicular Bisector Quadrilateral, respectively.

Apprentice Noah - a contrasting case of software use

Apprentice Noah's exploration was also facilitated by his use of the DGE. In preparation for his investigation, Apprentice Noah synthesized each definition by creating a dynamic model of each quadrilateral (ABQ, then MPQ, and finally PBQ), constructing each derived quadrilateral in the DGE off of a general parent quadrilateral.

Once he had generated these initial representations, he explored each definition in turn (MPQ, then ABQ, then PBQ) using wandering dragging with the parent quadrilateral's vertices. As with Dr. Sam, there was purpose to Apprentice Noah's dragging; he wanted to consider multiple and diverse examples and did so via his manipulations—he also included both concave and convex examples.

In addition, Apprentice Noah made a special construction during his investigations to investigate what happens when the parent quadrilateral is a square (Vignette). The previous time his geometry class met, they discussed circumscribed and inscribed triangles; remembering this, Apprentice Noah focused a portion of his investigation on determining if the MPQ could be circumscribed. He did this by constructing various circles and then dragging vertices until he generated this soft invariant. He noticed, at this point, that the parent quadrilateral was a square and posed this conjecture. He verified this by constructing a robust square along with its MPQ.

This finding influenced his subsequent investigations. Not only did Apprentice Noah use wandering dragging with a general parent quadrilateral to investigate the ABQ and PBQ definitions, but he considered the special case where the parent quadrilateral was a square. In each case he took the time to construct these derived quadrilaterals off of a robustly constructed square, each of which induced a degenerate case of the definitions.

By observation alone, it appeared that Apprentice Noah was using sophisticated strategies in his investigations. Like Dr. Sam, he made diverse and even extreme shapes and even narrowed his investigation to some special cases. Speaking with Apprentice Noah revealed, however, that his investigation was not directed by sophisticated or advanced problem-solving strategies; he explained:

> ``...kind of my approach, that I've taken every time I've done this, so far, this semester was: I'll make the thing and then I'll start moving it around, because maybe I'm hoping that if I did something accidentally, that I wouldn't think to do like deliberately, I would come up with a conjecture I wouldn't have otherwise come up with, so--and then after a while of doing that, just randomly moving and trying to make something happen, eventually I have to stop and start thinking, `Okay now. How can I move this to make it do something specific?''' (Interview)

This is not to say that Apprentice Noah had no strategies or intent in his work. He did try to cause things to happen in his manipulations (Interview), but his efforts lacked the sophistication, analysis, and intent that Dr. Sam's did.

This was further revealed by his response to the invariants that he *did* induce. When he came upon special cases, he did not deeply investigate the extent and conditions under which they occurred; instead of pushing the extents and boundaries, he simply noted the case and moved on. For example, Apprentice Noah induced a degenerate case of both the ABQ and PBQ definitions when he considered and constructed the special case where the parent quadrilateral is a square. Even though he could have performed further manipulations (like Dr. Sam), he made no efforts to explore the conditions that induce these degenerate cases; he simply observed it for squares and left it at that (Vignette)--a fact that can also be seen in his list of conjectures (see table 3). It is also of note that the degenerate cases he discovered were circumstantially obtained, not the byproducts of strategic investigation or manipulation of those cases.

Table 3: Apprentice Noah's Conjectures

Quadrilateral Type	Conjectures
MPQ Conjectures:	1. $\underline{A'D'} = \underline{B'C'}$ and $\underline{A'B'} = \underline{C'D'}$ for all midpoint quadrilaterals. 2. A circle inscribed in a square will circumscribe the midpoint quadrilateral of that square. 3. $\underline{A'C'}$ and *B'D'* always intersect at their midpoints 4. $< A'D'C' = < A'B'C'$ and $< D'A'B' = < B'C'D'$
ABQ Conjectures:	1. There will always be at least 2 vertices of *A'B'C'D'* inside of *ABCD*. If there are only 2, they will be either A' and C' or B' and D'(one each from opposite ends) 2. $A'B'C'D'$ will be a reflection of $ABCD$ (ie, if $ABCD$ is counterclockwise, $A'B'C'D'$ will be clockwise) (assuming convex or nonconvex whichever is the one where all segments of interior points are on the interior) 3. If $ABCD$ is a square then A', B', C', and D' will be concurrent. 4. At most 2 vertices can be on *ABCD* either vertices A' and C' or B' and D' (special case of #1)
PBQ Conjectures:	1. $A'B'C'D'$ will have the opposite orientation of $ABCD$ (clockwise vs. counterclockwise) 2. The vertices A', B', C', D' can be inside or outside $ABCD$ 3. The points A', B', C', D' will be concurrent at the center of $ABCD$ if $ABCD$ is a square

Novice Scott - a contrasting case of software use

Novice Scott's approach to the task (detailed in Belnap & Parrott, 2013; 2015) did *not* use the software. Using a pencil, ruler, and protractor, he carefully constructed a single prototype for each derived quadrilateral. Once created, he used each as an external source of information, treating it as a literal object by taking physical measurements of its angles and side lengths. Throughout his conjecturing process, he exhibited prototypical thinking, making all conjectures a generalization from these three prototypes; he did not consider the diversity of examples inherent in the definitions. As a result of his conjecturing process, he produced a small set of conjectures (in table 4), which primarily concerned superficial features of the quadrilaterals.

Novice Scott's work connected to the technology *during the interview*. When asked what would have improved his ability to perform the task, Novice Scott claimed that he would have had more success with the task, if he understood the software because he could create more than one drawing and be sure of its accuracy. To observe the technology's impact, we briefly instructed Novice Scott on the value, use, and dynamic capabilities of the software, then assisted

Table 4: Novice Scott's conjectures (written work)

Quadrilateral Type	Conjectures
ABQ Conjectures	ABCD -> A'B'C'D' - Original shape held - Stayed in same order A → A' … [rough diagram with labels shown] - rotated to the left
PBQ Conjectures	ABCD → A'B'C'D' - shape flipped & changed order - inverted shape - Slope A → B = -(A' → B) - positions of point changed order ABCD →A'D'C'B' [rough diagram with labels shown]
MPQ Conjectures	ABCD → A'B'C'D' [rough diagram with labels shown] - original shape held - stayed in same order - rotated to the right

him in constructing a dynamic PBQ. Concerned that the shape did not look at all like the one he drew, we helped him reorganize the vertices so that it appeared similar to his carefully constructed prototype.

To support his own investigative interests, we helped him construct the measures of the angles and side lengths (Vignette). Almost immediately, Novice Scott observed and added a stronger conjecture about PBQs, ``A + A' = 180 [degrees]; for all angles'' (Written work). This ability was a result of the precision afforded by the software, which allowed him both to see the numerical relationships *and* verify that it worked for each pair of angles.

While the software enabled him to more easily see these numeric relationships, it did *not* change his treatment of the objects. He treated the dynamic model as if it were a single, static, physical model. He used it as a virtual prototype, only manipulating it and considering multiple examples when pressed and encouraged to do so (video transcript and written work). Even with encouragement, he only made minimal manipulations, twice dragging a single vertex from one position on the screen to another nearby, essentially creating two additional static examples.

Discussion and Implications

Technology is an important and powerful tool for modern mathematical investigations. It provides many affordances and in some cases provides easy access to advanced investigative strategies. The cases we have shared suggest the need for care, planning, and thoughtful reflection both when studying technology and using it to facilitate student learning.

First, from a methodological standpoint, these cases suggest that studies aimed at understanding individuals' investigative strategies and approaches cannot rely solely on observation. From a purely observational standpoint, Apprentice Noah's work with the software had strong similarities to Dr. Sam's, from which one might infer that Apprentice Noah was using the same advanced strategies that Dr. Sam was in his investigation. Apprentice Noah's interview revealed, however, that while he *was* trying to examine diverse cases, his investigation was guided more by a hope at a chance encounter--an understanding of his process that was only revealed via discussion of his approaches during the interview.

Second, from a pedagogical standpoint, it is clear that technology can facilitate the use of advanced instantiation strategies in mathematical investigations; however, technological tools (by themselves) may prove insufficient to elicit them. Although software usage made certain relationships clearer for Novice Scott, it did not change the way he selected his examples or treated them, nor did it change his propensity to generalize from insufficient examples; however, combining it with intervention *did* motivate some change (at least at the moment). Furthermore, software usage helped Apprentice Noah *encounter* a degenerate case, yet he did not engage any efforts or strategies in investigating the phenomenon, even though the software usage made such an investigation easily accessible. For clarity, we are not saying that Noah could not have employed more advanced strategies, but that the accessibility of both the strategies and opportunity that the software provided did not on its own elicit them; he did not have the propensity to investigate the situation, even though he found it interesting enough to conjecture it (something he was picky about).

It is plausible that a different task *could* have elicited more advanced strategies. For instance, if we had given an *embedded*-conjecturing task, that is one where conjecturing was embedded in or subordinate to some overarching problem or goal (Belnap & Parrott, 2014), such as, "Determine when the ABQ is undefined or does not result in a quadrilateral." we would have expected different results from all participants--maybe even spontaneous use of the maintaining dragging strategy from some.

This is *our own* conjecture: Advanced instantiation strategies are *not* a direct byproduct of the use of technological tools, that is, not *spontaneously* generated by use of software tools. This could partially explain the difficulties novices experience when doing *overt*-conjecturing tasks (Belnap & Parrott, 2014). It appears that outside intervention or motivation is necessary to advance individuals' instantiation strategies.

Perhaps instructors can provide such intervention, as we did during our interview with Novice Scott. They can question students' certainty about their observations, challenging their propensity to generalize from too few examples. They can press students to generate and consider diverse and extreme examples in their investigations. In these and other ways, they may foster the development of values, perspectives, and habits that would enable students to explore mathematical situations.

Embedded-conjecturing tasks may also play a key role in the development of students' instantiation strategies. Perhaps embedded-conjecturing tasks could be designed which elicit specific strategies, strategies whose value, purpose, and role could be discussed overtly in the classroom.

There are certainly many questions to be answered. How can instructors facilitate student development of more advanced instantiation strategies? What role do tasks play in the development of mathematical practices? What types of experiences will elicit and facilitate the development, not only of advanced instantiation strategies, but the propensity to use them in undirected mathematical investigations? How does one internalize strategies to the point where they can draw upon them without external structure, pressure, or push? These are just a few of the questions that need to be addressed in order for us to best engage technology in the development of mathematical practices.

References

Alcock, L. (2010). Research in collegiate mathematics education vii. In Hitt, F., Holton, D., and Thompson, P., editors, Mathematicians' perspectives on the teaching and learning of proof, pages 63–92. New York, NY.

Baccaglini-Frank, A. & Mariotti, M. A. (2011) "Conjecture-generation through Dragging and Abduction in Dynamic Geometry". In A. Mendez-Vilas (Ed.) *Education in a technological world: communicating current and emerging research and technological efforts.* Formatex

Belnap, J. K. and Parrott, A. (2013). Understanding mathematical conjecturing. In Brown, S., Karakok, G., Hah Roh, K., and Oehrtman, M., editors, Proceedings of the 16th SIGMAA on Research in Undergraduate Mathematics Education Conference, volume 1, pages 573–580, Denver, CO.

Belnap, J. K. and Parrott, A. (2014). Mathematical enculturation through conjecturing. In Gold, B., Simons, R., and Behrens, C., editors, Using the philosophy of mathematics in teaching mathematics. Submitted to MAA Notes.

Belnap, J.K. and Parrott, A. (2015). Understanding conjecturing: the role and nature of examples. In progress.

Conference Board of the Mathematical Sciences (2012). The Mathematical Education of Teachers II. Providence RI and Washington DC: American Mathematical Society.

Harel, G., and Sowder, L. (1998). "Students' proof schemes." In Research on Collegiate Mathematics Education, Vol. III, edited by Ed Dubinsky, Alan Schoenfeld, and James Kaput, pp. 234 -283. American Mathematical Society.

Laborde, C., Kynigos, C., Hollebrands, K., & Strasser (2006) "Teaching and Learning Geometry with Technology." In A. Gutierrez, & P. Boero (Eds.) Handbook of research on the psychology of mathematics education: past, present and future. Sense Publishers The Netherlands.

Lockhart, P. (2002) A mathematician's lament. Retrieved from Devlin's Angle March 2008 column on MAA online. www.maa.org/external_archive/devlin/LockhartsLament.pdf

Lockwood, E., Ellis, A., and Knuth, E. (2013). Mathematicians' example-related activity when proving conjectures. In Proceedings of the Sixteenth SIGMAA on Research in Undergraduate Mathematics Education Conference. Denver, CO.

Lockwood, E., Lynch, A., Ellis, A., Dogan, M., Wiliams, C., and Knuth, E. (2014). Mathematicians' example-related activity in formulating conjectures. In VanZoest, L. and Kratky, J., editors, Proceedings of the 36th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education., Kalamazoo, MI.

Strauss, A., & Corbin, J. (Eds). (1998). Basics of Qualitative Research: Techniques and procedures for developing grounded theory (2nd ed.). Thousand Oaks, CA : SAGE Publications, Inc.

An analysis of proof-based final exams

Mindy Capaldi
Valparaiso University

Abstract: Evaluating final exams can give insight into what aspects of a course instructors value the most. This study examines whether final exams in proof-based courses accurately reflect instructors' perceptions of their exams. It also categorizes the questions on the exams in terms of type, format, and cognition level. The level of cognition was further distinguished by imitative reasoning and creative reasoning. Results indicate that instructors of proof-based classes are relatively aware of the content of their exams. The largest discrepancy between instructor perception and reality concerned the number of application problems. Analysis also showed that about a fifth of the questions and over half of the points are proof-related. The exams were almost entirely short or broad answer and required high cognition levels when proofs were assumed to require creativity.

Key words: [Proof, Exams, Assessment, Creative Reasoning, Imitative Reasoning]

Introduction

Final exams are prevalent and significant tools of assessment in college mathematics, including in proof-based courses. Such courses are notoriously difficult for students to master and for instructors to teach effectively. Studying the final exams of proof-based courses could at least reveal what instructors expect students to know and be able to do by the end of the semester. Exam design may influence students' beliefs about mathematics and what reasoning is essential in the subject (Bergqvist, 2007). Insights into proof-based final exams might be useful to new instructors of such classes as they grapple with designing a course that could be critical to the students' mathematics career. Experienced instructors could also benefit by comparing their exams to the results of this research and considering whether they personally have a misalignment between perception and reality.

This study was largely motivated by Tallman and Carlson's characterization of introductory calculus final exams (2012). Prior to that study, there was little previous research concerning calculus I exams in the United States, although Lithner had examined reasoning in calculus textbook exercises and found that most exercises required only superficial reasoning (Tallman & Carlson, 2012; Lithner, 2004). More recently, White and Mesa extended the classification of cognitive orientation for calculus I tasks by considering multiple types of coursework (White & Mesa, 2014).

A literature search related to proof-based exams did not uncover any relevant results. In terms of general mathematics exam research, some progress has been made. Marso and Pigge found that K-12 tests contained mostly multiple-choice, matching, and short answer questions and were at simply the knowledge cognition level (1991). Senk et. al. showed that high school mathematics test questions required low level reasoning and were not open-ended or connected to applications (1997). Bergqvist researched university exams in Swedish mathematics courses and found that 70% of the tasks were solvable, and 15 out of 16 exams were passable, by using only imitative reasoning. Imitative reasoning includes repeating memorized information or applying algorithms to solve problems (2007). Tallman and Carlson found conclusions similar to the previous research (2012). Their results indicated that calculus I exams included nominal cognitive demand and were largely computational. There was also a striking difference between instructors' perceptions and reality concerning how often students were asked to explain their thinking and the number of computational questions. Instructors believed they were asking for more explanation, and fewer computation, than they actually were (Tallman & Carlson, 2012).

An emphasis on computation and imitative reasoning seems prevalent throughout mathematics final exams from the K-12 level through calculus. One of the goals of this study was to examine whether those emphases persist through proof-based courses. Drawing upon the methodology of the calculus study, each proof-based exam item was coded within three categories: item representation, item format, and item orientation. Tallman and Carlson had to change previous frameworks to suit the characterizations needed for that study of calculus exams (2012). Similarly, a slightly altered framework was developed to better fit proof-based course exams. Instructors of the proof-based courses were surveyed to gauge how their expectations aligned with reality.

Research Questions

The four main research questions were:

1. What levels of cognition are required on proof-based final exams?
2. How do instructor's perceptions of their exams differ, if at all, from the actual exams?
3. What proportions of the test are creative versus imitative reasoning?
4. How do proof-based courses compare to calculus I?

Theoretical Framework

The primary framework for this study was an overview of the type of reasoning required on the exams, creative versus imitative. In addition to that framework were the exam characterizations of item orientation, item format, and item representation. Item orientation was based upon Bloom's taxonomy of cognitive demand (Anderson, Krathwohl, & Bloom, 2001; Bloom & Krathwohl, 1956). The lower levels of that characterization fit into the category of imitative reasoning, while the higher are creative reasoning (see Table 1). Previous research demonstrated that a significant difficulty for students' learning mathematics is the reliance on superficial reasoning (Lithner, 2008)

Creative Reasoning (CR). Creative mathematical thinking involves processes that are distinguished by flexibility and novelty. A solution that uses such thinking should contain reasoning that is new to the student, choices leading to a plausible solution which are supported by arguments, and is grounded in appropriate mathematical properties (Bergqvist, 2007). Cognitive levels requiring CR were identified as "analyze," "evaluate," and "create." The creativity level of orientation included such tasks as proving theorems or constructing an example. Instructors reported that none of the proofs on the exams had been seen previously by students, so they could not have simply regurgitated the proof. Therefore, the framework classified proof solutions as creative. Some tasks provided a correct or incorrect proof and required students to critique the proof; these were considered part of the evaluate level, but were still CR since the particular reasoning sequence required was novel to the student.

Imitative Reasoning (IR). Students use IR when they are basically reproducing task solutions. IR can be split into two subcategories: memorized and algorithmic reasoning. Reciting a definition uses memorized reasoning. Finding the derivative of a function would use algorithmic reasoning (Bergqvist, 2007). Proving theorems can incorporate both of these types of reasoning, as the prover must remember definitions and structures of proof. However, if the theorem is new to a student, then the framework will assume that some creativity is involved in the reasoning sequence. Item orientations "remember," "recall and apply procedure," "understand," and "apply understanding" were considered IR.

Item Representation. Each exam item task and solution were coded into one or more of the eight categories listed in Table 3. Knowing the representations that appear on final exams gives a snapshot of the variety of tasks and solutions that students were familiar with during the semester. If the exam included many applied problems, then we could conjecture that they solved numerous such problems in the course. Or if a large portion of solution items were coded as "explanation," then we would expect that students often needed to demonstrate their understanding of a concept by explaining it.

The item representation characterization used in this study slightly deviated from that of Tallman and Carlson (2012). Items were coded as "definition/theorem" not only if students were required to provide a statement or interpretation, but also if they were asked to apply a definition or theorem. The latter condition was added for exam questions like, "Use the [blank] algorithm to…," which necessitated an application of the algorithm but not a statement of it. Another change was the inclusion of a task statement category for "explanation." In some questions an example or proof, perhaps incorrect, was provided by the instructor. This was viewed as being given the explanation, which students then described or critiqued. Such an item differed from the "proof" category for task statements, in which only the theorem statement or conjecture was given to the students.

Item Format. Formats of exam items included categories like True/False, short answer, or broad open-ended (see Table 5). Additionally, those three categories were split depending on whether explanation was required. Item format characterization was similar to Tallman and Carlson, except that "word problem" was excluded in this characterization (2012). Any problem that would have been a word problem was coded as applied for item representation. Removing word problems from the list of item formats allowed each question to be coded as only a single format.

Each item requiring students to prove a theorem was coded as broad open-ended with explanation, since proofs are often correct despite being structured or worded differently.

Methodology

This study collected exams and surveys from 18 proof-based course instructors. The instructors were from a variety of institutions: two national universities, four regional universities, and 12 national liberal arts colleges.[1] Of the 18 instructors, 44% were tenure-track, 5.5% were visiting, 22.2% were associate, and 27.8% were full professors. About half of the classes were taught using a mostly lecture format, and half incorporated inquiry-based learning and/or flipped pedagogy. There was a total number 243 exam items, or questions, that were coded.

Each instructor submitted an exam and syllabus for their course. These were not anonymous, so that they could be matched with answers to the following questions.

1. Were any of the questions on the exam previously seen by students (on homework, in class, …)? If so, please specify which ones. Did the students know that that particular problem would be on the exam?
2. If the points distribution is not clear on your exam, could you briefly describe the breakdown of points?

All instructors reported that their exam questions had not been previously seen in class. Of course, some questions were similar to tasks completed throughout the semester, but none were exactly the same.

Each instructor also completed an anonymous survey. First, they were asked for information about their position, institution, and teaching style. Then the survey led participants to rank the importance of certain skills and estimate the frequency that some types of questions were asked on their exams. The ranking was for the following list of student skills:

- Know definitions or theorems;

[1] See Tallman and Carlson for a description of these institution levels (2012; p218).

- Fluency in mathematics/symbols;
- Demonstrate understanding through explanations or providing examples
- Understand proof structure and logic
- Ability to solve applied problems

The exams were analyzed by coding each question into an appropriate category for their item format, representation, and orientation. One item could fit into multiple representations or formats, but was only characterized in a single level of orientation. Under each of those three classifications, the frequency and weight of each category was calculated. Under item orientation, creative versus imitative reasoning was specifically considered.

Once the exam analysis was complete, instructor surveys were reviewed. The results of the survey were compared to the results of the exam analysis.

Results

Item Representation. Coding for item representation showed that 21.4% of tasks and 21.2% of solutions represented theorems and proofs, respectively. When asked "How frequently do you ask students to evaluate a statement or conjecture on an exam (for instance, 'prove or disprove')?" with a scale from 1-6 (1 being not at all and 6 very often), 77.8% of instructors picked levels 4-6. Thus, in terms of number of theorems and proofs, instructor perceptions and exams do not seem to align.

Additionally, 44.4% of instructors ranked the "ability to solve applied problems" as the most important skill for students to master, but only four task items out of 243 were coded as applied. Two of those were from a Linear Algebra II with Applications course, and two were from a Foundations and Structures of Mathematics class. It is possible that responders defined applied differently than the description "The task presents a physical or contextual situation," that was used in the study. Instructors were not given a definition of "applied" for the survey. Further results for item representation can be seen in Table 4.

Item Format. Every instructor picked a level from 4-6 (6 being very often) when asked how frequently students were required to explain their thinking. Coding results showed that there were no multiple-choice items, but the combination of true/false, short answer, and broad open-ended items which required explanation summed to 58.02% of the total formats. Thus, instructor beliefs more closely matched reality in this category. Approximately 6% of items were broad open-ended without explanation; most of those were questions that asked students to generate an example. About 38% of items and 30% of exam points were short answer without explanation, which were generally the purely computational solutions. This also corresponds to instructors' beliefs, since 16 out of 18 instructors reported that less than half of the points on their exams were for purely computational solutions. Further results for item format can be seen in Table 6.

Item Orientation. Over half (57.8%) of exam points were coded at the creative level of item orientation. Only a small number were coded as analyzing or evaluating, so the total percentage of points that were given for CR was 62.3%. Every proof question, whether it asked students to evaluate or analyze a given proof or prove/disprove a given theorem, was in the CR levels of item orientation. Not every IR item was computational and not every CR item was proof-related, but those types of questions made up the majority of the two categories. All but one instructor perceived that more than 40% of their exam points went to proof-writing, with the most common answer being that 70-79% of the exam grade had that emphasis. So, while instructors beliefs concerning how frequently they ask proof-related questions did not align with their exams, their understanding of how much proofs were worth did correspond to reality.

Discussion

This study found some discrepancies between instructors' perceptions of their proof-based exams and the actual exams concerning frequency of proof-related questions and applied items. However, the gap between the characterization of the exams and instructor beliefs was not as large or widespread as what was found in the study on calculus I exams. Also, proof-based course exams require less imitative and more creative reasoning, in contrast to the calculus exams (Tallman & Carlson, 2012).

Possibilities for further research include distinguishing between different proof-based courses. Additional studies could also investigate whether the cognitive intentions of a question match what level of cognition students actually achieve. A similar problem was studied concerning national tests in Sweden, which found that students usually tried to solve tasks that were not similar to textbook exercises by using creative reasoning (Boesen, Lithner, & Palm, 2010). White and Mesa also considered other coursework in calculus classes, and the cognitive demand required for those tasks (2014). Including homework, quizzes, etc. in a study of proof-based course tasks could extend our understanding of what students are asked to do and how much creative reasoning they are using on their final exams.

References

Anderson, L. W., Krathwohl, D. R., & Bloom, B. S. (2001). A Taxonomy for Learning, Teaching, and Assessing: A Revision of Bloom's Taxonomy of Educational Objectives. Boston, MA: Ally & Bacon.

Bergqvist, E. (2007). Types of reasoning required in university exams in mathematics. *The Journal of Mathematical Behavior*, *26*(4), 348–370. doi:10.1016/j.jmathb.2007.11.001

Bloom, B. S., & Krathwohl, D. R. (1956). Taxonomy of educational objectives: The classification of educational goals. Handbook I: Cognitive domain. NY, NY: Longmans, Green.

Boesen, J., Lithner, J., & Palm, T. (2010). The relation between types of assessment tasks and the mathematical reasoning students use. *Educational Studies in Mathematics*, *75*(1), 89–105. doi:10.1007/s10649-010-9242-9

Lithner, J. (2004). Mathematical reasoning in calculus textbook exercises. *The Journal of Mathematical Behavior*, *23*(4), 405–427. doi:10.1016/j.jmathb.2004.09.003

Lithner, J. (2008). A research framework for creative and imitative reasoning. *Educational Studies in Mathematics*, *67*(3), 255–276. doi:10.1007/s10649-007-9104-2

Marso, R. N., & Pigge, F. L. (1991). An analysis of teacher-made tests: Item types, cognitive demands, and item construction errors. *Contemporary Educational Psychology*, *16*(3), 279–286. doi:10.1016/0361-476X(91)90027-I

Senk, S. L., Beckmann, C. E., & Thompson, D. R. (1997). Assessment and grading in high school mathematics classrooms. *Journal for Research in Mathematics Education*, *28*(2), 187.

Tallman, M., & Carlson, M. (2012). A Characterization of Calculus I Final Exams in U.S. Colleges and Universities. In *Proceedings of the 15th Annual conference on Research in Undergraduate Mathematics Education* (Vol. 1, pp. 217–226). Portland, OR: SIGMAA on RUME. Retrieved from http://sigmaa.maa.org/rume/crume2012/RUME_Home/RUME_Conference_Papers_files/RUME_XV_Conference_Papers.pdf

White, N., & Mesa, V. (2014). Describing cognitive orientation of Calculus I tasks across different types of coursework. *The International Journal on Mathematics Education*. doi:10.1007/s11858-014-0588-9

Appendix

Table 1.
Adaptation of Cognitive Behavior (Table 1) from Tallman and Carlson (2012, p223).

Item Orientation	Description
IR: Remember	Students retrieve knowledge from memory of a definition or theorem.
IR: Recall and Apply Procedure	Students recognize what knowledge or procedures to use when directly prompted to do so (e.g., Find the converse, order, union, etc…)
IR: Understand	Students interpret, explain, justify, compare, or make inferences that require an understanding of a concept.
IR: Apply Understanding	Students recognize when to use (or apply) a concept without direct prompting or instructions, demonstrating an understanding of the concept.
CR: Analyze	Students determine relationships in the material by comparing, categorizing, deducing, etc…
CR: Evaluate	Students make judgments based on criteria and standards. Checking and critiquing are cognitive processes at this level.
CR: Create	Students generate, plan, and produce. Includes proving theorems or generating examples.

Table 2.
Coding results for item orientation.

Item Orientation	% of Items	% of Points
Remember	7.41	3.80
Recall and Apply Procedure	27.57	23.13
Understand	9.88	6.65
Apply Understanding	4.53	3.53
Analyze	1.23	1.16
Evaluate	4.94	3.38
Create	44.03	57.78

Table 3.
Adaptation of Item Representation (Table 2) from Tallman and Carlson (2012, p224).

Item Representation	Task Statement	Solicited Solution
Applied/Modeling	The task presents a physical or contextual situation.	Students define or use a model to describe the situation.
Symbolic	The task conveys information in the form of symbols.	Students manipulate, interpret, or represent through symbols.
Tabular	The task provides information in the form of a table.	Students organize data in a table.
Graphical	The task presents a graph or diagram.	Students generate a graph or illustrate a concept graphically.
Definition/theorem	The task asks the student to state, interpret, or apply a definition or theorem.	Students state, interpret, or apply a definition or theorem.

Proof	The task presents a conjecture or proposition.	Students demonstrate the truth of a conjecture or proposition by reasoning deductively.
Example/ Counterexample	The task presents a proposition or statement with the expectation that an example or counterexample be provided.	Students produce an example or counterexample.
Explanation	An example or proof, perhaps incorrect, is given and the student is asked to evaluate or explain it.	Students explain or evaluate a given example or proof.

Table 4.
Coding results for item representation.

Item Representation	% of Task Items	% of Task Points	% of Solution Items	% of Solution Points
Applied/Modeling	0.87	2.41	0.46	1.22
Symbolic	38.74	76.69	34.48	50.90
Tabular	0.22	0.59	1.15	1.36
Graphical	0.00	0.00	1.15	2.47
Definition/Theorem	33.33	60.13	34.71	58.52
Proof	21.43	57.70	21.15	53.82
Example/Counterex.	3.46	5.88	3.91	6.13
Explanation	1.95	2.28	2.99	4.15

Table 5.
Adaptation of Item Format (Table 3) from Tallman and Carlson (2012, p224).

Item Format	Description
Multiple Choice	One question posed with one or more correct answers in a list of choices. Student chooses from the list.
Multiple Choice (Explain)	Student chooses from a list and explains their choice.
True/False	A statement is presented and the student chooses whether it is true or false.
True/False (Explain)	A statement is presented and the student chooses whether it is true or false and explains their choice.
Short Answer	One question presented, which has one correct answer that the student must write.
Short Answer (Explain)	One question presented, which has one correct answer that the student must write and explain/justify.
Broad open-ended	One question presented, which has multiple correct answers. The student must write one of them.
Broad open-ended (Explain)/ Proofs	One question presented, which has multiple correct answers. The student must write one of them and explain/justify.

Table 6.
Coding results for item format.

Item Format	% of Items	% of Points
Multiple Choice	0	0
True/False	3.29	1.24
True/False (Explain)	0.41	0.44
Short Answer	38.27	29.41
Short Answer (Explain)	11.11	8.65
Broad open-ended	5.76	4.82
Broad (Explain)/Proofs	41.15	55.70

Pre-service teachers' conceptual understanding of numerals and arithmetic with numerals in base-ten and bases other than ten

Iwan Elstak
Valdosta State University

Ben Wescoatt
Valdosta State University

This preliminary study explores pre-service teachers' learning and understanding of arithmetic as they are confronted with numbers represented in different bases in order to add to the body of literature about teacher knowledge. Students in an initial mathematics content course for elementary teachers were interviewed as they solved problems related to representing whole numbers and related to arithmetic operations on numbers represented in various bases. Initial findings suggested that the areas of knowledge development during the course were still domain specific and the links, or concepts, in terms of semantic networks were still weakly connected. Students could hold contradictory views of concepts. Additionally, standard algorithms for operations in other bases were also not consolidated; they were not abstracted to the point that new structures had emerged as conceptualized.

Key words: Numeration, Teacher training, Declarative knowledge, Abstraction in context

Introduction

This study aims to investigate how pre-service teacher students understand numerals of different bases and to explore the obstacles faced while learning about numeration. How do they use standard algorithms of addition, subtraction, multiplication and long division with conceptual understanding, meaning knowing the mathematical basis of each step in the algorithms? We use a qualitative methodology with multiple interviews and pre-and post-tests to investigate this question.

Number Sense

Number sense is "a person's general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgments and to develop useful strategies for handling numbers and operations" (McIntosh, Reys, & Reys, 1992, p. 3). The inclusion of number sense as a foundational idea of mathematics in the National Council of Teachers of Mathematics' *Principles and Standards for School Mathematics* highlights the importance of the development of appropriate number sense in young learners. The NCTM suggests that students should not only understand numbers and number systems but how to represent numbers; that is, students should understand what numerals are (National Council of Teachers of Mathematics, 2000). The need to understand numbers and numerals is important as number and numeral properties are fundamental in understanding the arithmetic algorithms (McIntosh, Reys, & Reys, 1992).

Mathematical Knowledge for Teaching

Throughout the mathematics education literature, the suggestion is made that a teacher's mathematical knowledge influences what his or her students learn. For example, Tanase (2011) suggested that a teacher's knowledge of place value affected the students' conceptions. Thus, a student's success in learning basic arithmetic may depend on his or her teacher's understanding of numeration. Hiebert and Wearne (1992) found that students with a solid conceptual understanding of place value generally had better understanding of arithmetic operations. In fact, these students utilized numeration concepts when solving problems new to them; when approaching the addition of two-digit numbers for the first time, they were able to generate their own algorithms using grouping schemes without prior instruction.

While teaching for conceptual understanding is important, both pre-service and in-services teachers have been shown to be lacking in this conceptual understanding. Teachers

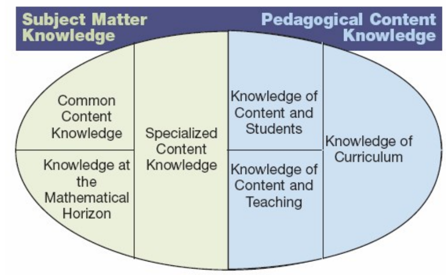


Figure 1: The Mathematical Knowledge for Teaching from Hill & Ball, 2009.

appear to have inadequate knowledge of the decimal system, being unable to adequately represent numbers or discuss place value (Stacey et al., 2001; Matthews & Ding, 2011). When representing numbers, their representations tend to lack structure that identifies the base of the numeral and the grouping concept necessary for the base (Matthews & Ding, 2011). Additionally, teachers have difficulties in identifying why students make errors in place value; in fact, teachers' errors are very similar to student errors (Stacey et al., 2001; Muir & Livy, 2012). Moreover, when teaching numeration concepts, teachers tend to rely on procedural rather than conceptual methods (Muir & Livy, 2012).

The development of teachers' number sense in a conceptual way is important in that numerous studies have suggested a relationship between teachers' knowledge for teaching mathematics and students' learning and understanding (e.g., Hill, Rowan, & Ball, 2005). Thus, students will be more likely to understand concepts in a conceptual rather than solely procedural way if their teachers have a deeper, conceptual understanding of the concept. In terms of modeling a teacher's knowledge for the teaching of mathematics, a common model used is the Mathematical Knowledge for Teaching model developed by Ball, Hill, Thames, schilling, Bass, et al. (e.g., Hill & Ball, 2009). According to the model (Figure 1), teachers of mathematics should possess knowledge different from a common person. For example, understanding the digits in a numeral as representing the number of groups of a certain size, as determined by the base, falls in specialized content knowledge. As another example, realizing that young learners may not fully understand how to write a numeral (writing "*1007*" for "one hundred seven") would be knowledge of content and students.

The Current Study

The aim of this current study is to explore and better understand the development of arithmetic concepts as pre-service teachers learn about the operations and are confronted with numerals of different bases. That is, the study will explore the mathematical knowledge for teaching as it applies to the development of number sense in pre-service teachers. A motivation for studying pre-service teachers as they experience numerals of different bases is explained by the following quote from Zazkis (1999):

> Considering non-conventional structures helps in gaining a better understanding and appreciation of those that were chosen as "conventional" and learned as a mother tongue. Working with non-conventional structures helps students in constructing richer and more abstract schemas, in which new knowledge will be assimilated. (p. 650)

That is, as students may no longer rely on algorithms learned long ago to operate with numerals, they must resort to foundational knowledge. In so doing, the belief is that students will then situate their number sense in a broader, better-connected scheme. Thus, when students are face with problems novel to them, they should rely on conceptual understanding to navigate the problem as they do not have a learned algorithm on hand. Zazkis (1999) claimed that exploring numerals in bases other than ten would dis-equilibrate learners in the Piagetian sense, helping the learners appreciate the need to reconstruct, or recalibrate, their number sense schema. Our study will use the standard algorithms for the operations, but situated in the novel context of new bases that will force the students to ground their reasoning on the mathematics of the numerals rather than on learned tricks and recipes.

Related Literature

Herman, Zilles, and Loui (2011) studied undergraduate students' conceptions of numeration systems in a computer organization course taken by computer science, electrical engineering, and computer engineering students. They found that as students learned about binary, octal, and hexadecimal numeration systems, the students developed strong procedural skill but had weak conceptual knowledge about numeration in spite of both procedures and concepts being taught. In the course sequence, the binary system was first, followed by octal and then hexadecimal. Students had the most difficulty with the hexadecimal system. In order to perform tasks such as number comparisons in hexadecimal, students would typically begin by first converting from hexadecimal to binary; this conversion was followed by conversion to decimal if the comparison was still not understood. Moreover, students who were proficient in decimal and binary could have difficulty interpreting hexadecimal numerals. Thus, students needed to convert numerals into a "mother tongue" in order to truly understand a problem. Overall, students tended to rely on procedural tricks rather than conceptual reasoning. Moreover, the researchers suggested that students had difficulties similar to those encountered by young learners approaching whole numbers for the first times, indicating that despite additional training, the undergraduate students were unable to move away from pre-existing conceptions.

Persistence of misconceptions was a finding of Bartolini Bussi (2011) in her exploration of teachers and pre-service teachers working tasks with manipulatives. Even though all of the participants had coursework covering the generalities of place value, when asked to perform arithmetic tasks with manipulatives, they were unable to connect the manipulatives to algorithms via conceptual understanding. For example, when pre-service teachers were using a spike abacus, they were willing to denote the value of a place through a particular color of a bead rather than through a position. In other words, external qualities of the manipulative not germane to the concept dictated utilization of the manipulative.

As pre-service teachers learn about place value, they must ignore facets immaterial the concept and set aside pre-existing, procedural knowledge of the decimal system to understand the richer concepts involved in a numeration system with place value. McClain (2003) studied a teaching treatment focused on supporting the development of conceptual understanding of place value and multidigit addition and subtraction. By using activities focusing on the packaging of units of candy, the treatment focused on the grouping concept and avoided extraneous issues such as color that could become an impediment. Overall, the pre-service teachers were able to move beyond just accepting algorithms because they worked as symbol manipulation. Through the use of pictorial representations, the teachers were able to understand and explain the important concepts involving grouping in the addition and subtraction algorithms. This result shows the important role that manipulatives can play in helping students attach meaning to symbols. That is, the symbols in addition and subtraction algorithms no longer had vacuous referents; students could connect the symbols to concrete referents if needed. Overall, the study demonstrated the ability of pre-service teachers to develop a conceptual understanding of important place value concepts.

Price (2011) explored pre-service teachers' conceptual understanding of place value as they engaged with a special positional numeral system, Orpda. Orpda was essentially a base-five system. However, rather than using the digits 0 through 4 to write numerals, the symbols ~, *, @, #, and ^ were used to represent no, one, two, three, or four objects. The Orpda system was developed to dissuade students from relying on the usual base ten system as numeration topics were studied. Price found that the students showed more understanding of place value and the concepts of grouping and regrouping after engaging with the Orpda system.

While Orpda was intended to avoid reliance on base ten, in an earlier study of in-service teachers and Orpda, the teachers needed to compare Orpda to the decimal system in order to understand how to write numerals in Orpda (Hopkins & Cady, 2007). During a workshop, after trying to represent a group of five objects with place value, for example, as @#, the workshop facilitators led the in-service teachers to consider the numerals in the decimal system. For example, to represent 5, one could write $2 + 3$ but not 23. Eventually, some teachers suggested *~, the correct representation; in order to convince those teachers skeptical about this representation, the facilitators compared it to the numeral 10 in base ten.

Expanding on earlier results, Cady, Hopkins, and Price (2014) discussed further results of their study of pre-service teachers learning about place value through Orpda. As mentioned, Orpda was created to dissuade students from converting to the more familiar decimal system; instead, students needed to associate the signified quantity with the symbol; this association would mimic the learning trajectory experienced by elementary students. Similarly, the teachers needed to associate the name of the symbol with the symbol. The intent of Orpda was to force the pre-service teachers to experience a disequilibrium concerning place value, separating them from their knowledge of the familiar decimal system, and to force the teachers to reflect and reconceptualize their understanding of place value. The researchers noted that the pre-service teachers began to associate quantity with the symbol after several meetings. The students commented on the use of manipulatives to facilitate this development, again highlighting the important role that appropriate manipulatives play in developing conceptual understanding of place value. Through the use of Orpda, teachers developed a deeper appreciation of how their future students learn and gained a more robust conceptual foundation to assist them as they facilitate learning with these students.

The literature highlights the important role that appropriate manipulatives and representations hold in facilitating the development of a more conceptually-based number sense. Furthermore, as students learn, they have a tendency to rely on prior knowledge of the decimal system. This tendency is problematic in that this knowledge is typically shallow and procedural and probably faulty. Thus, better understanding how number sense develops in pre-service teachers is important.

Theoretical Framework

Our general framework is the theory of complex declarative knowledge (Chi, 2005). There are two types of knowledge: procedural and declarative. Declarative knowledge refers to facts, descriptions, concepts, principles, ideas, schemas and theories. Procedural knowledge refers to knowledge of how to do things, associations between goals, situations, and actions to achieve them (or avoid them). Declarative knowledge is use-independent and descriptive. The theory of numbers and operations is such a system of knowledge.

Declarative knowledge is described with three constructs: semantic networks, theories, and schemas (Markman, 1999). Semantic networks are conceptualized as nodes that connect concepts by links (relations). Knowledge is accessed by traversing links. Concepts (and nodes) can be linked through multiple links. Therefore declarative knowledge is *grouped by domains.* Domains are formal or informal areas of knowledge that are similar in meaning and cluster together. The structure of domain representations depends on dominant relations in the domain. Representations of domains are locally structured by their dominant relations. Formal mathematical theories have strong relations. The existence of semantic structures does not mean that domains have an overarching, single, coherent structure. Concepts are components of domains, but domains may not have higher order organization. Domains can be represented as theories. Theories usually have a center-periphery structure, organized by core concepts or principles.

Schemas are structures that codify the notion of intuition rooted in patterns of experience. They are activated as units, and they are bounded. Activating one part of a schema also activates other parts. Schemas are abstract, because they are rooted in recurring patterns of experience (Chi, 2005). Numeration in base-ten is a domain of declarative knowledge (with procedural components) that contains multiple schemas. The various domains of numeration in base-ten are not strongly connected in the mind of the student. New knowledge on numbers and numeration with a base other than ten is semantically related to numeration with base-ten, but the overarching theory is not organized by core principles for every student. Therefore, we expect multiple moments of confusion as a student tries to incorporate new knowledge of numeration next to the old, base-ten variety. The existing base-ten domain of knowledge has areas such as number words that are not available when moving to other numeration concepts. This also can lead to confusion.

The second theory used in this study, one that augments what can be explained with declarative knowledge, is the theory of Abstraction in Context, based on activity theory and the notion of Freudenthal of vertically re-organizing existing mathematical (knowledge) structures to construct new knowledge (Dreyfus, 2012). It is called an activity theory because such new structures are created in real activity, with the following components. *Recognizing* is an action where the student identifies for herself a structure and then uses this to fulfill a goal, or understand a problem, and in the process, *construct, put together,* or *assemble* knowledge artifacts to *produce a new structure*. A refinement of the model is important to our paper, the idea that the abstraction (i.e., a new construction from the previous knowledge) is consolidated when it is used to create a new construction (Ozmantar & Monaghan, 2005). The understanding of numeration in other bases must be consolidated before it can be used together with algorithms for addition and subtraction, and in turn, the addition and subtraction structure must be recognized and understood before it can be used to build multiplication along the lines of the standard algorithms applied to alternative bases.

The coordination of these two frameworks forms the basis for our analysis of students' understanding of numeration and operations with new numerals.

Methodology

Our goal was to answer the following research questions.

1. What are pre-service student teachers concepts of numeration in bases other than ten, and
2. To what extent do these students understand standard (algorithmic) operations like addition, subtraction, multiplication and long division with numerals of bases other than ten in conceptual rather than procedural ways?

Setting

The data for this study were collected at a southern, mid-sized regional university. Participants in the study were recruited from two sections of an initial mathematics course taken by pre-service elementary and special education teachers. Students take the course prior to full admittance into their programs. The main topics covered in the course are sets, numeration, and whole number operations. The same instructor, one of the researchers, taught both sections of the course. Every participant was female, due in no small part to zero male students being enrolled in the course. This paper focuses on the data of four participants. The four students in the study were selected based on the content of their interviews in terms of revealing diverse patterns in the learning process. The pseudonyms for the participants are Anna, Beth, Cheryl, and Donna.

Instruments

The first interview explored students' understanding of numeration systems by requesting them to write numerals in specified bases for specified whole numbers. Example questions were: What is the numeral for eighteen in base eight? What is the numeral for eight times eight in base eight? The second interviews focused on students' reasoning in algorithms while they performed addition, subtraction, and multiplication on numerals in various bases. The third interviews focused on multiplication, long division, and number facts such even and odd and divisibility with numerals in various bases. An example question from the third interview was: If a numeral in base five is divisible by twenty-five, what are its last digits on the right? The interviews occurred during the class unit on numeration and operations. Thus, interview one occurred after students had finished the numerals subunit in class and while they were learning about addition and subtraction, interview two occurred while students covered multiplication, and interview three occurred after students finished the division subunit. In this way, the development of students' learning and understanding could be observed and probed. Each interview lasted between sixty and ninety minutes. During each interview, students were prompted to speak aloud their reasoning. The audio of each interview was captured for each interview and transcribed. For select problems during interview three, the video of students performing an action was captured. During all interviews, students wrote solutions on paper; this work was kept for further analysis.

A pre-test was administered to the participants before the first interview, with 13 questions on numeration and algorithms for addition, subtraction, multiplication and long division. The same test was given as a post-test to participants after the third interview to gauge if their responses would reflect sophistication, less mechanical explanations of their actions, or some form of heightened awareness of place value, connections between multiplications and powers of ten and the appearance of zeros in the product, and a more reasoned explanation of the long division process (explained extensively in class and discussed during interview three). In addition to these diagnostics, the participants took in-class exams that assessed their abilities with numeration concepts.

Data Analysis

The transcripts were analyzed with a focus on students' constructions of numerals in various bases and the associated reasoning, cognitive devices used, and obstacles identified and solved for that purpose. We also looked for patterns of how students dealt with structural numeration issues such as their understanding of why zeros appeared in algorithms or when multiplying a numeral by the base numeral, how addition and subtraction algorithms were explained, and how multiplication and long division in alternative numerals were conceptualized and understood.

Teaching Sequence

Before discussing the data and results, we will present the sequence of teaching followed in each class.

Unit 1. The basis of numeration in the decimal system; The digits 0,1,2,…, 9 to create numerals using the powers of ten: 1, 10, 100, 1000, etc; Systems with bases less than 10, such as base-five, and related digits: 0, 1, 2, 3, and 4; Numerals constructed with these bases and digits; The absence of number words for numerals in other bases such as in base-five.

Unit 2. Conversions from base-ten to base-N (N<10) and vice versa; Use of the list of powers of N for the conversion, e.g. converting to base five uses 1, 5, 25, 125, 625 etc.

Unit 3. Addition and subtraction and related algorithms in base-N.

Unit 4. Multiplication and long division algorithms in base-N; even and odd in base-N; Divisibility by three and five in base-N.

Results and Discussion

We begin by discussing students' concepts of numerals before moving on to discussions of students' reasoning with algorithms.

Numerals and Conversions

At the beginning of interview one, students were asked the question: What is a numeral?

Anna: Well…a number is abstract and it's just in your head. And …eh…numerals are signs and symbols that represent a number.

Beth: The number is something in your head. It's something abstract that you make up, I guess. A numeral is the actual written portion of it. So a numeral represents the number. So if I have the number four in my head, I would write four. And the four that I write down is the numeral. Um… a digit is the building blocks of um, a number, of the numeration system. It's a place holder for a number. So, like if I have the number 1-0-4, the one is a digit, the zero is a digit, and the four is a digit.

Cheryl: Both number and numeral represent a value. A number is in your head. A numeral is something like a sign. A digit is like a sign, like square root (sign).

Q: What do you mean?

C: It's a sign, just like you have the square root of 144. Um… that answer would be 12. But it is not like a digit per se. OK. The digit is in place value, kind of like you have like the ten, the ones, and the hundreds.

Donna: A digit is a number in a certain base…or a numeral that does not have to be re-grouped to determine its value.

The responses from the four students suggested that they understood in their own way that numbers and numerals are related in a semiotic way. That is, the numeral serves as a signifier of the number. Numbers are abstract concepts. Additionally, students understood that numbers and numerals have slightly different links to other notions such as digit, place value, and base.

Notions students held concerning digit and numeral were explored next. Students were asked about digits needed to write numerals in various bases.

Q: What are the digits for base-five?

A: For base-five there is: 0, 1, 2, 3, 4.

Q: What about digit 6?

A: Six exists, but it is not known as 6, it is known as 11.

Q: If we have a base-five system, what is the numeral for the number five itself?

A: That would be just a 1 and a 0.

Q: Why a 1 and a 0?

A: You have a copy of five and zero copies…zero units.

D: It is ten (uses the word "ten"). …one-zero!...oooh!

B: So base-five would be zero through five. So that's zero, one, two, three, four. Um…is five included? Zero, um…in base-five, five is included. Um… no, I don't know. Zero, one, two, three, four (whispers to self, writes numerals)…five is not included. I can't remember (voice goes in high pitch, laughs)…um.

Q: So you listed 0, 1, 2, 3, 4. Why did you pick those particular ones?

B: Because I can't go above five. And so between zero and five is 1, 2, 3, 4. And so, in order to represent five, I can use any of those numerals, and put them together to represent five.

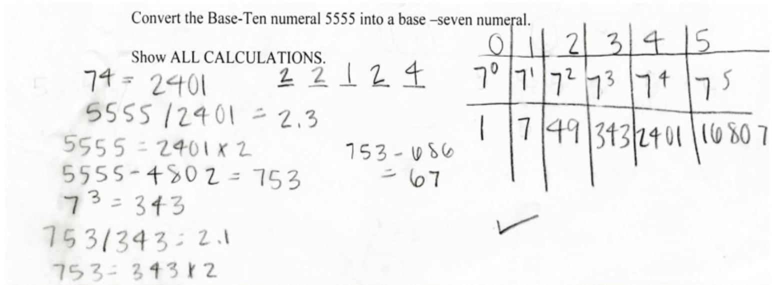


Figure 2. Conversions from base-ten to base-seven.

After more discussions about base digits, the following question is posed to Beth:

Q: Okay, what about if we are in a base-five numeration system; what is the numeral for five?

B: If you are in base five, um... This is where I can't remember…Then I feel like five can be included in base five, but if not, then you could do like four plus one, that would be five.

This episode suggested that Beth's understanding of numerals at this point was still weak. She needed special prompts to make the connection with the concept of the numeral for five in base five. Additionally, Beth did not recognize the role of place value in the numeral. She attempted to decompose the number five into an equivalent sum, but one that had no relationship to its base. This decomposition phenomenon was also observed by Hopkins and Cady (2007) as in-service teachers struggled with representing five in Orpda. In order to force Beth to commit to an answer, the interviewer asked her what should be written for five.

Q: I agree, three plus two is five. But if we then wanted to represent five as a numeral.

B: Then you would do one times five to the first plus zero times five to the zero. So you would write 1-0! Because you have one set of five and zero in the ones spot.

Q: So I have a question. What did I say that made you think of what you wrote down?

B: You said, "What would you actually write down to represent five?"

Q: So that is different from the question, "What is the numeral for the number five?"

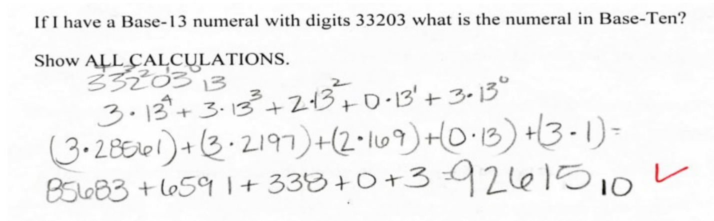
B: Numeral for the number five (muttered). In base five or just in general?

Q: Here, we are in base five. … My big question then is, the original question said, "What is the numeral?" I reworded it and said, "What would you write down?"

B: I don't know that there's a difference, but I guess if you said, what would I write down? I wrote down more than you asked for. So I guess I represented, I expanded it. And then wrote the actual numeral. So it's the same question, but I think when you asked me to write down something, I'm going to show my work more and show where I got that numeral from.

Beth seemed to be sensitive to the wording of the questions and was uncertain about digits for base five. Was 5 a digit or not? She tilted toward "no," although this understanding should be classified as tenuous. Her sudden ability to write "five" in base five as $1 \times 5^1 + 0 \times 5^0$, or as 10_{five}, suggested disconnected areas of knowledge in her mind at this point in the learning trajectory.

Most students displayed no problems using procedures for converting base-ten numerals to another base. For example, on a test, Beth converted 5555 into a base-seven numeral (Figure 2). What was significant was the use of the ordered list of powers of seven to find the numeral. The list had exponents including the order of the digits (from left to right). The new digits were calculated, with no errors. However, as will be argued, this part of the knowledge



If I have a Base-13 numeral with digits 33203 what is the numeral in Base-Ten?

Show ALL CALCULATIONS.

33203_{13}

$3 \cdot 13^4 + 3 \cdot 13^3 + 2 \cdot 13^2 + 0 \cdot 13^1 + 3 \cdot 13^0$

$(3 \cdot 28561) + (3 \cdot 2197) + (2 \cdot 169) + (0 \cdot 13) + (3 \cdot 1) =$

$85683 + 6591 + 338 + 0 + 3 = 92615_{10}$

Figure 3. Conversion from base-13 to base-ten.

and understanding of numerals did not seem to be integrated with the knowledge of how to apply powers of a base with operations like recognizing squares of bases as units in the list of place values. For example students struggled with the representation of 64 in base eight, or 49 in base seven.

Likewise, conversions of numerals from non-base-ten systems back to base-ten were not problematic. As an example, Cheryl converted a base-13 numeral to base-ten (Figure 3). The calculations were done with no apparent flaws or confusions. This suggested that when procedures were involved, the students seemed to have no serious problems.

Addition and Subtraction

The interview data on addition suggested that students had no serious problems with adding in bases other than ten when using the standard algorithm. However the test results suggested that adding simple units in a non-standard format could still be problematic. For example, Anna went wrong in row three after several correct steps (Figure 4).

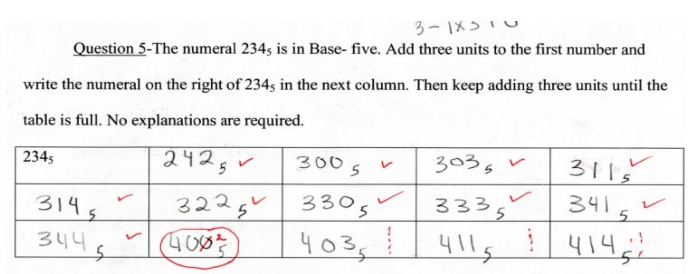
Working the same problem, Cheryl started by adding 3 units to 234_5. Her sum was recorded as 240_5. Accepting this result as correct (it is not!), then the next numeral should have been 243_5 (which is correct). The next numeral should be $243_5 + 3_5 = 301_5$. The next numeral was 303_5, which was the correct addition of 3 to 300_5. Again, if we accept the numeral 303_5 Cheryl has as correct, the next numeral she wrote, (312_5), cannot be correct. Thus, while displaying fluency with the standard algorithms, students generally struggled with addition, even of small quantities, when the addition was presented in a different format, suggesting that students may have relied on procedural tricks to get correct answers.

On a similar problem, Beth solved addition in base-nine, adding one unit to 338 to find 339 (Figure 5). The error of using the digit 9 in a base-nine numeral was repeated again later in the same problem. Proper usage of digits appeared to be a recurring error for Beth; she appeared to not comprehend what a digit represented for a numeral.

Multiplication

During the third interview, Anna was tasked with the following multiplication problem: $434_{\text{five}} \times 13_{\text{five}}$. The main issue that was probed in the interview was how to explain the zeros that appeared when using the standard algorithm for multiplication in base-five. For the sake of brevity we give one piece only.

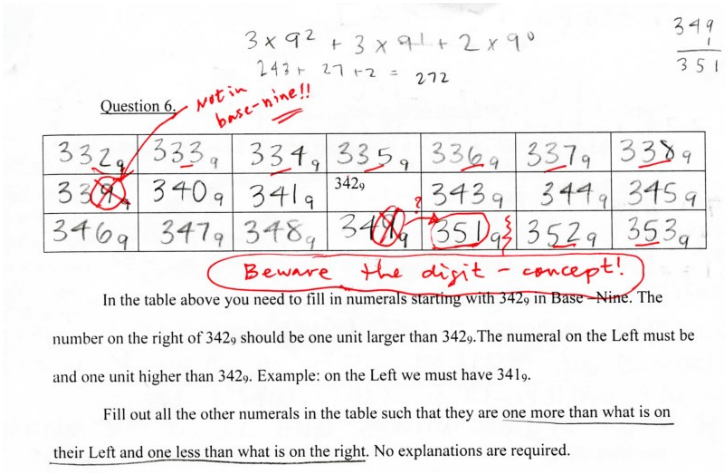
A: First I take the 3 and 4 because they are units, which is 12 in base-ten, and in base five, 22. So then you have the 2 …in units…and then the 2 goes into the tens. (She continues with the next digit.)…then you go to 3 times 3, well, that …since you are



Question 5-The numeral 234_5 is in Base- five. Add three units to the first number and write the numeral on the right of 234_5 in the next column. Then keep adding three units until the table is full. No explanations are required.

234_5	242_5	300_5	303_5	311_5
314_5	322_5	330_5	333_5	341_5
344_5	400_5	403_5	411_5	414_5

Figure 4. Addition in base-five.



$3 \times 9^2 + 3 \times 9^1 + 2 \times 9^0$

$243 + 27 + 2 = 272$

349
1
351

Question 6.

Not in base-nine!!

332_9	333_9	334_9	335_9	336_9	337_9	338_9
339_9	340_9	341_9	342_9	343_9	344_9	345_9
346_9	347_9	348_9	349_9	351_9	352_9	353_9

Beware the digit - concept!

In the table above you need to fill in numerals starting with 342_9 in Base ~~Nine~~. The number on the right of 342_9 should be one unit larger than 342_9.The numeral on the Left must be and one unit higher than 342_9. Example: on the Left we must have 341_9.

Fill out all the other numerals in the table such that they are one more than what is on their Left and one less than what is on the right. No explanations are required.

Figure 5. Addition in base-nine.

> moving from the units…to the tens…or rather the fives, because you are in base five, 3 times 3 in base five is 14. So…if you want to write it all out you would end up with no units, so that's a zero! And then the 4 goes into five and the 1 goes into 25 (She referred to the columns of that value).

Anna explained how the first zero in the standard multiplication, now in base five, occurred mathematically; however, her reasoning was based more on a procedure than a concept. Rather than acknowledging that the second digit of 3 represented three groups of five and reasoning about the implications for this, she relied on "moving" to the tens [sic] place as why a zero would appear. Thus, she relied on a procedural trick. She correctly stated that 3 times 3 was 14_{five}. However, she missed the fact that she was really multiplying 3 times 30 in base five, which is 140_{five}. At the end of the interview, Anna claimed to finally understand how the distributive property explained the occurrence of the zeros in multiplication. However, her claim should be taken within the context of her procedural reliance.

Grouping

One of the last tasks participants engage in was a task novel to them in that they had not first encountered it in the classroom. The participant was given a certain number of pattern blocks (yellow hexagons, red trapezoids, blue rhombi, and white diamonds). They were instructed that each block represented a unit and requested to organize the blocks in order to create a base-three numeral. In general, participants struggled with this task. For example, Beth was given sixteen blocks. She began by counting the sixteen blocks.

> **B:** So I have sixteen, and to put that into base three, hmm, it'd be five-one. So, I don't know. … Can I make the shape of a five and a one out of my objects?

As she did in her first interview, Beth struggled with representing numbers using digits appropriate to their bases; she was more than happy to use a 5 in base three. Moreover, she did not connect creating a numeral with the idea of grouping according to the size indicated by the base. After pausing, she set aside one shape, grouped five shapes, grouped another five shapes, and then another five shapes. Finally, she whispered to herself, "I need more objects," in order to form two more groups of five objects, giving her as she said, "five groups of something and one group of something." While she was forming groups of the same size to match the number of groups indicated by the digit, the base of the numeral was not present; instead, the groups were comprised of "something."

Upon being prompted to consider the base of the numeral and the digits to use, and after stating that the allowable digits were 0, 1, and 2, Beth still accepted 51 as the numeral. This error persisted. She finally realized that 5 wasn't a digit used in base-three and eventually

Figure 6. Beth forming the numeral 121_5.

wrote the numeral as 121_{five}. To then form the numeral with the objects, Beth grouped up five yellow hexagons, placed them to her left, and said, "One." Then, she grouped four red trapezoids and then grouped four blue rhombi and said, "Two," placing them in front of her, to the right of the hexagons. Finally, she grouped three white diamonds and placed them on her right, to the right of the pile of blue shapes and the pile of red shapes. She explained her representation (Figure 6) in the following way:

> **B:** So, my idea was that this is all one group, cuz it's all the same. So that represents the one in the, um, three to the second column. And then the two is represented by two different colors, or shapes, or whatever. So, that's the two, which is the three to the first column. And then this is one group of something in the units column.

Thus, while the digits were represented, the base of the numeral was still absent. In fact, Beth relied on inherent features of the manipulatives, the color, to distinguish where pieces are placed rather than relying solely on group size and place value (Bartolini Bussi, 2011). When asked to comment on how the base of the numeral features in the representation, Beth explained it in terms of placement.

> **B:** Because it's just one group of one thing. And it's the farthest to the left. So it would be my three to the second group.
>
> **Q:** So, if, but if I came to look at it, how would I know that that's supposed to be three to the second? Without you there to explain it?
>
> **B:** I don't know.
>
> **Q:** Is there a way that we can somehow group these? So that we could get across the idea of the place value?
>
> **B:** Ummm. I mean, keep them lined up like that.
>
> **Q:** But how would I know that it's not, like, base four or base five? Or even base ten?
>
> **B:** I don't know.

Initially, Beth relied on first converting the base ten numeral to the base three; in a sense, she needed to understand the problem in her "mother tongue" before proceeding to answer the question. She couldn't directly proceed to forming groups of three. However, she may not have understood that forming groups of three was necessary. That she did not connect the idea of "grouping" to numerals was a bit troubling as grouping is a fundamental concept of numeration. During the first interview, Beth explained her reasoning as she wrote ten as a base-five numeral. In her explanation, she mentioned "carrying" over a five.

> **Q:** If you could speak a little bit about what "carrying it all over" means.
>
> **B:** Um, I guess it means. So, we said I can't have a five. So four plus one is five. So I'm taking the whole, the one, the whole entire ones place and moving it over to the tens place. And so I'm taking that five and moving it over. But there's just one, it's just one five, so that's why I wrote one.

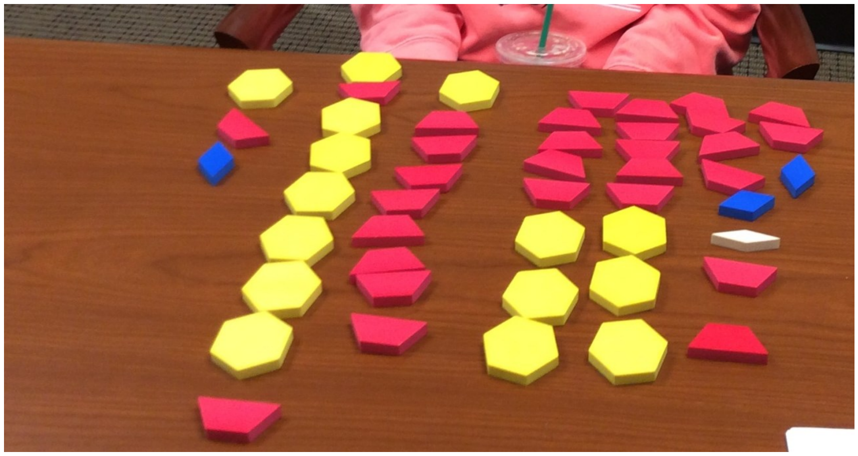


Figure 7. Donna creating the numeral 1210$_{three}$.

> **Q:** Okay. So I'm wondering. You're, you're talking about carrying it over. Bringing it over. Is there anything in your mind, mentally going on? Like are you envisioning anything?
>
> **B:** Mmmm. Not really. Just writing it out.
>
> **Q:** But it's not referring to anything in particular? It's not like you see five object in your mind?
>
> **B:** Um-mm.

Thus for Beth, the idea of a group did not appear to have a concrete referent; it appeared to have a vacuous referent. This could be problematic for Beth as concrete manipulatives are typically used to introduce concepts of groups leading to the writing of numerals in contemporary elementary curricula. Perhaps the concept of "grouping" was just a procedure/phrase to use instead of saying "borrow" or "carry." She indicated this sentiment when she explained what "regrouping" meant to her:

> **B:** I guess it means when you're taking away from another column and moving it to the next one, you're taking what you already have and adding it to what you just got. So that's regrouping.

In contrast to Beth's performance on the grouping task, Donna fluently constructed the numerals to represent the number of objects given to her. Unlike Beth, Donna did not first write the base-three numeral and then construct the grouping. Rather, she formed her groups and then read off the numeral that her groupings represented. For example, given forty-eight objects, she created one group of twenty-seven objects, two groups of nine objects, and one group of three objects (Figure 7). She arranged them from left to right, group of twenty-seven, groups of nine, and group of three. In explaining, she also clarified that there were zero units. She then read off the numeral as 1210. Like Beth, Donna arranged the pieces in a positional manner, mimicking the placement of the digits in a written numeral. However, unlike Beth, Donna relied on the concepts of the numeration system to inform her representation, namely, the base of numeral; Donna's representation actually showed groupings indicating the base.

Pre-/Post-tests

The tests showed no significant improvement in the way multiplication or long division was explained. Anna used the same language with the same unexplained steps. Beth described the procedure without any explanations. Cheryl executed the long division until she reached decimals, but left out all the reasoning. Donna did not answer that question in the post – test. In both tests the same language, unexplained use of the algorithm, and unexplained outcome was shown. There was no explanation of what "bring down XYZ" means mathematically in the post test. This suggested that the study of arithmetic with other bases was not yet matured enough in the mind of the students to achieve a "backward transfer" (Hohensee, 2011), some form of using the algorithm with more emphasis on concepts and reasoning than simple recipe-like steps.

Devices and Recurring Errors

In the interviews, Beth used the phrase "you can't do (or have) Y in base X" (X is one of the base values from 2 to 13) 15 times in her three interviews. Anna used a similar phrase only twice in her three interviews. As an example, when asked why seven cannot be used in base seven, Beth would reply, "You cannot have a seven in base seven…" When Donna was asked the same question she replied, "Seven would not be in the digit place because once you get to seven you would have to re-group to show its value; so it would be 10. Because it is one copy of seven." We analyzed this use of the phrase as a sign of incomplete understanding of place values. The student was aware of some mathematical reasoning, but her reflex, or her understanding took the form of the "voice" of the teacher, without stating the reasoning. The fact that Anna used it only twice in her interviews pointed to an intermediate understanding that was almost complete.

We also noted that students struggled when they had to spontaneously convert a square of a base into a numeral. For example, Question 13 of the first interview asked, "What is the numeral for 8 times 8 in base eight?" Three of the four students hesitated to answer this question. Only one (Donna) had an immediate and correct answer: 100_8. One student (Cheryl) had to calculate first that it was 64, and then counted in base-8 to find the numeral. Beth initially wrote the numeral 78_8. However, thinking to herself, she said, "But if you can't have eight in base eight," realizing her conundrum. She hesitantly decided upon 100.

> **B:** If you just did one times eight to the squared, plus zero times eight to the first, plus zero times eight to the zero, that would give you, one, that doesn't make sense though.
>
> **Q:** Is that the base eight representation for sixty-four?
>
> **B:** Umm … if I can't have eight in base eight, then I would say yes. But if I can, then I would say no. Because earlier we did the same thing, when you said I couldn't have a five in base five, and we came up with one-zero-zero as well. And so it doesn't make sense to me that one-zero-zero could be the same thing for different bases. But maybe it can be.

Beth appeared to be in a very fragile state of learning, unconvinced about the role digits played in numerals and the role the base served, as indicated by her hesitation to accept what digits were used and the fact that 100 in different bases could represent different quantities.

Findings and Conclusion

Our findings and answers to the research questions can be stated thus

1. Students in our study were able to create numerals with bases less than 10, with the knowledge they had developed in class.
2. The knowledge developed was sufficient to do addition with the standard algorithm. But when given problems of addition not necessarily in the standard format (similarly for subtraction), errors occurred frequently pointing to non-consolidated understanding.
3. Students are able to convert from one base to another and back using learned procedures. However when asked to write numerals of powers of the base, no clear understanding was shown.
4. Long division was studied extensively, with ample reasoning, but it did not result in more sophisticated algorithms when doing long division after the conclusion of the course as shown in the post-test. The students omitted the reasoning altogether.

5. While being proficient with algorithms for operations, students displayed a lack of understanding of fundamental numeration concepts, such as digit and grouping. Thus, students appeared to rely on memorized procedure rather than conceptual reasoning.

In terms of our theoretical framework, we explain our observations as follows. In the declarative knowledge framework, the areas of development during the course were still domain specific and the links, or concepts, in terms of semantic networks were still weakly connected. What was learned in conversions did not transfer or connect sufficiently with the domain of numerals and the domain of powers of the base. The standard algorithms for addition and multiplication in other bases were also not consolidated; they were not abstracted to the point that new structures had emerged as conceptualized in "abstraction in context" theory. It seemed that the activity connected with students' study of problems in multiplication and long division were understood but not really at the level of consolidated knowledge. To understand long division, one needs to deeply understand multiplication and subtraction in multiple contexts. This process was clearly not deeply known by most students when doing long division. The post-test suggests that they dropped the reasoning to explain the outcome.

Implications

Zazkis (1999) had suggested that studying numeration in bases other than ten could help students better appreciate the decimal system. However, when students insist on interpreting the numerals in base-ten so as to use the memorized, procedural tricks, these good intentions could be undermined. This study seems to implicitly validate treatments such as the Candy Factory (McClain, 2003) and Orpda (Cady, Hopkins, & Price, 2014) that disallow students from resorting to base-ten translation and force students to reflect on the fundamental concepts of numeration, such as the meaning of digit and the concrete meaning of the base. In order to develop a conceptual meaning of the role that the base has in a numeral, pre-service teachers should experience numeration with concrete manipulatives. Through manipulatives, students can better understand that "grouping" actually refers to something real and is not just a phrase to use instead of "carry" or "borrow" and at the same time, gain a better sense for what a numeral actually represents.

References

Bartolini Bussi, M. G. (2011). Artefacts and utilization schemes in mathematics teacher education: Place value in early childhood education. *Journal of Mathematics Teacher Education,14*(2), 93-112.

Cady, J. A., Hopkins, T. M., & Price, J. (2014). Impacting early childhood teachers' understanding of the complexities of place value. *Journal of Early Childhood Teacher Education, 35*(1), 79-97.

Chi, M.T.H., & Ohlsson, S. (2005). Complex declarative learning. In K.J. Holyoak & R.G. Morrison (Eds.). *Cambridge Handbook of Thinking and Reasoning* (pp 1-59). New York: Cambridge University Press.

Donovan, M. S., & Bransford, J. D. (Eds.). (2005). How students learn: Mathematics in the classroom. Washington, D.C.: The National Academies Press.

Dreyfus, T. (2012) Constructing abstract mathematical knowledge in context. Paper presented at 12th International Congress on Mathematical Education. Seoul , Korea. Retrieved from http://www.icme12.org/upload/submission/1953_F.pdf

Hiebert, J., & Wearne, D. (1992). Links between teaching and learning place value with understanding in first grade. *Journal for Research in Mathematics Education, 23*(2), 98-122.

Herman, G. L., Zilles, C., & Loui, M. C. (2011). How do students misunderstand number

representations? *Computer Science Education, 21*(3), 289-312.
Hill, H., & Ball, D. L. (2009). The curious – and crucial – case of mathematical knowledge for teaching. *Phi Delta Kappan, 91*(2), 68-71.
Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.
Hohensee, C. (2011). Backward transfer: How mathematical understanding changes as one builds upon it (Doctoral dissertation). Retrieved from http://hdl.handle.net/10211.10/1441
Hopkins, T. M., & Cady, J. A. (2007). What is the value of @*#?: Deepening teachers' understanding of place value. *Teaching Children Mathematics, 13*(8), 434-437.
McClain, K. (2003). Supporting Preservice teachers' understanding of place value and multidigit arithmetic. *Mathematical Thinking and Learning, 5*(4), 281-306.
McIntosh, A., Reys, B. J., & Reys, R. E. (1992). A proposed framework for examining basic number sense. *For the Learning of Mathematics, 12*(3), 2-8.
Markman, A. B. (1999). *Knowledge representation.* Mahwah, NJ: Lawrence Erlbaum Associates.
Matthews, M., & Ding, M. (2011). Common mathematical errors of pre-service elementary school teachers in an undergraduate course. *Mathematics and Computer Education, 45*(3), 186-196.
Muir, T., & Livy, S. (2012). What do they know? A comparison of pre-service teachers' and in-service teachers' decimal mathematical content knowledge. *International Journal for Mathematics Teaching and Learning.* Retrieved from http://www.cimt.plymouth.ac.uk/Journal/muir2.pdf
National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
Ozmantar M.F. & Monaghan, J.(2005). Voices in scaffolding mathematical constructions. Retrieved from http://fractus.uson.mx/Papers/CERME4/Papers%20definitius/8/Ozmanter.Monaghan.pdf
Price, J. H. (2011). *Exploring the relationship between Orpda and teachers' conceptual understanding of place value*. (Unpublished doctoral dissertation). University of Tennessee, Knoxville. Retrieved from http://trace.tennessee.edu/utk_graddiss/1013
Stacey, K., Helme, S., Steinle, V., Baturo, A., Irwin, K., & Bana, J. (2001). Preservice teachers' knowledge of difficulties in decimal numeration. *Journal of Mathematics Teacher Education, 4*(3), 205-225.
Tanase, M. (2011). Teaching place value concepts to first grade Romanian students: Teacher knowledge and its influence on student learning. *International Journal for Mathematics Teaching and Learning*. Retrieved from http://www.cimt.plymouth.ac.uk/journal/tanase.pdf
Zazkis, R. (1999). Challenging basic assumptions: Mathematical experiences for pre-service teachers. *International Journal of Mathematical Education in Science and Technology, 30*(5), 631-650.

The calculus concept inventory: A psychometric analysis and framework for a new instrument[1]

Jim Gleason, University of Alabama
Diana White, University of Colorado Denver
Matt Thomas, Ithaca College

Spencer Bagley, University of Northern Colorado
Lisa Rice, Arkansas State University

Concept inventories have become increasingly common in STEM disciplines as a means of assessing student conceptual understanding on a given topic, and overall they have led to significant reform in the teaching and learning of content in their respective disciplines. In mathematics, the use of the Calculus Concept Inventory seems, anecdotally and based on a review of the literature, to be growing. Yet peer-reviewed literature on its development and psychometric properties is lacking. Using data from approximately 1800 students across four institutions, we analyzed its content validity, internal structure validity, and reliability. We conclude that the data is consistent with a unidimensional model and that the instrument lacks sufficiently strong reliability for its intended use. Based on these findings, we argue the need for creating and validating a criterion-referenced concept inventory on differential calculus and outline a potential framework for such an instrument.

Key words: Calculus, Conceptual Understanding, Concept Inventory, Instrument Evaluation

The Committee on STEM Education has determined that a focus on improving STEM education during the first two years of undergraduate education is one of four priority areas of the Federal government (Federal Coordination in STEM Education Task Force, 2012). Since calculus is a key component of the first two years of a majority of undergraduate STEM majors, perhaps no course is as pervasive in STEM students' careers as calculus and deep conceptual understanding of calculus content provides a foundation for a majority of these majors.

As educators and educational researchers, we seek to develop calculus courses effective in building conceptual understanding in addition to procedural fluency, and continually investigate promising new pedagogical strategies. The Mathematical Association of America recommends that all math courses should build conceptual understanding by helping "all students progress in developing analytical, critical reasoning, problem-solving, and communication skills and acquiring mathematical habits of mind" (Barker et al., 2004, p. 13).

In an effort to provide feedback in the process of transitioning courses toward conceptual understanding, a genre of psychometric instruments called concept inventories have been developed over the past 25 years. Concept inventories are designed to measure basic conceptual understanding in science and mathematics courses. For introductory calculus courses, Epstein (2007, 2013) developed the Calculus Concept Inventory (CCI), but this instrument lacks a thorough psychometric analysis, and several recent studies suggest the CCI does not measure students' conceptual understanding (Bagley, 2014; Thomas & Lozano, 2013).

This paper conducts a psychometric analysis of the CCI using a content validity analysis of the instrument to compare the instrument and its stated goals, and a structure and reliability analysis using responses of approximately 1800 students at four institutions. The results suggest

1 We are thankful to Guada Lozano and Chris Rasmussen for contributions that have supported this work.

the CCI does not exhibit many of the psychometric properties its developers originally suggested. Thus we conclude with ideas pointing towards potential modifications and new instruments.

Literature Review

Measuring Conceptual Understanding

It is more challenging to measure conceptual knowledge than procedural knowledge. Students who have encountered problems and strategies for solving them can typically repeat known procedures without using the type of conceptual knowledge they need to solve novel problems and integrate different types of knowledge. The subject of introductory calculus is particularly well suited for exploring conceptual topics due to the presence of ideas such as limits and continuity, which can be treated either algebraically or graphically (Koirala, 1997), and conceptual understanding has been a key aspect of calculus reform (Hughes Hallett, 2006; Hughes Hallett, Robinson, & Lomen, 2005). The disconnect between high school and college mathematics classes, even high school calculus and college calculus, has been an active area of study, and differences in approach and style of thinking is often cited as a reason for the disconnect (Clark & Lovric, 2009; Long, Iatarola, & Conger, 2009; Mann, 1976; CUPM Panel, 1987; St. Jarre, 2008). A focus on conceptual understanding at the college level provides both a challenge and an opportunity of learning for students, including those who have seen the material before.

The difficulty in constructing useful conceptual questions illustrates why the construction of a high quality concept inventory is challenging. Research has shown, however, that the use of high quality conceptual questions can lead to greater student understanding. A project called the "Good Questions" project aims to "raise the visibility of key calculus concepts, promote a more active learning environment, support young instructors in their professional development in their early formative teaching experiences, and improve student learning" (Miller, Santana-Vega, & Terrell, 2006, p. 193). The project provides questions which have been used in college calculus classrooms to encourage active discussion of the content and lead to conceptual understanding by the students. An example of a "good question" is whether the statement "you were once exactly π feet tall" is true or false; students may claim the statement to be false despite holding a belief that height is a continuous function. These types of questions have been shown to improve student learning, though only when specifically used as a tool to encourage student discussion (Miller et al., 2006). The Good Questions project is based in part on Eric Mazur's Peer Instruction method (Mazur, 1997), developed for introductory physics classes (Terrell, 2003). Mazur's instructional ideas for interactive teaching were also extended by Pilzer (2001) to include other physics and mathematics settings, and have shown greater gains in conceptual knowledge among students compared with procedurally-focused lectures, with little to no change in procedural skill (Mazur, 1997).

Concept Inventories

One way to measure conceptual understanding in STEM education has been through multiple choice instruments called concept inventories. The first concept inventory to make a significant impact in the undergraduate education community was the Force Concept Inventory (FCI), written by Hestenes, Wells, and Swackhamer (1992). Identifying that students' commonsense beliefs were incompatible with Newtonian mechanics, the test was used to analyze students' thinking in introductory mechanics courses. Despite the fact that most physics professors considered the inventory questions "too trivial to be informative" (Hestenes et al., 1992, p. 2), students did poorly on the test, and comparisons of high-school students with university students showed only modest gains in both groups. Of the 1,500 high-school students and over 500 university students who took the test, high school students showed normalized gain scores between .2 and .23, indicating that they learned 20%-23% of the previously unknown concepts; college students showed gains of at most 32% (Hestenes et al., 1992, p. 6). Through a process of development and refinement, the test has become an accepted and widely used tool in the physics community, and has led to changes in the way introductory physics has been taught (Mazur, 1997).

Halloun and Hestenes (1985) define the knowledge required to successfully answer questions on the FCI as "common sense" ideas of mechanics, such as interacting forces. Frequently students think of interacting forces as a stronger force overpowering a weaker force, such as pushing a chair out of the way, instead of an interaction according to Newton's third law (Hestenes et al., 1992). The results of their studies suggest that a large proportion of the students who do well by traditional measures of procedural skill in introductory mechanics courses have common-sense beliefs which are in direct contradiction with Newtonian mechanics.

The FCI paved the way for the broad application of analyzing student conceptual understanding of the basic ideas in a STEM subject area (Hake, 1998, 2007; Hestenes & Wells, 1992; Hestenes et al., 1992); concept inventories have been written for biology, chemistry, and astronomy (see, e.g., Anderson, Fisher, & Norman, 2002; Mulford & Robinson, 2002; Bailey, 2008; Marbach-Ad et al., 2009). More recently, a concept inventory has been written for introductory calculus (Epstein, 2007, 2013); though the descriptions of the validation and analysis have been less clear than in other concept inventories. The rigorous validation process used to develop a similar instrument, the Precalculus Concept Assessment (PCA) (Carlson, Oehrtman, & Engelke, 2010), provides a model for the extensive validation that we aim to complete in this project. The Calculus Concept Readiness instrument (Carlson, Madison, & West, 2010) is a successor to the PCA.

The results of more quantitative studies, like those involving concept inventories, show positive results of interactive instruction on student learning, but they are not without controversy. In one series of articles, a debate ensued over the merits of the FCI, and how to interpret its results. A core component of the debate was how the results of the test should be used in practice (Heller & Huffman, 1995; Hestenes & Halloun, 1995; Hestenes et al., 1992; Huffman & Heller, 1995). In addition to improving understanding of how students think about a topic, concept inventories like the FCI provide a tool for comparing instructional techniques. Hake (1998) used the Force Concept Inventory to show that students in classes exhibiting what he called "Interactive Engagement" outperformed students in "Traditional Lecture" classrooms. We seek to develop a concept inventory in calculus to perform similar measurements. We also believe it will help instructors identify students who may need remediation, and provide researchers with tools to study the relationship between conceptual knowledge in different STEM subject areas by combining different concept inventories.

Calculus Concept Inventory

Content Validity

The main purpose of the CCI was to create an instrument that would measure student understanding of calculus concepts by calculating classroom normalized gains (i.e., change in the class average divided by the possible change in the class average). The developers of the instrument described measuring above random chance at the pre-test setting and avoiding "confusing wording" as key goals (Epstein, 2013, p. 7). However, a released sample CCI item uses terminology and notation that is not part of the vocabulary of a first-time calculus student, including the word "derivative" and the notation "f'(x)" (Epstein, n.d.). Such vocabulary and notation will give students who are repeating calculus a performance advantage in the pretest, regardless of their conceptual understanding of the subject, and therefore they will show lower normalized gains; previous studies of the CCI have found this (Epstein, 2013). To address this, we compared the vocabulary and notation used in each of the items in the instrument with vocabulary and notation included in the Common Core State Standards for Mathematics (NGACBP & CCSSO, 2010).

Nine of the 22 items on the CCI contained terminology or notation not included in any standards for prerequisite courses for calculus. These included notation such as f '(x), f ''(x), and dy/dx and the word "derivative". Two additional items contained language that is closely related to some precalculus topics. For example, some students may have exposure to the ideas behind linear approximations or to the relationship between velocity and acceleration, but others likely have not. For these students the pre-test measure of conceptual knowledge will be weakened by all 11 items using difficult terminology, and students repeating calculus may outperform them in these items as well as the other nine in the pre-test.

These issues confirm that the CCI fails to satisfy necessary conditions for measuring conceptual understanding for students entering their first calculus course. This makes the standard normalized gains between pre-test and post-test a poor approach to evaluating first semester calculus courses. However, these issues do not preclude the instrument from providing an accurate measure of conceptual understanding at the conclusion of the first semester of calculus.

Internal Structure Validity

The dimensionality of the CCI is unknown. The use of a total percent of correct answers to determine normalized gains implies that the instrument measures a single construct and that each of 22 items provides the same level of information about student mastery. However, Epstein (2013) states that the instrument has two primary components related to functions and derivatives, and a secondary area of inquiry related to limits, ratios, and the continuum. The creators have not described a mode of analysis of student performance that supports the three-component structure. Therefore, determining whether a unidimensional approach to scoring provides useful information requires a comprehensive analysis of the internal structure of the instrument.

Testing and pre-testing of approximately 2000 students at four universities was performed at the beginning and the end of a first semester calculus class. We cleaned the data by eliminating subjects with missing data, which yielded a sample size of 1792 students. We randomly selected either a pre-test or post-test for each, which yielded an even distribution of pre-tests and post tests.

To determine the expected number of factors related to the instrument, we next used the eigenvalues of the inter-item correlation matrix. We followed this with a confirmatory factor analysis based on the predicted number of factors, with a bent toward a unidimensional model. The eigenvalue analysis compared the results from the actual data to results from randomly generated data with the same sample size and with a 20% probability of correct answers, the latter important since almost all of the items on the CCI provided five options for responses.

The analysis of the eigenvalues from the factor analysis suggests the CCI has at most two components, rather than the three the creators described. Both the first and the second eigenvalue are above the 95% confidence interval for the randomly generated data. However, the closeness of the second eigenvalue (1.24) to the 95% confidence interval of the eigenvalue generated by random data 1.1765 +/- 0.04 calls into question whether this second component is present (since a large first eigenvalue will pull up the second eigenvalue).

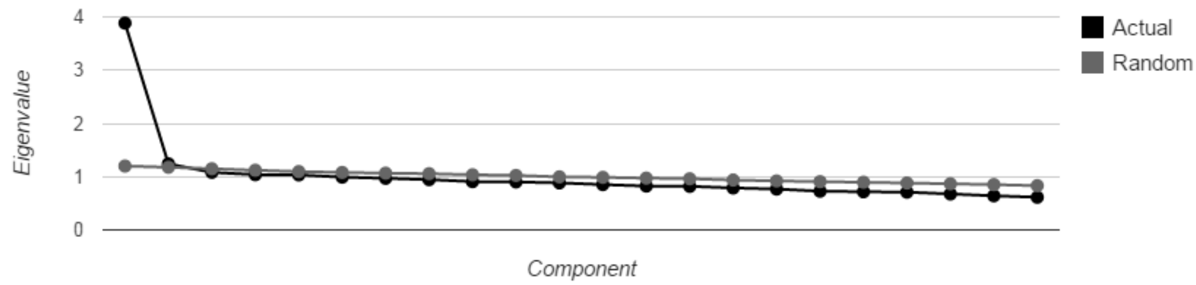


FIGURE 1: SCREE PLOT FOR CALCULUS CONCEPT INVENTORY

Table 1: Item CFA Estimates for CCI

	Full CCI		Abbreviated CCI	
Item	**Estimate**	**Standard Error**	**Estimate**	**Standard Error**
Question 1	1.000			
Question 2	5.776	1.313	1.000	
Question 3	5.537	1.264	0.961	0.065
Question 4	5.649	1.288	0.978	0.065
Question 5	4.574	1.058	0.802	0.062
Question 6	3.243	0.769	0.560	0.053
Question 7	3.497	0.825	0.604	0.055
Question 8	5.055	1.158	0.877	0.062
Question 9	4.792	1.103	0.830	0.062
Question 10	4.693	1.084	0.803	0.062
Question 11	0.816	0.349		
Question 12	2.380	0.610	0.414	0.056
Question 13	4.228	0.985	0.735	0.060
Question 14	3.386	0.803	0.587	0.055
Question 15	3.735	0.880	0.650	0.058
Question 16	2.928	0.704	0.504	0.052
Question 17	5.619	1.282	0.975	0.065
Question 18	1.535	0.412		
Question 19	3.322	0.790	0.570	0.055
Question 20	3.575	0.849	0.617	0.058
Question 21	3.857	0.917	0.661	0.064
Question 22	3.885	0.913	0.678	0.059

Since the scree plot and eigenvalue analysis suggests a unidimensional structure, and given the creators have at times conducted analyses as if it is unidimensional, we used a one-dimensional confirmatory factor analysis model to determine model-data fit. The model revealed

231 degrees of freedom, $p<0.001$. Item estimates are provided in Table 1. A Comparative Fit Index (CFI) of 0.936 and a Root Mean Square Error of Approximation (RMSEA) of 0.024 (Hu & Bentler, 1999) provided excellent fit indices. Therefore, we submit that the instrument has a unidimensional model. However, three of the items (1, 11, and 18) have significantly lower estimates that the remaining items. Removing these items maintains the unidimensionality of the instrument (CFI: 0.939 and RMSEA: 0.028) and leaves all estimates at approximately equal values. This renders percentage correct a more accurate estimate of an individual's ability. There is no need to scale the values of certain items.

Reliability

There is no standard of internal consistency necessary for an instrument, like the CCI, that intends to compare the normalized gains of two different groups. The CCI barely satisfies standards for an instrument designed to measure differences in means between groups of at least 25-50 individuals, with an internal consistency reliability alpha of 0.7 (Epstein, 2013). As Wallace and Bailey (2010) point out, using the normalized gain as a measurement parameter may not be appropriate. They propose using the same types of gains using ability estimates obtained through item response theory models. The current study uses such models to determine the internal consistency reliability of the CCI.

Using the results of the factor analysis, we used an appropriate unidimensional or multidimensional item response theory model. This allowed us to analyze the internal reliability of the instrument and to measure the test information and standard errors for the instrument.

Since the instrument satisfies the unidimensionality assumption, we can use one, two, or three parameter models in our data analysis. Since we believe the different items have different discrimination, we only used the two and three parameter models. The three parameter model had poor model-data fit on several of the items loading heavily on the construct; the c parameters for the majority of the items were significantly below random chance. Therefore, we deemed a two parameter model the best fit for the data and the theoretical construct of the inventory. In the analysis of the two parameter model, three items demonstrated a weak fit--Items 1, 11, and 18. These three items also had low loadings in the factor analysis. By removing them from the analysis we could determine if the remaining items have an improved fit. The remaining 19 items had a good fit (-2LL of 37258, $p<0.0001$) with the two parameter model. The standard error for the ability estimate of individuals is extremely high; the lowest value is 0.4128 logits and the average error is 0.7307 logits (See Figure 2). These figures imply that if an individual has the mean conceptual understanding of calculus, as measured by the CCI, his or her measured score by the inventory has a 68% chance of being within 0.42 logits of the mean. This suggests the inventory can only differentiate between samples of means if there is a substantial difference between the samples, or the sample size approaches 100 students each.

The results in Table 2 make it possible to transform the percent correct score into logit scores.

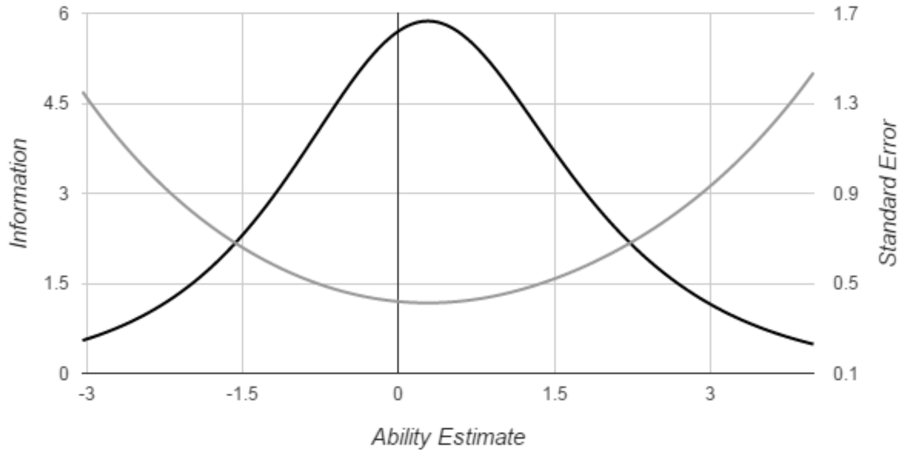


FIGURE 2: TEST INFORMATION FUNCTION AND STANDARD ERROR

Table 2: Transformation of Scores to Logits

Number Correct	Ability Estimate	Number Correct	Ability Estimate	Number Correct	Ability Estimate
0	-3.05	7	-0.13	14	1.47
1	-2.52	8	0.10	15	1.77
2	-1.75	9	0.31	16	2.13
3	-1.28	10	0.53	17	2.60
4	-0.93	11	0.74	18	3.36
5	-0.63	12	0.97	19	4
6	-0.37	13	1.21		

This study shows that the CCI does not conform to accepted standards for educational testing (American Educational Research Association, 2014; DeVellis, 2012). Therefore, given the centrality of calculus to STEM disciplines, we propose that a new assessment should be developed. The remainder of this paper proposes a framework for such an instrument.

Framework for a Differential Calculus Concept Inventory

Hiebert and Grouws (2007) define conceptual understanding as “mental connections among mathematical facts, procedures, and ideas” (p. 380). Thompson, Philip, Thompson, and Boyd (1994) describe computational and conceptual approaches as “two sharply contrasting orientations towards mathematics teaching,” while Sfard, Nesher, Streefland, Cobb, and Mason (1998) define computational and conceptual knowledge by considering their roles in discourse. These two types of discourse promote different types of thinking: computational discourse occurs when conversation focuses on calculation-based processes, but not on specific instances of procedural manipulation of symbols. For example, presenting a solution to a given problem may not be considered computational discourse, but explaining how to do certain types of problems would be. Conceptual discourse is dialogue which focuses on the motivations for the calculations, and the reason for employing particular strategies. The sociomathematical norms of the classroom heavily influence this dialogue, since the expectations of justification may include varying levels of conceptual analysis depending on a particular classroom. The preferences and orientations of the instructor as well as the students influence these norms (Thompson, Philip, Thompson & Boyd, 1994).

In Zandieh's (2000) framework for analyzing students' understanding of the derivative, the derivative function results from covarying input values with the values of the rate of change. This framework builds upon Sfard's (1991, 1992) account of the development of the function concept in terms of process-object pairs: Since the derivative of a function is a function defined as the limit of a ratio, understanding of the derivative requires understanding three layers of process-object pairs. The ratio process takes two objects (two differences) and acts by division to produce the difference quotient. The limiting process acts on the reified difference quotient, "passing through" infinitely many of the ratios approaching a particular value; the result is reified as the limit, and defines each value of the derivative function. The derivative function "passes through" infinitely many input values, producing output values at each point. Further, students can understand each of these process-object pairs in different contexts. In this framework, the number and quality of connections made between process-object pairs and contexts indicates the depth of a student's understanding of derivative.

We desire to extend this framework to include the conceptual understanding of most differential calculus concepts at the level of a first semester calculus student. In that regard, we need to extend the conceptual framework of Zandieh to include some concepts that are not directly tied to the derivative but are instrumental in success in calculus. These concepts include knowledge of other subjects that are assumed of students as they enter a first semester calculus course that are not directly related to the concepts of function, ratio, or limit. One example would be the concept that a ball has zero velocity at the top of its path when thrown. Another would include the ability to translate physical situations into mathematical language or diagrams, including the ability to draw a diagram of a ladder placed against a wall or the shadow of an individual in relation to a light source. By adding mathematical modeling as a concept in our framework for differential calculus, we no longer include the physical as a representation of the concepts as items that would fit within that dimension of Zandieh's framework are now included in a connection between the mathematical modeling and the corresponding concept. Instead, following the work of Roundy, Dray, Manogue, Wagner, and Weber (2015), we include a numerical/tabular representation to our framework.

Therefore, the framework for a Differential Calculus Concept Inventory is based on an extension of Zandieh's (2000) framework of conceptual understanding of the derivative and focuses on four concepts in differential calculus: Ratio, Function, Limit, and Mathematical Modeling (see Table 2). The remainder of this section will survey a selection of the literature discussing student understanding and common misconceptions in relation to these four concepts.

Table 3: Differential Calculus Concept Inventory Framework

		Representations			
		Graphical	Verbal	Symbolic	Numerical/ Tabular
Process- Object Layer	Ratio				
	Function				
	Limit				
	Mathematical Modeling				

Ratio. Carlson, Jacobs, Coe, Larsen, and Hsu (2002) define covariational reasoning as the coordination of two quantities related by a function, and reasoning about the ways they change together. They identify five levels of covariational reasoning: coordination, in which students realize that the two quantities are related; direction, in which students reason about how the direction of change of one variable relates to change in the other; quantitative coordination, in which students reason about how the amount of change of one variable relates to the other; average rate, in which students reason about how the output variable behaves on different small intervals in the input variable; and instantaneous rate, in which students reason about how continuous change in the input variable affects the output variable. This ability to coordinate continuous changes in the input variable and the output variable requires an understanding of the derivative as a rate of change. This concept of ratio also includes a conceptual understanding of rational expressions, including rational numbers and rational functions (Kalchman, Moss, & Case, 2001).

Function. Sfard (1991, 1992) describes the development of the function concept proceeding from operational to structural, passing through three stages called (a) interiorization, in which, for instance, students compute tables of functional values by explicitly evaluating functional expressions at particular numbers, (b) condensation, in which students increase in the ability to reason about the process as a coherent whole, and (c) reification, in which the process becomes an object in its own right, able to be operated upon by other processes. Similarly, APOS (action, process, object, schema) theory (e.g., Breidenbach et al., 1992) posits that students pass from action views (carrying out calculations on specific numbers or interpreting the graph of a function as a curve in the plane) through process views (thinking of a function as receiving inputs, performing operations thereon, and returning outputs) to object views of function (able to be operated on by other processes). Sfard (1992) notes that many students develop a "semantically debased" pseudostructural conception (p. 75), or an object view they cannot unpack to get at the underlying process from which it arose. Students with such a view may regard an algebraic formula as a thing in itself divorced from any underlying meaning, or a graph as detached from its algebraic representation or the function it represents. Other student difficulties, particularly with composition and inverting, reflect students' inability to go beyond an action conception of function (Dubinsky & Harel, 1992; Even, 1993).

Limit. Tall and Vinner (1981) describe several common components of students' concept images for limit that may conflict with the formal definition and thus cause problems in the development of sophisticated conceptual understanding. Some students view limit as a dynamic process with "a definite feeling of motion" (p. 161), in which input values approach a particular value, causing output values to approach a particular result. This view may cause students to believe that the limit is a value that is never reached; students holding this misconception will be unable to understand the formal definition of continuity. Other students conflate limit and bound, believing that a limit is a value that a function can never exceed (Williams, 1991). Student understanding of limit also significantly influences their understanding of derivative. In Zandieh's (2000) framework, limit is the second layer of process-object pairs: students must thus understand both the limiting process that acts on the difference quotient, "passing through" infinitely many ratios approaching a particular value, and the reified limit object that defines each value of the derivative function.

Mathematical modeling. Part of the power of differential calculus for STEM majors is the ability to describe real world situations. Many students, however, have difficulty modeling functional relationships (Carlson et al., 2002). While knowledge areas described above such as covariational reasoning are necessary for modeling, this particular skill requires additional ability in "mathematising, which means, turning a non-mathematical matter into mathematics, or a mathematically underdeveloped matter into more distinct mathematics" (Freudenthal, 1993, p. 72, italics in original). The ability to transform a real-world or non-mathematical context into a mathematical one provides challenges for students before calculus (Gerofsky, 1996), during calculus (Sofronas, DeFranco, Vinsonhaler, Gorgievski, Schroeder, & Hamelin, 2011) and in classes following calculus, such as differential equations (Rasmussen, 2000), where modeling problems have been used to assess differential equations students' conceptual knowledge (Kwon, Rasmussen, & Allen, 2005). The use of contexts in mathematics provides a challenge for students who are learning mathematics for a variety of reasons, including the language used (Ärlebäck, Doerr, & O'Neil, 2013), though understanding of context is important for students to be able to develop meaningful representations of the quantities in the problem (Moore & Carlson, 2012). To successfully utilize the ideas in calculus, students need to be able to model real-world situations. Understanding topics such as covariational reasoning, limits, and functions provides challenges for students, as described above. An additional skill necessary for applying the ideas of calculus is to situate this knowledge with context.

Conclusion

This study assesses the degree to which the CCI conforms to standards for important psychometric properties such as content validity, internal structure validity, and reliability. We further conclude that it does not conform to accepted standards for educational testing (American Educational Research Association, 2014; DeVellis, 2012). We thus propose that a new instrument should be developed that does meet these standards. Calculus instructors could use such an instrument for formative and summative assessment and use the results to improve student learning in their first two years of undergraduate study, with significant impact across STEM fields. Researchers and evaluators could use the instrument to compare instructional techniques, with additional gains for student learning.

References

American Educational Research Association. (2014). *Standards for educational and psychological testing*. Washington, DC: American Educational Research Association.

Anderson, D.L., Fisher, K.M., & Norman, G.J. (2002). Development and evaluation of the Concept Inventory of Natural Selection. *Journal of Research in Science Teaching, 39*(10), 952-978.

Ärlebäck, J. B., Doerr, H. M., & O'Neil, A. H. (2013). A Modeling Perspective on Interpreting Rates of Change in Context. *Mathematical Thinking and Learning*, *15*(4), 314–336. doi:10.1080/10986065.2013.834405

Bailey, J.M. (2008). Development of a concept inventory to assess students' understanding and reasoning difficulties about the properties and formation of stars. *The Astronomy Education Review, 6*(2), 133-139.

Bagley, S. (2014). *Improving student success in calculus: A comparison of four college calculus classes.* (Doctoral dissertation). Retrieved from ProQuest Dissertations and Theses.

Barker, W., Bressoud, D., Epp, S., Ganter, S., Haver, B., & Pollatsek, H. (2004). *Undergraduate programs and courses in the mathematical sciences: CUPM curriculum guide*. Washington, DC: Mathematical Association of America.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics, 23*, 247–285.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352–378.

Carlson, M., Madison, B., & West, R. (2010). The Calculus Concept Readiness (CCR) instrument: Assessing student readiness for calculus. *arXiv preprint arXiv:1010.2719.* Retrieved from http://arxiv.org/abs/1010.2719

Carlson, M., Oehrtman, M., & Engelke, N. (2010). The Precalculus Concept Assessment: A tool for assessing students' reasoning abilities and understandings. *Cognition and Instruction, 28*(2), 113-145.

Clark, M., & Lovric, M. (2009). Understanding secondary-tertiary transition in mathematics. *International Journal of Mathematical Education in Science and Technology*, *40*(6), 755–776. doi:10.1080/00207390902912878

CUPM Panel. 1987. Report of the CUPM panel on calculus articulation: Problems in transition from high school calculus to college calculus. *The American Mathematical Monthly, 94*(8), 776–785.

DeVellis, R.F. (2012). *Scale Development: Theory and applications* (3rd ed.). Thousand Oaks, CA: SAGE Publications.

Dubinsky, E., & Harel, G. (1992). The nature of the process conception of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy. MAA Notes, 25,* 85–106. Washington, DC: MAA.

Epstein, J. (2007). Development and validation of the Calculus Concept Inventory. In *Proceedings of the Ninth International Conference on Mathematics Education in a Global Community* (pp. 165–170).

Epstein, J. (2013). The calculus concept inventory - Measurement of the effect of teaching methodology in mathematics. *Notices of the American Mathematical Society, 60*(8), 2-10.

Epstein, J. (n.d.). Calculus concept inventory instrument. *Field-tested Learning Assessment Guide for science, math, engineering, and technology instructors: Tools*. Retrieved from http://www.flaguide.org/tools/diagnostic/calculus_concept_inventory.php

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education, 24*(2), 94-116.

Federal Coordination in STEM Education Task Force (2012). Coordinating Federal science technology, engineering, and mathematics (STEM) education investments: Progress report. Washington, DC: U.S. Government Printing Office.

Freudenthal, H. (1993). Thoughts in teaching mechanics didactical phenomenology of the concept of force. Educational Studies in Mathematics, 25(1&2), 71–87.

Gerofsky, S. (1996). A linguistic and narrative view of word problems in mathematics education. *For the Learning of Mathematics, 16*(2), 36-45.

Halloun, I.A., & Hestenes, D. (1985). The initial knowledge state of college physics students. *American Journal of Physics 53*(11), 1043-1055.

Hake, R. R. (1998). Interactive-engagement versus traditional methods: A six-thousand-student survey of mechanics test data for introductory physics courses. *American Journal of Physics, 66*, 64-74.

Hake, R. R. (2007). Six lessons from the physics education reform effort. *Latin American Journal of Physics Education*, *1*(1), 24–31.

Heller, P., & Huffman, D. (1995). Interpreting the force concept inventory: A reply to Hestneses and Halloun. *The Physics Teacher*, *33*, 507–511.

Hestenes, D., & Halloun, I. (1995). Interpreting the Force Concept Inventory: A response to March 1995 critique by Huffman and Heller. *Physics Teacher*, *33*(8), 502–504.

Hestenes, D., & Wells, M. (1992). A mechanics baseline test. *The Physics Teacher 30*, 159-166.

Hestenes, D., Wells, M., & Swackhamer, G. (1992). Force concept inventory. *The Physics Teacher*, *30*(3), 141–158. doi:10.1119/1.2343497

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 371-404). Reston, VA: National Council of Teachers of Mathematics.

Hu, L., & Bentler, P. M. (1999). Cutoff criteria for fit indices in covariance structure analysis: Conventional criteria versus new alternatives. *Structural Equation Modeling, 6*, 1–55.

Huffman, D., & Heller, P. (1995). What Does the Force Concept Inventory Actually Measure? *Physics Teacher*, *33*(3), 138–43.

Hughes Hallett, D. (2006). What have we learned from calculus reform? The road to conceptual understanding. *MAA NOTES*, *69*(July), 43.

Hughes Hallett, D., Robinson, M., & Lomen, D. (2005). Conceptests: Active learning in calculus. In A. Rogerson (Ed.), *Eighth International Conference Reform, Revolution and Paradigm Shifts in Mathematics Education* (pp. 112–114).

Kalchman, M., Moss, J., & Case, R. (2001). Psychological models of the development of mathematical understanding: Rational numbers and functions. In S. Carver & D. Klahr (Eds.), *Cognition and instruction: Twenty-five years of progress* (pp. 1–38). Mahwah, NJ: Erlbaum.

Koirala, H. P. (1997). Teaching of calculus for students' conceptual understanding. *The Mathematics Educator*, *2*(1), 52–62.

Kwon, O.N., Rasmussen, C., & Allen, K. (2005). Students' retention of mathematical knowledge and skills in differential equations. *School Science and Mathematics, 105*(5), 227-239.

Long, M. C., Iatarola, P., & Conger, D. (2009). Explaining Gaps in Readiness for College-Level Math: The Role of High School Courses. *Education Finance and Policy*, *4*(1), 1–33. doi:10.1162/edfp.2009.4.1.1

Mann, W. R. (1976). Some disquieting effects of calculus in high school. *The High School Journal*, *59*(6), 237–239.

Marbach-Ad, G. Briken, V., El-Sayed, N.M., Frauwirth, K., Fredericksen, B. Hutcheson, S. Gao, L.-Y., Joseph, S. Lee, V. McIver, K.S., Moser, D., Quimby, B.B., Shields, P., Song, W., Stein, D.C., Yuan, R.T., & Smith, A.C. (2009). Assessing student understanding of host pathogen interactions using a concept inventory. *Journal of Microbiology & Biology Education, 10*, 43-50.

Mazur, E. (1997). *Peer instruction: A user's manual*. Upper Saddle River, NJ: Prentice Hall.

Miller, R. L., Santana-Vega, E., & Terrell, M. (2006). Can good questions and peer discussion improve calculus instruction? *Primus*, *16*(3), 193–203. doi:10.1080/10511970608984146

Moore, K. C.,& Carlson, M. P. (2012). Students' images of problem contexts when solving applied problems. *The Journal of Mathematical Behavior, 31*(1), 48–59.

Mulford, D.R., & Robinson, W.R. (2002). An inventory for alternative conceptions among first-semester general chemistry students. *Journal of Chemical Education, 79*(6), 739-744.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards.* Washington, DC: Authors.

Pilzer, S. (2001). Peer instruction in physics and mathematics. *Primus*, *11*(2), 185–192. doi:10.1080/10511970108965987

Rasmussen, C. (2000). New directions in differential equations: A framework for interpreting students' understandings and difficulties. *Journal of Mathematical Behavior*, 20, 55-87.

Roundy, D., Dray, T., Manogue, C.A., Wagner, J.F., & Weber, E. (2015). An extended theoretical framework for the concept of derivative. Retrieved from http://timsdataserver.goodwin.drexel.edu/RUME-2015/rume18_submission_68.pdf

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1-36.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification – The case of function. In G. Harel & E. Dubinsky (Eds.), The concept of function: Aspects of epistemology and pedagogy (MAA Notes, no. 25, pp. 59-84). Washington, DC: MAA.

Sfard, A., Nesher, P., Streefland, L., Cobb, P., & Mason, J. (1998). Learning mathematics through conversation: Is it as good as they say? *For the Learning of Mathematics*, *18*(1), 41–51.

Sofronas, K. S., DeFranco, T. C., Vinsonhaler, C., Gorgievski, N., Schroeder, L., & Hamelin, C. (2011). What does it mean for a student to understand the first-year calculus? Perspectives of 24 experts. *The Journal of Mathematical Behavior*, *30*(2), 131–148. doi:10.1016/j.jmathb.2011.02.001

St. Jarre, K. (2008). They Knew Calculus when They Left: The Thinking Disconnect between High School and University. *Phi Delta Kappan*, *90*(2), 4.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151-169.

Terrell, M. (2003). Asking good questions in the mathematics classroom. In *Mathematicians and Education Reform Forum Newsletter* (Vol. 15, pp. 3–5).

Thomas, M., & Lozano, G. (2013). Analyzing Calculus Concept Inventory gains in introductory calculus. *Proceedings of the Sixteenth Annual Conference on Research in Undergraduate Mathematics Education*. Denver, CO.

Thompson, A. G., Philipp, R. A., Thompson, P. W., & Boyd, B. A. (1994). Calculational and conceptual orientations in teaching mathematics. In A. Coxford (Ed.), *1994 Yearbook of the NCTM* (pp. 79–92). Reston, VA: NCTM.

Wallace, C.S. & Bailey, J.M. (2010). Do concept inventories actually measure anything?. *Astronomy Education Review, 9*(1). doi:10.3847/AER2010024

Williams, S.R. (1991). Models of limits in college calculus students. *Journal for Research in Mathematics Education, 22*(3), 219-236.

Zandieh, M. J. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. Research in Collegiate Mathematics Education, IV, 103–127.

The use of examples in the learning and teaching of a transition-to-proof course

Sarah Hanusch
Texas State University

Abstract: This study investigates the ways that undergraduate students use examples in their transition to proof course, and the influence that the instructor had on the students' decisions to use examples. Data was collected from the instructor and a sample of students via observations and interviews to develop an model of effective example use in a transition-to-proof course. The results show that the students can often state the circumstances in which an example could provide insight during proof writing, but struggle during the implementation.

Key words: Examples, Transition-to-proof, Ground theory, Case study, Instruction

Introduction and Background

It is well documented that undergraduate students struggle when they start taking proof-based mathematics courses (Bills & Tall, 1998; Sowder & Harel, 2003; Weber, 2005; Weber & Alcock, 2004}. Weber (2001) classified the difficulties of undergraduates learning to prove into three categories: an inadequate conception of what constitutes mathematical proof, misunderstanding or misusing a definition or example during proof construction, and a lack of strategic knowledge, defined as "heuristic guidelines that they can use to recall actions that are likely to be useful or to choose which action to apply among several alternatives" (p. 111). Heuristics are difficult to teach, but students typically do not learn them unless an attempt was made to teach them (Lester, 1994). As such, there is a need to study the instruction of strategic knowledge, and to analyze the effect, if any, the instruction had on the students' strategic knowledge.

Using examples during proof writing and other related tasks is one type of strategic knowledge. A literature review shows that examples serve several purposes when provers form and prove conjectures and that using examples effectively is a difficult process (Alcock & Inglis, 2008; Alcock & Weber, 2010; Iannone, Inglis, Mejia-Ramos, Simpson & Weber, 2011; Lockwood, Ellis & Knuth, 2013). These authors theorize that if undergraduate mathematics students are introduced and instructed in generating and using examples, then some students may show improvements in their proof constructions, although the evidence is far from conclusive.

In mathematics there are many different types of examples, such as examples which demonstrate techniques, examples of types of problems with known solutions, examples of classes, examples of carrying out an algorithm, and examples satisfying a given definition (Watson & Mason, 2002). For the purposes of this study, the term example means a mathematical object which satisfies specific characteristics and illustrates a definition or concept (Moore, 1994). In particular, sample proofs are not considered to be examples for this study.

One reason to use examples in the study of mathematics is to extend conceptual knowledge of mathematical objects (Alcock, 2009; Alcock & Inglis, 2008; Watson & Mason, 2005). Examples may provide insight into the statement to be proved, and may generate the ideas for the proof, especially generic examples (Harel & Tall, 1991; Rowland, 2001). However, sometimes examples distract a prover from recognizing the central ideas. A prover must attempt to distinguish useful and useless examples from each other in the context of their own thinking.

Whether a prover finds an example useful largely depends on how the proves manages their knowledge. Resource management is a common theme in the research about problem-solving strategies (Carlson & Bloom, 2005; Polya, 1957; Schoenfeld, 1992). Schoenfeld says resource management is “not just what you know; it’s how, when, and whether you use it” (1992, p. 60). Viewing example use on proof-related tasks as a problem-solving strategy, any model of example use must include aspect of why to use examples, how to use examples, when to use examples, and assessing whether the example was a fruitful strategy for the task.

In order to use examples effectively, a prover must know how to verify or generate examples and counterexamples. Previous work on constructing examples establishes three strategies for constructing examples: trial and error, transformation, and analysis (Antonini, 2006). Trial and error is the process of starting with recalled examples and testing the conditions for an example. The transformations strategy consists of starting with an example that satisfies some of the desired characteristics and adjusting the example until it satisfies all of them. The analysis strategy starts by assuming an example exists and continues with an analysis of the required properties. This analysis allows the prover to deduce additional properties until the desired example was recalled, the prover develops a procedure that constructs an example, or a contradiction arises. Antonini (2006) observed that graduate students in mathematics typically use these strategies in succession beginning with the trial and error strategy, then the transformation strategy, and only moving onto the analysis strategy when the other strategies are ineffective. Additionally, Iannone, et al. (2011) found that undergraduate students primarily construct examples using trial and error, and occasionally use transformation. None of the students in their sample used analysis. Both techniques resulted in accurate examples with approximately the same relative frequency.

In this study, the following question will be addressed:

- In what ways did the students use examples effectively and ineffectively on tasks during their transition-to-proof course? What aspects served as barriers to using examples effectively?
- How did the instructor model effective example use? What did the instructor say about using examples effectively? How did the instructor design and teach the course to achieve these goals?

Method

The primary methodology for this study is that of an instrumental nested case study (Creswell, 2013; Patton, 2002). The case for this study is a single section of a transition-to-proof course at a large university in the southwest of the United States. All of the 27 students enrolled in the course were pursuing a major in pure mathematics, applied mathematics, mathematics for secondary teaching, or computer science with a minor in mathematics. All students enrolled in the course consented to participate.

The course met twice a week for 15 weeks and each session was 80 minutes long. A typical day began with the students writing homework problems on the blackboard prior to class time. After returning written homework and taking role, the instructor reviewed and corrected all of the student work on the board. The corrections always involved significant interaction with the students, especially the student who contributed the result. Although these were called student presentations, the students themselves did not have to stand and justify their work. The students also received credit whether or not this work was accurate. The review of the student work often took 30-40 minutes and the remainder of the class time was spent with instructor led lectures interspersed with many questions.

Data Collection. Data was collected from several sources in order to triangulate the evidence. One source of data collection is daily observations of the classroom that are documented via video recordings and field notes. The field notes were taken using a smart pen, which makes an audio recording linked to the written text. The purpose of this data is to observe the examples used by the instructor during lectures and the examples used during student presentations.

The interviews with the instructor were conducted to triangulate and the use of examples during instruction. Three interviews with the instructor provided insight into the motivations for the choices made by the instructor during class, and the instructor's expectations for the students. In addition, the sections of the book that were used during the class are analyzed for the example use contained therein.

Four students were selected to participate in three interviews throughout the semester, with an attempt to maximally vary the students' levels of academic success, mathematical preparation and degree specialization (Merriam, 2009). These four students are addressed by pseudonyms to protect their identities. Each interview consisted of three components: 1) a semi-structured portion addressing proof strategies and goals for the course, 2) a task-based portion where students attempted several proof related tasks and 3) a reflection on the decisions made while working on the tasks.

The tasks selected were aligned to the material from the course, and often selected from the textbook. Although all of the tasks can be said to be related to proof writing, several of the tasks did not simply ask the students to write a proof. The tasks asked the students to generate examples, evaluate the arguments of other, make a conjecture, determine whether a statement is true or false, finding a counterexample to a statement, and proving statements.

Data Analysis. The organization of this study is a grounded theory that uses a nested case study and a case comparison design (Patton, 2002). To determine how students use examples, the interviews with students were transcribed and coded using an open coding scheme and the constant comparative method (Merriam, 2009). A within case analysis of each student was completed, to form a detailed description of when and how each individual chooses to use examples. A cross-case analysis compared the individual cases to find commonalities and differences, which permits explanations which describe the students in the class as a whole (Merriam, 2009). From this analysis a theory of effective example use was developed.

For the instruction, the field notes and video recordings of the lectures were coded for every instance of example usage with the constant comparative method (Merriam, 2009), and revisions were made to the theory of effective example use. The interviews with the instructor clarified the motivations for decisions made during instruction, and provided triangulation and member checking (Patton, 2002).

The final level of analysis compares the results of the two cases, to draw conclusions about the connections between the behavior of the students and the instruction provided. These connections are drawn by comparing the examples usage by the students, instructor and contained within the text. The comparison is supported by interview questions where the instructor speculates about student performance, and the students recall the course instruction.

Results

The framework for using examples effectively during a transition-to-proof course was developed by looking at the 82 instances that the students used examples or counterexamples during the task-based interviews, and at the hundreds of examples presented during the course

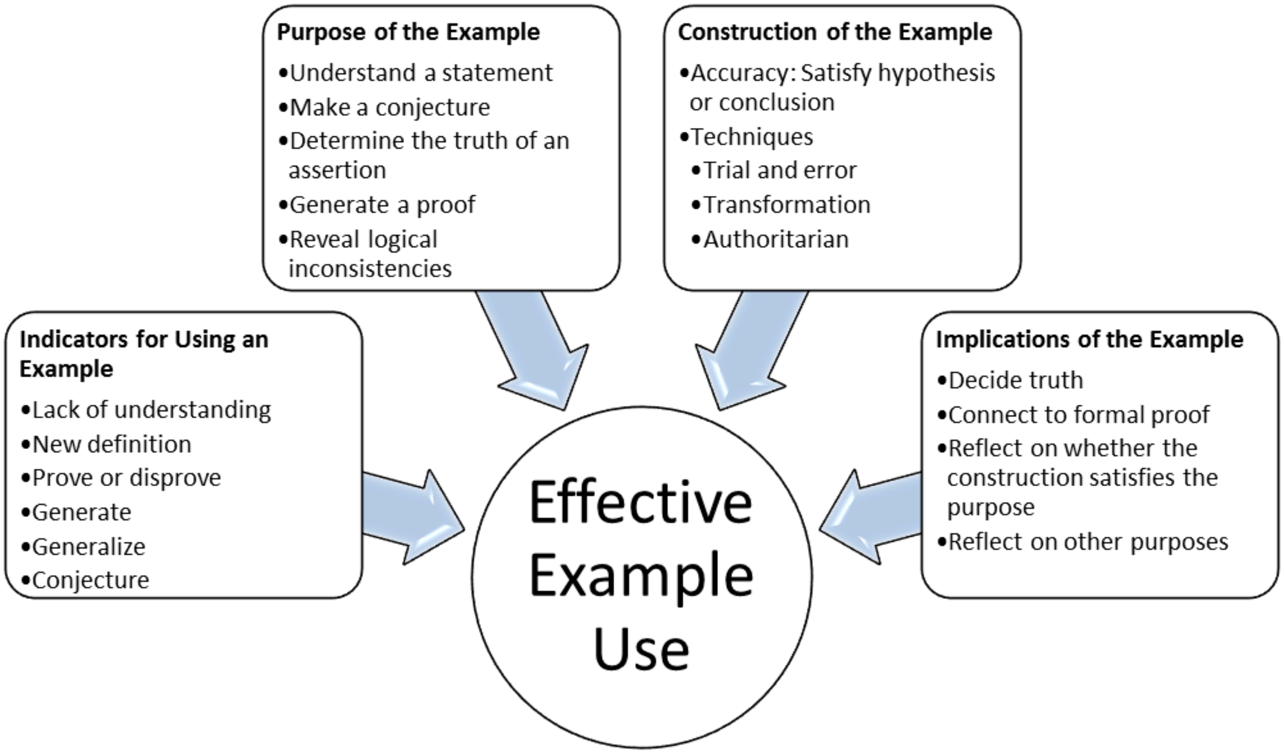


Figure 1. This shows a model for using examples effectively on the tasks asked in a transition-to-proof course. A prover uses examples effectively if they have implementable knowledge in all four categories.

lecture. The entire process of using an example was considered, and four phases emerged through the analysis: deciding to use an example (indicators), deciding the purpose of the example (purposes), finding or constructing the example (construction), and connecting the example to the larger task or proof (implications). Figure 1 shows a graphic that depicts all four phases.

The indicators of examples use are the aspects of a task or the solving process that inspire the prover to use an example as a strategy. The students generally decided to use examples either from the language of the task, or as a consequence of identifying weaknesses in their understanding. The students were frequently prompted into using examples the language of the task. For instance, after reading the directions *prove or disprove with a counterexamples* Carl stated “I want to start with a counterexample because you only need one.” Approximately half of the examples generated by students occurred on tasks that included the instruction “prove or disprove with a counterexample,” although this can be partially attributed to task selection.

The students were also prompted to use examples when they encountered a definition or statement that they did not fully understand. After Amy read a task that asked whether the composition of two decreasing functions is decreasing, she realized that she did not recall the definitions of increasing and decreasing functions. She then proceeded to look up the definition in her textbook and while reading the definition produced the sketch seen in Figure 2. Although this is not a fully concrete example, it helped Amy to understand the formal definition and to move forward through the task.

The instructor wanted the students to use examples when the task says *prove or disprove*,

Figure 2. Amy drew this sketch of increasing and decreasing functions after she recognized a gap in her knowledge of these concepts.

the task says *there exists*, the task asks the students to make a conjecture, they receive a new definition or they do not understand a new definition. Additionally, the instructor modeled using examples in each of these situations for the students. During an interview, the instructor summarized this by saying that she wants her students to consider examples “anytime they’re stuck and don’t know what else to do. Anytime they don’t thoroughly understand the definition. Anytime they see a new definition whether they think they thoroughly understand it or not.” The lectures expanded this aspect of the theory by including new definitions as an indicator of when to use examples.

The students mentioned four different purposes of examples during the interviews: understanding a statement, determining the truth of a claim (which includes generating a counterexample), making a conjecture, and generating a proof. All four of the students used examples to make the determination that a statement is true or false, and this was the most commonly stated purpose of examples. The students also recognized the value of using examples to understand a statement. Amy argued that “seeing the definition as … more concrete and less abstract is a lot more helpful.” Amy and Raul used examples for this purpose frequently through their interviews.

The instructor used examples for another purpose during the course, namely to reveal logical inconsistencies and underlying assumptions. This occurred primarily while correcting the students’ work on the blackboard. For instance, one student presentation included the equation $x+y=c+db=cb+db$, and concluded that $x=cb$ and $y=db$. The instructor helped the students to recognize the error in this conclusion with the counterexample $1+4=2+3$. Additionally, the instructor frequently introduced examples that revealed underlying assumptions. For instance, after introducing that for $x, y \in \mathbb{R}$, $xy=0$ implies that $x=0$ or $y=0$, the instructor warned the students that this statement does not hold in all worlds, namely in clock arithmetic or in matrix multiplication. Both of these purposes involve the use of counterexamples, but are different from using counterexamples to establish that a claim is false.

The next phase is accurately constructing the desired example or counterexample. On many of the tasks, the students struggled with generating examples and counterexamples, and often misidentified the results of their efforts. One instance of this occurred on a task from the first interview (see Figure 3), when Raul arbitrarily picked the values $a=5$, $b=10$, $c=6$ and $d=4$, Provide either a proof or a counterexample for the following statement. For integers a, b, c, and d, if a divides b-c and a divides c-d, then a divides b-d.

Figure 3. This is a task from the first round of interviews. Some students struggled when generating and identifying examples and counterexamples on this task.

and determined that these values formed a counterexample because $a=5$ does not divide $b-d=10-4=6$. Although Raul did compute $b-c=10-6=4$ and $c-d=6-4=2$ in an effort to address the hypotheses, he did not realize that a counterexample to the statement had to satisfy the hypotheses but not the conclusion. However, most of the inaccurate example constructions occurred during the first interview, which suggests that the students improved in this aspect throughout the course.

The students either took their example from an authoritarian source (the question, the textbook, their lecture notes), or constructed their examples using trial and error or transformation. None of the students attempted to use an analysis construction technique, which is consistent with the results of Iannone, et al. (2011). The students used trial and error almost exclusively during the first interview, and progressed to using transformation with greater frequency during the final interview. It is unclear if this progression is due to increased experience, the instruction received, or the change in mathematics content.

The instructor seldom discussed the trial and error approach of construction during the class. She did demonstrate how to verify various types of examples, and instead relied on correcting the students' attempts to teach the technique. Towards the end of the semester, she did demonstrate the transformation technique a few times. During a mid-semester interview, the instructor said "I would like to move them toward more directed examples where they are intentionally trying to go certain places, but I doubt that most of them are ready for that. Right now I'm happy if they try random examples to see what's going on, as long as they don't stop there." From this statement, it appears that the instructor had fairly accurate expectations of her students' abilities with regards to example construction.

The final phase of using examples effectively is determining the implications from the example. The implications involve attempting to fulfill the previously established purpose of the example, or deciding that the example fulfills another purpose. For instance, Carl originally tried to find a counterexample to the statement that the product of a fine function and another function is fine, see Figure 4. He proceeded to construct the example $\sin(\pi x)\cdot x2$, and observed that the product function is zero whenever one of the factors has a zero. Although his purpose had been to determine truth, Carl realized that this example provided the insight for him to generate a proof, and thus fulfilled another purpose.

After constructing an examples the most common behaviors exhibited by the students was to construct another example, decide the truth of the claim and to do nothing. With less

A real valued function is called fine if it has a zero at each integer.
Prove or disprove: The product of a fine function and another function is fine.

Figure 4. This is a task from the third round of interviews. Carl changed his purpose from determining truth to generating a proof.

SH: Okay. The question said, “Prove or disprove.” Right now what do you think you've done?
Amy: Disproven it? Well, I think I've done both!
SH: Okay. …You came up with two examples where it worked and one example where it doesn't.
Amy: Yeah, like this one it will never hold but this oh yeah it says it works. And that works. I don't know. I don't know.
SH: How do you disprove something?
Amy: Come up with one example where it doesn't work. Done.
Amy: Is that really all I needed? I say one ...
SH: That's really all you needed.
Amy: Cool.

Figure 5. This is an excerpt from the first interview with Amy, where she did not recognize the implications of her constructions. The underlined portions reveal the inconsistency between Amy’s stated purpose and her ability to recognize the fulfillment of said purpose.

frequency, the students connected the example to the formal proof (meaning they referenced the example while writing their proof) and made a conjecture. This phase seemed to provide the greatest struggle for the students. The students often did not understand what they could learn from their constructions. For instance in the first interview, Amy produced two examples and a counterexample to a statement, and then proceeded to write a proof of the statement that she believed to be valid, see the interview excerpt in Figure 5. At the end of the interview, I asked her whether she had proven or disproven the statement, and she argued that she had done both. However, after I asked how to disprove a statement, Amy immediately responded by saying a single counterexample was sufficient. She then paused for a moment, and realized that she could have finished working on the problem much earlier.

The instructor wanted to students to reflect on every example construction or proof attempt that they worked on, and to determine what they could learn from each attempt. She expressed this sentiment during a mid-semester interview when she said “I want them moving from example to proof. I don’t want them stuck in the example mode, where they can’t write a proof, but I don’t want them stuck in the theoretical mode, where they’re writing down words they don’t know… I’d like them to be bouncing back and forth.” However, she never modeled more than two iterations of this reflecting process during class time, due to time restraints in the classroom. Both she and the students reported that more repetitions were completed during office hours, however this was not observable.

When the instructor modeled the implications of using examples, she usually established the truth of a statement and connecting examples to proof language. When establishing the truth of a statement, the instructor tied the examples back to the quantifiers and determined whether the given example proved the statement or simply provided some evidence.

Discussion

Throughout the course, the instructor modeled and discussed with the students all four phases of using example effectively on proof-related tasks. However, not all four aspect were given equal class time; the instructor discussed indicators and purposes often, but constructions and implications with considerable less frequency. The instructor argued that she would have liked to spend more time on the constructions and implications, but within the constraints of the classroom time, she decided to spend class time on other aspects of the course. This is appropriate, since the primary purpose of the course is teaching students how to write proofs, not construct examples.

The students seemed to perform more consistently on the more behavioral aspect of the using examples. For instance, they were quick to recognize when the language of the task indicated using examples, but did not always recognize when an example would improve their understanding of the definitions.

Overall, this framework extends the work of Alcock and Weber (2010) on the purposes of examples, the work of Iannone, et al. (2011) on the construction of examples to create a theory of the entire process of using examples. Although this theory was established from a single transition-to-proof course, it is likely that it will have applicability to other courses and to the example use of mathematicians.

Reference

Alcock, L. (2009). Teaching proof to undergraduates: semantic and syntactic approaches. In F. Lin, F. Hanna, & M. de Villiers (Eds.), *Proceedings of the ICMI study 19 conference: Proof and proving*. (Vol. 1, p. 227-246). Taipei: National Taiwan Normal University.

Alcock, L., & Inglis, M. (2008). Doctoral students' use of examples in evaluating and proving conjectures. *Educational Studies in Mathematics, 2*, 111-129.

Alcock, L., & Weber, K. (2010). Undergraduates' example use in proof construction: Purposes and effectiveness. *Investigations in Mathematics Learning , 3* (1), 1-22.

Antonini, S. (2006). Graduate students' processes in generating examples of mathematical objects. In *Proceedings of the 30th international conference on the psychology of mathematics education* (Vol. 2, p. 57-64). Prague, Czech Republic.

Bills, L., & Tall, D. (1998). Operable definitions in advanced mathematics: The case of the least upper bound. In A. Olivier & K. Newstead (Eds.), *Proceedings of the twenty-second international conference for the psychology of mathematics education* (Vol. 2, p. 104-111). Stellenbosch, South Africa: University of Stellenbosch.

Carlson, M. P., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational Studies in Mathematics*, *58*, 45-75.

Creswell, J. W. (2013). *Qualitative inquiry and research design* (3rd ed.). Thousand Oaks, CA: Sage Publications Ltd.

Harel, G., & Tall, D. (1991). The general, the abstract, and the generic in advanced mathematics. *For the Learning of Mathematics, 11*(1), 38-42.

Iannone, P., Inglis, M., Meja-Ramos, J., Simpson, A., & Weber, K. (2011). Does generating examples aid proof production? *Educational Studies in Mathematics, 77*(1), 1-14.

Lester, F. K., Jr. (1994). Musings about mathematical problem-solving research: 1970-1994. *Journal for Research in Mathematics Education, 25*(6), 660-675.

Lockwood, E., Ellis, A., & Knuth, E. (2013). Mathematicians example-related activity when proving conjectures. *The Electronic Proceedings for the Sixteenth Special Interest Group of the MAA on Research on Undergraduate Mathematics Education*, 21-23.

Merriam, S. B. (2009). *Qualitative research: A guide to design and implementation.* San Francisco, CA: Jossey-Bass.

Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics, 27*(3), 249-266.

Patton, M. Q. (2002). *Qualitative research and evaluation methods* (3rd ed.). Thousand Oaks, CA: Sage Publications Ltd.

Polya, G. (1957). *How to solve it: a new aspect of mathematical method* (2nd ed.). Princeton, NJ: Princeton University Press.

Rowland, T. (2001). Generic proofs in number theory. In S. R. Campbell & R. Zazkis (Eds.), *Learning and teaching number theory: Research in cognition and instruction* (p. 157-183). Westport, CT: Albex Publishing.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (p. 334-370). New York: MacMillan.

Sowder, L., & Harel, G. (2003). Case studies of mathematics majors' proof understanding, production, and appreciation. *Canadian Journal of Science, Mathematics & Technology*

Education , 3 (2), 251-267.
Watson, A., & Mason, J. H. (2002). Extending example spaces as a learning strategy in mathematics. In A. Cockburn & E. Nardi (Eds.), (Vol. 4, p. 377-384). Norwich, UK: University of East Anglia.
Watson, A., & Mason, J. H. (2005). *Mathematics as a constructive activity: Learners generating examples.* Mahwah, NJ: Erlbaum.
Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics, 48*(1), 101-119.
Weber, K. (2005). A procedural route toward understanding aspects of proof: Case studies from real analysis. *Canadian Journal of Science, Mathematics, and Technology Education, 5*(4), 469-483. doi: 10.1080/14926150509556676
Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. Educational Studies in Mathematics, 56 (2), 209-234.

Examining students' proficiency with operations on irrational numbers

Sarah Hanusch — Texas State University

Sonalee Bhattacharyya — Texas State University

Abstract: Fluency with our number system is a critical part of mathematics. Understanding how rational and irrational numbers work and fit in to the number system as a whole is at the foundation of a good understanding of mathematics. In this study, we present developmental mathematics students with a task which tests understanding of the closure of irrational numbers under addition and multiplication. We analyze the data using the strands of proficiency framework from Adding It Up *(Kilpatrick, Swafford, & Findell, 2001), searching for evidence of each strand. The results show varied levels of proficiency within each strand and overall. We conclude with implications for teaching.*

Key words: Irrational Numbers, Developmental Mathematics Students, Strands of Proficiency, Qualitative Methods

Introduction and Review of Literature

Undergraduate students need proficient knowledge of the real numbers, because the real numbers form the foundation for more advanced mathematical concepts. To achieve numerical literacy, students must have some proficiency in the real number system (Fischbein, Jahiam, & Cohen, 1995; Guven, Cekmez, & Karatas, 2011). Studies have shown that the set of irrational numbers is difficult to grasp, because of challenges with the definitions of rational and irrational numbers (Fischbein et al., 1995), with the connection between irrational numbers and limits (Peled & Hershkovitz, 1999), and with moving between multiple representations of irrational numbers (Arcavi, Bruckheimer, & Ben-Zvi, 1987; Sirotic & Zazkis, 2007a, 2007b).

Irrational numbers literature. Few studies have focused on irrational number concepts, and most of these studies have looked at pre-service or in-service teachers (Arcavi, et al., 1987; Fischbein, et al., 1995; Guven, et al., 2011; Peled & Hershkovitz, 1999; Sirotic & Zazkis, 2007a, 2007b). Fischbein et al. (1995) found that high school students and pre-service teachers cannot correctly define rational and irrational numbers, which was confirmed by Guven, et al. (2011). Additionally, pre-service and in-service teachers may struggle with the classification of numbers (Arcavi, et al., 1987; Fischbein, et al., 1995).

Peled and Hershkovitz (1999) studied 70 pre-service teachers' conceptions of irrational numbers. The results from this study indicate that these teachers knew the definition of an irrational number, but they could not move between multiple representations flexibly. In particular, they found that many of the teachers in their study believed that *5* (and all irrational numbers) cannot be placed on the number line. These students failed to understand the connection between the graphical, decimal and radical representation of this number. Sirotic and Zazkis (2007a) followed up this study by asking 46 pre-service teachers to locate *5* exactly on the number line. Only 9 of these students were able to accomplish this, and most believed an approximation was the only way to accomplish the task. The pre-service teachers could not connect the radical form of the number to the number line.

Two studies have asked pre-service teachers about the closure of addition and subtraction on the irrational numbers (Guven, et al., 2011; Sirotic & Zazkis, 2007b). Guven et al. (2011) found that only 39% of first year pre-service teachers and 18% of fourth year students thought it false that the sum of two irrational numbers is always irrational. They had better results with the product of two irrational numbers, where 93% of first year and 72% of fourth year pre-service teachers answered that the product is not always irrational. Sirotic and Zazkis (2007b) found that much of the knowledge the students have regarding operations is procedural and rote, citing that the pre-service teachers could rationalize the denominator of a radical expression, but did not use the existence of conjugates to conclude that the product of two irrational numbers can be rational. Another finding from this study is that the pre-service teachers had difficulty viewing some irrational numbers, such as $7-2$ and 25 as single objects.

Developmental mathematics literature. In this study, we sought continue the work of Guven, et al. (2011) and Sirotic and Zazkis (2007b) regarding operations on irrational numbers. We chose developmental mathematics students at a large university for the sample because rational and irrational numbers are included in the curriculum for the course, and to differentiate from the previous studies. Developmental mathematics is a term for non-credit bearing courses taught at a college or university. The topics vary, but most consist of arithmetic and algebra reviews to prepare students for success in college algebra.

Few studies have focused on the mathematical knowledge of students at the developmental level (Givven, Stigler, & Thompson, 2011; Grubb & Cox, 2005; Stigler, Givven, & Thompson, 2010). In one large study, Stigler et al. (2010) and Givven et al. (2011) collected data from community college developmental mathematics students and concluded that most students at the developmental level suffer from "conceptual atrophy," meaning the students are unable to connect "basic intuitive ideas about mathematics" (Stigler et al., 2010, p.15) with procedures and concepts.

Theoretical framework. In this study, we consider an individual to be mathematically proficient if they demonstrate procedural fluency, conceptual understanding, strategic competence, adaptive reasoning and productive disposition, as described in Adding It Up (Kilpatrick, Swafford, & Findell, 2001). These five are titled the strands of proficiency, and they "are interwoven and interdependent in the development of proficiency in mathematics" (Kilpatrick et al., 2001, p. 137). The interconnectedness of these strands emphasizes the importance of making connections between mathematical topics and skills.

- Procedural fluency refers to the ability to accurately select, carry-out and interpret mathematical procedures. The term fluency indicates an ability to choose between multiple procedures to find the most convenient or useful choice.
- Conceptual understanding refers to understanding mathematical ideas, including definitions and logic. It also refers to making connections between mathematical ideas.
- Strategic competence is the ability to select a strategy to solve a problem, and revise the strategy as needed.
- Adaptive reasoning is the ability to apply logic to a problem and to reach justified conclusions. It also involves altering your assumptions when faced with new data.
- Productive disposition refers to having positive beliefs about mathematics, such as believing that studying mathematics is worthwhile or that mathematics makes sense. Another indicator of productive disposition is when an individual believes they are capable of learning mathematics.

Although this framework was originally designed to describe the mathematical knowledge of K-12 students, we felt that it could reasonable be extended to students at the developmental level. The previous studies on developmental mathematics students seem to indicate that students at this level will demonstrate difficulties in the strands of conceptual understanding, strategic competence and adaptive reasoning (Givven, et al., 2011; Grubb & Cox, 2005; Stigler, et al., 2010). Similarly, the results regarding irrational numbers seem to indicate problems with conceptual understanding and adaptive reasoning.

Our research questions for this study are:

- Within each strand of proficiency, what are the developmental mathematics student's ideas relating to operations on the irrational numbers?
- In which strands of proficiency do the students demonstrate strengths and weaknesses with regards to operations on the irrational numbers?
- Do developmental mathematics students demonstrate overall proficiency regarding operations on the irrational numbers?

Method

This study occurred in a developmental mathematics course at a large university in the southwest of the United States. This university offers two mathematics courses at the developmental level, meaning non-credit bearing courses, which serve as prerequisites for entry level courses, such as College Algebra. Students placed into level one must successfully pass level one and level two before enrolling in credit bearing courses. For this project, we invited all the students enrolled in the level one course.

The course is organized with large lecture sections and smaller laboratory sections. Each week, students spend one hour in a large lecture taught by the instructor of record, and three hours in smaller laboratory sections, taught by graduate assistants. A majority of the instruction occurs in the laboratory sections with a standardized curriculum among all sections. The weekly lecture is intended to review, expand, and elaborate on the topics discussed in lab. Although most assignments were assigned and collected in the laboratory, this semester the lecturer assigned weekly homework assignments that connected directly to the lecture.

During the semester in which data was collected, 77 students were enrolled in all sections of the level one course. The data collected was a portion of one lecture homework assignment. After the students turned the homework into their instructor, photocopies of the students' responses and the consent form were given to the researchers for analysis. Only 31 students provided consent for their homework to be analyzed. This study includes responses from two questions:

1. What can you conclude about the sum of two irrational numbers? Is it always irrational? Always rational? Sometime irrational and sometimes rational? Explain your reasoning.
2. What can you conclude about the product of two irrational numbers? Is it always irrational? Always rational? Sometime irrational and sometimes rational? Explain your reasoning.

The written homework assignments were analyzed for evidence of the five strands of proficiency from *Adding it Up* (2001). An open coding scheme was developed within the lens of each strand of proficiency, in the style of Glaser and Strauss (1967). A detailed description of each category is found in the results section.

Results

In the results that follow in the following sections, all counts are per problem. This means that there are 62 responses, two for each student. We made this decision because in several instances an individual student performed differently on the two questions, and because we are focused primarily on the qualitative aspects of the results rather than the quantitative. All names included are pseudonyms that reflect gender.

Several of the responses were either blank (3) or included responses with a claim but no justification for the claim (14). We included these responses in the presentation of the results because they spoke to productive disposition; however they do not provide evidence for the other strands. A consequence of this decision is that within the individual strands a significant proportion of the responses were classified in a no evidence category.

Procedural Fluency. To analyze procedural fluency we considered three criteria: 1) the response included the correct operation, 2) the response included correct computations, and 3) the response correctly identified numbers as rational or irrational. A student demonstrated strong procedural fluency if they satisfied all three criteria, moderate if they failed one criterion and weak if they failed two or more. Some responses did not reference any relevant procedures, and these were coded as no evidence.

Twenty-two of the responses exhibited strong procedural fluency, 10 exhibited moderate, 11 exhibited weak and 19 provided no evidence. Only two of the responses contained incorrect computations, and both of those responses came from the same student. Generally when the students attempted to add or multiply two numbers they were successful in this task. Most of the difficulties occurred with the classification of rational and irrational numbers rather than with the procedures themselves.

Several responses did not indicate the operations of addition or multiplication. One instance of a student whose response did not reference a specific operation is Olivia, who claims the sum of two irrational numbers "can be both, because an irrational number can be turned into a fraction which would be considered rational." Although Olivia does reference division, her explanation does not reference the relevant operation of addition. Because Olivia did not consider addition, and performed no computation, we were unable to determine whether or not she can use procedures fluently. The same is true of the other responses in the no evidence category; we could not classify these responses because they did not attempt to perform operations.

Conceptual Understanding. To analyze conceptual understanding we considered either the definitions provided for rational and irrational numbers, or the examples provided. Responses were classified into four categories: *correct, representation error, definition error* and *no evidence*. Twenty-six responses were classified as correct, meaning the student provided either a correct definition for rational or irrational numbers, or the students classified all numbers generated correctly as rational or irrational.

Nine responses were nearly correct, but had *representation errors*. One instance in this category is Nathan who wrote "irrational # is any number that isn't rational," but then used zero as both a rational and irrational number. The other responses in this category misclassified the square roots of perfect squares as irrational numbers. In some instances, the students believed that the rationality depends on the representation of the number, i.e. *4* is irrational, but *2* is rational; see Carl's response in figure 1. In four other responses, the students indicated that all fractions are rational, not just those with integers. For instance, Andy claimed that "if we put 1 under any number, it would be rational." These students clearly have misconceptions about rational numbers and fractions.

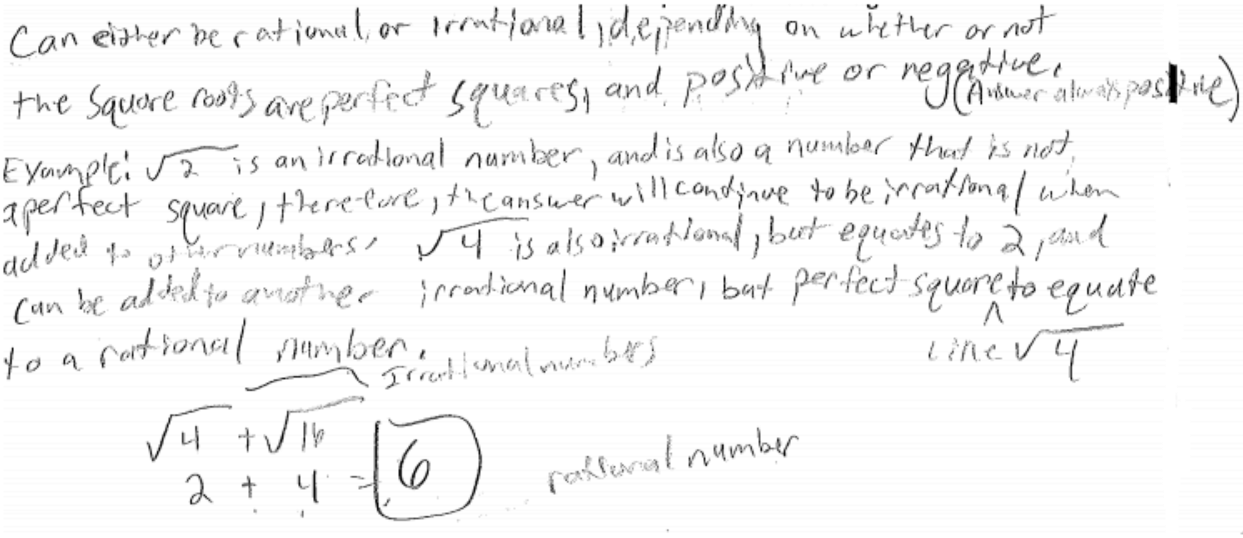


Figure 1 Carl's response to the sum problems where he classifies as irrational, and as rational.

Six responses were classified as having a *definition error*. In these responses, the student classified rational numbers as irrational numbers, and integers as rational numbers. These students clearly do not have a sufficient understanding of the definition of an irrational numbers.

The final category, *no evidence*, had 21 responses. These responses did not provide sufficient information to infer the students' understanding of the concepts. For instance, Yvette wrote "an irrational number remains irrational, the product will not change it to a rational answer." It is unclear whether Yvette actually understands the definition of a rational or irrational number from her work. Some of the students reached the correct conclusions to the problems, but many (including Yvette) reached incorrect conclusions.

Strategic Competence. In this category, the students were classified by the strategy the used to approach the problem. The first category, called *correct with examples*, a response had to indicate that both rational and irrational sums are possible, and they had to attempt to provide at least one example to support each of those situations. These examples were not always correctly identified. The second category, called *incomplete with examples*, a response included examples to justify their responses, but only included a rational or irrational example. The third category, called *properties*, the students attempted to use properties of irrational numbers to justify their claims. The final category is *no evidence* where the response includes no justification, or did not fit any other category.

Sixteen responses included the correct strategy, although not all of these identified the rational and irrational numbers correctly. Eighteen responses were classified as *incomplete with examples*. An incomplete strategy took one of two forms: either the participants claimed that the sum or product was always rational or always irrational and only provided examples that supported their claim, or the participants claimed that the sum or product could be rational or irrational but only provided examples to show one of the cases. We chose to use the convention that a claim was correct only if it distinctly mentioned that a sum (or product) can be rational or irrational. There were a few instances where the student hedged their response with conditions, such as when Monica says "Two irrational numbers very rarely add up to be a rational number",

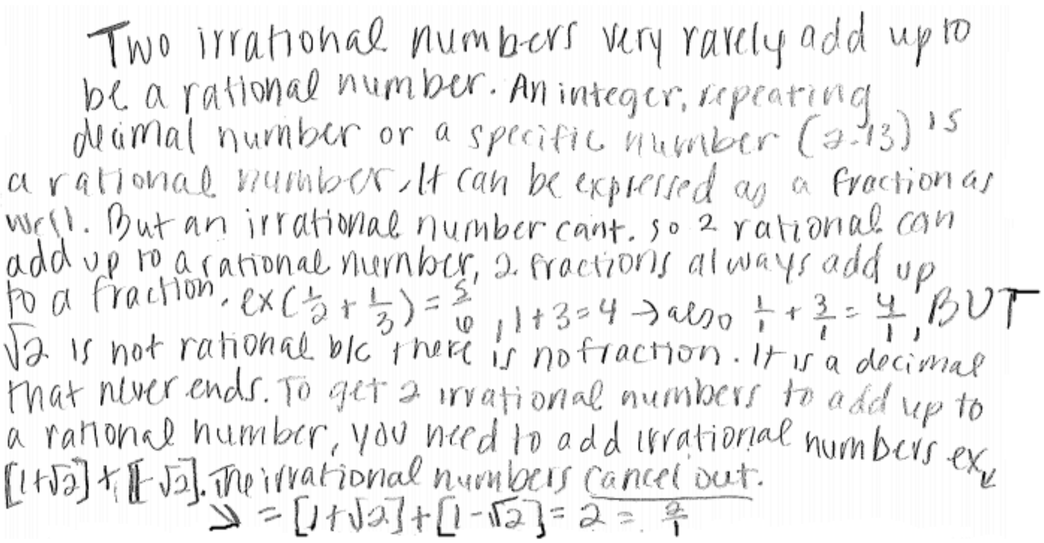


Figure 2 Monica demonstrated an understanding of rational and irrational numbers, but did not full justify her claim.

as shown in Figure 2. Although Monica seems to have a strong understanding of the concepts, she did not utilize a complete strategy since she only provided a rational example.

Eleven responses attempted to use *properties*. These responses were split between claiming the sum or product is always irrational and claiming the sum or product could be both. None of these arguments produced valid results, and most of the premises included are false statements, such as when Andy claimed "if we put a one under any [irrational] number it would be rational." Of the remaining 17 responses in the *no evidence* category, one response was particularly unusual. Heather attempted to use the variable x to create a general argument. She frequently wrote $x2$, and while we suspect she meant to use x there is no evidence to support that supposition. However, she did recognize that a number plus its negative is zero, and then attempted to use that to help her argument. She claimed "If two numbers sum to a rational number, both must be rational or both must be irrational," but she did not justify this claim any further.

Adaptive Reasoning. To analyze the strand of adaptive reasoning, we initially separated the responses into three categories. The categories were students who made a claim and provided *complete justification*, students who made a claim and had *incomplete justification*, and students who made a claim, but provided *no justification*. Three responses did not make any claim at all.

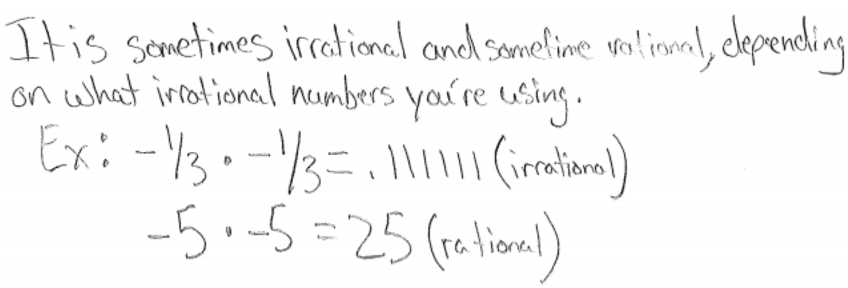


Figure 3 David had a complete justification for his claim, even though he misclassified the numbers in his examples.

Sixteen responses made a claim and provide a *complete justification*. Not every student in this category used correct definitions of rational and irrational, but they did include an example labeled as rational and another labeled as irrational as justification for their conclusion. For instance, David in Figure 3 thought that negative numbers and fractions were irrational, and only positive integers are rational, yet he backed his claim that the product can be rational or irrational with examples for each category.

The *incomplete justification* group contained 24 responses, and these responses were characterized by an attempt to justify a claim, but their reasoning erred at some point. For instance, Victor made a claim that the sum of two irrational numbers is always irrational, but his justification was just a few examples. We infer that Victor's reasoning was limited by insufficient examples. It is unclear whether or not Victor recognized that he could not prove a statement with examples.

Betsy is the only student to indicate indecision in a response, see Figure 4. She initially claimed that that the sum of two irrational numbers is sometimes rational and sometimes irrational, but then crossed out her response and changed it to say always rational. It seems that she was only able to produce a rational example, and then made a new claim.

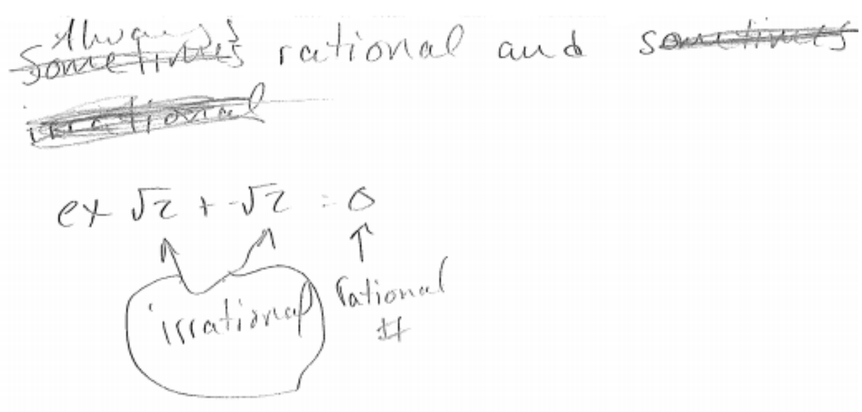


Figure 4 Betsy changed her claim after working examples

In the final category, most of the students either left a blank page or only included a phrase indicating their claim. A few attempted some justification, but these showed severe limitations in reasoning ability.

Productive Disposition. The data collected in this study is inadequate to provide deep insight into the productive disposition of the students in the sample. However, a few insights can be gleaned from this data. One fact that may indicate low productive disposition among this group of students is the fact that fewer than 50% of the students enrolled in the course participated in this study. While that number is not unusual in research studies, in this study data collection came from a single homework assignment. Anecdotally, we know that many students enrolled in this course do not participate in the lectures, and as such we suspect that many of the students did not turn in the assignment.

A few additional responses also indicated a low productive disposition. One student, Peter, turned in a completely blank paper. Fourteen responses included only a claim and did not provide a justification. On one of these responses, Rachel said “I'm not sure why, or I can't give you reasoning. Without really studying notes or referring back, it's just what I think.” It seems that either she felt that she was not allowed to use references, or she simply chose not to make the effort. She also included a frown face figure on the second question, indicating discontentment with the finished product.

Overall. Looking at the various responses as a whole we observed that 42 of the 62 responses stated the correct claim of *sometimes rational and sometimes irrational*, see Table 1. Of the 31 students, 19 made the correct claim about sums of irrational numbers and 23 made the claim about products. Some of the responses were difficult to categorize because the students wrote statements such as “the sum can be rational.” However, unless a response explicitly mentioned the sum or product can be irrational, then we classified the response as always rational. The analogous protocol was used for always irrational responses.

Although the majority of the claims were correct, many of the justifications were not. The most frequent justification scheme, 27% of the responses, included both rational and irrational examples. Tied at the same level were the responses that made a claim but did not provide any justification for their response. Attempting to use properties was the justification scheme used in 21% of the responses. This scheme is admirable because it could have led to sophisticated arguments, but unfortunately nearly all of the properties that the students used were false statements. The remaining responses provided examples that were either just rational, or just irrational.

Table 1 A table of the claims made on the responses and the nature of the justifications

Claim				Justification		
Rational or Irrational	42	68%		Two Examples	17	27%
Always Rational	8	13%		Rational Example	9	14%
Always Irrational	9	14%		Irrational Example	6	10%
				Properties or Other	13	21%
No Claim	3	5%		No Justification	17	27%

Discussion

The previous literature argues that developmental mathematics students struggle from conceptual atrophy, meaning the students undervalue concepts and focus exclusively on procedures (Givven, et al., 2011; Stigler, et al., 2010). However, we feel that our results do not support that claim, since a majority of the responses included the correct claim, which indicates an intuitive understanding of rational and irrational number concepts.

Although most of the students did make the correct claim, they struggled with their justifications. The responses of several students indicate that they have insufficient examples of irrational numbers to be able to answer the questions posed. Leslie, George and David were the most egregious instances of this since these students clearly could not correctly distinguish the definitions of rational and irrational numbers. Other students demonstrated subtler problems with their example space. For instance, Carl, Jill and Walter struggled with the classification of the square roots of perfect squares.

Several students provided only one example to justify their claims. Many of these students tried to justify a universally quantified statement using a single example, and none of the responses included any indicator that the examples were insufficient. These students did not seem to recognize the logic of their statements. The remaining students who used only one example claimed that the sum or product could be rational or irrational. In all of these cases, the student included an example for the rational case, but not the irrational case. Some of the students quantified their responses, which seems to indicate that they believed the rational case to be the difficult case, and thus the only one that needs justification.

Relating to the examples spaces of these students, nearly every student chose square roots of small positive integers for the irrational number examples. The only other irrational number chosen was π. This is consistent with the work of Sirotic and Zazkis (2007b), where they argue that a limited example space can lead to misconceptions of irrational numbers. Although some of the students mentioned a decimal representation of irrational numbers, none of the students chose irrational numbers in this representation. A few expressed the sum or product in such a representation, although this seems to be a reaction from using a calculator to compute. More students made the correct claim about the products than the sums, and we attribute this to the choice of radicals for the examples. The procedures for multiplying radicals are significantly easier than adding radicals, and may have contributed to this result.

We support some changes to the instruction given to students on irrational numbers during grade school, and in developmental mathematics courses. Specifically, we support instruction that connects multiple representations of irrational numbers, including decimal expansions, radicals, other constants and placement on the number line. Providing students with questions that involve reasoning, such as "find an irrational number that lies between 32 and 53 on the number line," can promote reasoning and translating between representations. Such a question also open a discussion about the abundance of irrational numbers without directly addressing the concepts related to the cardinality of sets. Clear understanding of representations of irrational numbers may aid students as they progress through algebra, especially with regards to identifying the x-intercepts of polynomials with irrational roots.

References

Arcavi, A., Bruckheimer, M., & Ben-Zvi, R. (1987). History of mathematics for teachers: The case of irrational numbers'. *For the Learning of Mathematics, 7*, 18-23.

Fischbein, E., Jahiam, R., & Cohen, D. (1995). The concept of irrational numbers in high-school students and prospective teachers. *Educational Studies in Mathematics,29*, 29-44.

Givven, K. B., Stigler, J. W., & Thompson, B. J. (2011). What community college developmental mathematics students understand about mathematics, part 2: The interviews. *MathAMATYC Educator, 2,* 4-18.

Grubb, N. W., & Cox, R. D. (2005). Pedagogical alignment and curricular consistency: The challenges for developmental education. *New Directions for Community College*, 93-103.

Guven, B., Cekmez, E., & Karatas, I. (2011). Examining preservice elementary mathematics teachers' understandings about irrational numbers. *PRIMUS, 21,* 401-416.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics.* National Academy Press.

Peled, I., & Hershkovitz, S. (1999). Difficulties in knowledge integration: revisiting Zeno's paradox with irrational numbers. *International Journal of Mathematical Education in Science and Technology, 30*, 39-46.

Sirotic, N. & Zazkis, R. (2007a). Irrational numbers on the number line-where are they? *International Journal of Mathematical Education in Science and Technology, 38,* 477-488.

Sirotic, N. & Zazkis, R. (2007b). Irrational numbers: The gap between formal and intuitive knowledge. *Educational Studies in Mathematics, 65,* 49-76.

Stigler, J. W., Givvin, K. B., & Thompson, B. J. (2010). What community college developmental mathematics students understand about mathematics. *MathAMATYC Educator, 1*, 4-16.

Discourse in mathematics pedagogical content knowledge

Shandy Hauk
WestEd &
U. of Northern Colorado

Allison Toney
University of North Carolina
Wilmington

Reshmi Nair
U. of Northern Colorado
& Hood College

Nissa Yestness
Colorado State University

Melissa Goss
University of Northern Colorado

What is happening for in-service teachers at the classroom intersection of mathematics, culture(s), teaching, and learning? How can knowing the answer to that question inform teacher preparation, induction, and development? In ongoing efforts to model and measure the intercultural and relational aspects of pedagogical content knowledge, we present a model and data analyses. The focus is teacher learning and intercultural orientation development. Data are pre- and post-program written tests, surveys, and classroom observations among four cohorts (70 in-service teachers) enrolled in a two-year master's program. The focus at the conference was harvesting the intellectual power of the audience to consider questions about the connections – qualitative, quantitative, and otherwise – among core constructs in pedagogical content knowledge, the thinking that teachers do in connecting them, and how knowing about intercultural orientation and how it plays out in the classroom can inform teacher education and professional development.

Key words: Pedagogical content knowledge, Discourse, Intercultural awareness

Background

What mathematical reasoning, insight, understanding, and skills are entailed when a person teaches mathematics well? Many have worked to develop theoretical models and measures to address this question (Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008; Shulman, 1986). In their work, Ball and colleagues have proposed three types of subject matter knowledge and three types of pedagogical content knowledge (PCK) as non-overlapping categories in the domain of mathematical knowledge for teaching (MKT, see Figure 1, next page).

Current U.S. educational policy requires evidence-based decisions about teacher preparation, induction, and development. Meeting this need calls for models and measures that are credible and transferable across at least a small range of mathematics instructional contexts. The MKT model and related instrument development for K-8 teachers have provided a reliable and useful foundation at these lower grades. Ongoing development of MKT models for grades 8 and higher is adding to that foundation (Hauk, Toney, Jackson, Nair, & Tsay, 2014; Speer, King, & Howell, 2014). These additions at the secondary and post-secondary level have focused on mathematical discourse and meaning-making for teaching (Hauk et al., 2014; Powers, Hauk, & Goss, 2013; Speer et al., 2014; Thompson & Carlson, 2013). Thought, speech, and context inform each other. In particular, struggling with the ambiguities introduced in learning to use technical vocabulary, in and out of classroom contexts, supports mathematical meaning-making

(Barwell, 2005). In parallel, developments in teacher education research have included calls for attention to the cultural and sociopolitical aspects of mathematics instruction (e.g., Gutiérrez, 2012, 2013). The knowing that happens in pedagogical content knowledge can be seen as both a set of connections among rather stable fact-sets and as contextualized, but dynamic, ways of thinking.

Discourse, as an aspect of teaching, is central in our effort to bring an explicit attention to the use of language and the dense set of values about mathematical appropriateness, clarity, and precision that are integral to thinking, learning, and communicating in mathematics both in and out of school settings. Our previous work has discussed the connection between Ball and colleagues' model of PCK and an additional aspect called *knowledge of discourse* that relies on ideas from intercultural orientation (Hauk, et al., 2014). Here we report on our continuing work to address the twin needs of measures that capture information about PCK and models that attend to the actively cross-cultural nature of most mathematics instruction in the U.S. Hinging on unpacking "discourse" and connecting it to the PCK model shown in Figure 1, this work has led to the model in Figure 2.

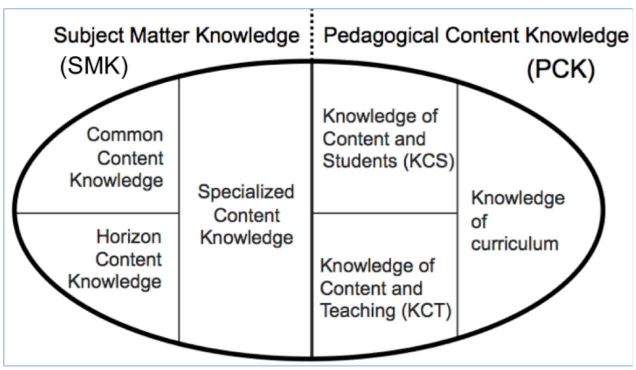


Figure 1. Dimensions of mathematical knowledge for teaching (MKT) from Hill, et al. (2008).

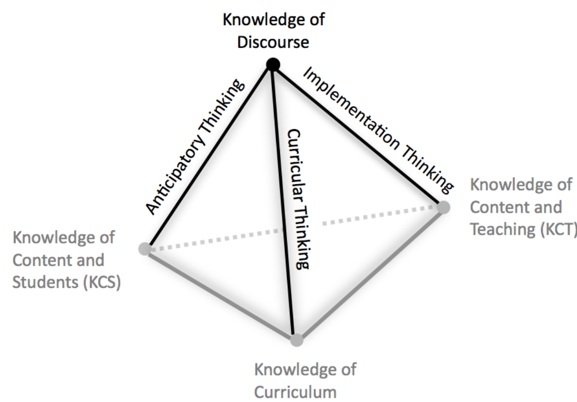


Figure 2. Extended model of PCK, from Hauk, et al. (2014).

The development of the model in Figure 2 has been grounded in classroom practice. The need for a construct like Knowledge of Discourse emerged early in our efforts to develop a measure of PCK that would capture growth in the kinds of knowledge valued as a mathematics teacher builds instructional effectiveness. Across our work, secondary and post-secondary teachers have said they know they are effective when students learn facts and, also, build a flexible understanding of mathematical ideas that can be brought to mind and actively used when needed. Early assessment and interview development led us to reuse that as a description of how to know that professional development was effective: We know professional development is effective when teachers learn facts and, also, build a flexible understanding of MKT ideas that can be brought to mind and actively used when needed. It was in getting at the "brought to mind and actively used" aspect that Knowledge of Discourse came to the foreground.

Throughout the revisions of the model summarized so briefly in Figure 2, we have iteratively visited three major strands of work:

Area 1. developing a written test that can capture change in PCK,

Area 2. advancing work on an observation-plus-interview protocol that can document bringing to mind and using PCK, in real time in the classroom, and

Area 3. refining a model of PCK to provide language, and examples, for our own further development as teacher educators and researchers in mathematics education. Each of these aspects has contributed to this report. Below, after providing some background on the model, we offer information on empirical results in a particular project related to Areas 1 and 2. These were shared at the conference presentation as background for a lively conversation about Area 3. We close with the fruits of the RUME conference discussions and some thoughts on next steps.

Theoretical Framework

In his review of over 100 research publications in mathematics education that reported on "discourse," Ryve (2011) found that the myriad conceptions of discourse offered by researchers could be understood through the work of Gee (1996), who distinguished between "little d" discourse and "big D" Discourse. "Little d" discourse is about language-in-use. In mathematics teaching and learning, this may include connected stretches of utterances and other agreed-upon ways of communicating mathematics such as symbolic statements or diagrams. Discourse (big D) is situated discourse, encompassing verbal and non-verbal aspects, from the subtleties of local vocabulary and symbolic or diagrammatic representation to the nuances of gesture, tone, hesitation, wait time, facial expression, hygiene, and other aspects that make for authenticity in an interaction (Gee, 1996). In what follows, our use of the term *discourse* is in the "big D" sense. Discourse, so defined, addresses Shulman's (1986) attention to semiotics:

> The syntactic structure of a discipline is the set of ways in which truth or falsehood, validity or invalidity, are established... Teachers must not only be capable of defining for students the accepted truths in a domain. They must also be able to explain why a particular proposition is deemed warranted, why it is worth knowing, and how it relates to other propositions, both within the discipline and without, both in theory and in practice… This will be important in subsequent pedagogical judgments. (p. 9)

As indicated in the excerpt above, Shulman's original statements about pedagogical content knowledge included knowledge for interacting effectively with the multiplicity of discourses students, teacher, curriculum, and school bring into the classroom. In particular, in the cultures of secondary and post-secondary academic and research mathematics, valued communication includes (among others) the sense-making discourse practices of description, explanation, and justification. These are also valued in school mathematics curriculum and instruction (e.g., the *Common Core* Standards for Mathematical Practice, National Governors Association, 2010).

The ways that teachers and learners are aware of and respond to valued forms of communication across multiple cultures is a consequence of their orientation towards cultural difference, their *intercultural orientation*. This is not a reference to teacher beliefs about the teaching and learning of mathematics. Rather, intercultural orientation is the perspectives about *difference* each person brings to interacting with other people, in context. For teachers, it includes perceptions about the differences between their own views and values around teaching and learning and the views of their students.

Gutiérrez (2013) refers to *conocimiento* to identify a relational, connected, way of knowing that is qualitatively different from declarative kinds of knowing (e.g., of facts and their contexts). Our work, too, relies on this idea and it is reflected in the "thinking" edges of the PCK model in Figure 2. What is more, Gutiérrez's (2012) *Nepantla* captures the aspects of professional learning Shulman described as "the exercise of judgment under conditions of unavoidable uncertainty" and the "need for learning from experience as theory and practice interact" (Shulman, 1998, p. 516), both of which are aspects of the interculturally informed discourse extension to the model of PCK. We join an already moving river of ideas. Various streams of research and development on mathematics teacher learning already spring from a research-practice synergy that views all people in a classroom as participants in learning. It is the question of the nature of that learning and of the interaction of the people in its support that is foundational (Schoenfeld, 2013).

Though some teachers work in largely monocultural classrooms – in the sense that most students share experience of a common set of culture-general norms and practices – the U.S. is shifting from such circumstances to cultural heterogeneity. For example, the 21st century version of multicultural can mean 2, 5, or even 10 different home language groups in a single classroom (Aud, Fox, & KewalRamani, 2010). Given the diversity of students in the nation's classrooms and the demographics of instructional staff in U.S. schools, teachers are destined to have regular opportunities for cross-cultural classroom experience that, for most, will be fraught with unavoidable uncertainty. Many new teachers leave, citing as a reason that they were not prepared for what the work is really like (Keigher, 2010).

What was recently explored in the project from which this research emerges is attention to this missing aspect of heterogeneity: dealing with the realities of navigating the multiple cross-cultural relationships in professional development and school contexts. Several frameworks exist for interacting and communicating with people across professional (and personal) cultures. In particular, healthcare and international relations have generated suggestions based on theories of intercultural sensitivity development and styles of conflict resolution communication (e.g., Bennett, 2004; Hammer, 2009). The developmental model of intercultural sensitivity centers on orientations towards cultural difference (Bennett, 2004). The core of this approach is building skill at establishing and maintaining relationships in, and exercising judgment relative to, interculturally-rich situations. The developmental continuum has five named milestone orientations to noticing and making sense of cultural difference: *denial*, *polarization*, *minimization*, *acceptance*, and *adaptation*. With mindful experience we develop from ethno-centric ignoring or *denial* of differences, moving through an equally ethno-centric *polarization* orientation that views the world through an us-versus-them mindset. With growing awareness of commonality, we enter the less ethno-centric orientation of *minimization*, which may, however, over-generalize sameness and commonalities. From there, development leads to an ethno-relative *acceptance* of the existence of intra- and intercultural differences, and on to a highly ethno-relative *adaptation* orientation.

Discourse is situated, in the present case it is situated in a mathematics class, and Knowledge of Discourse includes what a teacher may say. It also is used in how the teacher orchestrates conversation and discussion in the classroom. And, it is about what a teacher knows or anticipates about students' previous experiences and how to situate that in the classroom -- in the context of the mathematics goals in the classroom. For example, knowing how to establish, elicit, and respond to sociomathematical norms, would live in Knowledge of Discourse.

The lens of intercultural orientation development leverages powerful agents for improving teaching and collegial interaction. Teachers can build self-awareness and apply developmentally (for them) appropriate approaches to their own learning with colleagues and to student learning in their classrooms. We return to our exploration and development of examples of these ideas (Area 3 of our research program) after first sharing a brief summary of some empirical results related to Area 1 and Area 2. The empirical work is to provide some of the context in which the model in Figure 2 was developed (and continues to be revised) and a foundation for the reporting on the results of the intellectual work of the group at the RUME conference session.

Written Test and Observation/Interview Protocols - Measuring PCK

Methods

Setting: The setting was a blended face-to-face and online delivered master's degree program in mathematics for in-service secondary teachers. Designed to reach urban, suburban, and isolated teachers in rural areas, the program is conducted using a variety of technologies (e.g., *Collaborate* for synchronous meetings, *Edmodo* for asynchronous communication). Offered through a joint effort at two Rocky Mountain region universities, cohorts of 10 to 20 participants complete a 2-year master's program in mathematics with an emphasis in teaching (about half of the course credits in mathematics, half in mathematics education).

Participants: Participants for the quantitative results reported here were in-service secondary teachers who teach grades 6 to 12 mathematics. To date 71 teachers have entered the program, 33 have completed it, 18 are continuing, and 20 have dropped or taken leave from the program.

Instruments: The development of the written test of pedagogical content knowledge and real-time observation instrument is reported elsewhere (Hauk, Jackson, & Noblet, 2010; Jackson, Rice, & Noblet, 2011). The most important things to note here are that the written assessment included: released items from the LMT (Ball et al., 2008), new items with more complex mathematical ideas modeled on the LMT items, some secondary *Praxis* items, and open-ended extensions to these limited option items. Multi-year test development has included cognitive interviews with in-service teachers and mathematics teacher educators as they completed individual items or collections of items. In addition to the established face validity of the tests, tests of the constructs' internal consistency (Cronbach's alpha) indicate good overall reliability (α >.75 on each construct).

Constructs on the written instrument were curricular thinking, anticipatory thinking, and kinds of Knowledge of Discourse. While the written test of PCK has included items related to KCT, as a component of implementation thinking, testing this knowledge by self-report is problematic. So far, it has seemed that a better way to get rich information about implementation thinking is through observing a teacher in the classroom and interviewing about the observation later. The observation instrument documented in-class actions, utterances, and behaviors related to curricular thinking, anticipatory thinking, implementation thinking, and kinds of Knowledge of Discourse (e.g., observation categories included noting instances of mathematical description, mathematical explanation, mathematical justification – more on this below).

As of this writing, we have pre-tests for 70 teachers, first follow-up tests for 61 teachers (after 1 year in the program), and exit exams (post-program) for 33 teachers. Also at this writing, pre- and post-program observation data is complete for 17 teachers. The observation instrument, based on the LMT video observation protocol (see Learning Mathematics for Teaching website; development reported elsewhere) showed good reliability overall (α >.78 on each construct). Like the LMT protocol, the observation tool used samples called "segments" (6 minutes each: 3 minutes observed, 3 minutes to record notes; each class visit had 7 to 12 segments). An "observation" was three consecutive classroom visits. Experienced observers trained new observers to use the instrument; inter-rater reliabilities were greater than 0.8. To measure intercultural orientation and sensitivity development we used the established Intercultural Development Inventory (Hammer, 2009; idiinventory.com).

Empirical Results

We care about generating research-based and theory-grounded quantitative results because school leaders have to make evidence-based decisions about teacher learning. Current policy says "evidence" is based on test results. Reciprocally, what the empirical study is giving us is nuanced examination of teacher knowledge growth, in service of theory and model development.

Results after four years have indicated teacher knowledge growth for each of the constructs of interest. Paired samples *t*-tests on teachers' percent scores on the written test indicate statistically and practically meaningful growth in the desired direction in curricular content knowledge and discourse knowledge. Teachers' scores on items coded as Knowledge of Discourse (KofD) increased significantly (t=2.189, p=.047) from pre-test (M=56.82, SD=15.43) to post-test (M=66.22, SD=19.09).

For the observation data, to date there are two statistically significant results (Bonferroni correction applied). One was in the observation category "General language for expressing mathematical ideas (overall care and precision with language)." While such use of general language was seen, on average, in about 49% of pre-program classroom segments, by the end of the program it was present in more than 80% (M=80.34, SD=19.71). The other significant result was in "Mathematical descriptions (of steps)" (i.e., segments where the teacher or students accurately used mathematical language – in symbols, words, shapes, or diagrams – to describe the steps of some process). On average, across pre-program observations, this was seen in about 40% of class segments (M=40.28, SD=21.94), increasing to almost 70% of segments, post-program (M=68.10, SD=19.31). Though not statistically significant, there was also increase in the relative frequency of mathematical explanations (from 40% to 51%) and justifications (14% to 23%). Three other observed variables appeared to be approaching significance (i.e., p<.01): the percent of segments where (a) student voices were present in the room (increasing from 80% to 90% of segments), (b) teachers were observed to use conventional notation (increasing from 54% to 90% of segments), and (c) fewer mathematical errors occurred (decreasing from about 4% of the time to nearly 0%). Similarly, we have seen changes in the desired direction on the measure of intercultural competence development (e.g., see Hauk, Yestness, & Novak, 2011).

Examples - Communicating in and through PCK

The ways teachers responded to PCK test items and their extensions (on paper and in cognitive interviews) led to questions for us related to discourse (little d) and, eventually, to big D discourse. To illustrate, we give two examples. First, we present an example that highlights the connection between intercultural orientation and Knowledge of Discourse. Then, a second example takes the form of an annotated script, a fictionalized version, based on an actual conversation between two teachers (one a novice and one more experienced) as they worked through a task from the written test of PCK.

Example 1: Coexistence of Mathematics and Physics Discourses in Calculus

In our current work to unpack Knowledge of Discourse we consider the continuum of intercultural orientation, of ways of seeing differences between one's own values, view, and communication of the (mathematical) world and that of others. Central to this idea of intercultural awareness is ways of noticing. Perhaps the *denial* orientation might take the form (in the context of mathematics instruction): "I know the MATH, the math discourse, I don't really notice any other discourse." Such an orientation is not of denial in the sense of "I'm going to say it is not there" but denial as in "I can't even see it."

The *polarization* orientation towards orchestrating the conversation in a math class might be characterized as: "There's a RIGHT way to talk about things and there's a WRONG way to talk about things. And we're going to make sure we use the right way." Depending on the experience and values of a teacher, the "right" way to talk about applied related rates problems in calculus may or may not include physics discourse or associated engineering discourse. Nonetheless, enacting a polarized orientation in mathematics teaching would mean seeing, for instance, that a mathematical practice is happening or noticing a norm being developed. Perhaps, when a teacher strongly identifies with the mathematical culture, they are loyal to that culture. And, when focused on right ways and wrong ways of talking, do not attend to (may not really care) what is done in a physics class.

From a *minimization* orientation, minimizing differences and paying attention to similarities, teachers may also be very true to their mathematics knowledge, their mathematical culture, and valued ways of communicating. Yet, for someone mathematically trained, this might be characterized as, "Look how this is LIKE mathematics. Physics is like mathematics, the idea is similar even if the way it is said is a little different. Let's talk about how it is similar. Let's leverage the fact that students have seen this in physics before." Consider a basic example in the representation of vectors. Suppose the book represents vectors in the form $v=3i+5j$ and some students, who are also in physics, write $v=<3,5>$. It may be characteristic of a minimization orientation to write both representations on the board *once* and then note "But these are basically the same, so we'll use the one I know, the one common in math." In development towards an *acceptance* orientation, it might be more characteristic to notice and accept either representation on students' written work and suggest students use whichever makes most sense for them – anchored in the idea of a common goal, that vectors make sense to students. Pushing this small example even further, a well-developed acceptance orientation might be evidenced when a teacher alternated between the notations when talking with students and encouraged students to become fluent in both (i.e., modeling fluency in moving back and forth among the different representations while also encouraging students to accept and understand the difference in the representations).

More generally, an *acceptance* orientation might be characterized by a statements like: "I'm a mathematician, but I'm accepting the fact that all of my students are not going to be mathematicians" and "I'm accepting the fact that there may be other ways, physics ways or biology ways, of talking about this mathematical idea that are valuable, and maybe even more valuable to them [the students] than my math way of talking about it. I'm going to embrace that, those various ways, coming out in the conversation in the classroom." But a general intention of accepting the different ways in the classroom may not provide guidance to students about how to make decisions on which discourse(s) are useful in a given mathematical context (e.g., solving applied problems in biology may not be facilitated by an abstract mathematics vocabulary, and vice versa).

A further developmental orientation is *adaptation*. Now, not only does a person accept that there are these differences, the adaptation oriented teacher seeks out ways to give students opportunities in noticing, articulating, and responding to those differences. An adaptation orientation might be characterized by statements such as: "I seek out ways to have students pursue opportunities that arise from variety in approach or strategy. I don't have to give many, or even one method to them. They can go get it. I don't have to be in the loop. So math is a relative thing now. Learning math is still central but, while the goals are for learning about rigorous math and include the standard math language and representations, how I and students connect ideas and access, or organize, or value ideas is not necessarily strictly limited to the ways valued by a purely mathematical perspective."

Though not fully delineated by researchers, the theory of intercultural competence development also hypothesizes something called an *integration* orientation. This is something that is likely to be very rare. This perspective might be characterized by a statement like: "Okay, that physics approach to this problem is a whole other way of looking at the world. It's internally consistent. Which I, as a mathematician, value. So, it's okay. And I'm going to integrate what I can without violating my own truth to mathematics. I'm going to be myself as a mathematician, in that environment." We suspect such a view might be analogous to the ultimate mission behind much of theology: studying a variety of belief systems, without disagreement or approval of the system, while remaining authentic in one's own beliefs. In the research around intercultural competence development, examples of how an integration orientation might be realized come in the shape of expert and effective negotiators in high stakes endeavors (e.g., diplomat, hostage negotiator).

Example 2: Discourse During Use of Pedagogical Content Knowledge

Bringing to mind and using mathematical knowledge for teaching happens in many ways. An example of *curricular thinking* in the model in Figure 2 comes when mathematically situated discourse and knowledge of curriculum are brought to mind to create a rubric for grading a quiz. Among the items appearing on the PCK written test, was a task that asked teachers to do a mathematics item and then to generate a rubric for grading the item. The conversations that follow were based on actual teacher work and cognitive interview. First we generated a 2-column conversation of "little d" discourse – the actions and utterances of two teachers, Selma (experienced) and Jamie (novice) in solving the problem (this material can be seen in the table of the interaction below in column 1 and the bold face material in columns 2 and 3). Then, based on cognitively guided interviews on the task, we created the extensively annotated 3-column example, sketching the thoughts of each teacher. The first part of the interaction is focused on subject matter knowledge, SCK in particular. The balance is about their work to make a rubric. The purpose here is to formalize an example. It is based on the needs that emerged from conference attendees' wrestling with the ideas presented. The example is meant as an illustration of why it matters and can be useful to consider various aspects of Knowledge of Discourse in teacher education, induction, and professional development.

> Part 1: The Richter scale is a base 10 logarithmic scale used to measure the magnitude of earthquakes; i.e., an earthquake measuring 7 is ten times as strong as an earthquake measuring 6. An earthquake that measures 6.8 on the Richter scale has a magnitude that is approximately what percent of an earthquake measuring 6.6?
>
> Part 2: Provide a rubric that you could use for grading student answers.

Figure 3. Test item with extension.

Selma's ethno-centric approach to noticing and dealing with difference, a polarization orientation to difference, is represented in her view that her own knowledge of mathematics is paramount in solving the problem, and that she must compare whatever Jamie says to that foundation. For each of Jamie's contributions, Selma must determine whether Jamie is with her (therefore right, part of "us") or not (therefore wrong, part of a different group or "them"). Elements of this are evidenced in her "I" language in rows 5, 16, and 18, and in Selma regularly pausing the problem solving process to evaluate whether suggestions are right or wrong (rows 9, 12, 16, 18). Jamie, whose orientation is to minimize difference, views her knowledge as being essentially the same as Selma's. For Jamie, because they both "speak mathematics," it will not be difficult to work together to solve the problem. She interprets Selma's comment in row 5 as affirming "their" problem solving process, and shifts to "we" language (rows 10, 13, 17).

The interaction also has evidence of orientation in the approach each takes to (a) creating and (b) defending decisions about generating a rubric. Still focused on using her knowledge as the central reference, Selma asserts that how she awards points in her rubric is different from Jamie's method. Meanwhile, Jamie works to find commonality between the two (row 29). Jamie maintains that they have an important commonality, the language of mathematics, though the specific wording may be different.

Early in the conversation, Jamie decides she and Selma are "on the same page" (row 2). She spends the next few lines confirming they are thinking the same way about the problem, even while Selma considers whether they might be thinking differently (rows 6 and 7). In fact, Jamie spends much of the conversation looking for ways to affirm her convictions that she and Selma are thinking similarly about the problem-solving context (rows 5, 20, 21) and in creating a rubric (rows 28, 29, 30, 31, 39). Selma, on the other hand, looks to see *if* she and Jamie are like-minded. Jamie confirms for her they are like-minded in the problem solving context (rows 14, 18). Once they begin the rubric task, Selma must again decide whether she and Jamie are like-minded. Given their initial rubrics (see Figures 4 and 5), she quickly decides they are not (rows 27, 28). Pointing out those differences gives rise to some tension. When encountering conflict, as when the social or emotional stakes go up, people tend to fall back to an earlier developmental orientation. This is represented in the vignette when Selma and Jamie revert to denial and polarization, respectively (rows 31-37).

	Description of actions while working on prompt	*Selma*	*Jamie*
1	The prompt is written on the center of the whiteboard. Both stand at the board, the prompt visible between them, calculators in hands.		**I'm first thinking of using logs because it says "base 10 log scale." But then I'm thinking we want to make a ratio because it says "10 times as strong."**
2	Selma picks up a marker and writes the following on the board: 10^6 $10\times$ 10^7	*"10 times as strong": If that's the information in the prompt, then we also need information about* $10^{6.8}$ *and* $10^{6.6}$.	*She's writing the ratio. We're thinking about the problem the same way. We're on the same page, so we'll proceed together. I don't have to think about that part anymore.*
3	Selma punches on the keypad of her calculator. She writes the following on the board under her previous figure: $10^{6.8} = 6309573.445$ $10^{6.6} = 3981071.708$	**So, we need…**	*[continuing to make sense of the prompt] If I have to figure out a way to solve this problem, percent is also going to be important.*
4	Jamie points at the prompt.	*To find the percent change, I do this procedure.*	**It says "percent." So, greater than 100%.**
5	Selma gestures at $10^{6.8} = 6309573.445$ $10^{6.6} = 3981071.708$	**If I subtract these two. Oh wait.**	*We have a shared knowledge of how to compute percents. I'm continuing with your procedure.*
6	Jamie enters "$10^{6.8} - 10^{6.6} =$" into her calculator. Then she enters "$Ans \div 10^{6.6} =$"	*Something about the prompt saying log scale makes me uncomfortable. I'm worried your way is not the right way.*	**And divide it by the 6 one.**
7	Jamie writes .58489 next to Selma's calculations of $10^{6.8}$ and $10^{6.6}$.	*I'm not sure that's right, but I'm going to see what you do. Maybe you are doing it the right way.*	**So, ".58489." 58%.**
8	Jamie points to 10^6 $10\times$ 10^7	*Something about the nature of percents is giving me pause. Are we computing these correctly?*	**6 is 10% of the 7 one, right?**
9	Selma steps back from the board.	**Is it?**	*What is 10% of* 10^7*?*
10	Jamie points to 10^7.	*Okay, I'm listening to you. That's the right*	**Well, if we times this one by .1.**

		way to compute the percent.	
11	Jamie looks down at her calculator and enters $10^{6.8} \times .58493 =$	*Something about the nature of percents is still making me uncomfortable. I'm not sure this problem is right. Does it want percent increase? Or percent change? What is the right answer?*	**So, $10^{6.8}$ times .58493 is 3,981,071. Okay, so $10^{6.6}$.**
12	Selma points to the prompt.	**Is this worded correctly? It has to be over 100. So, that's the percent increase. Would it be 158%?**	*We subtracted to find what percent more $10^{6.8}$ is than $10^{6.6}$. But the question asks what percent is 6.8 of 6.6?*
13	Jamie enters $10^{6.8} \div 10^{6.6} =$ into her calculator.	*Okay, percent change is the right question.*	**Were we supposed to subtract? We found the difference. So maybe it's just $10^{6.8} \div 10^{6.6}$. So, it's 158.5, which makes sense.**
14	Selma points to 10^6 $10\times$ 10^7	**Yeah. Like that.**	*10^7 is 10% more than 10^6 because $10^7 \div 10^6 = .1$. So, $10^{6.8} \div 10^{6.6} = 1.58$ says $10^{6.8}$ is 158% of $10^{6.6}$. Okay, so it's essentially the same either way.*
15		*I'm still not sure this is the right way to do this, though. The prompt says log scale. I'm worried your way is not the right way.*	**Which makes sense.**
16		**Because it's a log scale, I feel like it's a log somewhere. So, I don't think we're right.**	*That's the same thing, isn't it?*
17	Jamie gestures at $10^{6.8} = 6309573.445$ $10^{6.6} = 3981071.708$ *[first points to right side of equal and then to two exponents]*	*Okay, I agree with you about the log thing. Is percent change really the question?*	**But it's log base 10 that converts it to magnitude. So, if we were to take the log of the magnitude, it would give us the Richter scale.**
18		**Yeah, that makes sense. I keep going back to percent change not increase.**	*Because they mean essentially the same thing. It's just how the question is worded.*
19		*You did it the right way.*	**It was a 58% increase, which means 158%.**
20	Selma writes 158.5% under Jamie's	**Okay. That makes sense.**	*We solved the problem together. Yay us!*
	.58489.		
21		**So, now it says to write a rubric.**	*We just solved the problem together. We're going to write a rubric together, too.*
22	Jamie points at Part 2 of the prompt and reads aloud.	*It will take way too long to try to do this together. We need to do it separately first.*	**"That you and your colleague could use for grading student answers." Does that mean we should make it together?**
23		**Well, yes. But let's start separate. You make yours and I'll make mine, and then we'll come together**	*Okay. That way we can use any small differences in our rubric to make the final one stronger.*
24	Both are quiet for several minutes as they write on separate sides of the white board. Selma writes [Figure X below] Jamie writes [Figure X below]	*Getting the answer wrong doesn't get you any points. Setting the problem up wrong doesn't get you any points. If you set up the first part of the problem correctly, you can get 1 point. If you set that part up correctly, and recognize the correct ratio between $10^{6.8}$ and $10^{6.6}$, you get 2 points. And of course, you get full credit when you do all of it right.*	*Right or wrong, I want them to be able to explain why they did what they did. If they can get the right answer and explain why it's correct, that should get full credit. If they can't do any of that, they should get 0 points. But they might be able to explain the whole problem right, but then have something fall apart in the math at the end. That should get a lot of credit because that's better than just guessing the right answer, but not really being able to say why. So, that should get 1 point and the other should get 2 points.*

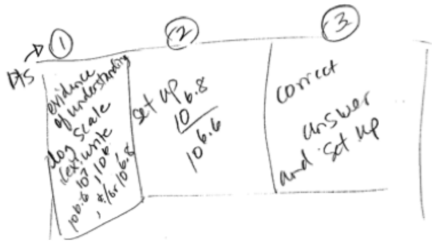


Figure 4. Selma's rubric.

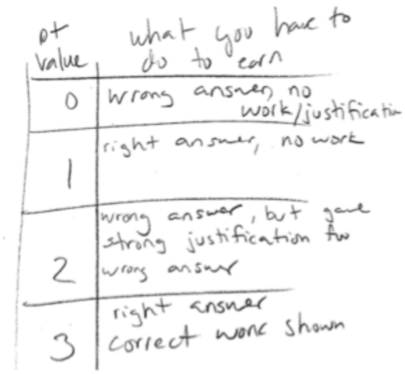


Figure 5. Jamie's rubric.

	Description of actions while working on prompt	*Selma*	*Jamie*
25	Selma steps back from the white board.	**Are you ready to talk?**	*We had the same idea about the math. We're probably thinking similarly about how to grade it.*
26	Jamie steps back from the white board and looks over at Selma's work.	*Okay, let's see what we did differently.*	**I think so.**
27	Selma looks over at Jamie's rubric.	*I'm already seeing big differences in these rubrics. She gives 2 points for a wrong answer and 1 point for a right answer. How can she give 2 points for wrong work?*	**Okay, this is what I did. I knew I wanted them to get the right answer.**
28	Jamie looks again at Selma's work and points to her 3-point column.	*Yes, those cells are the same, but there's still a lot of difference there.*	**Okay, like yours – and we both also want them to be able to explain it. Yeah, like you have "set up" and I have "correct work." So, a right answer with correct work or set up gets full credit.**
29	Selma points to the 1- and 2-point columns of her rubric.	**Right, but what I think is different is where we give 1 and 2 points. I'm basing all my points on how much of the problem they get right.**	*Really? You're showing what the "set-up" is on the rubric. That's essentially what I meant when I wrote "justification."*
30	Selma points to the 1- and 2-point rows on Jamie's rubric.	**You're giving points for a wrong answer. Why would you do that?**	*I don't think the wrong answer is what's important there. The justification is what's important. Like, if they wrote* $\frac{10^{6.8}}{10^{6.6}}$ *on their paper and then got the wrong answer for some reason. That's like what you wrote on your rubric.*
31		*But you still gave 2 points for the wrong answer!*	**We want to know what they *can* do. I know sometimes I start right, but then maybe I make a small mistake. But I knew what I was doing. I want to be able to give points to a student who made a small mistake.**
32		**Yeah. But why would a right answer get fewer points than a wrong answer?**	*I'm just saying a correct set-up might also sometimes get a wrong answer – like when they accidentally punch in the wrong numbers on their calculator.*
33	Jamie stands away from the white board and gestures back and forth between the two rubrics as she talks.	*Yeah. But my 1-point column doesn't have a place in your rubric. Just noticing powers of 10 isn't going to get 2 points when there's a lot more to the problem than that.*	**If I know they know how to set it up, I know they've got the foundations of the math we're teaching. That's what we're getting at in both of these rubrics. We're both looking at how they set up the problem. I'm just saying a correct set-up might also sometimes get a wrong answer – like when they accidentally punch in the wrong numbers on their calculator.**

34	Selma gestures at her rubric.	**Okay. But mine breaks down the steps of this problem into smaller chunks. With mine, the score is based on how much they know how to do. They get credit for doing each piece along the way.**	*What if they just write the right answer with no work?*
35		*That's not going to happen.*	**What if they just write the right answer with no work?**
36		**My students know better than to turn in a quiz without showing their work. If they got the right answer, the work will be right.**	*Eh, that's not an answer. We're going to have to agree to disagree here.*
37		*I don't think we're getting anywhere with this.*	*I don't think we're getting anywhere with this.*
38		*We need different rubrics because we're different teachers and we notice different things about this problem.*	**Maybe the rubrics don't have to be the same exactly if they're still getting at the same kinds of ideas.**
39		**Well, even when we give common tests and quizzes, we still grade our own stuff. I think we should have different rubrics.**	*What we have in common is that we agree that as different teachers we need different rubrics.*

Applications and Discussion

By adding Knowledge of Discourse as a variable to be described/measured, we include the interdependence of Knowledge of Discourse with KCS, KCT, and Knowledge of Curriculum in the extended model of PCK. The linking of these kinds of knowledge are represented through the connectors Anticipatory Thinking, Implementation Thinking, and Curricular Thinking, respectively, in Figure 2 (see Hauk et al., 2014 for more on these aspects).

Inevitably, there are both similarities and differences between teachers' own content-based acculturations, their own everyday cultures, prior mathematical enculturation of students, everyday culture of students, intended mathematical enculturation of the curriculum or school, and interim classroom cultures that combine all of these (and others, e.g., physics). The teacher having knowledge of these is mathematically important. Each has a mathematical component in terms of how one [student or teacher] sees mathematics or uses mathematics or values mathematics or communicates mathematically. And at the same time, for other disciplines it also is important. A rich Knowledge of Discourse in the context of calculus can include a knowledge of physics discourse (see, for example, the report in these proceedings by Firouzian & Speer, 2015). In fact, emergent from the conference presentation were conversations about the ways some knowledge of how those steeped in physics talk about and make sense of applied calculus problems is needed in order for a teacher to notice and point out to students the value of a physics approach (i.e., know and use the discourse of physics).

How teachers and learners approach (a) navigating different discourses, (b) establishing classroom mathematical discourse(s), and (c) the tools they have to do this, are all informed by their intercultural orientation. In pursuit of applications of this model and data analyses, we had several questions for RUME participants in the session.

Question 1 to attendees: What would make a compelling argument for you about the connections among these ideas? What kinds of data do you suggest we compare?

Attendee response 1: Session participants clearly wanted some rich examples in which the ideas were evidenced so that the evidence could be pointed to (and distinguished from evidence of other aspects of MKT). This call for examples led to the addition (the Area 3 result) of the annotated example conversation between Selma and Jamie.

Question 2 to attendees: Based on your experience, what would you expect about connections among the ideas in the model?

Attendee response 2: Attendees generally agreed that a substantive answer to this question would first require the examples called for in response to the first question.

Question 3 to attendees: How would knowing the answer to the questions we ask help teacher preparation, induction, and development? How would they inform collegiate practice of teaching with the adults who are in-service and pre-service teachers?

Attendee response 3: To get at a transition from theory to practice, participants in the session noted that knowing the answers, and having in hand some examples along with the model and ideas behind Figure 2, gives teacher educators tools and language for instruction (of both pre- and in-service teachers). Also, having an example that gets at the calculus/physics context could allow a contrasting cases approach to understanding the model for teachers. One might create a learning activity for teachers where they start with the calculus/physics discourse analysis (since the difference in the two professional discourses of math and physics may be more accessible to the highly mathematically trained). Then, have a second case where the nuances of analysis are applied to an examination of an example where there are similar professional cultures but differing intercultural orientations. The addition to this report of the Selma and Jamie case arose from the conference conversation. We have also begun development of a contrasting case about two teachers working on, and building a rubric for grading, an applied mathematics item with rich contrasts between physics and mathematical discourse.

Acknowledgements

First, our thanks to the 20 or so people who came to our session and shared their struggles, ideas, and sense-making about Knowledge of Discourse as a component in pedagogical content knowledge. Several conversations begun during the session continued through the conference. In particular, our thanks to Shawn Firouzian for very rewarding conversations. This material is based upon work supported by the National Science Foundation (NSF) under Grant Nos. DUE0832026, DUE 0832173, and the Institute of Education Sciences, U.S. Department of Education through Grant R305A100454. Any opinions, findings and conclusions or recommendations expressed are those of the authors and do not necessarily reflect the views of the NSF, the Institute, or the U.S. Department of Education.

References

Aud, S., Fox, M. A., & KewalRamani, A. (2010). Status and Trends in the Education of Racial and Ethnic Groups. NCES 2010-015. *National Center for Education Statistics*.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59,* 389-407.

Barwell, R. (2005). Ambiguity in the mathematics classroom. *Language and Education*, *19*(2), 117-125.

Bennett, M. J. (2004). Becoming interculturally competent. In J. Wurzel (Ed.), *Towards multiculturalism: A reader in multicultural education* (2nd ed., pp. 62–77). Newton, MA: Intercultural Resource Corporation.

Common Core State Standards - see National Governors Association (2010), below

Firouzian, S., & Speer, N. (2015). Integrated mathematics and science knowledge for teaching framework. *Proceedings of the Proceedings of the 18th conference on Research in Undergraduate Mathematics Education*, Pittsburgh, PA (this volume).

Gutiérrez, R. (2013). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education 44*(1), 37-68.

Gutiérrez, R. (2012). Embracing Nepantla: Rethinking "Knowledge" and its Use in Mathematics Teaching. *REDIMAT-Journal of Research in Mathematics Education*, *1*(1), 29-56.

Gee, J. P. (1996). *Social linguistics and literacies: Ideology in discourses (2nd Ed.)*. London: Taylor & Francis.

Hammer, M. (2009). The Intercultural Development Inventory: An approach for assessing and building intercultural competence. In M. A. Moodian (Ed.), *Contemporary leadership and intercultural competence* (pp. 203-217). Thousand Oaks, CA: Sage.

Hauk, S., Jackson, B., & Noblet, K. (2010). No teacher left behind: Assessment of secondary mathematics teachers' pedagogical content knowledge. In S. Brown (Ed.), *Proceedings of the 13th conference on Research in Undergraduate Mathematics Education*, Raleigh, NC. <http://sigmaa.maa.org/rume/crume2010/Archive/HaukNTLB2010_LONG.pdf>.

Hauk, S., Toney, A. F., Jackson, B., Nair, R., & Tsay, J.-J. (2014). Developing a model of pedagogical content knowledge for secondary and post-secondary mathematics instruction. *Dialogic Pedagogy: An International Online Journal, 2*, 16-40. DOI: 10.5195/dpj.2014.40

Hauk, S., Yestness, N. R., & Novak, J. (2011). Transitioning from cultural diversity to cultural competence in mathematics instruction. In S. Brown, S. Larsen, K. Marrongelle, and M. Oerhtman (Eds.), *Proceedings of the 14th conference on Research in Undergraduate Mathematics Education*, vol 1., pp. 128-142. Portland, OR. <http://sigmaa.maa.org/rume/RUME_XIV_Proceedings_Volume_1.pdf>.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39,* 372-400.

Jackson, B., Rice, L., & Noblet, K. (2011). What do we see? Real time assessment of middle and secondary teachers' pedagogical content knowledge. In S. Brown, S. Larsen, K. Marrongelle, and M. Oerhtman (Eds.), *Proceedings of the 14th conference on Research in Undergraduate Mathematics Education*, vol 1., pp. 143-151. Portland, OR. <http://sigmaa.maa.org/rume/RUME_XIV_Proceedings_Volume_1.pdf>.

Keigher, A. (2010). *Teacher attrition and mobility: results from the 2008-09 Teacher Follow-up Survey* (NCES 2010-353). U. S. Department of Education. Washington, DC: National Center for Education Statistics.

National Governors' Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Washington, DC: Authors.

Ryve, A. (2011). Discourse research in mathematics education: A critical evaluation of 108 journal articles. *Journal for Research in Mathematics Education*, *42*(2), 167-198.

Powers, R., Hauk, S., & Goss, M. (2013). Identifying change in secondary mathematics teachers' pedagogical content knowledge. In S. Brown, G. Karokok, H. Roh, and M. Oehrtman (Eds.), *Proceedings of the 16th Conference on Research in Undergraduate Mathematics Education*, vol. 1, pp. 248-257, Denver, CO. <http://sigmaa.maa.org/rume/RUME16Volume1>.

Schoenfeld, A. H. (2013). Schoenfeld, A. H. (2013). Classroom observations in theory and practice. ZDM: The International Journal on Mathematics Education, 45(4), 607-621, doi 10.1007/s11858-012-0483-1

Shulman, L. S. (1998). Theory, practice, and the education of professionals. *The Elementary School Journal, 98*(5), 511-526.
Shulman, L. (1986). Those who understand: Knowledge growth in teaching. Educational Researcher, 15(2), 4-14.
Speer, N. M., King, K. D., & Howell, H. (2014). Definitions of mathematical knowledge for teaching: using these constructs in research on secondary and college mathematics teachers. *Journal of Mathematics Teacher Education*, 1-18.
Thompson, P. W., & Carlson, M. (2013, January). An Assessment of Teachers' Mathematical Meaning for Teaching Secondary Mathematics and its Implications for MSPs. Presentation at the NSF Learning Network Conference (Washington, DC). Summary: <http://hub.mspnet.org/media/data/Session_Thompson.pdf?media_000000007934.pdf>.

How might students come to see a differential equation as a function of two variables?

George Kuster
Virginia Tech

Morgan Dominy
Virginia Tech

Using Tall and Vinner's notion of Concept Image (1981) we analyzed the concepts students used while working with the differential equation $P'=3P$ and the connections between these concepts. We identified five interconnected elements in the students image of P' in the context of differential equations that played an integral role in the students' reasoning while attempting to solve interview tasks. In this study we report our findings concerning a pair of students' beginning to treat a differential equation as a function of two variables by reasoning with their notions of slope and rate. Specifically we address the students' transition from seeing a differential equation as solely an algorithm for verifying a function is a solution to a differential equation, to treating the differential equation as a relationship between a function's rate of change and evaluated value.

Key words: Differential Equation, Slope, Rate, Function

Purpose and Background

The predictive power of differential equations gives them a key role within science, technology, engineering, and mathematics (STEM) fields. Given the large number of undergraduate STEM majors, and the recent efforts to increase these numbers (Engage to Excel Report, PCAST, 2012), the importance of improving students' understanding of differential equations is significant. Differential equations is an advanced topic in undergraduate mathematics that presents a unique challenge to students by invoking their understanding of mathematical concepts that were introduced in previous courses, and then building on them. The course is traditionally taught to students majoring in various STEM fields, that have completed a three-course calculus sequence (differential, integral and multi-variable) and usually serves as a gateway to more advanced topics, such as biological modeling, analysis, fluid dynamics, finite elements, and control systems. For these reasons, it is immensely important that STEM majors have a robust understanding of at least the fundamental properties of differential equations. As such, research that helps illuminate and support student learning in differential equations is also important.

In its simplest form, a differential equation is a relationship between a quantity of interest and that quantity's rate of change (Kohler & Johnson, 2006). Embedded within this definition are a great number of concepts, however, for our purposes we primarily focus on function and rate of change. Student understanding of function and rate of change have been the focus of extensive research (e.g. Carlson, 1998; Carlson, et. al, 2002; Tall &Vinner, 1981; Thompson, 1994; Zandieh, 2000), and the various complexities involved in understanding these concepts have been well documented. The existence of these concepts within the more advanced topic of differential equations serves to further increase the complexity of these concepts for students; it has been noted that not only are multiple ideas coming together to form a cohesive whole, but there is a "fundamental leap in the thinking required" (Rasmussen, 2001, p. 67) to understand various aspects of differential equations with regard to these concepts. With this in mind, a growing body of research has shown the concepts of function and rate of change to be necessary and important for students learning differential equations (Habre, 2000; Keene, 2007; Rasmussen & Blumenfeld, 2007; Rasmussen & King, 2000).

For instance, Stephan an Rasmussen (2002) document students utilizing the notion that *y*, in differential equations of the form $y'=ft,y$, is both a function and a variable to reason about differential equations and their solutions. Additionally, Rasmussen and Whitehead (2003) claim that this image is required for a robust understanding of differential equations, though that it is also one of the most difficult for students to attain. In their work, they assert that students with this image were able to almost effortlessly transition between substituting values for y (seeing y as a variable) and treating y as a continuously changing function, in an effort to extract information from the differential equation. Students that had this ability were able to construct solution functions by using their notion of rate of change to determine the nature of the solution function's behavior.

Tangentially related, Donovan (2007) used a card-sorting task to explore what it means to conceptualize a differential equation as a function itself. Donovan, using Sfard's (1991) reification theory, showed that students with the ability to act on differential equations as a function are afforded a more robust understanding of differential equations and their solutions as compared to students that do not conceptualize a differential equation as a function. More specifically, students that are able to transition between graphical and algebraic representations of differential equations of the form $y'=f(y)$, were better able to draw connections between differential equations and their solutions. Additionally, Donovan's work shows that this ability supports students in treating y as both a function and a variable in the differential equation, interpreting dy/dt as a variable, slope or derivative, and understanding t as an independent variable in both differential equations and their solutions. Donovan also noted that students with the ability to treat a differential equation as a function treated the differential equations as connected to and informative of their solutions. Based on these findings it is clear that students' understanding of function is not only utilized in differential equations, but actually built on: functions are now solutions to, and embedded within equations, something not often encountered in previous coursework.

In her research on student reasoning in differential equations, Keene (2008) found that students reason with their notion of derivative and use the idea of changing rates to qualitatively discuss the relationship between differential equations and their solutions. Further, students reasoned with these ideas across both graphical and symbolic representations of differential equations and their solutions (e.g. slope fields, solution spaces and symbolic forms of solution functions/differential equations). Connected to this, Rasmussen and Whitehead (2003) documented that students use rate to build images of prediction and function, and treat it as a quantity that determines the behavior of a function. Specifically, they note that students use the notion of rate to predict the values the solution function will attain at a certain point. Their work points out the interaction between students' images of rate and function, and shows that these two concepts come together to support student learning in differential equations.

Research also shows that student reasoning with rate is not limited to reasoning about only one solution function. Stephan and Rasmussen (2002) present findings on how students reason with slope. They found that students utilized the following ideas: reasoning about the way in which slopes change over time, slopes are invariant horizontally for autonomous differential equations, and infinitely many slopes are encountered in a slope field but only finitely many are visible. Each of these ideas relates to student thinking about slopes (a graphical representation of rate of change) and how they are useful in understanding solutions to differential equations. Additionally, they noted that students reasoned with dy/dt versus y graphs, which is precisely one of the graphical representations Donovan (2007) discusses in his research concerning understanding differential equations as functions.

While it has been shown that having the images of y as both a function and a variable, and a differential equation as a function itself are immensely important for a robust understanding of differential equations, how students construct these images is still unclear. We interpret the research findings above as indicating that students' notions of rate can support their progression toward understanding a differential equation as a function itself. Given the importance of the concepts of function and rate of change in student learning of differential equations, this study investigates how students might come to see the differential equation $'c$as a function of two variables. More specifically we examine the students images of P' and the role it plays in understanding a differential equation as function of two variables.

Mathematics Background

Much of what we discuss in this paper is related to what students understand about the ordinary differential equation $P'=3P$ and variants thereof. This relatively conventional differential equation encompasses a myriad of interconnected mathematical ideas and concepts. For instance, some of the concepts within this equation are the notion of function, vector field, rate, solution space, derivative and the additional ideas that come with them. Using various analytical methods one can find the general solution to this equation, which is $P(t)=Ce3t$. This general solution, in fact, represents an entire space, in this case an infinite set of solutions that satisfy the differential equation. Given an initial condition such as $P(t0)=P1$, one is able to find a unique solution satisfying both the equation and the condition. Further, the equation $P'=3P$ provides a relationship between the rate of change, P', and the quantity P, that holds for all functions in the solution space. That is, given a specific value of P one can find the rate of change for any solution function, at the instance when that function takes on the same evaluated value. This differential equation also establishes a relationship between functions and their derivative. That is, one can interpret $P'=3P$ as a mapping between two function spaces. Specifically, any solution satisfying this differential equation is such that operating on that function with the derivative is the same as multiplying the function by three. Combining these two notions means that P is both a function and a variable. The general and particular solutions can also be found using graphical methods. A tangent vector field for the differential equation, which is a map of the tP-plane that includes vectors tangent to the solutions at various points in the plane, can be used to construct a graphical image of the functions within the solution space. The construction of a tangent vector field requires treating $'$ as a function of two variables because one must use values for and to attain values for $'$. This means that the differential equation can also be interpreted as a statement about two rates: how $'$ changes as P changes, and how P changes as t changes. These concepts, ideas and techniques form a baseline for the mathematics underlying the equation $P'=3P$.

Theoretical framework

Due to the interweaving nature of the mathematical concepts involved in this study, Tall and Vinner's (1981) notion of *concept image* serves as an effective medium for which one may interpret the students' understanding of $P'=3P$ and the various pieces and concepts associated with it. The term concept image is defined by Tall and Vinner as "all the cognitive structure in the individual's mind that is associated with a given concept" (1981, p. 151). Further, one's concept image can include visual representations, mental imagery, beliefs and experiences associated with a given concept (Vinner, 1991). For example, when a student thinks of a differential equation, they may envision a written equation, a direction field, a collection of analytic solutions, a rendering of one or more solutions in the plane, etc. One's concept image is not static, it is ever growing and evolving as the number of interactions one has with a given concept increases. Additionally, this theoretical framework accounts for potential differences in the sets of elements within a concept image that are activated in different situations. In other words, only some elements within ones concept image may be triggered during certain situations, at a given time, while other elements are not; this portion is defined as the *evoked* concept image (Tall and Vinner, 1981). Because different elements may be activated in different situations, it is possible for a student's concept image to contain seemingly contradictory elements, and for the student to not be aware of these contradictions. It is, however, also possible that a student may encounter a situation in which these contradictory elements are evoked simultaneously, potentially creating a problematic situation for the student for which resolution is needed. The notion that seemingly contradictory elements may exist in ones understanding helps explain how the students we interviewed progressed toward seeing a differential equation as a function of two variables.

For the purposes of this work, the notion of concept image will serve as a way to not only catalog the concepts the students used while attempting to solve differential equations tasks, their understanding of these concepts, and the *connections* between these concepts, but also as a way to identify when it might be possible to promote positive change in students' understanding of these concepts. We take a connection between elements within the students concept image to be indicated by occurrences in which the students related two or more elements in the same expression. Our notion of connection is commensurate with the theoretical perspective, in that it can be thought of as an instance in which common elements between two separate images are evoked in the same situation. It should be noted that a student's concept image, like a nesting doll, can contain elements which are in and of themselves images. For instance, in our analysis we identified slope and derivative as two elements within the students' concept image of P'. These are two distinct, though non-disjoint concepts, and for the students each had its own image.

This theoretical framework addresses the research question and methods by providing a way to analyze the students' evoked image(s) of the various concepts they utilized when working with differential equations, as well as providing insight into how the individual notions differed throughout the interviews. Most importantly the framework provides a way to explain the events that unfolded during the second interview, the students began treating P as a variable and the differential equation as a function.

Participants and Methods

In order to better understand the students' concept images, two, one-hour semi-structured pair interviews were conducted with Alice and Jen (pseudonyms). Alice and Jen both earned A's in a differential equations course one semester prior to the interviews. The interviews were video recorded and analyzed using iterative video analysis (Lesh & Lehrer, 2000). The main goal of the interviews was to generate verbal and written work that could be used to identify the concepts the students used while reasoning about changes in the solution functions with regard to changes in initial conditions. To accomplish this the students were presented with tasks during the interview to engage them in problem solving activity. Due to the exploratory nature of the research, a decision was made to keep the questions as general as possible in an effort to better examine the students' mathematics (Steffe and Thompson, 2000) by providing opportunities for them add their own interpretations of the mathematics involved. Sample interview questions and potential follow up questions can be seen in Appendix A. During the second interview, the students began to reason about characteristics of the solution functions using their notions of slope and rate, but a more interesting story developed. These events changed the direction of the second interview, as it became more of a teaching experiment (Steffe and Thompson, 2000) towards the end.

The first interview was designed to evoke the concepts the students used while solving differential equations tasks so they could be identified through analysis and further explored during a second interview. After the first interview the authors watched the recording of the interview specifically looking for the concepts and reasoning students used while expressing the existence of a relationship between initial conditions and solution functions. These instances were then time stamped, transcribed, and coded further to aid in the development of the second interview protocol. Specifically, the researchers identified categories such as *rate*, *slope* and *relationship* (between changes in t and changes in P) as important notions in the students' discussions about $P'=3P$ with regard to changes in initial conditions. For example, when the students were asked to interpret the meaning of P' and P in the equation $P'=3P$ their replies included "P' is the relationship between changes in P and changes in time," and "it's saying [P'], is equivalent to the ratio [dP/dt]." Students also noted that changing the initial conditions changes P, as it changes where the solution function starts.

In an attempt to further explore the nature of these elements during the second interview, an unexpected event occurred for both the researchers and students that changed the direction of analysis during the second phase. While providing the students with an initial value problem aimed at gaining a better understanding of the students' images of slope and rate, the students began to treat the differential equation as a function of two variables. For instance, they began to attend to changes in P while keeping t constant, something they had not done previously. They also began to treat P as a dependant variable in the differential equation, substituting numerical values in the equation to calculate P', which was something they had earlier stated as being incorrect. Due to this, during the second phase, attention shifted to the students' images of rate and slope and how their reasoning about these concepts promoted an understanding of P' as a function of both P and t. The data collected during the first interview was then re-analyzed with this goal in mind. In the following sections we present the results of our analysis and describe how the various elements in the students' image of P' played a role in their progression towards seeing the differential equation as a function of two variables.

Results

Throughout the interviews the students expressed various interpretations of P' in the differential equation $P'=3P$, each of which shaped their interpretation of the differential equation and its various components. As a result of the analysis we discovered five elements within Alice and Jen's concept image of P': slope, relationship, rate, ratio, and derivative. To a greater extent we focus on slope and rate, as these played a significant role in the student's reasoning about $P'=3P$ as a function of two variables. Though each of these elements can be thought of as separate entities existing in the students' larger understanding of P', quite frequently the students discussed these elements in ways that indicated a more interconnected nature. Additionally, the relationship between the elements played a significant role in the students' reasoning at the end of the second interview, which lead to their treatment of P' as a function of both P and t. Specifically, the students had certain expectations regarding the value of ***P*′**, that were based on their notions of slope, rate and derivative. These expectations were not met during their work on the initial value problem presented during the second interview and to reconcile this, they began to account for changes in ***P*′** with respect to ***P***. This is something they had not expressed in any of their prior interactions.

The students' image of ***P*′** is shown in Figure 1; the solid lines represent elements within the students' concept image of ***P*′** in the context of a differential equation and the dashed lines depict connections between the elements. A connection was identified during the second interview; for example, when Alice drew a line tangent to the solution function she graphed earlier, while saying "this is gonna be the rate at which it's increasing." We took this as an indication of the existence of a connection between the students notions of slope and rate. Interestingly it was these connections that promoted the growth of the students understanding in a direction more commensurate with that of an expert. This is discussed in more detail in the conclusion.

The results are presented in three sections. First, we discuss some of the elements within the students' image of ***P*′** prior to their engagement with an initial value problem task. Then, we describe the students' reasoning while completing the initial value problem task from the second interview with an eye towards how they utilized the elements within their image of ***P*′**. Lastly, we outline how the students reasoned about and treated the differential equation as a function after the task.

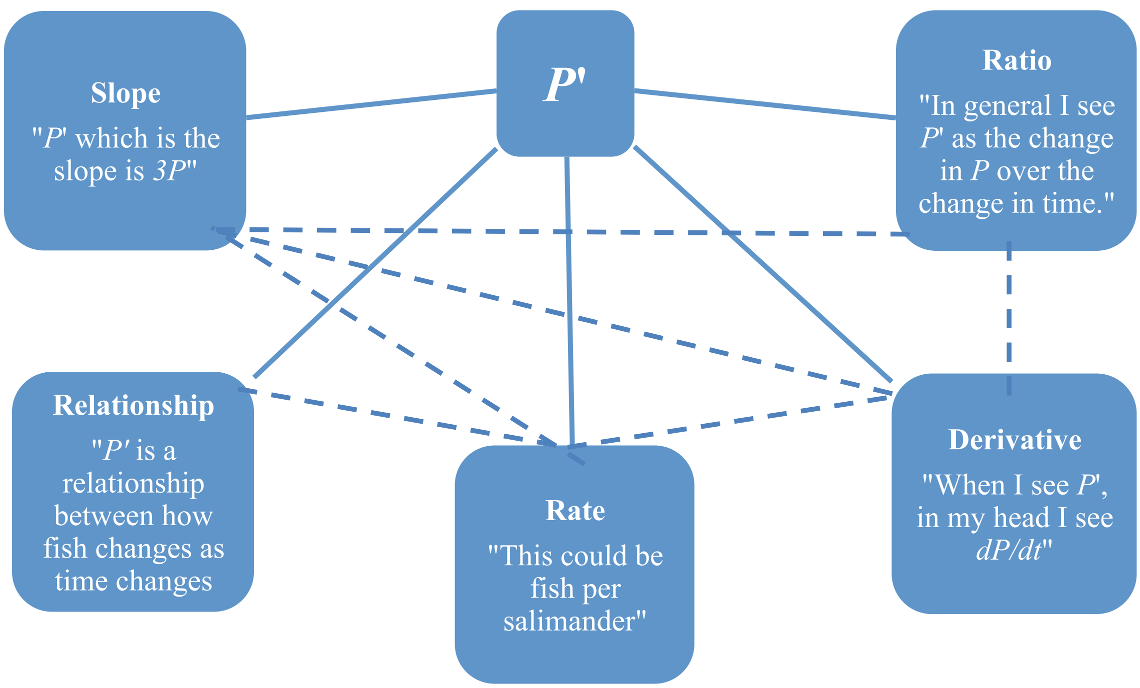


Figure 1: Students' Various Interpretations of ***P*′**

Image of Slope and Rate

In the first interview, Alice noted that when she sees ***P*′** (in ***P*′=3*P***) it signifies, "a relationship between changes in *P* and changes in time." From this statement alone it was not clear what the relationship was or what defined it. In the second interview Alice was asked to elaborate on this statement, the following is transcription of the conversation that followed.

1 A: I wouldn't say that from $P' = 3P$ necessarily, it doesn't explicitly say that's a relationship of something changing as something else changes. But when I see P', in my head I see dP/dt.

2 G: Ok and this thing saying $3P$ over here doesn't tell you anything about dP/dt?

3 A: It's saying [$3P$] is equivalent to the ratio [dP/dt] or the relationship.

4 G: And the relationship you mean is dP/dt?

5 A: um hum

6 G: So what does it mean for the relationship to be equivalent?

7 A: To me it says given the solution of what P is, that, if you plug it in or put it into that relationship spot P', then they have to equal each other.

8 …

9 Yeah but you have to find P to figure that out. So they are equal. So P' or the relationship is equal to P' is equal to $3P$ but you have to have P to understand that relationship.

Jen also expressed the notion that the relationship (how P changes as t changes) is not explicitly defined by the differential equation and noted you need to "solve for" or be given P (a solution function) first to meaningfully talk about P'. From the transcript above we can see that in Line 7 is in reference to the algebraic manipulations involved to verify that a function is a solution to a differential equation (this was brought up by the students in the first interview). This helps inform why the students, when initially asked to determine a numerical value for P', would solve the differential equation, compute the derivative of the solution function and then use this derivative to find a value for P'. Both of the students implied that the relationship (dP/dt or P') depends on P because, "you need P first," but based on the fact that the differential equation does not define this relationship, we do not interpret this to mean that the differential equation was a function of P for the students.

The students' notion of slope was strongly related to P' and played a large role in their interpretation of the differential equation in relation to solution functions. Specifically, the students noted that the slope of the solution function at any point was represented by P' and would be equal to $3P$. Further, they noted this would hold for any of the functions in the solution space. They concluded this by reasoning that taking the derivative of any of the solution functions would result in getting three times that solution function. Importantly, early in the interviews the scope of this idea seemed only to refer to the algebraic representation of the function, and not the function's values. In other words, the students noted that they could multiply the solution function by 3 to get $P'(t)$, but they did not express the idea that multiplying the value of P by three would result in getting the value of the rate of change of P. Additionally, they expressed that the value of P' would increase as t increased and that this was true for any solution. The following excerpt explicates some of their expressed understandings from the second interview.

10 A: When I think of P' equaling $3P$, $3P$ is going to be the graphical solution of the growth or whatever [draws a particular solution for $P'=3P$]. Then P' is gonna be your slope… Then if you take the slope here [points to the arbitrarily defined point (10, 30) on the solution function] this [draws a line tangent to the curve at that point] is gonna

be the rate at which it's increasing at that given time.

11 G: Take your arbitrary numbers. Can you put that stuff [the values] in there [the differential equation] and tell me what you think it [the slope] is?

12 A: P is an equation of sorts. P can't necessarily be a number because if you take the derivative of a number you get zero [pointing to a place on the graph where the slope is not zero].

13 …

14 J: P' which is the slope is $3P$

15 G: So what does that mean?

16 J: It means your slope is $3P$ which whatever that is I don't know.

17 G: So P is the problem when you say 'whatever that is I don't know' it's that you don't know what P is or that you don't know what P' is?

18 J: I know what P' is, I don't know what P is.

The students make an explicit connection between slope and P' in Lines 10 and 14. In Line 12, Alice expresses a connection between her notions of the derivative and slope which is also related to the algebraic manipulations she was referring to in Line 7. Further, Line 12 represents Alice's notion that the differential equation does not accept "numbers" as inputs for P and we interpret this as meaning that P is not a variable in the function $P'=f(P)=3P$ for the students. Most importantly, for our purposes, she makes a clear connection between the value of the derivative and its graphical representation as the slope of a tangent line. Alice may have very well utilized this connection to analyze what she believed a solution function would look like when making her claim about P' increasing as t increased.

Jen's statements in Lines 14-18 are closely related to her notions of rate in the context of differential equations. Earlier in the interview the students calculated the particular solution $P=Ce3t$, unprompted. Jen then noted that there were an infinite number of solutions that satisfied the differential equation. When asked why these were all solutions to the same differential equation she said, "they all have the same rate." Jen also noted that for functions to be solutions to the differential equation they had to be such that when taking the derivative it was the same as multiplying by three. She went on to say, "all of their rates are the same which is $3P$ and you get the $e3t$ as long as those [rates] are the same."

The students' notion of rate is what allowed a single differential equation to have numerous solutions. In this way it unified all of the functions in the solution space; they all had $3P$ as the rate. It was not clear, however, what the quantities their use of rate was referring to, how rate was different from slope and whether or not the students were attending to different values for P. For instance, in some cases it seemed as though rate was a property between changes in P and t, and at other times it seemed to be a property between in P' and P (comparing all of the solution functions and their derivatives or rate for the students). The slope of a line tangent to a solution function was also an element of the students image of P' and was closely related to the value of the derivative. The students expressed that the slope was $3P$, and that taking the derivative of P (a function) would result in 3 times P. If we consider Lines 1, 7, 12 and 18, the students express not being able to use values for P in the differential equation and not being able to understand P' without an explicit function for P. In other words, the differential equation is an equation used to verify that the solution satisfies certain characteristics defined by the differential equation, but, it is not used to determine the value of P'. For the students, finding P' is done by taking the derivative of the solution function and evaluating this at a certain t value. These images came to play a role in the students reasoning during the initial value problem that will now be discussed.

One of the interesting and important aspects of the students' images of slope, rate and derivative lies in the ways in which the students related each of these images. Namely, for the students, each of these was equated with $3P$. At various times during the interviews the students expressed that the slope was $3P$, the rate was $3P$ and the derivative was $3P$. Additionally at different times $3P$ represented a relationship between different quantities. At times it was also used to relate changes in P with changes in t, whereas at other times the students related changes in P with changes in P' (mostly when using their notion of rate). This, in and of itself, is not surprising, as to the expert, P' does indeed represent these three concepts, and each is equal to $3P$. For the students, the connections between these images, however, do not exist solely in the value they take on, and illustrated in Figure 2.

It should be noted that for the students, slope and derivative were qualities of a single solution function, whereas rate was also used to describe a property of each of the solutions in the entire solution space. These distinctions are important as these ideas play a crucial role in what transpired during the initial value problem task discussed in the following section. We assert that during this task the students' images of slope, rate and derivative were evoked in a way that made the students aware of contractions in their understanding of these concepts with regard to differential equations. Namely, the students expected the values for the slope, rate and derivative to be equal across various solution functions at the same t value.

Based on our analysis, we determined the existence of three main elements within the students' image of slope within the context of differential equations: the slope of a solution function at any point is represented by P' and is equal to $3P$, without an explicitly defined $P(t)$ one cannot infer information about the slope, and one cannot substitute numbers into the differential equation to find the value of the slope. Additionally, the ideas that solution functions all have the same rate and that the differential equation defines this rate (what P' equals), were elements within the students' image of rate. We assert that these ideas came together in ways that promoted the understanding of a differential equation as a function that relates the value of a function to that functions rate of change.

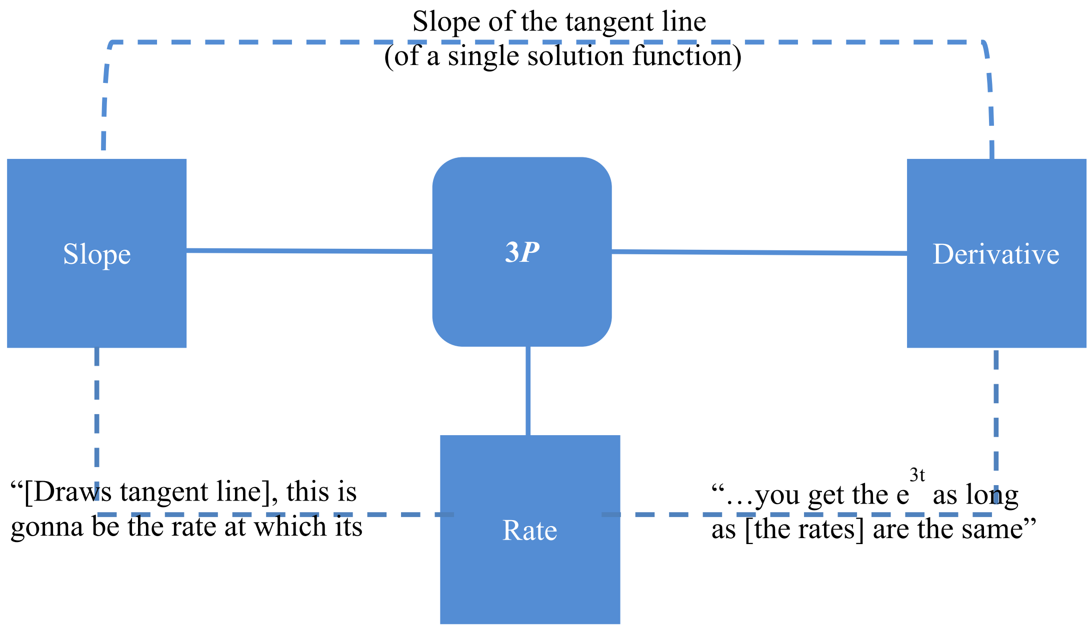


Figure 2: Connections Between the Elements Slope, Derivative and Rate

Initial Value Problem Task

In an effort to better understand the nature of the students' image of slope and rate, the students were asked to find the particular solutions for four initial value problems (IVP) all of which having the differential equation $P'=3P$ (See Appendix A, Interview 2, Question 2). The students used analytical methods to quickly find and then graph the four respective particular solutions. When asked why these solution functions all satisfied the same differential equation Jen replied, "You said that. You said the initial conditions are 5, 6, 7 and 8 and all of their rates are 3P." Here Jen interpreted the rate as being dictated by P' in the differential equation provided. The students then began to calculate the derivative of each of the particular solutions and showing that the result was the same as multiplying the original function by three and that each of the solutions shared this property. Based on this response, it was not clear exactly how their image of rate was different from their image of slope and if they were attending to changes in P, but it seemed that the students were attributing the word "rate" to a property they saw as unifying all of the solutions (e.g. P' is the same as three times P for all of the functions). Additionally, it was not clear what quantities the students were using when discussing the rate.

Alice then began to explain that for a given solution, the rate was increasing as time increased. To show this, she began calculating the derivative values of the particular solution $P=5e3t$ at increasing values of t, specifically 0, 5 and 10. She represented this increasing value by drawing tangent lines on the solution curve, each with steeper slopes and marking each slope with the respective values, *15, 15e15* and *15e30.* She then repeated this for the solution function $P=8e3t$ getting the derivative values of *24, 24e15* and *24e30.* Alice's work can be seen in Appendix B. This provided insight into the quantities this rate was referring to (namely P and t) and prompted a turning point in the second interview as the students then began to question their conclusion that all of the solutions had the same rate. Specifically, Jen stated that Alice's work "proves the solutions increase at different rates" at a fixed time. Immediately following this statement, Jen began referring to the differential equation as a model. Recall that earlier she interpreted the differential equation as indicating the solutions "all have the same rate" and that this rate was $3P$. When asked why she was drawing a distinction between the model and the rate, referring to the solution functions she said, "these all follow the same model but they have different rates."

We assert that at the moment Jen required drawing a distinction between the differential equation representing a rate, and the differential equation as a model, Jen's image of P' in the context of the differential equation $P'=3P$ expanded to include a relationship between P' and P. That is, Jen began to see the differential equation as a relationship between the values $P'(t)$ and Pt not just a way to verify a function satisfies the differential equation. Further, she began to attend to the different values $P(t)$ can attain even when holding t constant.

A Differential Equation as a Function of P and t

When Jen was asked to elaborate on the distinction between the model and the rate she began to incorporate ideas indicative of attending to changes in P and how those changes related to P' (in the sense of both the differential equation and the rate at which P changes as t changes). Jen said, "They [the model and the rate] are doing different things in my head, but the numbers follow the same formula." Here, Jen was comparing the numerical values Alice wrote down earlier (e.g. $15e15$ and $24e15$), to their respective solution functions noting that they both followed the formula $P'(t)=3P(t)$. For Jen, the model was a relationship between the solution function P and the P', and the rate was a relationship between changes in P and changes in t. Specifically, she was no longer reasoning about the differential equation solely as an relationship between two functions, P and its derivative. The image of model reflected the fact that the form of the derivative for each solution function, P, was three times P ($P'=3P$), but allowed for differences in $P(t)$. She was now comparing the numerical values attained across multiple solutions and relating them to the form of the differential equation as if the differential equation was a function between $P'(t)$ and $P(t)$. Additionally, she was doing this for multiple values of t.

Their work on this task made them realize that to find the value of P', they could simply determine the value of P and substitute this value into the differential equation. This was opposed to their method for calculating P' earlier, which was solving for $P(t)$, taking the derivative and substituting an appropriate value of t into the derivative function. They noted that solution functions were such that the derivative of the function was three times P, and this meant that one need not take the derivative to calculate P' as the differential equation already provided a way to calculate it. Namely, that evaluating P and multiplying it by three (as a result of taking the derivative), was the same as evaluating P and multiplying it by three (as dictated by the differential equation).

Conclusion

It is important to note that for the students we interviewed, P' represented multiple concepts that could be used interchangeably. It is our opinion that this occurred not because of the conflation of the concepts, but because of the numerical values associated with those concepts. Specifically, the rate at which the function changed, the derivative and the slope at any point in time on the solution functions were all thought to equal $3P$. For the expert, this may not seem problematic because the value of $P't$ at any point on a solution function (for $P'=3P$) is indeed $3P(t)$ and this value will be the same as the slope of the tangent line at that point. Further, this relationship holds for any point on any solution in the solution space. The key here is that this is in reference to a specific solution function, and that the value of $3P$ changes as P changes.

The relationship that the students expressed during this interview between rate and slope, relied on the notion that the slope, rate and derivative will each take on the same numerical value across all solution functions. We suspect that their images for each of the three concepts (perhaps, at times, with the exception of rate) did not allow for attending to changes in $P(t)$, rather the students by and large associated P in the term $3P$, with representing only one solution function. Half way through the second interview, the students were asked to determine four solution functions and use them to further explain what they meant by their comments from the first interview concerning slope and rate. This presented a problem for the students since the "slope" across the solution functions changed as $P(t)$ changed, but the "rate" (the unifying characteristic) did not. In other words the slope was still $3P$, but the functions (and, keeping t-values constant, the evaluated values of these functions) were not the same. For the students, the slope was a numerical value that was calculated by taking the derivative and evaluating the derivative at a certain time. When the numerical values of the slopes began to change across the various P-values (for the same t-value), the students began to second guess why their solutions satisfied the differential equation. In doing so they began to attend to the nature of the solution's derivative values in terms of the solutions respective values.

As such, we claim that the students transitioned from seeing the differential equation as an algorithm for verifying that a solution satisfies a differential equation to treating P in the differential equation as not just a function, but as a changing numerical value and related that numerical value to the value of P' at that same t-value. Recall, that initially the students rejected the idea of substituting values into the differential equation for P. Their reasoning was that P had to be a function, the students warranted this claim by stating that taking the derivative of a constant would yield zero, and this would not make the differential equation true. In this case, the students were treating P as a function, but not as a variable, in fact, they were outright rejecting it as a variable. During their work in the second interview, they realized that P took on multiple values, even without changing t, arguably adding to their understanding of P in the differential equation. Namely, they realized that P' took on multiple values at the same t-value due to the changing values for P. That is, the students came to see the differential equation as a relationship between P', the variable t and as a result of their work on the initial value problem task, the variable P. As such, they began to substitute values into the differential equation for P.

Implications

This work opens the door for asking how one might create tasks designed to evoke the students' notions of rate, slope and derivative in ways that specifically promote coming to see a differential equation as a function of two variables. Investigating if students enrolled in other differential equations courses share the ideas expressed by these students is also warranted. One aspect of our findings that seems most interesting is that individually, the students' images of rate, slope and derivative (with the exception of only a few ideas) were by and large correct. It was only when the students' ideas come together at a single instance of time that confusion arose. Further, this confusion led toward progressing the students mathematical understanding.

Additionally, the findings suggest more attention is needed with regard to students' coordination of quantities in differential equations. A single ordinary differential equation relates at least three quantities: the function's independent variable, the function's value, and the function's rate of change. How students coordinate these three quantities when reasoning about differential equations and their solutions is a relatively open area of study, one that may illuminate the issues our students were having concerning the ambiguity of the quantities they associated with rate.

Lastly, our findings show that students can learn/construct mathematical ideas in rather unconventional or surprising ways. Though admittedly accidental on our part, the students' mathematical ideas about rate, slope and derivative were leveraged in a way that afforded and promoted positive mathematical growth concerning differential equations as a function. This begs the question, what other mathematical ideas could be developed utilizing students' notions of slope, rate and derivative?

References

Carlson, M. (1998). A cross-sectional investigation of the development of the function concept *CBMS Issues in Mathematics Education*, 7.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 352-378.

Donovan, John E. (2007)The importance of the concept of function for developing understanding of first-order differential equations in multiple representations. In: Hauk, Shandy (ed), *Electronic Proceedings for the Tenth Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education.*

Habre, S. (2000). Exploring students' strategies to solve ordinary differential equations in a reformed setting. *Journal of Mathematical Behavior, 18*(4), 455–472.

Lesh, R., & Lehrer, R. (2000). Iterative refinement cycles for videotape analyses of conceptual change. In R. Lesh & A. E. Kelly (Eds*.), Handbook of research design in mathematics and science education*, 665-708. Hillsdale, NJ: Erlbaum

Keene, K. A. (2007). A characterization of dynamic reasoning: Reasoning with time as parameter. *The Journal of Mathematical Behavior*, *26*(3), 230-246.

Keene, K. A (2008). Ways of reasoning: Two case studies in an inquiry-oriented differential equations class. In *Proceedings of the Thirtieth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*.

Kohler, W. E., & Johnson, L. W. (2003). *Elementary differential equations*. (2nd ed.) Boston, MA: Addison-Wesley.

President's Council of Advisors on Science and Technology. (2012). *Engage to excel*. Washington, DC: White House.

Rasmussen, C. (2001). New directions in differential equations: A framework for interpreting students' understandings and difficulties. *Journal of Mathematical Behavior, 20*, 55-87.

Rasmussen, C., & Blumenfeld, H. (2007). Reinventing solutions to systems of linear differential equations: A case of emergent models involving analytic expressions. *The Journal of Mathematical Behavior*, *26*(3), 195-210.
Rasmussen, C. L., & King, K. D. (2000). Locating starting points in differential equations: A realistic mathematics education approach. *International Journal of Mathematical Education in Science and Technology*, *31*(2), 161-172.
Rasmussen,C., & Whitehead, K. (2003). Undergraduate students' mental operations in systems of differential equations. In *Proceedings of the Twenty-Third Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*.
Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22, 1-36.
Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds*.),Handbook of Research Design in Mathematics and Science Education*, 267 - 307. Hillsdale, NJ: Erlbaum.
Stephan, M., & Rasmussen, C. (2002). Classroom mathematical practices in differential equations. *The Journal of Mathematical Behavior*, *21*(4), 459-490.
Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational studies in mathematics*, 12(2), 151-169.
Thompson, P. W. (1994). Images of rate and operational understanding of the Fundamental Theorem of Calculus. Educational Studies in Mathematics, 26(2-3), 229–274.
Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 65–81). Dordrecht: Kluwer Academic.
Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. *CBMS Issues in Mathematics Education*, *8*, 103-122.

Appendix A - Sample Protocol Questions

Interview 1

1. What does the following differential equation mean to you?

$$P' = 3P$$

a. What is P?

b. What is P'?

c. What is the relationship between them?

2. Suppose the above equation(from Question 1) can be used to model the fish population in the duck pond. How might it be used to determine the number of fish in the pond at a given time?

a. Are there graphical ways to solve this problem?
b. Are there analytical ways to solve this problem?

3. Now suppose the population in the year 2014 ($t = 0$), is 6. What will the population be in 2024?

a. How do you know that?

b. Can you represent this in a different way?

Interview 2

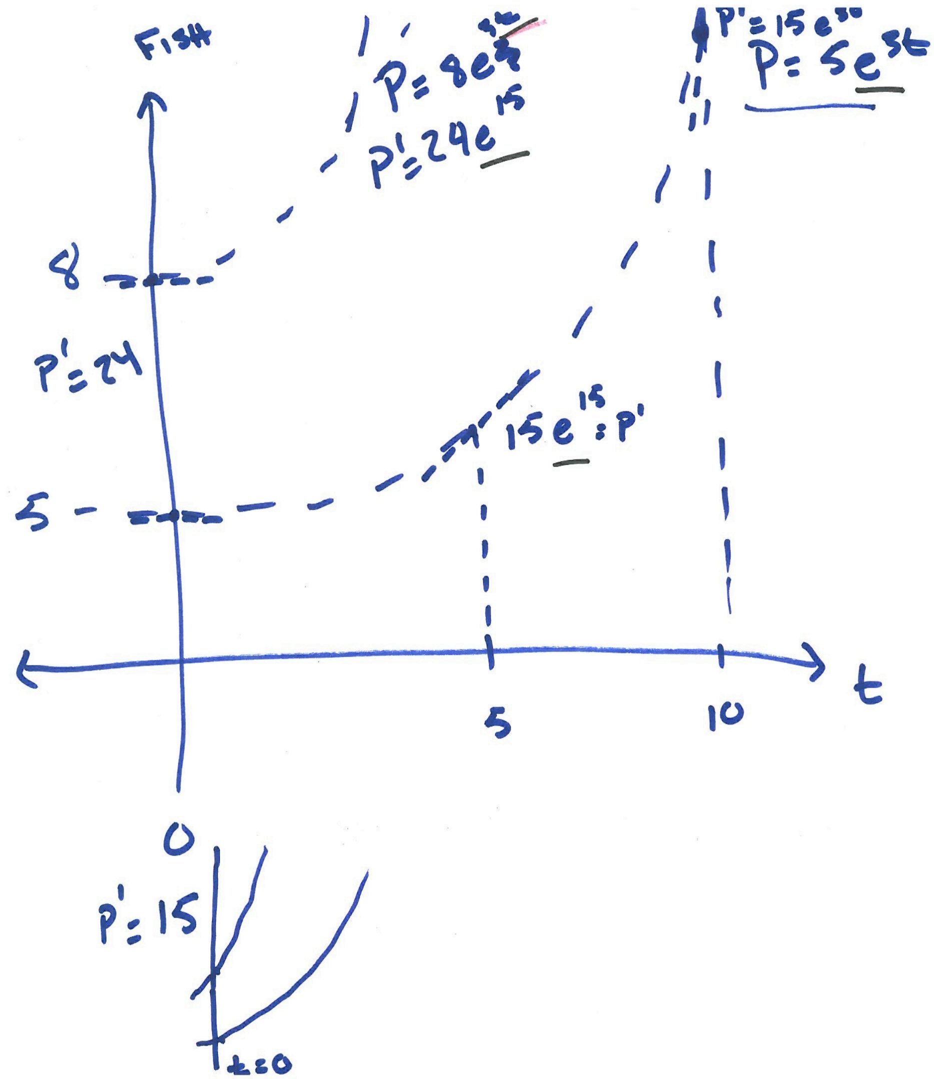
1. Recall the fish population problem from last time, $P' = 3P$, where the population in 2014 ($t=0$) was given as $P(0) = 6$. Now suppose that $P(0) = 10$.

   a. What does this change mean to you?
   b. Given the new conditions what will the population be in 2024?
   c. What if the initial condition changed to $P(0) = 30$, how would the solution change?
   d. Does every initial condition provide a distinct solution function?

2. Suppose there are four identical ponds and that the fish population in each ponds can be modeled by the equation $P'=3P$. Further, the initial populations for each of the respective ponds are 5, 6, 7, and 8. Find the equation that models the population for each pond.

   a. What does the differential equation tell you about the fish population in each of the ponds?
   b. What is the rate of change of the each of the populations at $t=5$?
   c. What are the slopes of the lines tangent to each of the solution curves?
   d. How is the slope different from the rate?

Appendix B - Alice's Work for the Initial Value Problem



Unconventional uses of mathematical language in undergraduate proof writing

Kristen Lew & Juan Pablo Mejía-Ramos
Rutgers University

Despite its perceived importance in the learning of mathematics, there is a dearth of research on students' use of mathematical language, particularly when writing proofs at the undergraduate level. In this exploratory study, we analyze written student exams from an introductory proof course and identify fourteen categories of unconventional uses of mathematical language found in undergraduate student generated proofs.

Key words: Proof writing, undergraduate mathematics, mathematical language

Proofs are an essential type of communication in professional mathematical practice. In an influential paper on the role of proof in mathematics, Rav (1999) wrote that proofs "are the heart of mathematics" that play an "intricate role […] in generating mathematical knowledge and understanding" (p.6). Thus, one of the primary goals of mathematics instruction at the advanced undergraduate level is fostering students' abilities to understand and construct valid proofs. However, research has shown evidence of undergraduate students' difficulties when reading and constructing proofs (e.g. Selden & Selden, 1987; Bills & Tall, 1998, Weber, 2003). In particular, Moore (1994) suggested that one of students' difficulties when constructing proofs is their inability to understand and use mathematical language and notation.

Literature Review

Mathematical language has been studied and interpreted in a variety of ways. For example, Kane (1968) viewed mathematical language as a combination of natural language and a system of mathematical symbols, whereas Ervynck (1968) viewed mathematical language as a foreign language composed almost entirely of technical symbolic representations. Further, Pimm (1987) described mathematical language as a set of meanings that is created and expanded with the formation of new terminology and the designation of technical definitions to words in natural language. One constant of these varied perspectives is that each notes significant differences between mathematical and natural language. For example, Veel (1999) discussed the precision necessary when implementing certain verb phrases in mathematical language (e.g. the verbs 'to be' and 'to have' are used in different relational processes in mathematics than in natural English) and Halliday (1978) noted the high degree of nominalization in mathematical language, in which a mathematical action or phenomenon becomes an object (e.g., the action of differentiating becomes differentiation). As these aspects of precision and rigor in mathematical writing may cause difficulties for students, a number of mathematics educators have suggested ways to improve students' use of mathematical language at the K-12 level (e.g. Veel, 1999; Moschkovich, 1999; Lemke, 2003).

At the university level, a number of mathematicians have described how to properly and effectively use written mathematical language in professional settings, such as when writing a mathematics paper, an expository article, or a mathematics textbook (AMS, 1962; Halmos, 1970; Gillman, 1987; Krantz, 1997; Higham, 1998). Other authors have written similar guides for undergraduate mathematics students, containing specific advise on mathematical writing at the undergraduate level (Houston, 2009; Alcock, 2013; Vivaldi, 2014). However, the advise on these guides is not informed on a systematic study of how mathematicians and undergraduate students understand the language of mathematics, but on these authors' own personal and professional experiences. For instance, in his influential essay on how to write mathematics, Halmos (1970) explained:

> The recommendations I have been making are based partly on what I do, more on what I regret not having done, and most on what I wish others had done for me. […] Do, please, as I say, and not as I do, and you'll do better. Then rewrite this essay and tell the next generation how to do better still. (p. 152)

Indeed, empirical research on how mathematicians and undergraduate students understand mathematical language is lacking. In particular, we only know of two studies (Konior, 1993; Burton & Morgan, 2000) that have explicitly and empirically investigated the language of mathematical proof writing, and neither study investigates how undergraduate students understand such language. Based on their teaching experience, Selden and Selden (1987) identified 17 types of reasoning errors undergraduate students made when writing proofs. One of these errors was related to the use of mathematical language and was described as a tendency to *overextend symbols*, where "one symbol is used for two distinct things" (p. 464). In this paper, we take a step towards better understanding mathematical proof writing at the undergraduate level by identifying a wider range of students' unconventional uses of mathematical language.

Furthermore, we believe that one possible reason for this gap in the research literature is the lack of an accessible point of entry for researchers interested in the language of mathematical proof writing at the undergraduate level. That is, as the learning and usage of this particular type of language is currently under theorized, it is difficult to systematically study it in an empirical manner. In this paper we describe a theoretical perspective that we believe could be useful to investigate the language of mathematical proof writing at the undergraduate level. In particular, we illustrate how this theoretical perspective helps us to understand undergraduate students' unconventional uses of mathematical language when writing proofs.

Theoretical Perspective

Academic English is a "register of English used in professional books and characterized by the specific linguistic features associated with academic disciplines" (Scarcella, 2003, p. 9). The conceptual framework of academic English, as developed by Scarcella (2003), describes the various components of academic English and how one might become literate in this register. In particular, Scarcella noted that sub-registers of academic English exist for different disciplines and that "one must 'do' discipline-specific work with academic and discipline-specific language" (p. 9). We consider mathematical proof writing to be a particular sub-register of academic English. Indeed, proofs are written in English[1], but in a register with features (such particular lexica and sentence structures) that are specific to the academic discipline of mathematics. Scarcella's (2003) framework has previously been applied to mathematics education, but only in studies of English language learners at the elementary school level (e.g. Silva et al., 2008; Heller, in press). We propose to apply this framework to investigate aspects of the mathematics sub-register of academic English, particularly of the language of proof writing at the undergraduate level.

Scarcella's framework is an expansion of Kern's (2000) model of literacy, originally developed for the study of foreign language instruction, which includes three dimensions of literacy: linguistic, cognitive, and sociocultural/psychological. Scarcella's (2003) framework includes these same three dimensions and further specifies various components within the

[1] Clearly mathematical proofs are also written in other languages. Nonetheless, we argue that proofs are written in a specific register of a natural, or common, language.

linguistic and cognitive dimensions. Scarcella suggested that learners must develop proficiency in diverse linguistic components in order to become literate in academic English. In particular, the linguistic dimension of academic English has phonological, lexical, grammatical, sociolinguistic, and discourse components. The cognitive dimension is comprised of knowledge, higher-order thinking, strategic, and metalinguistic awareness components. We will briefly describe the components of academic English and provide examples of how the components can be applied specifically to the academic language of mathematics.

The Linguistic Dimension

The *phonological* component "includ[es] stress, intonation, and sound patterns" (p. 11) and "knowledge of graphemes (symbols) and arbitrary sound-symbol correspondences" (p. 13). Given how much of mathematical language relies on the use of domain specific symbolism and notation, knowing these graphemes and having the ability to accurately pronounce them is particularly important in mathematical practice. Furthermore, as mathematical symbols carry precise, sometimes detailed meaning, proper use is particularly important to assure the correct meaning of a grapheme is communicated. Accordingly, Higham (1998), Krantz (1997), Houston (2009), Alcock (2013), and Vivaldi (2014) all suggested revising proof writing by reading the proofs aloud to identify misuses of language. Moreover, knowing these sound-symbol correspondences is vital for understanding advanced mathematics lectures and instruction.

The *lexical* component requires knowledge of the words used in a field. In particular, Scarcella (2003) distinguished between general words used in everyday language, academic words used across academic fields, and technical words that are field-dependent. She noted that knowledge of fixed expressions, which are "expressions that tend to stick together and cannot be changed in any way" (p. 14), is also part of the lexical component. In mathematics, academic words such as *therefore, thus, since,* and *hence* are often used to connect arguments of a proof. Further, words like *if* and *or* have mathematical meanings that differ from those in other English discourses (both common and academic), which must be understood by learners of mathematical language. The knowledge of many technical words (e.g. coprime, monic, Euclidean, abelian, and discriminant) and technical fixed expressions (e.g. relatively prime, absolutely convergent, continuously differentiable, Cartesian product, and Jordan block) is also necessary for successful communication. Moreover, as discussed above, knowledge of the various symbols and notation commonly used in practice is crucial for those who wish to use mathematical language. This knowledge would include understanding the meanings of the symbols what are the appropriate symbols to use in a given proof.

The *grammatical* component of academic English entails knowledge of "the grammatical co-occurrence relations that govern the use of nouns" (Scarcella, 2003, p.15). For instance, Scarcella noted students need to learn the associated grammatical features for these technical words, "certain nouns […] are generally followed by prepositional phrases" and that some "verb + preposition combinations […] cannot be changed" (p. 16). This is particularly common in mathematics where objects are often described in relation to other objects, e.g. a function is *continuous at a point*, an integer is *relatively prime to another integer*, a group is *isomorphic to another group*, and a function *maps from one space to another*. Omitting the appropriate prepositional phrases leads to semantic ambiguity, an undesirable trait of mathematical language. To this point, Scarcella (2003) also indicated that the reference system of academic English insists that students should not use "pronouns[,] such as it[,] that do not have identifiable noun referents" (p.16), which is consistent with Houston's (2009), Alcock's (2013), and Vivaldi's (2014) recommendations that students avoid using unclear referents in mathematical writing. Gillman (1987), Krantz (1997), Higham (1998), Houston

(2009), and Vivaldi (2014) all also noted the importance of using correct grammar in the language of mathematics, including using correct grammar with symbols and mathematical expressions.

The *sociolinguistic* component involves developing competence in a variety of functions of language, including an understanding of the appropriateness of a given sentence in a particular context. Scarcella (2003) noted that “signaling cause and effect, hypothesizing, generalizing, comparing, contrasting, explaining, describing, defining, justifying, giving examples, sequencing, and evaluating” (p. 18) are all examples of academic language functions. In mathematics, some important language functions would include proving, contradicting, generalizing, defining, justifying, and evaluating. Moreover, a student of mathematics would need to learn what uses of mathematical language are appropriate in what contexts. For example, the mathematical writing demonstrated by a mathematician in class may include abbreviations and shorthand notations that are not generally accepted in formal writing. Further, the language used by mathematicians to explain mathematical phenomena may be more informal than the language used in a written proof. Students must recognize these distinctions in order to be proficient in the sociolinguistic component of mathematical language.

The *discursive* component entails understanding and using linguistic forms necessary to communicate successfully and coherently. For instance, in every day language, greetings and parting phrases indicate to speakers the beginning and end of conversations. Scarcella (2003) noted academic English “includes specific introductory features and other organizational signals” (p. 19) and that “writers’ presentation of ideas must be orderly and convey a sense of direction” (p. 19). In mathematical proof writing, this relates to Konior’s (1993) finding that professionally written proofs share a common construction using organizational signals and frames to guide readers through the proof, as well as mathematicians’ suggestions to make the structure of a proof clear in mathematical writing (AMS, 1962; Halmos, 1970; Gillman, 1987; Higham, 1998; Houston, 2009; Vivaldi, 2014).

The Cognitive Dimension

In order to become literate in an academic language, Scarcella (2003) argued learners “must obtain factual information as well as what is often called *critical literacy*, the ability to read for intentions, to question sources, and to identify others’ and one’s own assumptions” (p. 22). In other words, academic English requires cognitive components in addition to the linguistic components described above.

The *knowledge* component entails accumulating facts, ideas, and definitions that aid students’ composition of texts. In particular, Scarcella (2003) referred to schema theory to explain that a student’s prior experience and knowledge affects his or her proficiency at language. Scarcella (2003) argued, since in schema theory the “comprehension process involves, among other things, assimilation of new knowledge into existing schemata and accommodation of existing schemata to fit new knowledge” (p. 23), so background knowledge is necessary for students to comprehend language. Since mathematical knowledge is of a particularly cumulative nature (Robert & Scharzenberger, 1991; Veel, 1999), it is natural that background knowledge is critically important to acquiring proficiency in mathematical language and proof writing.

The *higher-order thinking* component involves the critical thinking required to speak persuasively and write coherently. With respect to academic writing, this component includes the abilities to “determine which ideas are relevant to their texts, […] support thesis statements, remain focused on these statements, and frequently refer back to them” (p.23-24). In addition, Scarcella (2003) noted readers must be able to “interpret what the readings state and what they do not state” (p. 23) and writers must determine what ideas are relevant and how they contribute to the thesis of a text. In the mathematics education literature, Alcock and Weber (2005) found that undergraduate students failed to notice missing warrants when reading a mathematical proof. Moreover, the mathematical writing guides by Halmos (1970), Gillman (1987), Krantz (1997), and Higham (1998) all discussed the importance of excluding irrelevant information, yet choosing necessary arguments to conclude one's proofs – emphasizing the necessity of brevity and conciseness when writing proofs.

The *strategic* component of academic literacy involves "knowledge of cognitive and metacognitive strategies for reading, writing, speaking, and listening" (Scarcella, 2003, p. 24). Examples of such strategies when writing in academic English are "brainstorming, comparing their experiences with the texts they are writing about, giving examples, citing experiences, and providing evidence form other texts on the subject" (p. 25). In the realm of mathematical proof writing, Weber (2001) has suggested the importance of strategic knowledge when constructing proofs and Weber, Brophy, and Lin (2008) have identified some reading strategies used by successful mathematics students including: rephrasing part of the argument, justifying a step within a proof, anticipating arguments while reading the proof, and re-reading the proof.

The *metalinguistic* awareness component involves the capacity to reflect and think about the language one uses. With respect to academic English, Scarcella (2003) emphasized the importance of revising and editing in communicating successfully in academic situations. In particular, Scarcella suggested "identifying content that should be addressed or removed from a text, and moving text to make it more effective" (p. 25). The topics of revising, editing, and proofreading (and their importance) were addressed in some way in each of the mathematical guides discussed in the literature review (AMS, 1962; Halmos, 1970; Gillman, 1987; Krantz, 1997; Higham, 1998; Houston, 2009; Alcock, 2013; Vivaldi, 2014).

The Sociocultural/Psychological Dimension

Finally, the sociocultural/psychological dimension of academic English emphasizes the fact that academic English "arises not just from knowledge of the linguistic code and cognition, but also from social practices in which academic English is used to accomplish communicative goals" (p. 29). In this way, attributes such as cultural norms, beliefs, motivations, values, and practices all constitute the sociocultural/psychological dimension of academic language. Merely knowing the linguistic code and meeting the cognitive demands of learning academic English is not enough for a student to be literate in academic English. There is little point in learning a language that one does not know how to use to successfully communicate with others in practice.

Scarcella (2003) argued that competence in each of these three dimensions (linguistic, cognitive, and sociocultural/psychological) contributes to a learner's proficiency in academic English. Some of the existing literature concerning mathematical proof construction can be tied to the cognitive dimension of this particular type of academic language. For example, Weber (2001) found that undergraduate students lacked background knowledge, made irrelevant inferences when constructing proofs, and lacked strategic knowledge needed for constructing proofs. Each of these findings relates to a component of the cognitive dimension of academic language discussed above. However, a discussion of the linguistic dimension of learning the language of mathematical proof writing seems to have been left out of the conversation in the mathematics education research literature. In one exception, Selden and Selden (2003) made a brief comment that stylistic clarity is a valued norm in mathematical practice. Since the authors specifically note an instance of an undergraduate student participant's confusion experienced due to an inappropriate use of one variable being used in multiple ways, one may infer that these stylistic norms are related to the linguistic dimension of mathematical language. Expanding on this, Selden and Selden (2003) suggested, "it would be interesting to investigate how such norms are maintained and, in particular, adopted by novices in a wider mathematical community" (p. 21).

We are however, unaware of any studies of the type suggested by Selden and Selden, which consider the use of language in proof writing at the undergraduate level. As the language 'norms' mentioned by Selden and Selden (2003) have not be studied, we first to aim to identify the ways in which undergraduate students use mathematical language in what could be considered non-standard ways. As a result, in this study we investigated the following research question: What are some of the unconventional ways in which undergraduate students use mathematical language when writing proofs?

Methods

For this study, we examined written student exams that were collected from four different instructors of an introductory proof course at a public, research university in the US. It is noteworthy that due to the exploratory nature of this study, these exams were collected through a convenience sampling technique and therefore we make no claims of the representativeness of our findings in a wider population of undergraduate students. An introductory proof course is an undergraduate-level mathematics course that intends to introduce mathematics students to various types of mathematical proof and reasoning. Such a course is often a student's first exposure to formal mathematical proofs and a prerequisite to other proof-based mathematics courses. The exam tasks involving student proof writing were analyzed for use of mathematical language that we believed to be unconventional according to our own perceptions of the way that mathematicians communicate in formal writing settings.

In this exploratory study, data were analyzed using open coding in the style of Strauss and Corbin (1998). Fourteen categories of unconventional uses of mathematical language emerged from the data. Once we had refined descriptions for each one of these fourteen categories, we moved into a second stage of analysis in which we considered how the linguistic dimensions of Scarcella's conceptual framework for academic English related to each one of those categories.

Results

We found that the fourteen categories of unconventional use of mathematical language in undergraduate proof writing concerned students' difficulties constructing complete and unambiguous mathematical sentences, making clear the flow of the argument, introducing variables, and using mathematical symbols. Furthermore, we found that each of these categories can be mapped to one or more linguistic components as described in Scarcella's (2003) conceptual framework for academic English.

Constructing Complete and Unambiguous Mathematical Sentences

Four of the emerging categories of unconventional uses of mathematical language were related to the construction of complete and unambiguous sentences in mathematical proof writing. These four categories are: 1) lacks proper grammar and punctuation, 2) uses non-statements, 3) uses unclear referents, and 4) uses lay speak.

First, Scarcella argued that mathematical English is a subset of academic English. Since proof writing is still writing in the English language, a writer of a proof should use capital letters to begin a sentence and use proper punctuation to separate clauses and end the sentence. As such, a proof that *lacks proper grammar and punctuation* is an example of an unconventional use of mathematical language. For instance, one student wrote the passage shown in Figure 1. Based on this mathematical writing, it is unclear where the sentences begin and end. In particular, this category is related the grammatical component of mathematical proof writing, as complete sentences should be used throughout a proof.

> Because $g \circ f$ is a bijection $\exists z, w \in A$ such that $g \circ fz = d$ and $g \circ fw = g \circ fz$ and $w = z$ since f and $g \circ f$ are bijections then $x = y = w = z$ $g \circ fx = g \circ fy = ga = g(b)$

Figure 1. Mathematical writing that lacks proper grammar and punctuation.

Non-statements are a collection of words and symbols in mathematical language, which lack meaning and are not full sentences. Thus *using non-statements* entails that a student writes phrases, which are not complete sentences. Specifically, words in mathematical English such as "suppose", "let", and "assume" should precede both an object and a statement about the object. For example, "Suppose *n* is an integer" both introduces the variable *n* and describes the variable *n* as an integer, providing further information describing n. However, when one student wrote the non-statement "Suppose $(S \circ R)-1$.", this student has not supposed anything about the relation $(S \circ R)-1$. In English this would be akin to saying "Suppose a cat" – this statement says nothing about the cat, and a listener would likely ask "Suppose what about a cat?" Such non-statements are not grammatical sentences and are thus related to the grammatical component of mathematical proof writing.

Using unclear referents indicates that a student has used a pronoun such as "this", "these", "it", or "they" without specifying the necessary antecedent or antecedents. Such word use is unconventional as it introduces ambiguity into a written proof. For example, when one student wrote, "Thus, it is injective" without any explicit discussion of a function, or if there are multiple functions present within the proof, a reader would be unsure what or which object is injective. We note that using unclear referents in mathematical proof writing is related to the grammatical component of Scarcella's framework of academic English. While the inclusion of antecedents to avoid unclear references is suggested in English grammar, we stress that it is particularly important to mathematical proof writing – where precision and rigor are key.

A proof written *using lay speak*, or common language, means that the proof is written using informal and non-mathematical words. Such a proof can indicate a student's unfamiliarity with both the mathematical language and content. Moreover, using a non-standard vernacular for mathematical proofs can introduce confusion and ambiguity. For example, in proving that the sets $A0=\{0\}$, $A+o=\{1,3,5,\ldots\}$, $A+e=\{2,4,6,\ldots\}$, $A-o=\{-1,-3,-5,\ldots\}$, and $A-e=\{-2,-4,-6,\ldots\}$ form a partition of $\mathbb{Z}$, one student wrote the passage in Figure 2. The phrases "positive sets", "negative sets", "the evens", "the odds", and "share nothing" are too imprecise to prove that $A0$, $A+o$, $A+e$, $A-o$, and $A-e$ are pairwise disjoint; we link using lay speak in mathematical proofs to the lexical component of academic English. It is noteworthy that mathematicians use informal language when writing a proof sketch or a back-of-the-napkin proof. However using lay speak is unconventional in complete written proofs, so a writer must be cognizant of this distinction. Thus, we also note that this category of unconventional use of mathematical language is related to the sociolinguistic component of academic English.

> All are pairwise disjoint, since the positive sets share nothing the with negative sets and the evens share nothing with the odds and $\{0\}$

share nothing with the rest.

Figure 2. Mathematical writing that uses lay speak.

Making the Flow of the Argument Clear

Four of the emerging categories of unconventional uses of mathematical language were related to making the flow of the argument clear. These four categories are 1) lacks verbal connectives, 2) fails to state assumptions of hypothesis, 3) fails to make explicit the structure of the proof, and 4) includes statements of definitions.

A proof *lacks verbal connectives* when the author does not sufficiently use English words, in particular conjunctive adverbs (such as ‘therefore’, ‘since’, ‘so’, ‘thus’, ‘then’, and ‘hence’), to connect arguments and sentences in a proof. Without such words, a reader would be unsure whether a statement is being assumed or being concluded from previous assertions. Moreover, it would be unclear where the hypothesis of a statement ends and where the conclusion ends. The following work is an example of a student’s proof that lacks verbal connectives:

let $x, z \in A$ s.t.
$x,z \in R$
$R=S \circ S-1$
$x,z \in S \circ S-1$
$\exists y$ s.t. $x,y \in S-1$ and $y,z \in S$
$y,z \in S \therefore z,y \in S-1$
$z,y \in S-1 \therefore y,z \in S$

Figure 3. Mathematical writing that lacks verbal connectives.

Such a proof lacking verbal connectives would show the author’s lack of proficiency in both the discursive and sociological components of Scarcella’s (2003) framework. Since the proof lacked connecting words to indicate the direction of ideas or the structure of the argument, this proof indicates the author’s lack of proficiency in the discursive component. Moreover, constructing a proof lacking verbal connectives indicates that the author of the text may not be aware of the appropriate format of complete proofs. While a proof similar to the work shown in Figure 1 may be appropriate as a proof sketch, it would likely be inappropriate to write a complete proof in this way. Thus, writing proofs that lack verbal connectives would indicate that the author does not know in what context this type of presentation is appropriate and thus lacks proficiency in the sociolinguistic component.

A proof that *fails to state assumptions of hypothesis* does not clearly state what is being assumed in each part of the proof. For example, in a proof of the statement “Let R and S be relations on a set A. Prove: $S \circ R-1=R-1 \circ S-1$.”, one student began a proof as shown in Figure 4:

Suppose $S \circ R-1$ s.t. $x,z \in S \circ R-1$.
Since $x,z \in (S \circ R)-1$, then $z,x \in (S \circ R)$.
Since $z,x \in (S \circ R)$, then $y,x \in S$ and $z,y \in R$.

Figure 4. Mathematical writing that fails to state assumptions of the hypothesis.

Besides beginning the proof with the non-statement discussed above, the student did not make it clear to the reader what he or she was assuming about the relations R and S (i.e. that they were both relations on a given set A). Further, a proof of a claim of set equality, like the statement above, usually requires two steps, showing that one set is a subset of the other and vice-versa. However, based on the writing, the structure of argument of the proof is not made clear to the reader, so Figure 4 is also an example of a proof where the writer *needs to make structure the proof explicit.* Both of these unconventional uses of mathematical writing make reading and understanding the proof more difficult for the reader, as failing to state the assumptions of the hypothesis and failing to make explicit the structure of the proof hide the flow of the logic of the proof. As a result, these unconventional uses of mathematical writing are tied to the discursive component of Scarcella's (2003) framework for academic English.

Finally, *including stating of definitions*—that is, including entire statements of definitions within a proof—is another category of an unconventional use of mathematical language. Including lengthy definition statements can disrupt the flow of the proof and distract readers from the argument at hand. For example, in another student's proof of the statement "Let R and S be relations on a set A. Prove: $S\circ R-1=R-1\circ S-1$.", the student included these following statements as shown in Figure 5.

So $S\circ R = a,c \in A\times C \;\; \exists b\in B$ s.t. aRb and $bSc\}$
and $S\circ R-1 = c,a\in C\times A \;\; \exists b\in B$ s.t. cSb and $bRa\}$.

Figure 5. Mathematical writing that includes statements of definitions.

These statements of definitions may be superfluous and distract the reader from the argument of the proof. Naturally, keeping in mind the statement of a related definition may be crucial when constructing the argument for a proof; however, some mathematicians (e.g. Selden & Selden, 2004) disagree that the definition statement should be included in a written proof. Thus, a writer should be aware of the situations in which including an entire statement of a proof is or is not appropriate. That is to say, stating definitions in a proof shows that the writer may lack proficiency in the sociolinguistic component of the linguistic dimension of Scarcella's framework. While we agree that this category is likely to be controversial due to the importance of applying definitions when constructing proofs we believe that it is worthwhile to investigate the extent to which students include entire statements of definitions in their written proofs and the extent to which mathematicians agree or disagree that this is indeed an unconventional use of mathematical language.

Introducing Variables in Mathematical Proof Writing

Four categories of students' unconventional uses of mathematical language emerged that were related to the introduction of variables in mathematical proof writing. These categories are 1) overuses variable names, 2) uses unspecified variables, 3) uses the universal quantifier instead of "let", and 4) uses overquantified variables.

A proof that *overuses variable names* uses a single variable name to represent more than one value or object. For example, one student wrote "Suppose a divides b and a divides c, then $\exists k\in\mathbb{Z}$ such that $b=ak$ and $\exists k\in\mathbb{Z}$ such that $c=ak$", then the variable name k has been overused. In particular, as it stands this statement assumes that b and c are the same integer. Thus, it is necessary for the student to introduce separate integers as the divisors of b and c. This category of unconventional use of mathematical language can be mapped to both the lexical and sociolinguistic components. This unconventional use is tied to the lexical component as it shows that the student does not recognize how to choose and subsequently use the variables in an unambiguous manner. Further, overusing variable names is tied to the sociolinguistic component of mathematical language because one must be able to differentiate between situations in which repeating variables names may be acceptable and when it is unacceptable. For example, it may be appropriate to repeat the use of variable names in different sections of the same proof–for instance in a proof by cases, a variable may be used differently in the separate cases. It is also reasonable that a proof sketch may use the same variables for different purposes, which would be an acceptable use of variables as these sketches are not for public consumption. Thus, if a student overuses variable names within a proof, this suggests that the student may not understand in which contexts these uses of variable names are appropriate and may lack proficiency in the sociolinguistic component.

Using a universal quantifier instead of "let" suggests that a student has made an argument for every element of a set, rather than an arbitrary element of that set. Many proofs require manipulation of an arbitrary representative element of a set to show some property holds for the entire set; such proofs begin with a statement such as "Let $a \in \mathbb{N}$." stating that a is in a particular set of numbers. However, some students instead stated " $\forall a \in \mathbb{N}$.", which can be read "for all a in N". These two statements "Let $a \in \mathbb{N}$. Then for $b \in \mathbb{N}$, $a \sim b$." and " $\forall a \in \mathbb{N}$. Then for $b \in \mathbb{N}$, $a \sim b$." are semantically different based on the word/symbol choice between the word 'let' and ' $\forall$'. Using a universal quantifier instead of "let" or in conjunction with the word "let" when introducing a variable suggests that the writer lacks a proficiency in the lexical component of mathematical proof writing. While the phrase " $\forall a \in \mathbb{N}$" is not uncommon in mathematical notation, it is a prepositional phrase and thus is not a grammatical, complete sentence without a subject and a predicate. In this way, this unconventional use of mathematical language can be a particular case of a non-statement and is also related to the grammatical component of Scarcella's (2003) framework.

In a similar vein, using *overquantified variables* indicates that a student uses both mathematical quantifiers and words such as "let" or "suppose" immediately preceding them. For example, one student wrote "Let $\forall a,b,c \in \mathbb{Z}$.", we claim that this is an unconventional use of mathematical language as the variables are introduced both with the word "let" and with the universal quantifier. As this statement is not an appropriate use of the universal quantifier and is not a complete sentence, such a use of overquantified variables suggests that the writer lacks a proficiency in both the lexical and grammatical components of mathematical language.

Unspecified variables are variables used in a proof, but they have not been designated the set to which the variable belongs. Failing to do so can introduce ambiguity to a proof. For example, one student wrote "Let $x=ab$ " but fails to indicate what set a and b belong to, x could be a rational or irrational number. We note that this category can also include incomplete specifications of the set to which a variable belongs; for example, when a variable is simply introduced as an integer, whereas it necessarily should be introduced as an integer greater than 3. Using unspecified variables suggests that a student lacks proficiency in the grammatical component of the mathematical register. Using a variable without specifying the set, or without sufficiently specifying the set, to which it belongs is comparable to using a pronoun without a clear referent. While the use of the variables may not be incorrect lexically, their use with respect to the words and sentences surrounding them creates confusion.

Using Mathematical Symbols in Mathematical Proof Writing

Two categories of students' unconventional uses of mathematical language emerged that were related to using mathematical symbols in mathematical proof writing. These categories are 1) using formal propositional logic and 2) mixing mathematical notation and text.

Using formal propositional logic is another category of unconventional use of mathematical language. For example, in a proof of the task, "Let $x \in Q$. Let $y \in \mathbb{R}$. Show that $x+y \in \mathbb{Q}$ if and only if $y \in \mathbb{Q}$" one student wrote "So $\exists a \in \mathbb{Z} \wedge \exists b \in \mathbb{N}(ab=x+y)$." In this example, we claim that the student's focus on symbolic notation and use of the logical conjunction, $\wedge$, is unconventional of mathematical proof writing. While propositional logic certainly is mathematical, using phrases of propositional logic is non-standard for written proofs in most areas of mathematics[2]. Phrases of propositional logic are commonly used in early sessions of introduction to proof courses; however, they are not used in typical written proofs. Thus, a writer needs to recognize the situations in which using formal propositional logic is and is not appropriate. As such, this unconventional use of mathematical language is tied to the sociolinguistic component of mathematical language.

Categories	**Description of the Category**	**Linguistic Component**
Lacks proper grammar and punctuation	Student does not attend to general rules of English grammar and punctuation – which is important not only in English but also in mathematical language.	Grammatical
Uses non-statements	Student uses a collection of words and symbols that are not full sentences.	Grammatical
Uses unclear referents	Student uses "this", "these", "it", "they" and other pronouns without specifying to what they are referring.	Grammatical
Uses lay speak	Student uses informal/non-mathematical words in sentences within a proof.	Lexical and Sociolinguistic
Lacks verbal connectives	Student does not use verbal conjunctions, focusing on symbolic representation instead.	Discursive and Sociolinguistic
Fails to state assumptions of hypothesis	Student does not explicitly state what is being assumed about the mathematical objects in the proof.	Discursive
Fails to make explicit the structure the proof	Student does not indicate the general structure of the proof – such as introductory/closing statements or comments on the status of an argument.	Discursive
Includes statements of definitions	Student provides entire statements of definitions within a proof.	Sociolinguistic
Overuses variable names	Student uses the same variable name to represent different values.	Lexical and Sociolinguistic
Uses universal quantifier instead of "let"	Student uses the universal quantifier when selecting an arbitrary element of a set.	Lexical and Grammatical
Uses overquantified variables	Student uses both mathematical quantifiers and words such as "let" or "suppose" to introduce	Lexical and Grammatical

[2] Naturally, one exception is the field of logic.

	variables.	
Uses unspecified variables	Student uses variables without first specifying the set to which they belong.	Grammatical
Uses formal propositional logic	Student uses phrases of propositional logic in a proof – for example, using logical symbols.	Sociolinguistic
Mixes mathematical notation and text	Student uses mathematical symbols in prose inappropriately.	Lexical and Sociolinguistic

Table 1. Categories of unconventional use of mathematical language.

Mixing mathematical notation and text indicates that a student has inappropriately used mathematical symbols or notation within prose. For example, students wrote proofs including included statements "So there are 19 possible differences that are ≥ -9 and ≤ 9" or "since $X \subseteq A$ and $\subseteq B$". We claim that this is an unconventional use of mathematical language since binary operators such as ≥ and ⊆ require notation on both the left and right sides of the symbols. As knowledge of how mathematical operators function within language is related to the lexical component, we contend that a proof including these types of unconventional uses indicates the writer's lack of proficiency of the lexical component of the language of mathematical proof writing. One may note however, that it is not uncommon for mathematicians to use these symbols as shown in the examples of unconventional use when writing in informal settings. That is to say, some mathematicians may use binary operators in this way as a short hand when writing notes or even on the board in class to save time. Thus a student mixing mathematical notation and text in a written proof would indicate the student's lack of proficiency in the sociolinguistic component since the student may not be aware that these 'short hand' uses of notation are inappropriate in formal settings.

These fourteen categories describe different ways that undergraduate use mathematical language in unconventional ways when writing proofs. The categories, brief descriptions of each, and their relations to the linguistic components of academic English are presented in Table 1.

Discussion

The language of mathematical proof writing is a particularly rigorous subset of academic English. As such, we believe an undergraduate mathematics student's introduction to this mathematical language is somewhat similar to an English language learner's introduction to academic English. In this paper, we described fourteen unconventional uses of mathematical writing found in undergraduate student-generated proofs and related these unconventional uses to the lexical, grammatical, sociolinguistic, and discursive components of the linguistic dimension of academic English. Due to the exploratory nature of this study, we make no claims of the generality or the exhaustive nature of these categories of unconventional uses of mathematical language.

We have, however, found these components to be useful for understanding how the unconventional uses of mathematical language stray from what we find to be standard mathematical language at the advanced undergraduate level. Understanding how the different components of the linguistic dimension of academic English relate to these unconventional uses can help us to investigate how students understand and struggle with the linguistic aspects of mathematical language. In particular, understanding how students navigate these linguistic components can lead to improved teaching and learning of how to write proofs at the undergraduate level. By considering the related linguistic components of the types of unconventional uses of mathematical language, we can begin to understand some ways that undergraduate students may be struggling with mathematical proof writing.

For instance, the application of Scarcella's (2003) framework can help educators to improve upon instruction of mathematical language, since instruction to address a lack of competence in one linguistic component may differ significantly from instruction to address a lack of competence in another component. For example, teaching a student to correct the use of non-statements, an unconventional use of mathematical language that is related to the grammatical component of proof writing, might entail instructing the student about the necessary parts of a sentence, about the use of mathematical words such as "suppose", "let", and "assume", and why non-statements are ungrammatical and lack mathematical meaning. On the other hand, proofs and statements that are examples of unconventional uses of mathematical language that are related to the sociolinguistic component of proof writing are not necessarily incorrect. For instance, the example of a student using formal propositional logic described above makes perfect mathematical sense. Thus, teaching a student to correct unconventional uses of mathematical language related to the sociolinguistic component is not a matter of instructing the student how to write mathematical statements, it is a matter of teaching students to be aware of the particular situation and context in which they are working. This could involve providing examples of situations in which these uses are appropriate and inappropriate, and pointing out why this varies depending on the particular situation.

We agree with Scarcella (2003) that instructors should provide direct instruction to their students, including calling explicit attention to form and giving instructional feedback concerning their use of academic English. While introduction to proof courses intend to introduce mathematics students to various types of mathematical proof and reasoning, it may be the case that these courses should also include more explicit discussion of the linguistic components of mathematical language.

This application of Scarcella's (2003) conceptual framework for academic English to the investigation of mathematical proof writing opens the doors to new avenues of research in an area that is both important and under-researched in mathematics education. In particular, we claim that having a clearer view of the language of mathematical proof writing will enable researchers to understand students' difficulties with reading and writing mathematics. Further our discussion of unconventional uses of mathematical language in undergraduate proof writing lead to some interesting, future research questions: How do mathematicians view and describe the linguistic norms of mathematical proof writing at the undergraduate level? How do undergraduate mathematics students understand these norms? To what extend do students' perceived views of these norms align with those of the professional mathematical community? How do students' understandings of these norms develop and change throughout a semester of an introduction to proof course? We believe that answering these and other related questions would constitute a significant contribution to our current understanding of mathematical language in general, and the language of mathematical proof writing in particular.

References

Alcock, L. (2013). *How to Study as a Mathematics Major*. Oxford, UK: Oxford University Press.

Alcock, L., & Weber, K. (2005). Proof validation in real analysis: Inferring and checking warrants. *The Journal of Mathematical Behavior*, *24*(2), 125-134.

AMS (1962). Manual for authors of mathematical papers. *Bulletin of the American Mathematical Society*, *68*(5), 429-444.

Burton, L., & Morgan, C. (2000). Mathematicians writing. *Journal for Research in Mathematics Education*, 31(4), 429-453.
Ervynck, G. (1992). Mathematics as a foreign language. In W. Geelsin & K. Graham (Eds.), *Proceedings of the 16th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 217-233). Durham, NH: International Group for the Psychology of Mathematics Education.
Folland, G. (1998). *Handbook of Writing for the Mathematical Sciences* by Nicholas J. Higham; *A Primer of Mathematical Writing* by Steven G. Krantz. The American Mathematical Monthly, *105*(8), 779-781.
Halliday, M. A. K. (1978). *Language as social semiotic.* London: Edward Arnold.
Halmos, P. R. (1970). How to write mathematics. *L'Enseignement Mathématique*, *16*(123-152).
Heller, V. (in press). Academic discourse practices in action: Invoking discursive norms in mathematics and language lessons. To appear in *Linguistics and Education*.
Higham, N. J. (1998). *Handbook of writing for the mathematical sciences*. Philadelphia, PA: Society for Industrial and Applied Mathematics.
Houston, K. (2009). *How to think like a mathematician: a companion to undergraduate mathematics*. Cambridge, UK: Cambridge University Press.
Gillman, L. (1987). *Writing mathematics well: a manual for authors*. Washington, DC: Mathematical Association of America.
Kane, R. B. (1968). The readability of mathematical English. *Journal of Research in Science Teaching*, *5*(3), 296-298.
Kern, R. (2000). Notions of literacy. In R. Kern (Ed.), *Literacy and language teaching* (pp. 13-41). New York: Oxford University Press.
Konior, J. (1993). Research into the construction of mathematical texts. *Educational Studies in Mathematics*, *24*(3), 251-256.
Krantz, S. G. (1997). *A primer of mathematical writing: Being a disquisition on having your ideas recorded, typeset, published, read and appreciated*. Providence, RI: American Mathematical Society.
Lemke, J. L. (2003). Mathematics in the middle: Measure, picture, gesture, sign, and word. In M. Anderson, A. Sáenz-Ludlow, S. Zellweger, & V. V. Cifarelli (Eds.), *Educational perspectives on mathematics as semiosis: From thinking to interpreting to knowing*, (pp. 215-234). Ottawa, Ontario: Legas Publishing.
Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in mathematics*, *27*(3), 249-266..
Pimm, D. (1987). *Speaking mathematically: Communication in mathematics classrooms*. London: Routledge & Kegan Paul.
Rav, Y. (1999). Why do we prove theorems?. *Philosophia mathematica*, 7(1), 5-41.
Robert, A. and Scharzenberger, R. (1991). Research in teaching and learning mathematics at an advanced level. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 127-139). Springer: Netherlands.
Scarcella, R (2003). Academic English: A conceptual framework. University of California Linguistic Minority Institute. Retrieved January 20, 2015 from http://escholarship.org/uc/item/6pd082d4
Selden, A., & Selden, J. (1987, July). Errors and misconceptions in college level theorem proving. In J. Novak (Ed.), *Proceedings of the second international seminar on misconceptions and educational strategies in science and mathematics* (Vol. 3, pp. 457-470). Ithaca, NY: Cornell University.
Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can

undergraduates tell whether an argument proves a theorem? *Journal for research in mathematics education*, *34*(1), 4-36.
Selden, A. & Selden, J. (2014). The Genre of Proof. In T. Dreyfus (Ed.), *Mathematics & Mathematics Education: Searching for Common Ground* (pp. 248-251). Springer: Netherlands.
Silva, C., Weinburgh, M., Smith, K. H., Barreto, G., & Gabel, J. (2008). Partnering to develop academic language for English language learners through mathematics and science. *Childhood Education*, *85*(2), 107-112.
Strauss, A. & Corbin, J. (1998). *Basics of Qualitative Research. Techniques and Procedures for Developing Grounded Theory.* Thousand Oaks, USA: Sage Publications, Inc..
Veel, R. (1999). Language, knowledge and authority in school mathematics. In F. Christie (Ed.), *Pedagogy and the shaping of consciousness: Linguistic and social processes*, (pp. 185-216). New York, NY: Continuum International Publishing Group.
Vivaldi, F. (2014). *Introduction to Mathematical Writing.* London: The University of London.
Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, *48*(1), 101-119.
Weber, K. (2003). Students' difficulties with proof. *Research Sampler.* Mathematical Association of America. Retrieved July 30, 2014 from: http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof
Weber, K., Brophy, A., & Lin, K. (2008). Learning advanced mathematical concepts by reading text. In M. Zandieh, C. Rasmussen, & K. Marrongelle, *Proceedings of the 11th Conference on Research in Undergraduate Mathematics Education*. San Diego, CA: San Diego State University.

Exhaustive example generation:
Mathematicians' uses of examples when developing conjectures

Elise Lockwood
Oregon State University

Alison G. Lynch
University of Wisconsin-Madison

Amy B. Ellis
University of Wisconsin-Madison

This paper explores the role examples play as mathematicians formulate conjectures. Although previous research has examined example-related activity during the act of proving, less is known about how examples arise during the formulation of conjectures. We interviewed thirteen mathematicians as they explored tasks requiring the development of conjectures. During the interviews, mathematicians productively used examples as they formulated conjectures, particularly by creating systematic lists of examples that they examined for patterns, an activity that we call "Exhaustive Example Generation." The results suggest further research and pedagogical implications for explicitly targeting examples in conjecturing, and the study contributes to a body of literature that points to the benefits of exploring, identifying, and leveraging examples in proof-related activity.

Key words: Examples, Mathematicians, Conjecturing, Proof

Introduction and Motivation

Formulating and proving mathematical conjectures are key aspects of mathematical practice at all levels (Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002; Knuth, 2002; Sowder & Harel, 1998), and yet there is much evidence that gaining facility with proof-related activities is challenging for students (e.g., Healy & Hoyles, 2000; Kloosterman & Lester, 2004; Knuth, Choppin, & Bieda, 2009; Porteous, 1990). Following a tradition of research investigating mathematicians' thinking (Carlson & Bloom, 2005; Weber, 2008; Weber, Inglis, & Mejia-Ramos, 2014), we have previously studied the work of mathematicians, who are themselves adept at proof-related activities, in order to better understand how to help students formulate and prove conjectures (Lockwood, Ellis, Dogan, Williams & Knuth, 2012; Lockwood, Ellis, & Knuth, 2013). In those studies we found evidence that mathematicians select and use examples strategically in their proving activities, which concurs with others' findings (e.g., Epstein & Levy, 1995). To date, we have studied the roles of examples in mathematicians' *proving* of conjectures, but not in their *formulating* of conjectures. In this paper, we extend our previous work by studying mathematicians' example-related activity as they engage in conjecture development. We seek to answer the questions: *What aspects of mathematicians' example-related activity enable the productive formulation of mathematical conjectures? In what ways does exhaustive example generation facilitate mathematicians' formulation of mathematical conjectures?* Our examination details some of the ways in which mathematicians systematically generate and use examples in generating conjectures, and we discusses implications for the teaching and learning of proof.

Relevant Literature and Theoretical Perspective

We define an example as Bills and Watson (2008) do, as "any mathematics object from which it is expected to generalize" (p. 78). Although much of the existing literature emphasizes the limitations of example-based reasoning, particularly as a means of justification, a number of researchers have suggested the potential value examples may play in proof-related activity (e.g., Buchbinder & Zaslavsky, 2011; Iannone, Inglis, Mejia-Ramos, Simpson, & Weber, 2011; Weber, 2010). Other researchers have similarly reported that students and mathematicians display strategic uses of examples that benefit their proof-related activities (e.g., Antonini, 2006; Garuti, Boero & Lemut, 1998; Pedemonte, 2007; Sandefur, Mason, Stylianides, & Watson, 2013; Weber, 2008). As Epstein and Levy (1995) point out, "Most mathematicians spend a lot of time thinking about and analyzing particular examples….it is probably the case that most significant advances in mathematics have arisen from experimentation with examples" (p. 6). Likewise, Harel (2008) notes that, "Examples and non-examples can help to generate ideas or give insight [about the development of proofs]" (p. 7). Similarly, our previous work suggests that examples can be meaningful and helpful both for students (Ellis, et al., 2012) and for mathematicians (Lockwood, et al., 2012; Lockwood, et al., 2013). In focusing on mathematicians, we recognize that there may be some discrepancies between mathematicians' practice and how students themselves might develop a particular practice. However, we agree with a number of researchers who argue for the importance of being informed about mathematicians' thinking and activity in spite of such potential discrepancies (e.g., Carlson & Bloom, 2004; Weber, et al., 2014).

In this paper, we focus on one particular aspect of the role of examples in proof-related activity by examining the activity of conjecturing. While other researchers have studied mathematicians' conjecturing activities (such as Belnap & Parrott, 2013, who identify themes in comparing novice and expert conjecturers), here we focus on the role of examples in the development of conjectures. By seeking to identify potentially fruitful aspects of example-related activity in the development of conjectures, our work extends both research involving examples in proof and research involving conjecturing.

Lannin, Ellis, and Elliot (2011) define conjecturing as "reasoning about mathematical relationships to develop statements that are tentatively thought to be true but are not known to be true (to the conjecturer)" (p. 13). We adopt this definition and note that whether or not a statement is a conjecture may depend on who is conjecturing – for example, a middle school student might make a conjecture that would not be considered a conjecture by someone more mathematically mature. Lannin, et al. also suspect that the role of examples in conjecturing could be significant, noting "Conjectures may be developed through examining specific examples and then reasoning inferentially from specific situations…" (p. 14). Our interest in conjecturing arose during interviews in which we studied mathematicians' uses of examples in proving. In that work, we developed and refined a framework (Lockwood et al., 2012; Lockwood et al., 2013) categorizing *example types, example uses,* and *example strategies*. Although the framework is not presented here due to space, it served as a broader context to guide data analysis. In conducting interviews with this framework in mind, we realized that there might be value in asking mathematicians more open-ended questions, to see how their example-related activity might change as they *conjectured* rather than *proved*. We thus used the framework of example-related activity as a guiding lens with which to situate the current study.

Methods

We conducted a sequence of two hour-long interviews with mathematicians. Thirteen of these mathematicians participated in an initial interview in which they were presented with one or two mathematics tasks, and ten continued for a second interview in which they were presented with two different tasks. The participants were from a large Midwestern university and included seven professors, three postdocs, and three lecturers, with eight males and five females. Twelve participants hold a Ph.D. in mathematics, and one participant holds a Ph.D. in computer science. There were a variety of mathematical areas represented, including topology, number theory, and analysis. A member of the research team (an advanced mathematics PhD student, also the second author) conducted the interviews. During the interviews, the mathematicians were given time to work on the tasks on their own and were asked to think aloud; generally, the interviewer did not interrupt except to ask clarifying questions or to answer questions from the mathematicians. The mathematicians' audio and written work were recorded using Livescribe pens, which keep live records of the mathematicians' spoken words and written work.

For this paper, we report on data from two tasks: The *Interesting Numbers* task from Interview 1, and the *Fixed Point* task from Interview 2. The *Interesting Numbers* task states: *"Most positive integers can be expressed with the sum of two or more consecutive integers. For example, 24 = 7 + 8 + 9, and 51 = 25 + 26. A positive integer that cannot be expressed as a sum of two or more consecutive positive integers is therefore interesting. What are all of the interesting numbers?"* One approach to solving this task is as follows: It can be shown that the sum of any two or more consecutive positive integers has an odd factor greater than 1. Conversely, if a positive integer N has an odd factor $k > 1$, it can be shown that N can be written as the sum of either k or $2N/k$ consecutive positive integers, whichever is smaller. The interesting numbers are thus exactly those positive integers that have no odd factors greater than 1 (in other words, powers of 2).

The *Fixed Point* task states: *"For each positive integer n, let* $F(n) = \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{4} \right\rceil + \cdots + \left\lceil \frac{n}{2^k} \right\rceil$, *where* k *is the unique integer such that* $2^{k-1} \leq k < 2^k$ *and* $\lceil x \rceil$ *denotes the smallest integer greater than or equal to* x. *For which numbers* n *is* $F(n) = n$*?"* One approach to solving this task is as follows: It can be shown that $F(2^i - 1) = 2^i - 1$ for any positive integer i. It can also be shown that $F(2n) = n + F(n)$ for any positive integer n, so if $F(n) = n$, then $F(2n) = 2n$. It follows that $F(2^i - 2^j) = 2^i - 2^j$ for any positive integers $i > j$. Conversely, it can be shown that $F(n) > n$ for any integer n not of the form $2^i - 2^j$. Therefore $F(n) = n$ exactly when $n = 2^i - 2^j$ for positive integers $i > j$.

The tasks were chosen because: a) they were accessible to the mathematicians (i.e., they did not require specialized content knowledge and were easy to explore) but were not trivial (i.e., a solution was not well known or immediately available), b) they were accessible to the interviewer, allowing her to ask relevant questions and engage with the mathematicians, and c) they involved open-ended questions that would facilitate conjecturing. These were not presented as "prove or disprove" statements that already specified a conjecture, but rather these tasks required that certain numbers and sets be characterized. Through such activity, the mathematicians developed conjectures that they could then attempt to prove. The *Interesting Numbers* task came from Andreescu, Andrica, & Feng (2007), and the *Fixed Point* task came from Krusemeyer, Gilbert, & Larson (2012).

For analysis, the interviews were transcribed. In addition, the Livescribe pen yields both an audio record of the interview and a pdf document of the interviewee's written work that can be played back, so one can see and hear what was written and said in real time. To analyze these interviews, two members of the research team independently coded and then discussed the same four Interview 1 interviews using Lockwood, et al.'s (2012, 2013) framework for example types, uses, and strategies. In coding the interviews, the researchers also developed emergent codes that were not captured by the previous framework. After the four interviews were initially coded, compared, and discussed, the remaining nine Interview 1 interviews were divided up and coded.

Similarly, for Interview 2 two members of the research team each independently coded the transcripts and then came together to discuss each of the transcripts and come to consensus. The two researchers ultimately agreed (through discussion) on the codes for all of the interviews. After completing the coding of all the interviews, the researchers met to discuss phenomena and themes that pertained especially to conjecturing and revisited relevant episodes in the transcripts.

Results

In this paper we elaborate a phenomenon called *Exhaustive Example Generation*, in which the mathematicians systematically and, in some sense exhaustively, went through every example in a finite sequence in order to gather information. This was a new use of examples that had not previously emerged in our framework of example types, uses, and strategies. In *Exhaustive Example Generation*, the mathematicians sequentially exhausted a list of examples and reflected back on these organized example lists while working towards developing a conjecture. This subsequent reflection is very important, as it suggests that the example generation served to create an exhaustive set of examples from which the mathematicians could reason. Among the thirteen mathematicians we interviewed, ten of them engaged in *Exhaustive Example Generation* while working on the *Interesting Numbers* task, and four of them did so while working on the *Fixed Point* task. This activity was productive for some mathematicians, as we explore below, suggesting that there is potential value in the methodical generation of examples in formulating conjectures. We specify three main affordances of *Exhaustive Example Generation* (*Conjecture Development*, *Lemma Development*, and *Conjecture Breakin*g), which arose as consequences of three main underlying strategies (*Pattern Searching, Identifying Structure Between Examples,* and *Identifying Structure Across the Data Set*). We elaborate these ideas below.

Affordances of Exhaustive Example Generation

Many of the mathematicians used *Exhaustive Example Generation* to their benefit, but the specific affordances of the activity varied, depending on how the mathematician used it. Here we report on three different affordances that the mathematicians gained as they engaged in *Exhaustive Example Generation*.

Conjecture Development. For some of the mathematicians, *Exhaustive Example Generation* contributed directly to the development of a conjecture. To illustrate this phenomenon, we present Dr. Sullivan's (a professor) work on the *Interesting Numbers* task. Dr. Sullivan began by computing a sequence of small sums: 1+2=3, 2+3=5, 3+4=7, and 4+5=9. From these examples, he recognized that odd numbers greater than 1 could not be interesting. He proved this fact algebraically by showing that any odd number $2n+1$ is the sum of n and $n+1$. Continuing with algebra, he then looked at general sums of 3, 4, and 5 consecutive numbers beginning with n. Each case gave him an algebraic expression ($3n+3$, $4n+6$, $5n+10$) representing numbers that were not interesting, from which he tried to generalize.

After some time, Dr. Sullivan recognized that his algebraic manipulation had not illuminated a conjecture, and he said, "Okay. So at this point, I would start over and try and do something a little more visual." He then drew a number line and began to write out the numbers. Because Dr. Sullivan already knew that the odd numbers were not interesting, he crossed those out as he wrote. He then proceeded to go through the even numbers and cross out those of the form $3n+3$, $4n+6$, and $5n+10$ for some n (Figure 1). After working through the numbers 1 through 21, he concluded, "well, the answer does kind of pop out that it's the powers of 2, doesn't it?" By actually writing out the examples and then crossing out non-interesting numbers, the pattern of numbers not crossed out – 1, 2, 4, 8, and 16 – stood out in his figure. His construction of the complete table, and his subsequent reflection on it, suggest the *Exhaustive Example Generation* phenomenon – he systematically gathered a complete sequence of examples and deduced patterns from them.

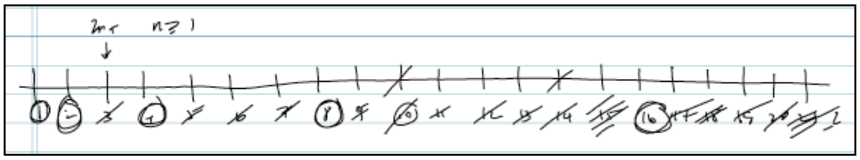


Figure 1: The "visual" list from which the powers of 2 conjecture emerges

We perceive that Dr. Sullivan's prior knowledge and experience made him attuned to this sequence of numbers as powers of 2. Dr. Sullivan continued to pursue the powers of 2, saying, "Okay, so, um, so at this point I would maybe try the next one, 32," and he proceeded to write a conjecture that interesting numbers are powers of 2. To us, Dr. Sullivan's careful construction of examples allowed for what was to Dr. Sullivan a common, familiar pattern to emerge visually on the page. His work suggests that the methodical generation of examples (*Exhaustive Example Generation*) directly facilitated the efficient formulation of the conjecture.

Lemma Development. Some of the mathematicians made observations about the examples they generated that led to the statement and proof of lemmas. One example of this can be seen in Dr. Sullivan's work described above. After generating a sequence of small sums (1+2=3, 2+3=5, 3+4=7, and 4+5=9), he observed that all of the odd numbers greater than 1 could be obtained as the sum of two consecutive numbers. He then formulated this as a lemma (*odd numbers are not interesting*) and produced an algebraic proof.

Another example can be seen in Dr. Taylor's (a postdoc) work on the *Interesting Numbers* task. Dr. Taylor first looked at the numbers 1 to 11 and tried to write each one as a sum of consecutive numbers (Figure 2a). He noticed that odd numbers were sums of 2 consecutive numbers and multiples of 6 were sums of 3 consecutive numbers. Dr. Taylor formalized these observations as lemmas and produced algebraic proofs of each statement (Figure 2b). Regarding multiples of 6, he said,

Dr. Taylor: So, if your number is divisible by 6, well, you divide by 3, you get some k. So like $3k$ is equal to whatever your number is, say A. And so, it's k+k+k equals A. And then you just make one of them bigger and one of the smaller."

In the final step of his argument, he replaced one of the k terms in the sum with $k+1$ and replaced one of the k terms in the sum with $k-1$, writing $(k-1)+k+(k+1) = A$. For Dr. Taylor, this proved that multiples of 6 can be written as sums of 3 consecutive numbers.

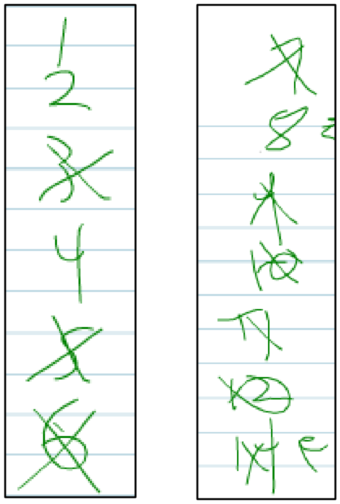


Figure 2a: Dr. Taylor’s lists of examples

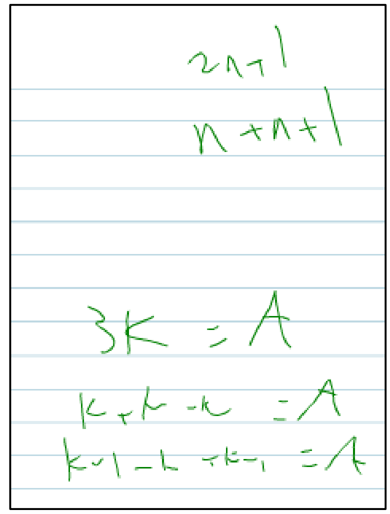


Figure 2b: Dr. Taylor’s algebraic proofs

At this point, Dr. Taylor tried 14 since it was the next smallest number not ruled out by his lemmas. Looking at 10 (written as 1+2+3+4) and 14 (written as 2+3+4+5), he recognized that two times an odd number can always be written as the sum of 4 consecutive numbers.

Dr. Taylor: So, yeah, it's true that if you have a number which you can write as the sum of two things - so every odd number you can write as the sum of two things - then twice that number, you should also be able to write as four things.

Dr. Taylor formalized this observation as well, proving a lemma that numbers congruent to 2 mod 4 other than 2 are sums of 4 consecutive numbers. These lemmas allowed Dr. Taylor to restrict his attention to multiples of 4, which led to the development of the full conjecture.

Conjecture Breaking. *Exhaustive Example Generation* also allowed the mathematicians to find examples that broke preliminary conjectures, which in turn led to the articulation of more accurate conjectures. This is seen in Dr. Hughes (a postdoc) as he worked on the *Interesting Numbers* task. Dr. Hughes initially conjectured that the interesting numbers were the non-primes after looking at the numbers 1 to 6 (and incorrectly deciding that 6 was interesting, Figure 3a). He continued on to look at the numbers 7 to 10 before he realized his mistake, saying about 6, “Oh, 1, 2... 1 plus 2 plus 3. Right. Revise conjecture. So far, so, the interesting numbers so far are 4, 8, [...] It looks like it’s the [multiples] of 4” (Figure 3b). As a result of collecting more data, Dr. Hughes found a counterexample to his initial conjecture. This counterexample prompted him to reflect on his (corrected) set of data and refine his conjecture.

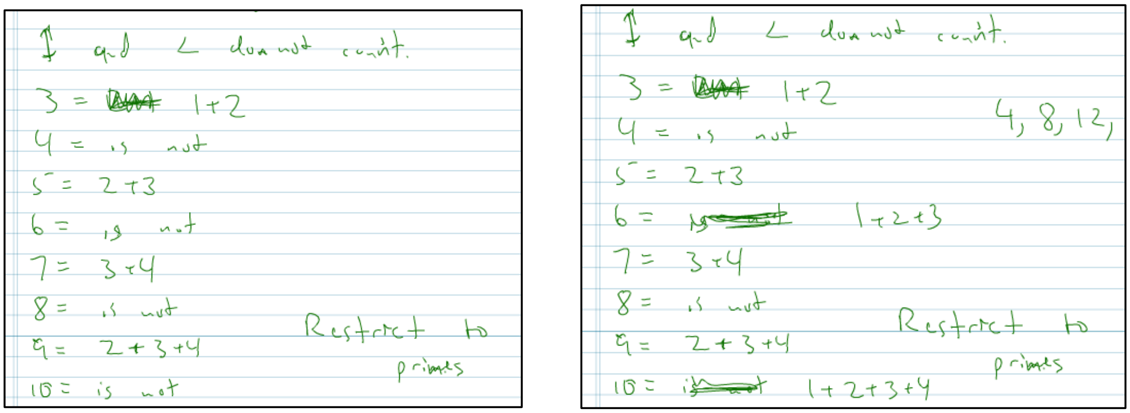


Figure 3: Dr. Hughes's initial (3a) and revised (3b) conjectures

Dr. Hughes revised his conjecture once more (to a correct conjecture) when he looked at 11, 12 and 13 and discovered that 12 was also not interesting. We note that Dr. Hughes was developing and breaking conjectures as he was in the process of generating examples, and not after he had already completed the example generation. This stands in contrast to some of the other mathematicians (such as Dr. Weisman, described below) who formulated conjectures once the exhaustive list of examples had been completed and could be reflected upon.

Strategies for engaging with Exhaustive Example Generation

The mathematicians varied in how they engaged in *Exhaustive Example Generation*, in the sense that they appeared to have different broad strategies about how they should engage with the set of examples they had made. Here we explore three different strategies that we observed among the mathematicians.

Pattern Searching. Most of the mathematicians used their *Exhaustive Example Generation* as a means by which to look for empirical patterns among the working or non-working examples. In this activity, the mathematicians searched for and identified a numerical pattern among the exhaustive list they had constructed. They then used the pattern they had identified to develop a conjecture, a lemma, or to find a counterexample (the three affordances described above).

The work of Dr. Sullivan and Dr. Hughes in the previous section provides examples of Pattern Searching behavior. In the case of Dr. Sullivan, he wrote out a number line with the numbers 1 to 21, crossed out all the non-interesting numbers, and circled the remaining interesting numbers. He then reflected on the compiled list of interesting numbers and identified a pattern among them – that they were all powers of 2 – which led him to a conjecture.

In the case of Dr. Hughes, he searched for patterns as he generated examples and made conjectures based on those patterns. When he thought 4 and 6 were the first interesting numbers, he conjectured that the interesting numbers were the non-primes. After correcting his work and noticing that 4 and 8 were the first interesting numbers, he conjectured that the interesting numbers were the multiples of 4. In both cases, he developed conjectures based on the patterns he identified in the interesting numbers.

As another example of Pattern Searching, we consider Dr. Weisman's (a professor) work on the *Interesting Numbers* task. Dr. Weisman had found that the non-interesting numbers were of the form (n-m)(n+m-1)/2 with n>m>0. He proceeded to make a table of the first few numbers of that form (Figure 4).

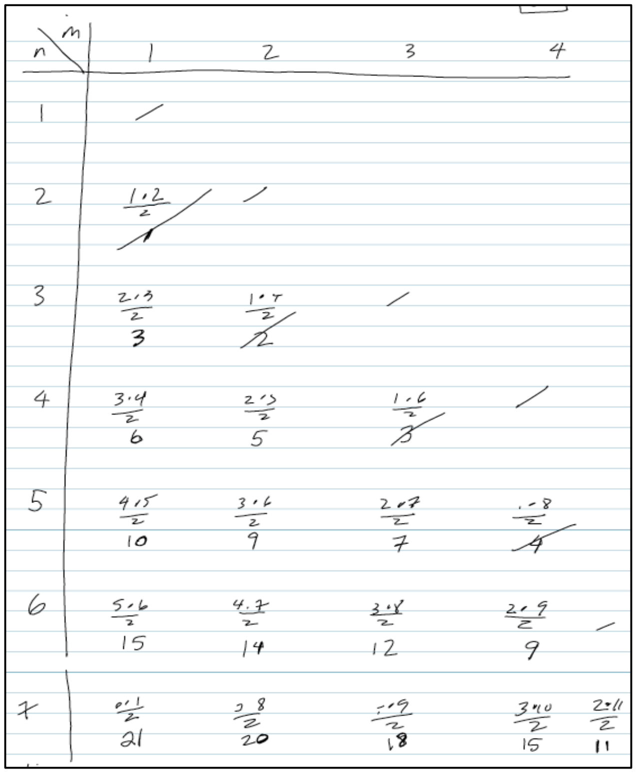


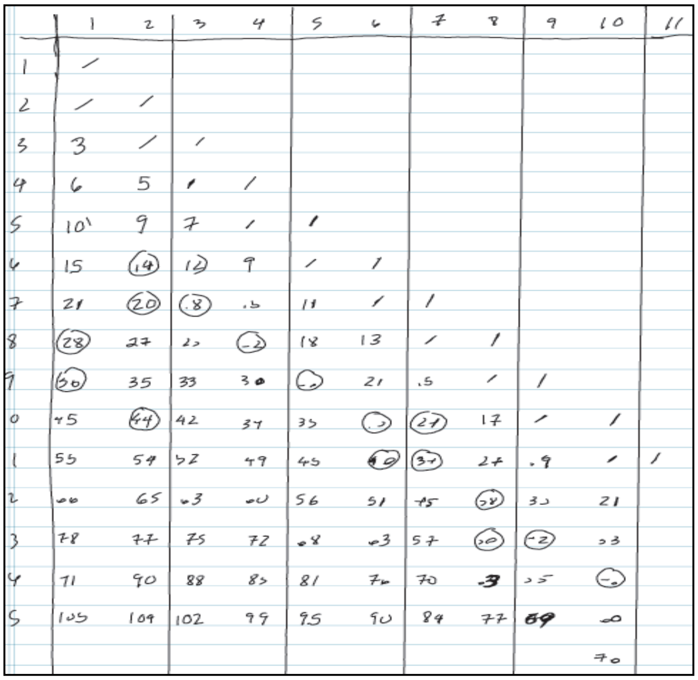
Figure 4: Dr. Weisman's initial table of examples

By examining his table, Dr. Weisman deduced some patterns in the non-interesting numbers, but he did not find anything conclusive. He then said the following:

Dr. Weisman: Okay, well, I'm a believer in generating some data…what I'm gonna do is make an even bigger version of this table. And just look to see what numbers show up.

Dr. Weisman then proceeded to create this larger table (Figure 5a), and his work displays a great deal of care in detailing out a large number of cases. By referencing this larger table, Dr. Weisman made a conclusive list of the interesting numbers up to 50 by crossing out the non-interesting ones (Figure 5b), making the following comments after he was finished.

Dr. Weisman: That's rather remarkable. So, wild conjecture at this point. Certainly quite surprising. Interesting corresponds to the powers of 2.



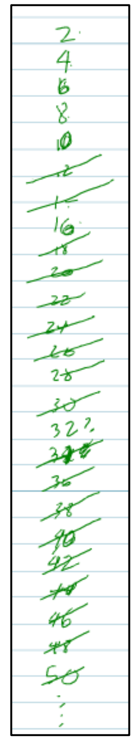


Figure 5a: Dr. Weisman's complete table   Figure 5b: Dr. Weisman's corresponding list

We want to point out that Dr. Weisman's remarks suggest that he was searching for a pattern among the interesting and non-interesting numbers. He used the table (Figure 5a) as a tool to systematically generate examples, then he used the list (Figure 5b) to organize his examples. His remarks also suggest that he was not completely sure of the conjecture as he was working, but rather the reflection on the list of examples helped him to record and recognize the salient property of interesting numbers. We thus see that the conjecture he developed was facilitated through the strategy of Pattern Searching among the exhaustive list of examples he had generated.

Leveraging Structure Between Examples. The mathematicians also at times identified structure between examples and used this structure productively in their conjecturing activity. When using this strategy, the mathematicians observed a common structure between two or more examples. By a common structure, we mean a mathematical property or relationship between two or more examples that can be used to explain why the examples do or do not satisfy the conditions of the task. The mathematicians used this common structure to classify the general case (Conjecture Development) or to identify key features or subcases of the general case (Lemma Development).

One example of this strategy appears in Dr. Taylor's work, described in the previous section. In that work, he developed lemmas based on a common structure he observed between examples. For his first lemma, Dr. Taylor observed that all of the odd numbers he generated were sums of 2 consecutive numbers. Moreover, for a given odd number N, he observed that the two consecutive numbers in the sum were the integers just below $N/2$ and just above $N/2$. From this observation, Dr. Taylor was able to prove that all odd numbers were non-interesting.

Dr. Taylor also observed a common structure between 6 and 12 (that they both could be written as the sum of 3 consecutive numbers) and between 10 and 14 (that they both could be written as the sum of 4 consecutive numbers). By leveraging the common structure between each of those sets of examples, Dr. Taylor proved lemmas stating that multiples of 6 and numbers that were two times an odd number were non-interesting.

Leveraging Structure Across the Data Set. On some occasions, the mathematicians identified structure across the entire set of examples they had generated. That is, rather than identifying structure in a particular example or among a couple of examples, they actually looked to the set of examples itself as a source of structure. An example of this is Dr. Jones' work on the *Fixed Point* task. She began by computing *F(n)* for *n*=1 through 10 (Figure 6).

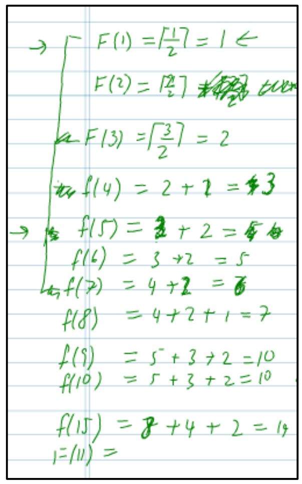


Figure 6: Dr. Jones' initial listing of examples

This activity did not allow her to see a pattern for which numbers that worked, and so she began to look at patterns in the sums themselves. She recognized a potential pattern and more systematically listed and arranged the sums for $F(2)$ through $F(16)$ so that the summands lined up in columns. Note that her language in the following excerpt, along with the written work in Figure 7, suggests that she is looking at a structural pattern across the entire list of examples:

Dr. Jones: The first [summands] go up every second one, the next one goes up every 4. The ones after that go up every 8.

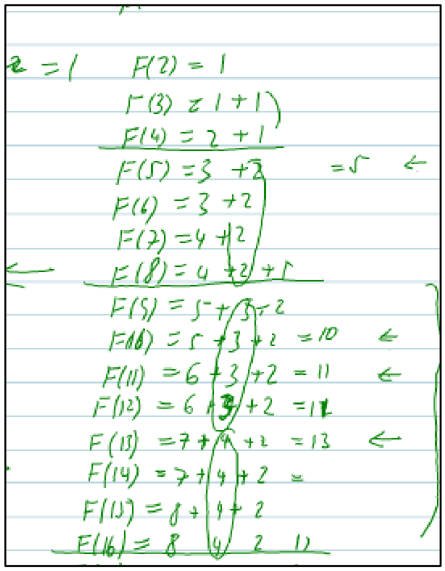


Figure 7: Dr. Jones notices a pattern among the summands

Her observation of this structure allowed Dr. Jones to generate examples more quickly through *F*(32) (Figure 8) and eventually up to *F*(40). This work demonstrates a case in which the mathematicians used the list of examples that was made during *Exhaustive Example Generation* in order to detect some broader structure in the entire list. The particulars of the examples were less important for her in this investigation, but rather she was focused on a structure that she identified by examining the collective set of examples.

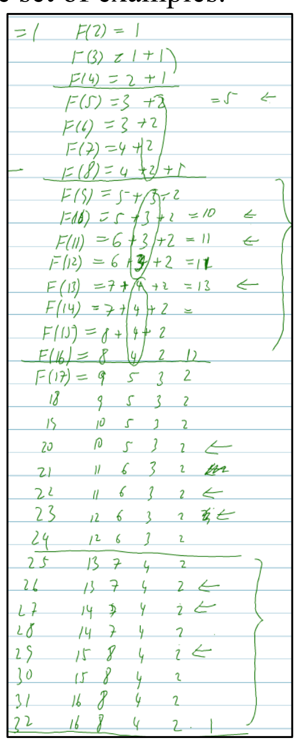


Figure 8: Dr. Jones uses the pattern to construct more examples quickly

It is unclear how helpful this particular activity was for her on this particular task, as she did not ultimately arrive at a correct conjecture of the problem. However, we consider her work on this task to be an instance of how one might use *Exhaustive Example Generation* productively, and it is conceivable that this strategy of identifying structure in a set of examples could be an effective and powerful tool for some conjectures and tasks.

Discussion and Conclusions

The results highlight ways in which example-related activity such as *Exhaustive Example Generation* may play a valuable role in the development of conjectures. Specifically, systematically generating exhaustive sets of examples and subsequently reflecting on them facilitated direct conjecture formulation, lemma development, and preliminary conjecture breaking. Such example-related activity was widespread among the mathematicians we interviewed, as ten of the 13 mathematicians used examples while conjecturing on the *Interesting Numbers* task. All ten who used examples formulated a correct conjecture, while the remaining three mathematicians used a purely algebraic approach toward the task. Of those three mathematicians, one formulated a correct conjecture through algebraic manipulation, one solved the task algebraically without ever formulating a working conjecture, and one never developed a correct conjecture. Although algebraic manipulation was an effective strategy for two of the mathematicians, we suspect that for conjecturing purposes, it did not so clearly illuminate potential patterns as the actual generation of concrete examples did for the other mathematicians.

It is not clear, however, that example use while formulating a conjecture affected the ultimate strategy and success in proving that conjecture. Of the ten mathematicians who used examples while conjecturing on the *Interesting Numbers* task, only four continued to use examples while developing a proof (three of those four mathematicians successfully proved the full conjecture, although only two of those three used examples to develop the final proof). Of the six mathematicians who used examples in conjecturing but not in proving, four successfully proved the full conjecture. Two of the three mathematicians who used algebra throughout successfully provided a full proof. Although the evidence suggests that examples helped in conjecturing, there is more to investigate as to the relationship between example use in conjecturing and proving or disproving those conjectures.

It is also important to note that some mathematicians (as seen with Dr. Sullivan and Dr. Weisman) took the time to painstakingly catalogue a number of examples. The generation of sets of examples and subsequent reflection on these examples enabled the mathematicians to formulate conjectures effectively and efficiently. Their extant knowledge clearly played a key role in their formulation of conjectures based on their lists of examples. Also notable is the fact that these mathematicians engaged in deliberate and strategic example generation, and this is in line with other studies that have demonstrated metacognitive aspects of mathematicians' activity (e.g., Carlson & Bloom, 2005; Lockwood, et al., 2013; Savic, 2012). These observations suggest that in developing conjectures, there was much to gain for mathematicians who were willing to engage carefully, systematically, and intentionally with sets of examples, and who could relate these examples to their prior mathematical knowledge.

These results suggest some preliminary pedagogical implications. There may be value in helping students learn to be more methodical in their use of examples, going beyond finding a few confirming examples that simply come to mind. Specifically, students may benefit from generating comprehensive sets of data that they can survey in search of patterns, although they must learn to relate such data with their own mathematical knowledge. Such activity could in turn illuminate conjectures, and it is important to emphasize for students that such work may take patience and care. Also, the tasks in our study were well suited to facilitate *Exhaustive Example Generation*. Other tasks might be more or less effective in fostering conjecturing. Instructors should be aware of what kinds of activity and thinking certain tasks elicit and should expose students to tasks that might encourage *Exhaustive Example Generation* activities.

References

Andreescu, T., Andrica, D., & Feng, Z. (2007). *104 Number Theory Problems: From the Training of the USA IMO Team*. Boston, MA: Birkhauser.

Antonini, S. (2006). Graduate students' processes in generating examples of mathematical objects. In J. Novotna, H. Moraova, M. Kratka, & N. Stehlikova (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, 57-64.

Ball, D., Hoyles, C., Jahnke, H., & Movshovitz-Hadar, N. (August, 2002). *The teaching of proof*. Paper presented at the International Congress of Mathematicians, Beijing, China.

Belnap, J., & Parrott, A. (2013). Understanding Mathematical Conjecturing. In the *Electronic Proceedings for the Sixteenth Special Interest Group of the MAA on Research on Undergraduate Mathematics Education*. Denver, CO: Northern Colorado University. February 21-23, 2013.

Bills, L., & Watson, A. (2008). Editorial introduction. *Educational Studies in Mathematics*, *69*, 77-79.

Buchbinder, O. & Zaslavsky, O. (2011). Is this a coincidence? The role of examples in fostering a need for proof. *ZDM – The International Journal on Mathematics Education, 43*(2), 269-281.

Carlson, M & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational Studies in Mathematics, 58,* 45-75.

Ellis, A. E., Lockwood, E., Williams, C. C. W., Dogan, M. F., & Knuth, E. (2012). Middle school students' example use in conjecture exploration and justification. In L.R. Van Zoest, J.J. Lo, & J.L. Kratky (Eds.), *Proceedings of the 34th Annual Meeting of the North American Chapter of the Psychology of Mathematics Education* (Kalamazoo, MI).

Epstein, D., & Levy, S. (1995). Experimentation and proof in mathematics. *Notice of the AMS*, *42*(6), 670–674.

Garuti, R., Boero, P., & Lemut, E. (1998). Cognitive unity of theorems and difficulty of proof. *Proceedings of the 22nd Annual Meeting of the Psychology of Mathematics Education*, Vol. 2, 345-352.

Harel, G. (2008). What is Mathematics? A Pedagogical Answer to a Philosophical Question. In R. B. Gold & R. Simons (Eds.), *Current Issues in the Philosophy of Mathematics From the Perspective of Mathematicians,* Mathematical American Association.

Healy, L. & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education*, *31*(4), 396–428.

Iannone, P., Inglis, M., Mejia-Ramos, J. P., Simpson, A., & Weber, K. (2011). Does generating examples aid proof production? *Educational Studies in Mathematics, 77*, 1–14.

Krusemeyer, M. I., Gilbert, G. T., & Larson, L. C. (2012). *A Mathematical Orchard: Problems and Solutions*. Washington, DC: Mathematical Association of America.

Kloosterman, P., & Lester, F. (2004). *Results and interpretations of the 1990 through 2000 mathematics assessments of the National Assessment of Educational Progress*. Reston, VA: National Council of Teachers of Mathematics.

Knuth, E. (2002). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education*, *33*(5), 379–405.

Knuth, E., Choppin, J., & Bieda, K. (2009). Middle school students' production of mathematical justifications. In D. Stylianou, M. Blanton, & E. Knuth (Eds.), *Teaching and learning proof across the grades: A K–16 perspective* (pp. 153–170). New York, NY: Routledge.

Lannin, J., Ellis, A.B., & Elliott, R. (2011). *Essential understandings project: Mathematical reasoning (Gr. K – 8).* Reston, VA: National Council of the Teachers of Mathematics.

Lockwood, E., Ellis, A.B., Dogan, M.F., Williams, C., & Knuth, E. (2012). A framework for mathematicians' example-related activity when exploring and proving mathematical conjectures. In L.R. Van Zoest, J.J. Lo, & J.L. Kratky (Eds.), *Proceedings of the 34th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 151 – 158). Kalamazoo, MI: Western Michigan University.

Lockwood, E., Ellis, A., & Knuth, E. (2013). Mathematicians' Example-Related Activity When Proving Conjectures. In the *Electronic Proceedings for the Sixteenth Special Interest Group of the MAA on Research on Undergraduate Mathematics Education*. Denver, CO: Northern Colorado University. February 21-23, 2013.

Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics, 66*, 23-41. Doi: 10.1007/s10649-006-9057-x.

Porteous, K. (1990). What do children really believe? *Educational Studies in Mathematics, 21*, 589–598.

Sandefur, J., Mason, J., Stylianides, G. J., & Watson, A. (2013). Generating and using examples in the proving process. *Educational Studies in Mathematics*. Doi: 10.1007/s10649-012-9459-x.

Savic, M. (2012). What do mathematicians do when they have a proving impasse? *In the Electronic Proceedings for the Fifteenth Special Interest Group of the MAA on Research on Undergraduate Mathematics Education*. Portland, OR: Portland State University. February 23-25, 2012.

Sowder, L., & Harel, G. (1998). Types of students' justifications. *Mathematics Teacher, 91,* 670–675.

Weber, K. (2008). How mathematicians determine if an argument is a valid proof. *Journal for research in Mathematics Education, 39*(4), 431-459.

Weber, K. (2010). Mathematics majors' perceptions of conviction, validity, and proof. *Mathematical Thinking and Learning*, 12, 306-336.

Weber, K., Inglis, M., & Mejia-Ramos, J. P. (2014). How mathematicians obtain conviction: Implications for mathematics instruction and research on epistemic cognition. *Educational Psychologist, 49*(1), 36-58. Doi:10.1080/00461520.2013.865527.

Business Faculty Perceptions of the Calculus Content Needed for Business Courses

Melissa Mills
Oklahoma State University

In 2000, the MAA composed a the Curriculum Reform Across the First Two Years report that outlined their recommendations for Business Mathematics, among other mathematical topics. For Business Mathematics, they recommended an emphasis on applications, modeling, and communication of mathematical results. Because of time constraints, adding in more applications and modeling must require cutting out some topics that are traditionally covered in such a Calculus course. This study aims to determine the specific Calculus content that is used in undergraduate level business courses to inform decisions about which content to cover in Business Calculus. This study uses an online survey instrument and interviews to explore what Calculus topics business faculty view as relevant to, and necessary for, various business specializations.

Key words: Business Calculus, Applied Calculus, Business Faculty, Calculus Concepts

Literature review

Calculus for business students (which is often referred to as Applied Calculus) is typically taught in mathematics departments by instructors who are unfamiliar with business applications and terminology. Because the course is a service course for other departments, the content covered and the way in which it is covered by mathematics professors may not be consistent with the needs of the students. Because of this dissonance, some institutions have dropped this mathematics requirement and have moved the content into a quantitative course taught by business faculty (Depaolo & Mclaren, 2006).

A significant body of research has examined Calculus for STEM students (Asiala, Cottrill, Dubinsky & Schwingendorf, 1997; Bressoud, Carlson, Mesa & Rasmussen, 2013; Carlson, Jacobs, Coe, Larsen & Hsu, 2002; Gonzalez-Martin & Camacho, 2004; Gravemeijer & Doorman, 1999; Oehrtman, 2009; Thompson, 1994; Zandieh, 2000), however, the research on Applied Calculus and Business Calculus in particular is far less extensive (Garner & Garner, 2001; Liang & Martin, 2008). In 2000, the MAA released the CRAFTY Curriculum Foundations Project, which outlined recommendations for the reform of many different courses including Calculus for Business students. Reforms include using realistic business problems, business technology, and modeling from real data sets. The report concludes that the cooperation and communication between the business and mathematics faculties is critical to the implementation of their recommendations (Lamoreux, 2000).

Calculus for business students typically covers topics similar to those covered in engineering calculus, but with less rigor and more applications. The Business and Management recommendations portion of the CRAFTY report were written by a team of 36 Business professors and 6 mathematics professors from across the country. They stated that "the business calculus curriculum should include an introduction to rates of change, the dynamic nature of real-world systems, constrained optimization, and interpretations of area under a graph." They called for MORE emphasis on realistic business problems, modeling with realistic data, use of business technology, development of mathematical communication skills and interpretation of solutions and LESS emphasis on computation and techniques of symbolic differentiation and integration.

They advise that "math departments can help prepare business students by stressing problem solving using business applications, conceptual understanding, quantitative reasoning, and communication skills. These aspects should not be sacrificed to breadth of coverage." They also state: "when in doubt, mathematics faculty should cover *less* material – and treat the material covered with respect – imparting to students a sense of the importance of mathematics as a necessary part of the development of successful business people." For instructors of these business mathematics courses, the reality is that adding more modeling and applications must require cutting out some topics that were previously covered, or possibly covering them in a different way.

Differentiation and integration are foundational calculus concepts, and business calculus tends to cover both topics and how they apply to business settings (Liang & Martin, 2008; Garner & Garner, 2001). However, in a recent survey of business faculty at a liberal arts college, May (2012) found that they viewed partial derivatives as more important than integration. He also found that business faculty value modeling data to fit functions and skill with using Excel for business problem solving more than they value theoretical understanding and computational techniques. However, the final exams in the business calculus course at that particular institution showed that the mathematics instructors placed more emphasis on things like symbolic integration than applications, for example. It is unclear whether these results were specific to liberal arts colleges or are more widespread.

As we move forward in reforming applied calculus, it is important to assess the needs of business students and the expectations of business faculty so that we can make informed decisions about the specific calculus content that should be covered. To do this, it is necessary to open the lines of communication between business faculty and the mathematics faculty who are teaching these courses. This leads to the research question addressed in this study:

What calculus concepts do business faculty members at a large comprehensive research institution perceive as necessary for their students to succeed in their subsequent business courses?

Methods

With the approval of the university's Institutional Review Board, I sent a recruitment email to all faculty members in the school of Business at a medium sized U.S. comprehensive research university. The email contained a link to an anonymous online questionnaire using the Surveymonkey website. The questionnaire contained six free-response items addressing the usage of calculus concepts in business courses (see appendix). I purposefully chose not to provide a list of calculus concepts, because I was interested in what the business faculty perceived as calculus concepts.

Table 1: The number of participants and number of instructors in each department

Department	Participants	Instructors
Accounting	2	20
Economics and Legal Studies	8	21
Entrepreneurship	1	13
Finance	5	16
Management	3	29
Marketing	5	21
MSIS	4	20

(20% overall response rate)

The university at which the study was conducted has seven departments within the School of Business. The departments are Accounting, Economics and Legal Studies, Entrepreneurship, Finance, Management, Marketing, and Management Science and Information Systems (MSIS). Out of the 140 faculty members and lecturers in the Business

School, there were 28 respondents. The respondents represented a wide range of departments, and all departments were represented by at least one participant, as shown in Table 1.

The respondents to the survey identified courses in Economics, Finance, and MSIS that utilized calculus concepts. Based on these recommendations, I contacted the faculty members who were currently teaching those courses or who had taught them in the previous two semesters to recruit them for interviews. Seven instructors agreed to participate: three from Economics, two from Finance, and two from Management Science Information Systems. In the interviews, I asked them to elaborate on the calculus content that they use in their courses and the applications of calculus content specific to the courses they teach.

Results

Instructors from four of the seven departments listed calculus concepts used in their courses. Instructors from Accounting, Entrepreneurship, and Management did not list any Calculus concepts at all. All of the concepts that were mentioned by instructors are listed in Table 2, organized by department. When more than one instructor was representing a department, I took the intersection of the responses. The most frequent Calculus concepts mentioned were optimization, derivatives, and rates of change. Three participants mentioned marginal functions or marginal analysis. Economics instructors seemed to emphasize multivariable functions, partial derivatives, and optimization of multivariable functions. Two finance instructors mentioned integration, though one specified that he was referring to numerical integration such as the normal distribution. One economics instructor mentioned that integration was used some by doctoral level students in his department.

Table 2: Calculus concepts needed in business courses by department

Economics and Legal Studies	Finance	Marketing	MSIS
Derivatives, marginal functions, constrained optimization, partial derivatives, Lagrange multipliers, integrals (for doctoral level students)	Rates of change, second derivatives, geometric series, marginal analysis, integration	marginal cost/revenue rates of change	optimization, derivative rules, Economic Order Quantity

The topics covered in Applied Calculus tend to be similar to topics covered in a first semester Calculus course, though not as rigorous. For example, derivative rules may be taught without a detailed study of limits. The CRAFTY report recommended that "the business calculus curriculum should include an introduction to rates of change, the dynamic nature of real-world systems, constrained optimization, and interpretations of area under a graph" (Lamoreux, 2000). The business instructors in this study did mention rates of change, derivatives, optimization, and integration. The Economics instructors believed that multivariable functions and partial derivatives were also important topics. It was unclear the extent to which integration was used by the instructors, as only three of the 28 participants mentioned integration.

Technology usage was also recorded. The instructors were asked to indicate both the technology that they use in the courses they teach as well as the technology with which their students should be familiar to succeed in their program. Sixty-eight percent of the instructors said that their students need to be able to use Excel and 54% said that their students need to be familiar with (non-graphing) calculators. Only 7% said students should be familiar with graphing calculators, and 7% said their students should be familiar with Wolfram Alpha. No respondents selected Maple, and 18% of respondents selected "Other" and specified SAS or SPSS, which were not specifically listed because they are statistics software.

The survey participants were asked to identify courses in their department whose content uses calculus concepts. The courses identified were: Macroeconomics, Microeconomics, Managerial Economics, Investments, Futures & Options, Financial Management, Operations Management, Management Science Methods, and Management Decision Theory. The instructors who were currently teaching these courses or who had taught the course in the previous two semesters were solicited for interviews, and seven instructors volunteered to participate. Three were from Economics (E1, E2, and E3), two from Finance (F1 and F2), and two from MSIS (M1 and M2).

From the interviews, I learned that business calculus is currently not a prerequisite for the economics or finance courses listed above. The MSIS department has recently added business calculus as a prerequisite for their upper-division courses. Professor E3 said, "I have switched to using more algebra and less calculus. But, of course, you cannot avoid calculus in Intermediate Micro." He later said, "I think I don't want my students to be penalized because they don't understand the math. So, if I get the feeling that my students are struggling with the math, I find a way to simplify it." Another economics professor, E2, lamented, "a lot of students don't have the mathematical background. It would be nice for my students to have Calculus first." One of the finance professors, F1, said, "I use quite a bit of math, but not a ton of Calculus." The next few paragraphs will outline the calculus concepts that the instructors from each department use in their courses.

Instructor E1 said, "In economics, everybody is maximizing or minimizing something. Whether it's a consumer trying to maximize their well-being subject to a budget constraint, a firm attempting to minimize production costs associated with producing some output, or a firm attempting to maximize their profit. In economics, life is constrained optimization." When asked what types of optimization problems are given to students, the instructor said that they typically give students optimization problems with linear constraints so that they can be brought back to single-variable optimization problems. They sometimes use Lagrange multipliers to solve more complicated optimization problems. E3 said, "If I have students who have some Calculus background, then I will use Lagrange multipliers."

All three economics professors said that they use derivatives to find marginal functions and that they do marginal analysis in their courses. They also verified that it was a standard convention in Economics to always graph using quantity as the independent variable and price as dependent. Professor E1 stated that he would use the second derivative test if the students had the mathematical background. They all said that they use the derivative when computing elasticity of demand. The professor who has taught macroeconomics said that he uses the first derivative of the logarithm of the Gross Domestic Product to determine the growth rate.

The final question for each of the economics professors was whether they thought that integration or partial derivatives were more important for their students. All three said that they thought partial derivatives were more important. They all said that they use integration in their graduate courses, but that the graduate students were required to have had a more traditional calculus sequence along with differential equations and linear algebra.

The finance professors said that they review elasticity, cost, revenue, profit, supply, and demand. They use mathematics to compute present and future values of exponential functions. Professor F1 said that he thought it was essential to emphasize applications. In his course, F1 said that he uses geometric series. He also said that they often maximize return for a given risk or minimize risk for a given return, but that they reason about these things graphically and don't usually derive them analytically. F1 also stated that using spreadsheets is important for Finance students, and that graphing calculators are not useful for them outside of their math classes. He said that when they take their licensing exam, they are not allowed to use programmable calculators, but may use a financial calculator.

Professor F2 said that there is an application of the derivative in bonds. The duration is the partial derivative of the price of a bond with respect to interest rates. He said that they usually reason about duration graphically. The convexity of the bond is the second derivative. When asked about integration, F2 said, “In my course, there’s really not any integration, but that doesn’t mean I don’t think it’s important. To get to the level of finance that you are using integrals, that will be Master’s or PhD level, because a lot of it is in probability where you’re integrating over the normal density, but we don’t do that in undergrad, at least as far as I know.” F1 also said that he does a lot with Excel in his courses, and does some optimization problems using the Solver tool in Excel.

The MSIS professors said that they didn’t use much calculus per se in their courses, but M1 stated, “It’s more of a mathematical maturity that we’re wanting background-wise before we do linear programming, for instance.” In one MSIS course, there is a chapter on non-linear programming which deals with optimization of two-variable functions using Lagrange multipliers, and M1 said that he would quickly review the derivative rules in that chapter, but that “if [the students] have never had that, then the little 5 minute time that I spend on it isn’t enough.” M2 stated that he didn’t use calculus in his course, but did expect students to be able to solve systems of linear equations. M1 also uses optimization when determining the Economic Order Quantity, which deals with finding the quantity that a business should order to minimize the total costs, including purchasing costs and carrying costs. Neither of the MSIS professors mentioned partial derivatives or integration.

Discussion

The survey showed that the concepts of differentiation, optimization, and rate of change were mentioned most frequently by business faculty as topics that would be useful to their students. Instructors in three departments: Accounting, Entrepreneurship, and Management, did not list any calculus concepts needed by their students.

In mathematics departments, courses tend to be taught with graphing calculators, but very few of the business instructors viewed graphing calculators as a useful technology for their students. Spreadsheets and non-graphing calculators were the most frequently selected type of technology used in business courses. Interviews with Finance professors revealed that their students are encouraged not to use graphing calculators because they are not allowed on their licensing exams. Thus, if we are trying to prepare students to succeed in their subsequent business courses, it makes sense to reconsider the technology that is used in the classroom.

Like May (2012) found, integration may not be a topic that is as important to business students as it is to mathematics instructors. The Economics and Finance professors specifically stated that integration was not used in their undergraduate courses, and that the masters and doctoral students who would use integration were required to have a more rigorous mathematical background (up through differential equations) before entering the program. Calculus of several variables, particularly partial derivatives and optimization of multivariable functions, is used in undergraduate Economics courses, and was also mentioned by one of the Finance instructors. Thus, it may be that partial derivatives are more directly applicable to business students than integration, at least for students who do not plan to attend graduate school.

Since the study has a small sample size and is restricted to one institution, it is not clear whether these results reflect the perspective of business faculty on a national scale. Also, this study only addressed the calculus content needed for students’ success in their subsequent business courses, which may differ from the content needed to succeed in their future careers. It is unclear whether the content goals of business faculty are consistent with the needs of the students' future employers. Further studies could investigate either the perspectives of business faculty at other institutions or the perspectives of managers in business industries.

This study is meant to begin a conversation about the needs of the students who are taking these courses and the goals of the departments who are being served by the Business Calculus course. Along with the addition of more problem solving, business applications, and business technology as suggested by the CRAFTY report (Lamoreux, 2000), mathematics departments may need to re-think the sequencing of the content in these courses to reflect the students' needs.

References

Asiala, M., Cottrill, J., Dubinsky, E., & Schwingendorf, K. (1997). The development of students' graphical understanding of the derivative. *Journal of Mathematical Behavior, 16*(4), 399-431.

Bressoud, D., Carlson, M., Mesa, V., & Rasmussen, C. (2013). The calculus student: insights from the Mathematical Association of America national study. *International Journal of Mathematical Education in Science and Technology, 44*(5), 685-698.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-378.

Depaolo, C. & Mclaren, C. (2006). The relationship between attitudes and performance in business calculus. *INFORMS Transactions on Education, 6*(2), 8-22.

Garner, B. & Garner, L. (2001). Retention of concepts and skills in traditional and reformed applied calculus. *Mathematics Education Research Journal, 13*(3), 165-184.

Gonzalez-Martin, A. & Camacho, M. (2004). What is first-year Mathematics students' actual knowledge about improper integrals? *International Journal of Mathematical Education in Science and Technology, 35*(1), 73-89.

Gravemeijer, K. & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics, 39*(1), 111-129.

Lamoreux, C. (2004). Report for business and management. In CRAFTY *The Curriculum Foundations Project: Voices of the Partner Disciplines.* Washington, DC: Mathematical Association of America, available online at http://www.maa.org/cupm/crafty/cf_project.html.

Liang, J. & Martin, L. (2008). An Excel-aided method for teaching calculus-based business mathematics. *College Teaching Methods & Styles Journal, 4*(11), 11-23.

May, M. (2012). Rethinking business calculus in the era of spreadsheets. *Proceedings of the conference on Research in Undergraduate Mathematics Education*

Oehrtman, M. (2009). Collapsing dimensions, physical limitation, and other student metaphors for limit concepts. *Journal for Research in Mathematics Education, 40*(4), 396-426.

Thompson, P. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics, 26*(2), 229-274.

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. *Research in Collegiate Mathematics Education. IV. CBMS Issues in Mathematics Education*, 103-127.

Appendix

Survey Questions for Business Faculty

Business Calculus (MATH 2013) is a required course for all business majors. The current study will investigate the usage of Calculus concepts in Business courses. The goal is to tailor the Business Calculus course to meet the needs of Business students and to help them transfer the concepts of Calculus to the various applications that they will see in their subsequent courses and in their future careers.

1. What department are you in?

2. Which courses in your department use calculus concepts?

3. What calculus concepts are used in these courses?

4. Are there other mathematics concepts that students need to succeed in courses in your department?

5. Do you use technology in your courses to investigate mathematical concepts?

6. What type of technology do you use in the courses you teach? (Check all that apply)

Calculators Graphing Calculators Maple Wolfram Alpha
Microsoft Excel Other (Please specify)

7. What type of technology do students in your department need to be successful in their business courses? (Check all that apply)

Calculators Graphing Calculators Maple Wolfram Alpha
Microsoft Excel Other (Please specify)

ABSTRACT ALGEBRA AND SECONDARY SCHOOL MATHEMATICS: IDENTIFYING MATHEMATICAL CONNECTIONS IN TEXTBOOKS

Ashley L. Suominen
University of Georgia

Many stakeholders concur that secondary teacher preparation programs should include study of abstract algebraic structures, and most certification programs require an abstract algebra course for prospective mathematics teachers. However, research has shown that undergraduate students learning abstract algebra for the first time often struggle to relate the new abstract concepts to previously learned concepts found in secondary school mathematics. In this study, I examined nine undergraduate abstract algebra textbooks to elaborate on the mathematical connections explicitly stated in the textbooks between abstract algebra and secondary school mathematics concepts, and categorized them according to an analytic framework based on categories of mathematical connections found in previous literature. In this report, I discuss three specific categories of connections: comparison through common features, generalization, and hierarchical or inclusion. I found that the connection types discussed in textbooks differ from past literature that highlight connections related to abstract algebra as generalization of school algebra.

Key words: Abstract algebra, Mathematical connections, Textbook analysis

Introduction

Zazkis and Leikin (2010) revealed that beginning undergraduate students experience great difficulty when starting their undergraduate mathematics courses because of their inabilities to build connections between these courses and the secondary school mathematics curriculum:

> Without connections students have to rely on their memory only and to remember many isolated concepts and procedures. To connect mathematical ideas means linking new ideas to related ideas considered previously and solving challenging mathematical tasks by thinking how familiar concepts and procedures may help in the new situations. (p.275)

Similarly, the first CBMS (2001) report on the mathematical education of teachers acknowledged, "Unfortunately, too many prospective high school teachers fail to understand connections between [abstract algebra and number theory] and the topics of school algebra" (p. 40). Further, Cuoco (2001) noted in an article about secondary teacher preparation programs, "Most teachers see very little connection between the mathematics they study as undergraduates and the mathematics they teach. This is especially true in algebra, where abstract algebra is seen as a completely different subject from school algebra" (p. 169). Cofer (2012) confirmed these assertions when she considered how prospective secondary mathematics teachers communicated the conceptual connections between abstract algebra and their teaching practices. She discovered that these prospective teachers were unable to link the two, despite having just finished an abstract algebra course. To put it simply, undergraduate mathematics students and especially prospective mathematics teachers are not recognizing the mathematical connections between abstract algebra and secondary school mathematics.

The purpose of this study was to examine abstract algebra textbooks in order to identify the mathematical connections explicitly stated between abstract algebra and secondary school mathematics concepts and classify how these connections were discussed. Few studies have explored these mathematical connections between secondary and tertiary mathematics, and the perspectives of textbooks have also not been considered. Thus, this research attempted to address these gaps in literature by considering the mathematical connections stated in abstract algebra textbooks. Textbooks were chosen as the chief source of data regarding the abstract algebra curriculum, because it is my belief that textbooks often drive the tertiary mathematics curriculum. Robitaille and Travers (1992) discovered that mathematics teachers rely heavily on their textbooks to help design the curriculum that they teach, and while Robitaille and Travers were referring to Grades K–12 mathematics teachers, I believe their observation is also true of tertiary mathematics professors. Further, I believe that if a textbook were to identify mathematical connections between abstract algebra and school mathematics, then professors will more likely discuss these connections in the course.

Theoretical Framework

Businskas (2008) and Singletary (2012) provided similar definitions of a *mathematical connection* in their dissertations: "a true relationship between two mathematical ideas" (Businskas, 2008, p. 18) and "a relationship between a mathematical entity and another mathematical or nonmathematical entity" (Singletary, 2012, p. 10). These relationships take different forms in existing literature: mathematics connections as a characteristic of mathematics, mathematical connections as an artifact of learning, and mathematical connections as an active process of doing mathematics. Mathematical connections as a characteristic of mathematics can be interpreted in different ways. For instance, Coxford (1995) described mathematical connections as unifying themes within the discipline (e.g., function and variable) or mathematical processes (e.g., proof and problem solving). Others (Businskas, 2008; Skemp, 1987; Zazkis, 2000) have explained this relationship as a concept-by-concept link in which two very specific concepts are linked in some way. Still other research (Businskas, 2008; Chappell & Strutchens, 2001; Hodgson, 1985) defined mathematical connections as equivalent representations across mathematical ideas. Despite these differences, these researchers shared an underlying belief that mathematical connections exist within the discipline of mathematics.

A second perspective of connections found in the literature is mathematical connections as an artifact of learning. These connections are described as "a process that occurs in the mind of the learner(s) and the connection is something that exists in the mind of the learner" (Businskas, 2008, p. 12-13). Mathematical connections in light of this perspective are a necessary component to learning because new mathematical ideas are connected to pre-existing schema or networks within the mind of the learner. Hazzan (1999) and Hiebert and Carpenter (1992) concluded that the teacher plays a pivotal role in helping the students to construct these connections among mathematical ideas. The final perspective, mathematical connections as a mathematical process or activity, blends the initial two perspectives in acknowledging that connections exist across the discipline and the learner should be involved in the activity of establishing or identifying these connections. Research from this perspective affirmed that the activity of making connections across mathematics is a significant aspect of doing mathematics (Boaler, 2002).

The theoretical perspective of mathematical connections as a characteristic of mathematics was utilized in this study because it best allows me to address my research question: What mathematical connections are explicitly stated between abstract algebra and secondary school mathematics in abstract algebra textbooks and how are these connections discussed? Previously established categories of mathematical concept-to-concept connections are displayed in Table 1 (Businskas, 2008; Singletary 2012). These categories aided me when analyzing the textbook and interview data.

Table 1

Categories of mathematical connections from research

Category	Description
Alternate representation	One concept is represented in different ways such as symbolic (algebraic), graphic (geometric), pictorial (diagram), manipulative (physical object), verbal description (spoken), or written description.
Comparison through common features	Two concepts share some features in common, which allows a comparison through the concepts being similar, exactly the same, or not the same.
Equivalent representation	One concept is represented in different ways but within the same form (i.e., one concept could be represented in different ways symbolically).
Generalization	One concept is an example of specific instance of another concept.
Hierarchical or inclusion	One concept is a component of or included in another concept. Since one concept is included or contained in the other concept, a hierarchical relationship exists between two concepts.
Logical implication	One concept logically dependences on another concept. Often an *if-then* relationship exists between the two concepts.
Procedural	One concept can be used to find another concept. The first concept could be a type of procedure or connecting method used when working with the other concept.
Real world application	One concept is an example of another concept in the real world (i.e., a concept refers to another concept outside the current mathematical context).

Relevant Literature

In one of the first articles to address the mathematical connections between abstract algebra and secondary school mathematics, Usiskin (1974) examined the generalization connection between the properties of addition and multiplication of real numbers and the structural properties of groups and fields. In this article he also described the isomorphic relationship between linear and exponential functions through the map $x \rightarrow a^x$, which forms an isomorphic group under composition. In a second article, Usiskin (1975) further discussed the connections between algebraic structures and familiar number systems and operators. He also described connections between the group structure and solving simple linear equations, the multiplicative group of invertible 2×2 matrices and solving systems of linear equations, and additive and multiplicative groups with familiar number systems and groups of geometric transformations. Usiskin classified the connections mentioned in these two articles as generalizations taught in abstract algebra.

The Conference Board of the Mathematical Sciences [CBMS] (2001, 2012) published two reports, *The Mathematical Education of Teachers* [MET] and *The Mathematical Education of Teachers II* [MET2], that made recommendations regarding the necessary mathematical knowledge for teaching. In these recommendations, CBMS provided specific connections that could be made between tertiary mathematics courses and secondary school mathematics. In the first MET report, the mathematical connections mentioned primarily focused on abstract algebra concepts being generalizations of secondary school mathematics concepts. For instance, the report suggested incorporating a task into an abstract algebra course that showed "explicitly how the number and algebra operations of secondary school can be explained by more general principles" (CBMS, 2001, p. 40). Another example was given in which the algebraic structures found in abstract algebra are the generalizations of familiar solving procedures of linear equations. The MET report also highlighted mathematical connections between abstract algebra and school geometry by discussing a symmetry group as generalization of the geometry of transformations of regular polygons. In the second MET report, CBMS (2012) recommended the study of ring and field structures as the underlying structures of operations with polynomials and rational functions, and a focus on mathematical concepts *inverse* and *identity.* That is, the abstract algebra concept *inverse* is clearly connected to the secondary school mathematics concepts *additive* and *multiplicative inverse*, *inverse matrix*, and *inverse function*. The MET2 report also identified the abstract algebra concept *isomorphism* to be important to draw connections through comparison of common features of secondary school mathematics concepts.

Two research studies have been conducted exploring the perspectives of prospective secondary mathematics teachers about the mathematical connections between their undergraduate mathematics courses and the mathematics they will teach at the secondary level. In a qualitative study in Turkey, Bukova-Güzel, Ugurel, Özgür, and Kula (2010) interviewed 36 prospective teachers during their last year of their teacher preparation program. They found that 83% of the prospective secondary mathematics teachers did not see connections between the undergraduate content courses they had taken and the secondary school mathematics curriculum. However, 25% of the participants did believe first year undergraduate courses, such as Calculus and Analytic Geometry, were coherent and related to the secondary curriculum. One participant's response stated:

> Since the courses are based on memorizing theorems and passing exams, it is really hard for us to apply even useful knowledge. At least on my own behalf, I was better at secondary school mathematics topics when I graduated from secondary school. (p. 2236)

In fact, 42% of the participants recommended designing new undergraduate courses that were directly related to secondary school mathematics. Cofer (in press), in an interview study with five prospective secondary mathematics teachers whom had recently completed coursework in abstract algebra, found similar results. The participants were unable to relate abstract algebra concepts to the school mathematics curriculum when asked about specific concepts such as division by zero and even numbers.

Methodology

In this study I examined nine abstract algebra textbooks to identify the explicitly stated mathematical connections between abstract algebra and secondary school mathematics and classify the ways in which these connections were discussed. Textbooks were selected based on a few criteria. First, I narrowed my focus to introductory undergraduate abstract algebra textbooks. While graduate textbooks may explicitly state mathematical connections between abstract algebra and secondary school mathematics, this study concentrated solely on undergraduate learning of abstract algebra. Second, all textbooks were published within the past 20 years from the start of the research study because they would be more likely to be used in undergraduate classrooms today. Next, I compiled a list of recently published abstract algebra textbooks by: (1) examining syllabi available online for introductory abstract algebra courses at more than 20 colleges and universities around the United States, (2) contacting five different textbook publishers about widely readily used abstract algebra textbooks, and (3) conducting online searches of textbook provider websites for textbooks that explicitly emphasis connections to school mathematics. The textbooks used in the study are listed in Table 2.

Table 2

Included abstract algebra textbooks

Author	Title	Publication year
Cuoco & Rotman	*Learning Modern Algebra: From Early Attempts to Prove Fermat's Last Theorem*	2013
Dummit & Foote	*Abstract Algebra (3rd ed.)*	2004
Fraleigh	*A First Course in Abstract Algebra (7th ed.)*	2003
Gallian	*Contemporary Abstract Algebra (8th ed.)*	2013
Hillman & Alexanderson	*Abstract Algebra: A First Undergraduate Course (5th ed.)*	1994
Hodge, Schlicker, & Sundstrom	*Abstract Algebra: An Inquiry-based Approach*	2014
Nicholson	*Introduction to Abstract Algebra (4th ed.)*	2012
Nicodemi, Sutherland, & Towsley	*An Introduction to Abstract Algebra with Notes to the Future Teachers*	2007
Shifrin	*Abstract algebra: A geometric approach*	1996

Each abstract algebra textbook was read and analyzed it in its entirely (explanation sections, homework problems, and any additional material). However, I primarily focused on concepts associated with the algebraic structures groups, rings, and fields. For each textbook, I created a list of every identified mathematical connection, noted where each connection was found in the text, and classified the type of connection based on my analytic connection framework. I used the previously established categories of mathematical connections (seen in Table 1) as my analytic framework but modified it slightly to better categorize the data. The finalized analytic connection framework can be seen in Table 3. Next, I constructed separate tables for each connection category detailing all the connections found in the textbooks. In order to ensure accuracy, I reexamined all of the mathematical connections found in the textbooks a second time using the finalized analytic framework.

Table 3

Analytic connection framework

Category	Description
Alternate representations	One concept is represented in different ways such as symbolic (algebraic), graphic (geometric), pictorial (diagram), manipulative (physical object), verbal description (spoken), or written description.
Comparison through common features	Two concepts share some features in common, which allows a comparison through the concepts being similar, exactly the same, or not the same.
Generalization	One concept is a generalization of another specific concept.
Hierarchical or inclusion	One concept is a component of or included in another concept. Since one concept is included or contained in the other concept, a hierarchical relationship exists between two concepts.
Real world applications	One concept is an example of another concept in the real world (i.e., a concept refers to another concept outside the current mathematical context).

Results

In this section, I will elaborate on the three most commonly stated mathematical connections categories found in abstract algebra textbooks: comparison through common features, generalization, and hierarchical or inclusion. Categorizing the written texts was not a simple task. The textbook authors, for instance, may have intended to make a generalization connection when in fact the written text is more of a comparison connection. In addition, there is a fine line between these types of connections. Some hierarchical connections could also be generalizations depending on how the authors presented the connection. In short, the findings presented in this paper are those that were most explicitly stated in the abstract algebra textbooks.

Comparison through common features

The former connection category allowed for two concepts to be compared as being similar, exactly the same, or not the same because the concepts share some common features. Table 4 summarizes the comparison connections I found in the abstract algebra textbooks. I then provide a detailed description of the most frequently made connections.

Table 4

Mathematical connections: Comparison through common features

Abstract algebra concept	**Secondary school mathematics concept**	**Number of textbooks**
Algebraic structures & properties	Number systems, arithmetic operators	8
Congruence	Solving linear equations	1
Fundamental theorem of algebra	Polynomial roots	2
Homomorphism, kernel, image	Mathematical modeling	1
Polynomial ring	Polynomial operators and vocabulary	8
Quaternions	Complex numbers	2
Unit	Invertible matrices	2

Eight of the nine abstract algebra textbooks introduced algebraic structures such as group, ring, or field by comparing their defining properties to those of familiar number systems and arithmetic operators. Five of these textbooks explicitly stated and described these connections. For instance, Cuoco and Rotman (2013) wrote, "The main idea is to abstract common features of integers, rational numbers, complex numbers, and congruences, as we did when we introduced the definition of commutative ring" (p. 191); Hodge, Schlicker, and Sundstrom (2014) wrote, "Rings are algebraic objects that share the same basic structure as the integers" (p. 89); and Nicodemi, Sutherland, and Towsley (2007) wrote, "The arithmetic of fields is similar to the arithmetic of the rational numbers" (p. 89). All of these textbooks also stated or explained that the set of integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$, real numbers $\mathbb{R}$, and complex numbers $\mathbb{C}$ each form a group under addition because these sets share the four common features with groups: closure under addition, associativity, zero as the identity element, and negative numbers as inverse elements.

Arithmetic operators such as addition and multiplication were also used in this comparison. For instance, the set of integers $\mathbb{Z}$ forms a group under addition but not under multiplication. The known arithmetic operators—addition, subtraction, multiplication, and division—were also compared to a field, given that it is a smallest algebraic structure in which all of these operators can be performed by nonzero set elements. Most textbooks also noted that known arithmetic operators have the same features as binary operators; namely, two set elements are combined to obtain one set element. Hillman and Alexanderson (1994) wrote, “Our notation for the operation has been the same as for multiplication in our familiar number system” (p. 74). This relationship is not surprising given that arithmetic operators are specific examples of binary operators. Focusing on the properties and characteristics of known number systems and arithmetic operators enabled the textbook authors to compare them to the newly introduced algebraic structures by drawing on shared and unshared features.

The polynomial ring was compared in eight of the nine abstract algebra textbooks to an abundance of polynomial information from secondary school mathematics. Much of the vocabulary used with polynomials in secondary algebra, such as *coefficient*, *degree*, and *polynomial equality*, was stated to be the same as the vocabulary used with polynomial rings. For instance, Nicodemi, Sutherland, and Towsley (2007) wrote, “In high school algebra, the polynomials studied usually had coefficients that were either integers or rational numbers. We will *extend the scope* of that investigation to consider polynomials with coefficients in other commutative rings” (p. 111, emphasis added). In fact, five textbooks explicitly mentioned the different types of polynomial coefficients found in secondary algebra and abstract algebra. Fraleigh (2003) also pointed out, “We will be working with polynomials from a slightly different viewpoint than the approach in high school algebra or calculus” (p. 198) when he contrasted the vocabulary used for the symbol x as a variable with polynomials in secondary algebra and indeterminate with polynomial rings in abstract algebra. Similarly, all of the textbooks mentioned the similar types of operations performed on polynomials and polynomial rings. For example, Dummit and Foote (2004) wrote, “The operations of addition and multiplication which make $R[x]$ into a ring are the same operations familiar from elementary algebra: addition is componentwise” (p. 234). The eight textbooks then discussed polynomial long division using a comparison of similar features and vocabulary with either numerical long division or polynomial long division.

Generalization

The generalization connection category designated when one concept is a generalization of another concept. Table 5 summarizes these connections found in the abstract algebra textbooks and then I provide a detailed description of the most frequently made connections.

Table 5

Mathematical connections: Generalization

Abstract algebra concept	**Secondary school mathematics concept**	**Number of textbooks**
Algebraic structures	Number systems	5
Binary operators	Arithmetic operators and number systems	4
Direct product	Cartesian plane and ordered pairs	2
Inverse	Negatives; Multiplicative reciprocal	5
Irreducibility	Factoring polynomials	5
Quotient Field	Fractions, operations with fractions	5
Sign rule in a ring	Product of two negative numbers is positive	5

One of the most commonly discussed connections in the abstract algebra textbooks was how various algebraic structures (i.e., group, ring, field) were generalizations of familiar number systems. Five of the nine textbooks explicitly mentioned that generalization. For instance, several textbooks noted that the integers $\mathbb{Z}$, the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, and the complex numbers $\mathbb{C}$ were familiar number systems that form groups under addition and the nonzero elements of $\mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ form groups under multiplication. Hillman and Alexanderson (1994) stated, "The most basic number systems are examples of groups, and we all learn to deal with these early on" (p. 40) and Nicodemi, Sutherland and Towsley (2007) called these number systems "protypes" under the "umbrella" of an arbitrary algebraic structure. Four textbooks then introduced the more general binary operator * for the familiar addition and multiplication operators. The definition of a group is then developed having that understanding of a binary operator. A similar approach was used to introduce rings, integral domains, and fields.

Five of the abstract algebra textbooks connected the concept of irreducible polynomials in a polynomial ring to prime polynomials and factoring. Upon introducing irreducibility, Fraleigh (2003) wrote, "The concept is probably already familiar. We really are doing high school algebra in a more general setting" (p. 213). In secondary school mathematics a polynomial is defined to be prime if it is unable to be factored, whereas in abstract algebra a polynomial is irreducible if it has no factorization of polynomials of lower degree than original polynomial. The latter is the generalization of the former for all polynomial rings.

The quotient field was introduced in five of the abstract algebra textbooks as the generalization of fractions and operations on fractions. Those textbooks reviewed concepts such as equivalent fractions, equating two fractions, and operating on the fractions (addition and multiplication) prior to introducing the more general understanding of a fraction. For example, Dummit and Foote (2004) noted, "In more precise terms, the fraction $\frac{a}{b}$ is the equivalence class of ordered pairs (a, b) of integers with $b \neq 0$ under the equivalence relation $(a, b) \sim (c, d)$ if and only if $ad = bc$" (pp. 260–261). With this definition in mind, the textbook then explored the properties, operations, and proven results of quotient fields.

Five of the textbooks also generalized the notion that the product of two negative numbers is positive. These textbooks utilized ring properties to prove four sign rules for negatives. Hillman and Alexanderson (1994) wrote, "The following result [sign proof] is a generalization of one of the rules of signs of elementary algebra" (p. 217). Nicodemi, Sutherland and Towsley (2007) noted, "It is interesting to see how to deduce these facts from the abstract properties of rings rather than from the elementary cookie-counting arguments that we usually use to explain the arithmetic of the natural numbers" (p. 84). These authors in particular allude to the importance of this connection in teaching by providing a ring based rationale for the positive result of multiplying two negative numbers and contrasting it with the rationale generally accepted in school mathematics.

Hierarchical or inclusion

A hierarchical or inclusion connection is one in which a concept is a component of or included in another concept. When one concept is included or contained in the other concept, a hierarchical relationship exists between the two concepts. Table 6 summarizes the comparison connections I found in the abstract algebra textbooks. I then provide a detailed description of the most frequently made connections.

Table 6

Mathematical connections: Hierarchical or inclusion

Abstract algebra concept	**Secondary school mathematics concept**	**Number of textbooks**
Algebraic structures & properties	Solving linear equations	6
Compass/geometric constructions	Ruler, circles, intersection, and other geometric concepts	7
Cyclic group	Division algorithm	1
Extension field/splitting field	Solving for the roots of a polynomial	5
Isomorphism	Function	3
Nilpotent	Geometric series and convergence	1
Permutation group	Function and function composition	3
Polynomial ring	Power series	1
Symmetry group	Rotation, reflection, function composition	7
Zero divisors	Solving quadratic equations by factoring	4

Seven of the nine abstract algebra textbooks included a chapter or section on compass or geometric constructions, which are geometric applications of field theory. The geometric constructions first experienced in high school geometry using a compass and straightedge were given an algebraic context. Thus, students should first have a basic understanding of geometric concepts such as angles, circles, distance, intersection, regular *n*-gons, and trisection in order to understand how their understanding relates to geometry. In fact, Dummit and Foote (2004) wrote:

> It is an elementary fact from geometry that if two lengths a and b are given one may construct using straightedge and compass the lengths $a \pm b$, ab, and $\frac{a}{b}$. It is also an elementary geometry construction to construct $\sqrt{a}$ if a is given: construct the circle with diameter $1 + a$ and erect the perpendicular [line] to the diameter. The length is $\sqrt{a}$. (p. 532)

Thus, all arithmetic operations can be constructed using a compass and straightedge, and additive or multiplicative inverses, the product of two numbers, and the square root of a number are all constructible numbers. Gallian (2013) defined a *constructible number* as a real number $\propto$ in which a line segment can be drawn with length $|\propto|$ in a finite number of steps (p. 400). He then provided three ways to construct such points: intersect two lines, intersect two circles, or intersect a line and a circle (p. 401). One chief result is to show the set of all constructible numbers then forms a subfield of $\mathbb{R}$ and any constructible number must be a field extension of $\mathbb{Q}$. Another result that is especially useful for future secondary mathematics teachers is that one cannot trisect an angle using compass and straightedge.

Similarly, seven of the abstract algebra textbooks introduced symmetry groups with geometric transformations. For instance, Nicodemi, Sutherland and Towsley (2007) defined symmetry of a regular figure as "a rotation of the figure around an axis of symmetry that takes the figure congruently onto itself" (p. 196). Gallian (2013) posed it this way:

> Suppose we remove a square region from a plane, move it in some way, then put the square back into the space it originally occupied. More specifically, we want to describe the possible relationships between the starting position of the square and its final position in terms of motion. (p. 31)

Thus, students can use their knowledge of high school geometry concepts of rotations, axes of symmetry, angle bisectors, and reflections in order to understand a symmetry group. Several of the textbooks illustrated the six symmetries (identity, 2 rotations, and 3 flips/reflections around the axes) for an equilateral triangle. Dummit and Foote (2004) generalized those findings for all regular n-gons: "There are exactly $2n$ symmetries of a regular n-gon" and "These symmetries are the n rotations about the center through $2\pi i/_{n}$ radian, $0 \leq i \leq n-1$, and the n reflections through the n lines of symmetry" (p. 24). In addition, these textbooks employed students' knowledge of function composition to discuss the operator used with symmetry groups. Gallian (2013) explained the operation order in this way, "In lower level math course function composition $f \circ g$ means g followed by f" (p. 33), meaning that the order of the symmetries moves right to left similarly to function composition.

Six of the abstract algebra textbooks seemed to rely on students' previous understandings of solving linear equations to serve as a foundation for the new algebraic structures. For instance, simple linear equations of the forms $a + x = b$ and $ax = b$ form groups under addition and under multiplication, respectively. Four of the textbooks asserted that the knowledge of solving such equations was an integral component in the definition of a group; namely, the properties used to solve a simple linear equation define the group structure. Table 7 illustrates this information with three different operators: addition, multiplication, and the more formal binary operator. Three of the nine textbooks introduced a field in a similar manner by walking through the properties needed to solve a linear equation of the form $ax + b = cx + d$. To solve this problem one must utilize additive and multiplicative inverses, additive and multiplicative identities, additive and multiplicative associativity, closure under addition and multiplication, and the distributive law. These properties along with commutativity are integral components of understanding the definition of a field. In both instances the properties used in solving linear equations are contained in the definitions of various algebraic structures.

Table 7

Simple linear equations form a group

1) Given		
$x + a = b$	$xa = b$	$x * a = b$
2) Determine the inverse of *a* under the operation and apply the operation with its inverse on the right of both sides of equation.		
$(x + a) + -a = b + -a$	$(xa) \cdot \frac{1}{a} = b \cdot \frac{1}{a}$	$(x * a) * a^{-1} = b * a^{-1}$
3) Use the associative law under the operation to regroup the left side of the equation.		
$x + (a + -a) = b + -a$	$x\left(a \cdot \frac{1}{a}\right) = b \cdot \frac{1}{a}$	$x * (a * a^{-1}) = b * a^{-1}$
4) The result of regrouping *a* with its inverse is the identity under the operation [let *e* be the unknown identity under the operation].		
$x + 0 = b + -a$	$x \cdot 1 = b \cdot \frac{1}{a}$	$x * e = b * a^{-1}$
5) Combining *x* with the identity under the operation results in *x* itself.		
$x = b + -a$	$x = b \cdot \frac{1}{a}$	$x = b * a^{-1}$

Similarly, five of the textbooks introduced the concept of extension field by building upon students' knowledge of solving for the roots of polynomial functions. For instance, if the given field is the real numbers $\mathbb{R}$, then the simple polynomial function is $f(x) = x^2 + 1$. Clearly, this polynomial does not have a solution in the field and is thus irreducible, so the question then arises whether or not a larger field that contains $\mathbb{R}$ would provide a root for the polynomial. Gauss answered that question by introducing the complex number system $\mathbb{C} = \mathbb{R} + \mathbb{R}i$ where $i^2 = -1$. Thus, to solve the polynomial $f(x) = x^2 + 1$, the domain needs to be extended to the complex number system.

Analogously, four of the textbooks made a connection between students' previous knowledge of solving quadratic equations by factoring and the abstract algebra concept of zero divisors. These textbooks illustrated that to solve a quadratic equation, say $x^2 + 2x - 15 = 0$ where $x \in \mathbb{R}$, the students learned to factor the equation, $(x + 5)(x - 3) = 0$, and conclude the only way that a product can equal zero is if one of the factors is zero, $x + 5 = 0$ or $x - 3 = 0$, so the only two possible solutions of the equation are $x = -5$ or $x = 3$. Students are then asked to utilize this previous knowledge to understand zero divisors. Several of these textbooks then problematized solving quadratics in a modular ring. Thus, the understanding of solving for the roots of a quadratic was included in the abstract algebra concept of zero divisors.

Conclusions

One interesting conclusion that emerged from this research is that despite the varying mathematical connection perspectives described in abstract algebra textbooks, there were similar key perspectives that could be categorized into specific groups. That is, the textbook authors discussed mathematical connections in five distinct ways: alternate representation, comparison through common features, generalization, hierarchical or inclusion, and real world application. However, textbooks primarily described specific mathematical connections using alternate representations, comparison through the common features, and hierarchical or inclusion as seen in Table 8. To be more specific, the sheer number of different connections explicitly made in textbooks was highest for the hierarchical or inclusion connection category, but the most commonly stated mathematical connection of specific concepts found in the largest number of textbooks used comparison of common features. These results are inconsistent with previous research that has suggested the primarily mathematical connection between abstract algebra and secondary school mathematics is generalization.

Table 8

Most commonly stated mathematical connections

Connection category	Abstract algebra concept	Secondary school mathematics concept	Number of textbooks
Alternate representation	Group	Geometric transformations, Solving linear equations	7
Comparison through common features	Algebraic structures & properties	Number systems, arithmetic operators	8
Comparison through common features	Polynomial ring	Polynomial operators and vocabulary	8
Hierarchical or inclusion	Compass/geometric constructions	Ruler, circles, intersection, and other geometric concepts	7
Hierarchical or inclusion	Symmetry group	Rotation, reflection, function composition	7

Another interesting conclusion that emerged from this research is that the mathematical connections stated in the abstract algebra textbooks were linked to secondary school geometry nearly as often as secondary school algebra. These results were also inconsistent with previous research that suggested that the importance of abstract algebra lies in its mathematical connections to school algebra. To be more specific, seven of the nine textbooks connected group theory to geometric transformations through alternate representation and hierarchical or inclusion connections. In addition, seven textbooks connected the abstract algebra concepts *compass or geometric constructions* to school geometry concepts *angles*, *circles*, *regular n-gons,* etc. A comparable number of abstract algebra textbooks explicitly made connections to secondary school algebra. That is, seven textbooks made an alternate representation connection between the abstract algebra concept *group theory* and secondary school algebra concept *solving linear equations*, and eight textbooks compared specific abstract algebra structures to secondary school algebra structures, operators, and vocabulary.

Implications

One major implication that can be drawn from the results of this research is the rationale for requiring prospective secondary mathematics teachers to take abstract algebra should be reevaluated. Stakeholders and policymakers' recommendations and previous research have often characterized abstract algebra as the generalization of school algebra. However, the findings of this study revealed a discrepancy between these held beliefs and the actual mathematical connections described in abstract algebra textbooks. In fact, other connections and connection types were discussed with greater frequency in this study. Abstract algebra can no longer be considered simply as the generalization of school algebra but rather it should be regarded as an extension of previous mathematical knowledge from algebra and geometry. This study's results revealed that textbook authors identified and discussed mathematical connections between abstract algebra and secondary school geometry nearly as often as those connections to secondary school algebra. Thus, abstract algebra provides prospective secondary mathematics teachers knowledge that is important to their understandings of school algebra as well as their understandings of school geometry.

Furthermore, the mathematical connections between abstract algebra and secondary school algebra in addition to the connections between abstract algebra and secondary school geometry are not simply generalizations. In fact, I discovered through this research that abstract algebra textbook mentioned connections of other types more frequently; namely, connections of the types: comparison of common features, hierarchical or inclusion, or alternate representations. The rationale for requiring prospective secondary mathematics teachers to take an abstract algebra course should then include abstract algebra as a further study of familiar mathematical ideas through studying structural comparisons, building upon previous mathematical concepts, and using alternate representations of algebraic concepts.

Another implication that can be drawn from the results of this study is that a variety of mathematical connections between abstract algebra and secondary school mathematics can be made, and these connections can be described in various ways. It is important for abstract algebra professors to recognize that not all abstract algebra textbooks identify the same mathematical connections nor do all the textbooks discuss mathematical connections in the same way. For instance, not all of the abstract algebra textbooks analyzed introduced polynomial rings using comparison of common features even though the majority of the texts did. By identifying the specific mathematical connections found in their assigned textbook, can build on these connections to help students develop more accurate concept images and concept definitions of abstract algebra content. For instance, professors can discuss connections with students in the classroom analogously to the ways in which connections are described in the assigned text. Coherently discussing mathematical connections may enhance student understanding of important connections or help students build new understandings from previous knowledge. Further, professors should be aware of the identified connections and connection types omitted from their assigned textbook so that they can discuss omitted connections in class or through supplementary materials. Abstract algebra professors may also want to talk about connections with students in ways that are not found in the textbook but would be beneficial to learning. For example, a professor may want to introduce alternate representations of a group when the assigned textbook only presents one representation of a group.

References

Boaler, J. (2002). Exploring the nature of mathematical activity: Using theory, research and "working hypotheses" to broaden conceptions of mathematics knowing. *Educational Studies in Mathematics*, *51*, 3–21.

Bukova-Güzel, E., Ugurel, I., Özgür, Z., & Kula, S. (2010). The review of undergraduate courses aimed at developing subject matter knowledge by mathematics student teachers. *Procedia: Social and Behavioral Sciences*, *2*, 2233–2238.

Businskas, A. M. (2008). *Conversations about connections: How secondary mathematics teachers conceptualize and contend with mathematical connections*. (Unpublished doctoral dissertation). Simon Fraser University, Burnaby, BC, Canada.

Chappell, M. F., & Strutchens, M. E. (2001). Creating connections: Promoting algebraic thinking with concrete models. *Mathematics Teaching in the Middle School, 7*(1), 20–25.

Cofer, T. (2012). *Mathematical explanatory strategies employed by prospective secondary teachers*. Manuscript submitted for publication.

Conference Board of the Mathematical Sciences. (2001). *The mathematical education of teachers* (Issues in Mathematics Education, Vol. 11). Providence, RI: American Mathematical Society.

Conference Board of the Mathematical Sciences. (2012). *The mathematical education of teachers II* (Issues in Mathematics Education, Vol. 17). Providence, RI: American Mathematical Society.

Cook, J. P. (2012). *A guided reinvention of ring, integral domain, and field*. (Doctoral dissertation, University of Oklahoma, Norman, Oklahoma). Available from ProQuest Dissertations and Theses databas. (UMI No. 3517320).

Coxford, A. F. (1995). The case for connections. In P. A. House & A. F. Coxford (Eds.), *Connecting mathematics across the curriculum* (pp. 3–12). Reston, VA: National Council for Teaching Mathematics.

Cuoco, A., & Rotman, J. J. (2013). *Learning modern algebra: From early attempts to prove Fermat's last theorem.* Washington, DC: Mathematical Association of America.

Dummit, D. S., & Foote, R. M. (2004). *Abstract algebra* (3rd ed.). Hoboken, NJ: John Wiley & Sons.

Fraleigh, J. B. (2003). *A first course in abstract algebra* (7th ed.). Reading, MA: Addison-Wesley.

Gallian, J. A. (2013). *Contemporary abstract algebra* (8th ed.). Boston, MA: Brooks/Cole, Cengage Learning.

Hazzan, O. (1999). Reducing abstraction level when learning abstract algebra concepts. *Educational Studies in Mathematics, 40*(1), 71–90.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127–146). New York, NY: Macmillan.

Hillman, A. P., & Alexanderson, G. L. (1994). *Abstract algebra: A first undergraduate course* (5th ed.). Prospect Heights, IL: Waveland Press.

Hodge, J. K., Schlicker, S., & Sundstrom, T. (2014). *Abstract algebra: An inquiry-based approach.* Boca Raton, FL: Taylor & Francis Group.
Hodgson, T. R. (1995). Connections as problem-solving tools. In P. A. House & A. F. Coxford (Eds.), *Connecting mathematics across the curriculum* (pp. 13–21). Reston, VI: National Council of Teachers of Mathematics.
Nicholson, W. K. (2012). *Introduction to abstract algebra* (4th ed.). Hoboken, NJ: John Wiley & Sons.
Nicodemi, O. E., Sutherland, M. A., & Towsley, G. W. (2007). *An introduction to abstract algebra with notes to the future teacher.* Upper Saddle River, NJ: Pearson Prentice Hall.
Robitaille, D. F., & Travers, K. J. (1992). International studies of achievement in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 687–709). New York, NY: Macmillan.
Singletary, L. M. (2012). *Mathematical connections made in practice: An examination of teachers' beliefs and practices*. (Unpublished dissertation). University of Georgia, Athens, GA.
Skemp, R. R. (1987). *The psychology of learning mathematics*. Harmondsworth, England: Penguin.
Usiskin, Z. (1974). Some corresponding properties of real numbers and implications for teaching. *Educational Studies in Mathematics, 5*, 279–290.
Usiskin, Z. (1975). Applications of groups and isomorphic groups to topics in the standard curriculum, grades 9–11. *The Mathematics Teacher*, *68*(3), 99–106, 235–246.
Zazkis, R. (2000). Factors, divisors, and multiples: Exploring the web of students' connections. *CBMS Issues in Mathematics Education, 8*, 210–238.
Zazkis, R., & Leikin, R. (2010). Advanced mathematical knowledge in teaching practice: Perceptions of secondary mathematics teachers. *Mathematical Thinking and Learning, 12*(4), 263–281.

PROOF EXPECTATIONS OF STUDENTS: THE EFFECTS ON PROOF VALIDATION

Ashley L. Suominen, Hyejin Park, & AnnaMarie Conner
University of Georgia

In this study, we examined how fifteen prospective secondary mathematics teachers validated five different arguments purported to be proofs that the sum of the first n odd natural numbers is n^2 in an interview setting. We also investigated the participants' stated expectations for middle student proof work. Our participants had varying evaluation criteria when validating arguments, but largely focused on generality and proof form or appearance. The majority of our participants expected only the empirical argument to be given by a middle school student. This was also the argument fewest participants accepted as a proof. In general, our participants were less likely to expect an argument from middle school students if they accepted it as a proof. Further research is needed to examine how argumentation experiences during their teacher preparation program influence prospective teachers' expectations of students' abilities to produce appropriate arguments.

Key words: Proof validation, Prospective secondary teachers, Teacher expectations, Middle school students

Proof plays a prominent role in the discipline of mathematics. Smith and Henderson (1959) wrote, "The idea of proof is one of the pivotal ideas in mathematics. It enables us to test the implication of ideas, thus establishing the relationship of the ideas and leading to the discovery of new knowledge" (p. 178). Current national recommendations (e.g., National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) suggest that *reasoning and proving* should become central to students' mathematical experiences across all grades. Consequently, teachers are asked to teach in ways that require a *robust* knowledge of proof, including proficiency for reading and analyzing students' arguments. Stylianides and Ball (2008) suggest that specific kinds of knowledge of proof and proving are necessary for teachers to analyze students' arguments and proofs.

Teachers' expectations for student arguments are one form of evidence of their pedagogical content knowledge specialized to their mathematical knowledge for teaching proof. Previous research on teachers' expectations has often focused on the relationships of expectations to self-fulfilling prophecies (e.g., Brophy, 1983; Jussim & Harper, 2005). However, for our purposes, the literature examining the accuracy of teachers' expectations for students is more relevant, particularly those studies that examine teachers' expectations for and analyses of students' proof attempts (e.g., Bergqvist, 2005) or other mathematical work (e.g., Pemberton & Galbraith, 2000).

Little research has been conducted on teachers' expectations for students' answers, particularly in the areas of argumentation and proof in mathematics. In this paper, we describe fifteen prospective secondary mathematics teachers' validations (after Knuth, 2002a; Selden & Selden, 2003; Weber, 2010) of mathematical arguments purported to prove that the sum of the first n odd natural numbers is n^2. We examined whether they found them to be mathematical proofs and whether or not they would expect each argument from middle school students.

Relevant Literature

Several researchers (e.g., de Villiers, 1999; Hanna, 1990; Knuth, 2002b) have discussed the types and roles of proof in mathematics and in mathematics education. However, research investigating students' conceptions of proof is complicated by a lack of consensus on a formal definition of proof within the field of mathematics (Hersh, 1993). Hanna (1990) proposed to distinguish between two kinds of proofs: *proofs that prove* and *proofs that explain* (p. 9). For the statement investigated in our study, Hanna stated that the proof by induction would be a proof that proves and Gauss' proof regarding the sum of the first n natural numbers would be a proof that explains. When considering the proof validation of practicing secondary teachers, Knuth (2002a) modified Hanna's proofs to construct similar arguments for the sum of the first n odd natural numbers and found that practicing teachers accepted empirical arguments in addition to deductive arguments as proofs. Other researchers (e.g. Selden & Selden, 2003; Weber, 2010) have investigated the process of proof validation in undergraduate mathematics classrooms by focusing on the process by which students determine whether an argument is a valid proof. In Weber's (2010) study, undergraduate students did not accept empirical arguments as proofs, although they did accept some faulty deductive arguments as proofs. This discrepancy raises an interesting question: Are differences in acceptance of empirical arguments due to the different sample populations or differing expectations of teachers and students in their respective communities of practice?

In general, the few studies reporting research on teachers' expectations for students' solutions have differing conclusions. In Bergqvist's (2005) follow-up study to how secondary mathematics students verify conjectures, eight upper secondary mathematics teachers in Sweden were interviewed to investigate their expectations of how students would verify conjectures. During the interview, participants were given three conjectures and asked how they thought students would approach each conjecture. They were then given four hypothetical student arguments for each conjecture and asked which solution matched their expectations for student work. Initially, these secondary mathematics teachers underestimated students' levels of reasoning based on the results from the previous study of student work. In fact, these participants expected only a few students to possess the high level of reasoning required to verify the conjecture. Algebraic solutions or the use of advanced mathematical notation also signaled to these participants that the argument came from a high performing student. Their expectations of students, however, were more accurate when they examined the hypothetical student arguments.

Conversely, Pemberton & Galbraith (2000) found that practicing secondary mathematics teachers overestimated what students would do. In this study, 340 students who had been accepted into undergraduate mathematics courses at the University of Queensland in 1997 were selected to complete a 24-item test during the first lecture session of the academic year. This test covered the content areas: algebra, graphs, indices and logarithms, trigonometry, and calculus. One hundred and twenty eight practicing secondary mathematics teachers in Queensland were then asked for each test item if they would expect students to successfully complete the question. In all but two of the test items, the teachers expected far greater success than the students' actual performance.

Tabach, Levenson, Barkai, Tsamir, Tirosh, and Dreyfus (2009) investigated 50 high school teachers' knowledge of students' correct and incorrect proof constructions within the context of elementary number theory (ENT). The teachers were given a questionnaire of six ENT statements and then asked to present correct and incorrect proofs that students would give to each of these statements. A total of 763 proofs were presented by the teachers. These were then categorized according to their modes of argumentation, modes of representation, and types of errors. The teachers in this study suggested that their students, when writing correct proofs, would use a general proof for two of the statements and a numeric example to validate or refute the other statements. Additionally, the teachers expected their students to use symbolic representations for correct proofs and numeric representations for incorrect proofs, which contradicts studies that students use numeric examples correctly and incorrectly in proofs. Few teachers also expected verbal representations. In a connected study, Tsamir, Tirosh, Dreyfus, Barkai, and Tabach (2009) investigated how one secondary practicing teacher evaluated students' arguments purported to prove six ENT statements. The teacher was asked to analyze 43 correct and incorrect arguments from hypothetical students and to determine their correctness. When determining whether students' arguments were acceptable, the teacher not only examined their correctness but also considered students' understanding of adequate ways of proving. Tsamir et al. also found that the teacher had very high expectations for students' proof work, expecting proofs to be not only correct but also minimal.

Theoretical Perspective

In our larger study, we combine a situative perspective on learning to teach mathematics (Peressini, Borko, Romagnano, Knuth, & Willis, 2004) with current research on teachers' beliefs about teaching, mathematics, and proof (e.g., Cooney, Shealy, & Arvold, 1998; Knuth, 2002a; Liljedahl, Rolka, & Rosken, 2007). From this perspective, we focus on the practices in which our participants participate as they attempt to position themselves as educators. As we narrowed our focus for this part of the study, we considered how prospective teachers situated themselves as secondary mathematics teachers in describing their expectations of what middle school students can or cannot do. Hammerness (2003) explained that these expectations are made by taking into consideration the images of students that make up their vision of teaching. In our study, we examined fifteen prospective secondary mathematics teachers' expectations for middle school students' arguments via their validations of purported proofs that the sum of the first n odd natural numbers is n^2, situated in the context of hypothetical middle school students' work.

Methodology

This paper presents a small part of the results of a larger study following fifteen prospective teachers through three semesters of mathematics education coursework and one semester of student teaching. Participants were asked to take part in five video-recorded semi-structured interviews (45–90 minutes long), one at the beginning of the larger study and one at the end of each semester. In this study, we focused our attention on one proof task from the third interview, which was conducted at the end of the participants' second semester in the mathematics education program. (Most participants were at the end of their junior year in college.) Our research questions for this research were: 1) How do prospective secondary mathematics teachers analyze arguments from students? and 2) What expectations do prospective secondary mathematics teachers have for middle school students' arguments?

During the interview we asked students to read the following claim and then reflect on five different hypothetical student arguments:

> *During an activity on pattern-seeking in your 8th grade class, Jesse, one of your students, said, "I think I found something. If you add the odd numbers, you get the perfect squares. You know, 1 is 1; 1 + 3 is 4; 1 + 3 + 5 = 9; see?" Jesse then asks, "Does this pattern keep going?" Rather than answer, you asked the students to work on answering Jesse's question.*

A summary of the different arguments (developed based on other studies) presented in the interview can be seen in Figure 1. For each argument in turn, we asked our participants their thoughts about the argument, if it was a proof, and if they expected the argument from a middle school student. Due to the semi-structured interview format, not all participants were asked every question about every argument; occasionally the interviewer did not push for an explicit answer for whether an argument was a proof or if it would be expected. Each interview was transcribed by one member of the research team and checked by another member to ensure accuracy.

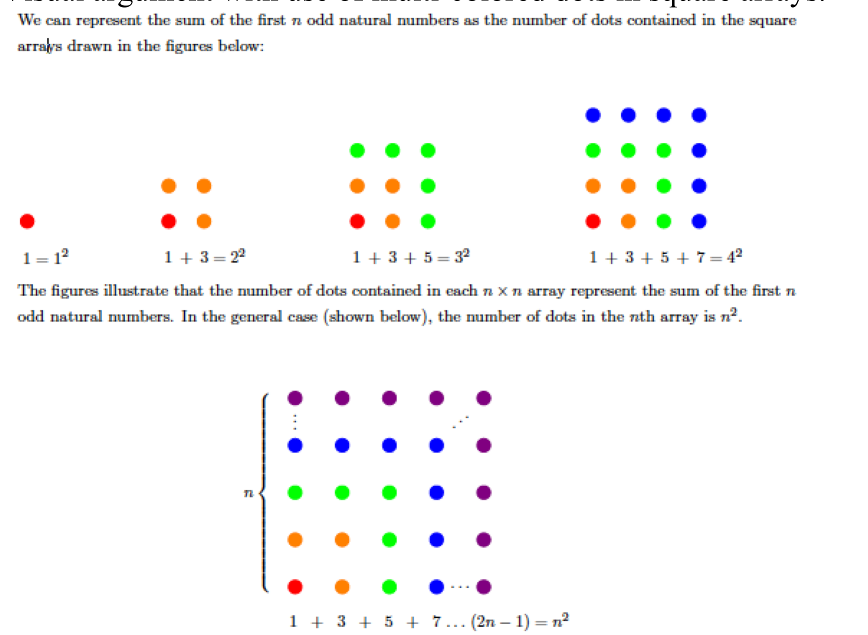
<table>
<tr><th>Arguments</th><th>Argument Summary</th></tr>
<tr><td>Bart</td><td>Examples: First 10 cases shown</td></tr>
<tr><td>Daphne</td><td>Visual argument with use of multi-colored dots in square arrays.

</td></tr>
<tr><td>Charlie</td><td>Gauss's proof using the property of symmetry
Algebraic manipulation of sum of the first n odd natural numbers:
$S(n) = 1 + 3 + \cdots + 2n - 1$</td></tr>
<tr><td>Eva</td><td>Algebraic manipulation of summation formula of the first n natural numbers: $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$</td></tr>
<tr><td>Archie</td><td>Proof by induction</td></tr>
</table>

Figure 1. Summary of proof validation tasks

To analyze our data, we first summarized each participant's proof validation work for each argument, noting specific actions and statements from each participant. We then coded their proof responses for the characteristics of proof used by the participants in their validations of the arguments. For instance, we coded references to the form of an argument, the generality of an argument, and the audience of an argument. Next, we created a data table with participants' responses to whether or not each argument was a proof and whether or not each participant expected the argument from a middle school student. We then summarized participants' rationales for their expectations of middle school students' arguments.

Results

Our analysis of the data was guided by the following questions: How do the prospective secondary mathematics teachers analyze arguments purported to be proofs that the sum of the first n odd natural numbers is n^2? What expectations for middle school students' solutions are reflected in their analyses? In this section, we describe our participants' validations of each argument and their accompanying expectations for middle school students' arguments.

Bart's Argument: Table of Examples

For Bart's argument, our participants focused primarily on the generality of a proof in determining whether or not it was a proof. For instance, Elliot said, "He [Bart] gives a lot of examples but it's also, I don't think it necessarily like proves it for every single term *n* and so I guess, I don't think you can make a conclusion or anything." Several participants classified this argument as a set of examples. Ultimately, eleven out of fifteen participants said this argument was not a proof, with rationales including that it did not prove it in general or did not prove for all *n*. Four participants identified Bart's argument to be a proof or part of a proof in the context of a middle school solution. These participants either overtly situated themselves as a middle school student or specifically considered how a middle school student would think about such a problem. For instance, one participant stated, "Let me try to get in my eighth grade class," and another said, "I feel like it is representative of an eighth grader."

Eight participants initially said they would expect this argument from middle school students. After seeing the other student arguments, six participants changed their answer from no to yes, for a total of fourteen participants expecting this argument from middle school students. These participants offered two main reasons for these expectations. Four participants based their expectations on students' prior mathematical knowledge. For instance, Vanessa would expect this argument because "they [middle school students] know what a perfect square is, they know what an odd number is, they know how to sum it, they know what consecutive odd numbers means." However, Robin would not expect it because a middle school student would just use numbers and not variables. Three participants based their expectations on what they would have done at that age. Reilly suggested that because he could see himself constructing a similar argument at that age, he expected the same from middle school students. Alex also mentioned what he would have done, discussing how he would move from a few examples to the general case, and concluding that an eighth grader would naturally begin with examples.

Daphne's Argument: Visual Dot Arrays

Daphne's argument generated the most diverse set of statements from our participants. Most seemed initially surprised about the appearance or form of Daphne's argument. For instance, Alex's initial response was, "It's a lot more abstract, or visually different." Vanessa was uncertain whether Daphne's argument was a complete proof, because she was used to seeing more writing and variables in a proof. However, fourteen participants ultimatelyclassified this argument as a proof, with one participant left unsure. Six of these participants considered the generality of the argument for validation. For instance, Josephine said, "She proved it for all N," and Rachel said, "They generalized it for N." Two other students relied on the computational accuracy of the argument to decide if Daphne's argument was a proof. For example, Jill verified the arrays were actually square with the correct numbers of dots, and William stated, "the reasoning is not as clear, but the computation works out."

Ten participants initially said they would not expect this argument from middle school students, four said they would, and one was unsure. However, three participants changed their answer from "no" to "yes" or "maybe" after seeing the other student arguments, for a total of seven participants expecting this argument from middle school students. The rationales for these expectations varied. Six of the participants based their expectations on what they would have done at that age. For instance, Elliot stated, "I mean I think it's really good. I don't think I would have done that in middle school though," so he would not expect this argument from a middle school student. Susan said it would be possible for a middle school student to produce Daphne's argument despite admitting she wasn't sure if she would be able to come up it as a college student. Susan was a unique participant in that she said most arguments were possible for middle school students. Four participants based their expectations on students' prior mathematical knowledge and classroom experiences. That is, if a middle school student had seen or been taught this type of argument before, then they might expect the student to produce Daphne's argument. Five participants appealed to the visual nature of the argument as being appropriate for middle school students. For instance, Rachel expected it from a middle school student because "I think they like to be really visual and draw things out."

Charlie's Argument: Adaptation of Gauss's proof

Fourteen participants classified Charlie's argument as a proof with only one remaining unsure. These participants provided two different rationales for validation: the form or appearance of the argument, and the generality of the argument. Five participants mentioned the form or appearance of the argument when classifying it as a proof. For example, Elliot said it "looks to be right," and Reilly said, "It looks like an algebraic proof." Susan said that Charlie's argument "models" something that she had seen before and would be how she would prove the number theory statement. Four participants pointed out that Charlie proved the general case, so this argument is a proof. Six participants provided no rationale for their decision and were not asked to explain further.

When asked whether they would expect this argument from a middle school student, nine participants initially said they would not expect it and three said it would be possible. Only one participant changed her answer to possible after being shown the other arguments. Three participants were not asked their expectations for this argument. Five participants based their expectations on students' prior mathematical knowledge or classroom experiences. For instance, Cathy observed that this argument relied on notation that students may have never seen before, which could cause confusion in understanding the argument. In addition, Elliot admitted his own confusion about the argument, "I just don't understand why they would add those two numbers," which led him to declare, "There's no way a middle schooler would do this." Elliot further based his expectations of middle and high school students on what he would have done in high school:

> I wouldn't have ever done that in high school. And I was one of the smarter ones in my high school. I mean there are people that can do this stuff, but…not a lot of people can do that.

Five participants based their expectations on what they would have done at that age, and two other participants mentioned that they would not expect this argument unless the middle school students were explicitly taught this approach.

Eva's Argument: Algebraic Manipulation of Formula for Sum of First n Natural Numbers

Thirteen participants classified Eva's argument as a proof with one remaining unsure. One participant was not explicitly asked whether or not this argument was a proof. Few participants provided their rationales or were asked to further elaborate their thoughts for their decision. However, three participants mentioned the form or appearance of the argument as one that resembles a proof. For instance, Charissa classified Eva's argument as a proof because it looked like one. One participant, Vanessa, seemed to consider the prior mathematical knowledge or classroom experiences of the audience when assessing if Eva's argument was a proof. She asserted that this argument would not be a proof to a middle school student even though it would be to a college student, because middle school students would not know the formula or necessary proof for the sum of the first n natural numbers is $\frac{n(n+1)}{2}$, so Eva's argument would not make sense to them.

Ten participants said they would not expect Eva's argument from a middle school student and four said it might be possible. One participant was not asked about his expectations for this argument. Also, no participants changed their expectations for this argument based on viewing other student arguments. When determining their expectations, nine participants pointed to the prior mathematical knowledge needed to complete this argument; namely, the proof of the sum of the first n natural numbers is $\frac{n(n+1)}{2}$. Seven of these participants would not expect middle school students to know this formula or have proven it previously since they themselves learned it in college, whereas two participants said it would be possible if the middle school student had been explicitly taught this prior necessary knowledge. Susan provided a unique rationale for stating it would be possible. That is, she discussed the more algebraic nature of Eva's argument as being appropriate for middle school students.

Archie's Argument: Proof by Induction

All fifteen participants classified Archie's argument as a proof. Twelve of these participants explicitly noted the proof technique used in the argument as proof by induction. Recognizing this technique was important for our participants when validating this argument. In fact, a few participants did not deeply examine the argument upon recognizing the induction technique. For instance, Josephine said Archie's argument was a proof after only ten seconds of reading. She then mentioned she didn't need to read the whole thing because Archie used induction, which seemed to be a favored proof technique of hers given that she compared nearly every other argument to induction.

The majority of our participants based their expectations of middle school students for Archie's argument on what they would have done in at that age. Because our participants did not learn proof by induction until college, thirteen participants said they would not expect this argument from middle school students. For instance, Cathy declared that induction is a college-level idea that would not be appropriate for a middle school student. Helen also asserted that this argument was too abstract and had too many variables for middle school students. Two participants, however, said it might be possible depending on the students' classroom experiences. Despite the fact that Susan learned induction in college, she discussed that it would be possible for a middle school student to produce this argument if "we had introduced like just basic proving ideas in elementary school."

In summary, the prospective secondary mathematics teachers that participated in this study seemed to value different things when validating arguments and determining their expectations of students. Our participants focused on generality when validating Bart and Daphne's arguments, but emphasized themselves as middle school students when discussing their expectations for students. The participants also highlighted the visual nature of Daphne's argument as being appropriate for middle school students, so many would expect it from a middle school student. However, the more complicated arguments (Eva, Charlie, Archie) were commonly accepted as proofs without rationales, perhaps due to the familiar proof techniques or forms of these arguments. In their expectations for middle school students, our participants often considered when they had learned the concept or proof technique. Our participants' expectations also changed upon viewing other arguments, which aligns with the findings of previous research (Bergqvist, 2005) that the accuracy of teacher expectations increased when given hypothetical student arguments. A summary of the findings can be seen in Table 1.

Table 1

Summary of participants' proof validations and expectations

Arguments	Proof?			Expect from middle school student?		
	Yes	No	Unsure	Yes	No	Maybe
Bart	4[1]	11		8 initial (14 final)	7 initial (6 changed to yes)	
Daphne	14		1	4 initial (7 final)	10 initial (3 changed to yes/maybe)	1
Charlie	14		1	3 initial (4 final)	9 initial (1 changed to possible)	
Eva	13		1	4	10	
Archie	15			2	13	

Note: Several participants adjusted their expectations upon seeing the other student arguments. This change is noted in the table.

[1]Participants classified this argument as a proof or part of a proof in the context of a middle school solution

Discussion

In this study, we noticed that our prospective secondary mathematics teachers often based their expectations on what they did or think they would have done in middle school. For instance, Cathy, when considering Archie's argument, said:

> Because I probably wouldn't have been able to do this until last year, which was my sophomore year of college. I wouldn't have, I don't think I would have been able to do this in high school much less middle school.

Many of our participants stated that since they couldn't produce Archie's proof until college, they would not expect it from a middle school student. Furthermore, since our participants had little formal proof writing experience in middle school, they did not expect middle school students to be able to develop arguments that were proofs. As a result, proof writing seemed to be an unfamiliar context or idea to think about in recollections of middle school. For instance, Alex said he wouldn't expect Charlie's argument, because "I don't think I would ever thought of this in high school. I don't even remember learning like proofs like this in high school." Similarly, Reilly mentioned:

> I said one of the things I want to do is build expectations for students, what to expect. I feel like these students probably expect and have like learned how to prove things when, when I was like I guess I hadn't done proofs until I got to college.

Our prospective secondary mathematics teachers expected middle school students to think empirically about arguments. Hence, most of our participants more readily expected Bart's argument from a middle school student and the majority did not expect Archie's argument. For instance, Reilly, when discussing his expectations for middle school students, stated, “I guess my expectations or what I would expect to see the most would be things that were more like this [referring to Bart’s argument], in essence is maybe prove more examples.” Many of our participants mentioned similar expectations by expecting the middle school students to use examples when formulating arguments. William provided a slightly different but related rationale for expecting empirical arguments when discussing Archie’s argument:

> I would definitely just like expect a little bit of exploration maybe trying to get to this [referring to Archie’s argument], you know playing around with even numbers odd numbers and making conclusions. Maybe that exploration, but I'm not, I don't know if I'd expect them to formulate the argument as middle schoolers.

In general, our participants did not believe the middle school students were able to apply the abstract thinking necessary in order to move from example usage and number exploration to formulating formal arguments. To them, the ability to think abstractly about mathematical problems is developed at a much later age, often at the tertiary level. A few participants also noted that the formality and structural nature of proofs require this abstract thinking, which is why they did not expect middle school students to write proofs.

The participants’ assessments in this study sometimes depended on what the specific students would have learned in the class or their prior mathematical knowledge. For instance, when considering Archie’s argument, William said, “I wouldn't expect this out of my high schoolers or middle schoolers unless we were in a class where induction was the main topic and we were really harping on that.” Only a few of our participants would expect middle school students to produce the more complicated arguments (Charlie, Eva, or Archie) and only they were in a class that explicitly discussed these proof writing approaches. Additionally, our participants primarily mentioned students’ prior mathematical knowledge with Eva’s argument. For instance, Cathy discussed her expectations about Eva’s argument:

> I would be surprised, but if you've already discussed what it means for something to be a sum, or for all the sums of natural numbers, I feel like if you've discussed that then students would be capable of doing it. But if you haven't even touched on what it means for, to find just the sum of all natural numbers then I feel like it would be a struggle for students to get to this point.

Cathy based her expectations of this argument on students’ prior mathematical knowledge. Two participants (Jason and Susan) possessed this attitude of openness towards expectations for student. They expressed opinions consistent with a belief that there is no limitation on what a student could do mathematically. For each argument, these participants discussed that if a student were taught the background knowledge necessary to understand the given argument, such as a certain proof form or certain mathematical ideas, then a middle school student could potentially construct any of the arguments. Thus, their expectations for middle school students were shaped by their beliefs about students rather than their proof validations.

Implications

The tendency of prospective secondary mathematics teachers to expect only empirical arguments from middle school students may not necessary be atypical; however, their primary rationale for these expectations, a comparison to themselves and their experiences, causes us to consider the experiences these teachers are given during their teacher preparation program. That is, what specific types of learning experiences are given to prospective secondary mathematics teachers regarding argumentation? From this study we see that prospective teachers need experiences in which they engage in argumentation, both as students and as teachers with different levels of students. Explicit discussions of what a valid argument is or what proof might look like in middle school or high school may be important for these prospective teachers as we consider how they will evaluate their students' arguments and what they will expect from their students. Future research should examine how these experiences influence prospective teachers' expectations of students' abilities to produce proof arguments.

Acknowledgements

This paper is based on work supported by the National Science Foundation under Grant No. 1149436. Opinions, findings, and conclusions in this paper are those of the authors and do not necessarily reflect the views of the funding agency.

References

Bergqvist, T. (2005). How students verify conjectures: Teachers' expectations. *Journal of Mathematics Teacher Education, 8,* 171-191.

Brophy, J. E. (1983). Research on the self-fulfilling prophecy and teacher expectations. *Journal of Educational Psychology, 75*(5), 631–661.

Cooney, T. J., Shealy, B. E., & Arvold, B. (1998). Conceptualizing belief structures of preservice secondary mathematics teachers. *Journal for Research in Mathematics Education, 29,* 306–333.

De Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras, 24*, 17–24.

Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange, 21*(1), 6–13.

Hammerness, K. (2003). Learning to hope, or hoping to learn? The role of vision in the early professional lives of teachers. *Journal of Teacher Education, 54*(1), 43–56.

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics, 24*(4), 389–399.

Jussim, L., & Harber, K. D. (2005). Teacher expectations and self-fulfilling prophecies: Knowns and unknowns, resolved and unresolved controversies. *Personality and Social Psychology Review, 9*(2), 131–155.

Knuth, E. J. (2002a). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education, 33*(5), 379–405.

Knuth, E. J. (2002b). Teachers' conceptions of proof in the context of secondary school mathematics. *Journal of Mathematics Teacher Education, 5*(1), 61–88.

Liljedahl, P., Rolka, K., & Rosken, B. (2007). Affecting affect: The reeducation of preservice teachers' beliefs about mathematics and mathematics learning and teaching. In W. G. Martin, M. E. Strutchens, & P. C. Elliott (Eds.), *The Learning of Mathematics* (pp. 319–330). Reston: National Council of Teachers of Mathematics.

National Governors Association Center for Best Practices and Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics*. Washington, DC: Author. Retrieved June 1, 2012 from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Pemberton, M., & Galbraith, P. (2000). Student knowledge and teacher perceptions of secondary mathematics graduates. *Australian Senior Mathematics Journal, 14*(2), 4-14.

Peressini, D., Borko, H., Romagnano, L., Knuth, E., & Willis, C. (2004), A conceptual framework for learning to teach secondary mathematics: A situative perspective. Educational Studies in Mathematics, 56(1), 67–96.

Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education,*

34(1), 4–36.
Smith, E. P., & Henderson, K. B. (1959). Proof. In P.S. Jones (Ed.), *The growth of mathematical ideas, grades K-12* (24th Yearbook of the NCTM, pp. 111–181). Washington, DC: NCTM.
Tabach, M., Levenson, E., Barkai, R., Tsamir, P., Tirosh, D., & Dreyfus, T. (2009). Teachers' knowledge of students' correct and incorrect proof constructions. In F. L. Lin, F. J. Hsieh, G. Hanna, & M. de Villiers (Eds.), *Proceedings of the ICMI Study 19 Conference: Proof and Proving in Mathematics Education* (Vol. 2, pp. 214–219). Taipei, Taiwan: National Taiwan Normal University.
Tsamir, P., Tirosh, D., Dreyfus, T., Barkai, R., & Tabach, M. (2009). Should proof be minimal? Ms T's evaluation of secondary school students' proofs. *Journal of Mathematical Behavior, 28,* 58-67.
Weber, K. (2010). Mathematics majors' perceptions of conviction, validity, and proof. *Mathematical Thinking and Learning, 12*(4), 306–336.

Calculus Students' Understanding of Interpreting Slope and Derivative and Using them Appropriately to Make Predictions

Jennifer G. Tyne
University of Maine

Studies have shown that students have difficulty with the concepts of slope and derivative, especially in the case of real-life contexts. Following up from a previous study, written surveys were used to collect data from 69 differential calculus students, and 7 clinical interviews were conducted. On the surveys, students answered questions about linear and nonlinear relationships and interpretations of slope and derivative. They also critiqued the reasoning and accuracy of a hypothetical person's predictions. In interviews, students explained their thought processes and reasoning while answering questions similar to the written survey but in a different context. Results indicate that students struggle with knowing what the derivative represents and how to use it appropriately to make predictions. These struggles might stem in part from students' incorrect interpretations of slope as a ratio of totals, as opposed to a ratio of changes over a particular interval.

Key words: Calculus, Derivative, Rates of Change, Slope, Student Understanding

Introduction and Research Questions

A robust understanding of derivatives and instantaneous rates of change in calculus requires an understanding of slope and average rates of change from precalculus (Hackworth, 1994). It is thus important for instructors to be alert to students' understanding of slope coming into calculus, and to design instruction that expands on that knowledge in teaching the derivative. Calculus students may not have the robust understanding of slope and rates of change that instructors assume, which likely impacts their learning of derivative. Furthermore, calculus requires students to leverage their slope and rates of change knowledge to understand instantaneous rates of change and continuously changing rates, requiring strong covariational reasoning on the part of students (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002).

The focus of this study is an investigation of the interpretation and use of slope and derivative in real life contexts, as well as students' abilities to critique the reasoning of others' predictions involving slope and derivative. Real life applications require students to translate from the context to the abstract level of calculus and then back to the context, a process that requires conceptual knowledge (White & Mitchelmore, 1996). Educators have emphasized the utility of these sorts of problems as providing "meaningful opportunities for students to develop their understanding of mathematics… [and] opportunities for students to communicate their understanding of mathematics" (Stump, 2001, p. 88).

The importance of modeling real life situations is reflected in the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices, 2010). One of the Standards for Mathematical Practice is to "model with mathematics," which focuses on the importance of solving problems that arise in everyday life (Standards for Mathematical Practice section, CCSS.MATH.PRACTICE.MP4). Likewise, research points to the use of problems in which "the problem situation is experientially real to the student" (Gravemeijer & Doorman, 1999, p. 111) as key to students' understanding of formal mathematics and their improved mathematical reasoning. The practice standards also call for students to be able to critique the reasoning of others, distinguish between correct and flawed logic and reasoning, and, if there is a flaw in an argument, explain what it is (Standards for Mathematical Practice section, CCSS.MATH.PRACTICE.MP3).

The present study builds on research about student understanding of rates of change (Beichner, 1994; Carlson, 1998; Hackworth, 1994; Hauger, 1995; Orton, 1983; Teuscher & Reys, 2007; Thompson & Thompson, 1992; Wilhelm & Confrey, 2003), slope (Barr, 1980; Crawford & Scott, 2000; Lobato & Thanheiser, 2002; Nagle, Moore-Russo, Viglietti, & Martin, 2013; Stump, 2001), and derivative (Asiala, Cottril, Dubinsky, & Schwingendorf, 1997; Bingolbali, Monaghan, & Roper, 2007; Ferrini-Mundy & Graham, 2004, 1991; Habre & Abboud, 2006; Park, 2013; White & Mitchelmore, 1996; Zandieh, 2000). Findings from these studies indicate that students have difficulty understanding slope as a constant and derivative as an instantaneous rate of change. However, there has not been much research on college students' verbal interpretation of slope and derivative, or students' understanding of the differences in making predictions involving constant and instantaneous rates of change. For example, do students know that a slope of 5 dollars per child means that "for each additional child, the cost increases by 5 dollars," and that a derivative of $C'(20) = 4$ dollars per day means "when the number of children are equal to 20, the cost is increasing at a rate of 4 dollars per child"? Do students understand that the slope can be used to calculate the increase in cost over any interval, but the derivative can be used only to estimate the cost increase close to the value of interest?

Recent research found that students with under-developed conceptions of slope and derivative were not able to correctly use rates of change to make valid predictions (Tyne, 2014). Almost two-thirds of the 74 students in that study incorrectly used the instantaneous rate of change as a constant rate of change, indicating confusion about the derivative's meaning. Results also pointed to an incomplete across-time understanding (Monk, 1994) of the derivative as a function. The present study extends this work by addressing the following research questions: (1) Can students interpret the slope and derivative in the context of the problem? (2) Can students appropriately critique the reasoning of someone else's use of slope and derivative to make valid predictions?

Student Understanding of Rates of Change, Slope, and Derivative

The current research most relevant to this study centers on students' understanding of rates of change in general and, more specifically, their understanding of slope and derivative. Rates of change are the overarching connection between the concepts of slope and derivative, and while slope is often considered a middle school and high school concept, it is an essential building block for students' success with the derivative.

Student Understanding of Rates of Change

With their basis in everyday experiences, rates of change are fundamental for understanding the relationships between various quantities (Confrey & Smith, 1994). Many researchers claim that students' success in higher level mathematics depends on a deep understanding of rate (Carlson et al., 2002; Zandieh, 2000). As rates of change play a significant part in describing and understanding changing quantities in biology, physics, chemistry, economics, and other areas, rates of change are a critical mathematical topic.

Research findings about rates of change relevant to this study fall into three categories: (1) students' underdeveloped concepts of rates, (2) students' difficulties interpreting rates of change, and (3) students' incorrect view of rates of change as the ratio of totals.

Thompson (1994) and Hackworth (1994) both speak to students' underdeveloped concepts of rates. In studying student understanding of the Fundamental Theorem of Calculus, difficulties in understanding were often tied to under-developed understanding of rates of change (Thompson, 1994). Hackworth (1994) found that instruction about derivatives failed to substantially change students' reasoning about rate situations, and that students who did poorly in calculus seemed to have an impoverished understanding of rates of change.

Orton (1983) suggests that rates of change "need to be introduced whilst the opportunity is there and long before any formal algebra/calculus treatment of the issues" (p. 26). Orton found that many calculus students do not think about rates of change in derivative problems, hence losing a connection that is useful as they moved on to higher-level mathematics.

Many studies point to students' difficulties with interpreting rates of change (Carlson, 1998; Teuscher & Reys, 2007; Wilhelm & Confrey, 2003). Teuscher and Reys (2007) studied Advanced Placement calculus students and concluded that students lack an understanding of the interpretation of rate of change (although their survey included only one interpretation question that had students use a graph to interpret a rate of change). Wilhelm and Confrey (2003) reported that most research on the rate of change concept involved motion and speed, the context most dealt with in calculus textbooks and courses. They promote teaching rates of change in multiple contexts, allowing the "learner the opportunity to see the 'like' in the contextually unlike situation, so that the learner might later be able to project these rates of change and accumulation concepts into novel situations" (p. 904). In Carlson's (1998) study, even the most talented calculus students had trouble interpreting rate of change information from a dynamic situation, as well as interpreting the covariant aspects of a real world situation. She found that current calculus curricula gave very little opportunity for students to interpret the covariant aspects and language of functions.

Rates of change can be viewed as a ratio of differences ($\frac{\Delta y}{\Delta x}$ or $\frac{y_2 - y_1}{x_2 - x_1}$) but some students misinterpret this and interpret it as the ratio of totals $\left(\frac{y}{x}\right)$, possibly because the expressions $\frac{\Delta y}{\Delta x}$ and $\frac{y}{x}$ are so similar (Hauger, 1995). Hauger concluded that "unless these subtle distinctions are made in the minds of students, it is a small wonder that they use $\frac{y}{x}$ when they should be using $\frac{\Delta y}{\Delta x}$"(p.27). Whether or not a line passes through the origin also results in difficulties for students regarding slopes and rates of change. Beichner (1994) found that students were much less successful in calculating slope when the line did not pass through the origin. Students would regularly divide a single ordinate value by a single abscissa value, forcing the relationship through the origin, in other words calculating slope as $\left(\frac{y}{x}\right)$ instead of $\left(\frac{\Delta y}{\Delta x}\right)$.

Student Understanding of Slope

Researchers have documented difficulties students have with the concept of slope (Barr, 1980, 1981; Crawford & Scott, 2000; Lobato & Thanheiser, 2002; Stump, 2001). Research findings fall into two categories relevant to this current study: (1) students' inabilities to interpret the slope as a rate of change, (2) students' under-developed conceptions of slope.

Researchers have called for instruction that allows for more opportunities for students to communicate about slope (Crawford & Scott, 2000; Stump, 2001). Stump (2001) found that while high school students demonstrated a better understanding of slope in functional situations, as opposed to physical situations, "many students had trouble interpreting slope as a measure of rate of change" (p. 81). She suggested instruction should focus on providing opportunities for students to communicate their understanding of slope. Crawford and Scott (2000) also call for having students communicate and reason about slope, as well as using real-world examples to introduce the concept of rates of change prior to introducing slope. Lobato and Thanheiser (2002) found that students can correctly calculate slope using the "rise over run" formula, but only view slope as a number, not as a measure of rate. They proposed ratio-as-measure tasks that can "help students develop an understanding of slope that is more general and applicable" (Lobato & Thanheiser, 2002, p.174).

Slope can be conceptualized in many different ways; researchers have focused on the conceptions both students and teachers bring to the classroom. Stump (1999) studied teachers' knowledge of slope. She found a substantial majority of teachers thought of slope as a geometric ratio, with less than 20% thinking of it as a functional concept that had no connection to rates of change. She notes that the teachers might have been capable of making the connection, but that they did not incorporate the connections into their definitions. Nagle, Moore-Russo, Viglietti, and Martin (2013) studied both college students' and instructors' responses to questions about slope, and classified their responses among 11 conceptualizations of slope. They found that while instructors demonstrated a multi-dimensional understanding of slope as a functional property, students rely on procedurally-based conceptualizations of slope and showed little evidence that they engaged in covariational reasoning. They conclude that it is imperative that instructors understand the conceptualizations commonly held by their students in order to build advanced ideas.

Student Understanding of Derivatives

Students' difficulties with the derivative are well documented in the literature (Asiala et al., 1997; Bingolbali et al., 2007; Ferrini-Mundy & Graham, 1994, 2004; Habre & Abboud, 2006; Park, 2013; White & Mitchelmore, 1996; Zandieh, 2000). Students are often able to compute derivatives using algorithms, but have very little conceptual knowledge about the derivative (Ferrini-Mundy & Graham, 1994; White & Mitchelmore, 1996). Research about derivative understanding relevant to this study falls into three categories: (1) student weaknesses with underlying concepts, (2) students' difficulties with covariational reasoning, and (3) the difficulties which stem from the multi-faceted nature of the derivative.

The underlying concepts of variable and function are difficult for students. White and Mitchelmore (1996) cite the need for a mature view of variable as a prerequisite to a successful study of calculus. Many of the students in their study did not hold abstract-general understanding of variable, instead taking a focus where they treated the variables as symbols to be manipulated instead of quantities to be related. Many students also come to calculus with a very primitive understanding of functions (Carlson, 1998; Ferrini-Mundy & Graham, 1994; Monk, 1994). Monk (1994) looked at students' understanding of functions from two approaches – pointwise and across-time. Point wise understanding is what students first attain in their learning about functions, thinking of particular values of the independent variable corresponding to particular values of the dependent variable. However, in calculus, students must have "across-time" understanding of functions, where changes in one variable lead to changes in another variable.

Similarly, rates of change knowledge is strongly linked to the notion of covariational reasoning, defined by Carlson et al. (2002) as "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354). Such reasoning requires someone to hold an image of two quantities' values simultaneously (Saldanha & Thompson, 1998). Researchers have found that students lack the understanding necessary to deal with these co-varying quantities efficiently, thus not grasping the across-time understanding (Bezuidenhout, 1998; Carlson, 1998). Students need to understand the covarying nature of the derivative function, not just the pointwise interpretation, in order to fully make sense of this key calculus concept. Park (2013) studied differential calculus students and found that most students use a point-specific understanding, and they do not have a fully developed function concept of the derivative.

The derivative is a multi-faceted idea, with students needing to connect many underlying concepts in order to think fully about the derivative (Ferrini-Mundy & Graham, 1994; Zandieh, 2000). The concept of the derivative can be represented graphically as the slope of a tangent line, verbally as the instantaneous rate of change, physically as velocity, and symbolically as the limit of the difference quotient (Zandieh, 2000). Researchers have found that students often do not connect a function's derivative with its rate of change, which leads to the inability to understand differentiation as an operator that measures a rate of change (Weber, Tallman, Byerley, & Thompson, 2012). Even when they can connect the derivative with rates of change, students often confuse derivative at a point with the derivative function (Ubuz, 2007). Habre and Abboud (2006) found that when students thought of functions in an analytic setting, they took a geometric approach concerning the concept of the derivative.

What is Not Yet Known about Student Understanding

Researchers have documented that students often have incomplete conceptions of rates of change, slope, and derivative, all key concepts in understanding the tasks in the current study. While some research has focused on students' abilities to interpret slope in real-life contexts, it has mostly involved high school students. Very little research has been done on students' verbal interpretation of the derivative as a rate of change. Maharaj (2013) calls for calculus instruction that focuses on "verbal approaches to applications on the derivative concept" (p. 15) that is related to the verbal interpretation of slope and derivative, a focus of the present study.

A second focus of the study, and one that is not covered in the literature, is student understanding of appropriate uses of slope and derivative to make predictions and to critique the reasoning of others. Though focused on functions in general, and not derivatives, Carlson (1998) found that second-semester calculus students were "unable to use information taught in early calculus and had difficulty interpreting and representing covariant aspects of a function situation" (p. 115). The current study also focuses on covariant aspects of functions, what Carlson (1998) defines as "recognizing and characterizing how change in one variable affects change in another" (p. 117). This study includes an examination of students' abilities to recognize how change in independent variables affects change in dependent variables.

Research Design

Theoretical Perspective

The present study was conducted within a cognitivist framework (Byrnes, 2000; Siegler, 2003), which posits that students make sense of the mathematics they are doing based on their experiences and that their answers are rational and subject to explanation (Ferrini-Mundy & Graham, 1994). Because a cognitive lens focuses on individuals' thinking, it is useful for investigating how students think about slope and derivative and use them to make predictions. The focus here is on detailed analyses of student understanding of a few key concepts, gained from direct student responses. Hence this study used a written survey instrument and follow-up interviews as data sources.

Setting

The data for this study were collected from 69 students enrolled in differential (e.g., first semester) calculus at a pubic university in the Northeast. Students completed the surveys during class time, approximately 80% through the course. Over 50% of the students had seen calculus in high school, and all needed to either pass a placement exam or complete precalculus at the University with a C or better to gain enrollment into differential calculus. Follow-up clinical interviews were done at the start of the following semester. Seven students participated, all of who completed the written survey the previous semester and were enrolled in Calculus 2 when the interviews were conducted. Interviews lasted 30-45 minutes and written work and audio were recorded with a Livescribe pen.

Data Collection

The survey instrument consisted of questions about slope and derivatives, including questions about linear and nonlinear relationships between the yield of a crop of corn (bushels) as a function of the amount of nitrogen put on the field (lbs.). The interview instrument was very similar to the written survey, except the context was the amount of drug given to a patient as a function of the patient's weight. The main focus of this paper is on the results from the interview tasks (Figure 1). The survey and interview questions are not mechanical in nature and therefore do not assess computational skills; instead, they are questions about students' interpretations of slope and derivative and their ability to critique others' reasoning, and therefore try to uncover their understanding about these topics.

For certain drugs, the amount of dose given to a patient, D (in milligrams), depends on the weight of the patient, w (in pounds).

A. Assume that D(w) is a linear function with a slope equal to 2 (m = 2).

   0. On the graph below, give a rough sketch of what the function D(w) looks like. Label the axes, but no need to scale them.
   1. What are the units on the slope, m = 2?
   2. Explain what this slope (m = 2) means in the context of the problem.
   3. Using the slope (m = 2), Nurse Jodi predicts that a patient's dose will increase by 2 mg when the patient's weight changes from 140 pounds to 141 pounds. How much confidence do you have in her reasoning? (circle one and provide explanation)
      Very Confident   Somewhat Confident   Not Confident
   4. Nurse Jodi accurately doses a 140-pound patient using the model. Her next patient is twenty pounds heavier and she reasons that she must increase the dose by 40 mg (2 mg for each pound of weight). How much confidence do you have in her reasoning? (circle one and provide explanation)
      Very Confident   Somewhat Confident   Not Confident

B. Now, assume D(w) is a non-linear function.

   0. On the graph below, give a rough sketch of what the function D(w) might look like.
   1. What are the units on $\frac{dD}{dw}$? (also known as $D'(w)$)
   2. Explain the meaning of the statement $D'(140) = 2$ in the context of the problem.
   3. Using the fact that $D'(140) = 2$, Nurse Jodi predicts that a patient's dose will increase by 2 mg when the patient's weight changes from 140 pounds to 141 pounds. How much confidence do you have in her reasoning? (circle one and provide explanation)
      Very Confident   Somewhat Confident   Not Confident
   4. Nurse Jodi accurately doses a 140-pound patient using the model. Her next patient is 160-pounds and she reasons that since $D'(140) = 2$, she must increase the dose by 40 mg (2 mg for each pound of weight). How much confidence do you have in her reasoning? (circle one and provide explanation).
      Very Confident   Somewhat Confident   Not Confident

Figure 1. Interview Instrument

The questions in the interview instrument were informed by the typical presentation of slope and derivative in textbooks, the Common Core Standards for Mathematical Practice, and the call for assessing students' across-time view of functions (Monk, 1994). In textbooks and in instruction, when focus is given to students' understanding of slope and derivative, usually the questions asked are similar to A1, A2, B1, and B2 (Figure 1). These questions address units (Bezuidenhout, 1998) and students' pointwise understanding of rates of change (Monk, 1994). Based on the Common Core's call for critiquing the reasoning of others, as well as students' across-time understanding of rate of change (Monk, 1994) the survey included questions A3, A4, B3, and B4. The linear questions (A3 and A4) were included to gain an understanding of students' knowledge of predictions based on linear change and are similar to typical textbook/instruction presentation of slope.

The focus on the present study is student responses for questions A2 and B2 (pointwise interpretation of slope and derivative, respectively), and questions A3-4 and B3-4 (across-time interpretation of slope and derivative, respectively; ability to critique the reasoning of others). In Figure 2, I give a hypothetical answer from an "ideal knower," what such a student would be thinking while solving the task, and what the question is designed to give information about. These descriptions were used to inform the data analysis.

Data Analysis

My approach to the interviews was informed by data analysis on the written surveys. I took a modified Grounded Theory (Strauss & Corbin, 1990) approach to analyzing the surveys. In pure Grounded Theory, the researcher does not look at literature until after the analysis. After an earlier literature review, I had an idea of possible categories that would emerge, but used Grounded Theory techniques to identify and refine my analysis categories.

I examined data from the written surveys by first categorizing answers from the pointwise questions (questions A2 and B2), and then categorizing answers from the across-time critiquing questions (questions A3-4 and B3-4). These categorizations helped in identifying themes to be addressed in interviews, and whether there were relationships between student responses on linear vs. non-linear and pointwise vs. across-time questions.

Interviews allowed me to probe student thinking more deeply, especially focusing on themes that emerged in the survey data analysis. I used categories from the survey data as my starting point, and if I saw a categories appear in the interview, I asked questions to get students to explain their reasoning. For example, one of the categories from the written survey data analysis was the need for another derivative to predict the change in dosage. If an interviewee engaged in this sort of reasoning, I asked, "Why do you need a different derivative to answer the question? What could you do with that information if you had it?"

Written Survey Findings

The written survey question context was different than the drug/weight interview context, but the questions were similar in content. The context was: *Let B(n) be the number of bushels of corn produced on a 10-acre tract of farmland that is treated with n pounds of nitrogen*.

For the purpose of this study, I examined four of the questions on the written survey: (1) pointwise interpretation of slope in the context of the problem, (2) pointwise interpretation of the derivative in the context of the problem, and (3) the two across-time derivative questions where the student is asked to critique the reasoning in Farmer Jim's predictions.

Pointwise Slope Interpretation

In examining student responses to the question of what a slope of 2 means in the context of the linear problem, three categories of responses emerged. First, some students responded correctly, saying something like "for each additional pound of nitrogen, two more bushels of corn are produced." The key language here is that students recognize that the slope represents a constant ratio in the *changes* in variables, thus the "additional" language.

Many more students responded using language that implied they were assuming a direct relationship that goes through the origin, in other words that the slope represents a constant ratio in the variable values $\left(\frac{y}{x}\right)$. The most common response was that "2 represents the number of bushels produced per pound of nitrogen" or "for every pound of nitrogen, 2 bushels of corn are produced," implying a direct proportional relationship of $B(n) = 2n$. This was coded as "ratio of totals" interpretation, and denoted it as $B(n) = 2n$.

A. Assume that D(w) is a linear function with a slope equal to 2 (m = 2).

A2. Explain what this slope (m = 2) means in the context of the problem.

The ideal knower would respond that the slope of 2 means that for each additional pound of weight, the patient gets an additional 2 mg of drug. This question assesses students' understanding of slope as a constant rate of change, where the ratio of changes in variables is constant. This is different than a directly proportional relationship where the ratio of amounts is constant, which implies a vertical intercept of zero.

A3. Using the slope (m = 2), Nurse Jodi predicts that a patient's dose will increase by 2 mg when the patient's weight changes from 140 pounds to 141 pounds. How much confidence do you have in her reasoning?

The ideal knower would respond "very confident" by understanding that a slope of 2 represents the increase in milligrams per pound, and that it is a constant rate of change. As the pounds increase by 1, the dosage increases by 2 mg. This question begins to assess students' across-time understanding of functions; they have to understand how the dependent variable changes as the independent variable increases by one.

A4. Nurse Jodi accurately doses a 140-pound patient using the model. Her next patient is twenty pounds heavier and she reasons that she must increase the dose by 40 mg (2 mg for each pound of weight). How much confidence do you have in her reasoning?

The ideal knower would respond, "very confident" and explain that the increase of 2 milligrams per pound is constant and would be applied to the twenty-pound increase. This question is designed to assess students' knowledge of the slope as a constant rate of change, and how it can therefore be applied to any change in the independent variable.

B. Now, assume D(w) is a non-linear function.

B2. Explain the meaning of the statement $D'(140) = 2$ in the context of the problem.

The ideal knower would respond that when a patient is 140 pounds, the patient's weight is increasing at a rate of 2 mg per pound. This question assesses students' understanding of the derivative in the context of the problem, and their ability to demonstrate a pointwise understanding of the derivative at a point.

B3. Using the fact that $D'(140) = 2$, Nurse Jodi predicts that a patient's dose will increase by 2 mg when the patient's weight changes from 140 lbs. to 141 lbs. How much confidence do you have in her reasoning?

The ideal knower would respond "somewhat confident," with some explanation of the instantaneous rate of change as an appropriate approximation for the marginal change, or for input values very close to the input value of the derivative. Students might also discuss linear approximation and how the tangent line is a good approximation for the function near the point of tangency. This problem is designed to assess students' understanding of the instantaneous rate of change for use in predicting marginal change. It is important that students demonstrate an understanding that the non-linear nature of the function means the derivative gives an estimate of the change (and because information is not given about the type of non-linear function, one cannot be sure how much error is involved).

B4. Nurse Jodi accurately doses a 140-pound patient using the model. Her next patient is 160-pounds and she reasons that since $D'(140) = 2$, she must increase the dose by 40 mg (2 mg for each pound of weight). How much confidence do you have in her reasoning?

The ideal knower would respond, "Not confident because 2 milligrams per pound is the instantaneous rate of change for a 140-pound person. Because the function is non-linear, one can not use the instantaneous rate of change to make a prediction so far away from 140-pounds." This ideal knower would understand that the instantaneous rate of change is not a constant rate of change, and cannot be used as an estimate of the rate of change except at or around the specific input value. This question is designed to assess students' across-time understanding of instantaneous rates of change.

Figure 2. Ideal knower responses.

Thirdly, there were incorrect or incomplete responses. For example, some gave answers out of context such as “the rate of increase of the function,” or gave vague answers such as “it tells us that the corn is increasing.” Table 1 summarizes the results.

Correct	“Ratio of Totals” Interpretation $B(n) = 2n$	Incorrect/ Incomplete
17%	39%	43%

Table 1. Student responses for the slope interpretation question, N=69

Pointwise Derivative Interpretation

In examining students’ responses to the question of what $B'(20) = 2$ means in the context of the non-linear problem, six categories of responses emerged (Table 2). First, some students responded correctly, saying something like “when the nitrogen is equal to 20 pounds, the corn yield is increasing at a rate of 2 bushels per pound of nitrogen.” The key language here is that students recognize that the derivative represents a rate of change at a point. Some students responded similarly, except with no units or incorrect units, answering, for example, “when the nitrogen is equal to 20 pounds, the corn is increasing at a rate of 2.”

Correct	Correct but no/wrong units	$B(n) = B'(n) * n$	$B'(n) = B(n)$	No context	Incorrect/ Incomplete
13%	10%	16%	10%	19%	32%

Table 2. Student responses for the derivative interpretation question, N=69

Even more students responded using language that implied they were assuming that the rate of change could be used to calculate the total yield, stating that the derivative means “that at 20 pounds, there are 2 bushels produced for each pound of nitrogen.” Some went on to conclude that equaled a total yield of 40 bushes, implying that $B(n) = B'(n) * n$.

Some other students interpreted the derivative as the function value, concluding that $B'(20) = 2$ means that when 20 pounds of nitrogen are applied, the total bushels are equal to 2. The last two categories were for students who gave a correct answer but not in the context of the problem (for example, “it is the slope of the tangent line when n = 20”) and those who gave incomplete or incorrect answers. Table 2 summarizes the results.

Of the 11 students (16%) who answered the derivative question using the “$B(n) = B'(n) * n$” interpretation, 8 of them answered the linear question using the incorrect ratio interpretation ($B = 2n$).

Across-Time Derivative Interpretation and Critiquing

Five categories emerged from the responses to the two across-time non-linear questions in which were asked to critique the reasoning of Farmer Jim. Some students answered correctly, stating that they were somewhat confident on the one-pound increase but not confident in the ten-pound increase, and gave an explanation about the derivative being a good approximation close to 20 pounds of nitrogen. Other students said they could not answer the question because they were not given the derivative at 21 or 30. Still others said they needed to know where the derivative was zero (or where the critical point of the function was located).

Another category of answers was for those students who stated they were not confident in both predictions, because the relationship is non-linear, and therefore the derivative is different at each pound. Some students answered “confident” on both, and gave reasoning such as “for each pound of nitrogen, 2 bushels are produced,” reasoning that would be appropriate for a linear function. Lastly, some students’ answers were incomplete or incorrect, and did not fall into the other five categories. Table 3 summarizes the results.

Correct	Need another derivative to make prediction	Need to know maximum/critical value to make prediction	Not confident on both predictions because relationship is non-linear	Confident on both because for each pound of nitrogen, 2 bushels are produced	Incorrect/ Incomplete
16%	14%	4%	19%	4%	42%

Table 3. Student responses for the critiquing Farmer Jim’s non-linear predictions, N=69

Interview Survey Findings

There were themes in the written surveys that I wanted to delve into in the interviews. First was the pointwise interpretation of the slope and derivative in the context of the drug problem. In the surveys, 39% of students gave a slope explanation using an incorrect “ratio of totals” interpretation ($B(n) = 2n$), thus implying a direct proportional relationship, and 16% gave a similar derivative explanation ($B(n) = B'(n) * n$). Second, the responses for Farmer Jim’s non-linear predictions brought some questions to the forefront. Twenty-three percent (23%) of the students did not distinguish between the 1-pound and the 10-pound increases, giving similar confidence levels and reasoning. Finally, 14% stated that they did not agree with Farmer Jim because they needed a different derivative to answer the question.

Of the seven interviewees, two gave what I consider ideal answers for all questions. For example, for $D'(140) = 2$, Brandon said that at 140 pounds, the weight “is increasing at a rate of 2 mg per pound.” He was very confident in Nurse Jodi’s 1-pound response, stating that the linear approximation is a good approximation of the total dosage near 140 pounds. He was not confident in the 20-pound increase, because it would be a huge under approximation (based on his concave up graph), and said it was different than the 141-pound answer because it was so much farther from 140 pounds.

Slope and Derivative Interpretation

For the interpretation of a slope of 2 in the context of the problem, 3 of the 7 interviewees gave correct responses. The other four gave “ratio of total” responses, using language such as “for every pound, they would need 2 mg of drug” or “dosage is twice the number of pounds.” In other words, these students thought of the slope as the ratio of totals (instead of the ratio of changes), that leads to an implied $D(n) = 2n$ relationship (where the y-intercept is zero).

For the derivative interpretation, three gave correct responses (at 140 pounds, the drug dosage is increasing at a rate of 2 mg per pound), and four gave responses stating that it allows you to calculate how much dosage to give per each pound. For example, Kelly stated that it is the slope of the tangent line and tells us “at that point, the dosage is twice the weight; so at 140 pounds the dosage is 280 mg.” In other words, she thinks that the total dosage is equivalent to the derivative at a point times that input value ($D(n) = D'(n) * n$).

Table 4 summarizes the slope and derivative interpretations. The three students who interpreted slope correctly went on to interpret the derivative correctly. All four who interpreted slope as the “ratio of totals” (thus implying a direct proportional relationship) uses the derivative to calculate the total dosage (multiplying derivative by x-value to get the total).

		Slope Interpretation	
		Correct	“Ratio of Totals” ($D = 2n.$)
Derivative Interpretation	Correct	3	0
	$D(n) = D'(n) * n$	0	4

Table 4. Interview responses for the slope and derivative interpretations, N=7

Critiquing of Nurse Jodi’s Non-linear Predictions

Two interviewees correctly critiqued Nurse Jodi’s predictions, using language about linear approximation, or language about the derivative being a good approximation for small increases in the independent variable.

Two interviewees used a derivative to calculate the change, whether it was a one-pound or twenty-pound increase. For example, Jackie was equally confident in both Nurse Jodi’s responses, saying that the derivative could be used to approximate the change in dosage. John also agreed that the derivative could be used to calculate the change in both one-pound and twenty-pound increases, but explained that he needed a different derivative (at 141 or at 160) to calculate the change. For example, he said if we knew the derivative at 160 was equal to 4 mg per pound, you would multiple that by the change in weight (20 pounds) to get the total increase of dosage of 80 mg.

The three remaining students stated that you needed another derivative (either at 141 or 160) to calculate the total dosage (not the change in dosage). For example, for the 20-pound increase, Harry said that he would need to know the derivative at 160. If that were 3 mg per pound, he would “multiply 160 by 3. That would give her the right dosage.” He went on to confirm it was the total dosage, not the increase in dosage. Similar to those who described the derivative as the amount of milligrams of drug to give for each pound, this interpretation is also one that concludes incorrectly that $D(n) = D'(n) * n$.

Summarizing Interview Findings

The two students who correctly critiqued Nurse Jodi’s predictions also interpreted both the slope and derivative correctly. For the three students who critiqued Nurse Jodi’s predictions by saying that they needed another derivative, and the derivative could be used to find the total dosage ($D(n) = D'(n) * n$), all three interpreted slope incorrectly as the ratio or totals, and the derivative incorrectly as meaning “2 pounds for every n.” Table 5 summarizes the students’ approaches to both the pointwise slope and derivative interpretation, and the across-time critiquing of Nurse Jodi’s predictions.

		Slope & Derivative Interpretation	
		Correct	Use $D = 2n$ for slope and $D(n) = D'(n) * n$ for derivative
Critiquing Jodi’s Predictions	Correct	2	0
	Use derivative to find change over any interval	1	1
	Use derivative to find the total dosage $D(n) = D'(n) * n$	0	3

Table 5. Interview responses for the slope and derivative interpretations, N=7

Conclusions and Implications

My research focused on student understanding of calculus concepts that required both pointwise and across-time understanding of functions (Monk, 1994). While slightly improved over Bezuidenhout's (1992) findings, where only 2 of 100 participants were able to interpret the meaning of a derivative in the context of a problem, it is still discouraging that only 17% of students interpreted the slope correctly in the context of the problem, and only 13% interpreted the derivative correctly. Similarly, only 17% of the students were able to use a valid argument when critiquing the reasoning of someone else's predictions.

The interviews revealed interesting connections between students' abilities to interpret the slope and derivative in the context of the problem, and their abilities to critique the reasoning of Nurse Jodi. The four interviewees who interpreted the slope using an incorrect "ratio as total" approach (implying $D(n) = 2n$) went on to interpret the derivative using an incorrect $D(n) = D'(n) * n$ approach. Three of these four students carried this incorrect interpretation on to their critiques, where they said that from the derivative at a point, one could figure out the total dosage ($D(n) = D'(n) * n$). These undesirable generalizations that students formed using their impoverished view of slope seem to be interfering with their understanding the derivative as a continuously varying rate of change.

Revisiting the Research Questions

Recall my two original research questions: (1) Can students interpret the slope and derivative in the context of the problem? (2) Can students appropriately critique the reasoning of someone else's use of slope and derivative to make valid predictions?

A large majority of students did not successfully interpret the slope and derivative in the context of the problem. We know students must understand rates of change in general to succeed in calculus (Hackworth, 1994), and my research adds to the set of findings that show that rates of change are not well-understood by calculus, many of whom may have fundamental misconceptions (Bezuidenhout, 1998). Over a third of the surveyed students took an approach to the slope interpretation where they interpreted slope as a ratio of the totals, which is much higher than the 12% of interviewees in Hauger (1995). As Hauger (1995) pointed out, the differences between $\left(\frac{y}{x}\right)$ and $\left(\frac{\Delta y}{\Delta x}\right)$ are very subtle. These are probably even more pronounced when verbally describing the slope, as just leaving out the word "additional" can alter the meaning. However, the four interviewees who used the incorrect interpretation of slope went on to interpret the derivative in a similar incorrect fashion, leading me to believe that it was not just a simple act of leaving out the word "additional."

We know that proportional reasoning is difficult for students (Hoffer & Hoffer, 1988; Lawton, 1993; Lesh, Post, & Behr, 1988; Tourniaire & Pulos, 1985), and perhaps students' misunderstandings of linear functions in general, and directly proportional relationships specifically (for which the slope *is* a ratio of totals) lead to an impoverished understanding of slope. Even with the pointwise interpretation questions, students performed poorly.

Considering the poor performance on the interpretation questions, it is not surprising that students were not able to critique the reasoning of someone else's predictions. Often their misunderstandings from the pointwise questions carried over to the critiquing questions. Only 16% of the surveyed students (11 out of 69), and 2 out of 7 of the interviewees, answered the non-linear critiquing questions correctly. Of the 11 surveyed students who answered them correctly, only one student used language about linear approximation.

Future Research and Teaching Implications

Expanding on previous findings that show students lack solid understanding of rates of change in general (Hackworth, 1994; Orton, 1984), findings suggest that students do not have full understanding of what slope and derivative mean in the context of modeling situations, nor do they understand appropriate uses of slope and derivative to make predictions.

More research must be done on students' incorrect interpretations of both slope and derivative. I am most interested in students' "ratio of totals" interpretation of slope ($f(x) = m * x$) and the connection to their incorrect interpretations of derivative to calculate the total ($f(x) = f'(x) * x$). From middle school, when direct proportional relationships are first covered, do students see slope as something to multiply the x-value by to get the y-value? If so, is that being carried over to the derivative (which they equate with slope)?

As mathematics instructors, we need to assess our students coming into our calculus courses. What is their understanding of slope, and their interpretation of slope in modeled contexts? We also need to look at the middle school curriculum. Can we do a better job introducing linear relationships, making clear that directly proportional relationships are a subset of all linear relationships, and that not all linear relationships are of the form $f(x) = m * x$? Lastly, we need to focus on not just what derivatives can be used for (linear approximation, marginal cost, etc.), but also stress their limitations in making predictions.

References

Asiala, M., Cottril, J., Dubinsky, E., & Schwingendorf, K. (1997). The development of students' graphical understanding of the derivative. *Journal of Mathematical Behavior*, *16*(4), 399–431.

Barr, G. (1980). Graphs, gradients, and intercepts. *Mathematics in School*, *9*(1), 5–6.

Barr, G. (1981). Some student ideas on the concept of gradient. *Mathematics in School*, *10*(1), 16–17.

Beichner, R. J. (1994). Testing student interpretation of kinematics graphs. *American Journal of Physics*, *62*(1994), 750. doi:10.1119/1.17449

Bezuidenhout, J. (1998). First-year university students ' understanding of rate of change. *International Journal of Mathematical Education in Science and Technology*, *29(3)*, 389–399.

Bingolbali, E., Monaghan, J., & Roper, T. (2007). Engineering students' conceptions of the derivative and some implications for their mathematical education. *International Journal of Mathematical Education in Science and Technology*, *38*(March 2015), 763–777. doi:10.1080/00207390701453579

Byrnes, J. B. (2000). *Cognitive Development and Learning in Instructional Contexts* (2nd editio.). Pearson Allyn & Bacon.

Carlson, M. (1998). A cross-sectional investigation of the development of the function concept. *CBMS Issues in Mathematics Education*, *7*, 114–162.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events : A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352–378.

Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics*, *26*(2), 135– 164.

Crawford, A. R., & Scott, W. E. (2000). Making sense of slope. *The Mathematics Teacher*, *93*(2), 114–118.

Ferrini-Mundy, J., & Graham, K. (1994). Research in calculus learning: Understanding of limits, derivatives, and integrals. In J. J. Kaput & E. Dubinsky (Eds.), *Research Issues in Undergraduate Mathematics Learning: Preliminary Analysis and Results* (Vol. 33, pp. 29–45). Washington, DC: The Mathematical Association of America.

Ferrini-Mundy, J., & Graham, K. (2004). The education of mathematics teachers in the United States after World War II: Goals, programs, and practices. In G. M. A. Stanic & J.

Kilpatrick (Eds.), *A History of School Mathematics* (pp. 1193–1310). Washington, D.C.: The National Council of Teachers of Mathematics.
Ferrini-Mundy, J., & Graham, K. G. (1991). An overview of the calculus curriculum reform effort: Issues for learning, teaching, and curriculum development. *American Mathematical Monthly*, *98*(7), 627–635.
Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: a calculus course as an example. *Educational Studies in Mathematics*, *39*, 111–129. doi:10.1023/A:1003749919816
Habre, S., & Abboud, M. (2006). Students' conceptual understanding of a function and its derivative in an experimental calculus course. *Journal of Mathematical Behavior*, *25*, 57–72. doi:10.1016/j.jmathb.2005.11.004
Hackworth, J. A. (1994). *Calculus students' understanding of rate*.
Hauger, G. (1995). Rate of change knowledge in high school and college students. In *Annual Meeting of the Americal Educational Research Association*. San Francisco, CA.
Hoffer, A., & Hoffer, S. (1988). Ratios and proportional thinking. In *Teaching mathematics in grades K-8: Research based methods* (pp. 285–313).
Lawton, C. A. (1993). Contextual factors affecting errors in proportional reasoning. *Journal for Research in Mathematics Education*, *24*(5), 460–466.
Lesh, R., Post, T., & Behr, M. (1988). Proportional Reasoning. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 93–118). Reston, VA: National Council of Teachers of Mathematics.
Lobato, J., & Thanheiser, E. (2002). Developing understanding of ratio-as-measure as a foundation for slope. In *Making sense of fractions, ratios, and proportions* (pp. 162–175).
Maharaj, A. (2013). An APOS analysis of natural science students' understanding of derivatives. *South African Journal of Education*, *33*(1), 1–19.
Monk, G. S. (1994). Students' understanding of functions in calculus courses. *Humanistic Mathematics Network Journal*, *9*, 21–27.
Nagle, C., Moore-Russo, D., Viglietti, J., & Martin, K. (2013). Calculus students' and instructors' conceptualizations of slope: A comparison across academic levels. *International Journal of Science and Mathematics Education*, *11*, 1491–1515. doi:10.1007/s10763-013-9411-2
National Governors Association Center for Best Practices, C. of C. S. S. O. (2010). *Common Core State Standards Mathematics*. Washington D.C.: National Governors Association Center for Best Practices, Council of Chief State School Officers.
Orton, A. (1983). Students' understanding of differentiation. *Educational Studies in Mathematics*, *15*, 235–250.
Park, J. (2013). Is the derivative a function? If so, how do students talk about it? *International Journal of Mathematical Education in Science and Technology*, *44*, 624–640. doi:10.1080/0020739X.2013.795248
Saldanha, L., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In *20th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 298–303).
Siegler, R. (2003). Implications of cognitive science research for mathematics education. In N. C. of T. of Mathematics (Ed.), *A research companion to principles and standards for school mathematics* (pp. 219–233). Reston, VA.
Stump, S. (1999). Secondary mathematics teachers ' knowledge of slope. *Mathematics Education Research Journal*, *11*(2), 124–144.
Stump, S. (2001). High school precalculus students ' understanding of slope as measure. *School Science & Mathematics*, *101*(February), 81–89.

Teuscher, D., & Reys, R. E. (2007). Rate of change: AP calculus students' understandings and misconceptions after completing different curricular paths. *School Science & Mathematics*, *112*, 359–376.
Thompson, P. W. (1994). Images of rate and operational understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics*, *26*(2-3), 229–274.
Thompson, P. W., & Thompson, A. G. (1992). Images of rate. In *Annual Meeting of the American Educational Research Association*. San Francisco, CA.
Tourniaire, F., & Pulos, S. (1985). Proportional reasoning : A review of the literature. *Educational Studies in Mathematics*, *16*(2), 181–204.
Tyne, J. (2014). Slope and derivative: Calculus students' understanding of rates of change. In T. Fukawa-Connolly, G. Karakok, K. Keene, & M. Zandieh (Eds.), *17th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 299–310). Denver, CO.
Ubuz, B. (2007). Interpreting a graph and constructing its derivative graph: Stability and change in students' conceptions. *International Journal of Mathematical Education in Science and Technology*, *38*(5), 609–637. doi:10.1080/00207390701359313
Weber, E., Tallman, M., Byerley, C., & Thompson, P. W. (2012). Introducing derivative via the calculus triangle. *Mathematics Teacher*, *104*, 274–278.
White, P., & Mitchelmore, M. (1996). Conceptual knowledge in introductory calculus. *Journal for Research in Mathematics Education*, *27*(1), 79–95.
Wilhelm, J. A., & Confrey, J. (2003). Projecting rate of change in the context of motion onto the context of money. *International Journal of Mathematical Education in Science and Technology*, *34*(March 2015), 887–904. doi:10.1080/0020739031000160666 0
Zandieh, M. J. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. *CBMS Issues in Mathematics Education*.

VIEWING MATH TEACHERS' CIRCLES THROUGH THE PRIME LEADERSHIP FRAMEWORK

Diana White
University of Colorado Denver
Diana.White@ucdenver.edu

Jan A. Yow
University of South Carolina
jyow@sc.edu

This study explored the effect of participation in Math Teachers' Circles (MTC) on aspects of teacher leadership. MTCs are a collaborative professional development model, aimed primarily at middle school mathematics teachers, focused on problem-solving and open-ended mathematical explorations. Prior studies have linked them to increased content knowledge and pedagogical content knowledge, particularly in the domains of number and operation. This exploratory study used self-report data gathered from a national survey across 13 MTC sites and consisting of 169 completed surveys from MTC participants. Analysis of the data showed that MTC participation can help teachers develop across the three stages of leadership presented in the PRIME leadership framework. Further research is needed to ascertain characteristics of teachers for whom this effect is particularly pronounced, as well as to reconcile some of the self-report data with actual practice.

Key words: Math Teachers' Circles, Professional Development, Middle School Education, Teacher Leadership, Teacher Knowledge

Introduction

The widely-implemented Common Core for State Standards in Mathematics (CCSS-M) is leading to a need for mathematics teachers to improve their own knowledge of mathematical content and how to teach that content as well as to lead students, parents, colleagues, administrators, and communities in understanding and meeting the new standards (CCSSI, 2010). Math Teachers' Circles (MTCs) are a relatively new and innovative form of professional development in which mathematics professionals, generally mathematicians and mathematics educators meet with math teachers to work on mathematics problems and discuss teaching strategies. Math Teachers' Circles offer potential to strengthen content knowledge (White et al., 2013; White et.al., 2014) while also developing teacher leadership skills (Yow and Lotter, 2014).

This study highlights the potential of MTCs to contribute substantially to the development of mathematics teacher leaders. This paper will provide background on MTCs, their history, and some problems MTCs have addressed. Both researchers are experienced MTC leaders, and this paper augments the resulting knowledge of MTCs, teacher leadership, and teacher professional development. Specifically, we examine quotes from national surveys of MTC participants using the National Council of Supervisors of Mathematics' PRIME Leadership Framework (NCSM, 2008) as a lens. Our specific research question is: *How does teacher involvement in a Math Teachers' Circle encourage their enactment of teacher leadership?* Concluding remarks highlight the potential of MTCs as an important component in the development of mathematics teacher leaders.

Math Teachers' Circles

MTCs are accessible entry point for mathematicians to work with mathematics teachers and for teachers interested expanding their mathematical knowledge and skill. The advent of the

CCSS makes this partnership particularly timely. MTCs tap into mathematicians' instinct to share their passion for mathematics to provide professional development, primarily to middle school math teachers. Mathematicians facilitate sessions, guiding the group, ask probing questions, modeling mathematical thinking, and in general facilitating mathematical conjecturing, exploring, communication, and discovery – all skills that a research mathematician uses on a daily basis in their work. Additionally, MTC leaders draw on their experience training graduate students, providing research experiences for undergraduates, and leading capstone experiences for mathematics majors, as these experiences all share some commonalities with MTCs.

MTCs began as an extension of Math Students' Circles, which have their roots in Eastern Europe, migrating to the United States in the 1990s. The first MTC began at the American Institute of Mathematics (AIM) in Palo Alto, California in 2006. It arose from teachers who took their own students to a Math Students' Circle in the local area deciding that they wanted their own venue to explore mathematics together. Today 71 active chapters in 36 states around the country host MTCs (AIM, 2015). For about seven years, AIM held two weeklong trainings for MTC leaders every summer, resulting in 10-12 new MTCs each year. MTC leaders were required to attend as a team, typically including two mathematicians, two middle school math teachers, and one administrator. After the summer training workshop, teams typically spend September through May securing funding and then launch their own MTC the following summer. Summer workshops are generally residential, lasting 4-5 days. Meetings then continue during the academic year with typically three per semester, each 2-2.5 hours in duration. Each MTC team tailors this basic model to meet the needs of its local setting. Many MTCs have been active for years, and the model makes it possible for circles to continue indefinitely.

Typical Session and Connecting to the Common Core

Sessions typically begin with the session facilitator, generally a mathematician, presenting a mathematically rich problem. MTCs select problems involving multiple levels of deep mathematical content to foster exploration. Participants work individually and in groups through various problem-solving strategies. Consider the following sample problem:

> Write numbers from 1 to 100 on the board. Each minute, you select any two of the numbers, erase them, and write on the board the sum plus the product of the two numbers. For example, if you erased 3 and 5, the sum plus the product is 8 plus 15, or 23, and so you write a 17 on the board. Now there are two 23s, but that's OK. Each minute, repeat this process of selecting two numbers and replacing them with their sum plus their product. What are the possible outcomes?

This problem also lends itself to discussing a variety of mathematical topics, to include, algebraic representations and functions, symmetry, and arithmetic and algebraic properties such as the associative, commutative, and distributive laws. From a problem-solving perspective, it readily lends itself to the techniques of "ask a simpler question" and "work backwards". In addition, this problem lends itself well to problem-posing, as a variety of other related questions can also be asked.

In addition, in working on this problem participants will naturally use most of the Standards of Mathematical Practice from the Common Core State Standards. As just a few examples, they must make sense of this problem, which to some is initially ambiguous, and persevere toward a solution (MP1); they must attend to precision (MP6), as calculation errors are extremely easy to make with so many computations involved; they must use appropriate tools strategically (MP5), as a calculator only helps minimally due to the large numbers that quickly arise; and they must look for and makes use of structure when they begin to use algebraic representations to facilitate their work on this problem (MP7).

This problem also has what many mathematics educators refer to as the "low-threshold, high-ceiling" property. That is, participants can understand it and begin to explore with a minimal mathematics background ("low threshold"), yet it can keep research mathematicians meaningfully challenged and connected to research-level mathematics ("high-ceiling"). One of the authors has found that this problem is so intriguing to mathematicians that when she uses it to describe MTCs she must preface her description by asking them not to spend the remainder of the talk engaged with the problem.

The sum plus the product problem prompts in-depth explorations; MTCs also investigate shorter problems, such as "What happens to the last (units) digit of 7^n as you substitute in consecutive natural numbers for n? What happens to the last two digits? The last three?". Computing either of these by hand would be quite cumbersome, and plugging them into a calculator provides an estimate in scientific notation, but does not help directly with finding the last few digits. However, an MTC can have a meaningful discussion of these problems in about 30 minutes. Participants are still asked to justify their reasoning, with an emphasis on understanding a pattern and being able to mathematically explain why it holds, as opposed to simply observing that it seems to exist.

In working through these problems, teachers work directly with content relevant to their students, while also developing their own mathematical reasoning skills. For considerable additional discussion of how MTC sessions may develop, see Fernandes, Koehler, and Reiter (2011), Donaldson et. al. (2014), Geddings, White, and Yow (2015), Taton (2015) and White (2015).

Connecting to the Classroom

Many MTCs also directly address teaching techniques, addressing diverse topics such as effective questioning strategies, how to translate lessons learned in MTCs to the classroom, and how to implement the CCSS. The authors have used both Boaler and Humprey's (2005) *Connecting Mathematical Ideas: Middle School Video Cases to Support Teaching and Learning* and Burago's (2010) *Mathematical Circle Diaries, Year 1: Complete Curriculum for Grades 5 to 7* to lead such sessions. By necessity, these sessions take on a different flavor than the mathematical problem solving sessions. For example, using a well-known problem such as the border problem (Boaler and Humprey's 2005), teachers watch a classroom video, discuss instructional practices that the instructor implemented, and make observations on student responses. Such sessions are often best led by a skilled mathematics educator with experience facilitating such discussions, thereby leveraging the partnerships between mathematicians and mathematics educators.

The first national survey of MTC participants in 2010 revealed that the benefits of MTCs went beyond the individual classroom. A number of participants made comments that indicated that since joining the MTC they had emerged as informal or formal leaders in their schools or districts. They attributed this at least in part to their participation in MTCs. The remainder of this paper discusses how MTCs connect directly with the PRIME Leadership Framework of the National Council of Supervisors of Mathematics (2008). As this preliminary data is entirely based on self-report data from participating teachers who elected to complete the survey, significant further research will be called for to deepen our understanding of the impact of and connections between MTCs and teacher leadership.

Literature Review & Theoretical Framework

Teacher Leadership

Dozier (2004) defined teacher leaders as "good teachers who influence others." Graham and Fennell (2001) identified the influence of teachers who believe they have the "skills and knowledge to act on a situation and improve it." Research suggests teacher leadership in mathematics education improves student performance broadly (Pellicer & Anderson, 2001).

Professional organizations define educational leaders in teaching and have only recently begun to identify teacher leadership that is subject-specific (NBPTS, 2010; NCTM, 1991). Several studies describe characteristics specific to mathematics teacher leaders (Yow, 2007; Langbort, 2001; Miller et al., 2000); however, few empirical studies about how to develop mathematics teacher leaders exist (Yow, 2010; Webb, Heck, & Tate, 1996).

CCSS-M emphasize K-12 students should be learning mathematics through problem solving (National Council of Teachers of Mathematics [NCTM], 2000; CCSSI, 2010). The incorporated Standards call for students to discuss, collaborate and justify their thinking through engaging tasks (NCTM, 2000; CCSSI, 2010). However, research shows that middle school mathematics teachers still often teach content in traditional didactic manners emphasizing textbooks and lecture (Grouws & Cebulla, 2000; Kent, Pligge, & Spence, 2003; Weiss, Pasley, Smith, Banilower, & Heck, 2003) and many barriers to instructional change exist (Anderson, 1996; Roehrig, Kruse & Kern, 2007). MTCs seek to empower teachers to make changes to their instruction in keeping with the current understanding of best practices (Fullan, 2001).

Professional Development

Professional development is one avenue for empowering teachers to make changes. Effective professional development of mathematics teachers should build their content knowledge, immerse them in authentic mathematical inquiry, address beliefs about mathematics, involve them in collaborative communities, and provide long-term support for pedagogical growth (Darling-Hammond, Chung Wei, Andree, Richardson, & Orphanos, 2009; Johnson, 2006; Loucks-Horsley, Hewson, Love, & Stiles, 2003). Flowers and Merten (2003) documented that middle school teachers have specific needs related to content and student learning. Further, teachers who attended professional development experiences connected to other school-based initiatives and that consisted of at least 8 total hours reported the experiences improved their teaching where shorter unconnected experiences had not (Flowers & Merten, 2003). Other studies show long-term professional development of more than 80 hours will prompt teachers to enact inquiry-based practices where shorter programs will not (Porter, Garet, Desimone, & Birman, 2003) and that professional development focused on mathematical problem solving improves teachers' pedagogical strategies and increases their content knowledge (Anderson & Hoffmeister, 2007). The design of MTCs incorporates characteristics this research supports: they are focused on building teacher content knowledge through problem solving strategies, long-term, and involve building collaborative communities between teachers, mathematics educators, and mathematicians.

MTCs are specifically recommended by the Conference Board of the Mathematical Sciences in their Mathematical Education of Teachers II document:

> Math teachers' circles [sic], in which teachers and mathematicians work together on interesting mathematics, provide ongoing opportunities for teachers to develop their mathematical habits of mind while deepening their understanding of mathematical connections and their appreciation of mathematics as a creative, open subject.… A substantial benefit of such programs is that they address the isolation of both teachers and practicing mathematicians: they establish communities of mathematical practice in which teachers and mathematicians can learn about each others' profession, culture, and work. (p. 68)

Other articles (White et. al., 2013) have aligned MTCs with Desimone's (2009) model for profession development, noting that it meets her five suggested criteria for effective professional development: content focus, active learning, coherence, duration, and collective participation.

Theoretical Framework

Our data analysis tool consists of the Principles and Indicators for Mathematics Education Leaders (PRIME). The National Council of Supervisors of Mathematics (NCSM) developed PRIME in 2008 to call attention to the importance of school leadership to improve teaching and learning. PRIME "aims to describe actions for mathematics education leaders across all settings, preK-12"; it notes the "complexity" of leaders' task (NCSM, 2008, p. 2). While PRIME addresses educational administrators as well, the current study focuses on teachers as those most closely connected to student learning.

PRIME lists a number of actions on a continuum of three stages of leadership growth. Stage 1, Leadership of Self, consists of knowing and modeling leadership; Stage 2, Leadership of Others, consists of collaborating and implementing structures for shared leadership on a local level; Stage 3, Leadership in the Extended Community, consists of advocating and systematizing improvements into the wider educational community (NCSM, 2008, p. 2). At each stage, NCSM notes, teachers have to build important attributes of self-knowledge, influence of others, and advocacy on a larger scale.

Methods

A national survey measuring MTC teacher impact was distributed to the leaders of 21 MTCs in 2010, with a request for them to disseminate it to their participants. The survey was modeled on the Student Assessment of their Learning Gains (SALG) survey (Seymour et al., 2000). It contained Likert scale and open-response items asking teachers to rate and address their gains in mathematical content knowledge, changes in attitudes and dispositions toward mathematics, growing knowledge of classroom instructional practices, and their professional activities before and after participation. Each of the open-ended prompts asked teacher respond with comments that were specific to the impact of MTCs. For example, one open-ended prompt read "Please comment on how your knowledge and understanding of mathematics has changed as a result of participating in your local Math Teachers' Circle." Another read "Please comment on how your attitudes and dispositions toward mathematics have changed as a result of participating in your local Math Teachers' Circle." A third read "Please comment on how participating in your local Math Teachers' Circle has affected your professional activities outside the classroom."

In keeping with the PRIME Leadership Framework (NCSM, 2008), responses to the open-response items were examined for evidence that they were exhibiting characteristics representative of Stage 1, Stage 2, or Stage 3 leaders.

Results and Discussion

In this section we integrate results and discussion, showing how participant responses that fell within each of the framework's three stages of leadership. These preliminary results were based on self-report data only. As such, further research would be needed to ascertain if their report of their classroom activities reflects the reality of their classroom. Moreover, this study makes no claims that MTCs will enable all teachers to transition through these three stages of leadership. Rather, we present evidence that some teachers who participate in MTCs perceive them as having effects that align with this continuum.

Fourteen MTCs had participant responses, ranging from three to 41, with a total of 169 responses. This represents both the variation in sizes of different MTCs as well as the length of time over which they have been in existence. Over 80% of these teachers had at least five years of teaching experience, and most (59%) had participated in and MTC for at least one full year. They taught in diverse settings, with 43% reporting that they taught in an urban setting, another 29% reporting that they taught in a suburban setting, and the remaining 28% reporting that they taught in a rural setting. Approximately 38% reported that they taught in high-needs schools. Both of the aforementioned categorizations were left for them to determine.

We now address each of the three stages.

Stage 1: Leadership of Self

MTCs encourage teachers to develop their content and pedagogical expertise, change classroom practices, and take risks (Yow, 2007), all of which are part of leadership of self. MTCs focus on building teacher content knowledge through problem-solving while also strengthening teachers' problem-solving skills and fluency with implementing the CCSS Standards of Mathematical Practice. Understanding the content and the disciplinary practices of mathematics provides a foundation for good teaching practices.

Participants have highlighted the power of taking on the role of learners of mathematics through MTCs. In the national survey, over 75% of respondents reported at least moderate gains, with approximately 60% reporting good or great gains, in their overall content knowledge of mathematics, their mathematical problem-solving skills, their understanding of various problem-solving strategies, and their understanding of connections between areas of mathematics. There was no formal measure of their gains in these areas. However, prior work (White et.al.. 2013; White et.al., 2014) using overlapping but different MTCs has demonstrated content gains related to the domain of number and operation as a result of summer workshops.

Teachers connected their perceived learning in the MTC to the underlying nature of mathematics with quotes such as "…given me the opportunity to see a bigger picture in mathematics", "… can see many more connections between various mathematical activities", and "The logic and order that mathematics creates seems to become more clear the deeper we explore."

In the context of CCSS-M, which asks many teachers to teach in ways that are often quite different from how they were taught, the opportunity to revisit the role of learner has particular power. As one teacher notes:

> MTCs also emphasize the habits of mind and the disciplinary practice of mathematics, in which a question does not necessarily have a single "answer" and exploration follows uncertainty. As one MTC participant noted, "I have not participated in a workshop where I as a person have to struggle through, and the presenter did not share the correct

> answer." Another comments, " [I] feel more comfortable with open ended questions… [I] feel comfortable with questions where I might not know the solution".

Another takes this a step further, noting how this has changed her mathematical sense of identity, "You encouraged me as a mathematician. I have never actually seen myself as one before."

In the MTC environment, leaders rarely give participants "answers" to the problems, even at the end of problem-solving session, which encourages them to continue to grapple and discuss the problem. We argue that this in turn leads to MTC participants feeling more comfortable in allowing their students time to explore mathematics and to hold off on immediately offering assistance. One teacher notes, "We have always done a lot of inquiry based lessons, but now I feel more comfortable allowing students to go deeper in their exploration.", while another comments that s/he now has the confidence "to allow students to explore deep mathematics." We argue that the teachers' experience in the MTCs and their report about engaging in more and deeper inquiry may mean that their student are more fully engaged in the first CCSS Standard of Mathematical Practice, perseverance.

Participants also describe the influence of the MTC on their own teaching, connecting also to the CCSS's third mathematical practice (constructing viable arguments and critiquing the reasoning of others):

> As a teacher I tell students that I don't have an answer key and we as a class have to decide if we solved a problem and whether our solution is reasonable. It is getting my students to understand the problem solving process and be able to reflect upon their thinking and justify their solution. It has helped build a community of problem solvers in the classroom.

MTCs allow teachers to work in community to learn mathematics and solve problems that they can then implement in their own classrooms. As another participant stated, "My classroom teaching has become more student-centered and engaging. Students are working together and discussing problems in groups, or exploring individually before sharing with a larger group." The skills MTC facilitators model and ask the teachers to engage in complement the development of content knowledge; through these elements MTCs encourage teachers to be Stage 1 Leaders.

Finally, teachers perceive that their MTC participation has increased their effectiveness in other mathematically intensive roles at their schools: "Math Circles training greatly increased my comfort level and confidence as a math competitions coach", "I had been involved with Math Team, MATHCOUNTS, A[merican] M[athematical] C[ompetitions] 8, and N[ew] Y[ork] M[ath] L[eague] in the past, but I have appreciated it more and done a better job at it since joining the Math Teachers Circle.", and "Math Circle was simultaneous to being asked to work on district curriculum development, and added to the process."

Stage 2: Leadership of Others

MTCs by their design lend themselves to Stage 2 Leadership of Others by "collaborating and implementing," encouraging participants to build community and learning how to solve problems alongside colleagues. Over 80% of respondents reported at least moderate gains, with over a third reporting great gains, in their enthusiasm for mathematics, their interest in discussing mathematics with colleagues, and their interest in discussing mathematics with professional mathematicians. One teacher commented, "I am inspired by a group of people who just want to get together and talk about math. I am not usually surrounded by people like that." Another notes the impact of collaboration "I have interacted with several participants that have approached problems differently than me, and have learned how to look at a problem from a different perspective than I would have otherwise."

Responses further reveal the MTC shows them how to collaborate with and hold mathematical discussions with their colleagues. A participant wrote: "I feel that working on mathematics with my colleagues gives me a wider perspective on how to view mathematics and what it means to teach mathematics," suggesting the ways in which MTCs reveal what it means to be a part of a mathematics education community. Another participant goes on to connect confidence in problem solving with collaborating with colleagues:

> My confidence with problem solving has increased a lot. I realize that just because I don't totally know a topic in math, I can still look at a problem and try to break it down to solve it. I'm not as intimidated. Also, being able to look at problems in this way enables me to be more likely to discuss the problem with other teachers or mathematicians who may suggest ways to solve the problem.

Finally, responses reveal how MTC have affected their direct and indirect leadership roles. One comments on the impact on attendance by colleagues, noting "I have shared with colleagues about the Math Circle and have more coming to each meeting." Others note direct mentorship or leadership: "I've been able to share as department chair these practices with other teachers" and "I've been able to mentor younger teachers and involve them in [mathematical content] discussions as well as help them structure more open ended lessons."

Stage 3: Leadership in the Extended Community

Because MTCs often span several schools and school districts, teachers are able to interact with an extended community of math teachers within a region and increase their ability to systematize the practices used in MTCs. This leads directly to preludes to Stage 3 Leadership, as teachers build a wider network of colleagues with whom to learn and collaborate.

For some teachers, this begins with an increase of awareness of and participation in the broader community. As one participant noted: "Participating in the MTC meetings has encouraged me to network with others and attend conferences with them." Others note an increase in interest in professional organizations and attendance at associated conferences: "I was more interested in going to the national NCTM this past April and I did.", "I decided to join NCTM for the first time.", and "Since attending Teacher's Circles [sic] I've been to the local Math Educator Conference every year."

Others commented directly about the connection to mathematicians and feeling like they are part of the same extended community of mathematics professionals. One notes "I enjoy the opportunity to discuss mathematics with my colleagues and especially with the professional mathematicians. I feel that they see us as peers and treat us as the math teachers/professionals that we are." One notes how this has led to further connections with higher education institutions, "Participating in the Math Teachers' Circle has made me more likely to participate in other math/science activities at the university." Finally, one connects this directly to mathematical leadership, "I have come into contact with more professional mathematicians, and I have been willing to step in and take more of a leadership role myself in the field of mathematics."

Through MTC participation, teachers recognize their role as part of a professional community. One notes "I'm much more connected to mathematics education community outside of my school, as I am now working with various other teachers." Another participant described the confidence this experience has provided, saying, that being a part of an MTC "has given me the confidence to step into more of a leadership role and a role in developing curriculum and lesson plans."

Teachers from MTCs have presented at state mathematics conferences as a result of their participation in MTCs. One teacher shared, "I have started giving presentations at meetings and conferences, have become the mentor for new math teachers, and am peer reviewer of math activities for the classroom at my level for the region." Presenters such as these use the example of MTC meetings to advocate for strong instructional practices, describing the rich problems that they themselves have spent time solving with their students in their turn. By describing the rich mathematical content of the problem and the mathematical directions they or their students may take to solve the problem, they provide a large number of teachers the opportunity to consider these methods, thereby engaging in tenets associated with Stage 3 Leaders.

One participant who has become an instructional coach credited her MTC with helping in her job transition. "My leadership role in our math circle has correlated directly with my role change in my district from classroom teacher to teacher leader (instructional coach)." These stories suggest that MTCs play an important role in empowering some teachers to be mathematics teacher ambassadors (Yow 2007); in this role they tell others about the work they are doing in the MTC and in their classrooms with students, which can wield broad influence.

Conclusion

MTCs provide a valuable professional development experience for mathematics teachers. This study documents the powerful sense of community and broad understanding teachers report gaining of mathematical concepts and the mathematics education community as a result of their participation in MTCs. By developing mathematics teachers' leadership of self (by developing their content and pedagogical expertise, changing their practices, and encouraging risk-taking), leadership of others (by encouraging them to recognize they are a part of a collaborative learning community), and leadership in the extended community (by empowering them to assume formal and informal leadership roles beyond their own classrooms and schools), MTCs have the potential to influence teacher education beyond their actual participants.

Research on MTCs reveals their influence on mathematical knowledge for teaching (White, 2013). Marle, Decker, and Khaliqi (2012) report that classroom observations of participants in one MTC revealed an increase in the use of inquiry-based learning and in pedagogical content knowledge after a year of participation. Future research might address how extended participation in a long-standing MTC aids the continued evolution of a mathematics teacher leader from Stage 1 leadership to Stage 3 leadership (NCSM, 2008). Through these influences, the national community and network of MTCs has opportunities to contribute toward transforming mathematics teacher leadership throughout the country.

References

American Institute of Mathematics. *Math Teachers' Circle network*. Retrieved on February 14, 2015, from http://www.mathteacherscircle.org/about/results

Anderson, R.D. (1996). *Study of curriculum reform. Vol. 1:Findings and conclusions*. Studies of Educational Reform. (ERIC Document Reproduction Service No. ED 397535)

Anderson, C., & Hoffmeister, A. (2007). Knowing and teaching middle school mathematics: A professional development course for in-service teachers. *School Science and Mathematics, 107*, 193-203.

Barth, R. S. (2001). The teacher leader. *Phi Delta Kappan, 82*, 443-449.

Boaler J., & Humphreys, C. (2005). *Connecting mathematical ideas: Middle school video cases to support teaching and learning.* Portsmouth, NH: Heinemann.

Burago, A. (2010). *Mathematical circle diaries, year 1: Complete curriculum for grades 5 to 7.* Providence, RI: Mathematical Sciences Research Institute/American Mathematical Society.

Common Core State Standards Initiative. (2010). Common core state standards for mathematics. Retrieved June 27, 2013, from http://www.corestandards.org/math

Conference Board of the Mathematical Sciences. (2012). *The mathematical education of teachers II.* Providence, RI: American Mathematical Society.

Darling-Hammond, L. (1999). *Teacher quality and student achievement: A review of state policy evidence.* University of Washington Center for the Study of Teaching and Policy. Retrieved August 2, 2010, from http://depts.washington.edu/ctpmail /PDFs/LDH_1999.pdf

Darling-Hammond, L., Chung Wei, R., Andree, A., Richardson, N., & Orphanos, S. (2009). *Professional learning in the learning profession: A status report on teacher development in the United States and abroad*. Oxford, OH: National Staff Development Council.

Desimone, L.M. (2009). Improving Impact Studies of Teachers' Professional Development: Toward Better Conceptualizations and Measures. *Education Researcher, 38(3);* 181-199.

Donaldson, B., Nakayame, M., Umland, K, and White, D., (2014) Math Teachers' Circles: Partnerships between Mathematicians and Teachers, *Notices of the American Mathematical Society*, 61(11), p. 1335-1341.

Dozier, T. (October, 2004). *Turning good teachers into great leaders.* PowerPoint presentation at the National Principals Forum, Washington, D.C.

Elmore, R. (2000). *Building a new structure for school leadership*. Washington, DC: The Albert Shanker Institute.

Fernandes, A., Koehler, J., & Reiter, H. (2011). Mathematics teachers circle around problem solving. *Mathematics Teaching in the Middle School, 17*(2), 108-115.

Flowers, N., & Mertens, S. B. (2003). Professional development for middle-grades teachers: Does one size fit all? In P. G. Andrews & V. Anfara, Jr. (Eds.), *Leaders for a movement: Professional preparation and development of middle level teachers and administrators* (pp. 145-160). Greenwich, CT: Information Age Publishing.

Fullan, M. (2001). *Leading in a culture of change.* San Francisco, CA: Jossey-Bass.

Geddings, D., White, D., & Yow, J. (2015). SCHEMaTC: The Beginnings of a Math Teachers' Circle, *Mathmate*, 37(1).

Graham, K. & Fennell, F. (2001). *Principles and Standards for School Mathematics* and teacher education: Preparing and empowering teachers. *School Science and Mathematics, 101*, 319-327.

Grouws. D. A., & Cebulla. K. (2000). Elementary and middle school mathematics at the crossroads. In T. L. Good (Ed.), *American education, yesterday, today, and tomorrow.* (Vol. 2. pp. 209- 255). Chicago: University Press.

Kent, L. B., Pligge, M., & Spence, M. (2003). Enhancing teacher knowledge through curriculum reform. *Middle School Journal, 34 (4),* 42-46.

Johnson, C. C. (2006). Effective professional development and change in practice: Barriers science teachers encounter and implications for reform. *School Science and Mathematics, 106*, 150-161.

Lieberman, A., & Miller, L. (2004). *Teacher leadership*. San Francisco, CA: Wiley & Sons.

Loucks-Horsley, S., Hewson, P. W., Love, N., & Stiles, K. E. (2003). *Designing professional development for teachers of science and mathematics*. Thousand Oaks: Corwin Press.

Marle, P. D., Decker, L. L., & Khaliqi, D. H. (2012). *An inquiry into Math Teachers' Circle: Findings from two year-long cohorts.* Paper presented at the School Science and Mathematics Association Annual Convention, Birmingham, AL.

National Board for Professional Teaching Standards (NBPTS). (2006). *What teachers should know and be able to do: The five core propositions of the National Board*. Retrieved on August 2, 2010, from http://www.nbpts.org/the_standards/the_five_core_proposition

National Council of Supervisors of Mathematics. (2008). *The PRIME leadership framework: Principles and indicators for mathematics education leaders.* Bloomington, IN: Solution Tree.

National Council of Teachers of Mathematics (NCTM). (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards of school mathematics.* Reston, VA: Author.

Pellicer, L. O. & Anderson, L. W. (2001). Teacher leadership: A promising paradigm for improving instruction in science and mathematics. *Science, Mathematics, and Environmental Education.* (ERIC Document Reproduction Services No. ED465586).

Porter, A. C., Garet, M. S., Desimone, L. M., & Birman, B. F. (2003). Providing effective professional development: Lessons from the Eisenhower Program. *Science Educator, 12*, 23-40.

Roehrig, G. H., Kruse, R. A., & Kern, A. (2007). Teacher and school characteristics and their influence on curriculum implementation. *Journal of Research in Science Teaching, 44*, 883-907.

Short, P. M. (1994). Defining teacher empowerment. *Education, 114*, 488-492.

Spillane, J., Halverson, R., & Diamond, J. (2004). Towards a theory of leadership practice: A distributed perspective. *Journal of Curriculum Studies, 36*, 3-34.

Taton, J.A. (2015). Much More Than It's Cooked Up To Be: Reflections on Doing Math and Teachers' Professional Learning. *Perspectives on Urban Education 12(1),* Retrieved on March 24, 2015 from http://www.urbanedjournal.org/archive/volume-12-issue-1-spring-2015

Webb, N. L., Heck, D. J., & Tate, W. F. (1996). The urban mathematics collaborative project: A study of teacher, community, and reform. In S. A. Raizen & E. D. Britton (Eds.), *Bold ventures: Case studies of U.S. innovations in mathematics education (Volume 3)*. Boston: Kluwer Academic Publishers.

Weiss, I., Pasley, J., Smith, S., Banilower, E. R., & Heck, D. (2003). *Looking Inside the Classroom: A Study of K-12 Mathematics and Science Education in the United States*. Chapel Hill: Horizon Research, Inc.

White, D., (2015) Math Teachers' Circles as a Relatively New Form of Professional Development: An In-Depth Look at One Model, *Journal of Mathematics Education Leadership,*16(1).

White, D., Donaldson, B., Hodge, A., & Ruff, A. (2013). Impact of Math Teachers' Circles on teachers' mathematical knowledge for teaching, *International Journal for Mathematics Teaching and Learning*, 28 pages. Retrieved from http://www.cimt.plymouth.ac.uk/journal/

White, D., Umland, K., Donaldson, B., Nakayame, M., & Conrey, B. (2014). Effects of problem solving workshops on mathematical knowledge for teaching. *Proceedings of the 38th International Conference on Psychology of Mathematics Education*.

White, D., & Yow, J. (2015). Math Teachers' Circles: Connections to teacher leadership. *Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education*.
Yow, J. A. (2007). A mathematics teacher leader profile: Attributes and actions to improve mathematics teaching & learning. *National Council of Supervisors of Mathematics Journal of Mathematics Education Leadership*, *9*(2), 45-55.
Yow, J. A. (2010). "Visible but not noisy:" A continuum of secondary mathematics teacher leadership. *International Journal of Teacher Leadership*, *3*(3), 43-78.
Yow, J. A., & Lotter, C. (2014). Teacher learning in a mathematics and science inquiry professional development program: first steps in emergent teacher leadership. *Professional Development in Education.* DOI:10.1080/19415257.2014.960593

A Model of the Structure of Proof Construction

Tetsuya Yamamoto
University of Oklahoma

This paper offers a model of the structure of proof construction. The model provides a comprehensive view of proof construction, which can encompass the aspects, factors, patterns, and features involved in cognitive processes in proof construction. The model also clarifies the skills and abilities necessary for proof construction. Moreover, the model provides an algorithm for advancing a reasoning process. Using some examples, this paper shows that the model can serve as metacognitive and methodological knowledge to help students construct a proof based on logical deduction. The model may help students not only grasp a view of proof construction but also develop their skills for proof construction.

Keywords: Structure of proof construction, Metacognitive knowledge

Introduction

Proof is a central and essential skill for mathematics. However, studies have shown that proving is challenging for students at all levels. Although this phenomenon has been well documented, there is still room for an investigation of students' difficulties with proof construction. Ball, Hoyles, Jahnke, and Movshovitz-Hadar (2002) emphasized the need for empirical studies of students' difficulties with proofs and the development of effective strategies to teach proofs. Dreyfus (2012) suggested that various questions should be answered regarding students' cognitive difficulties with proof. This paper presents part of the findings from my study, in which I examined students' cognitive difficulties with proof construction in light of the structure of proof construction. In particular, this paper focuses on answering one research questions in my study: What is a model of the structure of proof construction?

Literature Review

Logical deduction is a central aspect of proof construction. Weber and Alcock (2004) discussed two types of proof production: syntactic and semantic approaches. The former draws inferences by manipulating definitions through "unwrapping the definitions" and "pushing symbols." The latter guides the inferences by using instantiations of concepts with intuitive and non-formal representations. They indicated both approaches should concur for successful proof construction based on logical deduction. Selden and Selden (2007) considered that a proof consisted of two parts: the *formal-rhetorical* and *problem-centered* parts. The former is produced through syntactic approach, which involves manipulation of logic and definitions. The latter is produced through semantic approach, which involves conceptual understanding and mathematical intuition. They provided a method, as proof framework, of constructing the *formal-rhetorical* part. They suggested that students should first write a hypothesis, leave a blank space, put the conclusion at the end, and fill in the blank space by unpacking the conclusion. According to their method, students write a proof from both ends toward the middle. Their method may not help students write a proof from the top down. There seems to be room for an exploration of an effective method to help students practice syntactic and semantic approaches and write a proof from the top down.

Harel and Sowder (1998) stressed the necessity of fostering students' skills for logical deduction. Some researchers pointed out the significance and necessity of establishing a view of proof construction so that it can help students develop their skills for logical deduction. CadwalladerOlsker, Miller, and Hartmann (2013) attributed student's incomplete concepts of proofs to a source of their difficulties with proof construction. Knuth (2002) suggested the nature and components of proofs should be clarified to help students. Ayalon and Even (2008) claimed that views and approaches to deductive reasoning should be given more attention.

Considering the above gaps to fill and the needs to meet, I created a model of the structure of proof construction so that it could (a) encompass the aspects, factors, patterns, and features involved in cognitive processes in proof construction across mathematical subjects, (b) help to build a framework for analyzing sources of students' difficulties with proof construction in a clear, organized, and systematic way, and (c) provide metacognitive and methodological knowledge to help students enhance their skills for logical deduction.

Theoretical Perspectives

Harel and Sowder (2007) asserted that "a single factor usually is not sufficient to account for students' behaviors with proof." (p.4) Furinghetti and Morselli (2009) observed that mathematical thinking was not dominated by purely cognitive behavior but might be influenced by another factor such as affect. I considered the following four aspects as major aspects of proof construction: *reasoning activity* (cognitive actions or thinking operations for advancing a reasoning process), *background knowledge* (knowledge around a given proof problem); *mental attitudes* (tenacity, persistence, flexibility, carefulness, and precision); and *affect and beliefs* (emotions, self-confidence, beliefs toward mathematics, proof, and logic). Those aspects were considered not to be independent but to be intertwined to influence one another. This study focused on the first three aspects because the last aspect (*affect* and *beliefs*) depended on individuals.

Those aspects were found to agree with the categories of the theoretical framework for exploring mathematical cognition, which Schoenfeld (2010) presented. The following are the categories of the framework: (1) knowledge base (what students know); (2) problem-solving strategies (the tools or the techniques for solving problems); (3) self-regulation or monitoring (monitoring and assessing progress); (4) beliefs (one's understanding, feelings, perceptions, and decisions). The aspects of the structure of proof construction found some similarities and correspondences with those categories: (1) and *background knowledge*; (2) and *reasoning activity*; (3) and *mental attitudes*; (4) and *affect and beliefs*.

Method

To create a model of the structure of proof construction, I applied, as a variation of the think-aloud method, a self-analysis to my cognitive processes in proof construction. Think-aloud is a valid and effective research method to understand an individual's thinking process (Van Someren, Barnard, & Sandberg, 1994). I observed, described, abstracted, and organized my thinking processes while solving proofs for theorems and propositions. In particular, I focused on examining and organizing the operations used to generate a statement from the previous statement while exploring the patterns and features necessary for successfully advancing a reasoning process. The model was refined and established by going through 42 proofs. Those proofs ranged over several mathematical subjects: Algebra, Analysis, Topology; Calculus, Discrete Mathematics, and Trigonometry. The proofs examined to create a model included not only direct proofs but also proofs by contradiction, by contrapositive, and by mathematical induction, but did not include the proofs that asked to construct a counter example.

For inter-rater reliability, I had six mathematics professors review the model. Using the proofs that I made and that I had them make, I had them check if my model could be applicable to those proofs. The model was also compared with Newell and Simon's theoretical framework for problem solving (1972).

Results

The exploration of a model of the structure of proof construction, in particular the structure of the *reasoning activity*, provided the following findings: (1) types of operations for advancing a reasoning process; (2) roles of the operations; (3) an order of the operations to be tried; (4) similarities and correspondences with the theoretical framework for problem solving (Newell & Simon, 1972); (5) stages of proof construction; (6) types of proofs; (7) *mathematical language*; (8) types of variables; (9) *ignition phrases*; (10) starting variables; (11) major steps for the *opening stage*; (12) framework for describing the skills and abilities necessary for proof construction; and (13) inter-rater reliability.

Types of Operations

While examining the operations used to generate a statement from the previous statement, all the observed operations were categorized into four types: *rephrasing an object*; *combining objects*; *creating a cue*; and *checking and exploring* (Table 2). An object means a statement or a sentence for each step, and sometimes a phrase in a statement. The operations in Table 2 explained any cognitive action taken to move from one statement to the next statement in each of the proofs I examined.

Rephrasing an object has three subtypes: (R1) *rephrasing an object* by applying the definition or the property of a concept; (R2) *rephrasing an object* through interpretation; (R3) *rephrasing an object* through algebraic manipulation. The following are examples of the subtypes.

(R1) "$f: X \to Y$ is continuous" can be rephrased with "*for every open set U in Y*, $f^{-1}(U)$ *is open in X*."

(R2) "$a \in xZ(G)$" can be rephrased with "$a = xz_1$ for some $z_1 \in Z$."

(R3) "$\phi(a) = \phi(b)$, where ϕ is a ring homomorphism" can be rephrased with "$\phi(a-b) = 0$" or "$a - b \in Ker\ (\phi)$."

Combining objects is an operation to combine two or more pieces of given information to create a new object. The operation of *combining objects* is automatically followed by *rephrasing an object*. In other words, when the operation of *combining objects* is used, the new object is created through *rephrasing an object*. The following is an example.

The object (1) "*a sequence* (x_n) *converges to* x_0 *in X*" and the object (2) "$\exists$ an open set U that contains x_0" can be combined into the object (3) "$\exists N \in Z^+$ such that for every $n \geq N$, $x_n \in U$."

When (1) and (2) are combined into (3), (3) is obtained through *rephrasing an object* by applying the definition of convergence of a sequence to the open set *U*. The operation code used here is CO(A, B)R1, which means that the objects (1) and (2) were combined into (3) and that the new object (3) was obtained through applying the definition of the concept of convergence.

Creating a cue has 5 subtypes: (C1) setting a variable; (C2) recalling prior knowledge and applying it to an object; (C3) setting some cases; (C4) making a claim or creating a new object; and (C5) considering an object. The operation of *creating a cue* by recalling and applying prior knowledge to an object (C2) is automatically followed by the operations of *combining objects* and *rephrasing an object* successively. The following is an example of C2.

Suppose (1) "$\exists\ y \in G$" and (2) "A *coset is generated by* xZ." Recalling and applying the fact that (3) "*an element in G belongs to some coset*," the objects (1), (2), and (3) can be combined into (4) "*the element y belongs to some coset of* Z." The object (4) can be further rephrased with "$a \in x^n Z$ *for some* $x \in G$ *and* $n \in N$" through interpretation.

Exploring includes thinking by trial and error, intuiting, experimenting, and creating an example. *Checking* includes reviewing, testing, evaluating, adjusting, modifying, and correcting what has been done.

Roles of the Operations

I classified the four operations into two major cognitive actions: *the main actions* and *the supporting actions* (Table 2). *The main actions* included *rephrasing an object*, *combining objects*, and *creating a cue*. *Rephrasing an object* and *combining objects* play a role of transforming objects while *creating a cue* plays a role of igniting a process. *The supporting actions* included *checking* and *exploring*. *Rephrasing an object* and *combining objects* may contribute to syntactic approach while *creating a cue* and *checking and exploring* may contribute to semantic approach. The following are the differences between *the main actions* and *the supporting actions*. First, *the main actions* are taken by everyone while *the supporting actions* depend on individuals. Second, in order to convince others, the statements produced through *the main actions* must be explicitly expressed while the statements produced through *the supporting actions* do not have to be explicitly expressed. *The supporting actions* do not mean that they are less important than *the main actions*. *The supporting actions* are as crucial as *the main actions*, which work behind and support *the main actions*.

Order of the Operations

For the operations in *the main actions*, there is a hierarchy in the order of the operations to be tried in advancing a reasoning process. *Rephrasing an object* is the first operation to be tried. *Combining objects* is the second measure. *Creating a cue* is the last resort. In other words, in advancing a reasoning process, one should first try *rephrasing an object*. If *rephrasing an object* does not work, try *combining objects*. If *combining objects* does not work, try *creating a cue*. If creating a cue does not work, try *checking* and *exploring*.

Comparison with Theoretical Framework for Problem-Solving

I considered proof construction as a sort of problem-solving. The types of operations were found to have some similarities with the components of the standard theory for problem-solving, which Newell and Simon (1972) introduced. Langley and Trivedi (2013) agreed Newell and Simon's theory was one of the most robust and stable theories on high-level cognition. The following are the characteristics pertinent to problem solving in their theory: (a) representation, interpretation, and manipulation of symbolic structures; (b) search through a set of available information; (c) selective search through heuristics; and (d) reduction of the differences between current and desired states. In the model of the structure of the *reasoning activity*, *rephrasing an object* plays a major role of (a). *Combining objects* and *creating a cue* can function as (b). *Checking and exploring* corresponds to (c). The first three operations (*rephrasing an object*, *combining objects*, and *creating a cue*) contribute to realizing (d).

Stages of Proof Construction

Considering the importance of a start of a proof, two stages were set for proof construction: *the opening stage* and *the body construction stage*. The *opening stage* is a preparation stage, which has four major roles: (i) setting a major proving strategy (a direct proof, a proof by contradiction, a proof by contrapositive, mathematical induction, a proof by a counter example); (ii) making the goal of the proof clearer; (iii) preparing for setting a starting variable, and (iv) deciding the type of the proof. *The body construction stage* is the main part of a proof, in which one advances a reasoning process by making good use of the four types of operations (Table 2).

Mathematical Language

I made a distinction between mathematical language and *mathematical language*. *Mathematical language* is the mathematical language that is fine-grained enough to enable students to advance a reasoning process and to convince others without leaving any ambiguity. For example, "A topological space *X* is compact" is mathematical language. The *mathematical language* for this statement is "for every open cover of *X*, there exists a finite open subcover of *X*," which is obtained by applying the definition of continuity in the topological sense. The definitions of concepts are the most paradigmatic examples of *mathematical language*. The effectiveness of translating an object into *mathematical language* is not limited to proof problems. *Mathematical language* plays a significant role in other types of mathematical problem-solving as well. For example, if a Pre-calculus student is given a statement "vectors *u* and *v* are orthogonal," which is mathematical language, they may need to translate it into *mathematical language* in order to solve a given problem, which is "$u \cdot v = 0$."

Types of Variables

A variable is a fundamental and crucial element of the *mathematical language*. Using a variable in a mathematical argument enables students to practice logical deduction in advancing a reasoning process. There are two types of variables. One type of variable is the one which appears or is explicitly written in a given problem. I call this type of variable "a given variable." The other type of variable is the one which does not appear or is not explicitly written in a given problem. I call this type of variable a hidden variable. A hidden variable may need to be derived, revealed, and used for advancing a reasoning process. In particular, I call a hidden variable that students must derive and explicitly set "a *hidden variable*." Students advance a reasoning process in a proof by making good use of both given and *hidden* variables.

A hidden variable has four types: *controlling variables*, *trivial variables*, *conditioned variables*, and *non-conditioned variables*. Both *controlling* and *trivial* variables are derived from the phrases such as "for every …," "for all …," or "If …" A *controlling variable* can have the power to confine another variable and to change another variable when it is set, while *trivial variables* do not. Both *conditioned* and *non-conditioned variables* are derived from the phrases such as "for some …" or "there exists …" A *conditioned variable* is defined by a *controlling variable* while a n*on-conditioned variable* is not. For example, in the definition of continuity of a function $f: X \to Y$ at $x = x_0$, which is "For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$," the variable "$\varepsilon$,"is a *controlling variable* because it is derived from "For every $\varepsilon > 0$" and confines "δ" when it is set. The variable "δ" is a *conditioned variable* because it is derived from "there exists a $\delta > 0$" and is subject to the *controlling variable* "ε." In the definition of a bounded sequence, which is "For every $n \in Z^+$, $|a_n| \le M$ for some $M \in R$," $n \in Z^+$ is a *trivial variable* because it is derived from "For every $n \in Z^+$" and does not confine another variable. The word "trivial" does not mean "not important." It can play a significant role in the process of reasoning process. $M \in R$ is a *non-conditioned variable* because it is derived from "for some $M \in R$" and it is not confined by any *controlling variable*.

Types of Proofs

It is crucial to start a proof with a correct variable. I call the variable that students need to derive and set at the beginning of a proof a "starting variable." The proofs examined in this study were classified into three types by ways to derive a starting variable. Type I: Students were to derive and set a starting variable from the conclusion of the given statement. This type of proof contains an *ignition phrase* in the *mathematical language* for the conclusion. Type I had two sub-types of proofs. While Type I (a) did not ask students to show A = B, Type I (b) asked them to prove A = B. Type II: Students were to derive and set a starting variable from a hypothesis of the given statement. This type included the proofs in which the *mathematical language* for the conclusion did not include *an ignition phrase*. Proofs by contradiction belong to this type. The proofs that required students to construct an object might belong to this type. In both Types I and II, some proofs had more than one starting variable. Type III: Students did not have to derive a starting variable because it was already given in the problem. The proofs that asked students to prove A = B belonged to this type. Proofs by mathematical induction and proofs of trigonometric identities were such examples.

Ignition Phrases

I called the phrases in the *mathematical language* that provide hidden variables "ignition phrases." There are two types of ignition phrases: a primary ignition phrase and a second primary ignition phrase. A primary ignition phrase is "for any …" "for every …," or "for all …." The phrase "If …" also can be considered a primary ignition phrase if it is rephrased with "for any …" or "for every …." A second primary ignition phrase is "for some …" The phrase "there exists …" also can be considered a second primary ignition phrase if it is rephrased with "for some …." *Controlling* and *trivial variables* come from primary ignition phrases while conditioned and *non-conditioned variables* come from second primary ignition phrases.

An ignition phrase enables students to derive and set a starting variable. I call the ignition phrase that provides a starting variable "an *ignition phrase*." In a proof of Type I, the primary ignition phrase that provides a *controlling variable* in the *mathematical language* for the conclusion is an *ignition phrase*. However, if a primary ignition phrase provides a *trivial variable*, it may not be an *ignition phrase*. In a proof of Type I, a second primary ignition phrase in the *mathematical language* for the conclusion cannot be an *ignition phrase*. In other words, students should avoid deriving and setting a starting variable from a second primary ignition phrase in the *mathematical language* for the conclusion. If the *mathematical language* for a conclusion does not contain a *controlling variable*, the proof belongs to Type II, in which a starting variable should be found in the *mathematical language* for a hypothesis. In a proof of Type II, both primary and second primary ignition phrases can be *ignition phrases*.

The following are some examples of ignition phrases. Suppose students are asked to prove that a topological space *X* is Hausdorff. They can begin their proofs with setting a starting variable, which can be derived from an *ignition phrase* in the *mathematical language* for the conclusion. The *mathematical language* for the statement is "*For any two distinct points x and y in X, there exists an open neighborhood U of x and an open neighborhood V of y such that U and V are disjoint*." The *ignition phrase* is "*For any two distinct points x and y in X*". Students can start their proofs with "*Let x and y be distinct points in X*." The phrase "there exists an open neighborhood *U* of *x*" is a second primary ignition. Since it is in the conclusion, it cannot be an *ignition phrase*. Suppose students are asked to prove that a sequence $\{a_n\}$ is bounded. The *mathematical language* for the statement is "For every $n \in Z^+$, $|a_n| \le M$ for some $M \in R$." There are two ignition phrases, which are "For every $n \in Z^+$" and "for some $M \in R$." However, neither of them is an *ignition phrase*. The phrase "For every $n \in Z^+$" is not an *ignition phrase* because $n \in Z^+$ is not a *controlling variable* but a *trivial variable*. The phrase "for some $M \in R$" is not an *ignition phrase* either because an ignition phrase "for some ...." in the *mathematical language* for the conclusion is not considered an *ignition phrase*. Since the mathematical language for the conclusion does not provide an ignition phrase, this proof belongs to Type II, in which a starting variable should be found in the *mathematical language* for a hypothesis of the proof.

Major Steps for the Opening Stage

Model steps in the *opening stage* were established tentatively: (i) Decide a major proving strategy (a direct proof, a proof by contradiction, a proof by contrapositive, a proof by mathematical induction). If students choose a proof by contradiction or by contrapositive, rephrase the problem accordingly. (ii) Note the conclusion of the given statement. (iii) Translate the conclusion into *mathematical language*. By doing this, students can make the goal of the proof clearer and the distance between the beginning and the end of the proof shorter. (iv) Look for an '*ignition phrase*' in the *mathematical language*. By doing this, students can tell the type of the proof and prepare a starting variable. If there is an *ignition phrase* in the *mathematical language* for the conclusion, it belongs to Type I. If there is no *ignition phrase*, the proof belongs to either Type II or Type III.

Framework for Analyzing Students' Proofs

A framework for analyzing students' difficulties with proof construction (Table 1) was created based on the model of the structure of proof construction. The analysis framework also serves as a list of the skills and abilities necessary for proof construction.

Inter-rater reliability

For an inter-rater reliability, I had six mathematics professors review the model. Using the proofs I created and/or their own proofs I had them create, I had them confirm that the model of the structure of proof construction was applicable to those proofs. I received an agreement from all of them.

Table 1: Components of the Structure of Proof Construction

Reasoning Activity (O: Opening Stage, B: Body Construction Stage)			
O	OPS		Decide a major proving strategy (a direct proof, an indirect proof, etc).
	OTC		Set a goal: translate the conclusion of the given statement into *mathematical language*.
	ODT		Prepare a starting variable: Decide the type of the proof by noting an *ignition phrase*.
B	R	R1	Rephrase an object by applying a definition, a property, or a theorem.
		R2	Rephrasing an object through interpretation.
		R3	Rephrasing an object through algebraic manipulation.
	CO	CO(A,B)R	Combine objects to create a new object.
	C	C1	Set a variable.
		C2	Recall concepts, properties, theorems, propositions, problem-solving techniques.
		C3	Set some cases.
		C4	Make a claim or create a new object.
		C5	Consider an object.
	CE	Ch	Review, test, evaluate, adjust, modify, or correct what has been done.
		Ex	Search a clue, explore a solution by making a graph, a diagram, and an example, by intuiting, or by doing trial and error.
Background Knowledge			
KDF			Know definitions and properties of concepts.
KTH			Know theorems, propositions, and lemmas.
KNT			Know notations.
KTE			Know proving techniques or problem solving techniques.
Mental Attitudes			
MT			Have tenacity and persistence; not give up proving or solving problems easily.
MF			Have flexibility to give up an idea that does not work and to try a new or different method.
MC			Have carefulness, precision, or alertness.
Affect and Beliefs			
AF			Have self- confidence, have a mentality to not let negative emotions affect proving or solving performances.
BL			Have a right belief on proofs, logic, and mathematics.

Table 2: Structure of the Reasoning Activity

Reasoning Activity in Proof Construction	ACTIONS ⇨	MAIN									SUPPORTING	
	Roles of the Actions ⇨	Transforming objects				Igniting a process					Supporting the main actions	
STAGES ⇩	Operations ⇨ Steps ⇩	R Rephrasing Objects			CO Combining Objects	C Creating a Cue					Ch Checking (review, evaluate, test, modify, correct)	Ex Exploring (search, do trial and error illustrate, experi-ment, intuit)
		R 1	R 2	R 3	CO(A,B)R	C 1	C 2	C 3	C 4	C 5		
Opening Stage	Z: Choose a major proving strategy.	Decide which strategy to use (a direct proof, a proof by contradiction, a proof by contrapositive, a proof by finding a counter example, or a proof by mathematical induction. If necessary, rephrase the problem according to the chosen method.										
	X: Make sure of the goal.	Given. (Find the conclusion of the given statement.)										
	Y: Make the goal clearer with *mathematical language*.	R1, R2, R3 (Translate the conclusion into *mathematical language*.)										
	P: Make sure of the hypotheses.	Given (Find all the hypotheses of the given statement and translate them into *mathematical language* if necessary.)										
	S: Prepare a starting variable.											
Body Construction Stage	Step 1: State the hypothesis.	Given.										
	Step 2: Set a starting variable.					C1						
	…											
	…											
	Conclusion											

Footnotes

Actions	
Main Actions	The operations applied to a step to generate the next step, whose outcome must be explicitly expressed to convince others
Supporting Actions	The operations to produce side work, whose outcome does not necessarily have to appear in the proof to convince others

Rephrasing an object	
R1	Rephrase an object by translating a concept, a theorem, or a property of concept into mathematical language mainly through applying its definition.
R2	Rephrase an object through formal or informal interpretation.
R3	Rephrase an object through algebraic manipulation or calculation, including solving an equation.

Combining objects	
CO(S, T)R	Connect and combine different pieces of objects (S and T) to create a new object. This action is always followedd by an operation of rephrasing.

Creating a cue	
C1	Set a variable.
C2	Recall prior knowledge, including a theorem, a proposition, a property of concept, or a mathematical law.
C3	Set some cases.
C4	Make a claim or create a new object.
C5	Consider an object.

Discussion

The model of the structure of proof construction provides an algorithm for advancing a reasoning process in proof construction. This section introduces the algorithm and illustrates how the algorithm works.

Algorithm for Proof Construction
A: *Opening Stage*
A0: Read the problem
- If necessary, translate the whole problem into *mathematical language*. (A0.1)

A1: Decide a major strategy.
- Decide which proving strategy to use, a direct proof, by contrapositive, by contradiction, by counter example, or by mathematical induction. (A1.1)
- For a proof by contrapositive or contradiction, rephrase the problem. (A1.2)
- For Type III, skip to B0. (A1.3)

A2: Note the conclusion.
- Do not be tempted to note a hypothesis. (A2.1)

A3: Translate the conclusion into *mathematical language*.
- Rephrase the whole conclusion through R1 (See Table 1). (A3.1)
- Rephrase the conclusion more than once, if necessary. (A3.2)

A4: Find an *ignition phrase* in the *mathematical language* for the conclusion.
A5: Decide the type of the proof.
- If A4 is a primary ignition phrase, the proof belongs to Type I. (A5.1)
- If there is no *ignition phrase*, the proof belongs to Type II or Type III. (A5.2)
- If there is no *ignition phrase* and the problem asks to prove A = B, it belongs to Type III. (A5.3)

A6: Find a starting variable.
- For Type I, derive a starting variable from the *ignition phrase*. (A6.1)
- For Type II, note a hypothesis, translate it into *mathematical language*, and find an *ignition phrase*. (A6.2)
- For Type III, start the *body construction stage* by trying one of the followings: Work on either A or B to change it into B or A, work on both to obtain A = C = B, or show $A \subset B$ and $B \subset A$. This can work for the proofs in Type I (b). (A6.3)

TA: Supporting tips for the opening stage
TA1 (Type I): A starting variable should be first found in a primary ignition phrase in the *mathematical language* for the conclusion. However, if a variable in the primary ignition phrase is a trivial variable, it may not be a starting variable. A variable from a second primary ignition phrase in the *mathematical language* for the conclusion cannot be a starting variable. If there is not *ignition phrase* in the conclusion, derive a starting variable from a hypothesis.
TA2: (Type II) A starting variable can be derived from both a primary and a second primary ignition phrases in a hypothesis.
TA3: (Type I.b and Type III) Try one of the following methods. (i) Work on either A or B until you change it into B or A, (ii) Work on both A and B until you get A = C = B, or (iii) Show both A ⊂ B and B ⊂ A. For $A \cong B$, (iv) find an isomorphism between A and B.

B: *Body construction stage*
B0: State the hypothesis (hypotheses).

B1: Set a starting variable.

- For Type I, set a starting variable from the *ignition phrase* obtained in A4.1. (B1.1)
- For Type II, translate the hypothesis into *mathematical language*. (B1.2)
- For Type III, skip this step and start to work on part of the conclusion. (B1.3)

B2: Make sure of the new goal of the proof.
B3: Try *rephrasing an object*, recalling the three sub-types (See Table 3).

- Whenever seeing a sentence containing a mathematical concept, translate it into *mathematical language*, and make it as fine-grained as possible. (B3.1)

B4: If it does not work, try *combining objects*.

- Find a hypothesis and use it (B4.1).
- If there is more than one hypothesis, choose the one that has a connection with the object you would like to combine with. (B4.2)
- When the *mathematical language* for a hypothesis contains a controlling variable, use this operation (*combining objects*) to specify the controlling variable.

B5: If it does not work, try *creating a cue*, recalling the five sub-types (See Table 1).
B6: If it does not work, try *exploring and checking*.

T: Supporting Tips.
T1: For all types of proofs, whenever encountering a statement containing a mathematical concept, translate it into *mathematical language* and make it as fine-grained as possible.
T2: For Type II, when the *mathematical language* for a conclusion contains a *trivial variable* or when the *mathematical language* for a hypothesis contains a *controlling variable*, confine the variable to some specific object at a certain step.
T3: For type I(b) and Type III, try one of the followings. (i) Work on either A or B until you change it into B or A, (ii) Work on both A and B until you get A = C = B, or (iii) Show both A ⊂ B and B ⊂ A. For $A \cong B$, (iv) find an isomorphism between A and B.
T4: Always, keep the goal obtained in A3 in mind.

The following examples show how the above algorithm helps students to construct a proof. To make the algorithm more understandable, I will explain in the form of a dialogue between an instructor and students. In the dialogue, I assume that the students are fully equipped with not only the knowledge of the above algorithm but also the knowledge necessary for solving the given problems.

Example 1 (Type I)

"Suppose $G/Z(G)$ is cyclic, where $Z(G)$ is the center of G. Prove G is abelian. What should we do first?" "Decide the major proving strategy (A1)." "What strategy would you use?" "A direct proof." "What is the next step?" "Note the conclusion (A2), translate it into *mathematical language* (A3), and find an *ignition phrase* (A4)." "What is the conclusion?" "G is abelian." "What is the *mathematical language*?" "For any $a,b \in G$, $ab = ba$." "What is the *ignition phrase*?" "For any $a,b \in G$." "What is the type of this proof?" "Type I(b)." "How did you figure that out?" "The *mathematical language* for the conclusion contains a primary *ignition phrase* 'for any $a,b \in G$' and the goal of the proof is to show A= B, where A = ab and B = ba." "Let's begin the *body construction stage*. After stating the hypothesis (B0) 'Suppose $G/Z(G)$ is cyclic, where $Z(G)$ is the center of G,' what would you do?" "Set a starting variable from the *ignition phrase* 'for any $a, b \in G$' (B1.2)." "How?" "(1) Let $a, b \in G$." "Then?" "Work on the left hand side (2) 'ab'until it changes into the right hand side 'ba' so that we can show $ab = ba$." "Then?" "First, try r*ephrasing an object* (B2)" "Does that (B2) work for 'ab' or 'a' and 'b'?" "No." "What should we do?" "Try B3 (*combining objects*)." "How?" "Note the hypothesis and use it." "What is the hypothesis?" "(3) $G/Z(G)$ is cyclic." "Are we ready to combine the objects (2) 'ab' and (3) '$G/Z(G)$ is cyclic'?" "No." "Why not?" "Because (3) '$G/Z(G)$ is cyclic' contains a mathematical concept 'cyclic.' "So?" "Translate the object (3) '$G/Z(G)$ is cyclic' into *mathematical language*. (T1)" "What is the *mathematical language*?" "(4) 'A coset of $Z(G)$ is generated by $< xZ >$ for some $x \in G$.'" "Now, are we ready to combine the objects (2) 'ab' (or 'a' and 'b') and (4) 'a coset of $Z(G)$ is generated by $< xZ >$'?" "Not really." "What can we do?" "Since B3 (*combining objects*) does not work, try B4 (*creating a cue*)." "There are five types of *creating a cue* (Table 2). Which would you try?" "C2 (recalling and applying prior knowledge)." "What relevant fact can we use to combine the objects (2) 'ab' and (4) 'a coset of $Z(G)$ is generated by $< xZ >$ for some $x \in G$'?" " (5) 'Every element in a group belongs to some coset.'" "Now, can we combine these three objects (2) 'ab', (4) 'a coset of $Z(G)$ is generated by $< xZ >$', and (5) 'every element belongs to some coset'?" "Yes, we can combine them to obtain (5) $a \in x^m Z$ and $b \in x^n Z$ for some $x \in G$ and for some $m, n \in Z^+$." "Then?" "Since we have finished applying B4 (*creating a cue*), we can resume with B2 (*rephrasing an object*)." "Can we further rephrase the object (5) '$a \in x^m Z$ and $b \in x^n Z$'?" "Yes. $a = x^m z_1$ and $b = x^n z_2$ for some $z_1, z_2 \in Z$." "So?" "Using the commutative property of elements of the center Z of G, we obtain $ab = x^m z_1 x^n z_2 = x^{m+n} z_1 z_2 = x^{n+m} z_2 z_1 = x^n z_2 x^m z_1 = ba$."

Example 2 (Type II)

"Suppose that a sequence $\{a_n\}$ is convergent. Show $\{a_n\}$ is bounded." "What major strategy would you use? (A1)" "A direct proof." "How would you start the *opening stage*?" "Note the conclusion (A2), translate it into *mathematical language* (A3), and find an *ignition phrase* (A4)." "What is the conclusion?" "$\{a_n\}$ is bounded." "What is the *mathematical language*?" "For every $n \in Z^+$, $|a_n| \le M$ for some $M \in R$." "What is the *ignition phrase*?" "None." "Are not 'For every $n \in Z^+$' and 'for some $M \in R$' ignition phrases?" " The phrase 'For every $n \in Z^+$' is not an *ignition phrase* because $n \in Z^+$ is a *trivial variable*. A primary ignition phrase that provides a *trivial variable* is not considered as an *ignition phrase*. The phrase 'for some $M \in R$" is not an *ignition variable* because a phrase 'for some …" in the conclusion cannot be an *ignition phrase*." "Then, how would you set a starting variable?" "Since there is no *ignition phrase* in the conclusion, this proof belongs to Type II. So, after stating the hypothesis (B0), translate it into *mathematical language*. (B1.2)" "What is the hypothesis?" "$\{a_n\}$ is convergent." "What is the *mathematical language*?" "$\lim_{n \to \infty} a_n = L$ for some $L \in R$." "Then, what would you do?" "We can further rephrase it." "How?" "For every $\varepsilon > 0$, $\exists N \in Z^+$ such that for every $n \ge N$, $|a_n - L| < \varepsilon$." "Next?" " Derive a starting variable (A5)." "How would you do that?" "Find an *ignition phrase* (A6.2)" "What is an *ignition phrase*?" "'For every $\varepsilon > 0$." "So?" "We can set $\varepsilon > 0$ as a starting variable. However, since the variable is a *controlling variable* derived from a hypothesis, you might want to confine it to certain object by T2." "How would you do that?" "Let $\varepsilon = 1$." "What have we gotten so far?"

"(1) $\exists N \in Z^{+}$ and $L \in R$ such that for every $n \geq N$, $|a_n - L| < 1$." "How would you advance a reasoning process?" "First, try B2 (*rephrasing an object*)." "Does it work?" "Yes. (1) can be rephrased with (2) for every $n \geq N$, $|a_n| < |L| + 1$." "Can B2 (*rephrasing an object*) still work?" "No." "Then?" "Try B3 (*combining objects*)." "Would that work?" "No, there is nothing to combine with the object (2) for every $n \geq N$, $|a_n| < |L| + 1$." "Then, what would you do?" "Try B4 (*creating a cue*)." "There are five types of *creating a cue*. (See Table 2). Which type would you try?" "Create a new object (C4)." "What would you create?" "The *M* such that $|a_n| \leq M$ for every $n \in Z^{+}$." "How would you do that?" "(3) Let $M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |L| + 1\}$." "Can you rephrase it? (B2)" "No." "So?" "Combining the objects (2) and (3), conclude that for every $n \in Z^{+}$, $|a_n| \leq M$."

Example 3 (Type III)

"Let's solve the following problem. 'Suppose $a \equiv b \pmod{n}$ for $a, b \in Z$ and $n \in N$. Prove $a^3 \equiv b^3 \pmod{n}$." "What would you do first?" "Decide a proving strategy." "What strategy would you use?" "A direct proof." "Next?" "Note the conclusion (A2) and translate it into *mathematical language* (A3)." "What is the conclusion?" "$a^3 \equiv b^3 \pmod{n}$." "What is the *mathematical language*?" "$a^3 - b^3 = (a-b)(a^2 + ab + b^2) = nc$ for some $c \in Z$." "Are we going to find an *ignition phrase* (A4)?" "No." "Why not?" "Because this proof belongs to Type III, so you don't need to derive a starting variable. (B1.3)" "Then, after stating the hypothesis, how would you start the *body construction stage*?" "Consider the left hand side (1) $(a-b)(a^2 + ab + b^2)$ and work on it until it can be changed into nc. (A6.3)" "Then, what would you do?" "First, try *rephrasing an object* (B2)." "Does that work for (1) $(a-b)(a^2 + ab + b^2)$?" "No." "Then, what would you do?" "Try *combining objects*. (B3) " "How would you do that?" "Find a hypothesis and use it." "What is the hypothesis?" "(2) $a \equiv b \pmod{n}$ for $a, b \in Z$ and $n \in N$." "Can we combine (1) and (2)?" "No." "Why not?" "Because (2) $a \equiv b \pmod{n}$ contains a mathematical concept 'mod n.'" "Then?" "Translate (2) $a \equiv b \pmod{n}$ into *mathematical language*. (T1)" "What is the *mathematical language*?" "(3) $a - b = nd$ for some $d \in Z$." "Are we ready to combine (1) and (3)?" "Yes, we can combine them to obtain (4) $(a-b)(a^2 + ab + b^2) = nd(a^2 + ab + b^2) = nc$, where $c = a^2 + ab + b^2 \in Z$."

Conclusion

The model of the structure of proof construction elucidates the aspects, factors, patterns, and features involved in a cognitive process in proof construction. I viewed proof construction from four aspects: (1) *reasoning activity*; (2) *background knowledge*; (3) *mental attitudes*; and (4) *affect and beliefs*. Each aspect consists of multiple factors. The *reasoning activity* consists of four major factors: (i) *rephrasing an object*; (ii) *combining objects*; (iii) *creating a cue*; and (iv) *checking and exploring* (Table 2). *Background knowledge* includes students' knowledge of definitions, properties, notations, relevant facts, theorems, problem-solving strategies or techniques. *Mental attitudes* consist of three major factors: (i) persistence and tenacity; (ii) flexibility; (iii) carefulness, alertness, and precision. *Affect and Beliefs* consists of two major factors: (i) emotions, feelings, self-confidence; (ii) beliefs toward proofs, mathematics, and logic.

Those factors directly provide the skills and abilities necessary for proof construction (Table 1). Some patterns involved in cognitive processes were detected: the types of proofs that were classified according to the ways to manage *the opening stage*; the types of variables; the model steps for *the opening stages* for each type of proofs; and the algorithm for advancing a reasoning process in *the body construction stages*. The features of proof construction are found in the significance of the use of *mathematical language*, roles of the operations for advancing a reasoning process, roles of the stages of proof construction, and roles of ignition phrases.

This paper ends with two hypotheses. One is that the model of the structure of proof construction can help to analyze students' difficulties with proof construction in a clear, organized, and systematic way. Students' proofs can be examined in terms of the three aspects (*reasoning activity*, *background knowledge*, and *mental attitudes*). Students' difficulties can be defined by the aspect of *the reasoning activity*. The sources of their difficulties can be explained in terms of the other two aspects (*background knowledge* and *mental attitudes*). The other hypothesis is that the knowledge of the structure of proof construction itself can help students overcome their difficulties with proof construction. The model can serve as metacognitive and methodological knowledge to enable students to practice logical deduction in proof construction.

The limitation of this study was that the number of proofs that were examined was limited in constructing the model and algorithm. There is still room for adjusting, modifying, correcting, and improving the model, especially the types of the proofs, the features of variables and the roles of ignition phrases, and the algorithm for advancing a reasoning process. Many more proofs from various mathematical subjects will need to be examined to refine the model of the structure of proof construction.

References

Ayalon, M., & Even, R. (2008). Deductive reasoning: in the eye of the beholder. *Educational Studies in Mathematics*, *69*(*3*), 235-247.

Ball, D., Hoyles, C., Jahnke, H., & Movshovitz-Hadar, N. (2002). The teaching of proof. In L. I. Tatsien (Ed.), *Proceedings of the International Congress of Mathematics*, *3*, 907-920. Beijing: Higher Education Press.

CadwalladerOlsker, T., Miller, D., & Hartmann, K. (2013). Adapting model analysis for the study of proof scheme. *Proceedings of the Sixteenth Annual Conference on Research in Undergraduate Mathematics Education*.

Dreyfus, T. (2012). Advanced mathematical thinking. Mathematics and cognition: A research synthesis by the International Group for the Psychology of mathematics education, 6, 113-134.

Furinghetti, F., & Morselli, F. (2009). Every unsuccessful problem solver is unsuccessful in his or her own way: Affective and Cognitive Factors in Proving. *Mathematics Education 70*, 71-90.

Harel, G., & Sowder, L. (1998) Students' proof schemes: Results from an exploratory study. In A. H. Schoenfeld, J. Kaput & E. Dubinsky (Eds), *Research in College Mathematics Education III* (pp. 234-283).

Harel, G. & Sowder, L. (2007). Towards a comprehensive perspective on proof. In F. Lester (Ed.) *Second handbook of research on mathematical teaching and learning*. NCTM: Washington, DC.

Knuth, E. (2002). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education, 33(*5), 379-405.

Langley, P., & Trivedi, N. (2013). Elaborations on a theory of human problem solving. Poster Collection: *The second annual conference on advances in cognitive systems*. Baltimore, MD.

Newell, A., & Simon, H. (1972). Human problem solving. Englewood. Cliffs, NJ: Prentice-Hall. Schoenfeld, A. (2010). How we think: A theory of goal-oriented decision making and its educational applications. New York: Routledge.

Selden, A., & Selden, J. (2007). Teaching proving by coordinating aspects of proofs with students' abilities. *Technical report*, 2007-2. Tennessee Technological University.

Van Someren, M., Barnard, Y., & Sandberg, J. (1994). The think aloud method: A practical guide to modeling cognitive processes. London: Academic Press.

Wever, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics*, *56*, 209-234.

Analysis of Students' Difficulties with Starting a Proof

Tetsuya Yamamoto
University of Oklahoma

This paper examines students' difficulties with starting a proof. The target population was those students who were enrolled in undergraduate Algebra, Analysis, and Topology in a large research university. 81 proofs, which were collected from students' mid-term and final exams and in-class problem solving sessions, were analyzed. The results showed that 39 proofs (about 48%) were not successful because the early stages of those proofs had defects. This paper investigates the sources of students' difficulties with starting a proof and provides pedagogical suggestions to help them overcome their difficulties.

Keywords: Difficulties with Starting a Proof, Opening Stage of Proof Construction

Introduction

Researchers have agreed that proof is an essential and key component in mathematics at all grades (Baylis, 1983; Wu, 1996; Hanna, 2000; Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002). Researchers have also shown that proof is challenging to students at all levels (Paola & Inglis, 2011; Pfeiffer, 2009; Stylianides, Stylianides, and Phillippou, 2007; Harel and Sowder, 2007). Although students' difficulties with proof construction have been well-documented, there seem to be few studies that spotlighted students' difficulties with starting a proof. As part of the findings from the study for my dissertation, this paper attempts to answer the following two questions: What are possible sources of students' difficulties with starting a proof? What is an effective method to help students start a proof more successfully?

Literature Review

Different researchers examined students' difficulties with proof construction from different angles: mathematical language (Finlow-Bates, 1994; Selden and Selden, 1995; Thurston, 1994; Dreyfus, 1999); students' understanding and usage of definitions (Tall, 1991; Vinner 1991; Frid, 1994; Moore, 1994; Edward & Wards, 2004; Zaslavsky & Shir, 2005; Knapp, 2006; Alcock, 2007; Selden & Selden, 2007; Paramerswaran, 2010); logic (Weber, 2002; Stylianides & Stylianides, 2007; Selden & Selden, 2009; Savic, 2011); informal representations (Alcock, 2004; Alcock & Weber, 2010; Lew, Mejia-Ramos, & Weber, 2013); proving strategies (Weber, 2001); and proof schemes (Harel & Sowder, 1998; Racio & Godino, 2001; Housman & Porter, 2003; Weber & Alcock, 2004; Zaslavsky & Shir, 2005). However, it seems there is not much research that focused on students' difficulties with starting a proof.

Moore (1994) examined students' cognitive difficulties with proof construction and provided seven major sources of students' difficulties. He pointed out that students' inability to begin a proof was one of the major sources of their difficulties. There seems to be little research that explored exactly what difficulties students had with starting a proof and specifically what factors hindered students from starting a proof successfully.

Selden and Selden (2012) provided a *proof framework* as an instructional method to help students with their early-stage of proof construction. They suggested that students should first write the hypotheses at the beginning of their proofs, leave a space for the body, write the conclusion at the end, and fill the blank space through unpacking the conclusion of the given statement. They also indicated that the *proof framework* worked well for some proofs but not for all. There still seems to be room for exploring an effective instructional method for helping students with proof construction in their early-stage proof construction.

This paper attempts to fill those above gaps. In order to explore possible sources of students' difficulties with starting a proof, a framework was created, with which to analyze students' difficulties in a clear and organized way.

Framework

In order to create a framework for analyzing students' proofs, a model of the structure of proof construction (Tables 1) was created. I created the model so that it could encompass the aspects, factors, patterns, and features involved in proof construction across mathematical subjects. In particular, I focused on the operations used to generate one statement from the next statement, which produced the structure of the reasoning activity (Table 2). The model was tested and refined through solving more than 42 theorems and propositions. The proofs examined to create the model ranged over several subjects including undergraduate Algebra, Analysis, and Topology. For inter-rater reliability, I had 6 mathematics professors at a large research university review it. Using my proofs and their proofs, I had them confirm that the model can be applicable to those proofs and earned an agreement from them.

Two stages were set for proof construction in the model: *the opening stage* and *the body construction stage*. I defined the *opening stage*, which I focus on in this paper, to be a preparation stage at which students (1) choose a major proving strategy (a direct proof, a proof by contradiction, a proof by contrapositive, a proof by mathematical induction, or a proof by a counterexample), (2) make the goal of the proof clearer, and (3) prepare a starting variable.

First, I discuss the factors involved in the *opening stage*: *mathematical language*, *variables*, and *ignition phrases*. Then, I discuss the types of proofs and the features of the *opening stage*.

Mathematical language

I made a distinction between mathematical language and *mathematical language*. I defined *mathematical language* to be mathematical language that is fine-grained enough to help students further advance a reasoning process and convince others without leaving any ambiguity. For example, "$f : X \to Y$ is continuous in topological spaces *X* and *Y*" is mathematical language. The *mathematical language* for this statement is "For every open set U in *Y*, $f^{-1}(U)$ is open in *X*." The definition of a mathematical concept is the most paradigmatic example of *mathematical language*. *Mathematical language* is a key factor for constructing a proof based on logical deduction. The significance of the use of *mathematical language* is not limited to proof construction. *Mathematical language* plays a crucial role in solving a regular mathematical problem. For example, if a student is given the condition in a Calculus problem, which says "the function $f : X \to Y$ has a horizontal tangent line at $x = a$," the student may need to translate the statement into "$f'(a) = 0$" in solving the problem. The former statement "the function $f : X \to Y$ has a horizontal tangent line at $x = a$,"is mathematical language while the latter "$f'(a) = 0$" is *mathematical language*.

Types of variables

A variable is a principal and key element of *mathematical language*. Students can convey their mathematical thoughts rigorously by way of variables. The variables used in a proof can be classified into two types: a given variable and a hidden variable. A given variable is a variable that appears or is explicitly written in a given problem while a hidden variable is not. However, a hidden variable sometimes need to be derived, explicitly expressed, and used for advancing a reasoning process. I call that type of hidden variable a "*hidden variable*." A hidden variable can be found in some special phrases in *mathematical language*, which I call ignition phrases. I call those expressions "ignition phrases." There are two types of ignition phrases: a primary ignition phrase and a second primary ignition phrase. The former is "for every …," "for any …," or "for all …." The phrase "If …" can be a primary ignition phrase of this type if it is rephrased with "for every …," "for any …," or "for all …." The latter is "for some …" The phrase "There exists …" can be a second primary ignition phrase if it is rephrased with "for some …" According to the types of ignition phrases, variables can be classified into four sub-types: *controlling variables*, *trivial variables*, *conditional variables*, and *non-conditional variables*. (The naming and classification of variables are tentative.)

Controlling and *trivial* variables are derived from primary ignition phrases. A *controlling variable* can have the power to confine and change another variable or a given statement when it is set while a *trivial variables* is not. Both *conditional* and *non-conditional variables* are derived from second primary ignition phrases. A *conditioned variable* is confined by a *controlling variable* while a *non-conditioned variable* is not. For example, in the definition of compactness, which is "For every open cover *W* of *X*, there exists a finite open subcover." The variable "an open cover W " is a *controlling variable* because it is derived from the primary ignition phrase "For every open cover *W* of *X*" and confines "a finite open subcover " when it is set. The variable "a finite open subcover" is a *conditioned variable* because it is derived from the second primary ignition phrase "there exists a $\delta > 0$" and is subject to the *controlling variable* "an open cover *W*." In the definition of a bounded sequence, which is "For every $n \in Z^+$, $|a_n| \le M$ for some $M \in R$," "$n \in Z^+$" is a *trivial variable* because it is derived from "For every $n \in Z^+$" but does not affect another variable by its change. "$M \in R$" is a *non-conditioned variable* because it is derived from the primary ignition phrase "for some $M \in R$" but is not controlled by a *controlling variable*.

Ignition phrases

I made a distinction between ignition phrases and *ignition phrases*. *Ignition phrases* are the ignition phrases that enable students to derive and set a starting variable. A starting variable is the one with which students start a proof. The primary ignition phrase that provides a *controlling variable* in the *mathematical language* for the conclusion can be an ignition phrase while the one that provides a *trivial variable* may not. The second primary ignition phrase in the *mathematical language* for the conclusion cannot be an *ignition phrase*. In other words, students should not derive and introduce the hidden variable from the second primary ignition phrase as a starting variable if it comes from the conclusion. If the *mathematical language* for the conclusion does not provide a starting variable, a starting variable should be found in the *mathematical language* for a hypothesis. Both primary and second primary ignition phrases in the *mathematical language* for a hypothesis can be *ignition* phrases.

The following are some examples. Suppose that students are given a problem, which asks to prove that "a function $f : X \to Y$ is continuous at $x = x_0$." They need to set a starting variable to start a proof. They should derive a starting variable from an *ignition phrase*. A starting variable should be first found in a primary ignition phrase in the *mathematical language* for the conclusion. In this case, the *mathematical language* for the conclusion is "For every

$\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$." The phrase "For every $\varepsilon > 0$" is a primary ignition phrase, which provides a controlling variable. Therefore, "For every $\varepsilon > 0$" is an *ignition phrase*. Students can start a proof with "Let $\varepsilon > 0$." The phrase "there exists a $\delta > 0$" is a second primary ignition phrase, but it is in the conclusion of the given statement. Therefore, the phrase "there exists a $\delta > 0$" is not an *ignition phrase*. For another example, suppose that students are given a problem that asks them to prove that a sequence $\{a_n\}$ is bounded. The *mathematical language* for the statement is "For every $n \in Z^+$, $|a_n| \le M$ for some $M \in R$." There are two ignition phrases, which are "For every $n \in Z^+$" and "for some $M \in R$." However, neither of them is an *ignition phrase*. The phrase "For every $n \in Z^+$" is not an *ignition phrase* because $n \in Z^+$ is not a *controlling variable* but a *trivial variable*. The phrase "for some $M \in R$" is not an *ignition phrase* either because an ignition phrase "for some ...." in the *mathematical language* for the conclusion is not considered to be an *ignition phrase*.

Types of proofs

Proofs can be classified into three types by ways to derive a starting variable. Students can tell the type of a proof by examining the conclusion of the given statement. In the first type of proof (Type I), students derive and set a starting variable from the *mathematical language* for the conclusion of the given statement. This type of proof contains an *ignition phrase* in the *mathematical language* for the conclusion. In the second type of proof (Type II), students derive and set a starting variable from the *mathematical language* for a hypothesis. Students can tell that a given proof belongs to this type when they first note the conclusion, translate it into *mathematical language*, and find out that the conclusion does not include an *ignition phrase*. The proofs that ask students to construct an object may belong to this type of proof. In both Type I and Type II, a proof can have more than one starting variable. In the third type of proof (Type III), students do not have to derive and set a starting variable because a starting variable is given in the problem. A trivial example is a proof of a trigonometric identity. A proof by mathematical induction also belongs to this type. The proofs that ask students to prove A = B is typical in Type III. There are also proofs in Type I that ask students to prove A = B. In any case, in order to prove A = B, students can work on either A or B until they can change it into B or A, work on both sides until they get A = C = B, or show that $A \subset B$ and $B \subset A$.

Example 1 (Type I). "Prove that if $f'(x) > 0$ on (a,b), $f(x)$ is strictly increasing on (a,b)." The conclusion of the given statement is "$f(x)$ is strictly increasing on (a,b)." The *mathematical language* for the conclusion is " For every $x_1, x_2 \in (a,b)$ with $x_1 < x_2$, $f(x_1) < f(x_2)$." The *ignition phrase* is "For every $x_1, x_2 \in (a,b)$ with $x_1 < x_2$." Therefore, students may start their proofs with "Let $x_1, x_2 \in (a,b)$ with $x_1 < x_2$."

Example 2 (Type II). "Suppose that X, Y are topological spaces, Y is compact, $x_0 \in X$, and N is an open set containing $\{x_0\} \times Y$ in the product space $X \times Y$. Prove that there exists an open neighborhood $W \subset X$ of x_0 such that $W \times Y \subset N$." The conclusion of the given statement is "there exists an open neighborhood $W \subset X$ of x_0 such that $W \times Y \subset N$." The conclusion contains an ignition phrase "there exists," which is a second primary ignition phrase. However, since the primary ignition phrase is in the conclusion, it cannot be an *ignition phrase*. Therefore, students should note a hypothesis and translate it into *mathematical language* to derive a starting variable. In this case, students can note the hypothesis "*N* is an open set containing $\{x_0\} \times Y$ in the product space $X \times Y$" and translate it into *mathematical language*. The *mathematical language* for the hypothesis can be that "for each $(x_0, y) \in \{x_0\} \times Y$, there exists a basis open set $U_y \times V_y \subset N_y$ containing (x_0, y), in which U_y and V_y are open neighborhoods of *y* in *X* and in *Y* respectively. The *ignition phrase* for the hypothesis is "for each $(x_0, y) \in \{x_0\} \times Y$." So, students may start their proofs with "Let $(x_0, y) \in \{x_0\} \times Y$."

Example 3 (Type III). "Suppose $\lim_{n\to\infty} s_n = L$ and $\lim_{n\to\infty} t_n = M$. Prove $\lim_{n\to\infty}(s_n + t_n) = L + M$." The conclusion of the given statement is "$\lim_{n\to\infty}(s_n + t_n) = L + M$." At this point, students can tell this proof belongs to Type III. Students can work on the left hand side of the equation "$\lim_{n\to\infty}(s_n + t_n)$" until they can change it into $L + M$. They can start their proofs with "Consider $\lim_{n\to\infty}(s_n + t_n)$."

Features of the opening stage

The most important operation to be taken in the *opening stage* is to translate the conclusion of the given statement into *mathematical language*. By translating the conclusion into *mathematical language*, students can (1) make the goal of the proof clearer, (2) make the processing distance between the beginning and the end of the proof shorter, (3) tell the type of the proof by finding an *ignition phrase*, and (4) prepare a starting variable to be set to develop a proof.

Table 1: Components of the Structure of Proof Construction

Reasoning Activity (O: Opening Stage, B: Body Construction Stage)			
O	OPS		Decide a major proving strategy (a direct proof, an indirect proof, etc).
	OTC		Set a goal: translate the conclusion of the given statement into *mathematical language*.
	ODT		Prepare a starting variable: Decide the type of the proof by noting an *ignition phrase*.
B	R	R1	Rephrase an object by applying a definition, a property, or a theorem.
		R2	Rephrasing an object through interpretation.
		R3	Rephrasing an object through algebraic manipulation.
	CO	CO(A,B)R	Combine objects to create a new object.
	C	C1	Set a variable.
		C2	Recall concepts, properties, theorems, propositions, problem-solving techniques.
		C3	Set some cases.
		C4	Make a claim or create a new object.
		C5	Consider an object.
	CE	Ch	Review, test, evaluate, adjust, modify, or correct what has been done.
		Ex	Search a clue, explore a solution by making a graph, a diagram, and an example, by intuiting, or by doing trial and error.
Background Knowledge			
KDF			Know definitions and properties of concepts.
KTH			Know theorems, propositions, and lemmas.
KNT			Know notations.
KTE			Know proving techniques or problem solving techniques.
Mental Attitudes			
MT			Have tenacity and persistence; not give up proving or solving problems easily.
MF			Have flexibility to give up an idea that does not work and to try a new or different method.
MC			Have carefulness, precision, or alertness.
Affect and Beliefs			
AF			Have self- confidence, have a mentality to not let negative emotions affect proving or solving performances.
BL			Have a right belief on proofs, logic, and mathematics.

Table 2: Structure of the Reasoning Activity

Reasoning Activity in Proof Construction	ACTIONS	MAIN									SUPPORTING	
	Roles of the Actions	Transforming objects				Igniting a process					Supporting the main actions	
STAGES	Operations / Steps	R Rephrasing Objects			CO Combining Objects	C Creating a Cue					Ch Checking (review, evaluate, test, modify, correct)	Ex Exploring (search, do trial and error illustrate, experi-ment, intuit)
		R 1	R 2	R 3	CO(A,B)R	C 1	C 2	C 3	C 4	C 5		
Opening Stage	Z: Choose a major proving strategy.	Decide which strategy to use (a direct proof, a proof by contradiction, a proof by contrapositive, a proof by finding a counter example, or a proof by mathematical induction. If necessary, rephrase the problem according to the chosen method.										
	X: Make sure of the goal.	Given. (Find the conclusion of the given statement.)										
	Y: Make the goal clearer with *mathematical language*.	R1, R2, R3 (Translate the conclusion into *mathematical language*.)										
	P: Make sure of the hypotheses.	Given (Find all the hypotheses of the given statement and translate them into *mathematical language* if necessary.)										
	S: Prepare a starting variable.											
Body Construction Stage	Step 1: State the hypothesis.	Given.										
	Step 2: Set a starting variable.					C1						
	…											
	…											
	Conclusion											

Footnotes

Actions	
Main Actions	The operations applied to a step to generate the next step, whose outcome must be explicitly expressed to convince others
Supporting Actions	The operations to produce side work, whose outcome does not necessarily have to appear in the proof to convince others

Rephrasing an object	
R1	Rephrase an object by translating a concept, a theorem, or a property of concept into mathematical language mainly through applying its definition.
R2	Rephrase an object through formal or informal interpretation.
R3	Rephrase an object through algebraic manipulation or calculation, including solving an equation.

Combining objects	
CO(S, T)R	Connect and combine different pieces of objects (S and T) to create a new object. This action is always followedd by an operation of rephrasing.

Creating a cue	
C1	Set a variable.
C2	Recall prior knowledge, including a theorem, a proposition, a property of concept, or a mathematical law.
C3	Set some cases.
C4	Make a claim or create a new object.
C5	Consider an object.

Method

This study took place in a large research university in the Midwest in the Spring of 2013. The target population was those students who were enrolled in undergraduate Algebra, Analysis, and Topology. There were two types of data sources: (1) students' in-class midterm and final examinations and (2) students' in-class problem solving sessions. Students' examinations were conducted by their instructors. Students' problem solving sessions were conducted by the researcher. In total, 81 students' proofs over twelve problems were collected. Students' proofs were analyzed with the analysis framework (Table 1). In particular, their difficulties with *the opening stage* were investigated in terms of the following two factors: students' ability to successfully translate the conclusion of the given statement into *mathematical language*; students' ability to set a correct starting variable. For each problem, an analysis table was created, which showed a model proof, in order to detect where students had difficulties in the proof. The table also showed what type of operation was used to obtain a statement from the previous statement for each step in the proof (Tables 4, 5, and 6). Each mistake or impasse the student made was analyzed based on the analysis framework (Table 1) in terms of the three aspects: *reasoning activity*, *background knowledge*, and *mental attitudes* (Tables 1 and 2).

Results

Out of the 81 proofs that were analyzed, 59 proofs (about 73%) were incomplete or unsuccessful. Out of those incomplete or unsuccessful 59 proofs, there were 39 proofs (about 66%) that had defects in their opening stages. Overall, in about 48% of 81 proofs, students had defects in *the opening stage*. This section presents three examples, showing how students' mismanagement in *the opening stage* occurred and affected their proofs.

Example 1: Frank and Anthony (Algebra)

Frank's proof is a representative example showing that students let their proofs go off track because they do not pay proper attention to the goal of a proof. Both Frank's and Anthony's proofs were also representative examples showing that students made their proofs unsuccessful because they set a wrong starting variable. Their difficulties occurred because they did not note the conclusion of the given statement in *the opening stage*. Moreover, the root cause of their difficulties was that they first noted a hypothesis of the given statement and set a starting variable from it.

Table 4: Analysis of Frank's and Anthony's proofs

	Given P, prove X.	Code	Frank	Anthony
X	Show G is abelian.	Conclusion	Given	Given
Y	Show $ab = ba$ for any $a, b \in G$.	R1	N	N
P	$G/Z(G)$ is cyclic, where $Z(G)$ is the center of G.	Hypothesis	Given	Given
1	Let $a, b \in G$.	C1	N	N
2	Recall $a, b \in G$ are in some cosets.	C2	N	N
3	Then, $a \in x^m Z$ and $b \in x^n Z$ for some $x \in G$ and $m, n \in N$.	CO(P, 2)R1	N	N
4	Let $a = x^m z_1$ and $b = x^n z_2$ for some $z_1, z_2 \in Z$.	R1	N	N
5	Recall the property of elements of a center of G.	R2	N	N
6	Consider ab.	C5	N	N
7	Then, $ab = x^m z_1 x^n z_2 = x^m x^n z_1 z_2 = x^n x^m z_2 z_1 = x^n z_2 x^m z_1 = ba$.	CO(5,6)R3	N	N

"N" in the right columns represents "not successful."

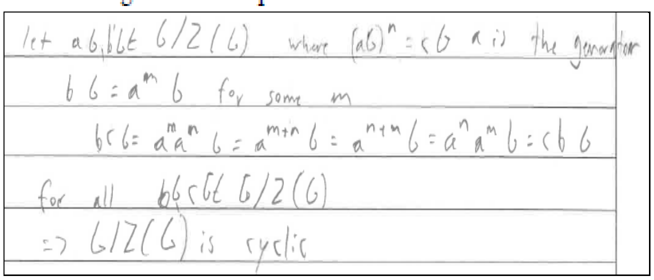


Figure 1: Frank's proof

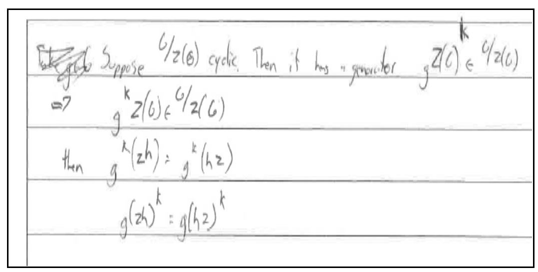


Figure 2: Anthony's proof

Frank first noted the hypothesis of the given statement "$G/Z(G)$ *is cyclic*" and derived starting variables from it instead of from the conclusion of the given statement. Having difficulties in dealing with the concepts and notations of cosets as well as $G/Z(G)$ being cyclic, his proof went off track and ended up with "$G/Z(G)$ *is cyclic*," which was given as a hypothesis at the beginning. He did not seem to make the goal of the proof "G is abelian" clear to himself. This might have caused him two problems. One was that he did not realize that his argument was going astray and ended up with a conclusion that he was not asked to prove. The other was that he was unable to set the correct starting variables "$a, b \in G$." Similarly, Anthony started to work on a given hypothesis, set starting variables from it, and tried to reach the goal by manipulating the variables, which made his proving argument unsuccessful.

Example 2: Ryan (Topology)

Ryan's proof is a representative example showing that students' failure to derive a starting variable from an *ignition phrase* in the *mathematical language* for the conclusion spoils their whole proofs.

Table 5: Analysis of Ryan's proof (Topology)

	Given P, prove X.	Code	Ryan
X	Show that $K = \{x_n : n = 0,1,2,\dots\}$ is compact.	Conclusion	Given
Y	Show that for any open cover of K, K has a finite open subcover.	R1	N
P	$\{x_n : n \in Z^+\}$ is a sequence in X converging to x_0, where X is Hausdorff.	Hypothesis	Given
1	Let $U = \{U_\alpha\}$ be an open cover of X.	C1	N
2	Construct an open cover of K by letting $V = \{V_\alpha = U_\alpha \cap K\}$.	C1	N
3	Note that $\exists U_{\alpha_0} \in U$ such that $x_0 \in U_{\alpha_0}$.	R2	S
4	Since x_n converges to x_0, $\exists N \in Z^+$ such that for all $n \geq N$, $x_n \in U_{\alpha_0}$.	CO(P,3)R1	S
5	Let $V_{\alpha_0} = U_{\alpha_0} \cap K$, where $V_{\alpha_0} \in V$.	C1	N
6	For each x_i with $i < N$, find an open set $V_{x_i} \in V$ such that $x_i \in V_{x_i}$.	C1	N
7	Note that $\{V_{x_1}, V_{x_2}, \dots V_{x_{N-1}}, V_{\alpha_0}\}$ is a desired finite open subcover of K.	CO(5,6)R2	N

"N" and "S" in the right column represent "not successful" and "successful."

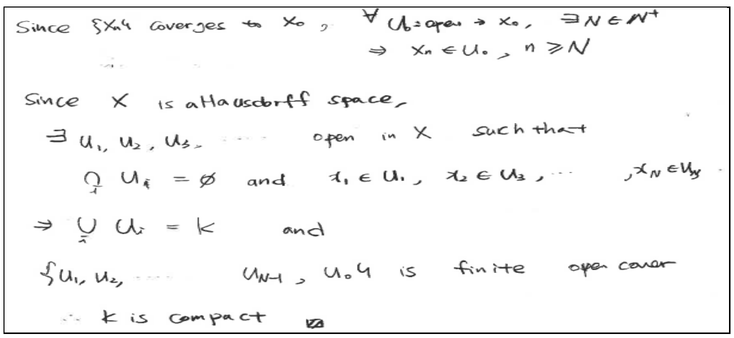


Figure 3: Ryan's proof

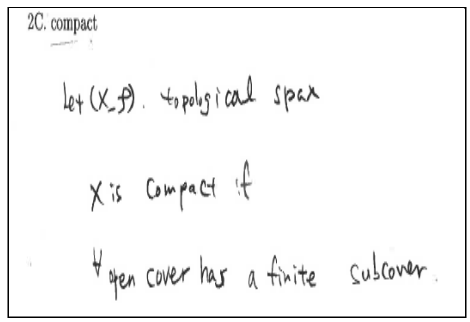


Figure 4: Ryan's statement

Ryan successfully made the goal of the proof clear to himself. As he indicated by trying to construct a finite open subcover of *K* (Figure 3), he noted the conclusion of the given statement "*K* is compact." He also knew the definition of compactness as he made a correct statement for the definition of compactness (Figure 4) in a previous problem. Moreover, he was knowledgeable enough to correctly translate the given hypothesis "$\{x_n\}$ converges to x_0" into *mathematical language* (Step 4), which was one of the key steps for the proof. However, he was still unable to make his proof completely successful because he failed to set a correct starting variable. In particular, he failed to note the *ignition phrase* "For any open cover of *K*" and to construct an open cover of *K*.

Example 3: Cade (Algebra)

Cade's proof is a representative example showing that students' failure to translate the whole sentence of the conclusion into *mathematical language* makes their proofs unsuccessful.

Table 6: Analysis of Cade's proof

	Given P, prove X.	Code	Cade
X	Show G is cyclic.	Conclusion	Given
Y	Show $G = \langle g \rangle$ for some $g \in G$ with $g \neq 1$.	R1	N
P	The order of G is a prime number.	Hypothesis	Given
1	Let $g \in G$ with $g \neq 1$.	C1	N
2	Consider $\langle g \rangle$.	C5	N
3	Note $\langle g \rangle$ is a subgroup of G.	C2	N
4	Recall the Lagrange's THM and apply it to $\langle g \rangle$.	C2	N
5	Then, by the Lagrange's THM, $\lvert\langle g \rangle\rvert = 1, p$.	CO(3,4,P)R1	N
6	Since $\lvert\langle g \rangle\rvert \neq 1$, $\lvert\langle g \rangle\rvert = p$.	CO(1,5)R2	N
7	Since $\lvert G\rvert = p$, $G = \langle g \rangle$.	CO(6, P)R2	N

"N" in the right column represents "not successful."

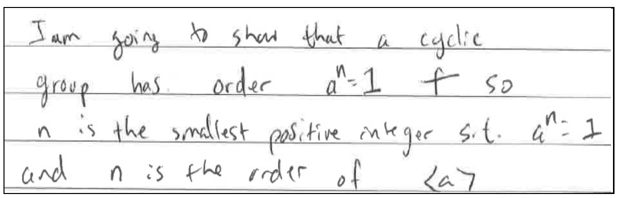
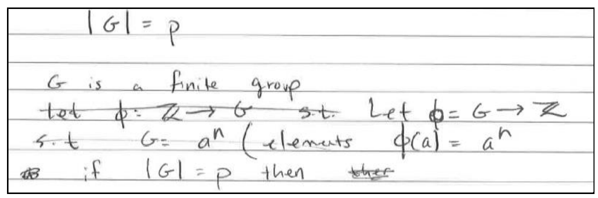

I am going to show that a cyclic group has order $a^n = 1$ f so n is the smallest positive integer s.t. $a^n = 1$ and n is the order of $\langle a \rangle$

Figure 5: Cade's strategy


$|G| = p$

G is a finite group

~~let $\phi: \mathbb{Z} \to G$ s.t.~~ Let $\phi = G \to \mathbb{Z}$

s.t $G = a^n$ (elements $\phi(a) = a^n$

if $|G| = p$ then ~~that~~

Figure 6: Cade's proof

As Figure 5 shows, Cade successfully noted the conclusion of the given statement "*G* is cyclic" as his goal for the proof. However, he was unable to translate it into "$G = \langle g \rangle$ for some $g \in G$ with $g \neq 1$" correctly, which seemed to result in his failure to develop his argument successfully (Figure 6). In particular, he focused on only part of the conclusion "cyclic" and did not translate the whole sentence "*G* is cyclic," which caused him to fail to derive a starting variable. Cade also depended on the hypothesis "$|G| = p$" in deriving a starting variable.

Discussion

As the above examples show, students' difficulties with *the opening stage* may occur in two ways: (1) Students failed to make the goal of the proof clear and (2) Students set a wrong starting variable. The sources of these difficulties can be attributed to their failure to note the conclusion of the given statement and to translate it into *mathematical language* at the beginning of a proof. The results also detected the causes that hindered students from noting the conclusion of the given statement: (a) students tended to start to work on a hypothesis and to set a starting variable from the hypothesis (Examples 1, 2, and 3); (b) students failed to pay attention to an *ignition phrase* in the *mathematical language* for the conclusion (Example 2); and (c) students focused on only part of the conclusion and failed to translate the whole sentence into *mathematical language* (Example 3). In particular, students' failure to set a correct starting variable was a crucial cause of their unsuccessful proofs. Students seemed to be tempted to start to work on a hypothesis and to derive a starting variable from it by the expression of a hypothesis, which often started with "Suppose …" Based on the model of the structure of the *opening stage* and the findings from the analysis of students' proofs, this section ends with presenting algorithm for dealing with the opening stage as pedagogical suggestions.

Algorithm for the Opening Stage
A: *Opening Stage*
A0: Read the problem.
- If necessary, translate the given problem into *mathematical language*. (A0.1)

A1: Decide a major proving strategy
- Decide which proving strategy to use, a direct proof, by contrapositive, by contradiction, by counter example, or by mathematical induction (A1.1)
- If using a proof by contrapositive or contradiction, rephrase the whole problem. (A1.2)
- If finding out that the proof belongs to Type III, skip to B0. (A1.3)

A2: Note the conclusion.
- Do not be tempted to note a hypothesis.

A3: Translate the conclusion into *mathematical language*.
- Rephrase the whole sentence of the conclusion through R1 (Table 2). (A3.1)
- Rephrase the conclusion more than once, if necessary. (A3.2)

A4: Find an *ignition phrase* in the *mathematical language* for the conclusion.
- A primary ignition phrase providing a *controlling variable* is an *ignition phrase*. (A4.1)
- A primary ignition phrase providing a trivial variable may not be an *ignition phrase*. (A4.2)
- A second primary ignition phrase from the conclusion cannot be an *ignition phrase*. (A4.3)
- If there is no ignition phrase in the conclusion, find one in a hypothesis. (A4.4)

A5: Decide the type of the proof.
- If there is a primary ignition phrase providing a controlling variable in A4, the proof belongs to Type I.
- If there is no *ignition phrase* in A4, the proof belongs to Type II or Type III.
- If there is no *ignition phrase* and the problem asks to prove A = B, it belongs to Type III.

A6: Derive a starting variable.
- For Type I, derive a starting variable from the *ignition phrase*. (A6.1)
- For Type II, note a hypothesis, translate it into *mathematical language*, and find an *ignition phrase*. (A6.2)
- For Type III, start the *body construction stage* by trying one of the following: Work on either A or B to change it into B or A, work on both to obtain A = C = B, or show $A \subset B$ and $B \subset A$. This also works for Type I (b). (A6.3)

B: *Body Construction Stage*
B0: State the hypothesis (hypotheses).
B1: Set a starting variable based on A6.
B2: Make sure of the goal of the proof obtained in A3.
B3: Start to apply the four types of operations for advancing a reasoning process in Table 2.

Conclusion

This paper started with creating a model of the structure of the *opening stage*, which encompassed the factors, patterns, and features involved in the *opening stage*. The *opening stage* has two major roles in proof construction: making the goal of the proof clear and setting a starting variable. Translation of the conclusion of a given statement into *mathematical language* is the key operation in the *opening stage*. By translating the conclusion into *mathematical language*, students can make the goal of the proof clear, tell the type of the proof, and set a starting variable. Then, this paper provided several sources of students' difficulties in staring a proof: students' failure to note the conclusion of the given statement; their inability to translate the conclusion into *mathematical language* due to lack of their knowledge of definitions; and their tendency to note a hypothesis rather than the conclusion in deriving a starting variable. Finally, this paper presented algorithm for dealing with the *opening stage* of each type of proof. This paper ends with hypothesizing that the knowledge of the structure of the *opening stage* serves as metacognitive and methodological knowledge to help students start a proof more successfully.

The number of the proofs that were examined to create the model of the structure of the *opening stage* was limited. There is still room for adjusting, modifying, correcting, refining and improving the model, especially the roles and features of variables and ignition phrases, the types of proofs, and the algorithm for dealing with the *opening stage*. More proofs from various mathematical subjects must be examined to improve the model.

References

Alcock, L. (2004). Uses of example objects in proving. In PME 28, Bergen, Norway.

Alcock, L. (2007). How do students think about proof? A DVD resource for mathematicians. *MSOR Connections*, *7*(2), 3-6.

Alcock, L., & Weber, K. (2010). Undergraduates' example use in proof construction: Purposes and effectiveness. Investigations in Mathematics Learning, 3,1-22.

Ball, D., Hoyles, C., Jahnke, H., & Movshovitz-Hadar, N. (2002). The teaching of proof. In L.I Tatsien (Ed.), *Proceedings of the International Congress of Mathematics*, *3*, 907-920. Beijing: Higher Education Press.

Baylis, J. (1983). Proof – The essence of mathematics, part 1. *International Journal of Mathematics Education and Science Technology 14*, 409-414.

Dreyfus, T. (1999). Why Johnny can't prove. *Educational Studies in Mathematics 38*, 86-109.

Edwards, B., & Ward, M.B. (2004). Surprises from mathematics education research: Student misuse of mathematical definition. *The American Mathematics Monthly*, *111*, 411-424.

Finlow-Bates, K. (1994). First year mathematics students' notions of the role of informal proof and examples. *Proceedings of the 18th Conference of the International Group for the Psychology of Mathematics Education*, 344-350.

Frid, S. (1994). Three approaches to undergraduate calculus instruction: Their nature and potential impact on students' language use and sources of conviction. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Research in collegiate mathematics education*, *1*, 69-100. Washington, DC: Conference Board of the Mathematical Sciences.

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics*, Special issue on "Proof in Dynamic Geometry Environments," 44(1-2), 5-23.

Harel, G., & Sowder, L. (1998) Students' proof schemes: Results from an exploratory study. In A. H. Schoenfeld, J. Kaput & E. Dubinsky (Eds), *Research in College Mathematics Education III* (pp. 234-283).

Harel, G. & Sowder, L. (2007). Towards a comprehensive perspective on proof. In F. Lester (Ed.) *Second handbook of research on mathematical teaching and learning*. NCTM: Washington, DC.

Housman, d., & Porter, M. (2003). Proof schemes and learning strategies of above-average mathematics students. *Educational Studies in Mathematics*, *53*, 139-158.

Knapp, J. (2006) A framework to examine definition use in proof. *Advanced mathematical thinking*, *2*, 15-22.

Lew, K., Mejia-Ramos, J., & Weber, K. (2013). Not all informal representations are created equal. *Proceedings of the 16th Annual Conference on Research in Undergraduate Mathematics Education*, *2*, 570-574

McKee, K., Savic, M., Selden, J., & Selden, A. (2010). Making actions in the proving process explicit, visible, and "reflectable." In *Proceedings of the 13th Annual Conference on Research in Undergraduate Mathematics Education*, Raleigh, NC.

Moore, R. (1994). Making the transition to formal proof. *Educational Studies in Mathematics 27*, 249-266.

Paola, I., & Inglis, M. (2011). Undergraduate students' use of deductive arguments to solve

"prove that …" tasks. *Proceedings of the 7th Congress of the European Society for Research in Mathematics Education*.
Paramerswaran, R. (2010). Expert mathematicians' approach to understanding definitions. *The Mathematics educator*, *20*, 43-51.
Recio, M. & Godino, D. (2001). Institutional and personal meanings of proof. *Educational Studies in Mathematics, 48*(1), 83-99.
Pfeiffer, K. (2009). The role of proof validation in students' mathematical learning. *Proceedings of the British Society for Research into Learning Mathematics*, *29*(3), 79-84.
Savic, M. (2011). Where is the logic in proofs? *Proceedings of the 14th Annual Conference on Research in Undergraduate Mathematics Education 2011*, *2*, 445-456.
Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*, *29*(2), 123-151.
Selden, A., & Selden, J. (2007). Teaching proving by coordinating aspects of proofs with students' abilities. *Technical report*, 2007-2.
Selden, J., & Selden, A. (2009). Teaching proof by coordinating aspects of proofs with students' abilities. In D. A. Stylianou, M. L. Blanton, & E.J. Knuth (Eds.), *Teaching and learning proof across the grades*. New York, NY: Rutledge.
Selden, A., & Selden, J. (2012). A belief affecting students' success in problem solving and proving. Proceedings of the 12th International Congress on Mathematical Education. International Comission on Mathematical Instruction (ICMI).
Stylianides, G., Stylianides, A., & Philippou, G. (2007). Preservice teachers' knowledge of proof by mathematical induction. *Journal of Mathematics Teacher Education*, *10*(3), 145-166.
Tall, D. (1991). The psychology of advanced mathematical thinking. In D. Tall (Eds). *Advanced mathematical thinking*. Dordrecht, The Netherlands: Kluwer.
Thurston, W. (1994). On proof and progress in mathematics. *Bulletin of the American Mathematical Society*, *30* (2), 161-177.
Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D.Tall (Ed.), *Advanced mathematical thinking, 11*,65-81. New York, Boston, Dordrecht, London, Moscow: Kluwer Academic Publishers.
Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, *48*(1), 101-119.
Weber, K. (2002). Instrumental and relational understanding of advanced mathematical concepts and their role in constructing proofs about group isomorphisms. *Proceedings of the 2nd International Conference on the Teaching of Mathematics*.
Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational studies in Mathematics, 56*, 209-234.
Wu, H. (1996). The role of Euclidean geometry in high school. *Journal of Mathematical Behavior*, *15*, 221-237.
Zaslavsky, O.,& Shir, K. (2005). Students' conceptions of a mathematical definition. *Journal for Research in Mathematics Education*, *36*(4), 317-346.

Analysis of Students' Difficulties with Proof Construction

Tetsuya Yamamoto
University of Oklahoma

Proof is a central and essential skill in mathematics. However, it is a challenging task for students at all levels. This paper presents the findings from the analysis of students' difficulties with proof construction, clarifies possible sources of their difficulties in light of the structure of proof construction, and offers an algorithm for proof construction as metacognitive knowledge for helping students with proof construction based on logical deduction.

Keywords: Structure of Proof Construction, Algorithm for Proof Construction

Introduction

Proof is an essential skill in mathematics and a central component in mathematics education: (Cirillo & Herbst, 2012; Kilpatrick, Swafford & Findell, 2001). However, studies have shown students encounter various difficulties with proofs at all levels (CadwalladerOlsker & Miller, 2013; Paola & Inglis, 2011). Proof is challenging not only for students to learn but also for instructors to teach (Hanna & Villers, 2007; Mariotti, 2006). There is still need for development of an effective teaching method to help students with proof construction (Harel & Sowder, 2007; Ball, Hoyles, Jahnke, & Movshoitz-Hadar, 2002). This paper presents part of the findings from my thesis, in which I examined students' cognitive difficulties in light of the structure of proof construction. This paper introduces the model of the structure of proof construction, clarifies students' difficulties with proof construction, and provides a practical method to help them overcome their difficulties.

Literature Review

Students' difficulties with proof construction have been well-documented. Many researchers spotlighted a particular aspect of proof construction: mathematical language (Dreyfus, 1999; Thurston, 1994) students' understanding and usage of definitions (Paramerswaran, 2010; Edwards & Ward, 2008; Knapp, 2006); logic (Stylianides & Stylianides, 2007; Savic, 2011); informal representations (Alcock, 2004; Lew, Mejia-Ramos, & Weber, 2013); and proving strategies (Weber, 2001). Several researchers provided a comprehensive error list and clarified various types of students' difficulties (Selden & Selden, 2003; Moore, 1994). CadwalladerOlsker, Miller, and Hartmann (2013) noted students' incomplete understanding of the components of a proof as a source of students' difficulties with proof construction. Kieran (1998) stressed the significance of establishing a model for describing observed phenomena in both theoretical and empirical research. However, there seems to be little research that examined students' difficulties based on a framework established by modeling the structure of proof construction.

Logical deduction is a key aspect of analytical proof scheme. Several researchers examined students' proof schemes to show their difficulties with practicing analytical proof scheme, while indicating the necessity of fostering students' skills for logical deduction (Stylianou, Chae, & Blanton, 2006; Harel and Sowder, 1998). Ayalon and Even (2008) claimed views and approaches to deductive reasoning should receive more attention. Papaleontiou-Louca (2003) stressed the importance of providing metacognitive knowledge (knowledge of one's processes and cognitive states) by modeling task completion for students' effective learning.

However, there seems to be little research that explored specific and practical instructional method to help students with logical deduction.

This paper attempts to fill the above gaps by the following ways: (1) offering a model of the structure of proof construction, (2) revealing what cognitive difficulties students had and what were possible sources of their difficulties; and (3) providing metacognitive knowledge for helping students with proof construction based on logical deduction.

Theoretical Perspectives

This study started with the creation of a model of the structure of proof construction. The model was built to (1) clarify the aspects, factors, patterns, and features involved in cognitive processes in proof construction, and (2) build a framework for analyzing students' proofs in a clear and organized manner. In creating the model, I used a self-analysis method as a variation of the think-aloud method while solving proof problems. Think-aloud is a valid and effective research method to understand an individual's thought process (Van Someren, Barnard, & Sandberg, 1994). I observed, described, abstracted, and organized my thought processes while proving more than 42 theorems and propositions. I investigated the operations used to generate one statement from the previous statement and categorized all the observed operations (Table 1). For inter-rater reliability, I had 6 mathematics professors review the model. Using my proofs and their proofs, I had them confirm that the model was applicable to those proofs.

The operations for the *reasoning activity* in the model were classified into the following four types: (a) *rephrasing an object* (through applying definitions and properties; interpretation; or algebraic manipulation); (b) *combining objects*; (c) *creating a cue* (setting a variable; applying prior knowledge; setting some cases; making a claim or an object; and considering an object) ; and (d) *checking and exploring*. Newell and Simon (1972) included the following four categories in their theoretical framework for problem solving: (i) representation, interpretation, and manipulation of symbolic structures; (ii) search through a set of available information; (iii) selective search through heuristics; (iv) reduction of the differences between current and desired states. The model seems to cover the categories that Newell and Simon (1972) set in their theoretical framework for problem solving: (a) plays a role of (i), (b) and (c) play a role of (ii), (d) plays a role of (iii), and a combination of (a), (b), and (c) realizes (iv).

The following tables show the structure of the *reasoning activity* (Table 1) and the aspects and factors of proof construction (Table 2). Table 2 serves not only as a framework for analyzing students' proofs but also as a list of the skills and abilities necessary for proof construction.

Table 1: Components of the Structure of Proof Construction

Reasoning Activity (O: Opening Stage, B: Body Construction Stage)			
O	OPS		Decide a major proving strategy (a direct proof, an indirect proof, etc).
	OTC		Set a goal: translate the conclusion of the given statement into *mathematical language*.
	ODT		Prepare a starting variable: Decide the type of the proof by noting an *ignition phrase*.
B	R	R1	Rephrase an object by applying a definition, a property, or a theorem.
		R2	Rephrasing an object through interpretation.
		R3	Rephrasing an object through algebraic manipulation.
	CO	CO(A,B)R	Combine objects to create a new object.
	C	C1	Set a variable.
		C2	Recall concepts, properties, theorems, propositions, problem-solving techniques.
		C3	Set some cases.

		C4	Make a claim or create a new object.
		C5	Consider an object.
	CE	Ch	Review, test, evaluate, adjust, modify, or correct what has been done.
		Ex	Search a clue, explore a solution by making a graph, a diagram, and an example, by intuiting, or by doing trial and error.
Background Knowledge			
KDF			Know definitions and properties of concepts.
KTH			Know theorems, propositions, and lemmas.
KNT			Know notations.
KTE			Know proving techniques or problem solving techniques.
Mental Attitudes			
MT			Have tenacity and persistence; not give up proving or solving problems easily.
MF			Have flexibility to give up an idea that does not work and to try a new or different method.
MC			Have carefulness, precision, or alertness.
Affect and Beliefs			
AF			Have self- confidence, have a mentality to not let negative emotions affect proving or solving performances.
BL			Have a right belief on proofs, logic, and mathematics.

Table 2: Structure of the Reasoning Activity

Reasoning Activity in Proof Construction	ACTIONS ⇨	MAIN									SUPPORTING	
	Roles of the Actions ⇨	Transforming objects				Igniting a process					Supporting the main actions	
STAGES ⇩	Operations ⇨ Steps ⇩	R Rephrasing Objects			CO Combining Objects	C Creating a Cue					Ch Checking (review, evaluate, test, modify, correct)	Ex Exploring (search, do trial and error illustrate, experi-ment, intuit)
		R 1	R 2	R 3	CO(A,B)R	C 1	C 2	C 3	C 4	C 5		
Opening Stage	Z: Choose a major proving strategy.	Decide which strategy to use (a direct proof, a proof by contradiction, a proof by contrapositive, a proof by finding a counter example, or a proof by mathematical induction. If necessary, rephrase the problem according to the chosen method.										
	X: Make sure of the goal.	Given. (Find the conclusion of the given statement.)										
	Y: Make the goal clearer with *mathematical language*.	R1, R2, R3 (Translate the conclusion into *mathematical language*.)										
	P: Make sure of the hypotheses.	Given (Find all the hypotheses of the given statement and translate them into *mathematical language* if necessary.)										
	S: Prepare a starting variable.											
Body Construction Stage	Step 1: State the hypothesis.	Given.										
	Step 2: Set a starting variable.					C1						
	…											
	…											
	Conclusion											

Footnotes

Actions	
Main Actions	The operations applied to a step to generate the next step, whose outcome must be explicitly expressed to convince others

Supporting Actions	The operations to produce side work, whose outcome does not necessarily have to appear in the proof to convince others

Rephrasing an object	
R1	Rephrase an object by translating a concept, a theorem, or a property of concept into mathematical language mainly through applying its definition.
R2	Rephrase an object through formal or informal interpretation.
R3	Rephrase an object through algebraic manipulation or calculation, including solving an equation.

Combining objects	
CO(S, T)R	Connect and combine different pieces of objects (S and T) to create a new object. This action is always followedd by an operation of rephrasing.

Creating a cue	
C1	Set a variable.
C2	Recall prior knowledge, including a theorem, a proposition, a property of concept, or a mathematical law.
C3	Set some cases.
C4	Make a claim or create a new object.
C5	Consider an object.

Method

This study took place in a large research university in the Midwest. The target population was students who were enrolled in undergraduate Algebra, Analysis, and Topology. There were two types of data sources: (1) students' in-class midterm and final exams, and (2) in-class problem solving sessions. The former was conducted by the students' instructors. The latter was conducted by the researcher. In total, 12 problems were collected. Five of them came from in-class problem solving sessions for Algebra I and II. Those 5 problems provided 39 students' proofs. The other 7 problems came from students' mid-term and final exams for Analysis I and Topology. Those 7 problems provided 42 students' proofs. In total, 81 students' proofs were reviewed. The problems used in in-class problem solving sessions were created by the researcher but chosen by the instructors of the courses. A model proof was created for each problem in the form of a table (Tables 5, 6, and 7). The table included the type of operation used to generate one statement for each step of the proof. This type of table helped to detect where students had difficulties and what operation students failed to use. Every mistake, impasse, or difficulty that a student made was examined to explore the sources of their difficulties. Table 2 was used to identify possible sources of each mistake or difficulty a student made.

Results

The analysis revealed that students had difficulties with setting variables, applying definitions and properties of concepts, interpreting an object, practicing algebraic manipulation, using a given information, recalling relevant facts, considering cases, creating a useful object, and considering an object. The analysis also indicated how greatly their difficulties were affected by their lack of knowledge of definitions, properties, notations, relevant facts and theorems, proving problem techniques, and their lack of persistence, flexibility, carefulness, alertness, and precision. The following are some examples.

Example 1 (Natalie) and Example 2 (Ed) (Topology)

Table 5: Analysis of Natalie's and Ed's proofs

	Given P1 and P2, prove X.	Operations	Natalie	Ed
X	Show $f : Y \to Z$ is continuous.	Conclusion	Given	Given
Y	Show that for an open set W in Z, $(f^{-1}(W))$ is open in Y.	R1	N	S
P1	$q : X \to Y$ is a quotient map.	Hypothesis	Given	Given
P2	$f \circ q$ is continuous.	Hypothesis	Given	Given
1	Let W be an open set in Z.	C1	N	S
2	Since $f \circ q$ is continuous, $q^{-1}(f^{-1}(W))$ is open in X.	CO(1, P2)R1	N	S
3	Noting P1, recall that if $q^{-1}(V)$ is open in X, then V is open in Y.	C2	N	N
4	Since q is a quotient map, $(f^{-1}(W))$ is open in Y.	CO(2,3)R2	N	I

"S", "N", and "I" in the right two columns represent "successful," "not successful," and "incomplete" respectively.

proof: let $q: X \to Y$ be a quotient map; Z be a space and let $f: Y \to Z$ be a map. Assume $f \circ q$ continuous, Then $f \circ q$ is a homeomorphism, so f continuous.

Now assume $f \circ q$ not continuous. Then $\exists$ V open in Y s.t. $f^{-1}(V)$ not open in X. So f not continuous.

Figure 1: Natalie's proof

(⇐) assume $f \circ g$ is continuous then f is continuous since $f \circ g$ is continuous, then $g^{-1}(f^{-1}(W)) = U$ where U is open in X since this is true we know that $f(U) = V$, V is open in Y. since $f \circ g$ is continuous then $f^{-1}(W) = V$ and W is open in Z $\Rightarrow$ f is continuous

Figure 2: Ed's proof

Natalie's proof was not successful mainly because she did not know how to start a proving argument. The following can be possible causes of her difficulties. (1) OTC: She failed to translate the conclusion "$f : Y \to Z$ is continuous" into *mathematical language*, that is, "for an open set W in Z, $(f^{-1}(W))$ is open in Y." (2) KDF: Her knowledge of "continuity" is not strong enough to express the concept in *mathematical language*. The knowledge that a starting variable "an open set W" can be derived from the *ignition phrase* "for any open set W in Z" could have helped her start a proof successfully.

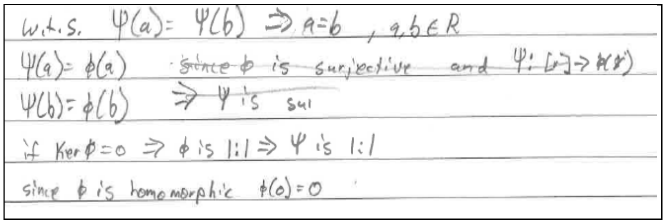
Ed failed in Step 3 (Table 5). The following can be possible causes of his difficulties. (1) C2: He was unable to recall the property of a quotient map "that if $q^{-1}(V)$ is open in *X*, then V is open in *Y*." and (2) KPR: He might not have known the property of a quotient map.

Example 3 (Eric) and Example 4 (Olivia)

Table 6: Analysis of Eric's and Olivia's proofs

	Given P1 and p2, prove X.	Operations	Eric	Olivia
X	Show $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is injective.	Conclusion	Given	Given
Y	Show that if $\psi([r]) = \psi([s])$, then $[r] = [s]$.	R1	S	I
P1	$\psi : R / Ker(\phi) \to S$ is a well-defined ring homomorphism.	Hypothesis	Given	Given
P2	$\phi : R \to S$ is a ring homomorphism.	Hypothesis	Given	Given
1	Suppose that $\psi([r]) = \psi([s])$.	C1	I	S
2	By the way to define $\psi : R / Ker(\phi) \to S$, $\phi(r) = \phi(s)$.	CO(1,P1)R1	I	S
3	Then, $0_S = \phi(r) - \phi(s)$.	R3	N	N
4	Since ϕ is a homomorphism, $0_S = \phi(r) - \phi(s) = \phi(r - s)$.	CO(2,P2)R1	N	N
5	Then, $r - s \in Ker(\phi)$.	R2	N	N
6	Then, $r = s + k$ for some $k \in Ker(\phi)$.	R2	N	N
7	Then, $r = s + k \in [s]$.	R2	N	N
8	Then, $[r] = [s]$.	R2	N	N

"S", "N", and "I" in the right two columns represent "successful," "not successful," and "incomplete" respectively.



w.t.s. $\psi(a) = \psi(b) \Rightarrow a = b$, $a, b \in R$

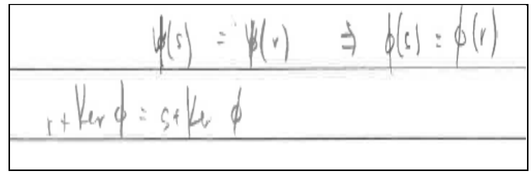
$\psi(a) = \phi(a)$ ~~since ϕ is surjective and $\psi: [r] \to \phi(r)$~~

$\psi(b) = \phi(b)$ ~~$\Rightarrow \psi$ is~~ su

if $Ker\,\phi = 0 \Rightarrow \phi$ is 1:1 $\Rightarrow \psi$ is 1:1

since ϕ is homomorphic $\phi(0) = 0$

Figure 3: Eric's proof



$\psi(s) = \psi(r) \Rightarrow \phi(s) = \phi(r)$

$r + Ker\,\phi = s + Ker\,\phi$

Figure 4: Olivia's proof

Eric's proof had mainly two defects. First, Eric had $\psi(a) = \psi(b)$ though he was supposed to have $\psi([a]) = \psi([b])$. The following can be possible causes. (1) MC: He might not have been careful to realize that $\psi(a) = \psi(b)$ had a domain error. (2) KNT: He might have lacked the knowledge of a coset, including the meaning and notation. Second, he did not reach Step 2 (Table 6). The following can be possible causes. (1) MF: He might have lacked flexibility to consider $\psi(a) = \psi(b)$ though he had $\psi([a]) = \phi(a)$ and $\psi([b]) = \phi(b)$. (2) CO: Apart from his notation problem, he failed to note the given hypothesis $\psi(R / Ker(\phi) :\to S$ "$:[r] \to \phi(r)$" so that he could have rephrased $\psi([a]) = \psi([b])$ with $\phi(a) = \phi(b)$.

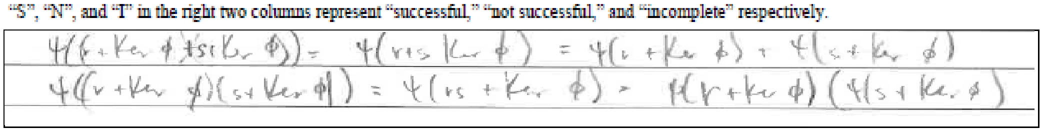
Olivia reached Step 2 (See Table 6), but was unable to show clearly how she can derive [r] = [s] from Step 2. The following can be possible causes. (1) MF: She might have lacked flexibility to change the expression of $\phi(r) = \phi(s)$ into $0_S = \phi(r) - \phi(s)$. (2) R3: She failed to rephrase $\phi(r) = \phi(s)$ with $0_S = \phi(r) - \phi(s)$ through algebraic manipulation.

Example 5 (Olivia)

Table 7: Analysis on Olivia's Proof

	Given P1 and P2, prove X.	Operations	Olivia
X	Show $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is a ring homomorphism.	Conclusion	Given
Y	Show (i) $\psi([r]+[s]) = \psi([r]) + \psi([s])$ and (ii) $\psi([r][s]) = \psi([r])\ \psi([s])$ for $[r], [s] \in R / K(\phi)$.	R1	S
P1	$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$.	Hypothesis	Given
P2	$\phi : R \to S$ is a ring homomorphism.	Hypothesis	Given
(i) 1	Let $[r], [s] \in R / K(\phi)$.	C1	S
2	Consider , $\psi([r]+[s])$.	C1	S
3	Note that $\psi([r]+[s]) = \psi([r+s])$.	R2	S
4	Since $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$, $\psi([r+s]) = \phi(r+s)$.	CO(3,P1)R1	N
5	Since $\phi : R \to S$ is a homomorphism, $\phi(r+s) = \phi(r) + \phi(s)$.	CO(4, P2)R1	N
6	Noting P1, $\phi(r) + \phi(s) = \psi([r]) + \psi([s])$.	CO(5,P1)R1	N
7	Therefore, $\psi([r]+[s]) = \psi([r]) + \psi([s])$.	CO(2-6)R2	I
(ii) 8	Consider $\psi([r][s])$.	C1	S
9	Note $\psi([r][s]) = \psi([rs])$.	R1	S
10	Then, $\psi([rs]) = \phi(rs)$.	CO(9,P1)R1	N
11	Then, $\phi(rs) = \phi(r)\ \phi(s)$.	CO(10, P2)R1	N
12	Noting P1, $\phi(r)\ \phi(s) = \psi([r])\ \psi([s])$.	CO(11, P1)R1	N
13	Therefore, $\psi([r][s]) = \psi([r])\ \psi([s])$.	CO(8-12)R2	I

"S", "N", and "I" in the right two columns represent "successful," "not successful," and "incomplete" respectively.



$\psi(r + Ker\,\phi + s + Ker\,\phi) = \psi(r + s\ Ker\,\phi) = \psi(r + Ker\,\phi) + \psi(s + Ker\,\phi)$

$\psi((r + Ker\,\phi)(s + Ker\,\phi)) = \psi(rs + Ker\,\phi) = \psi(r + Ker\,\phi)\,(\psi(s + Ker\,\phi))$

Figure 5: Olivia's proof

Olivia's proof was not successful because she was unable to show that $\psi([a][b]) = \psi([b])\psi([b])$ by way of $\phi(ab) = \phi(a)\phi(b)$. The following can be possible causes. (1) MC: She might not have been careful enough to realize she was unable to move from Step 3 to Step 7 and from Step 9 to Step 13 respectively. (2) MC: She was not careful enough to think about using the given hypothesis "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$." (3) CO: She missed using the hypothesis "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ to obtain Step 4 and Step 10.

Discussion

The model of the structure of proof construction led to defining students' difficulties to be those they had with the *reasoning activity*. Namely, in the model, students' difficulties with advancing a reasoning process were those with *rephrasing an object*, *combining objects*, *creating a cue*, and *checking and exploring*. There were two major factors that caused students' difficulties in the *reasoning activity*: (1) their lack of knowledge and (2) their lack of persistence, flexibility, and precision. For example, students' lack of knowledge may hinder them from *rephrasing an object* (Example 1) and *creating a cue* (Example 2), and may affect their use of notation (Example 3). Students' lack of flexibility and carefulness may hinder them from *rephrasing an object* (Example 4) and *combining objects* (Example 5). The results support that the aspects of proof construction (*background knowledge*, *reasoning activity*, *mental attitudes*) are intertwined to affect students' performances of proof construction.

The analysis of students' proofs led to the hypotheses: (1) the model of the structure of proof construction may help students grasp a comprehensive view of proof construction; (2) the knowledge of the structure of proof construction itself can help them advance a reasoning process. For example, the model suggests that in starting a proving argument, students should note the conclusion of the given statement, translate it into *mathematical language*, find *an ignition phrase*, and derive and set a starting variable from *the ignition phrase*. This knowledge might have helped Natalie (Example 1) start a proof. The model also suggests students should first try *rephrasing an object*, then, *combining objects*, and lastly *creating a cue* when they have impasses. This methodological knowledge could have helped, for example, Eric (Example 3) to rephrase $\psi([r]) = \psi([s])$ with $\phi(r) = \phi(s)$, which could have further led him to obtain $0_S = \phi(r) - \phi(s) = \phi(r - s)$.

Pedagogical Suggestions

This section provides an algorithm for proof construction as an instructional method which is derived from the analysis of students' proofs and the model of the structure of proof construction. First, I introduce some special terms used in the model of the structure of proof construction: *mathematical language*, *variables*, and *ignition phrases*. Then, I introduce the structure of proof construction in the model I have established: types of operations; types of proofs; stages of proof construction; and algorithm for proof construction.

Mathematical language. A distinction was made between mathematical language and *mathematical language*. *Mathematical language* is the mathematical language that is fine-grained enough to enable students to advance a reasoning process, to make a clear distinction from everyday language, and convince others without leaving any ambiguity. For example, "A group G is abelian" is mathematical language. The *mathematical language* for this statement is "for any two elements $a, b \in G$, $ab = ba$. The definitions of concepts are the most representative examples of *mathematical language*. Translation of an object into *mathematical language* is effective not only in proof construction but in any mathematical problem-solving. For example, if a Calculus student is given a statement "vectors *u* and *v* are parallel," which is mathematical language, they may need to translate it into *mathematical language* in order to solve a given problem, which is "$u \times v = 0$."

Variables. Variables are fundamental elements of *mathematical language*. It is crucial for students to be able to deal with variables correctly in advancing a reasoning process. There are two types of variables: a *given variable* and a *hidden variable*. The former is the one that appears in the problem while the latter does not. The latter sometimes needs to be derived and used for advancing a reasoning process. I call that type of variable "a *hidden variable*" to make a distinction from a hidden variable that students do not need to derive and set to advance a reasoning process. A *hidden variable* can be further classified into four types: *controlling variables*, *trivial variables*, *conditioned variables*, and *non-conditioned variables*. Both *controlling* and *trivial* variables are derived from the phrases such as "for every …," "for all …," or "If …, which I call primary ignition phrases. A *controlling variable* can have the power to confine, decide, and change another variable, while a *trivial variable* does not. The word "*trivial*" does not mean "not important." It happens that students can advance a reasoning process by considering a trivial variable. Both *conditioned* and *non-conditioned variables* are derived from the phrases such as "for some …" or "there exists …," which I call second primary ignition phrases. A *conditioned variable* is defined by a *controlling variable* while a *non-conditioned variable* is not. For example, in the definition of a sequence $\{a_n\}$ being convergent to $\{a_0\}$, which is "For every $\varepsilon > 0$, there exists an $N \in Z^+$ such that if $n \geq N$, then $|a_n - a_0| < \varepsilon$." the variable "$\varepsilon$,"is a *controlling variable* since it was derived from "For every $\varepsilon > 0$" and decides "N" when it is set. The variable "N" is a *conditioned variable* since it is derived from "there exists an $N \in Z^+$" and is defined by the *controlling variable* "ε." In the definition of $\{a_n\}$ being bounded, which is "For every $n \in Z^+$, $|a_n| \leq M$ for some $M \in R$." The variable "$n \in Z^+$" is a *trivial variable* because it is derived from "For every $n \in Z^+$" and does not confine another variable. The variable "$M \in R$" is a *non-conditioned variable* because it is derived from "for some $M \in R$" and it is not decided by any *controlling variable*.

Starting variables. I call the first hidden variable that students derive and set at the beginning of proof "a starting variable." It is crucial for students to set a correct starting variable to develop a poof. If a primary ignition phrase in the *mathematical language* for the conclusion provides a *controlling variable*, the variable should be a starting variable. If a primary ignition phrase provides a *trivial variable*, the variable may not be a starting variable. If the *mathematical language* for a conclusion does not contain a *controlling variable*, a starting variable should be found in the *mathematical language* for a hypothesis. A *controlling variable* in the *mathematical language* for a hypothesis can be a starting variable. After deriving a *controlling variable*, a second primary ignition phrase can be an *ignition phrase*. Namely, a *conditioned variable* may need to be set. A second primary ignition phrase in the *mathematical language* cannot be an ignition phrase. In other words, a conditioned variable in the *mathematical language* for a conclusion cannot be a starting variable. I call the ignition phrases from which students derive and set a starting variable "an *ignition phrase*" to make a distinction from the one from which students do not need to derive and set to advance a reasoning process.

Types of proofs. Proofs can be classified according to ways to derive and set a starting variable. There are three types. In the first type (Type I), students derive and set a starting variable from the conclusion of the given statement. This type of proof contains an *ignition* *phrase* in the *mathematical language* for the conclusion. Type I has two sub-types. While Type I (a) does not ask students to show A = B, Type I (b) asks to prove A = B. In the second type of proof (Type II), students derive and set a starting variable from a hypothesis of the given statement. The proof in this type does not contain an *ignition phrase* in the *mathematical language* for the conclusion. A proof by contradiction belongs to this type. The proof that asks to construct an object may belong to this type. The proofs in Types I and II can have more than one starting variable. For the third type of proof (Type III), students do not have to derive a starting variable because it is already given in the problem. The proofs that ask to prove $A = B$ belong to this type. Proofs by mathematical induction and proofs of trigonometric identities are such examples.

Stages of proof construction. There are two stages in proof construction: *opening stage* and *body construction stage*. The *opening stage* is a stage, at which students (i) set a major proving strategy (a direct proof, a proof by contradiction, a proof by contrapositive, mathematical induction, a proof by a counter example); (ii) make the goal of the proof clear; (iii) prepare for setting a starting variable, and (iv) decide the type of the proof. Students can make the goal of the proof clearer, prepare for setting a starting variable, and decide the type of a proof by translating the conclusion into *mathematical language* and examining an *ignition phrase*. *The body construction stage* is the main part of a proof, in which students advance a reasoning process by making good use of the four types of operations (Table 1).

Algorithm for Proof Construction

A: *Opening Stage*

A0: Read the problem

- If necessary, translate the whole problem into *mathematical language*. (A0.1)

A1: Decide a major strategy.

- Decide which proving strategy to use, a direct proof, by contrapositive, by contradiction, by counter example, or by mathematical induction. (A1.1)
- For a proof by contrapositive or contradiction, rephrase the problem. (A1.2)
- For Type III, skip to B0. (A1.3)

A2: Note the conclusion.

- Do not be tempted to note a hypothesis. (A2.1)

A3: Translate the conclusion into *mathematical language*.

- Rephrase the whole conclusion through R1 (See Table 1). (A3.1)
- Rephrase the conclusion more than once, if necessary. (A3.2)

A4: Find an *ignition phrase* in the *mathematical language* for the conclusion.

A5: Decide the type of the proof.

- If A4 is a primary ignition phrase, the proof belongs to Type I. (A5.1)
- If there is no *ignition phrase*, the proof belongs to Type II or Type III. (A5.2)
- If there is no *ignition phrase* and the problem asks to prove A = B, it belongs to Type III. (A5.3)

A6: Find a starting variable.

- For Type I, derive a starting variable from the *ignition phrase*. (A6.1)
- For Type II, note a hypothesis, translate it into *mathematical language*, and find an *ignition phrase*. (A6.2)

- For Type III, start the *body construction stage* by trying one of the followings: Work on either A or B to change it into B or A, work on both to obtain A = C = B, or show $A \subset B$ and $B \subset A$. This can work for the proofs in Type I (b). (A6.3)

TA: Supporting tips for the opening stage
TA1 (Type I): A starting variable should be first found in a primary ignition phrase in the *mathematical language* for the conclusion. However, if a variable in the primary ignition phrase is a trivial variable, it may not be a starting variable. A variable from a second primary ignition phrase in the *mathematical language* for the conclusion cannot be a starting variable. If there is not *ignition phrase* in the conclusion, derive a starting variable from a hypothesis.
TA2: (Type II) A starting variable can be derived from both a primary and a second primary ignition phrases in the *mathematical language* for a hypothesis.
TA3: (Type I.b and Type III) Try one of the following methods. (i) Work on either A or B until you change it into B or A, (ii) Work on both A and B until you get A = C = B, or (iii) Show both A ⊂ B and B ⊂ A. For $A \cong B$, (iv) find an isomorphism between A and B.

B: *Body construction stage*
B0: State the hypothesis (hypotheses).
B1: Set a starting variable.

- For Type I, set a starting variable from the *ignition phrase* obtained in A4.1. (B1.1)
- For Type II, translate the hypothesis into *mathematical language*. (B1.2)
- For Type III, skip this step and start to work on part of the conclusion. (B1.3)

B2: Make sure of the new goal of the proof.
B3: Try *rephrasing an object*, recalling the three sub-types (See Table 3).

- Whenever seeing a sentence containing a mathematical concept, translate it into mathematical language, and make it as fine-grained as possible. (B3.1)

B4: If it does not work, try *combining objects*.

- Find a hypothesis and use it (B4.1).
- If there is more than one hypothesis, choose the one that has a connection with the object you would like to combine with. (B4.2)
- When the *mathematical language* for a hypothesis contains a controlling variable, use this operation (*combining objects*) to specify the controlling variable.

B5: If it does not work, try *creating a cue*, recalling the five sub-types (See Table 1).
B6: If it does not work, try *exploring and checking*.

Examples

The following are the examples showing how the above algorithm works. To make the algorithm more understandable, I explain in the form of a dialogue between an instructor and students. In the dialogue, I assume that the students are fully equipped with not only the knowledge of the algorithm but also the knowledge necessary for solving a given problem.

Example 1 (Type I). "Suppose that $q: X \to Y$ is a quotient map and that $f: Y \to Z$ is a map such that $f \circ q: X \to Z$ is continuous. Prove $f: Y \to Z$ is continuous. Let's start the *opening stage*. What proving strategy would you use? (A1)" "A direct proof." "What is the next step?" "Note the conclusion. (A2)." "What is the conclusion?" "$f: Y \to Z$ is continuous." "Next?"

“Translate it into *mathematical language* (A3).” “What is the *mathematical language*?” “For any open set W in Z, $(f^{-1}(W))$ is open in *Y*.” “Then?” “Find an ignition phrase. (A4)” “What is the *ignition phrase*? (A3)” “‘For any open set W in Z.’” “What is the starting variable? (A4)” “An open set W in Z.” “Let’s start the *body construction stage*. After writing the hypothesis, what would you do?” “Set a starting variable (B1).” “So?” “Start with ‘Let *W* be an open set in Z’” “Then?” “Make sure of the new goal.” “What is that?” “To show $(f^{-1}(W))$ is open in *Y*.” “Next?” “Start to apply the four types of operations while keeping the supporting Tips (T1 – T2) in mind.” “We have gotten the object (1) an open set *W* in Z. What would you do?” “Apply *rephrasing an object* to the objet (1) an open set *W* in Z. (B3)” “ Does that work?” “No.” “Then, what would you do?” “Try the second operation ‘*combining objects*.’” “How would you do that?” “Find a hypothesis and use it. (B4.1)” “What is the hypothesis?” “There are two. (i) $q : X \to Y$ is a quotient map and (ii) $f \circ q : X \to Z$ is continuous.” “Which hypothesis should we use?” “Choose the one which has a connection with the object (1) ‘the open set *W* in *Z*.’ (B4.2)” Which hypothesis has a connection with the object (1) an open set *W* in *Z*?” “The second hypothesis (ii) $f \circ q : X \to Z$ is continuous.” “Why?” “Because both involve the space Z.” “Now are we ready to combine (1) ‘*W* is open in *Z*’ and (ii) ‘ $f \circ q : X \to Z$ is continuous’?” “No.” “Why not?” “Because the object (ii) ‘ $f \circ q : X \to Z$ is continuous’ contains a mathematical concept ‘continuous.’” “So?” “By T1, translate the object (ii) into *mathematical language*.” “What is the *mathematical language*?” “ (2) For any open set V *in* Z, $(f \circ q)^{-1}(V)$ is open in *X*.” “What do you observe in the object?” “The object (2) comes from the hypothesis of the given statement and the *mathematical language* for the statement contains a primary *ignition phrase* ‘for any open set in Z.’ So, By T2, we may want to specify the open set *V* in *Z* later.” “Now, are we ready to combine the objects (1) *W* is open in *Z* and (2) *for any open set* V *in* Z, $(f \circ q)^{-1}(V)$ *is open in X*?” “Yes, we can confine *V* by replacing *V* with *W* to obtain (3) $(f \circ q)^{-1}(W)$ *is open in X*.” “Then, what should we do?” “Try r*ephrasing an object* on the object (3) $(f \circ q)^{-1}(W)$ *is open in X* (B3).” “Does that work?” “Yes, the object (3) ‘$(f \circ q)^{-1}(W)$ *is open in X*’ can be rephrased with the object (4) ‘ $q^{-1}(f^{-1}(W))$ *is open in X*.’” “Can we further rephrase it?” “No” “Then?” “Try *combining objects*. (B4)” “How?” “Find a hypothesis and use it (B4.1).” “Do we have one?” “Yes, we have not used the first hypothesis (i) ‘ $q : X \to Y$ *is a quotient map*’ yet.” “Can we combine the objects (4) ‘ $q^{-1}(f^{-1}(W))$ *is open in X*’ and the hypothesis (i) ‘ $q : X \to Y$ *is a quotient map*’?” “No.” “Why not?” “Because (i) ‘ $q : X \to Y$ *is a quotient map*’ contains a mathematical concept ‘*a quotient map*.’ “So?” “Translate the hypothesis (i) into *mathematical language*. (T1)” “What is *the mathematical language*?” “ (5) ‘For any set *H* in *Y* that satisfies $q^{-1}(H)$ is open in *Z* for a quotient map $q : Y \to Z$, *H* is open in *Y*.’” “Now, are we ready to combine the objects (4) ‘ $q^{-1}(f^{-1}(W))$ *is open in X*’ and (5) ‘For any set *H* in *Y* that satisfies $q^{-1}(H)$ is open in *Z* for a quotient map $q : Y \to Z$, *H* is open in *Y*’” “Yes, since $f^{-1}(W)$ is a set in *Y*, we can specify the *H* by replacing *H* with *W* to obtain (6) ‘ $f^{-1}(W)$ *is open in Y*.’”

Example 2 (Type II). "Suppose $a \in Z$. Prove 4 does not divide $a^2 - 3$." "What proving strategy would you use? (A1)" "A proof by contradiction." "Then, what would you do?" "Rephrase the problem. (A1.2)" "What is the new statement?" "Suppose that 4 divides $a^2 - 3$ for every $a \in Z$" "What is next?" "Make sure of an *ignition phrase* in the new statement and start the *body construction stage* by directly working on the new statement to lead it to a contradiction. (A1.2)" "What is an *ignition phrase*?" "For every $a \in Z$, which is a *controlling variable*." "What does that imply?" "Since $a \in Z$ is a controlling variable derived from the *mathematical language* for a hypothesis, it may happen that we may want to confine the variable to a certain object (T2)." "Now, what would you do?" "Since it contains a mathematical concept 'divide,' translate it into *mathematical language* (T1)." "What is the *mathematical language*?" "(1) There exists $n \in Z$ such that $4n = a^2 - 3$." "Next?" "First, try B3 (*rephrasing an object*)." "Can you do that?" "Yes, rephrase the object (1) with, for example, (2) $3 = a^2 - 4n$, but I am not sure if that will work." "OK, then let's keep it to see what will happen. Then, what would you do?" "Since B3 (*rephrasing an object*) does not work anymore, try B4 (*combining objects*)." "Does that work?" "No, there is nothing to combine with (2) $3 = a^2 - 4n$." "Then, what would you do?" "Try B5 (*creating a cue*)." "There are five sub-types for *creating a cue* (See Table 2). Which would you try?" "C3 (set some cases)." "How would you use that?" "Set two cases, in which (i) $a \in Z$ is even and (ii) $a \in Z$ is odd. As expected, confine $a \in Z$ to a certain object (T2)." "Next?" "Consider the case (i). Suppose (3) $a \in Z$ is even." "Then?" "Since the statement contains a mathematical concept 'even,' translate it into *mathematical language* (T1)." "How?" "(4) Let $a = 2m$ for some $m \in Z$." "Then?" "First, try B3 (*rephrasing an object*)." "Does that work?" "Not anymore." "So?" "Try B4 (*combining objects*)." "How would you do that?" "Combine the objects (2) $3 = a^2 - 4n$ and (4) $a = 2m$ to obtain $3 = (2m)^2 - 4n = 4(m^2 - n)$, where $m^2 - n \in Z$." "Then?" "Since 4 does not divide 3, which is a contradiction." "Next?" "Work on the case (2) in a similar way. By letting (5) $a = 2m + 1$, combining the objects (2) $3 = a^2 - 4n$ and (5) $a = 2m + 1$, obtain $3 = (2m+1)^2 - 4n = 4(m^2 + m - n)$, where $m^2 + m - n \in Z$ (R1). It is a contradiction because 4 does not divide 3.

Example 3 (Type III). "Suppose $f : (a,b) \to R$ has a global maximum at some $c \in (a,b)$ and is differentiable at $c \in (a,b)$. Prove that $f'(c) = 0$. What proving strategy would you use? (A1)" "A direct proof." "Then?" "Note the conclusion. (A2)." "What is the conclusion?" "$f'(c) = 0$." "Next?" "Translate it into *mathematical language* (A3)." "What is the *mathematical language*?" "$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = 0$." "What do you observe in the object?" "This proof belongs to Type III." "Then, what would you do?" "Work on the left hand side of the equation $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ until we can change it into the right hand side, which is 0." "So?" "(1) Consider $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$." "Then?" "Apply *rephrasing an object* to the object (1) $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$. (B3)" "Can you do that?" "Yes, considering $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ means considering both (2) $\lim\limits_{x \to c^-} \frac{f(x)-f(c)}{x-c}$ and (3) $\lim\limits_{x \to c^+} \frac{f(x)-f(c)}{x-c}$. So, work on each separately."

"Next?" "Apply *rephrasing an object* to the object (2) $\lim\limits_{x \to c^-} \frac{f(x)-f(c)}{x-c}$. (B3)" "Does it work?" "No." "Then?" "Try *combining objects*. (B4)" "How?" "Find a hypothesis and use it. (B4.1)" "What hypothesis is available?" "(4) $f:(a,b) \to R$ has a global maximum at some $c \in (a,b)$." "How would you combine the objects (2) and (4)?" "We are not ready to combine them." "Why not?" "Because the object (4) $f:(a,b) \to R$ has a global maximum at some $c \in (a,b)$ contains a mathematical concept 'a global maximum.'" "So?" "Translate it into *mathematical language*. (T1)" "What is the mathematical language?" "(5) For all $x \in (a,b)$, $f(x) \le f(c)$." "Now, can we combine the objects (2) $\lim\limits_{x \to c^-} \frac{f(x)-f(c)}{x-c}$ and "(5) For all $x \in (a,b)$, $f(x) \le f(c)$?" " Yes. Since $f(x) \le f(c)$, $f(x)-f(c) \le 0$. Also, since $x \to c^-$, $x-c<0$. So, we can obtain the object (6) $\lim\limits_{x \to c^-} \frac{f(x)-f(c)}{x-c} \ge 0$." "Then?" "Work on the object (3) $\lim\limits_{x \to c^+} \frac{f(x)-f(c)}{x-c}$ in a similar way to obtain (7) $\lim\limits_{x \to c^+} \frac{f(x)-f(c)}{x-c} \le 0$." "Then?" "Since we cannot rephrase each object anymore, we try *combining objects*. (B4)." "How?" "Find a hypothesis and use it. (B4.1.)" "Do we have one?" "Yes, we have (8) $f:(a,b) \to R$ is differentiable at $c \in (a,b)$." "How would you combine them?" "The object (8) $f:(a,b) \to R$ is differentiable at $c \in (a,b)$" contains a mathematical concept, translate it into *mathematical language* (T1)." "What is the *mathematical language*?" "(9) $\lim\limits_{x \to c^-} \frac{f(x)-f(c)}{x-c} = \lim\limits_{x \to c^+} \frac{f(x)-f(c)}{x-c}$." "Are we ready to combine the objects (6) $\lim\limits_{x \to c^-} \frac{f(x)-f(c)}{x-c} \ge 0$, (7) $\lim\limits_{x \to c^+} \frac{f(x)-f(c)}{x-c} \le 0$, and (9) $\lim\limits_{x \to c^-} \frac{f(x)-f(c)}{x-c} = \lim\limits_{x \to c^+} \frac{f(x)-f(c)}{x-c}$?" "Yes." "How?" "$0 \le \lim\limits_{x \to c^-} \frac{f(x)-f(c)}{x-c} = \lim\limits_{x \to c^+} \frac{f(x)-f(c)}{x-c} \le 0$." "Then, what would you do?" "Try *rephrasing an object*. (B3)" "Does it work?" "Yes. $\lim\limits_{x \to c^-} \frac{f(x)-f(c)}{x-c} = 0 = \lim\limits_{x \to c^+} \frac{f(x)-f(c)}{x-c}$." "So?" "$\lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c} = 0$, which means $f'(c)=0$."

Conclusion

The model of the structure of proof construction can help students grasp a view of proof construction. It informs of not only the aspects of proof construction but the skills and abilities that are necessary for proof construction. Among the aspects of proof construction, the greatest factor that affects students' performances is their *background knowledge*. The analysis of students' proofs found how severely students' lack of knowledge affected their performances, in particular, their *reasoning activities*. Students should be encouraged and helped to acquire strong knowledge around concepts including definitions, notations, properties, relevant facts, theorems, propositions, and problem solving techniques and strategies. The analysis of their proofs also indicated that they needed metacognitive and methodological knowledge for advancing a reasoning process. The model of the structure of proof construction provides the metacognitive knowledge for advancing a reasoning process. The model provides not only the types of operations but also the order of the operations to be tried. The model also reveals the features of the structure of proof construction. *Mathematical language* is a key factor for the construction of a proof based on logical deduction. In particular, variables are principal elements of *mathematical language*. The model clarifies the types of variables, the relationships between variables and ignition phrases, and the ways to deal with variables according to their positions in a statement. Finally, the model offers algorithm for constructing a proof according to the types of proofs. An investigation of the effectiveness of the knowledge for the structure of proof construction can be a possible future project.

There is still room for improvement for the model of the structure of proof construction, in particular, the types of proofs, the types and roles of variables, and algorithm for proof construction. In order to improve the model, more proofs from various subjects must be examined. An exploration of stronger algorithm that can be applicable to various proofs can be another possible future project.

References

Alcock, L. (2004). Uses of example objects in proving. In PME 28, Bergen, Norway.

Ayalon, M., & Even, R. (2008). Deductive reasoning: in the eye of the beholder. *Educational Studies in Mathematics*, *69*(*3*), 235-247.

Ball, D., Hoyles, C., Jahnke, H., & Movshovitz-Hadar, N. (2002). The teaching of proof. In L. I. Tatsien (Ed.), *Proceedings of the International Congress of Mathematics* (Vol 3, pp. 907-920). Beijing: Higher Education Press.

CadwalladerOlsker, T., Miller, D., & Hartmann, K. (2013). Adapting model analysis for the study of proof scheme. *Proceedings of the Sixteenth Annual Conference on Research in Undergraduate Mathematics Education*.

Cirillo, M., & Herbst, P. (2012). Moving toward more authentic proof practices in geometry. *The Mathematics Educator, 21*, 11-33. Dreyfus, T. (1999). Why Johnny can't prove. *Educational Studies in Mathematics 38*, 86-109.

Edwards, B.,& Ward, M. (2008). The role of mathematical definition in mathematics and in undergraduate mathematics courses. In M. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 221-232). Washington, DC: Mathematical Association of America. Document: *ICMI Study* 1-12.

Harel, G., & Sowder, L. (1998) Students' proof schemes: Results from an exploratory study. In A. H. Schoenfeld, J. Kaput & E. Dubinsky (Eds), *Research in College Mathematics Education III* (pp. 234-283).
Harel, G. & Sowder, L. (2007). Towards a comprehensive perspective on proof. In F. Lester (Ed.) *Second handbook of research on mathematical teaching and learning*. NCTM:Washington, DC.

Kieran, C. (1998). The paradigm of modeling by iterative conceptualization in mathematics education research. In A. Sierpinska and J. Kilpatrick. (Eds.), Mathematics Education as a Research Domain. *A Search of Identity*, *2*, 213-215.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington, D.C.: National Academy Press.

Lew, K., Mejia-Ramos, J., & Weber, K. (2013). Not all informal representations are created equal. *Proceedings of the 16th Annual Conference on Research in Undergraduate Mathematics Education*, *2*, 570-574

Mariotti, A. (2006). Proof and proving in mathematics education. In A. Gutierrez, & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education (pp.173-*204). Rotterdam, The Netherlands: Sense Publishers.

Moore, R. (1994). Making the transition to formal proof. *Educational Studies in Mathematics 27*, 249-266. Newell, A., & Simon, H. (1972). Human problem solving. Englewood Cliffs, NJ: Prentice-Hall.

Paola,I.,& Inglis, M. (2011). Undergraduate students' use of deductive arguments to solve "prove that …" tasks. *Proceedings of the 7th Congress of the European Society for Research in Mathematics Education*.

Papaleontiou-Louca, E. (2003). The concept and instruction of metacognition. *Teacher Development*, *7*(*1*), 9-30.

Paramerswaran, R. (2010). Expert mathematicians' approach to understanding definitions. *The Mathematics educator*, *20*, 43-51.

Savic, Milos. (2011). Where is the logic in proofs? *Proceedings of the 14th Annual Conference on Research in Undergraduate Mathematics Education 2011*, *2*, 445-456.

Selden, A.,& Selden, J. (2003a). *Errors and misconceptions in college level theorem proving* (Technical Report, No.3), Cookeville, TN: Tennessee Technological University, Mathematics Department

Stylianides, G., Stylianides, A., & Philippou, G. (2007). Preservice teachers' knowledge of proof by mathematical induction. *Journal of Mathematics Teacher Education*, *10*, 145-166.

Stylianou, D., Chae, N, & Blanton, M. (2006). Students' proof schemes: A closer look at what characterizes students' proof conceptions. *PME-NA 2006 Proceedings*. 2, 54.

Van Someren, M., Barnard, Y., & Sandberg, J. (1994). The think aloud method: A practical guide to modeling cognitive processes. London: Academic Press.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, 48(1), 101-119.

Teachers' meanings for average rate of change in U.S.A. and Korea

Hyunkyoung Yoon
Arizona State University

Cameron Byerley
Arizona State University

Patrick W. Thompson
Arizona State University

This study explores teachers' meanings for average rate of change in U.S.A. and Korea. We believe that teachers convey their meanings to students and teachers who have productive mathematical meanings help students build coherent meanings. We administered a diagnostic instrument to 96 U.S. teachers and 66 Korean teachers. Some of teachers' responses revealed particular problematic meanings for average rate of change that should be addressed in professional development. Our analyses suggest that Korean teachers' meanings for average rate of change are substantially stronger than U.S. teachers' meanings.

Key words: Average rate of change, Mathematical meanings for teaching, Secondary teachers, International comparisons

There has been substantial interest in comparing student and teacher performance in the United States to other countries (Cai, 1995; Ma, 1999; Tatto, Ingvarson et al., 2008). Many people are aware that U.S. students are outperformed on mathematics assessments by students in many Asian countries. It is more surprising that according to PISA (Program for International Student Assessment) and NAEP (National Assessment of Educational Progress) that white students in our best performing state, Massachusetts, did not do as well as the average student in Korea (Hanushek, Peterson, & Woessmann, 2010). Furthermore, the average Korean student from any background outperformed students in Massachusetts who had at least one college educated parent. It is not easy to explain Korean students' superior performance by pointing to substantial diversity in the United States

Studies have demonstrated that there is a positive relationship between teacher knowledge and student performance (Baumert, Kunter et al., 2010; Hill, Ball et al., 2007). The TEDS-M (Teacher Education and Development Study in Mathematics) study investigated differences in teachers' mathematical content knowledge in seventeen countries to give further information about the relationship between teachers' knowledge and student performance internationally (Tatto, Peck et al., 2012). Although Korean teachers were not included in the TEDS-M study, secondary teachers in the United States did have lower scores than secondary teachers in other high performing Asian countries such as Singapore. TIMSS (Trend in International Mathematics and Science Study) and PISA scores indicated that Korean students outperformed other countries in international assessments. However, there are few studies that reveal Korean teachers' knowledge (Kim, 2007).

Our research team developed the *Mathematical Meanings for Teaching Secondary Mathematics* (MMTsm), a 44 item diagnostic instrument designed primarily to give professional developers insight into mathematical meanings with which teachers operate. We have piloted the MMTsm with 460 high school mathematics teachers in the United States. The MMTsm contains items that assess teachers' meanings for variation and covariation, function, proportionality, rate of change, and structure sense (Byerley & Thompson, 2014; Thompson, 2015; Yoon, Hatfield, & Thompson, 2014). In the summer of 2014 the first author translated the instrument into Korean and administered 42 items[1] to a convenience sample of 66 Korean teachers who taught 7th to 12th grades. The goal of the pilot in Korea was to understand how well the items revealed Korean teachers' meanings, to unearth any issues in the item translations, and to generate hypothesis about similarities and differences between Korean and U.S. teachers. As such, we have three research questions:

1) Do the translated versions of items make sense to Korean teachers in the way we intended?

2) What are the Korean teachers' mathematical meanings in the areas that the MMTsm asseses?

3) What are similarities and differences in U.S. and Korean teachers' meanings for average rate of change?

Our study of 7th to 12th grade teachers in Korea and the United States contributes to the investigation of international differences in teacher knowledge in two ways:

1) The MMTsm provides insight into the productive and unproductive meanings teachers operate with instead of categorizing responses as right or wrong.

2) Beyond a few studies with small sample sizes, little is published about secondary teachers' meanings of average rate of change in either the U.S. or Korea.

Literature Review

The third author (Thompson, 1994b) conducted a teaching experiment on the Fundamental Theorem of Calculus with 19 senior and graduate mathematics students, many of whom planned to teach secondary mathematics. He attended to the concept of average rate of change explicitly in his teaching experiment because of its centrality in understanding difference quotients and the Fundamental Theorem of Calculus. Thompson (1994b) described a typical mature meaning for average rate of change:

> *[By "average rate of change"] we typically mean that if a quantity were to grow in measure at a constant rate of change with respect to a uniformly changing quantity, then we would end up with the same amount of change in the dependent quantity as actually occurred. An average speed of 55 km/hr on a trip means that if we were to repeat the trip traveling at a constant rate of 55 km/hr, then we would travel precisely the same amount of distance in precisely the same amount of time as had been the case originally (p. 50).*

Based on quizzes and transcribed recordings of class discussions and tutoring sessions he concluded, the university mathematics students "apparently did not have operational schemes for average rate of change" (p. 49).

Coe (2007) conducted an interview-based study of three secondary teachers' meanings for rate of change. Peggy, an experienced teacher with an undergraduate degree in mathematics, was unable to provide a definition for average rate of change and became confused about how to account for varying speeds in the middle of a trip. The study found, "not one of the [three] teachers evidenced a fully coherent model of thinking that allowed them to work with the average rate tasks" (Coe, 2007, p. 237). If one thinks of speed as a multiplicative comparison of the changes in distance and time, it is possible to imagine an average speed as the constant speed that one must travel to go the same distance in the same amount of time. However, Peggy was

[1] We removed two items that did not work because of differences in language.

inclined to think that speed is an index of "fastness", so all of the changes in speed throughout the trip might seem important to take into consideration.

Additional U.S. studies of calculus students and secondary teachers are related to teachers' understandings of rate of change (Bowers & Doerr, 2001; Stump, 1999; Weber & Dorko, 2014). These studies suggest that teachers' meanings for rate of change might be inadequate for making sense of average rate of change. For example, Bowers and Doerr (2001) investigated 26 secondary teachers' thinking about the "mathematics of change" in two university technology based mathematics classes. They designed the first two instructional sequences to help the participants understand the Fundamental Theorem of Calculus by exploring relationships between linked velocity and position graphs (Bowers & Doerr, 2001, p. 120). Given a non-constant velocity versus time graph, more than seven teachers found the total distance traveled by simply multiplying time elapsed by the velocity at the end of the time interval using the formula $d=rt$. The formula $d=rt$ only works in situations with constant rates of change because the formula reflects a proportional relationship between distance traveled and time elapsed. Technically, this formula should be written "$\Delta d=r\Delta t$", because "$d=rt$" is only true if distance and time are both measured from zero. This misapplication of $d=rt$ suggests that teachers do not have an image of constant speed as a proportional relationship between changes in distance traveled and changes in elapsed time—an understanding of constant speed that is productive in developing a mature meaning of average rate of change.

Weber & Dorko (2014) investigated calculus students' and professors' descriptions of rate of change in various calculus situations. The meanings students displayed did not depend on making multiplicative comparisons of the change in one quantity to the associated change in another. For example, students conveyed meanings such as "rate as the process of differentiating a function, defined algebraically, using rules (e.g. product rule)", "rate as the slant or steepness of a graph" and "rate as something a function (or object) possesses (e.g. weight)" (p. 23). These meanings for rate do not involve relative size of changes and do not support a mature meaning of average rate of change. It is by thinking about the proportional relationship between changes in distance and changes in time that one sees why an understanding of "average rate of change" as arithmetic mean does not work. The mathematics professors in Weber & Dorko's study were much more likely to describe rate as "measuring the simultaneous variation of variable, or how fast variables change with respect to each other" (p. 23).

The first author did not find any studies about Korean teachers' meanings for rate of change or average rate of change after searching in Korean and English. However, Cho (2010) found that 36 Korean high school mathematics teachers showed high Mathematical Knowledge for Teaching (MKT) for Differentiation. The Korean teachers demonstrated particularly high subject matter knowledge on Cho's instrument that included a task on average rate of change. TEDS-M did not release any information on secondary teachers' understandings of rate of change (Tatto & Senk, 2011).

Theoretical Perspective

Coherent mathematical meanings serve as a foundation for future learning, so it is important that students build useful and robust meanings. One way students develop meanings is by trying to make sense of what their teacher say and do in the classroom. Before discussing how meanings are conveyed in the classroom, we will explain what we mean by *meanings*. According to Piaget, to understand is to assimilate to a scheme (Skemp, 1962, 1971; Thompson, 2013;

Thompson & Saldanha, 2003). Thus, the phrase “a person attached a meaning to a word, symbol, expression, or statement” means that the person assimilated the word, symbol, expression, or statement to a scheme. A scheme is an organization of ways of thinking, images, and schemes. When we say *assimilate* we mean the ways in which an individual interprets and make sense of a text, utterance, or self-generated thought. According to Piaget, repeated assimilation is the source of schemes, and new schemes emerge through repeated assimilations, which early on require functional accommodations and eventually entail metamorphic accommodations (Steffe, 1991).

We focus on teachers’ mathematical meanings because of their centrality in students’ construction of meaning. In classrooms, students might construct their meanings from their peers, from prior schemes, from resources the teacher selects for them or resources they find on their own. However, we suspect that a main source of students’ mathematical meanings lies in what teachers say and do. Students try to assimilate what the teacher says and does using their understandings of what is being taught. In doing so, the students will adjust what they say and do according to their understanding of what their teacher intends. In this sense, conversations in the classroom between a teacher and students entail mutual attempts by the teacher and students to understand each other. We suspect that teachers exert less effort in this regard than do students, and hence teachers have a greater impact on students’ meanings than do students have on the teacher’s meanings.

Our theory of meaning, and of ways meanings are conveyed through mutual interpretation, allows us to bridge theoretically what teachers know, what they teach, and what their students learn. While we cannot access the teachers’ mathematical meanings directly, we can delimit categories of responses according to particular mathematical meanings that we discern from them. We categorize teachers’ response based on meanings we believe might underlie the response based on the best available evidence of interviews and prior qualitative work. We assumed that, for the most part, meanings that teachers used to construct their responses to an item are meanings that would guide their decisions in the classroom.

We believe that meanings students construct are related to but not identical to a teachers’ meanings. In other words, there might be some gap between what teachers have in mind and what students understand. For example, a teacher might define average speed for her students by writing down a formula, but understand that the formula is related to finding a constant speed a hypothetical object would need to travel to go the same distance in the same amount of time. However, her students might understand that average rate of change is a formula to be applied in situations with the key words “average rate of change”, failing to develop a quantitatively rich meaning for average rate of change that helps them use it in a variety of contexts. Knowing a normatively correct mathematical formula for average rate of change is not the same as having a productive meaning for the formula. We found many teachers who were able to write a normatively correct formula to compute average rate of change on one item (not discussed in this paper), but who were unable to use it productively to answer the two items discussed later in this paper. Our focus on teachers’ meanings as a root for their actions allows us to think of meanings we think students might construct based on meanings we attribute to teachers. For example, if the teacher conveys the meaning that average rate of change is a formula, we believe the students might only construct a meaning for average rate of change as a formula that should be used in particular situations.

Methodology

Thompson (2015, p. 979) explained the process of creating items and rubrics for the *MMTsm*. We summarize the steps of a three year process below:

1) Draft items, interview teachers, and give item to mathematicians and math educators for review.
2) Revise items, interview teachers again.
3) Administer items to large sample of teachers and analyze responses in terms of the meanings they revealed.
4) Retire unusable items.
5) Interview teachers to understand why they gave the response that they did.
6) Revise items, potentially using teacher responses to make items multiple choice.
7) Administer revised items to large sample
8) Develop scoring rubrics.

After a first round of data collection in 2012, we categorized the responses from 144 teachers using a modified grounded theory approach (Corbin & Strauss, 2007). The modification was that we began our data analysis with strong theories of understanding magnitudes and rates of change, and of the nature of mathematical meanings and of characteristics that make them productive in instruction. After the 2013 pilot with revised items we developed a scoring rubric for each item by grouping grounded codes into levels based on the quality of the mathematical meanings expressed. The 96 U.S. high school teachers' responses reported in this paper are from the 2013 pilot. During team discussions of rubrics and responses, we continually asked ourselves "how productive would meanings we can discern from the teacher's response be for a student were the teacher to convey it?"

The first author translated each item into Korean. A Korean mathematics Ph.D. student, who taught high school mathematics in Korea for 7 years and wrote items for the Korean version of the practice SAT, translated the items back into English. The Ph.D. student had never seen the English versions. The first author and the third author reviewed the back translations and the first author made adjustments to the Korean versions (Behling & Law, 2000; Harkness, Van de Vijver et al., 2003).

The first author recruited Korean teachers from three groups: 13 peers of the first author from her undergraduate school, 32 teachers who were taking a qualification program[2] in eastern South Korea, and 21 teachers who were taking a graduate mathematics education class. The 96 U.S. high school teachers signed up voluntarily to participate in summer professional development projects taking place in two different states. They took the MMTsm as part of their professional development. U.S. and Korean teachers had similar years of teaching experience. The Korean teachers taught for an average of 4.5 years. This time included time teaching both middle school and high school mathematics. The 96 teachers in the U.S. sample analyzed in this paper taught at least one high school math class (algebra and above). We asked high school teachers how many times they had taught each subject and recorded the total number of high school classes taught. On average the U.S. teachers had taught 17.3 classes, which corresponds to approximately 4-5

[2] In Korea, all teachers who have taught more than three years must take a qualification training program to earn "1st class" teacher certificates.

years teaching. We also recorded the undergraduate major of teachers in the U.S. and Korea (See Table 1 and Table 2).

Table 1. U.S. teachers' undergraduate majors.

	Math	MathEd	STE	Other	Total
Bachelor's	10	14	6	8	38
Master's	17	22	5	14	58
Total	27	36	11	22	96

Table 2. Korean teachers' undergraduate majors.

	Math	MathEd	Stat	Other	Total
Bachelor's	8	45	1	1	55
Master's	1	10	0	0	11
Total	9	55	1	1	66

Two Average Rate of Change Tasks

Korean teachers saw 7 items on rate of change in MMTsm in the Summer of 2014. This report highlights the responses to two of these items. The item in Figure 1 is about a function's average rate of change over an interval. One can answer this item's question by joining two meanings: (1) an average rate of change is a constant rate of change, and therefore that it tells how many times as large a change in y is as an associated change in x, and (2) that a difference between two values of a function is the amount the function changed between those two values.

Consider a non-linear function defined on the interval 7.3 to 7.6. The function's average rate of change over that interval is 4. What is the difference between the value of the function at $x = 7.6$ and the value of the function at $x = 7.3$?

Select the best answer.

a. 0.3×4
b. 4
c. $0.3\ /4$
d. $4\ /\ 0.3$
e. $7.6 - 7.3$
f. Not enough information.

Figure 1. *The item named "difference from rate."* © 2014 Arizona Board of Regents. Used with permission.

We constructed the multiple choice options from teachers' answers to an earlier open-ended version and from teacher interviews. If the teacher thinks of average rate of change as "how many times as large a change in y is as a change in x over an interval", we believe that they will select choice (a). During an interview one teacher explained that the answer is 4 because the interval of 0.3 as identical to an interval of 1. We suspect that teachers who picked a response with a quotient (choices (c) and (d)) thought that rate of change should involve a formula that has a quotient. In this sense, we hypothesized that if a teacher had a calculational approach such as "average rate of change is dividing change in y by change in x" he would select (c) or (d). We included choice (e) to attract teachers who focused on the word "difference", so we think (e) reflects considering only change in x values. Additionally, we anticipated that the teacher who believes the function must be linear to determine the answer would select (f).

We used the item "San Diego to El Centro" (Hackworth, 1994) to reveal teachers' meanings for the idea of average rate of change (see Figure 2).

A car went from San Diego to El Centro, a distance of 90 miles, at 40 miles per hour. At what speed would it need to return to San Diego if it were to have an average speed of 60 miles per hour over the round trip?

Figure 2. *Part A of the item called "San Diego to El Centro"*. © 2014 Arizona Board of Regents. Used with permission.

The item "San Diego to El Centro" is composed of two parts. Part A (shown in Figure 2) and Part B (whether the teacher's answer was consistent with the fact that it would take 3 hours at a constant speed to go 180 miles). We put Part B on a separate page to guard against teachers looking ahead to Part B before answering Part A.

We found this item to be particularly useful for revealing the meaning that an average rate of change is an arithmetic mean of rates. Teachers with this meaning typically solve the equation $\frac{40+x}{2}=60$, ending with an answer of 80 miles per hour. However, the desired meaning of average speed is the constant speed that the car would need to travel to go the same distance in the same amount of time as the actual trip. If the car were to travel 180 miles at a constant speed of 60 mi/hr, it would travel for 180/60 hours (3 hours). The car spent 2.25 hours traveling from San Diego to El Centro. It therefore has 0.75 hours remaining to travel the rest of the trip. So it must have an average speed of 120 mi/hr in the second leg of the trip to have an overall average speed of 60 mi/hr.

Part B. A round trip of 180 miles at an average speed of 60 mi/hr will take 3 hours. Is this fact consistent with your answer on the prior page? Explain. If you would like to rework the problem, do so on this page. Please do not cross out your prior work.

Figure 3. Part B of the item called "San Diego to El Centro". © 2014 Arizona Board of Regents. Used with permission.

We added Part B so as to see whether teachers can understand the inconsistency of an answer found from thinking of average speed as the arithmetic mean of two speeds. We added "please do not cross out your prior work" because, in earlier trials, some teachers crossed out their work on Part A after reading Part B. We focus later on whether Part B perturbed teachers' meanings for Part A.

Responses to "San Diego to El Centro" Part A were scored with a rubric. Responses to "San Diego to El Centro" Part B were scored in terms of whether the teacher thought the answer on Part A is consistent with the fact stated in Part B. The first author scored the Korean responses with the English rubric. The item “San Diego to El Centro” Part A was scored with the following rubric:

Level	Level description	Sample response
Level 3 Response:	The response determined the return speed is 120mph by finding how much time remains for the second leg of the trip and computing the return speed accordingly. We ignore small computational errors if the response demonstrated a Level 3 type of reasoning.	SD EC $\frac{90\text{ mi}}{40\text{ m/hr}} = 2\frac{1}{4}$ hrs $\frac{90\text{ miles}}{}$ 180 miles at 60 miles/hr. = 3hrs. $\frac{90\text{ miles}}{3/4\text{ hr.}}$ $\cancel{90}^{30} \cdot \frac{4}{\cancel{3}} = \boxed{120\text{ miles/hr}}$
Level 2 Response:	The response first wrote 80 mph, and then ultimately found a return speed of 120 mph. (*Note*: We believe teachers whose first instinct is incorrect are less likely to have strong meanings for average rate of change than a teacher who immediately uses a productive meaning for average rate of change.)	120 ~~80~~ mi/hr $\frac{90}{40}$ $2\frac{1}{4}$ hrs $\frac{180\text{ mi}}{60\frac{\text{mi}}{\text{hr}}} = 3$ hrs $\frac{90}{80}$ (0, 0) (2.25, 90) (3, 180) 60 mi/hr S 90 E 90 S $\frac{90}{3/4}$
Level 1 Response:	Any of the following: • The response found an arithmetic mean of two speeds (e.g. *(40 + S)/2 = 60*) • The response found 80 mph without explicitly showing that they were using an arithmetic mean of two speeds.	SD → EC = 90mi at 40mph $\frac{40+S}{2} = 60$ $40+S = 120$ $S = 80$mph
Level 0 Response:	Any of the following: • I don’t know, scorer can’t interpret, work doesn’t address question. • The response does not fit level 1 to 3	$\frac{40}{90} = \frac{60}{180}$ I don't know how to do this accurately. I think it would be somewhere around 80-100 mph.

We categorized responses that changed from an arithmetic mean of two speeds to a productive meaning for average speed at Level 2. As mentioned in the theoretical perspective, we focus on meanings that teachers might convey in classrooms. We imagine that students might construct mixed meanings for average rate of change (arithmetic mean and desired meaning) from the Level 2 teachers. In this sense, we think Level 2 responses are less productive than Level 3 responses.

We did not attend to computational errors when placing responses at Level 3. We only focused on whether the meanings we could discern from a teacher's response fit the item's purpose. In this sense, we ignored minor computational errors.

Results

Teachers' responses to the item "difference from rate" are shown in Table 3.

Table 3. Responses to "difference from rate."

Response	U.S. (any degree)	U.S. w/ Math Degree	U.S. w/ Math Ed. Degree	Korea
*0.3 * 4*	47	14	18	61
4	9	1	6	1
0.3 / 4	12	2	6	0
4 / 0.3	11	3	3	1
7.6 - 7.3	6	2	2	0
Not enough information	19	3	1	3
I don't know	1	1	0	0
No answer	1	1	0	0
Total	96	27	36	66

About 49% of high school teachers from the United States gave the highest-level response, 0.3 times 4, to "difference from rate". About half of the teachers (51%) whose degree is mathematics or mathematics education from the United States gave the highest-level response. On the other hand, 61 out of 66 grade 7 to 12 teachers from Korea gave the highest-level response to this item. Since almost all Korean teachers had math or math education degree we did not distinguish the responses of Korean teachers by major.

The responses to "San Diego to El Centro" Part A show disparity between performance between U.S. and Korean teachers.

Table 4. Responses to "San Diego to El Centro" Part A

Response	U.S. (any degree)	U.S. w/ Math Degree	U.S. w/ Math Ed. Degree	Korea
Level 3	42	12	15	64
Level 2	8	3	4	0
Level 1 (80)	31	7	13	2
Level 0	14	4	4	0
No answer	1	1	0	0
Total	96	27	36	66

Approximately one third of teachers (31/96) in the United States revealed a meaning for average rate of change as an arithmetic mean of rates. Having a mathematics or mathematics education degree did not appear to be correlated to stronger meanings for average rate of change because 26% (7/27) of teachers with mathematics degree and 36% (13/36) of teachers with mathematics education degree showed a meaning for average rate of change as an arithmetic mean of rates. Only 2 out of 66 teachers in Korea had a meaning for average rate of change as an arithmetic mean of rates.

The responses to “San Diego to El Centro” Part B in U.S. show that most of the teachers in the United States that revealed a meaning for an arithmetic mean as average rate of change realized that it is not consistent with the fact given in Part B (Table 5).

Table 5. United States Teachers Responses to "San Diego to El Centro" Part B

Response U.S. Teachers	*Consistent*	*Not Consistent*	*Scorer Cannot Tell*	No answer	Total
Level 3	41	0	0	1	42
Level 2	6	2	0	0	8
Level 1 (80)	5	22	4	0	31
Level 0	7	4	1	2	14
No answer	0	0	0	1	1
Total	59	28	5	4	96

Results from “San Diego to El Centro” Part B show that Part B perturbed teachers because about 71% of teachers who revealed a meaning for average rate of change as an arithmetic mean of rates wrote that their answer is not consistent with the fact that the trip will take 3 hours on Part B. However, five teachers who revealed a meaning for average rate of change as an arithmetic mean of rates stuck to the their original response of arithmetic mean of rates arguing that it is consistent with the given fact on Part B (Table 6).

Table 6. Two sample responses in Level 1 and Consistent category

	Part A	Part B
Mr. Adams	SD El Centro. 40 miles at 40 m/hr. Speed = $\frac{dist}{time}$. $\frac{40+x}{2}=60$ $40+x=120$ $x=80$ 80 miles/hr.	This fact is consistent. 40 m/hr ~~1st~~ 1.5 hour = ~~40 miles~~ 60 miles 80 m/hr ~~2nd~~ 2nd 1.5 hour = ~~80 miles~~ ~~120 miles~~ $\frac{120}{180}$ Yes!

Ms. Augusta	San Diego → El Centro 90 miles @ 40 miles/h 180 miles @ 60 miles/h $\boxed{\text{80 miles/hr}}$	yes - the average between 80 miles/hr. + 40 mile/h is 60 miles/hr.

Both Mr. Adams and Ms. Augusta revealed an arithmetic mean of rates meaning for average rate of change. However, their responses to Part B show that Mr. Adams and Ms. Augusta's meanings for average rate of change are not identical. Mr. Adams applied the arithmetic mean of rates to the fact that the total trip will take 3 hours. He thought that the first leg of trip and the second leg of trip both took 1.5 hours. On the other hand, Ms. Augusta's response suggests that she did not check the fact that the total trip will take 3 hours. Rather, her response confirmed that "average" in average rate of change is no more than an arithmetic mean.

The responses to "San Diego to El Centro" Part B in Korea are in Table 7.

Table 7. Korean Teachers Responses to "San Diego to El Centro" Part B.

Response	*Consistent*	*Not Consistent*	*Undecided*	No answer	Total
Level 3	56	5	3	0	64
Level 2	0	0	0	0	0
Level 1 (80)	1	0	1	0	2
Level 0	0	0	0	0	0
No answer	0	0	0	0	0
Total	57	5	4	0	66

The reason why five Korean teachers in Level 3 wrote "not consistent" is that they made a computational error in Part A. Because we do not consider computational errors as part of our categorization system, the responses that show a Level 3 type of reasoning were categorized at Level 3. Three teachers in Level 3 wrote "I don't know", but we do not know why they wrote "I don't know". One possibility is that they did not carefully read Part B. One teacher in Level 1 stuck to an arithmetic mean as average rate of change in Part B, and the other in Level 1 wrote "I don't know".

Conclusion

The results show that, in our convenience samples, Korean teachers' meanings for average rate of change are substantially stronger than U.S. teachers' meanings. Almost all Korean teachers knew that average rate of change tells us that a change in y is a number of times as large as a change in x over an interval and did not confound "average" with "arithmetic mean". We believe that the meanings teachers hold, such as average rate of change as an arithmetic mean, are the meanings they will operate with during instruction. It is likely that students will develop meanings for average rate of change that are similar to their teachers' meanings. Thus, if a teacher has incoherent meanings the probability is high that his students will develop incoherent meanings. Because average rate of change is a Common Core Mathematics Standard it is critical that teachers' have opportunities to learn this standard.

We could not investigate why teachers' in Korea have stronger meanings for average rate of change. However, some studies suggest a plausible possibility if we consider that Chinese, Hong Kong, and Korean students have similar performance on international tests. Ma (1999) identified that Chinese elementary teachers showed more profound understanding than U.S. elementary teachers even though U.S. teachers have longer formal schooling and higher degree. Leung (2006) also suggested that Hong Kong and Korean elementary teachers already acquired mathematics competence when they were students in school. Thus, we suspect that the disparity in U.S. and Korean teachers' meanings is because Korean teachers developed stronger meanings while students than did U.S. teachers while students. Put another way, teachers in Korea were students in Korea, and teachers in the U.S. were students in the U.S. It is possible that Korean teachers developed meanings as students that U.S. teachers did not develop. We agree with Stigler and Hiebert (1999) that teaching is a cultural activity and that teachers' experiences as students are highly influential in their later career as teachers.

Our results suggest that if school students are to develop strong meanings for average rate of change, pre-service teacher preparation programs in the U.S must ensure that their graduates develop strong meanings for average rate of change. We emphasize that a strong meaning for average rate of change involves other major ideas in the school curriculum. A strong meaning for average rate of change entails strong meanings for constant rate of change, which itself entails concepts of variation, covariation, and proportionality (Thompson & Thompson, 1996; Thompson, 1994a, 1994b; Thompson & Thompson, 1992, 1994).

One obvious limitation of our study is that the sample is not random and thus generalization of the results to the larger populations in either U.S. or Korea is not possible. However, the convenience samples did provide evidence that the Korean teachers understood the translated items in the way we intended and that the rubrics written based on grounded coding of U.S. teachers' responses were sufficient to categorize the range of Korean teacher's responses. The two items reported here required understandings of mathematics useful for teaching, but are not specifically related to student thinking. We anticipate that our analyses of rate of change items that asked Korean and U.S. teachers to respond to a teaching situation will allow us to speculate about teachers' attention to student thinking regarding average rate of change.

Reference

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A*., et al.* (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal, 47*, 133-180.

Behling, O., & Law, K. S. (2000). *Translating questionnaires and other research instruments: Problems and solutions* (Vol. 133): Sage.

Bowers, J., & Doerr, H. M. (2001). An analysis of prospective teachers' dual roles in understanding the mathematics of change: Eliciting growth with technology. *Journal of Mathematics Teacher Education, 4*, 115-137. Retrieved from files/783/A1011488100551.html.

Byerley, C., & Thompson, P. W. (2014). Secondary teachers' relative size schemes. In P. Liljedahl & C. Nicol (Eds.), *Proceedings of the Proceedings of the 38th Meeting of the International Group for the Psychology of Mathematics Education*. Vancouver, BC: PME.

Cai, J. (1995). A cognitive analysis of US and Chinese students' mathematical performance on tasks involving computation, simple problem solving, and complex problem solving. *Journal for Research in Mathematics Education. Monograph*, i-151.

Cho, S. (2010). *A Study on the Mathematical Knowledge for Teaching (MKT) in Differentiation for Second grade in High School.* Master's thesis, The Graduate School of Education Ewha Womans University, Seoul.

Coe, E. E. (2007). *Modeling Teachers' Ways of Thinking About Rate of Change.* Arizona State University. Available from Google Scholar

Corbin, J., & Strauss, A. (2007). *Basics of Qualitative Research: Techniques and Procedures for Developing Grounded Theory* (3rd ed.): Sage Publications, Inc. Retrieved from http://www.amazon.com/dp/141290644X.

Hackworth, J. A. (1994). *Calculus students' understanding of rate.* Masters Thesis, San Diego State University, Department of Mathematical Sciences. Retrieved from http://pat-thompson.net/PDFversions/1994Hackworth.pdf

Hanushek, E. A., Peterson, P. E., & Woessmann, L. (2010). *US Math Performance in Global Perspective: How Well Does Each State Do at Producing High-Achieving Students? PEPG Report No.: 10-19.*: Harvard University

Harkness, J. A., Van de Vijver, F. J., Mohler, P. P., & fur Umfragen, Z. (2003). *Cross-cultural survey methods* (Vol. 325): Wiley-Interscience Hoboken.

Hill, H. C., Ball, D. L., Blunk, M., Goffney, I. M., & Rowan, B. (2007). Validating the ecological assumption: The relationship of measure scores to classroom teaching and student learning. *Measurement, 5*, 107-118.

Kim, Y.-O. (2007). *Middle school mathematics teachers' subject matter knowledge for teaching in China and Korea*: ProQuest.

Leung, K. S. F. (2006). Mathematics education in East Asia and the West: Does culture matter? In *Mathematics education in different cultural traditions-a comparative study of east asia and the west* (pp. 21-46): Springer.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Erlbaum.

Skemp, R. R. (1962). The need for a schematic learning theory. *The British Journal of Educational Psychology, 32*, 133–142.

Skemp, R. R. (1971). *The psychology of learning mathematics*. [Harmondsworth, Eng., Baltimore]: Penguin Books.

Steffe, L. P. (1991). The learning paradox. In L. P. Steffe (Ed.), *Epistemological foundations of mathematical experience* (pp. 26–44). New York: Springer-Verlag.

Stigler, J. W., & Hiebert, J. (1999). *The Teaching Gap: Best Ideas from the World's Teachers for Improving Education in the Classroom* (Vol. 1st (updated)). New York: Free Pess.

Stump, S. (1999). Secondary mathematics teachers' knowledge of slope. *Mathematics Education Research Journal, 11*, 124-144.

Tatto, M. T., Ingvarson, L., Schwille, J., Peck, R., Senk, S. L., & Rowley, G. (2008). *Teacher Education and Development Study in Mathematics (TEDS-M): Policy, Practice, and*

Readiness to Teach Primary and Secondary Mathematics. Conceptual Framework: ERIC.

Tatto, M. T., Peck, R., Schwille, J., Bankov, K., Senk, S. L., Rodriguez, M., *et al.* (2012). *Policy, Practice, and Readiness to Teach Primary and Secondary Mathematics in 17 Countries: Findings from the IEA Teacher Education and Development Study in Mathematics (TEDS-MM)*: ERIC.

Tatto, M. T., & Senk, S. (2011). The mathematics education of future primary and secondary teachers: Methods and findings from the Teacher Education and Development Study in Mathematics. *Journal of Teacher Education, 62*, 121-137.

Thompson, A. G., & Thompson, P. W. (1996). Talking about rates conceptually, Part II: Mathematical knowledge for teaching. *Journal for Research in Mathematics Education, 27*, 2-24.

Thompson, P. W. (1994a). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 179–234). Albany, NY: SUNY Press.

Thompson, P. W. (1994b). Images of rate and operational understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics, 26*, 229–274.

Thompson, P. W. (2013). In the absence of meaning…. In *Vital directions for mathematics education research* (pp. 57-93): Springer.

Thompson, P. W. (2015). Researching mathematical meanings for teaching. In L. D. English & D. Kirshner (Eds.), *Third handbook of international research in mathematics education* (pp. 968-1002). New York: Taylor & Francis.

Thompson, P. W. (in press). Researching mathematical meanings for teaching. In L. D. English & D. Kirshner (Eds.), *Third Handbook of International Research in Mathematics Education*. London: Taylor and Francis.

Thompson, P. W., & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, G. Martin & D. Schifter (Eds.), *Research companion to the Principles and Standards for School Mathematics* (pp. 95-114). Reston, VA: National Council of Teachers of Mathematics.

Thompson, P. W., & Thompson, A. G. (1992). *Images of rate*. Paper presented at the Annual Meeting of the American Educational Research Association, San Francisco, CA. Retrieved from http://bit.ly/17oJe30

Thompson, P. W., & Thompson, A. G. (1994). Talking about rates conceptually, Part I: A teacher's struggle. *Journal for Research in Mathematics Education, 25*, 279–303.

Weber, E., & Dorko, A. (2014). Students' and experts' schemes for rate of change and its representations. *The Journal of Mathematical Behavior, 34*, 14-32.

Yoon, H., Hatfield, N., & Thompson, P. W. (2014). Teachers' meanings for function notation. In P. Liljedahl & C. Nicol (Eds.), *Proceedings of the Proceedings of the 38th Meeting of the International Group for the Psychology of Mathematics Education*. Vancouver, BC: PME.

-

RUME Working Group: Research on Community College Mathematics

Attendees

Ann Sitomer (Oregon State University), Irene M. Duranczyk (University of Minnesota), John Smith (Pellissippi State Community College), Martha Makowski (University of Illinois at Urbana-Champaign), April Strom (Scottsdale Community College), Gabriel Tarr and Matt Weber (Arizona State University), Donna Bassett (Roane State Community College), and Rebecca Walker (Guttman Community College).

Working Group Report

The primary goal for the working group this year was to write a proposal to request funding for a conference with a focus on mathematics education research that addresses questions unique to the community college setting. This research is particularly critical now as so many curriculum decisions are being made, especially with respect to developmental or remedial mathematics. Consistent with the mission of two-year colleges, developmental mathematics is a fundamental part of community college mathematics. Frequently these curriculum decisions are being made without considering the research base that could better inform our understanding of mathematics learning and teaching in the community college setting (Mesa, Wladis, & Watkins, 2014). The discussion at the working group helped to identify two primary reasons for making the work of education researchers within the community college setting more visible: (1) This work should be a prominent subdomain of mathematics education research, and (2) Mathematics education research should inform mathematics learning and teaching at community colleges. For this reason, the group started work on *two* proposals at RUME. The first proposal is for a symposium to be held at the Research Session at the annual meeting of the National Council of Teachers of Mathematics, one of the major conferences for disseminating research in mathematics education. The focus of the symposium will be research on community college mathematics teaching. The RUME working group organizers are currently inviting mathematics education researchers focused on community college teaching to collaborate on this proposal next summer.

The second proposal is a request for NSF funding for a one-day conference to be held prior to the annual meeting of the American Mathematical Association of Two-Year Colleges with the purpose of bringing together researchers and practitioners. Similar to the design of the

Mentoring and Partnerships for Women in RUME (MPWR) seminar, the structure of the one-day conference will be panel discussions and conversations centered on research at the community colleges. We chose three areas in mathematics education research that would be of interest to community college practitioners: proportional reasoning, statistical reasoning and students' understanding of functions. We propose to invite panelists working in these areas to discuss their research. The conversations with practitioners will be guided by activities so that participants have an opportunity to discuss the ways research findings might impact practice. Ann Sitomer and April Strom are writing the proposal; John Smith and Kelly Mercer (a working group Participant from a previous year) will help organize the conference if funded.

Other working group activities included sharing accomplishments over the last year by current and previous working group participants. These accomplishments include a revision of an IES proposal submitted by Vilma Mesa, April Strom, Irene Duranczyk and Laura Watkins to be submitted to the NSF; the working title of this proposal is *An Exploration of Characteristics of Algebra Instruction at Community Colleges and their Relationship with Student Learning and Performance/Success*. Working group members have at least five articles published or in press:

- Mesa, V., & Lande, E. (2014). Methodological considerations in the analysis of classroom interaction in community college trigonometry. In Y. Li, E. A. Silver & S. Li (Eds.), *Transforming mathematics instruction: Multiple approaches and practices* (pp. 475-500). The Netherlands: Springer.
- Mesa, V., Wladis, C., & Watkins, L. (2014). Research problems in community college mathematics education: Testing the boundaries of K-12 research. *Journal for Research in Mathematics Education, 45*(2), 173-192.
- Mesa, V. (In press). Mathematics education in community colleges. In J. Cai (Ed.), *Third Handbook in Research in Mathematics Education*. Reston, VA: National Council of Teachers of Mathematics.
- Mesa, V., Burn, H., White, N. (In press). Basic good teaching in the *Characteristics of Successful Programs in College Calculus*. In D. Bressoud, V. Mesa, C. Rasmussen & H. Burn (Eds.) Findings from the *Characteristics of Successful Programs in College Calculus*.
- Mesa, V. (In press). Curriculum in the *Characteristics of Successful Programs in College Calculus*. In D. Bressoud, V. Mesa, C. Rasmussen & H. Burn (Eds.) Findings from the *Characteristics of Successful Programs in College Calculus*.

One member of the working group (Irene Duranczyk) is undertaking with a colleague a meta-analysis of quantitative studies conducted between 2001 and 2010 that focus on questions in community college mathematics education. This research is in progress and should be completed this year. We considered expanding this project to analyzes qualitative and mixed methods studies. Frequently decisions are made without regards to contextual differences. The experiences of community college students and faculty may best be revealed through their personal stories that are prevalent within qualitative and mixed methods community college mathematics education research base.

The primary purpose of our working group is to build a community of researchers working in the domain of community college mathematics education. We have been very successful towards achieving this goal. Some working group participants have participated each year, but *every* year we are introduced to new colleagues working in the domain who become collaborators on future projects. A secondary purpose is to link research to practice in mathematics teaching and learning at community colleges. The outcomes from the working group that convened at RUME 2015 moved us closer to this goal.

REFERENCES

Mesa, V., Wladis, C., & Watkins, L. (2014). Research problems in community college mathematics education: Testing the boundaries of K-12 research. *Journal for Research in Mathematics Education, 45*(2), 173-192.

Working Group on Education Research at the Interface of Mathematics and Physics

Warren Christensen
North Dakota State University

Megan Wawro
Virginia Tech

The area of focus for the 2015 Mathematics & Physics Working Group was the interface of student understanding of linear algebra and its application in physics courses, such as Quantum Mechanics. This working group brought together researchers in Undergraduate Mathematics Education and Physics Education to analyze data, compare methods for analysis, and highlight areas of interest or focus from within the data on student thinking. This focus was intended to foster discussions and possible collaborations about research at the interface of RUME and PER, in linear algebra and beyond, broadly conceived.

Keywords: Physics, Linear algebra, Interdisciplinary Research, quantum mechanics, eigentheory

Overview

The NRC's report *Discipline-Based Education Research: Understanding and Improving Learning in Undergraduate Science and Engineering* (2012) charges that the U.S. must improve STEM education, specifically recommending "interdisciplinary studies of cross-cutting concepts and cognitive processes" (p. 3) in undergraduate STEM courses. It further states that "gaps remain in the understanding of student learning in upper division courses" (p. 199), and that interdisciplinary studies "could help to increase the coherence of students' learning experience across disciplines … and could facilitate an understanding of how to promote the transfer of knowledge from one setting to another" (p. 202). This working group contributes towards this national need for basic research by investigating students' reasoning about and use of mathematics within upper-division physics (specifically, linear algebra).

Mathematics is often referred to as the language of science, and its use as a theoretical modeling tool for physics goes back centuries. Despite the significant amount of overlap in content and goals for instruction among courses in calculus, linear algebra and differential equations in Mathematics Departments and courses in Physics Departments that use these tools to solve physics problems, such as Quantum Mechanics, Electricity and Magnetism, Mechanics, Thermal Physics, there are very few published studies on the learning and teaching of these topics that touch on both the mathematics and physics (e.g., Christensen & Thompson, 2012; Wilcox, Caballero, Rehn, & Pollock, 2013; Pepper, Chasteen, Pollock, & Perkins, 2012). The community of PER and RUME faculty who are interested in and committed to collaborative research at the boundary of mathematics and physics need a space to exchange and develop ideas for collaboration and research.

In 2013, the first working group on RUME/PER was lead by Wawro and Christensen to start the conversation between researchers in physics and mathematics education researchers. There was been a demonstrable interest from the community, as evidenced by the interest in the 2014 working group on RUME/PER. Led by Eric Weber (RUME), Corinne Manogue (PER), and Aaron Wangberg (RUME), the 2014 Working Group on RUME/PER focused on curriculum development and implementation related to issues in multivariable calculus. It created a great deal of important discussion about the content of physics and mathematics, as well as the teaching of this content. At RUME 2014, the various researchers agreed that the 2015 focus would be on data analysis regarding mathematics and physics thinking, with a crosscutting content focus of linear algebra. Thus, the 2013 Working Group began a conversation of interdisciplinary issues, the 2014 Working Group focused on curriculum development and implementation, and the 2015 Working Group was focused on research methodology and analysis. We envision an ongoing series of such working groups, with a slightly different focus and leadership team each year.

The area of focus for the 2015 Mathematics and Physics Working Group was the interface of student understanding of linear algebra and its application in physics courses, such as Quantum Mechanics. This working group brought together researchers in Undergraduate Mathematics Education and Physics Education to analyze data, compare methods for analysis, and highlight areas of interest within a specific theme. The aim of this group was to use this specific and detailed discussion to educate and broaden the depth of understanding of research on the learning and teaching of math and physics for both PER and RUME faculty. We specifically engaged in discussion about various ways to use qualitative analysis methodologies to understand how people think and reason within and across disciplines. While the focus of the working group was focused linear algebra and quantum mechanics, the methods discussed are broadly applicable to research within either field (math or physics education research) and across most any content within either discipline.

Summary of Major Activities and Discussions

In order to create a focused discussion on research methodology and analysis, the central content for discussion at the working group was linear algebra and its uses in quantum mechanics. Specifically, the organizers chose a 20-minute video clip of a problem-solving interview with a physics professor, known as AC, from Christensen's research around which to focus during the working group. Approximately one week prior to the working group, Christensen provided access to the video for those who had indicated they would attend the working group and requested they watch it prior to the working group.

Part One: Introductions and Overview of the Content (8-9:30am)

The primary goal for the working group was to focus discussion on research. While discussions in previous working groups concerning pedagogical innovation and curriculum development were fruitful, the organizers wanted to limit discussion to content and research on faculty/student thinking about the content at hand. Additionally, discussion of research methods, theoretical frameworks and analysis were appreciated in so far as they applied to the study of mathematics and physics. The goal was to create a rich discussion on research without the oft-followed path of "When I teach _____ in my class." Also stressed was the importance of being sensitive to all individuals in the room, knowing that due to the differences in content backgrounds and research backgrounds, there was a strong possibility of misunderstandings and necessary clarifications.

The start of the Working Group featured brief presentations about relevant content from Linear Algebra and Quantum Mechanics. We recognized that few faculty in the room will be familiar with the perspectives of both mathematics and physics, and therefore having a conversation about content to start the session was crucial to a rich discussion of analyzing the subsequent video data.

After introductions, the working group began with Michelle Zandieh, Arizona State University, summarizing some of her relevant work in student understanding of linear algebra. In particular, Zandieh focused on a framework for student interpretation of the matrix equation $Ax = b$, which was a result of work with Christine Andrews-Larson of Florida State University (Larson & Zandieh, 2013). This framework details three prominent interpretations of the equation $Ax = b$, highlighting the role of the vector x in each interpretation. The linear combination interpretation is when b is a linear combination of the column vectors of the matrix A and x is the set of weights on the column vectors of A. The systems interpretation is when x is interpreted as the set of values that satisfy the system of equations corresponding to $Ax = b$, and the transformation interpretation of $Ax = b$ corresponds to a view in which an input vector x is transformed into the output vector b via multiplication by the matrix A. The utility of this framework is that it offers an analytic lens through which to make sense of a person's thinking via the variety of both correct and incorrect ways that a person blends and coordinates ideas. This interactive presentation helped prepare the working group participants for the forthcoming time of analyzing a physicist's interview by introducing a method for interpreting and analyzing interview data for nuanced aspects of the interviewee's reasoning.

John Thompson, Associate Professor at the University of Maine, presented on the use of eigentheory as a mathematical tool for understanding the Ising Model for spin ½ particles (such as electrons) in Quantum Mechanics. He gave an overview of the experimental findings of the Stern-Gerlach experiment, which demonstrated the peculiar behavior of certain particles passing through a series of strategically aligned magnetic fields. He showcased the Ising Model can be modeled using vectors and matrix in $\mathbb{R}^2$. The matrix acting on an eigenvector produces an eigenvalue and that matrix, just as in mathematics. However, in physics (within the Ising Model), the matrix is an operator, and has the physical significance of being a measurement of the spin of a system, (or perhaps the z-component of the spin, depending on the measurement taken). The eigenvectors for this measurement are the two possible quantum states of the particle, i.e., spin up or spin down. The eigenvalues are energy eigenvalues, as they represent the energy of the measured particle. The goal of this content lecture was to motivate why physicists care about eigentheory, and the nuances of the language used by physicists to talk about physical systems that may well look like pure mathematical expressions to mathematicians

Part Two: Video Analysis and Discussion (9:30-11:00am)

After the content discussion, the Working Group broke into small groups of three per group, with at least one math and physics person in each group, to watch the interview in its entirety. Each group had a computer to watch the video, and even though the video was made available in advance many attendees were viewing the video for the first time. Participants were asked to develop research questions, alternative lines of inquiry and other theoretical frameworks. The goal was to provide participants with a sense of the content at play –how it's taught in math classes and how it's taught and used in physics classes– and then to dig deeply into a rich piece of data that will create conversation. Following the viewing, significant time was made available for questions and comments on the interview protocol and other aspects of the research. Participants shared out points of interest from the video with the entire group.

Part Three: Prepared Analyses and Wrap Up (11:00am-12:00pm)

Christensen presented research on a different interview with an undergraduate using the same protocol, using a popular theoretical framework from PER, Framing and Resources (Hammer et al., 2006). This framework considers that student ideas come from a set of interconnected resources that are accessible (or inaccessible) to the student depending on how the student frames the problem (often unconsciously). The study used a lexicon analysis to mark to the frame in which the student was. The student seemed to use words that were consistent with physics in certain scenes where he was making appropriate sense of the task at hand, however this language was not used immediately thereafter when the student was struggling to make sense of something that was directly related. The findings were of interest to the attending group but additionally the articulation and use of the framework was novel for many of the RUME faculty in the audience.

Wawro concluded the working group's time together of considering analyses of the given video by summarizing the approach she used towards analysis. The guiding question, "What are any research questions that emerge for you as you watch the video?" prompted her to consider the request question, "What are AC's notions of "solution" in the data at hand?" She explained to the working group that her approach included watching the video and reading the transcript multiple times in order to notice a theme of interest with respect to AC's mathematics evidenced in the interview. In general, her process was informed by a sensitivity and tendency towards considering learners' concept images (Tall & Vinner, 1981) of a particular concept, which can lead to grounded analysis of the data and organization of learners' notions around various emerging themes. It can also lead to adoption and adaptation of pre-existing frameworks through which to further make sense of nuances within the data at hand. As such, the aforementioned framework by Larson and Zandieh (2013) provided a useful orientation towards the notion of "solution" for $Ax = b$, and the conference paper by Henderson, Rasmussen, Sweeney, Wawro, and Zandieh (2010) introduced the idea of "solution sense" when looking at students' interpretations of the matrix equation $Ax = 2x$. Because the time for the working group to come to a close was drawing near, Wawro discussed a selection of pre-prepared slides that contained transcript and accompanying analysis of AC's notions of "solution." One theme in her analysis was AC's notion that "to solve" seemed to involve determining specific values for x and y in $\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$, evidenced in statements such as "we don't have enough different equations here to actually solve numerically for x and y I suppose" and "which leaves us with an expression with two unknowns, which of course we can't solve" when converting it to a system of equations and noticing the second equation is redundant. The conversation concluded by discussing implications for what students might think "solution" means and the various possibilities for infinite solution sets for matrix equations in linear algebra.

The session wrapped with a brief discussion of future working group sessions including other in-depth analysis of video covering different topics and possibly review of other developed curriculum. One of the outcomes of the working group was a special session featuring PER and RUME faculty presenting at the 2015 Physics Education Research Conference at College Park, MD in July. This session has aided faculty in establishing points of compatible collaborations and deepening existing ties, and the participants look forward to pursuing these opportunities.

Conclusion and Future Directions

This working group was an excellent setting for a number of groups to network and collaborate, including: (a) qualitative researchers from RUME, (b) RUME members that research the teaching and learning of linear algebra, (c) qualitative researchers from PER, (d) PER members that research the teaching and learning of upper-division physics content, particularly quantum mechanics; and (e) any faculty member (physics or math) looking to engage in a discussion on student thinking, theoretical frameworks, and research methods. The combination of presentations, interactive data analysis, and discussions combined both novice and experienced researchers to begin collaborations and provide mentoring for those new to research in these areas. The 2015 Working Group organizers and participants, in conjunction with members of the previous 2013 and 2014 Math-Physics Working Groups, feel strongly that this conversation between PER and RUME faculty is not one that will end soon. Indeed, these conversations continued throughout the RUME conference, during meal times and research presentations. Plans are underway for the 2016 Math & Physics Working Group, to be led by other members of this community.

References

Hammer, D., Elby, A., Scherr, R. E., & Redish, E. F. (2005). Resources, framing, and transfer. In J. P. Mestre (Ed.), *Transfer of learning from a modern multidisciplinary perspective* (pp. 89-120). Greenwich, CT: Information Age Publishing.

National Research Council. (2012). *Discipline-Based Education Research: Understanding and Improving Learning in Undergraduate Science and Engineering.* S. R. Singer, N. R. Nielsen, and H. A. Schweingruber (Eds.). Committee on the Status, Contributions, and Future Direction of Discipline Based Education Research. Board on Science Education, Division of Behavioral and Social Sciences and Education. Washington, DC: The National Academies Press.

Pepper, R. E., Chasteen, S. V., Pollock, S. J., & Perkins, K. K. (2012). Observations on student difficulties with mathematics in upper-division electricity and magnetism. *Physical Review Special Topics - Physics Education Research, 8*, 010111.

Christensen, W. M., and Thompson, J.R. (2012). Investigating graphical representations of slope and derivative without a physics context. *Physical Review Special Topics - Physics Education Research, 8*, 023101.

Henderson, F., Rasmussen, C., Sweeney, G., Wawro, M, & Zandieh, M. (2010, February). Symbol sense in linear algebra. Paper presented at the Thirteenth Conference on Research in Undergraduate Mathematics Education, Raleigh, NC.

Larson, C. & Zandieh, M. (2013). Three interpretations of the matrix equation Ax=b. *For the Learning of Mathematics, 33*(2), 11-17.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151–169.

Wilcox, B. R., Cabellero, M. D., Rehn, D. A., and Pollock, S. J. (2013). Analytic Framework for Students' Use of Mathematics in Upper-Division Physics *Physical Review Special Topics - Physics Education Research, 9,* 020119.

Working Group: Research on College Mathematics Instructor Professional Growth – 2015 Conference Report

Shandy Hauk, Jessica Deshler, Natasha Speer

Abstract. The Research on College Mathematics Instructor Professional Development working group meets online several times per year and annually at the RUME conference. We solicit proposals from researchers in all areas of the professional development of college mathematics instructors across institutional types (e.g., community college, university). This includes, but is not limited to, research on factors that shape instructional practices and the experiences of instructors as they attend to student thinking in their instruction. The group's goals, historically and as they have evolved, continue to drive the focus of annual meetings. They include interaction that offers (1) informed support and feedback for researchers, (2) opportunities for networking and collaboration among mathematics educators interested in research and development of materials, processes, models, and theories to support the professional development of collegiate mathematics instructors, (3) continuing discussion of issues central to the field and ways to address them. The intended participants of this group include researchers in all of these areas, whether new to the field, to research in general (early career researchers) or experienced. Researchers need not present their work to participate in the group or provide feedback to others. Group meeting time is structured to allow feedback on research projects that are in progress. The working group is not meant to be a forum for presenting completed studies, but rather an opportunity to get feedback from peers on projects in any stage; from the refinement of research questions to study design, data collection and analysis, to discussion of venues for future presentation and proposals for funding of projects. We also discuss strategies for sharing our work with the practice-oriented college mathematics instructor professional development community, the needs of the working group, and ways of sustaining collaborations and communication among group participants during the year.

Background and Significance. Begun with a focus on the early professional development of graduate students who work as teaching assistants (TAs), this group has evolved to include research and research-to-practice work across the college instructional population. For the 2015 Conference on RUME, the group continued its many-threaded investigations and dissemination about the professional growth of post-secondary mathematics instructors, from novice to veteran. The definition for "novice" varies in mathematics education research. In current reports, a novice is someone whose teaching experience at the college level involves either a limited amount of time or of autonomy. For our purposes, a "novice college mathematics instructor" is a person with fewer than 1000 instructional classroom-contact hours with undergraduates (e.g., about 3 years experience teaching as instructor of record for three courses per semester) or - in terms of autonomy – a novice is someone having experience only as a discussion section leader who does not have responsibility for the design and delivery of syllabus, lesson content, and course grades. Most mathematics graduate student teaching assistants will complete their doctorates and still be "novices" in this sense. Novice instructors play a critical role in undergraduate mathematics education: graduate students are instructor of record for as many as a half-million undergraduates each year in the U.S. (Lutzer, Rodi, Kirkman, & Maxwell, 2007). What is more, Belnap and Allred (2009) found that this figure may underreport novice involvement in mathematics instruction, since 40% of graduate student TAs teach labs or discussion sections, without being the instructor of record. Speer, Murphy, and Gutmann (2009) estimated that more than a third of undergraduates will have a graduate student as a mathematics instructor. Other "novices" involved in instruction outside the classroom can be found among those graduate and advanced undergraduate students who tutor individuals and small groups.

Rationale and Purpose. While researchers in the group are committed to conveying their findings to practitioners, this working group conference forum is focused on research. Though such research may go hand in hand with development of curriculum or activities for those who provide professional development to college mathematics instructors, the RUME meeting is not a venue for disseminating curricular resources. At the same time, the working group is not a place to present mature research results, such as might be appropriate for regular presentation sessions at the RUME conference. Our purpose is to allow for early feedback on research projects, the tuning of design and analysis strategies, and the building of collaborations. Participating in the group and receiving feedback on emergent projects will scaffold work that can be shared later, at future conferences and through grant project development, as well as in research-to-practice dissemination. For some, having a formal way to participate in the conference by being a part of the working group may enable them to secure funds to attend. Guided by the mantra "the purpose of the group is to serve the needs of its members" we have three priorities: (a) to provide critical, informed support and feedback for those conducting research on or about college mathematics instructors; (b) to help mathematics educators interested in the experience of college mathematics instructors and their professional development connect and collaborate; (c) to consider over-arching issues related to individual research agendas and to endeavor to contribute to these common concerns.

History. The working group originated in 2002. From 2002-2007, the group convened at PME-NA and since 2009 the group has convened at the Conference on RUME. From the beginning, a main goal of the group has been to help develop a community of researchers with interests regarding novice instructors of college mathematics. Over the years, we have identified central issues in our area of research and extended the population of interest to include more experienced instructors, with the group primarily serving as a venue for building collaboration and getting constructive feedback on independent research projects. The group's work has provided valuable networking and collaborative opportunities, producing tangible outcomes. In addition to papers in the proceedings of the PME-NA and RUME conferences, some of the organizers published a paper based on the 2002 conference proceedings in College Teaching (Speer, Gutmann, & Murphy, 2005) and more recently the co-chairs published an overview of the current state of research in the field (Deshler, Hauk & Speer, 2015). Many working group members became authors of peer-reviewed papers in a themed volume of Studies in Graduate and Professional Student Development (Border, 2009) – seven of the eight articles are authored by researchers who have participated in the working group (e.g. Belnap & Allred, 2009; Gutman, 2009; Hauk et al., 2009; Kung & Speer, 2009; Latulippe, 2009; Meel, 2009; Speer et al., 2009). Several productive and ongoing research collaborations have developed among the group's participants (e.g., Hauk, Toney, Jackson, Nair, & Tsay, 2014). The group's current set of leaders (authors of this report), come from a variety of backgrounds, have been regular group participants, and have been involved in various related groups (e.g., MAA-AMS Joint Committee on Teaching Assistants and Part-Time Faculty, MAA Committee on Professional Development), have conducted grant-funded research in the area, and have presented at RUME previously. With increasing regularity since 2010, the group has met on-line during the year.

Working Group 2015 RUME Session Summary

This year the online registration for the RUME conference included an option for expressing interest in a pre-conference working group. As a result, many new people found out about the working group. At the working group meeting on February 19 there were 49 attendees (47 in person and 2 by Skype). About half of the attendees were new to the research work of the group. This year's meeting time was used for introdution, design, and research resource development:

Activity 1 (20 minutes). Introductions and an overview of developments in the past year for the group, including upcoming publication of the *AMS Notices* article by group members Deshler, Hauk, and Speer (2015).

Activity 2 (15 mintues). Given the large number of new attendees, we arranged a set of five research capsule reports by working group members about a past or current research project (Jess Ellis, Kitty Roach, Kim Rogers, Mary Beisiegel, Jason Belnap).

Activity 3 – Work in Progress Session 1 (50 minutes). An activity-based research design session. This small group activity provided feedback on study design and defining *technological pedagogical content knowledge* instrument development for “Design Challenge: Web-based Activity and Test System (WATS) – Randomized Controlled Trial of Online Tools in Developmental Algebra Instruction” a newly grant-funded research project that Shandy Hauk shared with the group.

Activity 4 – Work in Progress Session 2 (60 minutes). A set of small group conversations orchestrated by Natasha Speer and Jessica Deshler about the research and development resources to be made available through the web portal for a newly funded project: “Improving the Preparation of Graduate Students to Teach Undergraduate Mathematics” (temporarily known as the TAPD project). The attendees naturally fell into the two categories the new portal is meant to serve: active researchers in CMI development and consumers of that research. The activities focused on Design of the “Researchers’ Corner” and the “Instructional Resources” sections of the site and suggesting potential project short name or acronym.

Activity 5 (45 minutes) Next Steps. Paired and whole group conversations to generate ideas for how the group can continue to serve the needs of its members (now and in the future). This included announcements (e.g., the summer web portal development TAPD meeting planned for June), and identifying potential days/times and topics for the next online Collaborate sessions.

References

Border, 2. L. L. B., Speer, N. & Murphy, T. J. (Eds.)(2009). *Research on graduate students as teachers of undergraduate mathematics.* Studies in graduate and professional student development (Vol. 12). Stillwater, OK: New Forums Press.

- Educational research on mathematics graduate student teaching assistants: A decade of substantial progress, N. Speer, T. J. Murphy, & T. Gutmann.
- Mathematics teaching assistants: Their instructional involvement and preparation opportunities, J. K. Belnap & K. Allred.
- A case story: Reflections on the experiences of a mathematics teaching assistant, S. Hauk, M. Chamberlin, R. D. Cribari, A. B. Judd, R. Deon, A. Tisi, & H. Khakakhail.

- Beginning graduate student teaching assistants talk about mathematics and who can learn mathematics, T. Gutmann.
- Encouraging excellence in teaching mathematics: TAs' descriptions of departmental support, C. Latulippe.
- Planning practices of mathematics teaching assistants: Procedures and resources, D. Winter, M. Delong, & J. Wesley.
- Visions of acculturation: Using case stories to educate international teaching assistants in mathematics, D. Meel.
- Mathematics teaching assistants learning to teach: Recasting early teaching experiences as rich learning opportunities, D. Kung & N. Speer.

Deshler, J. M., Hauk, S. & Speer, N., (2015). Professional Development in Teaching for Mathematics Graduate Students. *Notices of the American Mathematical Society,* 62(6).

Hauk, S., Toney, A., Jackson, B., Nair, R., & Tsay, J.-J. (2014). Developing a model of pedagogical content knowledge for secondary and post-secondary mathematics instruction. *Dialogic Pedagogy: An International Online Journal, 2*, 16-40.

Lutzer, D.J., Rodi, S.B., Kirkman, E. E., & Maxwell, J. W. (2007). *Statistical abstract of undergraduate programs in the mathematical sciences in the United States: Fall 2005 CBMS Survey.* Washington, DC: American Mathematical Society.

Speer, N. M., Gutmann T., & Murphy, T. J. (2005), Mathematics teaching assistant preparation and development. *College Teaching, 52*(2), 75-80.

Research on College Mathematics Instructor Professional Growth – 2015 Participant Report

Facilitators:
Shandy Hauk <shauk@wested.org>,
Jessica Deshler <deshler@math.wvu.edu>,
Natasha Speer <speer@math.umaine.edu>

Participants Indicated in the table below.

	First Name	Last Name	Occupation/ Title	University	Email Address
1	Alison	Lynch	Student	University of Wisconsin-Madison	gordon@math.wisc.edu
2	Allison	Toney	Assistant Professor	U. North Carolina Wilmington	Toneyaf@uncw.edu
3	Alon	Pinto	Postdoc	UC Berkeley	alonp@berkeley.edu
4	Amy	Ellis	Associate Professor	University of Wisconsin-Madison	aellis1@wisc.edu
5	Andrew	Tonge	Professor	Kent State	tonge@math.kent.edu
6	Anna	Titova	Assistant Professor	Becker College	anna.titova@becker.edu
7	Annie	Selden	Adjunct Professor	New Mexico State University	aselden@math.nmsu.edu
8	Ashley	Duncan	Graduate TA	Arizona State University	ashley.duncan.1@asu.edu
9	Beverly	Reed	Associate Professor	Kent State	breed1@kent.edu
10	Christina	Starkey	Doctoral IA	Texas State University	cs1721@txstate.edu
11	Christine	Andrews-Larson	Assistant Professor	Florida State University	cjlarson@fsu.edu
12	Doug	Squire	faculty	West Virginia University	dsquire@math.wvu.edu
13	Edgar	Fuller	Professor and Chair	West Virginia University	ef@math.wvu.edu
14	Estrella	Johnson	Assis. Prof	Virginia Tech	strej@vt.edu
15	Houssein	El Turkey	Assistant Professor	University of New Haven	helturkey@newhaven.edu
16	Hyunkyoung	Yoon	Grad Student	Arizona State University	hyoon14@asu.edu
17	Iwan	Elstak	Professor/Dr	Valdosta State University	irelstak@valdosta.edu
18	Jacqueline	Dewar	Professor Emerita	Loyola Marymount University	jdewar@lmu.edu

19	James	Epperson	Associate Professor	The University of Texas at Arlington	epperson@uta.edu
20	Jason	Belnap	Assistant Professor	University of Wisconsin Oshkosh	belnapj@uwosh.edu
21	Jeffrey	Truman	Graduate Student	Simon Fraser University	jtruman@sfu.ca
22	Jennifer	Kaplan	Assoc Prof	UGA	
23	Jennifer	Tyne	Lecturer	University of Maine	jennifer.tyne@maine.edu
24	Jessica **Facilitator	Deshler	Assistant Professor	West Virginia University	deshler@math.wvu.edu
25	Jessica	Ellis	Asst. Professor	Colorado State University	jess.ellis84@gmail.com
26	Jessica Brooke	Ernest	doctoral candidate/lecturer	San Diego State University	brookeernest@gmail.com
27	John	Selden	Adjunct Professor	New Mexico State University	js9484@usit.net
28	Joshua	Case	Student	University of Maine	joshua.case@maine.edu
29	Kedar	Nepal	Assistant Professor	Mercer	nepal_k@mercer.edu
30	Kimberly	Rogers	Assistant Professor	Bowling Green State University	kcroger@bgsu.edu
31	Kitty	Roach	Grad Student	University of Northern Colorado	kitty.roach@unco.edu
32	Kristin	Noblet	Graduate Student	University of Northern Colorado	kristin.noblet@gmail.com
33	Kyeong Hah	Roh	Associate Professor	Arizona State University	khroh@asu.edu
34	Mary	Beisiegel	Assistant Professor	Oregon State University	mary.beisiegel@oregonstate.edu
35	Nadia	Hardy	Associate Professor	Concordia University	nadia.hardy@concordia.ca
36	Natasha **Facilitator	Speer	Associate Professor	The University of Maine	speer@math.umaine.edu
37	Nathan	Clements	Calculus Coordinator	University of Wyoming	nathan.clements@uwyo.edu
38	Nathan	Wakefield	Professor	University of Nebraska - Lincoln	nathan.wakefield@unl.edu
39	Nissa	Yestness	Postdoctoral Researcher	Colorado State University	Nissa.yestness@colostate.edu
40	Paul	Regier	GRA	The University of Oklahoma	paulrregier@gmail.com

41	Sarah	Bleiler	Assistant Professor	Middle Tennessee State University	sarah.bleiler@mtsu.edu
42	Shahram	Firouzan	Student	San Diego State University	sfirouzi@ucsd.edu
43	Shandy **Facilitator	Hauk	Researcher	WestEd/U. Northern Colorado	shauk@wested.org
44	Silvia	Saccon	Dean's Fellow	The University of Texas at Dallas	Silvia.Saccon@utdallas.edu
45	Sonalee	Bhattacharyya	Doctoral Teaching Assistant	Texas State University	sb1212@txstate.edu
46	Sue	Jong		Minnesota State University	
47	Teri Jo	Murphy	Program Officer	NSF	
48	Valerie	Kasper	Research Assistant	Florida State University	vpk05@my.fsu.edu
49	Younhee	Lee	Graduate student	The Pennsylvania State University	yul182@psu.edu

Undergraduate students reading and using mathematical definitions: Generating examples, constructing proofs, and responding to true/false statements

Valeria Aguirre Holguín
New Mexico State University

University students often encounter difficulties making correct use of definitions. This is partly because they are not familiar with the difference between dictionary definitions and mathematical definitions (Edwards & Ward, 2004). However, in order to succeed in upper-level mathematics courses, they must often construct original (to them) mathematical proofs, which intrinsically require the correct use of definitions. Only a little research has been conducted to discover how university students handle definitions new to them (e.g., Dahlberg & Housman, 1997). Our research question is: How do university students use definitions, to evaluate and justify examples and non-examples, to prove results, and to evaluate and justify true/false statements? Data were collected through individual task-based interviews with 23 volunteer students from a transition-to-proof course. There were five definitions but each student considered only one of the five. Content analysis and grounded theory are being used for analysis. Preliminary results are presented.

Introduction and research questions

The role of definitions in mathematics is fundamental. Such a statement might sound obvious and almost natural for mathematicians. Professors apparently expect students to be able to grasp definitions and then proceed to do something with them (Alcock, 2010). Nevertheless, "Many students do not categorize mathematical definitions the way mathematicians do; many students do not use definitions the way mathematicians do, even when the students can correctly state and explain the definitions; many students do not use definitions the way mathematicians do, even in the apparent absence of any other course of action." (Edwards & Ward, 2004).

This study is aiming to shed some light on how undergraduate students proceed when they are presented with new (to them) mathematical definitions. We have narrowed our questions to the following:

- How do students make use of mathematical definitions new to them? What are their natural reactions?
- What contributes to their difficulties with "unpacking" and using abstract mathematical definitions?
- How do they use a definition in three different settings: examples, proofs, and true/false questions?

This research falls within the scope of a framework developed by Selden and Selden (1995, 2008, 2009). I particularly focus on the part of the framework referring to operable interpretations of statements and formal and informal forms of a mathematical statement.

Literature Review

Research has revealed that students have a variety of difficulties understanding and using definitions (Furina, 1994; Zazkis & Leikin, 2008; Fernandez, 2004; Roh, 2008), many of which could be attributed to the "structure of mathematics as conceived by mathematicians and the cognitive processes involved in concept acquisition" (Vinner, 1991). Pierie and Kieren (1994) developed a nested model of understanding. Their theory suggests that understanding is not always a linear or continuous process, rather learners move back and forth within various levels of understanding.

According to Tall (1980), “concept image is regarded as the cognitive structure consisting of the mental picture and the properties and processes associated with the concept…Quite distinct from the complex structure of the concept image is the concept definition which is the form of words used to describe the concept.” A mismatch between the concept image developed by an individual and the actual implications of the concept definition often leads to obstacles in learning. The work of several researchers has confirmed this (Alcock & Simpson 2002a, 2002b; Wihelmi, Godino & Lacasta, 2007; Zaslavsky & Shir, 2005).

Parameswaran (2010) has addressed how mathematicians approach new definitions; her research shows that examples and non-examples play a very important role in the process of learning a new definition. However, students are not frequently asked to generate examples, most of the time they are provided with a worked-out example or an illustration (Reimann & Schult, 1996; Watson & Mason, 2002). It is also infrequent for students to discuss and analyze the reason why a proposed example is actually an example or a non-example. “The relationship between examples, pedagogy and learning is under-researched, but it is known that learners can make inappropriate generalizations from sets of examples” (Bills, Dreyfus, Mason, Tsamir, Watson & Zaslavsky, 2006). Watson and Mason introduced the concept of example spaces as sets of examples developed according to one’s previous experience and to accomplish certain objectives (Watson & Mason, 2005). Zazkis and Leikin (2007) observed the distinction between examples of mathematical concepts and examples of mathematical tasks; my research involves the former.

Although there are some studies addressing students’ and mathematicians’ use of definitions in the construction of proofs, there seems to be a lot more to investigate in respect to students’ perceptions of mathematical definitions, particularly about the competencies needed for them to make correct use of such definitions.

Methodology

I designed and conducted a series of semi-structured task-based interviews with voluntary participants taking a transition-to-proof course during Fall 2013. An interview protocol was designed for each of five selected definitions, but all the interviews followed the same format. These definitions were: function, continuity, semigroup, ideal, isomorphism, and group, spanning most of the course. I interviewed 23 volunteer students, which was almost the entire cohort. The interviews were conducted individually in a small seminar room. For each definition, 4-5 students were interviewed approximately two weeks before that particular definition came up in the course. The interviews were audio-recorded, and the students used LiveScribe pens in order that their real-time responses could be analyzed.

Based on my research interests and with the support of the existing literature, I wanted the interviews to address the following four main points. First, I wanted the students to be presented with a definition for the first time, that is, they were interviewed about a definition that they had not yet seen in class. Second, I also wanted to test their ability to interpret the definition, so I asked the participants if they were able to come up with some examples that could illustrate the definition. Third, I looked for information on their ability to make use of the definition in the construction of a proof that required no more than the definition itself. And fourth, I was interested in the way they could reason about true/false statements involving the definition. These four points were addressed by the design of five handouts, presented one after the other to each student individually in a 60 to 90 minute-interview. This design was partly inspired by the work of Dahlberg and Housman (1997) and the work of Housman and Porter (2003).

Students were also allowed to use their course notes during the interview, and the handouts already given to them. I was an assistant in the course, so the interaction between the participants and me was naturalistic, more than an interview it could be seen as a conversation about a particular mathematical definition. I kept track of the possible divergent paths followed by the participants.

The first analysis of the data was done considering each of the five definitions separately. I concentrated on only one definition at a time, analyzing all the data on the five handouts for the four to five respective students individually considering that definition. The second analysis was done by handout. I looked at the general performance across all handouts, considering one handout at time, across all participants.

Preliminary results

Some results that have emerged from the analyses are confirming how sturdy one's previous knowledge can be when trying to understand new definitions. Such knowledge can strongly influence, not necessarily in a beneficial way, the acquisition of new concepts. I found supporting evidence suggesting that the newer the definition to the student, and the less related to everyday language, the less was the interference of inappropriate previous knowledge. Function and continuity, for instance, are related to Calculus and Algebra classes taken previously, whereas ideal and isomorphism did not bring any previous mathematical knowledge to mind. Interestingly, the concept of group brought to mind the everyday usage of this word. I also observed that students tended to neglect the details of a definition, if not the complete definition, in constructing a proof, and were not fully aware that these details are provided for a reason. Very few of them seemed to try hard to follow what the definition states. Another observation is that students were initially reluctant to provide examples of a newly provided definition, but if I probed a little further and provided time, students were often able to provide an example, or at least they came to realize that their example or their ideas were inappropriate. Therefore my observations suggest that the newer the concept (to them), the harder to provide examples or non-examples but also the newer the concept the less the interference of inappropriate previous knowledge.

From my observations I have developed a preliminary conjecture on the different stages a student goes through in order to learn to correctly use mathematical definitions in a given context (which in this case is to construct a proof of a given statement). There seemed to be four stages: (1) Understand there is a difference between dictionary/everyday definitions and mathematical definitions (as Edwards and Ward (2004) suggested). (2) Understand when, and where, to use mathematical definitions. (3) Recall, look for, and attempt to use/follow definitions, not necessarily with success. (4) Use the definition successfully (within the given context).

Some possible questions for the audience

- Are there any other interesting behaviors we should account for when observing students working with definitions new to them?
- Thinking of designing a future intervention, how should definitions be taught in such courses? Should we provide students with more opportunities to grasp definitions before they have to use them?

References

Alcock, L. (2010). Mathematicians' perspectives on the teaching and learning of proof. *Research in collegiate mathematics education VII*, 63-91.

Alcock, L., & Simpson, A. (2002a). Definitions: Dealing with categories mathematically. *For the Learning of Mathematics*, *22*(2), 28-34.

Alcock, L. J., & Simpson, A. P. (2002b). Two components in learning to reason using definitions. In *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the Undergraduate Level)*.

Bills, L., Dreyfus, T., Mason, J., Tsamir, P., Watson, A., & Zaslavsky, O. (2006). Exemplification in mathematics education. In *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* 1, 126-154.

Dahlberg, R. P., & Housman, D. L. (1997). Facilitating learning events through example generation. *Educational Studies in Mathematics*, *33*(3), 283-299.

Edwards, B. S., & Ward, M. B. (2004). Surprises from mathematics education research: Student (mis) use of mathematical definitions. *The American Mathematical Monthly*, *111*(5), 411-424.

Fernández, E. (2004). The students' take on the epsilon-delta definition of a limit. *Problems, Resources, and Issues in Mathematics Undergraduate Studies*, *14*(1), 43-54.

Furina, G. (1994). Personal reconstruction of concept definitions: Limits. Challenges in mathematics education: constraints on construction: *Proceedings of the 17th Annual Conference of the Mathematics Education Research Group of Australasia held at Southern Cross University.*

Housman, D., & Porter, M. (2003). Proof schemes and learning strategies of above-average mathematics students. *Educational Studies in Mathematics*, *53*(2), 139-158.

Parameswaran, R. (2010). Expert mathematicians' approach to understanding definitions. *The Mathematics Educator*, *20*(1), 43-51.

Reimann P., & Schult, T. (1996). Turning examples into cases: Acquiring knowledge structures for analogical problem-solving. *Educational Psychologist*, *31*(2), 123-140.

Roh, K. H. (2008). Students' images and their understanding of definitions of the limit of a sequence. *Educational Studies in Mathematics*, *69*(3), 217-233.

Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*, *29*, 123-151.

Selden, J. & Selden, A., (2008) Consciousness in enacting procedural knowledge. In *Proceedings of the Conference on Research in Undergraduate Mathematics Education* (2008). Available at http://cresmet.asu.edu/crume2008/Proceedings/Proceedings.html.

Selden, J., & Selden, A. (2009a). Teaching proving by coordinating aspects of proofs with students' abilities. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and learning proof across the grades: A K-16 perspective (pp. 339-354)* . Routledge /Taylor & Francis: New York and London.

Selden, J., & Selden, A. (2009b). Understanding the proof construction process. In F.-L. Lin, F.-J. Hsieh, G. Hanna, & M. de Villiers (Eds.), *Proceedings of the ICMI 19 Study Conference: Proof and Proving in Mathematics Education, Vol 2* (pp. 196- 201). Taipei, Taiwan: Department of Mathematics, National Taiwan Normal University.

Tall, D., & Vinner, S. (1981). Concept image and concept definition with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*, 151-169.

Tall, D. O. (1980). *Mathematical intuition with special reference to limiting processes*. Paper presented at the meeting of the Fourth International Conference for the Psychology of Mathematical Education, Berkeley, CA.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced Mathematical Thinking,* 65-81. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Watson, A., & Mason, J. (2002). Student-generated examples in the learning of mathematics. *Canadian Journal of Science, Mathematics and Technology Education, 2*(2), 237-249.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples*. Mahwah, NJ, USA: Erlbaum.

Wilhelmi, M., Godino, J., & Lacasta, E. (2007). Didactic effectiveness of mathematical definitions: The case of the absolute value. *International Electronic Journal of Mathematics Education*, *2*(2), 72-90.

Zaslavsky, O., & Shir, K. (2005). Students' conceptions of a mathematical definition. *Journal for Research in Mathematics Education*, *36*(4), 317-346.

Zazkis, R., & Leikin, R. (2007). Generating examples: From pedagogical tool to a research tool. *For the Learning of Mathematics*, *27*(2), 15.

Zazkis, R., & Leikin, R. (2008). Exemplifying definitions: A case of a square. *Educational Studies in Mathematics*, *69*(2), 131-148.

Social networks among communities of undergraduate mathematics instructors at PhD-granting institutions

Naneh Apkarian
San Diego State University

Calculus is typically the first undergraduate mathematics course for science, technology, engineering, and mathematics (STEM) majors in the United States. Internationally as well as domestically, first year mathematics courses are credited with preventing students from continuing along STEM paths. A recent study of the features that characterize exemplary calculus programs from five PhD-granting institutions highlighted several common characteristics, one of which was the existence of a well-established system for coordinating Calculus I. This coordination of courses and instructors seems to engender a community of practice. This study aims to expand on this finding by leveraging social network theory to map the underlying structure of the social ties between instructors of lower-division undergraduate mathematics courses, to compare informal and formal organizational structures in each case, and to compare the communities across the five selected institutions. Here I report on the results from one of the five selected institutions.

Key words: Social Network Analysis, Communities of Practice, Calculus, Social Capital

Calculus is typically the first undergraduate mathematics course for science, technology, engineering, and mathematics (STEM) majors in the United States. Internationally as well as domestically, first year mathematics courses are credited with preventing students from continuing along STEM paths (Seymour & Hewitt, 1997) a fact which has led to increased research and attention by professional societies. Complicating an understanding of the situation in Calculus I is the fact that most PhD-granting universities offer many sections of calculus each semester. These sections tend to be taught by a wide range of instructors including visiting faculty, postdocs, adjunct lecturers, graduate students, as well as ladder rank faculty. The tremendous variation in who is teaching calculus makes for a situation where different students taking calculus the same semester at the same university may not be taught the same core material. This is particularly problematic for calculus since it is a fundamental prerequisite for subsequent STEM courses. Moreover, the quality of instruction may vary considerably, which can affect what students actually learn, even if the same content is being covered and assessed.

As part of a large national study of Calculus I programs, the Characteristics of Successful Calculus Programs (CSPCC) conducted case studies at five PhD-granting institutions selected for having a relatively more successful Calculus I program. At each of the five selected institutions, there was a central individual, the calculus Coordinator, who organized and led the enactment of the uniform aspects of calculus instruction. While the background of the Coordinator varied, what was common among the five Coordinators was their disposition toward their role. Each of the five Coordinators viewed themselves as a resource and facilitator rather than as the owner of the calculus program or the authority on how to teach. This finding suggests that further research needs to examine the extent to which faculty involved in teaching the calculus sequence communicate and interact with each other. Accordingly, the study reported is a first step in addressing the following goals:

1. To map and characterize the social network (informal structure) that exists among the actors within each community.
2. To compare the informal structure derived from the social network with the formal structure of the departmental hierarchy.
3. To compare and contrast the social networks across the five selected institutions.

The first goal aims to fill a gap in the research literature. Social network analysis admits quantitative measures to the description of a community of practitioners in a way that has not been seen at the undergraduate level, although it has been used extensively with K-12 communities (Daly, 2010). Such analyses will enable us to say more about *how* these communities support the successful calculus programs at the selected institutions. The second goal, comparing formal and informal structures, will be used to determine whether the instructors actually interact in the ways implied by the case studies. Specifically, I wish to discover whether the Coordinators are truly central actors who function as hubs for the dissemination of social capital, and if they are other brokers in the community. The third goal, which is beyond the scope of the current analysis, aims to describe differences between the communities at each of the five selected institutions.

Theoretical Perspective

This research is grounded in two complementary perspectives, the first of which draws on the *community of practice* perspective put forth by Wenger and colleagues (Lave & Wenger, 1991; Wenger, 2000). A community of practice is a collective construct in which the joint enterprise of achieving particular goals evolves and is sustained within the social connections of that particular group. In achieving a particular joint enterprise, such as the teaching and learning of calculus, a community of practice point of view highlights the role of brokers and boundary objects. A broker is someone who has membership status in more than one community and is in a position to infuse some element of one practice into another. The act of doing so is referred to as brokering (Wenger, 2000).

The community of practice perspective is well aligned with the perspective of *social capital theory*. This theory, which places value on social connections, has been leveraged in a wide variety of contexts, informing studies of "families, youth behavior problems, schooling and education, public health, community life, democracy and governance, economic development, and general problems of collective action" (Adler & Kwon, 2002, p. 17). The concept of social capital has also gained traction in organization studies, and it is in this area that our contribution falls. As social capital has been used in a wide variety of concepts, it has been conceptualized of in a wide variety of ways. However, common to all definitions is the notion that social capital consists of "resources embedded in social relations and social structure, which can be mobilized when an actor wishes to increase the likelihood of success in purposive actions" (Lin, 2002, p. 24). In a sense, social capital refers to the human capital that an actor can access through his or her social ties. In some cases, central actors, referred to as *hubs*, facilitate the flow of capital between otherwise unconnected actors. In these cases the hub functions in a way similar to that of a broker. I believe that the overlap in characteristics will help to identify potential brokers in the observed networks.

Methods

Following up from the CSPCC study, data collection for this study has commenced at the five selected institutions identified as having more successful Calculus I programs, with all data collection to be completed this spring. Data from one institution has been collected at the level required for network analysis (Daly, 2010). Social network surveys are being distributed to individuals at the selected institutions who have recently taught lower-division undergraduate courses, including Pre-Calculus, Calculus I, II, III, Linear Algebra, and Differential Equations. Network questions are used to ascertain the ties that exist between members of the community of calculus instructors, as well as the strength of those ties, and a variety of Likert scale and demographic questions are being used to characterize the actors between whom ties do or do not exist (Coburn & Russell, 2008).

Since this study aims to map the social network of a community of practice, I embarked on a *whole network analysis*. This type of analysis is performed by selecting a set of actors and measuring the ties between them. The standard approach for whole network analysis is to collect information regarding a few types of ties between many pairs of nodes (Daly, 2010). This study encompasses two types of group level analyses, those concerning *network structuring* and *group social capital*. That is, I am looking both to determine the structure of these communities and how they compare, as well as to see how social capital flows through each network (Daly, 2010).

In this case, the actors selected are instructors of lower-division undergraduate mathematics courses, gleaned from course catalogs, as well as all members of the department who have administrative roles relating to undergraduate students and courses. The CSPCC results hinted that a community of practice might exist within this larger community. The network ties being measured in this survey relate to advice, influence, and friendship. The survey also includes Likert scales designed to characterize the individuals, subgroups, and the larger community in terms of trust, innovative climate, professional learning community collaboration and involvement, as well as mathematical affect and beliefs. The general design of the study has been used widely, with success, for this type of analysis, though not among this type of community (Adler & Kwon, 2002; Coburn & Russell, 2008; Daly, 2010; Tichy, Tushman, & Fombrun, 1979). The questions themselves have also been adapted from the K-12 literature, reworded to reflect the difference in the institution type (Antonakis, Avolio, & Sivasubramaniam, 2003; Daly, Der-Martirosian, Moolenaar, & Liou, 2014; Daly, Moolenaar, Bolivar, & Burke, 2010; Daly, 2010; Moolenaar, 2012).

Preliminary Results

Based on the case study analyses from the CSPCC study, which identified coordination as a key feature across the five selected institutions, Rasmussen and Ellis (2014) argued that an important part of the story is the role that calculus Coordinator, among others, plays in creating and sustaining a community of practice around the joint enterprise of teaching and learning of calculus. In other words, the conjecture is that calculus is *not* seen as being under the purview of one person, such as the Coordinator, but rather that at these institutions, calculus is viewed as community property.

To explore this conjecture, I have begun to analyze the social networks that exist within the posited communities of practice at one of the selected institutions, using *social network analysis* methodology. The data collection for this project is ongoing, in part because high response rates are required for conclusive social network analysis (Daly, 2010). Toward research goal 1, as data is collected each participant becomes a node on a graph, and each connection becomes an edge. The frequency of communication, as well as the variety of connections between two actors, are used to weight these edges. Graph theoretic approaches can then be leveraged to analyze network density and centrality, in order to characterize the community as a whole. It is also possible to identify central actors in the network by locating hubs, which will allow the identification of those members of the community who act ask brokers. Of further interest are any existent subgroups within the community, located by identifying cliques in the graph. At the first institution, we already see the emergence of subgroups characterized by experience level.

Data collection at the first institution to be investigated has begun to yield hints for research goal 2. It appears that the hypotheses from Phases I and II regarding Coordinators are being supported. Preliminary analysis reveals that the official calculus Coordinator is in fact a central actor in the network, a main conduit for social capital, and therefore appears to be a hub – matching his formal job description.

Questions for Audience

1. To what extent might the general culture of mathematics departments foster or inhibit the existence of social networks revolving around issues of teaching and learning?
2. This study analyzes the existence and structure of social networks, but does not provide insight into *how* existing social networks came to be. What follow-up studies are needed to address this goal?
3. In what ways do these social networks provide informal professional development opportunities for mathematicians?

References

Adler, P. S., & Kwon, S.-W. (2002). Social Capital: Prospects for a new concept. *The Academy of Management Review*, *27*(1), 17–40.

Antonakis, J., Avolio, B. J., & Sivasubramaniam, N. (2003). Context and leadership: an examination of the nine-factor full-range leadership theory using the Multifactor Leadership Questionnaire. *The Leadership Quarterly*, *14*(3), 261–295. doi:10.1016/S1048-9843(03)00030-4

Coburn, C. E., & Russell, J. L. (2008). District policy and teachers' social networks. *Educational Evaluation and Policy Analysis*, *30*(3), 203–235.

Daly, A. J. (Ed.). (2010). *Social Network Theory and Educational Change*. Cambridge, MA: Harvard Education Press.

Daly, A. J., Der-Martirosian, C., Moolenaar, N. M., & Liou, Y.-H. (2014). Accessing capital resources: Investigating the effects of teacher human and social capital on student achievement. *Teachers College Record*, *116*(7).

Daly, A. J., Moolenaar, N. M., Bolivar, J. M., & Burke, P. (2010). Relationships in reform: The role of teachers' social networks. *Journal of Educational Administration*, *48*(3), 359–391.

Lave, J., & Wenger, E. (1991). *Situated Learning: Legitimate Peripheral Participation* (1st edition.). Cambridge England ; New York: Cambridge University Press.

Lin, N. (2002). *Social Capital: A Theory of Social Structure and Action*. Cambridge; New York: Cambridge University Press.

Moolenaar, N. M. (2012). A social network perspective on teacher collaboration in schools: Theory, methodology, and applications. *American Journal of Education*, *119*(1), 7–39.

Rasmussen, C., & Ellis, J. (2014). *Calculus coordination at PhD-granting universities: More than just using the same syllabus, textbook, and final exam.*

Seymour, E., & Hewitt, N. M. (1997). *Talking About Leaving: Why Undergraduates Leave The Sciences*. Boulder, CO: Westview Press.

Tichy, N. M., Tushman, M. L., & Fombrun, C. (1979). Social network analysis for organizations. *The Academy of Management Review*, *4*(4), 507–519.

Wenger, E. (2000). *Communities of Practice: Learning, Meaning, and Identity* (1 edition.). Cambridge, U.K.; New York, N.Y.: Cambridge University Press.

The Multiple Representations of the Group Concept

Annie Bergman, Portland State University
Kathleen Melhuish, Portland State University
Dana Kirin, Portland State University

This poster will explore the various representations of groups found within introductory abstract algebra textbooks. Representations play an essential role in students understanding of mathematics (Goldin, 2002). Textbooks provide one source for analyzing the intended curriculum and what representations students may have access to within their introductory course.

Key words: Abstract Algebra, Group theory, Textbook Analysis, Representations

As part of a larger study aimed to develop a validated assessment in introductory group theory, we conducted an in-depth textbook analysis of the most frequently used introductory abstract algebra texts. The texts where identified through a random sample of 294 institutions of the 1,244 schools offering mathematics majors. In schools where the textbook is not uniform, the textbook most recently utilized was included. Any textbook used by at least 20 schools was included for analysis. (This number was eventually lowered to include the 4th most popular textbook.)

In order to answer the question, what representations are being used in introductory group theory textbooks to express a group, we began with Mesa's framework for identifying representations of functions including symbolic, verbal, tabular, and other representations. The research team used a thematic analysis (Braun & Clarke, 2006) approach to address additional representations that emerged within the group theory context.

After analyzing both the narrative and exercises for all sections relevant to introductory group theory, we had identified 11 different representations for the group topic. The three most frequently used representations were documented over a hundred times each, Group Name (297), Verbal Description (128), and List of Elements (107). The poster will illustrate various representations, their respective frequencies, and potential nuanced relationships. The table below provides an example of the Klein four-group in each of the representations.

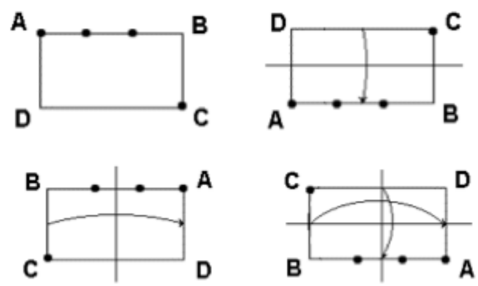
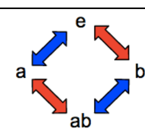
Mesa argues, while what students learn from textbooks is mediated by the school context (teacher, peers, instruction, assignments), the textbook provides a source of potential learning (Mesa, 2004, p. 1). Rich representations can be an asset to deep conceptual understanding. Our initial textbook analysis indicated a huge variety of representations available for reasoning about groups. However, textbook analysis provides a starting point. Further research is needed to explore the role of representations in the enacted curriculum and how they may help (or even potentially hinder) student understanding.

References

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*(2), 77–101.

Goldin, G. A. (2002). Representation in mathematical learning and problem solving. *Handbook of International Research in Mathematics Education*, 197–218.

Mesa, V. (2004). Characterizing practices associated with functions in middle school textbooks: An empirical approach. *Educational Studies in Mathematics*, *56*(2-3), 255–286.

<table>
<tr><th>Type of Representation</th><th>Example</th></tr>
<tr><td>Group Name</td><td>The Klein four-group or $\mathbb{Z}_2 \times \mathbb{Z}_2$</td></tr>
<tr><td>Set Builder Notation</td><td>$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(a,b) | a \in \mathbb{Z}_2, b \in \mathbb{Z}_2\}$</td></tr>
<tr><td>Group Presentation (generators)</td><td>$\mathbb{Z}_2 \times \mathbb{Z}_2 =< a, b | a^2 = b^2 = e, ab = ba >$</td></tr>
<tr><td>Cayley Table</td><td>
<table>
<tr><th>*</th><th>1</th><th>2</th><th>3</th><th>4</th></tr>
<tr><th>1</th><td>1</td><td>2</td><td>3</td><td>4</td></tr>
<tr><th>2</th><td>2</td><td>1</td><td>4</td><td>3</td></tr>
<tr><th>3</th><td>3</td><td>4</td><td>1</td><td>2</td></tr>
<tr><th>4</th><td>4</td><td>3</td><td>2</td><td>1</td></tr>
</table>
</td></tr>
<tr><td>List of Elements</td><td>$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0), (0,1), (1,0), (1,1)\}$</td></tr>
<tr><td>Verbal Description</td><td>The group of order 4 where each element is his own inverse.</td></tr>
<tr><td>Geometric Image</td><td>
</td></tr>
<tr><td>Left Regular</td><td>$T_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}, T_2 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{bmatrix}$
$T_3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 3 \end{bmatrix}, T_4 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$</td></tr>
<tr><td>Cayley Diagraph</td><td>
</td></tr>
<tr><td>Permutation</td><td>$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(1234), (12)(34), (13)(24), (14)(23)\}$</td></tr>
<tr><td>Matrices</td><td>$I = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, A = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}$
$B = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}, D = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}$</td></tr>
</table>

Roles of proof in an undergraduate inquiry-based transition to proof course

Sarah K. Bleiler
Middle Tennessee State University

Jeffrey D. Pair
Middle Tennessee State University

De Villiers (1990) suggested five roles of proof important in the professional mathematics community that may also serve to meaningfully engage students in learning proof: verification, explanation, systematization, discovery, and communication. We investigate written reflections on an end-of-semester assignment from undergraduates in an inquiry-based transition to proof course, where students reflected on instances during the semester when they engaged in the five roles of proof. We present (a) student rankings of role engagement, (b) the types of activities students recalled as influential to their engagement in the roles of proof, and (c) how students perceived they engaged in the five roles. Students in this course reflected on activities distinctive of the inquiry-based environment (such as discussing, presenting, conjecturing, and critiquing) as influential to their engagement in the roles of proof. We provide student quotations highlighting these activities and offer implications for both research and practice.

Key words: Introduction to Proof, Student Perceptions, Inquiry-Based-Learning

Proving is a central activity to the discipline of mathematics as it is the means by which mathematicians develop and communicate knowledge (Hemmi, 2010; Schoenfeld, 2009). Despite educational recommendations that students learn to prove (National Council of Teachers of Mathematics [NCTM], 2000; Conference Board of Mathematical Sciences [CBMS], 2012), proof is still largely absent from K-12 classrooms (Stylianou, Blanton, & Knuth, 2009). Even university mathematics majors report varied, and often limited, experiences in terms of their exposure to proof at the undergraduate level (Boyle, Bleiler, Ko, & Yee, under review). Additionally, undergraduate mathematics majors have difficulty constructing and validating proofs (Bleiler, Thompson, & Krajcevski, 2014; Selden & Selden, 2003; Weber, 2001). De Villiers (1990) noted that while proof serves a variety of important functions for mathematicians, students often do not see a need for proof. He posits that this may be due to an overemphasis on one function of proof in the classroom, namely, verification. While a central function of proof in the mathematics community is verification (i.e., obtaining conviction that a mathematical statement is true) students may already be convinced that instructor-selected theorems are true.

De Villiers argued that in order for students to learn proof in a meaningful way, additional roles of proof must be considered. This notion was echoed by CBMS (2012) in their recommendations for the preparation of undergraduate prospective mathematics teachers, "teachers must know that proof and deduction are used not only to convince but also to solve problems and gain insights" (p. 59). De Villiers suggested five roles of proof important in the professional mathematics community that may also serve to meaningfully engage students in learning proof: verification, explanation, systematization, discovery, and communication. De Villiers posits that students may better understand the need for proof if they understand and are able to engage in these additional roles.

Knuth (2002) and Hemmi (2010) argued that more research is needed to understand how the various roles of proof could be incorporated into mathematics instruction. Recent research has demonstrated the benefits of active-learning in undergraduate classrooms (Freeman et al., 2014). In mathematics, active inquiry-based learning (IBL) (Yoshinobu & Jones, 2012) approaches are often constituted by a focus on discourse and student-to-student interaction in the classroom (e.g., Stein, Engle, Smith, & Hughes, 2008). Some researchers (e.g. Blanton & Stylianou, 2014; Smith, 2006) have demonstrated the benefits of approaches that allow for student discourse in learning mathematical proof. However, no research has been conducted into the opportunities students have to engage in the different roles of proof during IBL instruction. An investigation into those opportunities is important so the field can gain greater insight into the ways IBL instruction may or may not support students' engagement in the discipline of mathematics as practiced by mathematicians. In this study, we seek to contribute to this research area by answering the following research questions: **RQ1.** Which of the five roles/functions of proof do students in an IBL proofs course perceive they engaged in the most? The least? **RQ2.** During which types of activities do students in an IBL proofs course recall engaging in the five roles/functions of proof? **RQ3.** How do students perceive that the activities they identify engage them in the five roles/functions of proof?

Theoretical Framework

The theory of situated learning (Lave & Wenger, 1991; Wenger, 1998) frames our study. Under this perspective, mathematical learning occurs when students have the opportunity to participate in the legitimate practices of the mathematical community. Proof is an activity central to the discipline of mathematics (Hemmi, 2010; Schoenfeld, 2009), and the following five roles of proof provide us with further insight into how mathematicians use proof (de Villiers, 1990). A mathematician engages in **verification** when a proof convinces the mathematician of the truth of a mathematical statement. The reason why the mathematical statement is true may be illuminated as a mathematician engages in the **explanation** role of proof. **Systematization** refers to proof's role in organizing and creating a deductive system of axioms, definitions, and theorems. A mathematician engaged in **discovery** may deduce an unanticipated result during the completion of a proof. Proof also provides a means for **communication** among mathematicians as they transmit mathematical knowledge. We use these roles of proof as a lens to qualitatively understand the different ways in which students engage with proof in an IBL classroom.

Methodology

Setting

The 13 undergraduate student participants (8 seniors, 2 juniors, 3 sophomores) in this research study were all enrolled in a single section of "Foundations of Higher Mathematics," which serves as the transition to proof course at a large southeastern U.S. institution. Nine were mathematics majors (seven of whom were prospective secondary mathematics teachers); the remaining four were mathematics minors. In the university catalog, the course describes a class that introduces students to set theory, proof, the language of mathematics, number systems, mathematical structures, and relations and functions. The overall goal of the course is to aid in students' transition from lower- to upper-level undergraduate mathematics courses, with a focus on proof/reasoning, problem solving, and informal and technical writing.

The instructor (first author) designed the course in an effort to achieve the aims in the university course description. The instructor also added some additional objectives that were stated on the course syllabus (see Table 1). Particularly relevant to this research is the objective "[Students will] gain an appreciation of the many roles of proof and reasoning in the discipline of mathematics (e.g., verification, explanation, systematization, discovery, communication)." Although gaining an appreciation of these five roles of proof was an objective of the course and was an implicit focus of instruction throughout the semester, students were not explicitly introduced to the terms describing the five roles until the last week of the semester.

Table 1. Course objectives for IBL Foundations of Higher Mathematics Course

Students will…
Develop a deeper understanding of key concepts from prior mathematics courses through the informal and formal justification of mathematical propositions.
Develop knowledge of new mathematics content, including concepts related to set theory, relations and functions, number systems, and mathematical structures.
Develop knowledge of the working practices of professional mathematicians, including conventions of language and writing.
Gain an appreciation of the many roles of proof and reasoning in the discipline of mathematics (e.g., verification, explanation, systematization, discovery, communication) and come to understand proof as inextricably connected to community norms and expectations in a particular socio-historical context.
Be able to think about mathematics in flexible ways (using different representations of mathematical objects) to develop intuitive and oftentimes informal arguments around a mathematics concept.
Be able to translate informal arguments into formal written proofs.
Be able to describe what constitutes (and what does not constitute) a valid proof, according to the proof-writing expectations defined by our classroom community (and drawing on the community expectations of professional mathematicians).
Be able to read mathematical arguments, provide constructive critiques of arguments, and revise/refine arguments to better align with the proof-writing expectations defined by our classroom community.
Be able to construct valid arguments using varying modes of argumentation (e.g., direct proof, indirect proof, proof by cases, mathematical induction, construction of counterexamples) and varying modes of argument representation (e.g., symbolic, pictorial, narrative).
Recognize the importance of inductive reasoning (e.g, for conjecture formation) and the importance of deductive reasoning (e.g., for systematization and formal communication of mathematics), and be able to move fluently between inductive and deductive reasoning to formulate mathematical proofs.

This IBL course was structured so that students worked individually on problems outside of class, and then during class they worked collaboratively to either solve new problems or refine proofs to problems they had already worked individually. The instructor served as a facilitator of class activities and discussions, and used little direct instruction or lecture. The course differed from what one might expect in a Modified Moore Method (MMM) classroom (Coppin, Mahavier, May, and Parker, 2009). In a typical MMM classroom, students individually work on their problem sets, and then take turns presenting their individual results to the whole class throughout the majority of the class period. Alternatively, in this IBL course, students rarely presented individual work to the whole class. Instead, students discussed their individual work in small-groups, and then spent a large amount of time in class refining their ideas with their teammates so that the small group could present their co-constructed ideas to the whole class.

Data Collection

Students in this class had a two-part final exam. The first part was an in-class content-based exam (worth 80% of final exam grade) and the second part was an at-home reading/reflection assignment (worth 20% of final exam grade). The at-home assignment constitutes the primary data source for this research study. For this assignment, students read de Villiers' (1990) paper, summarized each of the five roles in their own words, described events related to the class in which they recalled engaging in each of the five roles, and ranked the five roles of proof according to perceived level of engagement. See Figure 1 for the full directions provided to students for this assignment. Data in the form of classroom videos was also collected every class meeting. We consult this video data as a secondary source for our analysis, to inform and clarify the descriptions of activities that engaged students in the five roles of proof.

Data Analysis

We conducted a mixed methods analysis of students' responses on their reading/reflective assignment, where they ranked the five roles of proof in terms of level of engagement (quantitative analysis/RQ1) and recollected an event in which they engaged in each of the five roles (qualitative analysis/RQ2 and RQ3). Here, we describe the data analysis process that we used to analyze responses for all five roles of proof.

> Read the article by de Villiers (1990) titled, "The Role and Function of Proof in Mathematics." In this article you will find five roles/functions of proof that are discussed, namely, (1) verification, (2) explanation, (3) systematization, (4) discovery, and (5) communication. After reading carefully through the article, write 2-3 sentences to describe in your own words each of the five roles/functions of proof. Then, think back on your experience in this course and identify a different time when you believe you were engaged in each of the five roles/functions of proof. You might review our course PowerPoint slides and/or problem sets in order to jog your memory about particular instances when you engaged in the roles/functions of proof. Describe clearly and completely your recollection of this event, for example, by providing background information on (a) the problem/proof on which you (or a group of you and your peers) were working, (b) how your work on that problem engaged you in the identified role/function of proof, (c) why you think that experience engaged you most in the identified role/function of proof rather than other possible roles/functions, and (d) any other information that would help me to understand how you see yourself and your peers engaging in the identified role/function of proof. If there is a particular role/function of proof for which you cannot think of any experience that represents that role/function, please state so, and then describe an activity that could be included in next semester's Foundations course to make sure that those students have the opportunity to engage in the relevant role/function of proof. You may handwrite or type your assignment. Please try to think of your own examples for each category, rather than discussing with peers before completing the assignment. At the end of your assignment, please rank order the roles/function of proof according to which you believe you engaged in the most (1) to that which you believe you engaged in the least (5) throughout this semester.

Figure 1. Instructions for students' end-of-semester reading/reflective assignment

For the quantitative component of our analysis we examined the sum, mean, and median of the students' engagement rankings (see RQ1). For the qualitative component of our analysis, 65 descriptions of recollected events form our units of analysis (each about one paragraph in length). The analysis occurred as a four-step process. In step 1, two researchers independently read the collection of students' written responses and engaged in open process coding (Saldaña, 2009). In step 2, we used our individual lists to help us compile one list of process codes that could be used to describe the activities that students recalled when reflecting upon the five roles of proof. Through comparison of our individual lists and reference back to students' written reflections, we identified six broad activities to which students referred: *presenting, discussing, conjecturing, working on problem sets, critiquing, and constructing/developing proofs* (see RQ2). In Step 3, we returned to the data to conduct a more precise second-cycle coding (Saldaña, 2009), individually assigning each unit of analysis to the relevant activity codes from Step 2. As an example, we coded the following reflection from Stephanie (on the **communication** role of proof) as related to the activities of *discussing* and *critiquing*:

> We engaged in the communication role/function of proofs throughout the whole semester…. In our groups, we were given proofs already worked out and we would critique those proofs using our rubric. There would be a lot of discussion/debate within our groups as well as in the class that involved noticing errors in the proof or the good aspects of that proof. It helped with giving a better understanding of how proofs work.

Finally, in Step 4, we restricted our focus to the role and activity pairings for which each of us coded four or more student responses (see shaded cells in Table 2). For these pairings, we returned to the original data and identified themes in terms of how those students perceived the relevant activity (e.g., *critiquing*) engaged them in the role (e.g., **communication**) (see RQ3).

Table 2. Types and frequency of activities students recalled when reflecting on engagement in the five roles of proof. The number inside each cell represents the number of student responses (out of 13) that both researchers coded for a particular activity/role pairing. Shaded cells represent the activity/role pairing where both researchers coded four or more student responses.

	Verification	Explanation	Systematization	Discovery	Communication
Presenting	2	2	1	0	5
Discussing	0	5	1	1	7
Conjecturing	4	1	1	7	0
Working on Problem Sets	6	7	5	6	8
Critiquing	0	0	2	0	4
Constructing/ Developing	5	5	8	4	2

Results

RQ1: Which of the five roles/functions of proof do students in an IBL proofs course perceive they engaged in the most? The least?

The results of students' ranking of their engagement in the five roles of proof are presented in Table 3. Recall, a student assigned a rank of "1" to the role they perceived they engaged in the most and a "5" to the role they perceived they engaged in the least. Thus, according to the mean rankings, the role students perceived they engaged in the most overall was the **communication** role, followed by **verification**, **explanation**, **systematization**, and **discovery**, respectively.

Table 3. Results of student rankings: Engaged the most (rank of 1) to least (rank of 5). (N=13)

Role/Function	Sum	Median	Mean	S.D.
Verification	33	2	2.54	1.34
Explanation	37	3	2.85	1.41
Systematization	43	3	3.31	1.26
Discovery	52	4	4.00	1.04
Communication	30	2	2.31	1.32

Students' quantitative rankings aligned with their qualitative reflections. Many students wrote of constant engagement in the **communication** role of proof, such as Carla's reflection, "I experience this every day in class" (Carla). Moreover, for the least engaged role (i.e., discovery), three students explicitly mentioned not recalling an experience when they engaged in **discovery.** For example, Jeb wrote, "I don't think I was directly involved with finding some new mathematical result." (Note: Discovery was the only role where any students mentioned that they could not recall an experience related to that role).

RQ2: During which types of activities do students in an IBL proofs course recall engaging in the five roles/functions of proof?

The six activities students recalled were *presenting, discussing, conjecturing, working on problem sets, critiquing,* and *constructing/developing proofs.* In a longer version of this article, we will describe these activities and how they were implemented in the IBL course.

RQ3: How do students perceive that the activities they identify engage them in the five roles/functions of proof?

We present a selection of the results for three of the five roles of proof, with intentions of expanding on these results in a longer version of this paper.

As students reflected on their engagement in the **verification** role of proof, four recalled *conjecturing* as a relevant activity. In fact, each of these students specifically mentioned "Carla's conjecture" or "Susan's conjecture." These two conjectures were formed based on ideas of students in the class, and aptly named after those students. Both conjectures stemmed from students' exploration of content that extended beyond the boundaries of the planned course curriculum (as written in the problem sets). Our data suggest that for students, being presented with a conjecture posed by one of their peers caused them to legitimately consider the veracity of the conjecture. David's reflection provides us insight into his perceived engagement in **verification** through *conjecturing*:

> Just recently, when learning about unions and intersections of sets in class, Carla had a question regarding sets and intersections and unions of sets. She wondered if $A \cup B$ could equal $A \cap B$. You had us reflect, create a conjecture, and prove that conjecture. By us getting the chance to create and prove a conjecture based on this situation, we were making a hypothesis regarding sets and verifying that hypothesis.

Five of the students recalled *discussing* proofs as meaningful to their engagement in the **explanation** role of proof. Three of these students described how discussing proofs with peers allowed them to share and learn about why a particular proof technique works (i.e., understanding the proof process). The other two students described how discussing examples of a mathematical statement allowed them insight into why the statement was true (i.e., understanding the mathematics). Millie's reflection serves as an example of when proof was used as a means to gain insight into the underlying mathematics:

> For example, in Problem Set 9, we had to prove by induction that if $|S| = n$, then $|P(S)| = 2^n$….we started working on that one by first doing some examples. After we did those examples and examined how the number of elements in the power set changed from one example to the next, Susan said that each time the cardinality of the power set doubled. We then discussed why that happened and decided that it was because each time you added a new element to S, the element also had to be added to each of the subsets of S (or each element of the power set). Then we had the original set of subsets (which are still subsets so we have to keep them) and the new set with the new element added. So, as we talked it through and thought about how to write the proof, we discovered WHY the statement was true. The validity wasn't so much in question, but the why behind it was.

As a final example of our results, when students reflected on their experiences with the **communication** role of proof, four reflected on *critiquing* activities (recall Stephanie's reflection in the Data Analysis section of this paper) and five reflected on *presenting* activities. Notably, two students mentioned a specific presenting activity where students would share their individual arguments with their group members and then the whole group worked together to create a new product that best communicates their argument to present to the class. Both *presenting* and *critiquing* seemed to engage students in an active consideration of what counts as mathematical proof and exchange of strategies and ideas for approaching new arguments. For example, Krissy reflected:

> I found the most significant examples of engagement with the "communication" role of proof as the times in class that we evaluate multiple arguments for the same proof…. Every time we have engaged in this exercise, I have found new ideas and techniques for proof-writing that I eagerly attempted to use in my own proofs.

Discussion and Implications

Students in this IBL course reflected on many opportunities they had to meaningfully engage in five roles of proof. In addition to activities that might be considered common in traditional undergraduate proof classes, such as *constructing/developing* proofs or *working on problem sets*, students in this course reflected on activities such as *discussing*, *presenting*, *conjecturing*, and *critiquing* as influential to their engagement in the roles of proof. Student-generated *conjectures* fostered an environment where the validity of the conjectures was truly in question, and students saw the need for proof as a means to convince (i.e., **verification**). *Discussions* in the classroom were instrumental in helping students gain a "sense of illumination" (de Villiers, 1990, p. 18) into why a particular mathematical statement was true, or why a particular mode of argumentation (Stylianides, 2007) was appropriate (i.e., **explanation**). Moreover, students recalled *presenting* and *critiquing* activities as influential to their engagement in **communication**, where proof is "a form of social interaction…[that] involves the subjective negotiation of not only the meanings of concepts concerned, but implicitly also of the criteria for an acceptable argument" (de Villiers, 1990, p. 22).

Student reflections suggested that the roles of proof for them as newcomers in the mathematics community (Lave and Wenger, 1991) go above and beyond the roles of proof for mathematicians. For example, with respect to the **explanation** role of proof, students used proof as a means to make sense not only of the underlying mathematics but also of the proof process in general. Similarly, Yopp (2011) found that mathematicians identified a variety of roles of proof in undergraduate mathematics instruction beyond those mentioned by de Villiers (1990). Moreover, student reflections on roles such as **explanation** or **discover**y were not always restricted to deduction, but also included quasi-empirical methods (deVilliers, 2004).

In future research, we would like to use a broader lens to consider how such quasi-empirical methods foster students' engagement in practices of the mathematics community. We contend that to create a learning environment where students have opportunities to engage in authentic mathematical practices, more attention needs to be given to activities such as discussing, presenting, critiquing, and conjecturing in the mathematics classroom. Moreover, because we found that **discovery** was the least engaged role of proof for students in this class, our next step for course development is to design activities that foster students' engagement in the discovery role of proof. As an implication for practice, we believe that it would be beneficial for instructors to engage their students in a similar reflective assignment so that (a) students can more explicitly reflect on the roles of proof, and (b) instructors can gain insight into the roles of proof that are most prominent for students, and the areas for which more learning opportunities are needed.

References

Blanton, M. L., & Stylianou, D. A. (2014). Understanding the role of transactive reasoning in classroom discourse as students learn to construct proofs. *The Journal of Mathematical Behavior, 34,* 76-98.

Bleiler, S. K., Thompson, D. R., & Krajcevski, M. (2014). Providing written feedback on students' mathematical arguments: Proof validations of prospective secondary mathematics teachers. *Journal of Mathematics Teacher Education, 17,* 105-127.

Boyle, J. D., Bleiler, S. K., Yee, S. P., & Ko, Y-Y. (under review). Constructing and critiquing arguments: A case of *how* to determine what is proof.

Conference Board of the Mathematical Sciences. (2012). *The mathematical education of teachers II.* Providence, RI and Washington, DC: American Mathematical Society and Mathematical Association of America.

Coppin, C. A., Mahavier, W. T., May, E. L., & Parker, G. E. (2009). *The Moore Method: A pathway to learner-centered instruction (MAA Notes No. 75).* Washington, DC: Mathematical Association of America.

de Villiers, M. D. (1990). The role and function of proof in mathematics. *Pythagoras, 24*, 17–24.

de Villiers, M. D. (2004). The role and function of quasi-empirical methods in mathematics. *Canadian Journal of Science, Mathematics, and Technology Education, 4,* 397-418.

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences.* Retrieved from: http://www.pnas.org/content/early/2014/05/08/1319030111.full.pdf

Hemmi, K. (2010). Three styles characterizing mathematicians' pedagogical perspectives on proof. *Educational Studies in Mathematics, 75,* 271-291.

Knuth, E. J. (2002). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education, 33*, 379-405.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation.* Cambridge: Cambridge University Press.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Saldaña, J. (2009). *The coding manual for qualitative researchers.* Los Angeles, CA: Sage Publications.

Schoenfeld, A. H. (2009). Series editor's foreword: The soul of mathematics. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and learning proof across the grades: A K-16 perspective* (pp. xii-xvi). New York: Routledge.

Selden, A. & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education, 34,* 4-36.

Smith, J. C. (2006). A sense-making approach to proof: Strategies of students in traditional and problem-based number theory courses. *Journal of Mathematical Behavior, 25,* 73-90.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10*(4), 313-340.

Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education, 38*, 289-321.

Stylianou, D. A., Blanton, M. L., & Knuth, E. J. (Eds.). (2009). *Teaching and learning proof across the grades: A K-16 perspective.* New York, NY: Routledge.
Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics, 48,* 101-119.
Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity.* Cambridge: Cambridge University Press.
Yopp, D. A. (2011). How some research mathematicians and statisticians use proof in undergraduate mathematics. *Journal of Mathematical Behavior, 30*, 115-130.
Yoshinobu, S., & Jones, M. G. (2012). The coverage issue. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies, 22,* 303-316.

Poster: Can Mathematics Be a STEM Pump?

Student Trajectories in Engineering, Mathematics, and the Sciences (STEMS)

The national goal of increasing STEM majors is well-established. As noted in "Rising Above the Gathering Storm" (National Academies, 2008); "Preparing the Next Generation of STEM Innovators: Identifying and Developing Our Nation's Human" Capital (National Science Board, 2010) and several other recent publications (National Academy of Science, 2011; CRS Report, 2008; National Academy of Science, 2010; National Academy of Science, 2012), it is widely recognized that sustaining the economic leadership of the United States over the next two decades will require a robust supply of bachelor's degree graduates in the STEM disciplines. Certain specific courses within the first two years in mathematics and the sciences are rate-limiting for students that have a realistic chance of success in a major in science, technology, engineering, or mathematics (STEM), but "bog down" at a particular point in their trajectory of courses. Identification of such points, and application of resources to improve student success at those points, can dramatically improve retention in STEM.

Table 1: Comparison of grades in MA125 based on first math course at UAB for Freshman

First MA course	N	Mean	Standard Deviation	Std. Error Mean	Signif. level
125	1115	2.676	1.43191	0.04288	
98	101	0.931	1.28263	0.12763	0.000
102	204	1.427	1.42098	0.09949	0.000
105	299	1.977	1.54011	0.08907	0.000
107	118	2.203	1.33691	0.12307	0.326
106	128	2.648	1.45586	0.12868	0.836
Coding: A=4, B=3, C=2, D=1, W/F=0					

Pilot Study. The UAB Department of Mathematics together with the NSF-funded Greater Birmingham Mathematics Partnership conducted a preliminary trajectory study (Bond, 2013) of student success in Calculus 1 (all STEM majors at UAB require Calculus 1 or higher) as a function of the first UAB mathematics course taken. Table 1 is looking at just the starting point of the trajectories of first time freshmen in entry-level mathematics courses through Calculus 1 (MA125) in the years 2006-2012. Grades being compared are all MA125 grades (Chart 1): those of freshman placing directly into MA125 (first row of Table 1) versus the MA125 grade (subsequent rows) of those placing originally into a lower-level course and eventually making it to MA125. (The courses considered are, in order of mathematical level, Basic Algebra—MA098, Intermediate Algebra—MA102, Pre-Calculus Algebra—MA105, Pre-Calculus Algebra and Trigonometry Combined—MA107, and Pre-Calculus Trigonometry—MA106.)

This study shows that MA105 (Pre-Calculus Algebra) is a critical entry-level course. It is the highest level course below MA125 where students have a statistically significantly WORSE chance of succeeding in MA125 than those who are originally placed into MA125. (The significance results were confirmed with non-parametric Mann-Whitney analysis. Also, the significant differences remain when the students with higher ACT mathematics sub-scores, above 27, were eliminated from the data pool.)

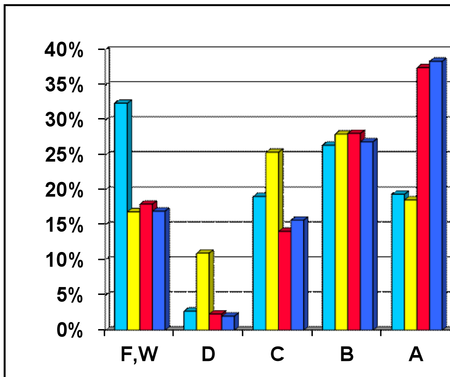


Chart 1

Over the past decade, STEM departments at UAB have conducted isolated studies on a departmental basis. Such studies are useful, but the conclusions are limited in scope since they typically only consider choosing one of several possible predictors. If mathematics is to be a STEM pump, then you have to understand the flow!

Proposed STEMS Study. The proposed STEMS study of Student Trajectories in Engineering, Mathematics and the Sciences would examine student grade, admission testing, and demographic data in the same mathematics courses, but not just the starting course: the data on all intermediate courses taken by each student would be captured and analyzed. In addition, the same data for entry-level courses in biology, chemistry, computer science, engineering, and physics would be captured and analyzed in parallel with the mathematics courses. Trends and commonalities would be identified in student trajectories with the goal of identifying critical rate-limiting points for success in STEM majors, such as exemplified by the MA105 starting point in student mathematics trajectories.

This poster reports both on the Pilot Study (Bond, 2013) of two points in mathematics student trajectories, and an outline of a planned study of STEM trajectories more generally.

References

Bond, William O., *UAB Mathematics Department Technical Report: Success Rates in Calculus*. January, 2013. http://www2.math.uab.edu/GBMP/SuccessInCalculus.pdf

Cochran, R., Mayer, J., & Mullins, B. (2007). The Impact of Inquiry-Based Mathematics Courses on Content Knowledge and Classroom Practice. *Electronic proceedings of the 2007 SIGMAA conference on Research in Undergraduate Mathematics Education.* Retrieved February 27, 2012 from http://www.rume.org/crume2007/papers/cochran-mayer-mullins.pdf.

Cochran, R., Mullins, B., Fulmore, J., & Mayer, J. (2009). *Challenging Courses, Student Achievement, and Barriers to Implementation.* Abstract available at http://hub.mspnet.org/index.cfm/lnc09_cochran/page/index.

Engler, John, *STEM Education Is the Key to the U.S.'s Economic Future.* US News and World Report, June 15, 2012. http://www.usnews.com/opinion/articles/2012/06/15/stem-education-is-the-key-to-the-uss-economic-future.

Mayer, J., Cochran. R., Stansell, L.R., Land, H.A., Bond, W.O., Fulmore, J.S., & Argo, J.H. (2009). *Incorporating Inquiry-Based Class Sessions with Computer-Assisted Instruction.* Abstract available at http://hub.mspnet.org/index.cfm/lnc09_mayer/page/index.

Mayer, J., Cochran, R., Stansell, L., Land, H., Fulmore, J., Bond, W., Argo, J., and Rehm, A. (2009), *Incorporating Inquiry-Based Class Sessions with Computer Assisted Instruction*. Electronic Proceedings of Research in Undergraduate Mathematics Education Conference, Raleigh, NC, February, 2009. http://sigmaa.maa.org/rume/crume2009/Mayer_LONG.pdf.

Mayer, J., Cochran, R., Fulmore, J., Ingram, T., Stansell, L., Argo, J., and Bond, W. (2010). *Blending Inquiry-Based and Computer Assisted Instruction in a Basic Algebra Course: a Quasi-Experimental Study*. Electronic Proceedings of Research in Undergraduate Mathematics Education Conference, Raleigh, NC, February, 2010. http://sigmaa.maa.org/rume/crume2010/Archive/Mayer.pdf.

Mayer, J., Cochran, R., Mullins, B., Dominick, A., Clark, F., & Fulmore, J. (2011) Perspectives on Deepening Teachers' Mathematics Content Knowledge: The Case of the Greater Birmingham Mathematics Partnership. In E. M. Gordon, D. J. Heck, K. A. Malzahn, J. D. Pasley, & I. R. Weiss (Eds.), *Deepening teachers' mathematics and science content knowledge: Lessons from NSF Math and Science Partnerships*.

National Academics, *Report of a Workshop on Science, Technology, Engineering, and Mathematics (STEM) Workforce Needs for the U.S. Department of Defense and the U.S. Defense Industrial Base.* Committee on Science, Technology, Engineering and Mathematics Workforce Needs for the U.S. Department of Defense and the U.S. Industrial Base (2012). http://www.nap.edu/catalog.php?record_id=13318

National Academies, *Rising Above the Gathering Storm: Energizing and Employing America for a Brighter Economic Future*, Committee on Prospering in the Global Economy of the 21st Century: An Agenda for American Science and Technology (2006, revised 2008). http://www.nap.edu/catalog.php?record_id=12999

National Science Board, *Preparing The Next Generation of STEM Innovators: Identifying and Developing our Nation's Human Capital* (2010) NSB-10-33. http://www.nsf.gov/nsb/publications/pub_summ.jsp?ods_key=nsb1033

PCAST, Report to the President: Engage to Excel: Producing One Million Additional College Graduates with Degrees in Science, Technology, Engineering, and Mathematics, February, 2012. http://www.whitehouse.gov/sites/default/files/microsites/ostp/pcast-executive-report-final_2-13-12.pdf

University of Alabama at Birmingham, Office of Planning and Analysis, "Retention Report 2012."

US Department of Commerce, July, 2011. http://www.esa.doc.gov/sites/default/files/reports/documents/stemfinalyjuly14_1.pdf

US Joint Economic Committee, *STEM Education: Preparing for the Jobs of the Future*, April, 2012. http://www.jec.senate.gov/public/index.cfm?a=Files.Serve&File_id=6aaa7e1f-9586-47be-82e7-326f47658320

Learning in one classroom: Developmental mathematics students and prospective mathematics teachers

Kenneth Bradfield, Raven McCrory, Aditya Viswanathan, & Kristen Bieda
Michigan State University

Developmental mathematics courses in the United States continue to lack in curriculum and instructional practices that promote students' mathematical proficiency. The instructional practices that researchers argue can promote students' mathematical proficiency in K-12 classrooms can apply to undergraduate classrooms as well. This poster will discuss an NSF-funded research project that facilitates students' mathematical development in a non-credit-bearing developmental mathematics course, in concert with providing prospective mathematics teachers an opportunity to learn to teach for mathematical proficiency. The project team collected quantitative data that compared the intervention students to their peers before and after participation in the course. Results indicated that developmental mathematics students who participated in our intervention started behind, caught up, and experienced more success than their peers in their subsequent mathematics course.

Keywords: Developmental mathematics, mathematical proficiency, pedagogy

Across the country, post-secondary institutions design courses to meet the needs of students that are underprepared for their mainstream sequence of mathematics courses. Although mathematics departments attend to the diversity of the methods and their participants, developmental courses continue to lack in teacher preparation (Conference Board, 2012; Schmidt, Blömeke, & Tatto, 2011) and cognitively-demanding curriculum (Attewell, Lavin, Domina, & Levey, 2006; Fairweather, 2008; Larnell & Smith, 2010). Larnell (2013) argued that these conditions hinder the development of students' mathematical proficiency.

Researchers in K-12 mathematics education have articulated the best pedagogical practices to promote the five strands of students' mathematical proficiency which are applicable to this undergraduate student population (Hodara, 2011). In the National Research Council's report *Adding It Up,* Kilpatrick, Swafford, and Findell (2001) identified the five strands for mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. To promote these strands, undergraduate instructors can engage students in doing, talking, and thinking about mathematics by encouraging students to participate in class discussions and focusing on students' justification of their own mathematical ideas (Fuson, Kalchman, & Bransford, 1999; Kilpatrick et al., 2001; Stein, Engle, Smith, & Hughes, 2008; Hodara, 2011).

This project investigated collaboration between a mathematics department and a university teacher education program where prospective mathematics teachers provided instruction, and received mentoring to learn to teach, in a developmental mathematics course. Developmental mathematics students enrolled in a face-to-face section that met twice a week for one hour and forty minutes to supplement their work in an online-based tutoring and assessment program. In our intervention section, pre-service mathematics teachers took turns co-teaching lessons in pairs who employed research-based instructional methods to promote students' mathematical proficiency under the guidance of mathematics education researchers.

Quantitative data demonstrated that students in the intervention started behind and caught up to their peers in the control group and in addition surpassed their peers in the non-intervention face-to-face sections. Research assistants collected test scores before (standardized tests, math department placement exam, and ALEKS pre-score), and after the course (final exam, ALEKS post-score) as well as course grades in the online course and their subsequent mathematics course. Pre-intervention test scores indicated that the intervention students started behind and caught up to their peers on the online-only control group. Based on post-intervention test scores, the intervention students passed the online course and also their subsequent mathematics course at higher rates than their peers in the non-intervention face-to-face sections.

Currently in its third year, the project continues to generate data with a new cohort of developmental mathematics students and pre-service mathematics teachers. Project members will use this new data along with attitudes and belief survey data to generate multi-level models of the intervention students, in conjunction with qualitative data that includes video observations and interviews. Project leaders continue to ensure that our curriculum aligns with minor changes in the pacing of the course and course design meets the needs of a new group of developmental mathematics students.

References

Attewell, P., Lavin, D., Domina, T., & Levey, T. (2006). New evidence on college remediation. *Journal of Higher Education, 77*(5), 886-924.

Bai, H., Wang, L., Pan, W., & Frey, M. (2009). Measuring mathematics anxiety: Psychometric analysis of a bidimensional affective scale. *Journal of Instructional Psychology, 36*(3), 185-193.

Conference Board of the Mathematical Sciences. (2012). *The mathematical education of teachers II: Draft for public discussion*. Providence, RI: American Mathematical Society.

Fairweather, J. (2008). *Linking evidence and promising practices in science, technology, engineering, and mathematics (STEM) undergraduate education: A status report for the National Academies National Research Council Board of Science Education*. Washington, DC: National Academy Press.

Fennema, E. (1976). Fennema-Sherman mathematics attitudes scales: Instruments designed to measure attitudes toward the learning of mathematics by females and males. *Journal for Research in Mathematics Education, 7*(5), 324-326.

Fuson, K. C., Kalchman, M., & Bransford, J. D. (1999). Mathematical understanding: An introduction. In J. D. Bransford, A. L. Brown & R. R. Cocking (Eds.), *How people learn: Brain, mind, experience, and school* (pp. 217-256). Washington, DC: National Academy of Sciences, National Research Council.

Hodara, M. (2011). Improving Pedagogy in the Developmental Mathematics Classroom. *CCRC Brief, 51,* (pp. 1-4). Community College Research Center, Columbia University.

Kilpatrick, J., Swafford, J., Findell, B., & National Research Council. Mathematics Learning Study Committee. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Larnell, G. V. (2013). *Toward reforming non-credit-bearing remedial math courses*. Chicago, IL: UIC Research on Urban Education Policy Initiative, University of Illinois at Chicago. Retrieved from http://www.scribd.com/doc/149829260/Toward-Reforming-Non-Credit-Bearing-Remedial-Math-Courses

Larnell, G. V. & Smith, III, J. P., (2010). Verb use and cognitive demand in K–8 geometry and measurement grade-level expectations. In J. P. Smith, III (Ed.), *Variability is the rule: A companion analysis of K–8 state mathematics standards*. Charlotte, NC: Information Age Publishing.

Schmidt, W., Blömeke, S., & Tatto, M. T. (2011). *Teacher education matters: A study of middle school mathematics teacher preparation in six countries*. New York: Teachers College Press.

Stein, M.K., Engle, R.A., Smith, M.S. & Hughes, E.K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10*(4), 313-340.

Seeking Solid Ground: A Study of Novices Indirect Proof Preferences

Stacy A. Brown
California State Polytechnic University, Pomona

The aims of this study are two-fold: (1) to investigate novices' proof preferences as indicated by novices' selection of the "most convincing" argument when engaging in proof comparison tasks involving an indirect and a direct proof; and, (2) to explore the criteria students' bring to bear on proofs as they engage in proof comparisons. Informed by the cK¢ model of conceptions proposed by Balacheff (2010), analyses indicate that directness was not a primary criterion used for the select of a proof during proof comparisons, even though this criterion was suggested by prior research. Instead, the primary criteria identified in students' rationales were familiarity and the degree of certainty in one's understanding of the given proofs, which in turn suggests that it is the subjective sense of being on solid conceptual grounds that determine students' preferences. These findings are considered in light of the cK¢ model of conceptions.

Key words: Indirect proof, Proof preferences, Proof by contradiction

Researchers interested in students' conceptions of indirect proofs, which include proofs by contraposition and proof by contradiction, have observed that students do not find indirect proofs convincing (Harel & Sowder, 1998) and experience difficulties accepting indirect proofs (Antonini & Mariotti, 2008). For instance, Harel and Sowder (1998) assert that in their teaching experiments, "Students seldom used proof by counterexample and they did not seem to be convinced by it; nor were they convinced by proof by contradiction" (p. 253). Furthermore, they argue that students' lack of preference for proofs by contradiction is a particular manifestation of the constructive proof scheme. Leron (1985), reflecting on the implementation of an instructional innovation, also argued that students do not find proofs by contradiction convincing and prefer constructive proofs. In an earlier study of students' levels of confusion with regard to the standard proof for the irrationality of $\sqrt{2}$, Tall (1979) noted that while the proof is "aesthetically pleasing … learners often feel a sense of emptiness and lack of explanation as to why $\sqrt{2}$ is not irrational" (p. 206). Working from the Cognitive Unity perspective, Antonini and Mariotti (2008) have highlighted how indirect proofs require learners to be able to shift between logically equivalent statements (e.g, from a principal statement, $p \rightarrow q$, to a secondary statement, $\neg q \rightarrow \neg p$, in a proof by contraposition). They argue that students' experience difficulties accepting the proof of the secondary statement as a proof of the principal statement and note that doing so may require "mental efforts that not all students seem to be able to make" (p. 411).

Given the amount of research indicating students' lack of preference for and difficulties deriving a sense of conviction from indirect proofs, it is surprising that none of the research discussed engaged students in comparative tasks involving both an indirect and a direct proof. This aspect of the research is especially surprising since several well-known studies, which explored the types of arguments students and teachers find convincing, have employed comparative tasks (Healy & Hoyles, 2000; Knuth, Choppin, & Bieda, 2009). Moreover, while Tall engaged students in comparisons his tasks only involved indirect proofs, for he sought to understand if certain indirect proof structures created less confusion for students. Thus, comparisons between indirect and direct were not employed. Other researchers, such as Harel and Sowder (1998), Leron (1985), and in some investigations by Antonini and Mariotti, inferred students' preferences from students' reactions to an indirect proof during a teaching experiment;

that is, students were reacting to a single argument. Thus, there is a need for research that explores students' proof preferences in the context of comparative tasks. This study aims to address this need by exploring the following questions:

1. *Do novice proof writers find indirect proofs less persuasive than direct proofs when engaging in proof comparisons?*
2. *Which rationales do students' provide for their selection of a more convincing proof during proof comparisons involving both direct and indirect proofs?*

These questions are of importance for several reasons. Research on indirect proofs is unique in that it represents the only area of research on students' proof conceptions that posits that students' difficulties stem from a desire for alternative forms of proof. Indeed, if you look to research on students' difficulties with other specific forms of proof, such as mathematical induction (Harel & Brown, 2008; Harel, 2001, Brown, 2003) or combinatorial proofs (Lockwood, 2011; Maher & Martino, 1996), you will not find such rationales. Moreover, it may be the case that when an actual alternative is present, other aspects become salient to students.[1] Lastly, as noted above, Antonini and Mariotti (2008) have made progress on identifying features that are unique to indirect proofs, both from a mathematical and cognitive perspective; namely, indirect proofs' use of shifts between principal and secondary statements. Yet, researchers have not documented whether or not it is these very features that students' attend to when engaging in comparative tasks. It is for these reasons that the current study was conducted.

Theoretical Perspective

This work is part of a larger research program generally focused on tertiary students: development of a *conception* of indirect proof, where conception is used in the sense of Balacheff's cK¢ model; the emergence of hypothetico-deductive reasoning (Piaget, 1968/1964); and, the cohabitation *knowings* (Balacheff, 2010) related to the verification and explanation functions of proof. While space limitations do not permit a full description of the cK¢ theory, a general comment and a discussion of key features are needed. The general comment is that the cK¢ model was designed as a theory to model individuals' development of conceptions and is a complementary theory to the *Theory of Didactical Situations* (Brousseau, 1997). The key features are best understood by examining the meaning of the symbols cK¢, which stand for *conception*, *knowing*, and *concept*. A *conception* is understood as the quadruplet (P, R, L, Σ), where P is a set of problems, R is a set of operators, L is a representational system, and Σ is the control structure. Roughly speaking, the set of operators, R, are the set of allowable actions with the learner's milieu. The control structure, Σ, consist of "the tools needed to make decisions, make choices, and express judgments on the use of an operator or on the state of a problem (i.e., solved or not)" (Balacheff, 2010, p. 127). One way of thinking about control structures is that they are the means that students bring to bear on a concept during acts of validation. A *knowing* is "the set of conceptions which can be activated by different situations the observer considers conceptually the same" (Balacheff, 2013, p. 4). Lastly, a *concept* is defined as follows, "Let's now claim the existence of a conception Cμ more general than any other conception to which it can be compared … A 'concept" is the set of all conceptions having the same object with respect to Cμ" (Balacheff, 2013, p. 10). The perspective taken in this paper is that the cK¢ model is appropriate for the research conducted since the research sought to understand students' judgments and their relation to the problem situations encountered. Indeed, this research can be

[1] This is not to say an alternative is always available; e.g., there is not a well-known direct proof of the irrationality of √2.

viewed as asking, which criteria do students' bring to bear on indirect proofs? Which control structures are employed to decide if a given proof is the most convincing proof to the learner?

The Study

The research reported occurred in two phases. Phase I involved the administration of an electronic survey to *novice proof writers*; that is, second and third year mathematics majors who were either concurrently enrolled in or had taken during the same academic year an introduction to proof course. The survey consisted of 3 comparison tasks and a statement selection task. Students were asked in the survey, "Which proof, in your opinion, is the most convincing? In other words, which proof better persuades you of the truth of the theorem?" Phase 2 of the study involved interviews with 20 novice proof writers. The interview tasks were the same as the survey tasks, with the exception that students were asked to verbally explain their selection and to respond to two proof comprehension questions. Specifically, students were provided with two proofs, given time to read the proofs, asked the preference question and then the comprehension questions. The first comprehension question asked students to classify the given proofs, using the most appropriate proof type, after having been given a list including the following types: direct proof, proof by cases, proof by contradiction, proof by mathematical induction, proof by contraposition, and other. The second comprehension question asked students to engage in the following hypothetical scenario: "Imagine that another student comes to you and says, 'Hey, I don't understand this proof. Can you explain it to me?' What would you say?" Data collection for the interviews included all of the students' written work and a video-recording of the interview. For the purposes of this paper, we will consider students responses to the direct/indirect comparison tasks, which where the Theorem 1 and Theorem 3 tasks.

The Theorem 1 task involved the comparison of an unfamiliar direct proof, Proof 1A, and a proof by contradiction, Proof 1B (see Appendix A), with familiar content. The inclusion of this task was motivated by pilot work in which students were asked to compare a direct proof involving the Principle of Mathematical Induction (PMI) to a proof by contradiction involving the Well-ordering Principle. In the pilot trials, which involved surveys and interviews, it was found that the majority of students (11:4 in *Trial 1* and 12:3 in *Trial 2*) selected the direct proof as the "most convincing." However, when asked to explain their selections students' written and interview responses were overwhelmingly focused on the students' familiarity with PMI and their lack of familiarity with the Well-ordering Principle. Indeed, there was no evidence of students' attention to directness (or a lack thereof) as a criterion for making judgments. To see if directness could be invoked as a criterion, the Theorem 1 task was design to present the less familiar content with the direct proof (1A) and the more familiar content with the indirect proof (1B).[2] It was posited that if students prefer direct proofs then that preference might override the lack of familiarity in 1A. If students did not exhibit a preference for 1A, then either: directness is among those criteria that are subordinate to other criteria, such as familiarity; or, directness is only a criterion in problem situations with specific characteristics, which were not evident.

The Theorem 3 task was design to control for familiarity and length in that the proofs provided were nearly identical in length and involved content presented in the same chapter in multiple texts. Proof 3A was an indirect proof (proof by contraposition) and Proof 3B was a direct proof (Appendix B). It was hypothesized that if students lack a preference for indirect proofs, then 3B would be selected by the majority of students and directness would be employed as a criterion for justifying one's selection of the "most convincing" proof. If 3B was not

[2] The unfamiliar content in 1A required more text, so the unfamiliar argument (1A) was slightly longer than the familiar argument (1B), which may be a complicating factor.

selected by the majority of novices, then the findings would provide grounds for questioning the extent to which directness is a general criteria for novices and, furthermore, if there are comparative situations in which a lack of preference is exhibited among novices.

Results

We will report findings from the selection tasks of Phase 1 and Phase 2 together (n = 33). Data from the Theorem 1 task indicate that the students found Argument 1B, the familiar indirect proof (proof by contradiction), more convincing than Argument 1A, the unfamiliar direct proof. The *indirect:direct* (1B:1A) selection ratio was 30:3. Data from the Theorem 3 task, however, do not suggest that students viewed the direct proof (3B) as more convincing that the indirect proof (3A). The *indirect:direct* (3A:3B) selection ratio was 15:17.[3] In pilot trials, the Theorem 3 task, which had minor modifications for the study, produced similar results. The selection ratio was 12:9. Taken together the pilot and study data yield an *indirect:direct* selection ratio of 27:26.

Findings from the 20 interviews with novice proof writers indicate that multiple criteria other than directness were used to judge the most convincing proof. Due to space limitations, the three most common criteria will be discussed. One of the most common criteria used in the Theorem 1 task for students' selection of 1A was familiarity. Below is an illustrative interview excerpt.

Nicholas: It's just the familiarity of the structure … to see it as the way we define odd. I am familiar with this (gestures to Argument 1B), as compared to … they jump into … series … and a … it's a little bit rough with my memory right now.

When responding to the classification task, Nicholas immediately pointed out that the first sentence of 1B was the negation of Theorem 1 and that this indicated that 1B was a proof by contradiction, suggesting his awareness of the proof type. Whereas, when asked to classify 1A, he was quite hesitant, which suggests a proof type was not of readily available to him.

Nicholas: And, here (gestures to Argument 1A), … (long pause) … [re-reading argument] I feel like this is more of a … more of a direct proof.

Another common criterion expressed by students was the complexity of the content. This concern manifested itself through comments focused on the complexity of the notations (symbols) and equations used in 1A, such as the summation notation.

Anthony: B definitely. The structure of 1B seems a lot more straight forward and it looks a lot more simpler and less symbols … so less symbols is always a preference for me.

When Anthony was asked to classify the proofs he noted that in 1B the statement, "a^2 cannot be both even and odd," indicated a contradiction. However, when he was asked to explain the first line of 1B, he became confused and began to question if the proof was correct. After re-reading the proof, he was asked again about the first sentence. He noted that it was the negation of Theorem 1 and reiterated his preference for 1B, indicating that even though he was aware of having some confusion, he wanted to avoid the "complicated" argument, 1A.

The third criterion that was common was distinct from the others in that it was a highly subjective criterion: *the degree of certainty in one's understanding* of the proof. This criterion manifested itself during the Theorem 1 task through students' remarks regarding their certainty in the validity of proof 1B, and in relation to concerns about their own understanding of the validity of proof 1A. For instance, Tina noted that she thought 1A was "too complicated" because of the summation notation and then justified her selection of 1B.

Tina: I feel that this (Argument 1B) is way more easier than this one (1A) and … it's shorter … and I don't know … I just understand that one better (gestures to 1B).

[3] One student responded by selecting both proofs.

Responses to the Theorem 3 task were similar to the Theorem 1 task in terms of the criteria students brought to bear on the proofs when selecting the "most convincing" proof, with three exceptions. First, familiarity was not used as a criterion to distinguish the arguments in any of the interviews. A second distinction was that it was common for students to consider the Theorem 3 proofs in terms of their alignment with the novices' own way of thinking.

Oliver: "I do not know why but I choose 3A … it's kind of logically like … in life you know, you're talking to your friend and you say 'but if this happens' then …"

A third distinction is that since proofs 3A and 3B were similar in their level of complexity (from an expert's perspective), considerations of complexity were highly dependent on the students' understanding of the related content. In other words, while students uniformly viewed 1A as more complex that 1B, this was not the case with the Theorem 3 proofs. Also, inquiries into students' rationales revealed that for many students the criterion, *the degree of certainty in one's understanding*, was the primary criterion by which students selected the most convincing proof. For instance, Kurt selected 3B (direct) and rationalized his choice by stating, "I just like this one better." Yet, when asked to explain 3A (indirect), responded: "To be honest, I'm a little confused by this one." Similarly Marianne, selected 3A, stating "they explain it better in this one (3A) than this one (3B)." When asked the comprehension questions she immediately classified 3A as a proof by contraposition and was able to clearly explain the sequence of statements. However, with 3B (direct) she proceeded hesitantly and then remarked, "I don't understand why they said this here" in reference to "By the given property, $A \subseteq \varnothing$." Both of these examples illustrate students' awareness of a point of personal confusion and their selection of the proof for which they experienced a higher degree of certainty in their understanding of the proof.

Discussion

The purpose of this study was two-fold: (1) to examine if novice proof writers find indirect proofs less persuasive than direct proofs when engaging in proof comparison; and (2) to identify the criteria students' provide for their selection of a more convincing proof during proof comparisons involving both direct and indirect proofs. The students' responses to the Theorem 1 proof comparison task indicate that when a direct proof is viewed as more complex and less familiar, students will prefer an indirect argument and that they may do so even if some level of confusion occurs in relation to the indirect argument. The students' responses to the Theorem 3 proof comparison task indicate that even when familiarity, length, and relative complexity were controlled for, novices do not use directness as a criterion for their selection of the "most convincing" proof. Furthermore, closer examination of the students' rationales indicates that a primary criterion is *the degree of certainty in one's understanding*. Taken together these findings support two conclusions: (1) directness is not a primary criterion for comparisons; and, (2) familiarity, complexity, alignment with one's own thinking, and the degree of certainty in one's understanding are the primary criteria used during comparison tasks, with the latter criteria being the most critical. Furthermore, if directness is a criterion for students, then the characteristics of the situations that evoke this criterion must be different than those of the tasks used in the study.

In relation to existing research, this study found, as have others (Healy & Hoyles, 2000; Knuth, Choppin, & Bieda, 2009), that familiarity impacts students' selection of the "most convincing" proof. Also, in terms of the theoretical perspective of this work, considerations of familiarity are not unexpected. Specifically, if the student is able to recognize similarities to a set of problems, P, a set of operators, R, (acceptable actions), and/or the representational system, L, and is able to comprehend the expression of judgments, which are indicative of a control structure, then the learner is viewed as having evoked an emerging conception. Thus, from the cK¢ perspective, students gravitate towards what they have *conceptions* of; that is, towards what they know. Moreover, according to the cK¢ model, a conception is defined in terms of a quadruplet (P, R, L, Σ), rather than the triplet (P, R, L) because the control structure component is essential. Specifically, Balacheff argues, "a conception is validation dependent," as is *knowing,* since "no one can claim to know without a commitment to and a responsibility for the validity of the claimed knowledge" (p. 126). In other words, conceptions require solid grounds. Thus, students' gravitation towards the proof for which they've experienced a higher *degree of certainty in their understanding* can be seen as indicative of students' need to recognize and agree with the control structure employed. Viewed in this way, it is students' emergent *conceptions* of the proofs that predicts students' selection of the most convincing proof, rather than the particular proof form, be it direct or indirect.

References

Antonini, S., & Mariotti, M. A. (2008). Indirect Proof: What is specific to this way of proving? *ZDM – The International Journal on Mathematics Education*, *40*, 401 – 412.

Balacheff, N. (2010). Bridging knowing and proving in mathematics: A didactical Perspective. In G. Hanna, H. N. Jahnke, and H. Pelte (Eds.), *Explanations and Proof in Mathematics: Philosophical and Educational Perspectives*. (pp. 115 – 135). Dordrecht, Netherlands: Springer.

Balacheff, N. (2013). cK¢, A model to reason on learners' conceptions. Retreived from http://www.researchgate.net/publication/256097317_cK_a_model_to_reason_on_learners'_conceptions on December 12, 2013.

Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics*. Dordrecht, Netherlands: Kluwer.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.) *Research on Collegiate Mathematics Education III*. (pp. 234-283). Providence, RI: American Mathematical Society.

Brown, S. (2003). *The evolution of students understanding of mathematical induction: A teaching experiment*. Unpublished doctoral dissertation, University of California, San Diego and San Diego State University, CA.

Harel, G. (2001). The development of mathematical induction as a proof scheme: A model for DNR-based instruction. In S.R. Campbell & R. Zazkis (Eds.), *Learning and teaching number theory: Research in cognition and instruction.* Monograph Series of the Journal of Mathematical Behavior, 2, 185–212.

Harel, G. & Brown, S. (2008). Mathematical Induction: Cognitive and Instructional Considerations. In M. Carlson & C. Rasmussen, *Making the Connection: Research and Practice in Undergraduate Mathematics*. Mathematical Association of America, 111-123.

Harel, G., & Sowder, L. (1998). Students' Proof Schemes. In E. Dubinsky, A. Schoenfeld, and J. Kaput (Eds.) *Research on Collegiate Mathematics Education*. (pp. 234-283). USA: American Mathematical Society.

Healy, L. & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education*, *31*, 396 – 428.

Knuth, E., Choppin, J., & Bieda, K. (2009). Middle school students' productions of mathematical justification. In M.Blanton, D. Stylianou, & E. Knuth (Eds.) *Teaching and learning proof across the grades: A K-16 perspective* (pp. 153-212). NY: Routledge.

Leron, U. (1985). A direct approach to indirect proofs. *Educational Studies in Mathematics*,

16(3), 321 – 325.
Lockwood, E. (2011). *Student Approaches to Combinatorial Enumeration: The Role of Set-Oriented Thinking.* Unpublished dissertation, Portland State University, OR.
Maher, C. A. & Martino, A. M. (1996). The development of the idea of mathematical proof: A 5-year case study. *Journal for Research in Mathematics Education, 27(2),* 194 – 214.
Piaget, J. (1968/1964). *Six psychological studies*. New York, NY: Random House.

Appendix A

Theorem 1: If a^2 is even, then a is even.	
Proof A:	**Proof B**:
Suppose a^2 is even. Since the sum n odd integers is n^2, we can say that $a^2 = 1 + 3 + \ldots + (2a - 1)$. In other words, $a^2 = \sum_{j=1}^{a}(2j-1)$. Therefore, $a^2 = (2a-1) + \sum_{j=1}^{a-1}(2j-1)$. By the properties of series, we see that $a^2 = (2a-1) + 2\sum_{j=1}^{a-1} j - \sum_{j=1}^{a-1} 1 = (2a-1) + 2\sum_{j=1}^{a-1} j - (a-1)$ Thus, $a = a^2 - 2\sum_{j=1}^{a-1} j$ Recall that by assumption a^2 is even. It follows that $a^2 = 2k$, for some integer k. Hence, $a = a^2 - 2\sum_{j=1}^{a-1} j = 2k - 2\sum_{j=1}^{a-1} j = 2(k - \sum_{j=1}^{a-1} j)$. Since $(k - \sum_{j=1}^{a-1} j)$ is an integer, a is even.	Suppose, a^2 is even and that a is odd. It follows that $a = 2k+1$ for some positive integer k. Hence, $a^2 = (2k+1)^2$ $= 4k^2 + 4k + 1$ $= 2(2k^2+2k) + 1$. Since $(2k^2+2k)$ is an integer, a^2 is odd. However, a^2 cannot be both even and odd. Thus, a is even.

Appendix B

Theorem 3: Suppose a set A has the property, for any subset B, A $\subseteq$ B. Then, A = $\varnothing$.	
Proof A:	**Proof B:**
Suppose A $\neq \varnothing$. Then there exists an a, such that $a \in$ A. Hence, A$\not\subseteq \varnothing$. Thus, there exists a subset B for which A$\not\subseteq$ B.	Assume A has the stated property. Recall, that $\varnothing$ is a subset of every set. Thus, $\varnothing \subseteq$ A. By the given property, A $\subseteq \varnothing$. Thus, A = $\varnothing$.

Conditions for cognitive unity in the proving process

Kelly M. Bubp
Ohio University

Although a mathematical proof is a syntactic product, the proving process often entails other reasoning types, such as semantic or intuitive, that contribute to the evaluation of conjectures and the construction of supporting arguments. Cognitive unity and rupture refer to the possible continuity or discontinuity, respectively, between various reasoning types, argumentation and mathematical proof, and the processes of evaluating and proving conjectures. Undergraduate students struggle with mathematical proof, but it is hypothesized that cognitive unity facilitates the proving process. In this study, task-based interviews were conducted with undergraduate students who completed three prove-or-disprove tasks. The goals are to determine conditions under which students experience cognitive unity or rupture when evaluating and proving the conjectures and conditions under which cognitive unity and rupture aided or hindered the proving process. Preliminary findings suggest that various factors affect cognitive unity, cognitive unity can hinder proving, and cognitive rupture can facilitate proving.

Key words: Cognitive Unity, Semantic and Syntactic Reasoning, Argumentation, Mathematical Proof, Task-Based Interviews

Proof is an essential aspect of doing mathematics. The proving process encompasses a multitude of activities including exploring and identifying patterns and relationships, generating conjectures and generalizations, and testing, refining, and proving conjectures (Committee on the Undergraduate Program in Mathematics (CUPM), 2004; de Villiers, 2010; Durand-Guerrier, Boero, Douek, Epp, & Tanguay, 2012). This complex process often involves potentially conflicting components: reasoning inside and outside the representation system of mathematical proof, argumentation and mathematical proof, and producing or evaluating conjectures and constructing proofs of such conjectures. It has been suggested that continuity, or *cognitive unity*, between these sets of components facilitates proving whereas discontinuity, or *cognitive rupture*, hinders it (Alcock & Weber, 2010; Garuti, Boero, & Lemut, 1998).

Research continues to show that many undergraduate students struggle with numerous aspects of mathematical reasoning and have limited facility in constructing mathematical proofs and counterexamples (Weber & Alcock, 2009; Harel & Sowder, 1998). It is imperative that we continue searching for ways to alleviate these difficulties. With this goal in mind, this study explores the following research questions: Under what conditions do undergraduate students experience cognitive unity or rupture? Under what conditions does cognitive unity facilitate or hinder the proving process? Under what conditions does cognitive rupture facilitate or hinder the proving process?

Literature Review

Undergraduate students are typically expected to construct mathematical proofs that belong to the representation system of mathematical proof (Weber & Alcock, 2009). Reasoning within the representation system of mathematical proof is called syntactic reasoning, and it has unique features that distinguish it from reasoning in other representation systems (Weber & Alcock, 2009). First, syntactic reasoning requires the precise use of language, notation, symbols, and definitions (CUPM, 2004; Weber & Alcock, 2009). Second, a syntactic proof must be unambiguous and have an apparent proof framework (CUPM, 2004; Weber & Alcock, 2009). Third, acceptable proof frameworks, such as direct or indirect proof, structure the proof and specify admissible assumptions and proper conclusions (Weber & Alcock, 2009). Fourth, syntactic proofs contain only mathematical statements employing some combination of the precise use of the English language and first-order logic. Finally, all reasoning in a syntactic proof must be based on definitions, assumptions, theorems, and the use of logical deduction. Informal representations such as graphs or examples, as well as reasoning based on informal representations, are not permitted as a basis for conclusions (Weber & Alcock, 2009).

Although a mathematical proof resides in the representation system of mathematical proof, the proving process often involves argumentation based on reasoning from other representation systems, such as semantic or intuitive reasoning. Semantic reasoning focuses on general understanding guided by examples, diagrams, or other informal representations, and intuitive reasoning is rooted in acquired knowledge and experience. Semantic and intuitive reasoning can help students evaluate conjectures and develop informal arguments by (a) suggesting a direction to pursue, (b) revealing similarities to see a "common global situation," (c) supporting empirical inferences, or (d) exposing underlying structure or patterns (Burton, 2004; de Villiers, 2010; Fischbein, 1987, p. 53; Weber & Alcock, 2009). However, students often struggle to link their argumentation outside and proving inside the representation system of mathematical proof (Raman, 2003; Weber & Alcock, 2009).

Cognitive unity occurs when arguments that are developed while evaluating (or producing) conjectures are transformed into a mathematical proof or counterexample (Garuti et al., 1998). Thus, cognitive unity represents continuity between the exploration and evaluation of a conjecture and the construction of an associated proof or counterexample. Additionally, because the evaluation of a conjecture often involves intuitive and semantic reasoning, cognitive unity can represent cohesion between reasoning outside and inside the representation system of mathematical proof. When such continuities are not achieved, cognitive rupture occurs. Garuti, Boero, and Lemut (1998) hypothesize that the greater the gap between the arguments for the truth value of the conjecture and the arguments which can be translated into mathematical proofs, the greater the difficulty in constructing a mathematical proof.

Method of Inquiry

The data in this paper come from a larger study that (a) investigated the types of reasoning students use to evaluate conjectures, (b) identified systematic errors students made during the proving process, and (c) examined connections between students' evaluation of conjectures and their success in constructing associated proofs and counterexamples.

Participants

Twelve participants were selected from the main campus of a public university in Ohio who met the criteria of being an undergraduate student who had passed at least one proof-based mathematics course with a B or better. Ten participants were in their fourth year of study at the university, and eleven participants were mathematics or secondary mathematics education majors.

Procedures

I conducted two task-based interviews with each participant which were audio-recorded and transcribed. Participants were asked to think aloud during the completion of three tasks and to clarify or expand on their thinking as necessary. Each task was provided one at a time on a separate sheet of paper. Participants were provided with a list of definitions of terms in the tasks, but no other materials were allowed. Participants used a LiveScribe Pen and paper that recorded synchronously audio and writing. After each task, I asked follow-up questions regarding the participants' work on the task, and specifically asked them to recall and identify when they made decisions about the truth value of the task.

Tasks

Each of the three tasks required the participants to determine the truth value of a given mathematical statement and prove or disprove the statement accordingly. The tasks dealt with basic information on functions and were chosen to be accessible to the participants. In line with Alcock and Weber (2010), each of the tasks referred to general objects and their properties and should have been amenable to semantic or syntactic reasoning strategies. Finally, the tasks provided opportunities to construct both proofs and counterexamples. The following three tasks will be discussed in this paper:

Injective Function Task: Let $f: A \to B$ be a function and suppose that $a_0 \in A$ and $b_0 \in B$ satisfy $f(a_0) = b_0$. Prove or disprove: If $f(a) = b$ and $a \neq a_0$, then $b \neq b_0$.

Monotonicity Task. Prove or disprove: If $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ decreasing on an interval I, then the composite function $f \circ g$ is increasing on I.

Composite Function Task. Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be functions. Prove or disprove: If the composite function $f \circ g$ is one-to-one, then g is one-to-one.

Analysis

The analysis for this study includes distinguishing between (a) reasoning in different representation systems, (b) argumentation and mathematical proof, and (c) the processes of evaluating a conjecture (leading to a decision with regards to its truth value) and constructing a proof or counterexample for the conjecture. Semantic and syntactic reasoning will be classified according to their definitions in the literature review, with subtypes identified and created as necessary. Each instance of reasoning will be classified separately in order to capture the use of multiple reasoning types throughout the decision process. Any argument that contains semantic reasoning will be classified as argumentation whereas an argument that includes only syntactic reasoning will be considered a proof. In order to distinguish between the evaluation and construction processes, reasoning that precedes the decision on the truth value of a conjecture will be classified as part of the evaluation process. Reasoning that is offered as support of the decision will be classified as part of the construction process, regardless of whether it precedes or succeeds the decision itself. Finally, the decision itself will be identified through either its statement during the proving process or its establishment by the participant during post-task questioning.

Preliminary Results

Analysis is ongoing, but preliminary results indicate that cognitive unity may depend on a variety of factors such as subtypes of semantic and syntactic reasoning used, the complexity of a correct proof or counterexample for a task statement, and whether the task statement was true or false. Additionally, I have distinguished two distinct types of cognitive unity and rupture (Table 1).

Table 1

Types of Cognitive Unity and Rupture

Cognitive Unity (CU)		Cognitive Rupture (CR)	
CU1	Construction based on evaluation	CR1	Construction not based on evaluation
CU2	Simultaneous construction and evaluation	CR2	No evaluation

The students in this study attained cognitive unity in the majority of cases on the Injective Function and Monotonicity Tasks, but there was only one case of cognitive unity on the Composite Function Task. Across all three tasks, cognitive unity was equally linked to correct and incorrect solutions. Cognitive rupture was mostly connected to incorrect solutions, but there were situations in which it corresponded to correct solutions.

Examination of the links between reasoning inside and outside the representation system of mathematical proof suggests that success in achieving cognitive unity between these reasoning types may be task-dependent. On the Injective Function Task, most cases of cognitive unity occurred between the representation system of mathematical proof and another representation system. However, on the Composite Function Task, most cases of cognitive rupture occurred between the representation system of mathematical proof and another representation system. Additionally, there were cases of cognitive rupture within the representation system of mathematical proof on the Composite Function Task.

Continued analysis will delve into (a) the effects of the above factors on cognitive unity and rupture, (b) the significance of distinguishing types of cognitive unity and rupture, and (c) the correlation between cognitive unity and rupture and correctness. Such analysis should contribute to the questions of what conditions may lead to cognitive unity and rupture and under which conditions cognitive unity facilitates proving and cognitive rupture hinders it.

Questions for the Audience

How can this research be used to effect curriculum change in which students are engaged in activities that promote cognitive unity? How might cognitive rupture be used to provoke cognitive conflict to help students identify misconceptions or inconsistent concepts images?

References

Alcock, L., & Weber, K. (2010). Referential and syntactic approaches to proving: Case studies from a transition-to-proof course. In F. Hitt, D. A. Holten, & P. Thompson (Eds.), *Research in collegiate mathematics education*. *VII* (pp. 93–114). Providence, RI: American Mathematical Society.

Burton, L. (2004). *Mathematicians as enquirers: Learning about learning mathematics*. Boston, MA: Kluwer.

Committee on the Undergraduate Program in Mathematics. (2004). *Undergraduate programs and courses in the mathematical sciences: CUPM curriculum guide*. Washington, DC: Mathematical Association of America.

de Villiers, M. (2010). Experimentation and proof in mathematics. In G. Hanna, H. N. Jahnke & H. Pulte (Eds.) *Explanation and proof in mathematics: Philosophical and educational perspectives* (pp. 205–221). New York, NY: Springer.

Durand-Guerrier, V., Boero, P., Douek, N., Epp, S., & Tanguay, D. (2012). Argumentation and proof in the mathematics classroom. In G. Hanna & M. de Villiers (Eds) *Proof and proving in mathematics education: The 19th ICMI study* (pp. 349–367). Dordrecht, Netherlands: Springer.

Fischbein, E. (1987). *Intuition in science and mathematics: An educational approach*. Dordrecht, Netherlands: Kluwer.

Garuti, R., Boero, P., & Lemut, E. (1998). Cognitive unity of theorems and difficulty of proof. In A. Olivier, & K. Newstead (Eds.) *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 2, pp. 345–352). Stellenbosch, South Africa.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education*. *III* (pp. 234–283). Providence, RI: American Mathematical Society.

Raman, M. (2003). Key ideas: What are they and how can they help us understand how people view proof? *Educational Studies in Mathematics, 52,* 319–325.

Weber, K., & Alcock, L. (2009). Proof in advanced mathematics classes: Semantic and syntactic reasoning in the representation system of proof. In D. A. Stylianou, M. L.

Blanton, & E. J. Knuth (Eds.), *Teaching and learning proof across the grades: A K–16 perspective* (pp. 323–338). New York, NY: Routledge.

The effects of supplemental instruction on content knowledge and attitude changes

There are many reports on the impact of Supplemental Instruction (SI) on grades and retention of students within STEM disciplines. Although we know that many factors contribute to higher academic success (e.g., increased content knowledge, motivation toward subject, and/or time on task), there is little evidence for the mechanism by which SI improves student performance in any of these factors. This report is a preliminary analysis of an ongoing, multi-disciplinary study, using preexisting valid and reliable content knowledge assessments and attitude surveys, to gain insight into the effects of SI on STEM students.

Supplemental Instruction (SI) was developed at University of Missouri – Kansas City (UMKC) in the early 1970's, in which SI workshops, led by students, provide opportunities for students in historically difficult classes to work together on additional problems. The SI leader is typically a junior or senior level student who has succeeded in the same class. There is a large body of work showing that students who participate in SI workshops typically have higher grades and retention rates (e.g; Fayowski & McMillan, 2008; Malm, Bryngfors, & Mörner, 2011). Prior data indicates that the SI program at our institution, in place since 2007, also have the same strong positive effects. However, the specific ways in which SI workshops help students improve content knowledge or attitudes towards the subject are less clear.

Our research project is working to rectify this deficiency in the literature by conducting a large-scale, multidisciplinary study measuring the effects of SI in terms of students' content knowledge and attitudes and beliefs about science and mathematics learning. Our specific research questions are:

1: To what extent is attendance in SI sessions associated with improved performance on course specific surveys of conceptual understanding?
2: Do the attitudes and beliefs about learning science and math of students who participate in SI sessions differ from the views of their peers who did not attend SI sessions? How, if at all, do these attitudes and beliefs shift from beginning to end of a course?

This project is interdisciplinary, but the poster presented will focus on our mathematics SI classes. In mathematics, we will focus on precalculus and first-semester calculus classes. Our methodology is twofold: first, we will measure the impact of SI on student learning via pre-/post- discipline-based concept inventories; in mathematics, we will use the Precalculus Concept Assessment (PCA) developed by Carlson, Oehrtman, and Engelke (2010). Second, we will study the impact of SI on student attitudes and beliefs about learning though a pre/post study of scores on the Mathematics Attitudes and Perceptions Survey (MAPS), developed by Code, Merchant, and Lo (in preparation). The science classes will use the Colorado Learning Attitude about Science Survey, or CLASS, and its derivatives.

This poster will present our preliminary analysis of the first semester of data collection, which consists of data collected at the end of the semester only. At the time of this proposal, data is still being collected and analyzed, but the data collection and analysis will be complete by the end of 2014.

References

Carlson, M.; Oehrtman, M.; Engelke, N., (2010) "The Precalculus Concept Assessment: A tool for assessing students' reasoning abilities and understandings," *Cog. & Instr., 28*(2), 113-145.

Code, W. J., Merchant, S., & Lo, J. (in preparation). "Measurement of Student Perceptions and Attitudes in Mathematics."

Fayowski, V.; MacMillan, P.D., (2008) "An evaluation of the Supplemental Instruction programme in a first year calculus course," *Int. J. Math. Educ. Sci. Tech., 39*(7), 843-855.

Malm, J.; Bryngfors, L.; Mörner, L., (2011) "Supplemental Instruction: Whom Does it Serve?" *Int. J. Teach. Learn. Higher. Educ., 23*(3), 282-291.

Calculus students' understanding of logical implication and its relationship to their understanding of calculus theorems

Joshua Case
University of Maine

In undergraduate mathematics, deductive reasoning is an important skill in the learning of theoretical ideas. Deductive reasoning is primarily characterized by the concept of logical implication (inferring what follows from a given premise). This plays a role whenever mathematical theorems are applied, i.e. one must first check if a theorem's hypothesis is satisfied and then make a correct inference. In Calculus, students must learn how to apply theorems. However, most undergraduates have not yet received extensive training in propositional logic. How do these students comprehend the notion of logical implication and how does it relate to their understanding of theorems? Results from a pilot study indicated that students struggled with the notion of logical implication in both symbolic and Calculus contexts. However, findings were inconclusive regarding the relationship between the two areas. Background on the current literature, results of the pilot study, and further avenues of inquiry are discussed.

Key words: [Logic, Implication, Calculus, Theorems, Converse]

Background and Research Question

Calculus plays a fundamental role in many STEM areas such as physics and engineering. Thus, many STEM majors will have to take at least one semester of Calculus in order to be successful in their specialization. During their Calculus journey, these students will encounter – in some form or another – propositions, lemmas, and theorems. These statements are used to infer further statements, which help to build up and make coherent any area of mathematical investigation. For example, Calculus students must first understand the basic algebraic properties of the real numbers (commutativity, associativity, etc.) in order to infer the rest of the results that will be presented to them throughout the semester. This deductive process, characterized by the notion of logical implication, is the hallmark of all mathematical thinking. Thus, in using a theorem effectively, a student must first comprehend logical implication, which requires the understanding of four reasoning patterns. These patterns are provided below with the assumption that the rule "A implies B" holds.

Modus ponens
Suppose A is True. Then B is True.

Inverse
Suppose A is False. Then it is not known whether B is True or False.

Contrapositive
Suppose B is False. Then A is False.

Converse
Suppose B is True. Then it is not known whether A is True or False.

Past research has shown that both children and adults struggle to understand these reasoning patterns and that the *Inverse* and *Converse* tasks are known to be particularly difficult (O'Brien, Shapiro, and Reali, 1971; Wason, 1968). Additionally, individuals appear to be less successful when tasks are placed in abstract settings (A. Stylianides, G. Stylianides, Philippou, 2004). Also, it is well known that students struggle with Calculus ideas such as limits, differentiation, and integration (e.g., Carlson & Rasmussen, 2008; Tall, 1993; Orton, 1983a; Orton, 1983b; Zandieh, 2000). The instruction students receive about these key ideas often includes theorem or theorem-like statements that make use of the above reasoning patterns. Although much work has been done on the issue of logical implication and Calculus learning, little (if any) research has been done concerning whether the understanding of logical implication is associated with the understanding of Calculus theorems. This research project was designed to answer the following research question: How do Calculus students comprehend the notion of logical implication and how does it relate to their learning of Calculus theorems? Similar to much of the prior work on student thinking about Calculus, this study was done from a cognitive theoretical perspective and thus students' written statements were used as data on their thinking and understanding of the ideas.

Research Methods

First, I describe the data collection that has taken place so far. Two surveys (Survey A and Survey B) were given in a first semester differential Calculus I class at a university in New England near the end of the spring 2014 semester. These surveys were distributed in the three recitation sections associated with the course. There were a total of 61 participants. Both surveys are structurally the same and have two parts. Part I (consisting of eight tasks) was designed to gauge student understanding of logical implication in an abstract setting. Many of these tasks resemble syllogisms (ex: All men are mortal. Socrates is a man. Therefore, Socrates is mortal) but are stated in a formal context. Part II (consisting of one task) was designed to gauge student understanding of Calculus theorems. Figure 1 shows examples of survey tasks from Part I and Part II.

Consider the following proposition:

Proposition 2. *For integers a and b, if $a \sim b$ then $aba \sim bab$.*

Use Proposition 2 to answer questions 3-6. For each question circle a, b, or c.

3) Suppose $2 \sim 13$ is false. Then

a. $(2)(13)(2) \sim (13)(2)(13)$ is true.

b. $(2)(13)(2) \sim (13)(2)(13)$ is false.

c. Not enough information to decide if $(2)(13)(2) \sim (13)(2)(13)$ is true or false.

Consider the following theorem from Calculus:

Theorem. *If f is continuous on the closed interval* [a,b] *and k is any number between f(a) and f(b), then there is at least one number c in* [a,b] *such that f(c) = k.*

7) Consider the function $f(x) = x^3$ on the closed interval [-3,2]. Given the above theorem and that

- f is continuous on $[-3, 2]$
- $f(-3) = -27$ and $f(2) = 8$
- -5 is between -27 and 8

what can we conclude?

Figure 1. (Left) A sample task from Part I. (Right) A sample task from Part II.

Data Analysis

Part I was coded for correctness only. For example, the solution to task 3) above is **c**, not enough information to decide if (2)(13)(2) ~ (13)(2)(13) is true or false. Part II was coded using the scheme below:

0 – An incorrect and unclear response with little or no attempt at applying the theorem.
1 – A partially correct or clear response with an attempt to apply the theorem.
2 – A mostly correct and clear response with an attempt to apply the theorem.
3 – A fully correct and clear response. The theorem was applied successfully.

Below are examples to illustrate the above scores used to code the task from Part II (the task on the right in Figure 1):

Response coded 0: *That this is a problem I don't understand, sorry I couldn't be more useful.*
Response coded 1: *That -5 is k, which is the number Between f(a) and f(b) and f(a) = -27 and f(b) = 8.*
Response coded 2: *That there is some value of x between [-3, 2] on the graph that represents -5 on the y-axis.*
Response coded 3: *There is at least one number in [-3,2] such that f(c) = -5.*

Results

Table 1 gives quantitative results for tasks 3-6 of Part I (both Survey A and B).

Success rates for the four reasoning pattern tasks in Part I

Task	Survey A (% of correct responses)	Survey B (% of correct responses)
3	12.5	10
4	62.5	93
5	81.25	79
6	12.5	11

Table 1. Results for Part I (both Survey A and Survey B).

These results indicate that students struggled with tasks resembling *inverse* and *converse* patterns (tasks 3 and 6), but struggled less with tasks resembling *modus ponens* and *contrapositive* patterns (tasks 4 and 5). Table 2 gives the results for Part II of both Survey A and B. Since the vast majority of participants were given a score of either 0 or 1, it appears that these Calculus students struggled to apply theorems clearly and correctly. Additionally, there did not appear to be a clear correlation between student performance on Parts I and II (see Table 3). For example, if a strong correlation existed between performance on the logical tasks (Part I) and the theorem interpretation task (Part II), the percentages on the diagonals of Table 3 would be much larger. The majority of students could correctly answer most problems from Part I, but struggled to give a coherent answer in Part II.

Score rates for the Calculus theorem task in Part II

Survey A

Score	0	1	2	3
%	62.5	25	9.375	3.125

Survey B

Score	0	1	2	3
%	55	45	0	0

Table 2. Results for Part II (both Survey A and Survey B).

Implications and Further Avenues of Inquiry

Based on the preliminary findings, it seems that undergraduate Calculus students do not have a complete grasp of logical implication and struggle to apply Calculus theorems. It also appears that there is not a clear relationship between student performances on the two parts. These results suggest that undergraduate Calculus students may need more training in logic, but that this preparation may not improve student ability to apply theorems learned in class. Future research plans include administering surveys with theorem questions that more closely match the structure of the questions in Part I to examine possible connections more directly. Also, interviews will be conducted to further explore students' interpretation and understanding of logic-based statements in Calculus theorems.

Relationship Between Scores on Part I and Part II (Survey A)

		Part I	
		% of students who correctly answered less than 50% of problems	% of students who correctly answered at least 50% of problems
Part II	% of students who obtained a score of 2 or 3	0	12.5
	% of students who obtained a score of 0 or 1	12.5	75

Relationship Between Scores on Part I and Part II (Survey B)

		Part I	
		% of students who correctly answered less than 50% of problems	% of students who correctly answered at least 50% of problems
Part II	% of students who obtained a score of 2 or 3	0	0
	% of students who obtained a score of 0 or 1	10.3	89.7

Table 3. Comparing student success on Part I with student success on Part II.

Questions posed to the audience will include:

1. If I were to use a different theorem for the task in Part II, which theorem would you suggest I use?
2. I plan to interview students to study their understanding of theorems. How should I design the interview? If the student struggles to express him or herself, what can I say to help elicit their thinking?

References

Carlson, M., & Rasmussen, C. (2008). Making the connection: Research and teaching in undergraduate mathematics education. *MAA Notes*. Washington, DC: Mathematical Association of America.

O'Brien, T. C., Shapiro, B. J., & Reali, N. C. (1971). Logical Thinking - Language and Context. *Educational Studies in Mathematics*, *4*(2), 201–219.

Orton, A. (1983a). Students' Understanding of Differentiation. *Educational Studies in Mathematics*, *14*(3), 235–250.

Orton, A. (1983b). Students' Understanding of Integration. *Educational Studies in Mathematics*, *14*(1), 1–18.

Stylianides, A. J., Stylianides, G. J., & Philippou, G. N. (2004). Undergraduate Students' Understanding of the Contraposition Equivalence Rule in Symbolic and Verbal Contexts. *Educational Studies in Mathematics*, *55*(1-3), 133–162.

Tall, D. (1993). Students ' Difficulties in Calculus. In *Proceedings of Working Group 3 on Students' Difficulties in Calculus* (pp. 13–28). Québec: ICME-7.

Wason, P. C. (1968). Reasoning about a rule. *Quarterly Journal of Experimental Psychology*, *20*(3), 273–281.

Zandieh, M. J. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *CBMS Issues in Mathematics: Research in Collegiate Mathematics Education* (Vol. IV(8), pp. 103–127).

Students' perceptions of the disciplinary appropriateness of their approximation strategies

Danielle Champney, David Kato, Jordan Spies, Kelsea Weber
California Polytehnic State University

Within the context of Taylor series expansions as approximations, we illustrate the context dependence of student reasoning about these approximations - specifically the ways in which students' notions of what is appropriate in mathematics, physics, or engineering, drive how they engage in and reflect on the solutions they produce. Using data from semi-structured interviews, we build on previous work to argue that students' epistemological framing not only plays a role in their choice of solution strategies, but also how they feel those solution strategies would be perceived within various disciplines.

Keywords: Taylor series, interdisciplinary, calculus, approximation

While traditionally, much of the literature and emphasis around interdisciplinary curricula has focused on the alignment of disciplinary content, there has been a recent push to look at other contributing factors, such as the epistemological differences between disciplines, and how students view and take those up when reasoning with problems across contexts (Stevens, et al., 2005; Watkins et al., 2012; Champney & Kuo, 2012, 2014). This research has suggested the importance of both understanding how students perceive and manage these epistemological differences between disciplines (Kuo & Champney, 2012), and of studying how these perceived epistemological differences could potentially offer alternate diagnoses to situations that the previous state-of-the-art might refer to as 'failed transfer' (Kuo & Champney, 2014). In this study, we intend to expand on the importance of understanding students' perceived epistemological differences, extending the analysis beyond the students' actual solution strategies to mathematics and physics tasks, toward their rhetoric around who they think the 'target audience' of their work is, and why their work and solutions are 'appropriate' for that audience, but not others. Therefore, we aim to demonstrate, with examples, how students' epistemological framing not only plays a role in their choice of solution strategies, but also how they feel those solution strategies would be perceived within various disciplines, such as physics, mathematics, and engineering.

Theoretical Grounding

We pursue this line of work drawing heavily on the notions of resource activation (e.g. Hammer et al., 2005) and the importance of attending to students' epistemological framing. Hammer, Elby, Scherr, and Redish (2005) argue that what has been described as transfer is the activation of similar knowledge resources in various situations, and that this activation depends not only on content knowledge and problem features, but also on epistemological stances towards what kinds of knowledge are appropriate in different situations. That is, there are factors beyond simply the students' content knowledge that can contribute to why they apply their math knowledge differently on similar problems, and one of those factors can be students' epistemological framing – the *epistemology diagnosis* (Kuo & Champney, 2014).

Champney & Kuo have drawn on this perspective (2012), to examine the ways that a single student's epistemological framing of two isomorphic tasks, which differed in context, dictated entirely different solution strategies, to the point where he denied any commonalities in what the tasks required of him, even though the underlying mathematics was virtually the same. Here we will expand on the 2012 findings, to demonstrate how three different students' epistemological framings of the same task not only dictate *how* they solve the problem, but for whom they feel their solution is most appropriate.

Data collection and methods

In the spring of 2011, 15 students participated in semi-structured interviews to investigate their reasoning about approximations on introductory physics and calculus content (see also Champney & Kuo (2012, 2014)). Interviews consisted of several questions, during which students reasoned aloud and an interviewer asked clarifying questions. Students were able to use a calculator on these tasks. The cases we present here are three undergraduate students (Joanne, Chris, and Brad), all of whom had recently completed introductory calculus and physics sequences at the same institution. These students were selected to highlight in this proposal because they represent broader categories of students in the larger data set, which is currently being analyzed for a final report.

Each student completed 3-4 tasks, including ARCTAN (see Figure 1), and a similar problem (PENDULUM), in which they were asked a similar question, within the context of calculating period of a pendulum. At the time of the interview, all students had studied Taylor series approximations. Because "bad approximation" is not well defined, in the task or by the interviewer, these tasks were designed to reveal how a student makes judgments about approximations, not whether students can solve these tasks "correctly."

The Taylor series about x=0 for arctan(x) is given by: $\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + ...$

How big a value can x be, before stopping after the second term is a bad approximation?

Figure 1. ARCTAN problem.

Results and ongoing analyses

The data described here are incomplete, as this is a preliminary report. However, included are highlights of the aforementioned students' interviews, and their work that is currently being analyzed and coded for presentation. We hope that the snapshot provided illuminates the type of contribution that we will make with this work in February 2015.

Joanne. Joanne solved the ARCTAN task first, and produced a quick and confident answer. To Joanne, this problem was no different than any other problem in which she was asked to approximate something (solutions to ODEs, root finding, etc…), and she had a scheme for performing a numerical approximation that looked like:

$$|Actual\text{-}Approximation| \leq Tolerance.$$

Joanne's choice of mathematical method was not ambiguous in any way – she utilized what she viewed as a flexible method, one she had used countless times in computer science, mathematics, engineering, and other courses, and applied it to the given problem. To Joanne, while the method is flexible enough to handle this problem, the role of context only came to bear when considering 'how precise' she wished to be. That is, Joanne's use of mathematical tools was not dictated by the context, but the outcome that she was attempting *was* bound by her interpretation of 'who would care' about this problem. In her own words,

J: Or whatever tolerance ... Well I was just when I did the first approximation [I used a tolerance of] ten to the negative 2 so I just arbitrarily set it, but if I were to, I'm trying to think… In my class, our tolerance is usually 10 to the negative 6 or something like that. But that's because for engineering it's a little more uh … Let's just change that to 10 to the negative 6.

I: And so all you did was change that right hand number?

J: Yeah you can interchange your tolerance to whatever you need it to be set to.

Joanne's method for PENDULUM was essentially identical to her work on ARCTAN, though her value for the acceptable 'tolerance' was adjusted:

J: Well, since this is an... It's a real life, still kind of, a real life problem … Because you're using this to model this to model something that is actually happening, you might be using it in an engineering context. So I would end up doing is for this one I would have a smaller tolerance value.

I: Smaller than the previous one?

J: Yeah, well the previous one was ten to the negative 2 or negative 6. So I would just make this ten to the negative 7 or 8.

We find it worthwhile to note that, (a) for Joanne, the process she uses to reason with the Taylor series is not uniquely tied to any particular context – and she in fact uses it across multiple contexts, but (b) Joanne's interpretation of an 'engineering audience' for her work dictated a stricter tolerance than she would employ in a mathematics or physics context. Thus, her framing of the problem as one in a much larger class of 'approximation problems,' for which this particular, flexible method is appropriate, permits her to make progress toward reasoning with the task itself, and to appropriate any desired mathematical tools.

Chris. To contrast Joanne, Chris settles on a 'data-driven' approach, through which he produces a table of values for $arctan(x)$, its series approximation, and the difference between the two. Upon discussing his solution, Chris articulates that the approach is satisfactory, but different audiences may view it as more or less successful:

C: I feel like a math professor would want you to do some sort of math thing to it rather than just looking at a bunch of data points and being like, "oh, that's where it's too far away."

I: So you felt like you didn't do a 'math thing'?

C: It wasn't like a math approach. It was kind of like more a physics approach, which is approximation. If you're building something and you need an approximation, you just decide when it's outside the tolerance and that's it.

I: Does that mean a physicist would like the way you thought about it, probably?

C: I think an engineer would. If you're building something and you need an approximation, you just decide when it's outside the tolerance and that's it.

I: But you said that you didn't do something very 'mathy.'

C: Yeah, I don't know. That's just kind of a feeling. When you integrate something as opposed to just guessing or just looking at something... If you don't do any math, it doesn't feel like a math problem. Even though it's presented like one. I think the presentation is the math problem but my approach wasn't a math problem's approach.

Thus, while Chris had a successful approach that led him to an answer with which he was comfortable, he viewed his approach as not being suitable within particular disciplines. Chris frames the task as one that may be appropriate for many disciplines, while his solution method may only be appreciated by some subset of those disciplines – that is, the type of knowledge required for an engineering context would permit him to pursue a more basic, comparison type of approach, while other disciplines, in his opinion, would desire either more 'rigorous' or "mathy" approaches in order to be acceptable.

Brad. One final point of contrast, Brad, made no progress with the ARCTAN task whatsoever – a phenomenon that we at least partially attribute to his framing of the task as one of a highly specialized class of tasks, for which more information is required in order to produce any reasonable results. Rather than quickly finding an appropriate method of attack and pursuing a solution, Brad's time with the ARCTAN task is mired in conversation about how these problems are all specific and require an authority figure (professor, TA, etc…) to provide a target error or method by which he 'should' solve the problem:

B: Cuz I'm an engineering major, so therefore I like ... you're usually calculating to determine the tolerance or the stress levels or whatever of some material. Therefore I like to be super, super exact so I'd choose a really, really small tolerance. But, if this is just like a *math problem, I mean you know, we are usually just given numbers and just work it out to get an answer.*

For Brad, both the method and the desired accuracy are driven by the perceived context, and therefore no amount of mathematical tools that he may or may not have at his disposal will make any difference, until he is provided more information. To reinforce this point, Brad's framing of this problem as one of a highly specialized class of problems leads him to abandon his attempts to utilize the Alternating Series Remainder Theorem, a graphical approach, and other approaches that he suggests and then immediately discounts.

Next steps, and Questions for the audience

In our final report, we will demonstrate that these three students represent broader groupings of students from the larger study – Joanne, representing those students who have flexible mathematical tools that they can appropriate for whatever audience they feel is appropriate; Chris, representing those students who are able to produce an answer and approach, but do not feel that it is suitable for all audiences; and Brad, representing those students who have a variety of mathematical tools available, but do not perceive the context as permitting them to use said tools without knowing more about their audience. These classifications are interestingly tied to the students' epistemological framings of the tasks with which they are presented - a phenomenon that can be more substantially demonstrated outside a 3-page proposal, but was briefly discussed in the previous section.

The audience of the preliminary report session can assist this work by helping the research team here to think through the following points of interest: (1) How may these students' respective framings of the ARCTAN task inform initial instruction of this topic in calculus classes, or reviews of this topic in physics/engineering classes? and (2) How can this speak to a broader program of helping students distinguish between an *approach* being appropriate for problem solving in a particular discipline vs. the *result* of problem solving be acceptable to a particular discipline.

References

Champney, D. & Kuo, E. (2012). Disciplinary dependence of student reasoning about approximation. Poster presented at Transforming Research in Undergraduate STEM Education Conference. St. Paul, MN, June 2012.

Kuo, E., Champney, D., & Little, A. (2012). Considering factors beyond transfer of k nowledge. Proceedings for Physics Education Research Conference (PERC). Philadelphia, PA, August 2012.

Champney, D. & Kuo, E. (2012). Beyond the physics classroom: Exploring disciplinary factors that influence students' reasoning about approximation, through video data. Proceedings for Physics Education Research Conference (PERC). Philadelphia, PA, August 2012.

Kuo, E. & Champney, D. (2014). Three Diagnoses of Why Transfer Across Disciplines Can Fail and Their Implications for Interdisciplinary Education. Polman, J. L., Kyza, E.A., O'Neill, D. K., Tabak, I., Penuel, W. R., Jurow, A. S., O'Connor, K., Lee, T., and D'Amico, L. (Eds.). (2014). Learning and becoming in practice: The International Conference of the Learning Sciences (ICLS) 2014, Volume 1. Boulder, CO: International Society of the Learning Sciences.

Hammer, D., Elby, A., Scherr, R. E., & Redish, E. F. (2005). Resources, framing, and transfer. In J. Mestre (Ed.), *Transfer of learning from a modern multidisciplinary perspective* (pp. 89–120). Greenwich, CT: Information Age Publishing.

Stevens, R., Wineburg, S., Rupert Kerrenkohl, L. R. & Bell, P. (2005). Comparative understanding of school subjects: Past, present, and future. *Review of Educational Research 75*(2), 125-157.

Watkins, J., Coffey, J., Redish, E. F., & Cooke, T. (2012). Disciplinary authenticity: Enriching the reforms of introductory physics courses for life-science students. Phys. Rev. ST Phys. Educ. Res., Vol. 8.

Acknowledgements

This work was partially supported by a sub-award under NSF-CCLI-0941191.

Developing abstract knowledge in advanced mathematics: Continuous functions and the transition to topology

Daniel Cheshire
Texas State University

Despite intuitive foundations, the nature of the transition to abstract topology often results in students' reliance on dissociated collections of definitions and theorems, without any integrated cognitive structure. In recent decades, there have been numerous analyses of proof, symbols, and the encapsulation of processes as factors in student comprehension, as well as content-specific studies examining which mental constructions support the development of coherent schemata for particular topics. I will expand this research by categorizing students' understanding in the domain of topology. In a year-long, mixed-methods study, I will analyze the components involved in the development of an axiomatic schema for continuous functions in topological contexts. I will compare this model with actual student constructions in an introductory topology course, collected through task-based interviews and a path-analysis on the coded data. The goal is to confirm the theoretical model, or to provide support for altering the model to increase its validity.

Key words: Topology, Mathematics Education, Continuous Functions, Open Sets, Abstraction

The field of topology is rooted in the abstraction and generalization of intuitive notions, such as continuity and connectedness, derived from the familiar settings of the real numbers and Euclidean space. These attributes make topology an ideal candidate for studying the intermediary processes between students' perceptual models of understanding and their development of formal conceptual schemas. Recent trends in research into abstract thinking have provided rich insights into processes involved in learning advanced mathematics, supporting the development of theoretical learning models for many abstract topics (Tall & Vinner, 1981; Sfard, 1991; Pirie & Kieren, 1994; Gray & Tall, 2007). However, few of these studies have measured the relation of these models to the development of topology-specific constructions. I will investigate concept development in this field through a synthesis of theories concerning genetic decomposition (Piaget, 1989; von Glasersfeld, 1995; Arnon et al., 2014), example use and instantiation of mental objects (Alcock, 2004; Pimm & Mason, 2004), and proceptual compression (Gray & Tall, 1994). The results of this analysis will be a starting point for the creation of instructional strategies to aid in the introduction to this challenging field of study.

Research Questions

To develop and expand the new theoretical learning models noted above, as well as to address the noted domain-specific deficit in the research, I wish to answer the following questions concerning students' understanding in an introductory topology course:

1) How, and to what extent, does a learner's formal schema for the continuity of functions depend on the existence, coherence and developmental level of these conceptual schemas?

   a) set theory and cardinality
   b) functions, images and pre-images on abstract domains
   c) sequences, limits, and related notions
   d) open/closed sets in metric and topological spaces

2) For each of these schemas, how do learners instantiate and use the requisite empirical, proceptual, and axiomatic objects; and what influence does this have on the development of formal, definition-based objects and an axiomatic level of understanding of continuity?

Methodology

The investigation will consist of a combination of qualitative and quantitative methods, and will be conducted at a large emerging-research university in central Texas. Classroom observations, artifact analyses, knowledge assessments, and task-based interviews with students in an introductory topology class, will be used to construct case studies of 3-5 students' experiences. In addition, all of the interviews will be coded with a rubric based on recent theoretical research and a factor analysis of the assessment results. These will serve as inputs into a structural equation model to determine the contributions of the factors proposed above. Finally, the model will be used to evaluate a series of preliminary concept decompositions for several key topological concepts, developed in light of current research and the expertise of two topologists.

In order to detect confounding variables, I will also examine how, and to what extent the students' construction of a formal schema for continuous functions is mediated by the students': *a)* notational fluency and proof construction/validation skills, and *b)* levels of self-efficacy in mathematics, especially in analysis and topology.

References

Alcock, L. (2004). Uses of example objects in proving. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, *2*, 17-24).

Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Roa Fuentes, S., Trigueros, M., & Weller, K. (2014). *APOS theory: A framework for research and curriculum development in mathematics education*. New York: Springer.

Gray, E., & Tall, D. (1994). Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education, 25*(2), 116-140.

Gray, E., & Tall, D. (2007). Abstraction as a natural process of mental compression. *Mathematics Education Research Journal, 19*(2), 23-40.

Piaget, J. & R (1989). *Psychogenesis and the History of Science*. New York: Columbia University Press.

Pimm, D., & Mason, J. (1984). Generic examples: Seeing the general in the particular. *Educational Studies in Mathematics, 15*(3), 277-289.

Pirie, S., & Kieren, T. (1994). Beyond metaphor: Formalising in mathematical understanding within constructivist environments. *For the Learning of Mathematics, 14*(1), 39-43.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1-36.

Tall, D., & Vinner S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151-169.

von Glasersfeld, E. (1995). *Radical Constructivism: A Way of Knowing and Learning*. London; New York: Routledge/Falmer.

The equation has particles! How calculus students construct definite integral models

Michael Oehrtman and Kritika Chhetri
University of Northern Colorado

This research characterizes the cognitive challenges that students encounter while constructing definite integrals to model physical quantities and relationships, how students resolve those challenges, and the resulting conceptual artifacts. We video-recorded four groups of second-semester calculus students working with definite integral models during two 50-minute labs. This paper focuses on how two groups of students that worked on the same problem reasoned about and constructed an integral representing the gravitational attraction between a co-linear thin rod and point mass. Prior research on students' understanding of definite integrals posits the multiplicative structure $f(x)\cdot\Delta x$ *in a Riemann sum as an essential component in conceiving an integral. Our findings indicate that in many contexts, other symbolic forms subsume the simple product in this essential conceptual role and that they interact significantly with students' symbolic forms for the definite integral.*

Keywords: Definite integral, Riemann sum, adding up pieces, symbolic form

Introduction and Research Question

Selden, Selden, Hauk and Mason (2000) witnessed that when presented with non-routine problems, more than half of the students who had completed a year and half of traditional calculus "were unable to solve even one problem and more than a third made no substantial progress towards any solution" (p. 128). Prevailing research on students' understanding of integration, one of the core concepts of calculus, posits that students have difficulty understanding how definite integrals may model actual quantities (Orton, 1983; Chhetri & Martin, 2014; Sealey, 2006; Sealey & Oehrtman, 2007; Von Korff & Rebello, 2012). Sealey (2006) decomposed the understanding of a Riemann integral into layers based on the order of mathematical operations involved in its definition. One conceives of i) quantities represented by an evaluated function $f(x)$ and an increment in its domain Δx, ii) their product, iii) a summation of these products, and iv) a limiting process applied to this sum in which $\Delta x \to 0$. Jones (2014) proposed the symbolic form of "adding up pieces" is helpful for students conceiving the definite integral. All existing research on definite integral frames the pieces being added as entailing the multiplicative structure $f(x)\cdot\Delta x$, but we hypothesized that that many definite integral models are not productively conceived in terms of this multiplicative structure. The example we explore in this paper asked students to express the gravitational force between a thin uniform rod of mass *M* and length *L* and a particle of mass *m* lying in the same line as the rod at a distance *a* from one end as

$$\int_a^{a+L} \frac{GMm}{Lr^2}\,dr$$

where *G* is the universal gravitational constant. Although we can always mathematically express the terms in the Riemann sum as a product, in this case as $\frac{GMm}{Lr^2}\cdot\Delta r$, it is unlikely that these factors are useful quantities to conceive either in the interpretation of the integral or in the modeling process. Our research question is:

Given that students have already conceptualized the Riemann integral as a model of accumulation, how do they reason about a novel problem in which the integral does not clearly a multiplicative quantitative structure?

In this paper, we detail how two groups of students in a second semester calculus course reasoned about and constructed this integral.

Theoretical Perspective

Sherin (2006) introduced the construct of symbolic forms to explore the ways people create or interpret novel equations, and hypothesized that we accumulate schemes of numerous types of mathematical expressions, which we discern almost as a Gestalt. For instance, when people see an equation of the form [ ] = [ ] + [ ], they may immediately interpret it through a schema of parts of a whole, considering quantities in the location of the brackets on the right hand side as parts and the quantity located on the left hand side as the whole, and where the equality indicates a balance of two (Jones, under review). Conversely, a situation involving a quantity decomposed into parts may invoke an equation of this form through activation of the same schema. Sherin (2006) defined a symbolic form as consisting of a symbol template and an associated a conceptual schema. The symbol template defines the arrangement of symbols in a mathematical expression. An associated conceptual schema is comprised of a network of meanings that people may use to bind particular symbols to the slots in the symbolic template (Sherin, 2006). Often a conceptual schema is driven by p-prims, basic ideas about how things work (diSessa, 1993).

For the purpose of this study, we consider the ways in which symbolic forms influence calculus students' construction of definite integrals to model physical situations. Jones (2014) characterized three symbolic forms for integrals with the template $\int_{[\,]}^{[\,]}[\,]d[\,]$ or $\int_{[\,]}[\,]d[\,]$: function matching, perimeter and area, and adding up pieces. Function matching refers to conceiving of integrals in terms of an antiderivative, where the slots determine the function, variable of integration, and (possibly) limits at which to evaluate the antiderivative. The perimeter and area schema associates each box with part of a perimeter in the *x*,*y*-plane where the variable inside *d*[ ] determines the horizontal axis, the limits of integration represent either vertical sides or the length of the horizontal side, and the slot for the function represents a fourth side along its graph. These boundaries form the perimeter of a region and the value of the integral represents the area of that region (Jones, under review). In the adding up pieces scheme, students break a quantity into smaller pieces formed as products $[\,]\cdot d[\,]$ throughout a region determined by the limits of integration then add those pieces to form a whole, represented by the entire integral.

Methods

We videotaped four groups (approximately four students in each group) of second-semester calculus students during two 50-minute labs about applications of the definite integral. In this paper, we present data from two groups of students working on the following problem:

The gravitational attraction between two particles of mass m_1 and m_2 at a distance *r* apart is

$$F(r)=\frac{Gm_1m_2}{r^2}.$$

> Write a definite integral that that gives the gravitational attraction between a thin uniform rod of mass M and length l and a particle of mass m lying in the same line as the rod at a distance a from one end.

While analyzing the videos, we paid close attention to instances that were problematic to students and carefully observed how students resolved their problems, coding for both active symbol templates and schemas.

Results

Five students from Group A (Aaron, Andy, Amber, Alexis and Austin) engaged in three key episodes of reasoning as they constructed a definite integral to model the gravitational force of attraction between a rod and point mass. The first episode constitutes their realization that the equation for gravitational force, $F = \frac{GMm}{r^2}$, cannot be used to directly compute the force in this situation. Initially, this formula emerged as a symbolic template, $[\,] = \frac{G[\,][\,]}{[\,]^2}$ where the meanings of the slots (schema) were $[\text{force}] = \frac{G[\text{mass}][\text{mass}]}{[\text{separation}]^2}$. The students associated masses M and m into the mass slots and a into the separation. But when Amber said, "because we have a rod and particle not 2 particles," they recognized that the formula was not applicable when one of the masses was distributed along a rod rather than being located at a single point. The second episode comprises their activities of reconceiving the problem in terms of an "adding up pieces" scheme.

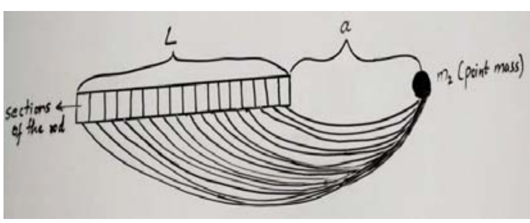


Figure 1. An artifact of students splitting the rod into pieces.

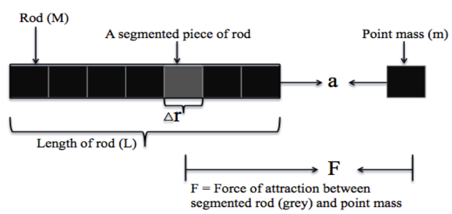


Figure 2. An illustration of students' breaking up idea.

They then drew a rod (as seen in Figure 1) and broke it into pieces to form point masses to which the formula $F = \frac{GMm}{r^2}$ could apply. The students drew the curved lines in Figure 1 to represent the force of attraction between each segment of the rod and the point mass. As they drew and reasoned about their diagram, they referenced elements in the template $[\,] = \frac{G[\,][\,]}{[\,]^2}$ with their revised schema focused on point masses to make several decisions about their activity. This symbolic form focused their attention on relevant quantities and forced them to represent and organize these quantities into a larger structure required to develop an appropriate model.

The adding up pieces symbolic form for the integral influenced the students' modeling process in two primary ways. First, it triggered and supported their shift to conceiving the rod as composed of numerous point masses and focusing on the formula as applying to one of these segmented point masses and the point mass in the original problem statement. Second, after the students developed an appropriate expression for these pieces of force, the symbolic form guided its incorporation into a definite integral with appropriate limits and separation of integrand and differential.

Developing an expression for the force between one piece of the rod and the point mass required the most time in the students' modeling process. The third episode involved the students working to fit the pieces into the slots. Even though they recognized the mass of segments of the rod conceptually, they struggled to represent this in terms of the given quantities. After several minutes and initial non-quantitative reasoning, they eventually realized that if the rod were to be broken into n pieces, the mass and length of each piece would be $\frac{M}{n}$ and $\frac{L}{n}$, respectively. Some quick algebra motivated by a need to express the eventual integral in terms of Δr rather than n allowed them to rewrite the mass of each piece as $\frac{\Delta r}{L} \cdot M$. At this point the students were able to appropriately represent all of the slots in the template $[\text{force}] = \frac{G[\text{mass}][\text{mass}]}{[\text{separation}]^2}$. Again driven by a need to develop a definite integral, the students identified the distance between each segment and the point mass as ranging from a to $a+L$ and bound these quantities into a definite integral $\text{whole} = \int_{[\text{start}]}^{[\text{end}]} [\text{pieces}]$, where $[\text{pieces}] = [\,]\, d[\text{variable}]$ resulting the final expression, $\int_a^{a+L} \frac{G\frac{M}{L}m}{r^2} dr$. They summarized their work by elaborating the meaning of each piece of their integral as seen in Figure 3.

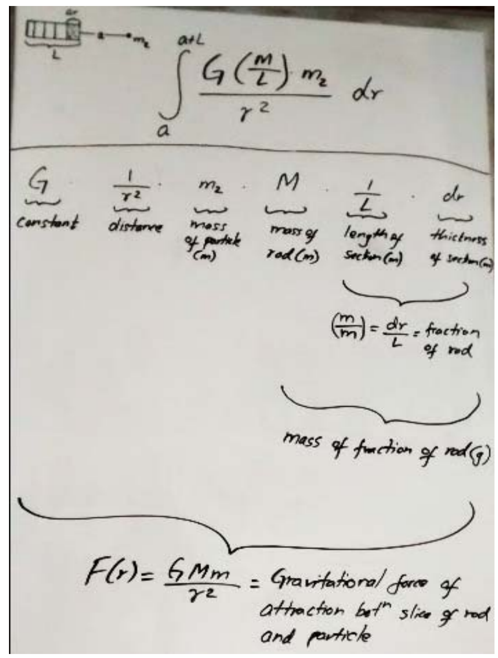


Figure 3. Final expression of students' work.

The four students (Cameron, Colt, Colby and Caesar) from Group C charted a different path solving this problem, but ultimately relied on the same symbolic forms to make progress. They engaged in three episodes of reasoning. Their first episode comprised of breaking the rod into segments without evoking the idea of "adding up pieces." Immediately after reading the problem, they drew a rod and point mass then broke the rod into bunch of little segments (Figure 4) and wrote the integral as $Gm\int_a^{a+L} \frac{M}{r^2}$. When the instructor inquired, none of the students were able to justify why they needed to break the rod into segments. Colby indicated that breaking the rod could help them identify the various parts of the definite integral template $\int_{[\text{start}]}^{[\text{end}]} [\text{something involving the variable}]$ but did not explain how 'breaking' the rod and integral related. In one of their several attempts they accidentally wrote the correct integral, $\int_a^{a+L} \frac{GMm}{Lr^2} dr$, but changed it immediately. For a while their discussion shifted from deciding what to write in the slot for the integrand and differential to discussing if and how the mass of the segments would vary, but still without attention to an adding up pieces schema.

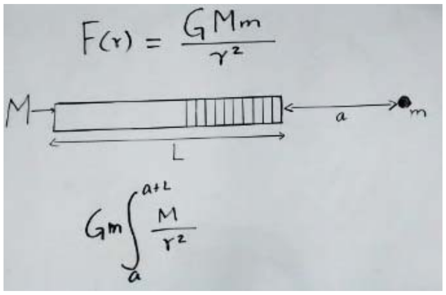


Figure 4. Students' initial illustration of their context.

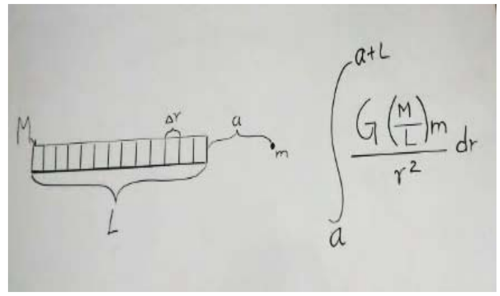


Figure 5. Students' final illustration of force between a rod and point mass.

During the second episode, the adding up pieces idea emerged after the teacher inquired if they had accomplished anything by segmenting the rod. Caesar introduced the idea of adding up pieces saying, "When we break it [the rod] up, the gravity at the end closest to the particle is gonna be greater than the other end…So then you add them [the force of attraction between the segments and point mass] all up and that's gonna give you total gravity of the whole rod since its one end is further away from the particle." The instructor then asked the group whether they could apply the formula $F = \frac{GMm}{r^2}$ to each segment of the rod that Caesar described. Everyone except Caesar instead continued attempting to identify in the components of the template $\int_{[\text{start}]}^{[\text{end}]} [\text{something involving the variable}]$ for another fourteen minutes. When the instructor returned and inquired about the evolution of Caesar's idea of adding up pieces of force, Cameron did not mention anything about adding up pieces. Caesar reasserted that they needed to add the forces of attraction between the segments of the rod and point mass to obtain the total force of attraction, i.e. $whole = [piece] + ... + [piece]$. At this point, everyone agreed with Caesar's idea and began to split the rod into segments (as seen in Figure 5), and they began identifying the slots of the new symbolic form, $\text{whole} = \int_{[\text{start}]}^{[\text{end}]} [\text{pieces}]$, where $[\text{pieces}] = []\, d[\text{variable}]$. With the instructor's prodding, Colt stated that by treating the segments of the rod as point masses, they would be able to apply the formula $F = \frac{GMm}{r^2}$. By this point, they had interpreted this as the template $[\text{force}] = \frac{G[\text{mass}][\text{mass}]}{[\text{separation}]^2}$ and were aware that they had to add the force of attraction between segments of the rod and point mass. Similar to Group A, this group also struggled to find mass of the segments. With their instructor's help, they determined that the proportion of the mass in each segment is $\frac{\Delta r}{L}$ so their mass is $\frac{\Delta r}{L} \cdot M$, which they then associated with a mass slot in the template $[\text{force}] = \dfrac{G[\text{mass}][\text{mass}]}{[\text{separation}]^2}$. Finally, they expressed the total gravitational force of attraction between the rod and point mass as $\int_a^{a+L} \dfrac{G\frac{M}{L}m}{r^2}\,dr$ (Figure 5).

Discussion

Sealey's (2006) work on students' understanding of definite integrals emphasized the layers of Riemann sum as fundamental for students to understand the need for definite integral. Her framework decomposed students' reasoning based on the mathematical operations involved on the right hand side of the definition $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_i)\Delta x$. Alternately, Jones (2014) posited the adding up pieces as a critical symbolic form for students to understand the definite integral, focusing on the left hand side of the equation. Our investigation illustrated critical interactions between these two basic structures. Our data reveals that students' interpretations of a symbolic form for the integrand can inhibit or trigger an adding up pieces scheme for the integral. Students in neither group invoked an adding up pieces scheme until they focused on the meaning of Newton's law of gravitation as applying specifically to two point masses (even though the students in Group C had already segmented the rod in the problem). The adding up pieces symbolic form alternately influenced student's interpretation of elements of Newton's law by focusing their attention on expressing the integral in terms of *r*, coaxing it into a form involving Δr as a factor, and considering the range of variation of *r*.

The definite integral modeling task discussed in this paper requires conceptually "breaking apart" and "adding up" pieces quantitatively more complicated than a product of the form $f(x_i)\cdot\Delta x$. We thus generalize Sealey's product layer to include more general symbolic forms, in this case drawn from the inverse square law in Newton's law of gravitation, $[\;] = \dfrac{G[\;][\;]}{[\;]^2}$. Unlike the groups that we have discussed in this paper, one of our other group of students worked on a relatively similar routine problem (finding the force of attraction between magnets) and they were able to construct the integral rather quickly and easily since the integrand was directly conceived as a product, $[\text{energy}] = [\text{force}]\cdot[\text{distance}] = f(x)\cdot\Delta x$. Another group worked on a task that was easily conceived as a product but required the group an intermediate amount of effort to reconceive with an appropriate integrand, $[\text{mass}] = [\text{density}]\cdot[\text{area}] = \delta(r)\cdot A(r) = \delta(r)\cdot C(r)\cdot\Delta r$.

We observed that students' ability to solve this non-routine problem was limited by the narrowness of the schemas that students applied. They achieved success upon being able to consider alternate, and eventually more appropriate symbolic forms for the situation.

Acknowledgment

This material is based upon work supported by the National Science Foundation under Grant Numbers 1245021 and 1245178.

References

Chhetri, K., & Martin, J. (2013). Challenges while modeling using Riemann sum. In S. Brown, G. Karakok, K. H. Roh, & M. Oehrtman (Eds.), *Proceedings of the Sixteenth Conference On Research in Undergraduate Mathematics Education* (Vol. 1, pp. 131-145), Denver, CO: University of Northern Colorado.

Jones, S. R. (2014). Three conceptualizations of the Definite integral in mathematics and physics contexts. *Electronic Proceedings for the 16th Annual Conference on Research in Undergraduate Mathematics Education*: Denver, CO.

Jones, S. R. (under review). Interpretation of the definite integral by a general U.S. calculus student population.

Orton, A. (1983). Students' understanding of integration. *Educational Studies in Mathematics*, *14*(1), 1-18.

Sealey, V. (2006). Definite integrals, Riemann sums, and area under a curve: What is necessary and sufficient. In *Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, p. 46).

Sealey, V., & Oehrtman, M. (2007). Calculus students' assimilation of the Riemann integral. In Proceedings of the 10th Conference on Research in Undergraduate Mathematics Education. San Diego, CA: San Diego State University.

Selden, A., Selden, J., Hauk, S., & Mason, A. (2000). Why can't calculus students access their knowledge to solve non-routine problems. *Issues in mathematics education*, *8*, 128-153.

Sherin, B.L. (2001). How students understand physics equations. Cognition and Instruction, 19(4), 479-541.

Von Korff, J., & Rebello, N. S. (2012). Teaching integration with layers and representations: A case study. *Physical Review Special Topics-Physics Education Research*, *8*(1), 010125.

Impacts on learning and attitudes in an inverted introductory statistics course

Emily Cilli-Turner
Salve Regina University

Recent studies have highlighted the positive effects on learning and retention rates that active learning brings to the classroom. A flipped classroom is a type of active learning where transmission of content occurs outside of the classroom environment and problem solving and learning activities become the focus of classroom time. This article reports on results of a study conducted in flipped and non-flipped introductory statistics classroom environments measuring student achievement in both classrooms on traditional assessments as well as measuring student attitudes toward the flipped classroom environment.

Key words: flipped classroom, introductory statistics, students' attitudes, learning outcomes

Introduction & Background

Although there continues to be debate amongst educators about the most effective way to teach students mathematics, several studies (e.g. Freeman et al., 2014; Lage et al., 2000) have demonstrated that lecturing may be doing a disservice to our undergraduate students. A meta-analysis of studies in STEM fields by Freeman et al. (2014) determined that "active learning increases examination performance by just under half a SD and that lecturing increases failure rates by 55%" (pg. 3). Thus, educators owe it to students to implement active learning, such as the inverted classroom, to increase their learning and performance in mathematics.

Models of the flipped classroom exist in many forms (e.g. Dove, 2013; McGivney-Burelle & Xue, 2013; Wilson, 2013), but there exist common elements that define this pedagogy. For one, the flipped or inverted classroom requires students to participate in instruction outside of class, where much of the content is introduced, usually in the form of videos, readings or other computer-based modules. Class time is then reserved for engaging with the content through small-group problem solving and discussions.

Teaching undergraduate introductory statistics courses present many unique obstacles for instructors and thus may be especially suited for inverted pedagogy. Statistics is usually a required course for a wide range of undergraduate majors, which can make it difficult to engage students in the material (Connors et al., 1998) Additionally, students often enter the course with a wide variety of previous instruction in statistics (Hudak & Anderson, 1990); with some students completing a statistics course in high school and others having no training at all. Recent studies (Dove, 2013; Strayer, 2012) have used flipped pedagogy to teach this course and found that this method allowed students to work at their own pace and contributed to an increase in student engagement.

The effectiveness of the inverted pedagogy has been identified in other disciplines (e.g. Hake, 1998; Lage et al., 2000; Moravec et al., 2010), yet there are only a handful of studies on the use of the inverted classroom in statistics. This study aims to build upon previous studies (Strayer, 2012; Wilson, 2013) of flipped pedagogy in introductory statistics courses. by answering the research questions:

1) What are the impacts of teaching introductory statistics in a flipped classroom format on students' learning?
2) What are introductory statistics students' attitudes towards the flipped classroom format and do these attitudes impact their performance in the course?

In this report, results of a completed study will be discussed as well as methodology for a follow-up study to further measure learning outcomes in statistics due to a flipped classroom.

Methodology

An exploratory study measuring the impact on introductory statistics students' attitudes and learning outcomes using flipped pedagogy was completed over two subsequent semesters at a small liberal arts university in the Northeastern United States. The undergraduate introductory statistics course included topics in descriptive and inferential statistics as well as probability. This course is broadly focused (i.e. content is not restricted to any one major or discipline) and is a required course at this university for students in numerous majors.

Course Design & Participants

During the first semester, two sections of the course were taught in a traditional lecture format. Class time was primarily spent lecturing and, although students were encouraged to ask questions during lecture, there was little dedicated class time for small-group or whole-group discussion of the material. Homework problems from the textbook were collected weekly and turned in on paper.

The second semester, three sections of the course were taught using a flipped classroom format. No more than five minutes of lecture was provided in class with some class days having no dedicated lecture time. The content was delivered in videos that the students were required to watch outside of class and classroom time was spent with students working in small groups on conceptual and problem solving activities related to the material. The videos students watched were created by the textbook publisher and ranged in length from 10 to 20 minutes. Students were required to take notes on the video as they watched it and a short, open-note quiz on the video was given each class day to encourage the students to watch the videos and to attend class. Homework was assigned each class day and was completed online using the MyStatLab (www.mystatlab.com) course management system.

The activities that the inverted classroom students were given to work on during class time ranged from a set of problems to solve as a group to conceptual activities meant to deepen their understanding of the material. Classroom activities were designed to be completed in groups of 3 – 4 and students were encouraged to use their notes and each other as resources. While students were completing the activities, the instructor would go around the room to answer questions as well as monitor student work and identify and correct misconceptions.

The flipped classroom group consisted of 78 students across three sections, of which 20 were male and 58 were female. The control group, that experienced the traditional classroom, consisted of 56 students across two sections, of which 27 were male and 29 were female.

Data Collection

Students in the flipped classroom group were given a survey at the end of the semester to assess their attitudes toward the flipped pedagogy. The survey required them to rate their agreement on a 5-point Likert scale with several questions regarding the flipped classroom. Additionally, the survey asked students to self-report what percentage of the assigned videos they actually watched during the semester, what percentage of the videos they re-watched and what percentage of the videos they took notes on. Course grades and grades on the final exam were collected for both the flipped group of students and the control group.

Results

From the attitudinal survey, the majority of students reported learning a lot from the videos (59%) and enjoying working on problems in class with their group members (50%), however only 31 of the 78 (40%) student participants indicated "Agree" or "Strongly Agree" with the statement "Overall, I like the flipped classroom method" as shown in Table 1.

Additionally, only 27 (35%) students responded that they "Agree" or "Strongly Agree" with the statement "I enjoyed the flipped classroom more than traditional teaching". This is somewhat perplexing since the results show that some students who reported enjoyment of the flipped classroom methods did not report that they preferred this way of teaching to a traditional lecture format. Strayer (2012) found similar attitudes toward the flipped classroom and writes about the "disequilibrium or unsettledness that students face in an inverted classroom" due to the inversion of traditional classroom roles.

			Likert Scale Response (%)		
Statement	Mean	SD	Agree	Neutral	Disagree
I learned a lot from watching the videos.	3.41	1.13	59	22	19
I often watched the videos or parts of the videos more than once.	2.82	1.29	35	18	47
I found the videos confusing and not helpful.	2.40	1.01	17	21	63
I enjoyed working on problems in groups in class.	3.38	1.10	50	32	18
Class time did not help my understanding of the material.	2.78	1.09	23	36	41
Overall, I like the flipped classroom method.	2.92	1.31	40	23	37
I enjoyed the flipped classroom teaching more than traditional teaching.	2.88	1.31	35	29	36

Table 1: Descriptive statistics for survey items focusing on experience of the flipped classroom.

Regardless of students overall attitudes toward the flipped classroom, results show a high level of participation in the activities of the flipped classroom. On average, students reported watching 86% of the videos that were assigned and taking notes on 94% of the videos that they watched. Also, approximately 35% of students reported that they watched all of the assigned videos. These results may be partly due to the daily quizzes that were given in class; as students were told that they could notes on the quizzes and they knew that the quiz would contain content presented in the video.

Although student attitudes were not wholly positive about the flipped classroom, positive impacts on student learning as measured through traditional assessments were present. Overall course grades did improve significantly, $t(132) = 4.96$, $p < 0.001$, $d = 0.87$ when the course was taught using the flipped classroom. However, since the grading structure was different in the flipped classroom and the homework was completed online, an increase in average course grade is perhaps not that surprising. The final exam administered to both groups was very similar and can serve as an accurate measure of learning gains. The final exam grades were significantly higher, $t(132) = 6.57$, $p < 0.001$, $d = 1.15$, in the flipped class than the traditional class. Furthermore, several common final exam questions were given to both groups in order to measure and compare student learning on important topics in introductory statistics. The flipped class group did better on all of the common final exam questions and significantly better on questions about writing hypothesis for a hypothesis test and finding sample size given an acceptable margin of error.

This study also wished to measure the relationship between statistics students' perceptions about the flipped classroom and students' performance in this learning environment. To address this, each student in the flipped classroom group was given a FC+ score based on their responses to statements on the attitudinal survey. The purpose of calculating the FC+ score is to determine if a student's attitudes about the flipped classroom have an impact on that student's performance in this learning environment. A correlation of FC+ scores with student grades on the final exam showed a weak relationship, $r = 0.18$, $p = 0.1087$ between these variables. A correlation of FC+ scores with student course grades also showed a weak relationship, $r = 0.27$, $P = 0.0152$. Thus, it seems that a statistics student's perception of the flipped classroom is not a reliable indicator of how well that student will perform in a flipped classroom course.

Follow-Up Study

During the Fall 2014 semester a follow-up study will be conducted using a Statistics Concept Inventory developed by Allen (2006). This inventory will be given to students in a flipped classroom group and a control group as a pre- and post-assessment. This will allow for a determination of topics and concepts in which the flipped classroom teaching is most effective for introductory statistics.

Questions for the Audience

- How can the results of this study best be presented to students to convince them of the utility of the flipped classroom?
- Survey data will also be collected about how students learn best. How can this data be used in conjunction with the results presented here?

References

Allen, K. 2006. The statistics concept inventory: Development and analysis of a cognitive assessment instrument in statistics. *Ph.D. Thesis*, University of Oklahoma.

Conners, F., S. Mccown and B. Roskos-Ewoldsen. 1998. Unique challenges in teaching undergraduate statistics. *Teaching of Psychology*. 25: 40–42.

Dove, A. 2013. Students' perceptions of learning in a flipped statistics class. In R. McBride and M. Searson (Eds.), *Proceedings of Society for Information Technology & Teacher Education International Conference 2013* (pp. 393-398). Chesapeake, VA: AACE.

Freeman, S., S. Eddy, M. McDonough, et al. 2014. Active learning increases student performance in science, engineering, and mathematics. To appear in: *PNAS*. Retrieved from: www.pnas.org/cgi/doi/10.1073/pnas.1319030111.

Hake, R. 1998. Interactive-engagement versus traditional methods: A six-thousand-student survey of mechanics test data for introductory physics courses. *American Journal of Physics*. 66(1): 64–74.

Hudak, M. and D. Anderson. 1990. Formal operations and learning styles predict success in statistics and computer science courses. *Teaching of Psychology*. 17: 231–234.

Lage, M., G. Platt and M. Treglia. 2000. Inverting the classroom: A gateway to creating an inclusive learning environment. *The Journal of Economic Education*. 31(1): 30–43.

McGivney-Burelle, J. and F. Xue. 2013. Flipping calculus. *PRIMUS*. 23(5): 477–486.

Moravec, M., A. Williams, N. Aguilar-Roca, and D. O'Dowd. 2010. Learn before lecture: A strategy that improves learning outcomes in a large introductory biology class. *CBE Life Sciences Education.* 9(4): 473–481.

Strayer, J. 2012. How learning in an inverted classroom influences cooperation, innovation and task orientation. *Learning Environments Research*. 15: 171–193.

Wilson, S. 2013. The flipped class: A method to address the challenges of an undergraduate statistics course. *Teaching of Psychology*. 40(3): 193–199.

The Transfer of Knowledge from Groups to Rings: An Exploratory Study

Typical undergraduate course sequences in abstract algebra initiate with group theory before proceeding to ring theory. This sequencing, along with the structural similarities between groups and rings, enables many ring-theoretic concepts to be formulated in terms of results from group theory. What remains to be seen, however, is the extent to which students are able to transfer their knowledge of groups while studying topics in ring theory. Using Wagner's transfer in pieces framework, we conducted an exploratory study to investigate how students in an inquiry-oriented classroom capitalized on their knowledge of groups to make sense of rings. Preliminary results indicate both instances of obvious transfer (e.g. subgroup to subring) and also more creative approaches that might lend insight into how students think about ring structure (e.g. characterizing field-like structures as 'abelian grouprings').

Key words: student thinking, abstract algebra, ring theory, transfer

Introduction

A course sequence in undergraduate abstract algebra devotes significant amounts of time to both group theory and ring theory. These courses are typically sequenced so that group theory precedes ring theory. An interesting consequence of the groups-first approach is that many concepts in group theory have direct mathematical analogues in ring theory. Indeed, from an expert's perspective, the potential for formulating ring-theoretic ideas in terms of groups is abundant. The rings content in some prominent abstract algebra texts illustrates some of this potential. For example, Fraleigh (1982) outright defined a ring in terms of an abelian group, and Gallian (2002), defining a ring in terms of each of its axioms, immediately thereafter explains that a ring is essentially "an Abelian group under addition, also having an associative multiplication that is left and right distributive over addition" (p. 230). Such characterizations set the tone for the remainder of the content to capitalize on these similarities whenever appropriate, including motivating ideals by drawing analogies to normal subgroups (Fraleigh, 1982, p. 250) and quotient rings by starting with the group of additive cosets (ibid; Gallian, 2002, p. 254). The similarities, of course, do not end there.

These analogies, presumably included in textbooks to help students make sense of rings, elicit an important pedagogical question: to what extent are students transferring their knowledge of group theory as they come to understand rings? It should be noted that the above attestations of expert mathematicians do not necessarily indicate definite avenues for student transfer. In fact, Wagner (2010) cautioned against basing judgments about students' ability to transfer structural knowledge based on the capabilities of experts. Research on student learning of algebraic structure corroborates this warning as well: students struggle (and perhaps fail) to access their formal knowledge of abstract algebra in situations that clearly call for it (Simpson & Stehlikova, 2006). As such, in this paper we address the following exploratory research questions: (1) Which group-theoretic concepts do students intuitively recognize and leverage as they come to understand ring-theoretic concepts? (2) What knowledge 'cues' from ring-theoretic concepts do students notice that enable them to frame their understanding of these concepts in terms of groups?

Theoretical Perspective

We employ Wagner's (2006) transfer-in-pieces (TIP) framework, which "demonstrates how means of attending to (reading out) and coordinating information available in particular instances may vary as a single mathematical principle is perceived as relevant in different contextual circumstances" (Wagner, 2006, p. 1). This is accomplished by investigating "the conditions under which particular resources and knowledge elements are most likely to be cued and used" (p. 6). The TIP approach is an adaptation of diSessa's (1988, 1993) knowledge-in-pieces framework, a constructivist lens for analyzing knowledge acquisition based upon "the fragmented system of intuitive knowledge we find in our students" (1988, p. 70). Indeed, the TIP framework aims to investigate evidence of knowledge transfer from a novice's (and not an expert's) perspective. This involves identifying those aspects of the problem scenario that *cue* various knowledge resources. The supports offered by these knowledge resources are called *affordances*. To investigate the implications of these knowledge resources and their corresponding affordances, we adopt diSessa and Wagner's (2005) related heuristics of *readout strategies* and *causal net.* Readout strategies "determine how characteristic attributes of a concept are attended to or seen" (Wagner, 2006, p. 7). The causal net "consists of knowledge and reasoning strategies that determine how what is observed is related to the desired information" (ibid.). These heuristics guided our identification and analysis of instances of transfer from groups to rings.

Methods

An abstract algebra classroom, both the students and the instructor, from a small Midwestern liberal arts college comprise the participants for this study. The instructor, Dr. North (pseudonym), was an experienced inquiry practitioner. The students had previously completed an inquiry-based unit on group theory; 8 class periods of 75 minutes apiece at the end of the course were devoted to ring theory. Each of these 8 classes was video recorded (portions of the class devoted to small group work focused on one small group)). The portions of the course dedicated to group theory were audio recorded using a Livescribe pen. Dr. North met regularly with members of the research team during this time to prepare for and debrief from the week's classes. Dr. North used an instructional sequence derivative of those developed by Author (2012) to support the guided reinvention (Gravemeijer & Doorman, 1999) of ring, integral domain, and field. In this way, the transfer-in-pieces framework is compatible with and ideally suited for the constructivist orientation of the classroom studied in this paper.

In particular, the data was subject to Cobb and Whitenack's (2000) iterative analysis method for analyzing classroom video data in which we viewed the video data multiple times, each time incorporating more detail. In alignment with our research questions and theoretical perspective, in the first viewing we identified those particular concepts (or aspects of a particular concept) in ring theory that cued concept projections from group theory. In the second and third viewings we determined how the characteristics of a group are attended to in these contexts, and examined how these observations are related to the intended concepts to be learned, respectively.

Results

While there were several prominent instances of transfer that arose as the students investigated rings, in this brief preliminary report we focus specifically on those related to the commutativity and inverse axioms and their roles in the students' self-appointed names for new structures. Each of these episodes resulted from an open-ended task in class in which Dr. North challenged the students to make conjectures related to their newly reinvented definition of ring. These episodes occurred after the students had reinvented the formal definition of ring (but before defining any other ring structures like commutative ring or field).

Commutativity and 'abelian rings': One small group conjectured about the existence and nature of subrings (an obvious transfer of the notion of subgroup in itself). In doing so, they first defined an 'abelian ring' as a ring in which "both operations are commutative," listing R, Z and Q amongst the examples. Their subrings conjecture was "if R is an abelian ring, then any subring S of R is an abelian ring."

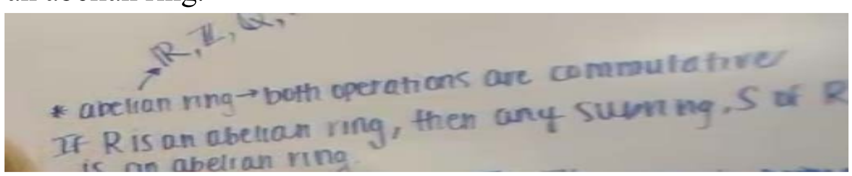


Figure 1: a small group's definition of an 'abelian ring.'

The cue for this instance of transfer appears to be noticing that a ring's multiplication is not required to be commutative (yet they are able to cite several examples in which the multiplication is, indeed, commutative). Their readout strategies in this regard appear to be grounded in their understanding of the utility of the distinction between group and abelian group. Notice that their proposed definition of an 'abelian ring' exactly parallels that of the conventionally-named commutative ring, while also indicating that students have decided that commutativity is an important axiom for characterizing different types of rings. We may frame these developments as affordances of the students' knowledge resources related to group and abelian group.

Multiplicative inverse. Along the same lines, a different group defined a 'group ring' as "a ring that contains a multiplicative inverse and identity, but does not mean that the multiplicative [sic] is commutative." A student in the group goes on to explain that "add [sic] is basically abelian but multiplication is now all the properties of a group, is the way I think of it." The *cue* for this attempt to frame their concept image of fields in terms of groups appears to rely primarily on the presence of multiplicative inverses and the multiplicative identity (using the real numbers as a reference), as evident in the ensuing class discussion led by Dr. North:

Dr. North:	So what's a group ring mean?
Student:	It's a ring that contains the multiplicative inverses and the multiplicative identity.
Dr. North:	OK, can anyone think of an example?

Student: The real numbers under addition and multiplication.

As such, their readout strategies in this regard seem to be rooted in the prevalence of multiplicative inverses. Though these affordances are not entirely correct (as the additive identity, for example, never has a multiplicative inverse), this link certainly does convey a productive initial understanding of field. Additionally, this might be suggestive of a tendency for students to characterize field-like structures as a group under both addition and multiplication.

Questions

(1) What other types of evidence of transfer (aside from direct mentions of 'group') should we look for in such an analysis? (2) What are some other, less-obvious instances of transfer that have emerged in your own classroom experience? (3) What other aspects of Wagner's *transfer in pieces* framework might help us expound upon identified instances of transfer?

References

Author. (2012).

Cobb, P., & Whitenack, J. W. (1996). A method for conducting longitudinal analyses of classroom videorecordings and transcripts. *Educational studies in mathematics*, *30*(3), 213-228.

Disessa, A. A. (1988). Knowledge in pieces. *Constructivism in the computer age, 49.*

DiSessa, A. A. (1993). Toward an epistemology of physics. *Cognition and instruction*, *10*(2-3), 105-225.

DiSessa, A. A., & Wagner, J. F. (2005). What coordination has to say about transfer. *Transfer of learning from a modern multi-disciplinary perspective*, 121-154.

Fraleigh, J. B. (1982). *A first course in abstract algebra*. Massachusetts: Addison-Wesley.

Gallian, J. (2002). *Contemporary Abstract Algebra (5th edition).* Massachusetts: Houghton Mifflin.

Gravemeijer, K., & Doorman, M. (1999). Content problems in realistic mathematics education: a calculus course as an example. *Educational Studies in Mathematics*, 39, 111-129.

Simpson, S. & Stehlikova, N. (2006). Apprehending mathematical structure: a case study of coming to understand a commutative ring. *Educational Studies in Mathematics*, 61, 347-371.

Wagner, J. F. (2006). Transfer in pieces. *Cognition and Instruction*, *24*(1), 1-71.

Wagner, J. F. (2010). A transfer-in-pieces consideration of the perception of structure in the transfer of learning. *the journal of the learning sciences*, *19*(4), 443-479.

Semantic and logical negation: Students' interpretations of negative predicates

Paul Christian Dawkins
Northern Illinois University

John Paul Cook
University of Science and Arts of Oklahoma

During exploratory teaching experiments intended to guide students to reinvent basic truthfunctional definitions for basic logical connectives, it unexpectedly emerged that undergraduate students reasoned about negation and negative properties in ways incompatible with the conventions and norms of mathematical practice. Specifically, our study participants often unpacked negative properties (not a rectangle) in terms of positive properties (is a parallelogram), which we call semantic negation. In this way, students did not readily adopt the mathematical assumption that the logical negation of a predicate designates the complement of the set of examples that satisfy that predicate. Students especially understood geometric sets of objects as being partitioned by familiar categories rather than those stipulated in a given statement. Semantic negation inhibited students systematizing activity regarding linguistic interpretation because their reasoning about disjunctions in various mathematical contexts depended intrinsically upon their understanding of particular topics so as to preclude abstractions approximating normative logical tools.

Keywords: truth-functional logic, guided reinvention, negation, disjunctions, reasoning about logic

One common justification provided for mathematics instruction in general, and prooforiented instruction in particular, is that it promotes students development of logical or deductive reasoning (Gonzalez & Herbst, 2006; Inglis & Simpson, 2008, 2009). Though professional mathematicians rarely use formal logic in their work (Hanna & De Villiers, 2008; Thurston, 1994), their mathematical reasoning conforms rather faithfully to formal logical norms (Azzouni, 2009; MacKenzie, 2001). This could suggest that formal logical structure can emerge from mathematical content itself, inasmuch as formal logic grew out of mathematics historically (Durand-Guerrier, 2008). Unfortunately, in many proof-oriented classrooms matters of logic (both norms of linguistic interpretation and normative modes of argumentation) pose persistent barriers to students apprenticeship in mathematical proving (e.g. Durand-Guerrier et al., 2012; Epp, 2003). Matters of linguistic interpretation are especially challenging because students preconscious reasoning processes can lead them to understand mathematical statements in ways quite different from their professors (e.g. Dubinsky & Yiparaki, 2000; Durand-Guerrier, 2003).

To address such difficulties, we conducted a sequence of exploratory teaching experiments (Steffe & Thompson, 2000) intended to guide students to reinvent (Gravemeijer, 1994) basic structures of predicate logic including truth-functional definitions of common logical connectives (or, if, then). We thereby intend to simultaneously 1) investigate how students untrained interpretations mathematical language diverge from the norms and conventions of the mathematical community and 2) develop learning trajectories (Clements & Samara, 2004) that support students identification and imposition of such norms and conventions upon their own mathematical activity. In this paper, we report about one (unexpected) emergent phenomenon that distinguished students untrained interpretations of mathematical statements from the normative interpretations used by mathematicians and suggest directions for guiding students systematizing activity. We use the term systematizing activity because our task sequence led students to 1) attend to and problematize their interpretations of mathematical language before 2) pursuing systematic tools for interpreting and assessing various mathematical statements of the same logical form (disjunctions and conditionals, both quantified and non-quantified). This emphasis placed upon students' own *reasoning about logic* (compatible with Stenning & van Lambalgen's, 2004, *reasoning for an interpretation*) distinguishes our analytical and instructional approach from much previous logic or deductive reasoning literature, which primarily investigated the emergent logical form of students' pre-conscious reasoning processes (e.g. Evans, 2005, 2007).

Mathematical Property/Category Relations

An extensive body of literature in mathematics education documents how students reason about property/category relations in non-normative ways (e.g. Alcock & Simpson, 2002; Edwards & Ward, 2008; Vinner, 1991). While mathematicians generally treat ratified definitions as *stipulated* (necessary and sufficient conditions for category membership), students often treat mathematical definitions as *extracted*, meaning they may include false examples or exclude desired examples (Edwards & Ward, 2008). For instance, students may not think of squares as rectangles even though they have all the defining properties of rectangles. This is partly because the terms *square* and *rectangle* are also shape names and because such categories in students' minds are more prototype-driven than property-driven (Murphy & Hoffman, 2012).

Such matters of category/property relations are essential to the normative logical models of mathematical language inasmuch as properties as diverse as "not a rectangle," "greater than 5," or "isosceles" may all be represented as predicates $P(x)$ where x could be a quadrilateral, number, or triangle, respectively. In the tradition that treats *logic* as "what remains when all meaning has been removed" (Sentilles, 1975, p. 12), this reduces each property to a function that maps examples to either 0 ("false") or 1 ("true"). One traditional tool for evaluating compound statements in predicate logic (e.g. "Given any quadrilateral, it is a rectangle or it is not a square") is the Venn diagram (Figure 1). In this view, quadrilaterals can be imagined as points in a rectangular region and the two predicates partition the region according to those examples that do or do not have the property P. By stipulating a region by the predicate P the Venn diagram assumes agreement between the property and a related category and that every example is in exactly one group (the "law of excluded middle"). Thus, the "not" operator or a property being "false" are understood to correspond to the complement of the set of examples that satisfy P. The value of this representation is that it can accommodate any disjunction rendered in the form "Given any $x \in S$, $P(x)$ or $Q(x)$" where P and Q represent predicates quantified over the set of examples S. In this interpretation, the disjunction is true only if every element of S is in at least one of the two circles, which also means there are no examples for which P and Q are both false. While the example statement above could also be modeled using P as "is a rectangle" and Q as "is a square" as in Figure 1, the models' compatibility relies upon the identification of logical "not" with the complement of the set partitioned by Q. Also, because this interpretation quantifies P and Q over all of S, for the purposes of evaluating the truth-value of the disjunction "Given any $x \in S$" is considered equivalent to "For every $x \in S$."

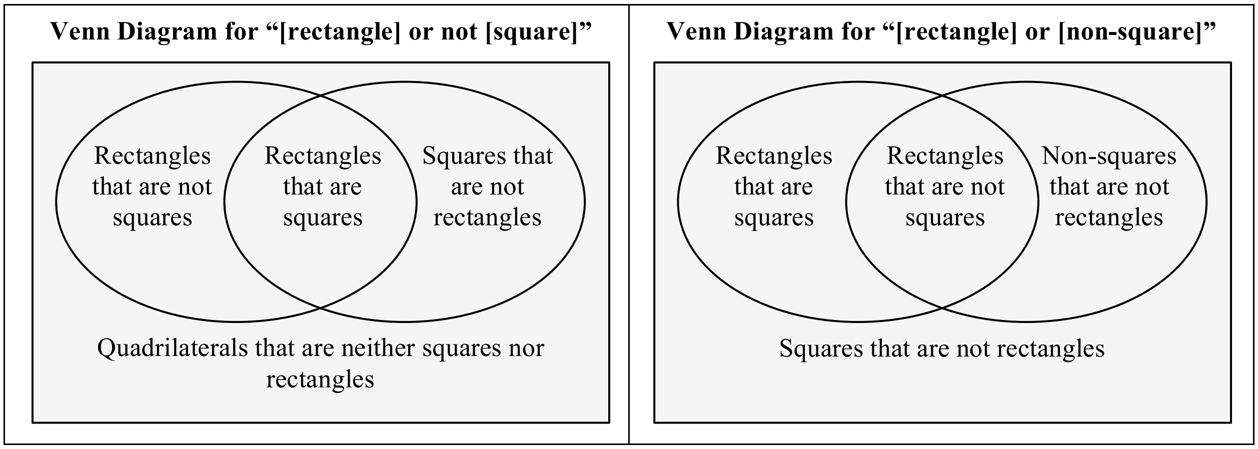


Figure 1: Alternative Venn diagrams partitioning examples by the predicates in a disjunction.

Study and Methods

The normative model for the logic of mathematical disjunctions achieves generality by being heavily abstracted from particular mathematical content. To replace "not a rectangle" with the Boolean function $P(x)$ entails a large "loss of information" in line with the assumption that *logic* ignores meaning. However, such views of *logic* take them far afield from (even expert) *reasoning* (Dawkins, 2014) and seem contrary to the afore-mentioned view that logic can emerge from mathematical structure. Furthermore, it is unclear how directly teaching students such pre-abstracted tools (as is common in many introduction to proof courses; Selden, 2012) will help them revise their pre-conscious modes of linguistic interpretation when reasoning about particular mathematical content. We anticipate that, for many students, logic answers a set of questions (about language, meaning, and reference) that they have not yet asked.

Because of these concerns, we restricted the activities in our teaching experiments to providing students with meaningful mathematical disjunctions to interpret, assess (determine true or false), and negate. We intended any logical structure to emerge from their reflections on their own reasoning and any abstract*ness* to result from their abstract*ions* and anticipations (Simon et al., 2013) regarding the truth-values of mathematical disjunctions. In this way, we anticipate that the resulting modes of *reasoning about logic* may be embedded within their mathematical (semantic) reasoning so as to influence their proof-oriented mathematical activity in context. We did not explicitly instruct students that they were learning logic and we tried to avoid introducing normative models or structures for predicate logic until students imposed them spontaneously.

We recruited Calculus 3 students from a medium-sized university in the Mid-Western United States. We selected this population because they were 1) mathematically proficient (their reasoning about basic topics is rich enough to display formalizable structure), 2) untrained in collegiate proving and mathematical logic, and 3) likely to benefit in future courses from the content to be reinvented. The data presented in this paper comes from the first three teaching sessions (all on disjunctions) with one pair of participants. Eric and Ovid had received no prior instruction in formal logic. Table 1 presents the disjunctions provided to study participants on the first two days. Their tasks over the three days were respectively to 1) determine whether each was true or false and explain any patterns in why the statements were true or false, 2) develop a "How-to guide" for assessing the truth of "or" statements, and 3) develop a method for negating disjunctions (formulating a related statement that will always have the opposite truth-value).

Day 1 Disjunctions	Day 2 Disjunctions: How-to Guide
A1. Given an integer number x, x is even or x is odd.	B1. Given an integer x, x is an even number or x+1

A2. The integer 15 is even or 15 is odd. A3. Given any two real numbers x and y, $x < y$ or $y < x$. A4. Given any two real numbers x and y, $x \leq y$ or $y \leq x$. A5. Given any real number y, y has a reciprocal $\frac{1}{y}$ such that $y \ast \frac{1}{y} = 1$ or y=0. A6. The real number π has a reciprocal $\frac{1}{\pi}$ such that $\pi \ast \frac{1}{\pi} = 1$ or $\pi = 0$. A7. The real number 0 has a reciprocal $\frac{1}{0}$ such that $0 \ast \frac{1}{0} = 1$ or 0=0. A8. Given any real number x, x is even or x is odd. A9. Given any even number z, z is divisible by 2 or z is divisible by 3. A10. Given any even number z, z is divisible by 4 or z is divisible by 3. A11. Given any even number z, z is divisible by 2 or z is divisible by 4. A12. Given any even number z, z is divisible by 4 or z+2 is divisible by 4.	is an even number. B2. 10 is an even number or 20 is an even number. B3. 13 is an even number or 6 is an even number. B4. 5 is an even number or 7 is an even number. B5. 8 is an even number or 37 is an even number. B6. Given any triangle, it is equilateral or it is not acute. B7. Given any triangle, it is acute, or it is not equilateral. B8. Given any triangle, the sum of the measures of the interior angles is 185.7° or the sum of the measure of the interior angles is 180°. B9. Given any quadrilateral, it is a square or it is not a rectangle. B10. Given any quadrilateral, it is not a square or it is a rectangle. B11. Given any rectangle, the interior angles are all right angles or the interior angles are all obtuse. B12. Given any two integer numbers x and y with x<y, there is an integer between x and y or x+1=y. B13. Given any two real numbers x and y with x<y, there is a real number between x and y or x+1=y. B14. Given any two natural numbers x and y with x<y, there is a natural number between x and y or x+1=y. B15. Given any quadrilateral, it is a rhombus or it is not a parallelogram.* B16. Given any quadrilateral, it is not a rhombus or it is a parallelogram.*
*: *Disjunctions introduced by interviewer during teaching session.*	

Table 1: Sample disjunctions from the first two instructional sessions.

The first author served as the teacher-researcher (Cobb & Steffe, 1983) and the second author served as the outside observer (Steffe & Thompson, 2000) for the exploratory teaching experiments. All teaching sessions were video recorded. Consistent with the teaching experiment methodology, the researchers met between each teaching session to reflect, review and analyze the teaching session recording, form hypotheses about student understanding, and formulate tasks and hypotheses for prompting student learning in the subsequent session. These hypotheses formed the initial codes and categories for retrospective analysis after the conclusion of the experiment. The teaching sessions were later analyzed and coded after the method of grounded theory analysis (Strauss & Corbin, 1998). In this paper, we report primarily upon the strongly emergent phenomena surrounding how students interpreted negative propositions and how they negated various kinds of mathematical properties. We use the description "strongly emergent" to mean: 1) common among the many study participants, 2) often resistant to change, and 3) inhibitive to students' systematizing activity and thus the learning goals of the study.

Results

Throughout the study, it became clear that students' reasoning about the mathematical disjunctions varied greatly depending upon the particular mathematical content of the statement. Thus, if *logic* means the structure that is left when all meaning is removed from the statements, then it might be said students did not have a single or consistent logic of mathematical disjunctions. Instead, the way students interpreted each disjunction depended greatly upon their reasoning about the statement's mathematical content (semantic meaning), and students' interpretation would even vary for a given statement as they would read, discuss, and reread the statement trying on various methods of interpretation. For instance, Eric and Ovid had the following discussion regarding B9 and B10 during the second teaching session:

E: [After reading B9] If it's a square, it's not a rectangle. Well, squares are rectangles, but…

O: "Is not a rectangle," that could mean it's a parallelogram or anything like that too, right so I would say it's true.

E: There's a square. There's not a rectangle. It could be the rectangle. I don't think a rectangle is considered a square. A square is, they're all even sides… But a square is a rectangle.

O: With equal sides.

E: But it's a specific rectangle, yeah. So I'd say it's false.

O: Umm, but for a quadrilateral it doesn't mean they all have to be right angles. You could have a parallelogram that is also not a rectangle.

I: So [Eric], what was your reasoning for saying it was false?

E: Well it could be a square, or it could be a rectangle that isn't a square… So it can be a square or it can be a rectangle or it can be anything else.

I: So that makes it false because?

E: It's saying, "If it's not a square it can't be a rectangle." But it could really be anything…

O: Yeah cause "not a rectangle" that's just a parallelogram then, or a square. So I would say that it's true.

E: Or it could be angled or it could be 90 degrees, it could be anything. It could be a rectangle if it's not a square. So, like, if it's not a square, it could still be a rectangle. This is saying, "it's either a square or it's not a rectangle." It could be a square, it could be a rectangle, it could be, like, an angled quadrilateral. So it's giving you, yeah like the "or statement" is like, "If it's not a square, it can't be a rectangle either," but it could be a rectangle if it's not a square.

O: So "it's either a square or a parallelogram," which is not a rectangle. So the only, so actually the only way that this is false if the "any quadrilateral" is a rectangle… [Reads B10 aloud quietly] Okay, so then I would say that's false too.

E: Yeah. Cause it's basically like a third thing it could be that doesn't satisfy those two… If they included all possible quadrilaterals, but this is pretty much saying there is only two types of quadrilaterals, when there could be a third.

As described to above, study participants often analyzed the provided disjunctions by iteratively paraphrasing or restating the given claims, as they understood them. In this episode, Eric provided multiple paraphrases for what B9 was "saying." He used both "either… or… " language and an "if not… then…" formulation. Regarding B10, he abstracted his interpretation in terms of his trichotomy of examples claiming that there are three groups and the disjunctions only mention two (though this was inaccurate for this statement). Though some of these particular interpretive trends recurred, students clearly did not interpret these mathematical disjunctions (all of the form "Given any $x \in S, P(x)$ or $Q(x)$") in a uniform way.

Semantic Negation and Logical Negation

Some of Ovid's reasoning reflects one primary reason their interpretations varied with the mathematical context. In this episode, Ovid employed a strategy of choosing examples ("given any") and testing whether they satisfied either condition in the statement. It is important that while Eric was satisfied to treat "not a rectangle" as a group of quadrilaterals, Ovid replaced the negative predicate with an alternative shape name. Ovid's behavior is one instance of a recurrent pattern of how participants in our experiments interpreted negative properties. The standard *logical negation* of being a rectangle is being a non-rectangle, which can be understood as a predicate that parallelograms satisfy. Our participants instead often treated negative predicates ("not a rectangle") as needing "unpacking" in terms of positive properties ("is a parallelogram"). This move constitutes what we call *semantic negation* rather than *logical negation*. This was most obviously problematic when the alternative property did not correspond to the complement of the original class, as when students replaced "not acute" with "obtuse," interpreted "not even" as "odd," or negated "<" with ">." Even after discussing that π is neither even nor odd, students persisted to negate "even" with "odd" and vise versa. Semantic negation inhibited students' systematizing activity because negations that depended strongly upon mathematical context did not display repeated structure for students to abstract.

Ovid's strategy of testing various examples in a quantified disjunction was quite common, but the students' strategy for example generation varied with the mathematical context. For statements like A9-12, students enumerated examples serially (and were generally convinced after 3-5 examples). Order relations (e.g. A3-4) led students to trichotomize examples into <, >, and = to a given example. In geometric contexts, students relied on familiar shape names or categories. In the episode above, though Eric partitioned the example space into the three normatively relevant groups (similar to a Venn diagram), he clearly did so with reference to the shape properties (even sides, right angles). In this case, he accepted negative properties ("not a rectangle") more in line with logical conventions. This afforded his quite generalized explanation that "this is pretty much saying there is only two types of quadrilaterals, when there could be a third." In this way, he anticipated that the two predicates in B10 would each capture only one of the three categories of quadrilaterals and thus one must be left out. However, assimilating B10 to his understanding of B9 led to a misconception that he was very slow to revise. Eric's accurate interpretation of B9 relied on the fact that its predicates eacj corresponded to one of the three relevant groups. The predicates in B10 (the negations of those in B9) thus each corresponded to two relevant groups in his trichotomy. It is clear from his rejection of B10 (which he defended in various ways for at least 10 minutes) that Eric did not perceive this negation/complement relation. Ovid's case-based strategy led him to affirm B10 more quickly.

A further consequence of Ovid's semantic negation and Eric's lack of negation/complement correspondence appeared later in the same session. We provided B15 and B16 to encourage them to reflect on the repeated structure of the set of examples across the geometric disjunctions we provided. They identified three relevant examples (as before) and Eric classified them according to whether they were or were not parallelograms and rhombi (see Figure 2). While the researchers anticipated that their classifications in terms of these properties relevant to the statements would convince them that they did not need to check any other examples to verify that B16 was true (i.e. these two properties trichotomize all quadrilaterals), the students instead began introducing other categories such as parallelograms and kites. This suggests they were still attending to semantic categories instead of the categories introduced by the propositions in the disjunction. Though they were initially convinced, their reasoning did not afford any justification for all examples fitting into three (or four) possible categories as suggested by the Venn diagram organization of examples. Only after some guided reflection did their attention shift from the familiar shapes to the four possibilities for any quadrilateral: is (not) a rhombus and is (not) a parallelogram. Categorizing the various examples in this way allowed them to begin to recognize the quadrichotomy structure available in a disjunction of two predicates.

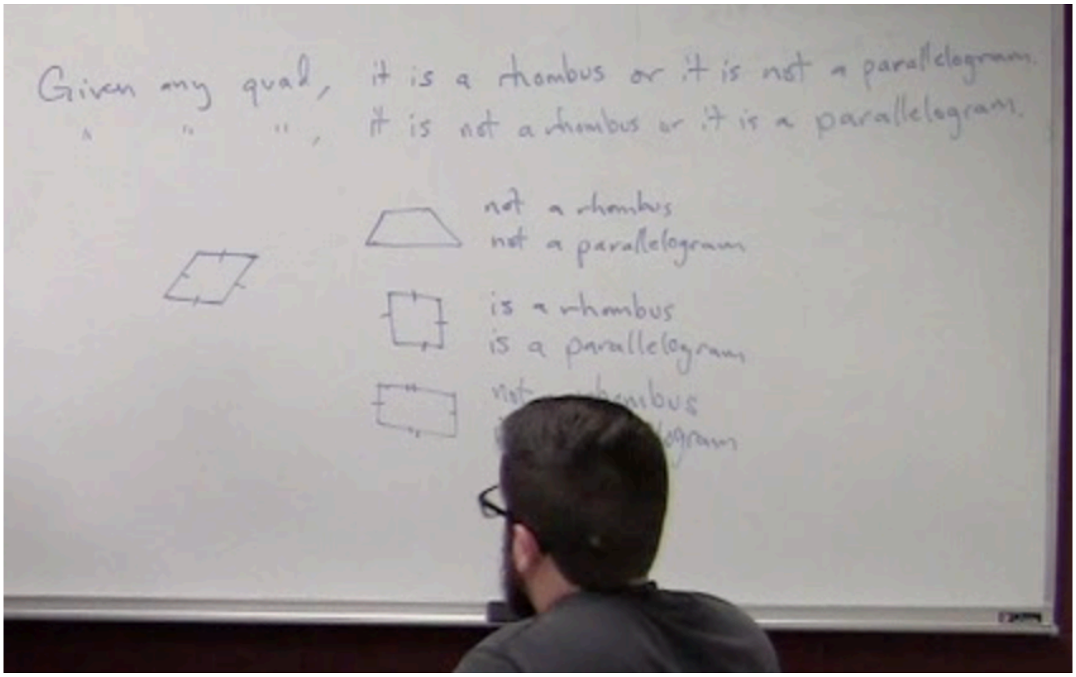


Figure 2: Classifying examples according to the predicates in a disjunction.

One useful strategy that emerged in several students' reasoning is represented by Eric's paraphrase of B9 as saying, "it can't be a rectangle." For at least one other student, this kind of *ontic paraphrase* allowed him to interpret a negative category properly without unpacking it in terms of another geometric category. Psychological literature (e.g. Evans, 2005) on conditional reasoning similarly found that adults reason more normatively about negative conditions in an ontic frame (what is allowed or what is possible). In our study, such pragmatic paraphrases sometimes helped students adopt more conventional interpretations of mathematical parlance.

Summary and Implications

Our teaching experiments succeeded over time in guiding students to reason about logical structure within their semantic (mathematical) reasoning. They adopted normative interpretations of non-quantified disjunctions and developed some useful heuristics for quantified disjunctions. However, their progress toward normative logical structures for quantified disjunctions was strongly inhibited by the prevalence of semantic negation. It seems that for students like Ovid, properties *described* shapes, but it would be inaccurate to say that "not a rectangle" served as a *predicate* (non-rectangle) in his untrained thinking, as predicates are understood in normative mathematical logic. As such, the way Ovid read and interpreted such mathematical disjunctions differed importantly from mathematicians' intended meanings. We did not anticipate before the study that the logical operator "not" would be so problematic for students' reinvention of conventional logic. Our study participants clearly did not automatically partition example spaces in a manner stipulated by the given statement (as in Venn diagrams), but rather thought of examples as being divided by more familiar semantic categories. This suggests that further attention to property/category relations and negative property/complement relations might benefit students being trained in mathematical proving. We anticipate that guiding students to *attend to* and *reason about logic* in context will provide further insight into students' untrained interpretations of mathematical parlance and actionable tools for apprenticeship in proving.

References

Alcock, L. & Simpson, A. (2002). Definitions: dealing with categories mathematically. *For the Learning of Mathematics, 22(2),* 28-34.

Azzouni, J. (2009). Why do informal proofs conform to formal norms?. *Foundations of Science, 14*, 9-26.

Clements, D., & Samara, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning, 6*(2), 81 – 89.

Cobb, P. & Steffe, L. (1983). The constructivist researcher as teacher and model builder. *Journal for Research in Mathematics Education, 14(2),* 83-94.

Dawkins, P.C. (2014). Disambiguating research on *logic* as it pertains to advanced mathematical practice. In (Eds.) T. Fukawa-Connelly, G. Karakok, K. Keene, and M. Zandieh, *Proceedings of the 17th Annual Conference on Research in Undergraduate Mathematics Education*, Denver, Colorado.

Dubinsky, E. & Yiparaki, O. (2000). On student understanding of AE and EA quantification. *Research in Collegiate Mathematics Education, IV,* 239-289.

Durand-Guerrier, V. (2003). Which notion of implication is the right one? From logical considerations to a didactic perspective. *Educational Studies in Mathematics, 53*, 5–34.

Durand-Guerrier, V. (2008). Truth versus validity in mathematical proof. *ZDM*, *40*(3), 373-384.

Durand-Guerrier, V., Boero, P., Douek, N., Epp, S.S., & Tanguay, D. (2012). Examining the Role of Logic in Teaching Proof. In Hanna, G. & De Villiers, M. (Eds.) *Proof and Proving in Mathematics Education* (pp. 369-389). Netherlands: Springer.

Edwards, B., & Ward, M. (2008). The role of mathematical definitions in mathematics and in undergraduate mathematics courses. In M. Carlson, & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education MAA notes #73* (pp. 223–232). Washington, DC: Mathematics Association of America.

Epp, S. (2003). The role of logic in teaching proof. *The American Mathematical Monthly, 110*, 886-899.

Evans, J. (2005). Deductive reasoning. In Holyoak, K.J. & Morrison, R.G. (Eds.) *Cambridge Handbook of Thinking and Reasoning*. (pp. 169-184) Cambridge, NY: Cambridge University Press.

Evans, J.St.B.T. (2007). *Hypothetical thinking: Dual processes in reasoning and judgement*. Hove, UK: Psychology Press.

González, G., & Herbst, P. G. (2006). Competing arguments for the geometry course: Why were American high school students supposed to study geometry in the twentieth century?. *International Journal for the History of Mathematics Education*, *1*(1), 7-33.

Gravemeijer, K. (1994). *Developing Realistic Mathematics Education*. Utrecht: CD-β Press.

Hanna, G., & de Villiers, M. (2008). ICMI Study 19: Proof and proving in mathematics education (Discussion document). *ZDM-The International Journal of Mathematics Education, 40*, 329-336.

Inglis, M., & Simpson, A. (2008). Conditional inference and advanced mathematical study. *Educational Studies in Mathematics*, *67*(3), 187-204.

Inglis, M., & Simpson, A. (2009). Conditional inference and advanced mathematical study: Further evidence. *Educational Studies in Mathematics*, *72*(2), 185-198.

MacKenzie, D. (2001). *Mechanizing proof: Computing, risk, and trust*. Cambridge, MA: The MIT Press.

Murphy, G. & Hoffman, A. (2012). Concepts. In K. Frankish & W. Ramsey (Eds.) *The Cambridge Handbook of Cognitive Science*. (pp. 151-170). Cambridge University Press: New York.

Selden, A. (2012). Transitions and proof and proving at tertiary level. In Hanna, G. & De Villiers, M. (Eds.) *Proof and Proving in Mathematics Education: The 19th ICMI Study*, pp. 391-420. Dordrecht, The Netherlands: Springer.

Sentilles, D. (1975). A Bridge to Advanced Mathematics. The Williams and Wilkins Co.: Baltimore, MD.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267-307). Hillsdale, NJ: Erlbaum.

Stenning, K. & van Lambalgen M. (2004). A little logic goes a long way: basing experiment on semantic theory in the cognitive science of conditional reasoning. *Cognitive Science, 28*, 481-529.

Strauss, A. & Corbin, J. (1998). *Basics of Qualitative Research: Techniques and Procedures for Developing Grounded Theory (2nd edition).* Thousand Oaks, CA: Sage Publications.

Thurston, W. (1994). On proof and progress in mathematics. *Bulletin of the American Mathematical Society, 30*, 161-177.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In Tall, D. (Ed.) *Advanced Mathematical Thinking* (pp. 65-80). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Value judgments attached to mathematical proofs

Eyob Demeke
University of New Hampshire

In mathematics, it is a common phenomenon to find several proofs for a single result. This inevitably leads us to believe that proofs are far more than a convincing argument. Indeed, it appears that there is considerable interest in the insight that is gained from the reasoning utilized in a proof. With the existence of several proofs of the same theorem, we are then confronted with a question of value judgment, as it is not necessarily the case that one values all proofs of a given theorem equally. In this theoretical report, I attempt to provide a framework that contributes to the discussion regarding value judgments about proofs by providing a comparative language to systematically talk about judgments one may attach to a proof. I argue that proofs can be valued for reasons such as (1) comprehensibility, (2) explanatory power, (3) originality and surprises, and (4) generalizability.

Key words: Proof; Value Judgment; Purpose of Proof.

In mathematical research, it is not unusual to see different published proofs of a single result. For instance, we have several published proofs of the Pythagorean theorem, the irrationality of $\sqrt{2}$, and the Fundamental Theorem of Algebra (FTA) – the last of these asserts that the field of complex numbers is algebraically closed. In fact, journals have published proofs for the FTA using theory from different mathematical perspectives (e.g. complex-analytic, topological, and algebraic). Dawson (2006) provides several reasons for why mathematicians produce new proofs of already proven theorems; they are, however, neither exhaustive nor necessary to seek new proofs of already proven theorems.

The existence of several proofs for a single result naturally leads to a question about value judgments regarding proofs. In this paper, I argue that mathematicians may value a proof for reasons such as: (1) comprehensibility, (2) explanatory power, (3) originality, and (4) generalizability. Before I begin my discussion of the above list, I must note that this list is certainly not exhaustive. Moreover, the qualitative values assigned to elements of the list are not objective; among other things, they depend on the mathematician's background. For example, it could be certainly the case that a proof deemed explanatory by one mathematician might not been seen as such by another.

Comprehensibility

Arguments that are complex, convoluted, and lengthy can be tedious and difficult to comprehend. Proofs that are short in length are usually preferred since shorter proofs are usually easier to follow. There is no reason to make an argument longer than it should be. Recall that one of the purposes of a proof is to *convince* others that a theorem is true; therefore, it is reasonable to prefer shorter proofs since they may be easier to follow. This notion of comprehensibility is related to Gowers' (2007) idea of *memorability*. Proofs that are easier to recall might be more valued over those that are more difficult. According to Gowers, the concept of *memorability* has to do with the number of 'key ideas' one has to keep in mind when following a proof. More specifically, proofs that are easier to remember have low *width* (Gowers, 2007).

It is important that one distinguishes the *width* of a proof from the length of a proof, the latter of which being the number of deductive steps required to complete the proof. It is possible to have long proofs with low *width*. In fact, this is common for proofs in measure theory. Suppose that one wanted to show some property, say X, holds for a positive measurable function *f*. One common technique used to prove theorems in measure theory is to show that X holds for characteristic functions, simple functions, and by way of approximation theorems, positive measurable functions, in that order. Proofs like this might be long when fully written, but one usually only needs to remember the common technique. In summary, proofs that are relatively short and have low *width* are easier to comprehend and, for this reason, they are usually preferred. In contrast, Dawson (2006) makes the point that formal proofs that are common in computer science or mathematical logic, when written out fully, are typically long and difficult to comprehend, hence they are less desirable. To elaborate this notion of *comprehensibility*, I will use two proofs showing the irrationality of $\sqrt{2}$.

Theorem 1.1: $\sqrt{2}$ *is irrational.*

Proof A: Suppose not. Consider the set $\mathrm{W} = \{\mathrm{a} + \mathrm{b}\sqrt{2} : \mathrm{a}, \mathrm{b} \in \mathbb{Z}\}$. W is closed under multiplication and addition. Define $\alpha = (\sqrt{2} - 1) \in W$. Since $0 < \alpha < 1$, we have that $\alpha^{\mathrm{k}} \longrightarrow 0$ as $k \longrightarrow \infty$. Assume $\sqrt{2} = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$. Since $\alpha^{\mathrm{k}} \in W$, $\alpha^k = x + y\sqrt{2}$ *for some* $x, y \in \mathbb{Z}$. So $\alpha^{\mathrm{k}} = \frac{xq+yp}{q} \geq \frac{1}{q}$, contradicting the fact that $\alpha^k \rightarrow 0$ as $k \rightarrow \infty$. Q.E.D.

Proof B: Suppose not. Then, $\exists p, q \in \mathbb{Z}^+$ such that $\sqrt{2} = \frac{p}{q}$. Further, suppose without loss of generality, that gcd*(p,q)*=1. Then, $p^2 = 2q^2$. This would eventually imply that both *p* and *q* are even, contradicting the supposition that $\gcd(p, q) = 1$. Q.E.D.

Both proofs establish that $\sqrt{2}$ is irrational. One can also argue that neither proof is significantly longer than the other. However, if one were to reproduce these proofs, one observes that there are more key ideas to remember in the first proof than there are in the second; thus, the second proof has lower *width* than the first. It is therefore conceivable to see why one might value the second proof over the first. There is even a shorter proof using the rational root theorem. This proof uses the fact that any rational root of a monic polynomial is necessarily an integer. Then the argument proceeds by showing that $p(x) = x^2 - 2$ is a monic polynomial whose root is not an integer. We then conclude that $\sqrt{2}$ is irrational. Certainly this proof is shorter in length and it is relatively easier to remember, but one has to first know the rational root theorem.

Explanatory power

Even though a proof may establish a cogent argument for one to believe a claim is true, it might fail to provide the sort of illumination or insight mathematicians hope for. Mathematicians prefer proofs that convey understanding or provide insight (Bell, 1976; Hanna, 1990; Hersh, 1993; Weber, 2008, 2010). Proofs are more desirable when they provide an explanation why a theorem is true, and less desirable when it fails to provide a "psychological satisfactory sense of illumination." (de Villiers, 1990). The notion that some proofs provide an explanation for why a theorem is true while others do not is debatable amongst philosophers of mathematics. For instance, according to the Aristotle-Pólya tradition argues that within deductive or demonstrative reasoning "there is a sharp contrast between two kinds of reasoning, the reasoning which shows why something is the case and the reasoning which only shows that something is the case"

(Cellucci, 2008, p. 202). Based on this tradition, a proof that shows not only that something is the case but also why it must be the case is desirable because it provides a sense of illumination why something is the case. In contrast, philosophers such as Popper (1934) and Balacheff (1987) argued that all deductive reasoning is essentially explanatory in the sense that such reasoning establishes both that something is the case and why it must be the case (Cellucci, 2008). According to this tradition, which goes from Popper to Balacheff, to explain X is to provide a deductive argument from given principles. Of course, all proofs do just that. Therefore, according to this school of thought, to distinguish various proofs of the same result in terms of their explanatory power makes little sense. However, the Popper-Balacheff perspective on mathematical explanation is problematic. For instance, philosophers such Cellucci (2008) argued that the Popper-Balacheff perspective on mathematical explanation is neither necessary nor sufficient. He offered, as an example, a proof of the Pythagorean theorem that reasons from the axioms of set theory. Such a proof, he argued, qualifies as an explanation according to Popper-Balacheff perspective; however, it does not give an explanation of the Pythagorean theorem because "it deduces the theorem from very general principles which have no special connection with the theorem"(Cellucci, 2008, p. 204).

Besides, if one were to seriously look at how mathematics is being practiced, I argue that we have good reasons to believe that talking about proofs in terms of their *explanatory power* is appropriate and it may be necessary. In fact, mathematicians often distinguish proofs that only demonstrate something is true from those that also show why something must be true. Take, for instance, the controversy surrounding Appel and Haken's joint proof of the Four-Color theorem (Thurston, 1995). Their computer-assisted proof drew criticism from eminent mathematicians such as Paul Halmos because he and others believed it failed to provide insight into why the theorem must be true (Thurston, 1995). Also, it is not unusual for mathematics educators to talk about proofs in terms of their *explanatory power.* Weber (2010), for instance, characterizes *explanatory proof* from a reader's point of view as a "proof that reconceives a domain of mathematics" (p.34). To further elaborate the notion of *explanatory power*, I will provide two examples. First, consider the proofs of the following theorem:

$Theorem$ 1.2: $2^{2^5} + 1$ $is\ not\ prime$. The following two proofs appeared in Avigad (2006).

Proof A: A calculation shows that $2^{2^5} + 1 = (641)(6700417)$. Q.E.D.

Proof B: First, note that $641 = (5)(2^7) + 1 \Longrightarrow (5)(2^7) \equiv -1 (mod\ 641) \Longrightarrow (5^4)(2^{28}) \equiv 1 (mod\ 641)$. On thoe other hand, we have that $641 = 5^4 + 2^4 \Longrightarrow 5^4 \equiv -2^4 (mod\ 641)$. Then $(5^4)(2^{28}) \equiv -2^{32} (mod\ 641)$. So we have that $1 \equiv -2^{32} (mod\ 641)$. Q.E.D.

Once again, both proofs verify that $2^{2^5} + 1$ is not prime. However, one may still value one proof over the other. For example, the first proof essentially says nothing besides stating that $2^{2^5} + 1$ is the product of 641 and 6700417. As a result, it may be more difficult to recall the first proof than the second, which means the first is less memorable. In addition, in the first proof it is not clear how one initially thought of the factors 641 and 6700417. However, the second proof at least illustrates that $2^{2^5} + 1$ could not be prime because 641 can be written both as $5^4 + 2^4$ and $(5)(2^7) + 1$. Next, let us consider two proofs, modified from Hanna (1990), regarding the sum of the first *n* positive integers.

$Theorem$ 1.3: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Proof A: We write out the terms first forward (as in *) and then backward (as in **)

1 + 2 + 3 +… + (n-1) + n ……………………………………….(*)

n + *(n-1)* + *(n-2)* + ... + 2 + *1* ……………………………………..(**)

The conclusion follows from the fact that the sum of the two terms in each column is *(n+1)* and we have *n* columns. Q.E.D.

Proof B: We will proceed by induction. Let $P(n)$ be the sum of the first *n* natural number is $\frac{n(n+1)}{2}$, we will show that $P(n)$ holds for all natural numbers n. The case for *n*=1 is immediate. For the inductive step, assume that for some $k, P(k)$ holds, so $1+2+\cdots+k=\frac{k(k+1)}{2}$. We must show that $P(k+1)$ holds. Since $1+2+\cdots k+(k+1)=\frac{k(k+1)}{2}+(k+1)=\frac{(k+1)(k+2)}{2}$. We conclude that $P(k)$ holds when $P(k+1)$ holds. Q.E.D.

There is no doubt that both proofs are certainly valid; hence, they are both convincing to a knowledgeable audience. Yet one might prefer the first proof over the second because the first proof provides us with some explanation as where $\frac{n(n+1)}{2}$ comes from. In contrast, the proof by induction does not provide any insight regarding where $\frac{n(n+1)}{2}$ comes from. In fact, philosophers such as Steiner (1978) and Lange (2006) argued that proofs by mathematical induction are generally non-explanatory. For Lange (2006), proofs by induction lack explanatory power because they run into what he calls "explanatory circularity".

Generalizability, fruitfulness, and illustrating significance of a theory or theorem

Proofs that succeed in proving more than the theorem are desirable. Consider, for instance, the intermediate value theorem which asserts that for any continuous function *f* over closed interval [*a, b*] such that *f(a)<y<f(b)* for some real number *y* there exists a real number *c* such that $c \in [a,b]$ and *f(c)=y.* One common proof of the theorem that appears in most undergraduate real analysis books uses the completeness axiom and properties of continuous functions. There is also an elementary proof that uses the nested interval property. It uses concepts about real numbers and techniques of trapping a real number inside of a sequence of nested intervals. The proof I present next, which is independent of the preceding techniques, is shorter and involves topological arguments.

Proof: Suppose not. Then one observes that the sets $S=\{x \in [a,b]: f(x)<y\}$ and $S'=\{x \in [a,b]: f(x) \leq y\}$ are the same. Therefore, S is both an open and closed set. Since $a \in S, b \notin S$ and S is a non-empty proper subset of $[a,b]$ that is both open and closed. This contradicts the fact that $[a,b]$ is connected. Q.E.D.

This proof is certainly shorter and more generalizable than the previous proof. Indeed, the argument in this proof shows that the intermediate value theorem holds for any continuous functions from a connected space X to the real numbers (Renz, 1981).

Surprises, originality, and connecting different domains of mathematical ideas

Given two different proofs (A and B) of a theorem X, there are times where proof A is more explanatory and more generalizable than proof B, but some may still view the former as less interesting. This usually occurs when an unexpected technique is used to prove existing and/or new problems. Consider the following two proofs (A and B) of the theorem below.

$Theorem\ 1.4$: $2^{\frac{1}{n}}\ is\ irrational\ for\ all\ n \geq 3.$

Proof A: Suppose not. Then $\exists p, q$ such that $2^{\frac{1}{n}} = \frac{p}{q}$. Further suppose that $\gcd(p, q) = 1$.

Then $p^n = 2q^n$. We will eventually get both p and q are even, which contradicts the supposition $\gcd(p, q) = 1$. Q.E.D.

Proof B: Suppose not. $\exists a, b \in \mathbb{Z}^+$ such that $2^{\frac{1}{n}} = \frac{a}{b}$. Now we have $2b^n = a^n \Longrightarrow a^n = b^n + b^n$, contradicting Fermat's Last Theorem. Q.E.D.

Clearly, proof A is more generalizable because it shows the result holds for all $n \geq 2$; however, the second proof only applies when $n \geq 3$. In addition, it is both essential and clear in proof A that if a prime p (in our case p=2) divides the product of two integers, then p will divide at least one of them. As a result, *proof A* provides some sense of insight or explanation why the result must hold (Cellucci, 2008). However, one could find *proof B* more interesting because, in a somewhat unexpected way, it uses a famous theorem, Fermat's Last Theorem. This suggests that some proofs have a natural shape and pop out of a comparatively narrow search. Other proofs – for instance, proof B – have an unnatural shape. Interestingly, mathematicians seem to value such "unnatural" proofs. Thus, the preceding shows that one could find a proof more valuable and/or interesting not necessarily due to its explanatory power or generalizability, but for reasons such as when a proof uses theorems or techniques in somewhat unexpected ways.

Implication for mathematics pedagogy

The above discussion on value judgments of proofs is naturally connected to mathematics pedagogy. I will therefore conclude this paper by elucidating some implications this notion of value judgment has in mathematics education pedagogy. In undergraduate mathematics, while there are various proofs of a given theorem, one cannot afford to present all existing proofs; indeed, it is not necessary to present all of them. This fact alone forces professors to choose some proofs over others. A natural question then follows: what should be the basis of the decision to present some proofs over others? There is no one single answer to this question, but it will certainly incorporate a subset of the criteria for value judgments I put forward earlier. Consider, for instance, a formal proof of the Pythagorean theorem from Hilbert's axioms that can easily be nearly 80 pages long. This proof, as Renz (1981) argued, might not be appropriate for a high school geometry course, or even for any undergraduate mathematics course. For starters, students may not comprehend the proof and the presentation of the proof may not be practical given the time constraints.

Other than temporal issues, there are issues such as instructional goals the professor has to consider when deciding which proofs to present. Take, for instance, the proofs about the sum of the first *k* positive integers in *Theorem 1.3* discussed above. In an introductory proof class where a professor might be interested in covering particular proof technique of mathematical induction, it is more appropriate that he chooses the inductive argument over the other. In summary, there are many other considerations one has to account for when choosing which proof of a theorem to present during lectures. These include, but are not limited to: available class time; the mathematical knowledge that students possess (professors should present only proofs that their students are able to comprehend); and the instructor's goal (this may include proof techniques that instructors want to communicate to their students).

References

Avigad, J. (2006). Mathematical method and proof. *Synthese, 153*(1), 105-159.

Bell, A. W. (1976). A study of pupils' proof-explanations in mathematical situations. *Educational Studies in Mathematics, 7*(1), 23-40.

Cellucci, C. (2008). The nature of mathematical explanation. *Studies in History and Philosophy of Science, 39*(2), 202-210.

Dawson, J. W. (2006). Why do mathematicians re-prove theorems? *Philosophia Mathematica, 14*(3), 269-286.

de Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras, 24*, 17-24.

Gowers, W. T. (2007). Mathematics, memory and mental arithmetic. In A. P. Leng, & M. Potter (Eds.), *Mathematical knowledge* (pp. 33-58). Oxford, GBR: Oxford University press.

Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange, 21*(1), 6-13.

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics, 44*(1-2), 5-23.

Hersh, R. (1993). Proving is convincing and explaining. Educational Studies in Mathematics, 24(4), 389-99.

Lange, M. (2009). Why proofs by mathematical induction are generally not explanatory. *Analysis, 69*(2), 203-211.

Renz, P. (1981). Mathematical proof: What it is and what it ought to be. *Two-Year College Mathematics Journal, 12*(2), 83-103.

Steiner, M. (1978). Mathematical explanation. *Philosophical Studies, 34*, 135-151.

Thurston, W. P. (1995). On proof and progress in mathematics. *For the Learning of Mathematics, 15*(1), 29-37.

Weber, K. (2008). How mathematicians determine if an argument is a valid proof. *Journal for Research in Mathematics Education, 39*(4), 431-459.

Weber, K. (2010). Proofs that develop insight. *For the Learning of Mathematics, 30*(1), 32-36.

The purpose of reading a proof: A case study of Lagrange's theorem

Eyob Demeke
University of New Hampshire

May Chaar
University of New Hampshire

In typical undergraduate advanced mathematics courses, professors spend ample class time presenting proofs; however little is know with regards to what students actually gain from these experiences. In our preliminary report we will attempt to address this gap, specifically with respect to a proof of Lagrange's theorem. We used Mejia-Ramos and colleagues' (2012) model in designing a proof comprehension test and task-based interviews to shed light on (1) the extent to which undergraduates comprehend the proof and (2) what undergraduates gain or learn from reading the proof. Initial examination of our data reveals that although the participants could follow the proof line by line, they had difficulty identifying key ideas and summarizing the proof. Participants acknowledged their responsibility to fill in gaps in proofs; yet they had trouble justifying non-trivial assertions. Despite participants' superficial comprehension of the proof, we still observed that participants gained conviction and learned new definitions.

Key words: Proof, Role of Proof, Proof Comprehension, Abstract Algebra.

Undergraduates in most upper-level mathematics courses are expected to spend ample time reading and writing proofs; however, the indisputable conclusion from the literature on proof is that students do struggle in courses that require proofs; in particular, students have difficulty grasping the concept of proof and the role logic and definitions play in mathematical argumentation (Harel & Sowder, 1998; Inglis & Alcock, 2012; Moore, 1994; A. Selden & Selden, 2003).

Although there are many studies on undergraduates' experience with respect to proof construction and validation, there are significantly fewer studies on undergraduates' reading comprehension of proof (J. Mejia-Ramos, Fuller, Weber, Rhoads, & Samkoff, 2012; J. P. Mejia-Ramos & Inglis, 2009). As a result, researchers (Mamona-Downs & Downs, 2005; Roy, Alcock, & Inglis, 2010; A. Selden & Selden, 2003) have called for more empirical studies on students' proof comprehension. In this study we applied a proof assessment model suggested by Mejia-Ramos et. al (2012) to address the following research questions: (1) To what extent do undergraduates comprehend proofs? (2) What do undergraduates gain or learn from reading proofs that are typically found in their textbook? In particular, we addressed these questions with respect to a proof of Lagrange's theorem in a typical undergraduate abstract algebra course.

Theory

The two most important roles of proof discussed in the literature are: (1) *conviction* or *verification*, and (2) *explanation*. *Convincing* is the idea that a proof demonstrates that a theorem is true. Although undergraduates and surprisingly mathematicians (e.g., Weber, Mejia-Ramos, & Inglis, 2014) are sometimes convinced without proof, De Villiers (1990) writes that "the well-known limitations of intuition and quasi-empirical methods" underscore the vitality of proof as a useful means of *verification* (p.19). Convincing may be the primary goal of any published proof; however, there is a consensus that the functionality of a proof is not, and should not, be limited to verifying that a theorem is true (De Villiers, 1990; Hersh, 1993). Indeed, it appears that there is considerable interest in the insight that is gained from the reasoning utilized in a proof. For a mathematician, a proof—beyond convincing—also functions as an explanatory argument. To explain is to provide insight as to why a theorem is true (De Villiers, 1990; Hersh, 1993; Thurston, 1995; Weber, 2008). Explanatory proofs are insightful precisely because they make "reference to a characterizing property of an entity or structure mentioned in the theorem, such that from the proof it is evident that the result depended upon the property" (Steiner, 1978). According to De Villiers (1990), explanatory proofs provide "psychological satisfactory sense of illumination" (p.19). In fact, all eight mathematicians interviewed in Weber's (2008) study reported that the primary reason they read published proofs is to gain insight.

Proofs, beyond convincing and explaining, can function as a tool to communicate techniques or ways of reasoning that can later be used to tackle other problems (Weber, 2008, Thurston, 1995). Thurston (1995) maintains that mathematicians sometimes use proofs to communicate a developed body of common knowledge or new techniques in the case of truly novel proofs. For example, mathematicians interviewed in Weber's (2010) study stated that when reading a proof, they would hope to learn new techniques that might eventually help them prove conjectures or problems they have been thinking about in their research.

Previous research on proof comprehension

Research on students proof comprehension is rare. Studies by Conradie and Frith (2000) and Weber (2012) report that mathematicians evaluate their students' understanding of proofs superficially. They usually ask students to reproduce proofs, a task that requires little beyond recalling facts and procedural fluency. For example, mathematicians in Weber's (2012) study said that they measured their students' understanding of proofs by (1) asking students to construct a proof of a theorem similar to one that was proven in class, and/or (2) asking them to reproduce a proof; some actually said they do not assess their students' understanding of a proof at all. Conradie and Frith (2000) argue for the development of a proof comprehension test aimed at assessing students' understanding of a proof rather than their ability to merely recall it. In their comprehension test, they provide the proof and pose several questions to their students. These questions vary from stating items such as the proof's form, hypothesis, and conclusion to filling in and/or explaining specific details in the proof.

Building on the work of Conradie and Frith (2000) and Yang and Lin (2008), Mejia-Ramos and colleagues (2012) developed a framework for a proof comprehension assessment. In this theoretical framework, the authors propose seven dimensions for understanding a proof: (1) meaning of terms and statements; (2) logical status of statements and proof framework; (3) justification of claims; (4) summarizing via high-level ideas; (5) identifying the modular structure; (6) transferring the general ideas or methods to another context; and (7) illustrating with examples. Dimensions one through three are directed towards assessment of students' comprehension of local aspects of a proof. By this they mean "understanding that can be discerned either by studying a specific statement in the proof or how that statement relates to a small number of other statements within the proof" (p.5). Whereas dimensions four through seven are holistic in nature, which means they are geared toward assessing students' overall comprehension of a proof. It is important to note that each of the dimensions should not be understood hierarchically. In our own study, we designed both our proof comprehension test and task-based interview questionnaires based on Mejia-Ramos and colleagues' (2012) framework for proof comprehension assessment.

Research Methodology

The targeted population for this study is undergraduate students enrolled in an introductory abstract algebra course in a large public university in the northeastern United States. We also conducted a video-recorded semi-structured interview with an algebraist who has over 15 years of experience teaching abstract algebra. Our student participants were asked to (1) complete a survey of background information, (2) complete a preliminary pre-proof written task, (3) read a proof, (4) respond to written items assessing his or her comprehension of this proof, and finally (5) respond to questions aloud asked by the interviewer to further assess their comprehension and mathematical learning gained by reading the proof. Altogether each task-based interview took approximately one and a half hours.

Recall that the two main research aims in this study are: (1) to examine to what extent undergraduates comprehend proofs, and (2) to investigate what they gain or learn from reading a proof. In order to address the first research goal, our initial analysis focused on how participants responded to the written comprehension test. To address what undergraduates gain from a proof, our preliminary analysis used roles of proofs documented in the proof literature such as *conviction or verification, explanation,* and *communicating techniques*.

Preliminary results and discussion questions

In what follows we will present results of a preliminary analysis of one participant, Amy, specifically with regards to her comprehension of Lagrange's theorem. Amy explained that the she understands a proof when it "makes sense from one step to another". Her statement was consistent with the linear approach she took to reading the proof as well as her inability to identify the key ideas of the proof when providing a summary. For example, she did not seem aware of the importance of creating a bijection to show equivalent cardinality. It had been our hope that she would notice this aspect of the proof, because (i) prior to reading the proof Amy stated that she did not know how to prove two sets have the same cardinality and (ii) she noted that whenever she reads a proof she hopes to learn "the strategy that they use in proof - in abstract they use a lot of different strategy in how they do it". Post-proof, Amy did report that she learned new strategies such as proving an equivalence relation; however she had exhibited knowledge of these strategies prior to reading the proof. Thus Amy did not seem to have acquired any new strategies or techniques from reading the proof. Furthermore, although Amy acknowledged her responsibility to fill in gaps in the proof, she was only able to justify trivial assertions. Despite the fact that Amy did not appear to completely comprehend the key arguments of the proof, she did suggest that she was convinced after reading the proof stating, "It makes sense to show it"; whereas, "before the proof I didn't believe it". We plan to analyze the interview transcripts of our remaining participants, and prior to this we would like to use our presentation to receive feedback regarding the following questions:

(1) Are there non-mathematical benefits that one can acquire from reading a proof, and if so, to what extent do we care about them?

(2) What methodological suggestions might you offer us to examine what undergraduate students do when reading a proof (without the use of eye tracking)?

Implications for further research and mathematics pedagogy

Given that a recent systematic investigation of a sample of 131 studies on proofs only yielded three studies that focused specifically on proof comprehension (J. P. Mejia-Ramos & Inglis, 2009), this study will be a welcome addition to the paucity of the proof literature in undergraduate mathematics education. Moreover, our further analysis of students' difficulties with respect to learning mathematics from a proof will perhaps suggest ways to help them better understand the higher level key ideas in proofs. Consequently, we will develop supportive materials for students' proof comprehension and disseminate among mathematicians who typically teach advanced undergraduate courses.

References

Conradie, J., & Frith, J. (2000). Comprehension tests in mathematics. Educational Studies in Mathematics, 42(3), 225-35.

De Villiers, M. (1990). The role and function of proof in mathematics. Pythagoras, 24, 17-24.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from an exploratory study. In A. H. Schoenfeld, J. Kaput & E. Dubinsky (Eds.), Research in collegiate mathematics education (3rd ed., pp. 234-283). Providence, RI: American Mathematical Society.

Hersh, R. (1993). Proving is convincing and explaining. Educational Studies in Mathematics, 24(4), 389-99.

Mamona-Downs, J., & Downs, M. (2005). The identity of problem solving. The Journal of Mathematical Behavior, 24(3–4), 385-401.

Mejia-Ramos, J. P., & Inglis, M. (2009). Argumentative and proving activities in mathematics education research. In F. Lin, F. Hsieh, G. Hanna & d. M. Villiers (Eds.), Proceedings of the ICMI study 19 conference: Proof and proving in mathematics education (Taipei, Taiwan ed., pp. 88-93)

Mejia-Ramos, J., Fuller, E., Weber, K., Rhoads, K., & Samkoff, A. (2012). An assessment model for proof comprehension in undergraduate mathematics. Educational Studies in Mathematics, 79(1), 3-18.

Moore, R. C. (1994). Making the transition to formal proof. Educational Studies in Mathematics, 27(3), 249-66.

Roy, S., Alcock, L., & Inglis, M. (2010). Undergraduate proof comprehension: A comparative study of three forms of proof presentation. Proceedings of the 13th Conference for Research in Undergraduate Mathematics Education,

Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? Journal for Research in Mathematics Education, 34(1), 4-36.

Steiner, M. (1978). Mathematical explanation. Philosophical Studies, 34, 135-151.

Thurston, W. P. (1995). On proof and progress in mathematics. For the Learning of Mathematics, 15(1), 29-37.

Weber, K. (2010). Proofs that develop insight. For the Learning of Mathematics, 30(1), 32-36.

Weber, K., Inglis, M., & Mejia-Ramos, J. P. (2014). How mathematicians obtain conviction: Implications for mathematics instruction and research on epistemic cognition. Educational Psychologist, 49(1), 36-58.

Weber, K. (2008). How mathematicians determine if an argument is a valid proof. Journal for Research in Mathematics Education, 39(4), 431-459.

Weber, K. (2010). Mathematics majors' perceptions of conviction, validity, and proof. Mathematical Thinking and Learning: An International Journal, 12(4), 306-336.

Weber, K. (2012). Mathematicians' perspectives on their pedagogical practice with respect to proof. International Journal of Mathematical Education in Science and Technology, 43(4), 463-482.

Yang, K., & Lin, F. (2008). A model of reading comprehension of geometry proof. Educational Studies in Mathematics, 67(1), 59-76.

Formative assessment and classroom community in calculus for life sciences

Rebecca-Anne Dibbs
Texas A&M-Commerce

Brian Christopher
University of Northern Colorado

Most of the attrition from STEM majors occurs between the first two semesters of calculus, and prospective life science majors are one of the groups with the highest attrition rate. One of the largest factors for students that persist in STEM major beyond the first semester of calculus was a sense of community and a perceived connection with their instructor. Since building a sense of community is one of the stated purposes of formative assessment, we investigated to what extent formative assessments could help build a sense of community in a calculus for life science majors course. Two cases of formative assessment used in two sections of this course will be discussed. When implemented as intended, the formative assessments completed weekly by the students made a positive contribution to students' sense of classroom community and their perceived connection with their instructor.

Key words: attribution, calculus, formative assessment

Students who have a poor perception of their quality of relationships with their instructors are more anxious, earn lower grades, and are more likely to cheat on assignments during their first year (Kurland & Siegel, 2013; Nadelson et al, 2013). The first year of college is also where the largest number of students switch out of a STEM major, and this switch is most likely to occur after calculus (Bressoud, Carlson, Mesa, & Rasmussen, 2013). Biology majors are most likely to switch majors after calculus (Bressoud, Carlson, Mesa, & Rasmussen, 2013), but students who passed calculus who perceive a personal connection with their instructor are less likely to switch. Formative assessment can create a communication loop between instructor and student, even in large classes, and is a low labor intensive way to address post-calculus STEM attrition (Clark, 2011; Shute, 2008; Wiliam, 2009). The purpose of this study was to investigate how different implementations of the same formative assessments influenced students' observable attribution behaviors in otherwise identical calculus courses. Without explicit mention of formative assessment during class by the instructor, students tended to display more entity-orientated behaviors.

Black and Wiliam's (2009) formative assessment framework and Vygotsky's (1987) Zone of Proximal Development (ZPD) were used as the theoretical perspective of the larger project. This report will focus on the scaffolding characterization of ZPD; where a learner is in their ZPD if they can complete a problem with assistance they could not complete independently. This characterization of ZPD dovetails with the fifth purpose of Black & Wiliam's framework (2009): increasing student ownership of learning, because students that feel they own their own learning are more likely to have incremental attribution.

The specific aspect of ownership investigated in this study was attribution (Dweck, 2006). According to Dweck (2006), attribution is the implicit beliefs that students have about intelligence. There is a continuum of attributions, with the two extreme cases being entity and incremental attribution (Figure 1). In either case, attribution is a pattern of thoughts and behaviors that is not entirely conscious; these patterns are easiest to observe when students struggle or fail with new material. Students with entity attribution believe that intelligence is a fixed quantity. These students are focused on performance goals, like grades. On material students with the entity attribution find easy, these students will generally outperform students with the incremental theory of intelligence; however, students with the entity attribution tend not to persist on difficult material. Since students in this category believe that intelligence is fixed, having any difficulty with material means that you cannot learn the content. Students who have the incremental theory believe understanding the material is the main reason for learning. These students will show high persistence on material, regardless of the difficulty level, because effort is how learning occurs.

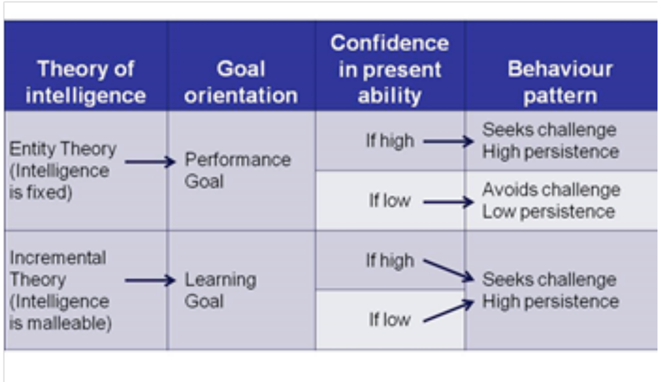


Figure 1. Consequences of each extreme attribution view (adapted from Dweck, 2006)

Methods

The data was collected as part of a larger educational ethnography (Wolcott, 2005); this study may be considered a case study (Patton, 1990) bounded by one semester of calculus for life science majors. There were two sections of calculus for life science majors held at the same time[1] were observed for one semester. Both of the instructors teaching were teaching the course for the first time and used a common schedule and assignments. Students in both classes were asked to complete a formative assessment related to the forthcoming content; this assignment was intended to be used as a planning tool for the instructors. Before a section of the textbook was covered in class, students were asked to read the section, define all major terms, write down all formulas, attempt a sample problem, and state what questions they had about the section; these assignments were graded on completion and worth 5% of the course grade. In Class 1 these formative assessments were collected weekly and not referenced in class by the instructor while in Class 2 these assignments were collected before every new section and referenced at the start of every new section by the instructor.

There were 33 students randomly assigned to each instructor after unexpected demand for the course required that the original roster be split in half and a second section created. All of the students taking the course were biology or biochemistry majors. There is no formal pre-health major at the institution, but the majority of the students enrolled in the course intended to apply for medical schools at the end of their undergraduate careers. Since this course is recommended as a first year course, 75% of the students in each section were freshmen participating in a first year experience; these students took several classes together and lived in the same dorm. The remaining students in each section were upperclassmen.

Each researcher chose a section as their primary observation responsibility. A researcher was present every day in class except for three exam days and three class days from each section where both researchers observed the same class to triangulate the observations. Informal interviews took place throughout the semester with students. At the end of the semester, 12 students were formally interviewed; eight from Section 1 and three from Section 2. During data analysis, the observation notes were coded for differences in instructor use of the formative assessments and students' persistence during challenging tasks; this data was used to triangulate the students' formal and informal interviews. The student interviews were coded for statements about students' beliefs about learning calculus as well as statements about the formative assessments.

[1] In order to accommodate the biology labs, this 4 credit course met 100 minutes on Monday and 50 minutes Wednesday and Friday.

Findings

Although the implementation of the same formative assessments (called reading sheets in the class) was different in the two classes, all of the interviewed students identified these assignments were a helpful tool for learning the material. Students found the reading sheets helpful because they made the learning objectives for each section clear and helped identify which parts of the content students found most difficult before instruction; this is one of the five purposes of formative assessment identified in Black and Wiliam's (2009) theoretical framework; this was one of the two major purposes students ideally identify as a use for formative assessment. Robert in Class 1 explained, "They [the reading sheets] give a warning for what's coming up next. I know ahead of time which parts of class I have to listen closely ahead of time. This makes a big difference, especially on Monday." Pat in Class 2 concurred: "I think that they're a good tool to using cause it a little bit of an overview of what you'll be going over that day usually…"

While students from both classes found that the formative assessments clarified the objectives of the upcoming content, students in each class used the information differently. In Class 1, students used the identified objectives to seek outside sources of supplemental instruction; half of the students interviewed from this class regularly watched YouTube videos based on keyword searches gleaned from the formative reading sheets. The other four students from Class 1 used the reading sheets to identify when they needed to pay attention in class like Robert mentioned in his interview. The Class 2 students did not watch YouTube videos outside of class; they considered the reading sheets as a preview of the upcoming material and a chance to ask for help.

In Class 1, the students' behavior showed low confidence, and entity theory of intelligence. Of the 189 times during the semester when the instructor asked a question, three students accounted for 133 of the answers. When asked about her low participation in the class, Gloria, a student who spoke twice in class all semester, explained, "The class is hard for me, and I'm not going to talk unless I'm sure that I'm right." Darcy, who responded to 68/189 questions throughout the semester, had performance orientated reasons for her participation:

> I'm a junior, and med school isn't that far off anymore. I only have a 3.69 right now. I need to pull my GPA up over a 3.7 or I lose my scholarship, and to do that I need A's in all of my classes this semester. I don't care if I look dumb in front of freshmen I don't know as long as [my instructor] gives me my A.

Five of the eight (62.5%) of the Class 1 interviewees stated that the main reason that they wanted an A because they had medical school aspirations, and these students showed beliefs and behavior patterns indicative of low confidence and an entity theory of intelligence. 75% (6/8) of the interviewees said that the most important part of the class was to memorize procedures, and that being asked to solve story problems or applications where they had not seen a prior example exactly like it was unfair. All eight participants felt that the instructor did not do enough examples. When students encountered what they considered a novel problem[2] on an exam, 62.5% of the interviewees left the answer blank. During group work, if a group did not succeed on their first attempt, 58% of all groups observed during the semester would wait and wait for the instructor to come by and ask if they needed help. Only the students that regularly participated in class would call the instructor to their group when they had questions. The data for Class 2 is sparser, but the interview participants showed less entity beliefs than the students in Class 1. Most students would participate in Class 2 by talking with peers or instructor about confusing topics which was made easy by the

[2] Students defined a novel problem as one that was not a question on the study guide with the numbers changed, or if there were "hard" numbers, like fractions.

instructor's use of group work in most classes excluding review and test days. If group work is too hard, all of the interview participants would try what multiple strategies and then ask for help when out of ideas. This behavior was also observed in 11 of 16 group work situations during class. Also, the instructor would have students approach him right after class or during office hours to better understand the material that the class went over, and had at least four students working in groups during office hours after the first test.

Discussion

Students in all classes found the formative assessments to be a valuable learning tool, albeit in different ways, which suggests there were benefits to the formative assessments regardless of implementation. The more frequent mention of formative assessment as well as explicit mentions of the formative assessments by the instructor appeared to support more incremental attribution behaviors and increase students' sense of connection with their instructor. However, the class with the less frequent collection of formative assessment appeared to exhibit more performance goals and was reluctant to seek help from their instructor. Although the literature on best practices in formative assessment does not indicate that weekly collection of formative assessments is necessarily problematic (Shute, 2008), the lack of feedback from instructor to students did appear to contribute to students' feelings of disconnection with their instructor in Class 1.

Since the students in these classes were almost all freshmen life science majors randomly assigned to these two instructors in moderately sized classes, we did not anticipate that students in the two classes would exhibit large differences in goal orientation and persistence on the more challenging group work problems. Without quantitative data, it is unclear if these differences were significant, but the findings of this investigation suggest that further inquiry into the relationship between formative assessment and attribution for first year undergraduates may be warranted.

1. Have any quantitative data instruments beside P.A.L.S been developed that could aid in further exploration of this topic?
2. What does 'ownership of material' mean for undergraduates?
3. How would the formative assessment being submitted electronically affect its usefulness?

References

Black, P., & Wiliam, D. (2009). Developing the theory of formative assessment. *Educational Assessment, Evaluation, and Accountability, 21*(1), 5-31.

Bressoud, D. M., Carlson, M. P., Mesa, V., & Rasmussen, C. (2013). The calculus student: insights from the Mathematical Association of America national study. *International Journal of Mathematical Education in Science and Technology*, 1-15.

Chickering, A. (1993). *Education and identity* (2nd ed.). San Francisco: Jossey-Bass.

Clark, I. (2011). Formative assessment: Policy, perspectives and practice. *Florida Journal of Education Administration and Policy, 4*(2), 158-180.

Dweck, C. S. (2006). Mindset: The new psychology of success. New York: Random House.

Kurland, R. M., & Siegel, H. I. (2013). Attachment and student success During the transition to college. *NACADA Journal*, *33*(2), 16-28.

Nadelson, L. S., Semmelroth, C., Martinez, G., Featherstone, M., Fuhriman, C. A., & Sell, A. (2013). Why Did They Come Here? The Influences and Expectations of First-Year Students' College Experience. *Higher Education Studies*.

Shute, V. (2008). Focus on formative feedback. *Review of Educational Research, 78(1),* 153-189.

Patton, M. Q. (1990). *Qualitative evaluation and research methods* (2nd ed.). Thousand Oaks, CA: Sage Publications.

Vygotsky, L. (1987). The development of scientific concepts in childhood. In R.W. Rieber and A.S. Carton (Eds.) *The collected works of L.S. Vygotsky, Volume 1: Problems of general psychology* (pp. 167-241). New York, NY: Plenium Press.

Wiliam, D. (2009). An Integrative Summary of the Research Literature and Implications for a New Theory of Formative Assessment. In: *Handbook of Formative Assessment*. Routledge, London.

Wolcott, H. F. (2005). *The art of fieldwork* (2nd ed.). New York, NY: Altamira Press.

Generalization in undergraduate mathematics education

Allison Dorko
Oregon State University

Steven R. Jones
Brigham Young University

Generalization is a critical component of mathematical thought and thus a goal of instruction. However, most generalization research has focused on pattern generalization and generalization in algebra, not undergraduate mathematics. It is not known whether the generalization frameworks derived from such work adequately describe generalization in undergraduate mathematics, where the sorts of generalizations students must make are not regarding patterns, but rather the meanings of concepts and ideas. In this paper, we propose elaborations on existing generalization frameworks in order to take into account issues in learning advanced mathematics. We use how students generalize their notion of integration from single to multiple integrals as an illustrative case study.

Key words: generalization, calculus, multivariable calculus, integrals, and integration

Introduction

Generalization is a critical component of mathematics and mathematical thought and is thus a goal of k-12 mathematics instruction (NGA & CCSSO, 2010). While there exists a body of knowledge regarding generalization in k-12 mathematics topics, such as pattern generalization (Becker & Rivera, 2007; García-Cruz & Martinón, 1998), algebraic reasoning (Amit & Klass-Tsirulnikov, 2005; Becker & Rivera, 2006; Carpenter & Franke, 2001; Carraher, 2008; Cooper & Warren, 2008) and functional thinking (Ellis, 2007), fewer studies have explored how students generalize undergraduate mathematics topics. It is not known whether the generalization frameworks derived from work in k-12 mathematics adequately describe generalization in undergraduate mathematics, where the sorts of generalizations students make are not regarding patterns, but rather the meaning of ideas.

Undergraduate mathematics topics require students to increasingly generalize their notions of function, limits, derivatives, and integrals. For example, students must generalize their notion of function from single to multivariable functions, then to considering entire classes of functions. In this paper, we use how students generalize their notion of integration from single to multiple integrals as an illustrative case study to explore the ways in which current generalization frameworks capture the sorts of generalizations undergraduate students make, and ways of generalizing that existing frameworks do not adequately describe. In this preliminary report, we describe our current work using two generalization frameworks to explore generalization in undergraduate mathematics. So far we have found that Harel and Tall's (1991) expansive, reconstructive, and disjunctive generalizations framework works well for categorizing generalizations at a large grain size. The second, Ellis' (2007) generalizing actions and reflection generalizations, works well for talking about generalization at a smaller grain size, but does not contain categories that describe all of the ways in which undergraduate students generalize. Thus the main contribution of our work is to adapt and expand these two frameworks to better describe generalization at the undergraduate level.

Comparison of Various Generalization Frameworks

In this section, we explore Harel and Tall's (1991) and Ellis' (2007) generalization frameworks. We chose Harel and Tall's framework because it was developed specifically for describing generalization in undergraduate mathematics, and Ellis' because it has been used in the undergraduate mathematics context (Dorko & Weber, 2014). A second reason for choosing Ellis' framework was the actor-oriented nature. That is, Ellis (2007) defines generalization as what students see as similar across situations. We like the actor-oriented perspective because we think that studying how students think necessitates taking their perspective.

Ellis' (2007) generalization taxonomy was developed from a teaching experiment with middle school algebra students. The framework, shown in Table 1[1], distinguishes between generalizing actions, or students' mental activity as they generalize as inferred through their actions and talk, and reflection generalizations, which are students' final statements of generalization. Harel and Tall's (1991) framework was developed from studying students learning' of linear algebra, and distinguishes between expansive, reconstructive, and disjunctive generalizations (Table 2).

Table 1. Adaptation of Ellis' (2007) generalization taxonomy

Generalizing actions	
Type I: Relating	1. Relating situations: association between two or more problem situations. (a) connecting back (b) creating new 2. Relating objects: association between two or more present objects. (a) property (b) form
Type II: Searching	1. Searching for the same *relationship* 2. Searching for the same *procedure* 3. Searching for the same *pattern* 4. Searching for the same *solution* or *result*
Type III: Expanding	1. Expanding the range of applicability 2. Removing particulars 3. Operating 4. Continuing
Reflection Generalizations	
Type IV: Identification or Statement	1. Identifying a phenomena that continues beyond a specific case 2. Making a statement of commonality or similarity regarding a common (a) property (b) objects or representations (c) situations 3. Stating a general phenomenon. (a) rule (b) pattern (c) global rule
Type V: Definition	1. Class of objects: The definition of a class of objects all satisfying a given relationship, pattern, or other phenomenon
Type VI: Influence	1. Prior idea or strategy 2. Modified idea or strategy

[1] Space considerations prohibit us from including all the details of the framework. What is shown in Table 1 is the category titles. The reader is encouraged to read the category descriptions in Ellis (2007).

Table 2. Harel and Tall's generalization framework (Harel & Tall, 1991, p. 1)

Type	Definition
Expansive	Expansion of the applicability range of an existing schema without reconstructing it
Reconstructive	Reconstruction of an existing schema in order to widen its range of applicability
Disjunctive	Construction of a new, disjoint schema to deal with the new context. The student adds the new schema to the array of schemas available.

Data Collection and Analysis

Throughout the paper, we use how students generalize integration from single- to multivariable integrals to illustrate some instances of generalization in higher mathematics. We chose to study integration because it is a core concept in undergraduate mathematics, spanning single and multivariable calculus as well as analysis. Our data comes from semistructured interviews (Hunting, 1997) with twelve students from multivariable calculus and undergraduate real analysis. Each student completed two interviews that lasted 45 minutes to an hour. We asked students to talk about their meanings for $\int_a^b f(x)dx$, $\int_c^d \int_a^b g(x,y)dydx$, and $\int_{x_{n_a}}^{x_{n_b}} \dots \int_{x_{1_a}}^{x_{1_b}} g(x_1, \dots, x_n)dx_1 \dots dx_n$. Specific questions included the meaning of the *dx*, *dydx*, and dx_n terms, and what it meant if the integral equaled some specific value (e.g., if $\int_a^b f(x)dx = 5.2$, what does the 5.2 mean?). The interviews were recorded using LiveScribe technology, which provides a synched, real-time recording of audio and students' written work.

What Current Frameworks Do Not Capture

While current frameworks do capture some of the ways in which students generalize integration (e.g., pattern matching such as interpreting the meaning of *dydx* and dx_1 based on their similar appearance to *dx*), there are ways of generalizing that fall outside of these frameworks.

Analogous reasoning. One of the ways undergraduate students generalize is through use of analogy[2], and this is something that is not captured well in Ellis' (2007) framework. For instance, students talked about the n-integral as "*n* volume, or *n*+1 volume", "*n* space", "volume in some sense", or "something in higher dimensional space; an *n*-dimensional surface that's bounding some, whatever that concept is in *n*+1 dimensions". Using Harel and Tall's (1991) terms, this is a reconstructive generalization because while the students expanded the idea of volume, they must reconstruct it to be able to talk about a "volume-like" measure of higher dimensional space. Ellis' (2007) framework does not include a category that adequately describes this sort of generalization. The two that are the closest fit, *expanding the applicability* of a phenomenon, or stating a *general rule* (the phenomenon/rule being that integrals represent a measure of space) do not capture the analogical reasoning that students seem

[2] We use 'analogy' to mean a similarity used to make comparisons, not as a reference to the large body of psychological literature about analogical reasoning.

to use when they call the n integral measuring “volume in some sense” or “whatever that [measurement] concept is in $n+1$ dimensions”.

Students may reason by analogy and make a generalization that is mathematically incorrect. For example, Melissa thought that a double integral found surface area. She drew the graph of a paraboloid and said “when we’re dealing with a flat picture, like $f(x)$, we just find the area of that flat picture, so it’s a regular area, but because we’re dealing with a three dimensional graph and $g(x,y)$, we would be finding the area of the entire surface.” She seemed to think that surface area is a measure of three dimensional space (a not uncommon belief among calculus students; see Dorko & Speer (2014). Thus Melissa generalized that area is to R^2 as surface area is to R^3. It is unclear how to label this using Ellis’ (2007) framework. It would perhaps fit under extending, but it is not clear whether it is expanding the range of applicability of a phenomenon (the phenomenon being that integrals find area) or stating a general rule (the rule being that integrals find area). Using Harel and Tall’s (1991) terms, this is an expansive generalization because the student expanded the area conception to a different context.

We think that merely calling generalizations reconstructive, expansive, or disjunctive is not descriptive enough. In contrast, saying that students generalize through use of analogy is far more informative in terms of the specific actions in which students engage as they generalize.

Representative rectangles to representative rectangular prisms. Many students generalized a representative rectangle in a Riemann sum to a representative rectangular prism in a sort of ‘Riemann sum’ for double integrals. For instance, one student said “we could take rectangles of width dx, depth dy, and height $g(x,y)$ [draws a thin prism]. Yeah, a rectangular prism. And then add all those together in this rectangle from a to b and c to d [draws a rectangle to represent the domain] and we’re measuring the rectangular prisms underneath the surface.” It is clear that the student extended the idea of a representative rectangle as something useful beyond the context of single integrals, but it is not clear exactly how to categorize this using Ellis’ (2007) framework. Did he relate situations, one being a single integral and a Riemann sum, and the other being a double integral? Or did he expand the range of applicability of a Riemann sum? Both seem to fit. Using Harel and Tall’s (1991) categories, we would classify this as a reconstructive generalization because the student adapted his notion of Riemann sums to use prisms rather than rectangles.

The above two examples illustrate our finding, thus far, that Harel and Tall’s (1991) framework describes how undergraduate students generalize, but due to the larger grain size it subsumes many different generalizing actions into single categories. In contrast, Ellis’ (2007) framework is of a smaller grain size yet it is sometimes unclear into which category students’ generalizations fit. Further, sometimes generalizations fit multiple categories. While mutually-exclusive categories is not necessarily a goal of Ellis’ framework, we think that the fact that students’ generalizations often have many types of generalizing actions reveals the complex nature of generalization in undergraduate mathematics, and thus illuminates a need to explore generalization in this area.

Discussion

Our main finding is that Harel and Tall’s (1991) framework captures students’ generalizations but does so at a large grain size that is not always useful, while Ellis’ (2007) framework captures, some, but not all, of students’ generalizations at a smaller grain size. We believe that this is a result of using a framework that emerged from a work with middle school students. Specifically, we suspect that the sorts of pattern-generating activity involved in her teaching experiment are dissimilar to generalizing integration, which we see as a more nuanced and multifaceted concept.

We wish to create an actor-oriented framework at the grain size of Ellis' with which to describe generalization in undergraduate mathematics. Such a framework might use something like Harel and Tall's (1991) three categories as broad categories, with subcategories similar to those in Ellis' (2007) framework. We suspect that if integration data alone suggests a need to elaborate generalization frameworks, then other advanced mathematical topics (e.g., derivatives, limits) will likely also reveal other ways in which current frameworks need to be expanded. This provides a starting point for future research.

References

Amit, M., & Klass-Tsirulnikov, B. (2005). Paving a Way to Algebraic Word Problems Using a Nonalgebraic Route. *Mathematics Teaching in the Middle School, 10*(6), 271-276.

Becker, J. R., & Rivera, F. (2006). *Sixth graders' figural and numerical strategies for generalizing patterns in algebra*. Paper presented at the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Mérida, México.

Becker, J. R., & Rivera, F. (2007). *Factors Affecting Seventh Graders' Cognitive Perceptions of Patterns Involving Constructive and Deconstructive Generalizations.* Paper presented at the 31st Conference of the International Group for the Psychology of Mathematics Education, Seoul, Korea.

Carpenter, T. P., & Franke, M. (2001). *Developing algebraic reasoning in the elementary school: Generalization and proof.* Paper presented at the Proceedings of the 12th ICMI study conference. The future of the teaching and learning of algebra.

Carraher, D. W., Martinez, M.V., & Schliemann, A.D. (2008). Early algebra and mathematical generalization. *ZDM Mathematics Education, 40*, 3-22.

Cooper, T. J., & Warren, E. (2008). The effect of different representations on Years 3 to 5 students' ability to generalise. *ZDM, 40*(1), 23-37.

Dorko, A., & Weber, E. (2014). Generalising calculus ideas from two dimensions to three: how multivariable calculus students think about domain and range *Research in Mathematics Education*. doi: 10.1080/14794802.2014.919873

Ellis, A. (2007). A taxonomy for categorizing generalizations: generalizing actions and reflection generalizations. *The Journal of the Learning Sciences, 16*(2), 221-262.

García-Cruz, J. A., & Martinón, A. (1998). *Levels of generalization in linear patterns.* Paper presented at the 22nd Conference of the International Group for the Psychology of Mathematics Education Stellenbosch, South Africa

Harel, G., & Tall, D. (1991). The general, the abstract, and the generic in advanced mathematics. *For the Learning of Mathematics*, 38-42.

Hunting, R. P. (1997). Clinical interview methods in mathematics education research and practice. *The Journal of Mathematical Behavior, 16*(2), 145-165.

NGA, & CCSSO. (2010). *Common Core State Standards* Washington, DC: Authors

Studying the understanding process of derivative based on representations used by students

Sarah Dufour
Université du Québec à Montréal

The research presented in this paper aimed to construct models of understanding processes of students learning the derivative concepts. The models are constructed following Duval's theoretical framework and Hitt's ideas on representations. A teaching experiment was designed to observe students in action while they were participating to teaching episodes on the introduction of the derivative. Preliminary results for one participant are presented.

Key words: [calculus, understanding, representations]

Introduction

Calculus has been a subject well studied in our field for many years. A common concern among different studies results is about the lack of conceptual understanding of students in calculus. Namely, studies of Engelbrecht, Harding & Potgieter (2005), Carlson, Oehrtman & Elgelke (2010), Zerr (2010), Haciomeroglu, Aspinwall & Presmeg (2010) and, Baker, Cooley & Trigueros (2000) point out this concern. To cope with this problem, these researchers propose ideas and paths to follow that could lead to a better conceptual understanding of calculus. From different theoretical frameworks, they suggest, among other, that different teaching approaches adopted by teachers, types of tasks given to students and the coordination of different representations could lead to the development of a conceptual understanding. Following this line of thought, Zandieh (2000) and Hähkiöniemi (2006) proposed a model and a hypothetical learning path, respectively, for the understanding of the derivative.

Moreover, of all the work done on the difficulties of students in calculus (e.g. Tall, 1992), epistemological obstacles (e.g. Sierpinska 1985) that could be related to the study of these concepts or conceptions that students may encounter about concepts in calculus is a strong base to develop tasks, activities or whole teaching sequences of calculus. All these aspects are essential and have to be considered to observe the development of students' understanding processes and in a context designed to promote a conceptual understanding.

Objectives

The main aim of this project is to analyze students' discussions and productions related to the solution of tasks, in order to study the understanding processes of the derivative, in particular, the construction of a conceptual understanding, in a context in which the teaching sequence aims to promote this kind of understanding by reinvesting different aspects proposed in the literature (such as difficulties and conceptions already pointed out by different studies).

Theoretical framework

Because conceptual understanding is the main object of this study, it is important to determine how it is defined in this project. Particularly in the mathematics education field, different visions or paradigms exist to describe what understanding is. I share Haciomeroglu et al. (2010)'s view that understanding is "the ability to recognize the idea embedded in different representations, to manipulate the idea within given representations, and to translate the idea from one representation to another" (p. 153). I will present the theoretical framework about the notion of register of representations of Duval (1993) and the notions of functional representation of Hitt (2003) related to the construction of knowledge.

Duval describes the understanding of a concept by the ability of the student to pursue different actions on semiotic representations belonging to different registers. Registers are defined as systems that are described by different inherent rules and respond to signs that have an interpretation proper to a particular system (Duval, 1995). These rules and signs are commonly accepted by a community such as the mathematics community or more precisely here, the “college level” mathematics community. Therefore, in the context of this project, an introduction to calculus course, the different registers we are dealing with are graphic register, algebraic register, verbal register (written or spoken), table register and schematic register.

Furthermore, one of the most important aspects to consider in Duval’s theory is the actions that students do with the representations: production, treatment, conversion or coordination. Indeed, Duval argues that for a student having an *integrative understanding*, he must be able to not only produce or recognize a representation of a concept in different register but also be able to convert from a representation in a register to a new representation in another register using the “significant units” proper to the representations (Duval, 2007). Ultimately, this conversion must be done in all the possible ways and Duval identifies it as the coordination of the representations from different registers.

Moreover, the representations described by Duval are quite “formal” since they belong to a register so they follow different rules and criteria. However, since I want to study the understanding of students as a process, I need to consider all kinds of representations even if they are more intuitive or do not follow every single rule associated to a register. Following Hitt (2006), I refer to this kind of representations as functional representations. Therefore, the aim is to formulate models of the understanding processes of students by describing the representations they use and how they use it through the learning of the concept of derivative.

Method

Taking into account the theoretical aspects given above, the research took a perspective on understanding as a construction process that develops through several actions (produce, treat, converse, coordinate) taken on representations of various types (formal or functional) and in various registers (algebraic, graphic, verbal, schematic, table). To observe an evolution of this understanding process, it is necessary to pay attention to the actions students do when facing a task that was previously elaborated with a specific learning intention.

Under these conditions, the teaching experiment appeared to be indicated to reach the objectives of the research. Indeed, this methodology aims to document, through the production of a model, the mathematical development of students by observing, for example, their learning processes in a “teaching” context. The teaching experiment “involves experimentation with the ways and means of influencing students’ mathematical knowledge” (Steffe and Thompson, 2000, p.7).

A teaching experiment includes teaching episodes that are constructed as the experimentation progresses and which are based on what students did in the precedent episode. In this project, I favor teaching episodes with a group of students outside their regular calculus course.

The experiment

Eight students volunteered to participate to the teaching experiment. There were five teaching episodes, one every week, of 1 hour 30 minutes long. I conducted the teaching episodes as the teacher/researcher.

I will focus, in this paper, on the first episode in which the task below was proposed to students (figure 1). It is an open problem proposed only in the verbal register. First, students had 10 minutes to think about the problem individually. After this period, they formed teams and started to share their ideas for almost an hour. Finally, we discuss the problem, the whole group together, for 30 minutes.

A car is driving on the highway. The driver sees an animal about 70 meters in front of him and immediately pressed on the brakes. He stopped after 4 seconds and narrowly did not hit the animal. If the animal had been closer, say 35 meters, and the driver could not avoid it, at which speed would he hit the animal?

Figure 1: Task proposed for the first teaching episode

Preliminary results

For this paper, I chose to write about what one participant, Julie, did during the first episode.

Throughout the teaching episode, Julie presents some representations around the concept of derivative. First, Julie assigns specific role to the variables involved in the situation. Indeed, distance and time do not have the same status as speed in the representation she produces. For example, she says: "We are supposed to have a line [the relation between time and distance] whose slope is speed" (translation from the episode 1). Here, speed is depending on the two other variables.

In addition, she shows a certain "structure" involving relations between these variables, as if she sees a kind of hierarchy:

- On the one hand, Julie talks about the distance and the time as objects (variables) and the speed as a relation depending on these variables (she actually talks about the speed as a slope);
- On the other hand, she can also treat the speed and the time as objects (variables) that are related to "give" the acceleration (which is also a slope).

Besides, she also gives some clues that this "structure" also exists in the graphic register. For example, she says: "The first slope is always…the distance! Oh no! This is the reverse". She also suggests that the structure, in the graphical register, is recognizable by graphics with a slope and another one with a curve. However, she cannot tell which graphic is associated with which relations. Finally, she uses three verbal representations that could conceptually be connected to each other that are "slope", "rate of change" and "derivative". However, in this first episode, Julie uses these three representations really distinctly, in different contexts (that is to talk about different objects). Maybe, she will connect them somehow in the next episodes.

This is another indication that Julie is not really "coordinating" the representations, which means she does not necessarily analyses *significant units* or information given by a representation to produce another. She is more in a posture in which she tries to remember something that she already saw in class. It is a way to access to different representations that can help to produce others and then maybe, along a process, be able to finally coordinate the representations of a concept from different registers. That will be really interesting to see how Julie's understanding will evolve from these representations that she seems here to simply remember.

Questions for discussion

The analysis has only started. The first episode is really rich and is providing some information about how students convert or not from one representation to another. Although the theoretical framework presented earlier seems very useful and operational for detailed analysis of students' learning processes, I sometimes wonder if it would be necessary to call another concept or framework that would describe more broadly the understanding processes. Moreover, since the teaching experiment was done with a group of students, it is sometimes a challenge to construct individual model of students' understanding while considering the interactions that happened during the episodes. Would it be necessary or relevant to produce individual models and a more general model to shed light on interactions?

References

Baker, B., Cooley, L. & Trigueros, M. (2000). A calculus graphing schema. *Journal for the research in Mathematics Education*, *31*(5), 557–578.

Carlson, M., Oehrtman, M. & Engelke, N. (2010). The Precalculus Concept Assessment: A Tool for Assessing Students' Reasoning Abilities and Understandings. *Cognition and Instruction*, *28*(2), 113–145.

Duval, R. (1993). Registres de représentation sémiotique et fonctionnement cognitif de la pensée. *Annales de Didactique et de Sciences Cognitives*, *5*, 37–65.

Duval, R. (1995). *Sémiosis et pensée humaine: registres sémiotique et apprentissages intellectuels*. Peter Lang.

Engelbrecht, J., Harding, A. & Potgieter, M. (2005). Undergraduate students' performance and confidence in procedural and conceptual mathematics. *International Journal of Mathematical Education in Science and Technology*, *36*(7), 701–712.

Duval, R. (2007). La conversion des représentations : un des deux processus fondamentaux de la pensée. *In* J. Baillé (Ed.), *Du mot au concept Conversion* (pp. 9–45). Presses universitatires de Grenoble.

Haciomeroglu, E. S., Aspinwall, L. & Presmeg, N. C. (2010). Contrasting cases of calculus students' understanding of derivative graphs. *Mathematical Thinking and Learning: An International Journal*, *12*(2), 152–176.

Hähkiöniemi, M. (2006). *Hypothetical learning path to the derivative*. University of Jyväskylä.

Hitt, F. (2003). Le caractère fonctionnel des représentations. *Annales de Didactique et de Sciences Cognitives*, *8*, 255–271.

Sierpinska, A. (1985). Obstacles épistémologiques relatifs à la notion de limite. *Recherches en didactique des mathématiques1*, *6*(1), 5–67.

Steffe, L. P. & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. *In* R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267–307). Hillsdale, NJ : Erlbaum.

Tall, D. (1992). Students' difficulties in calculus. *Proceedings of working group 3 on Students' Difficulties in Calculus, International Conference on Mathematics Education 7 (ICME 7)* (pp. 13–28). Québec, Canada.

Zandieh, M. J. (2000). A theoritical framework for analyzing student understanding of the concept of derivative. *CBSM Issues in Mathematics Education*, *8*, 103–127.

Zerr, R. J. (2010). Promoting students' ability to think conceptually in calculus. *Primus*, *20*(1), 1–20.

An intended meaning for the argument of a function

Ashley Duncan
Arizona State University

This poster describes an intended meaning for the argument of a function when reasoning about a function covariationally. An instructional investigation was designed to promote this meaning with students in a college-level precalculus course using an instructional task and a didactic object relating to modeling the relationship between the time elapsed since Tommy threw a rock off of a bridge in seconds and the height above a lake in feet of the rock. Students were formally introduced to the term argument after the task and asked to evaluate and explain the meaning of functions defined with an argument as the function input. After engaging with the task, these students demonstrated larger learning gains than students in previous studies on the same questions that involved evaluating a function for an argument expression.

Key words: Function, Quantity, Argument

Introduction

An overarching goal of most precalculus courses should be to study the relationships between two quantities. Most of the time, students study these relationships in the context of a function with one quantity defined as the input of the function and the second quantity defined as the output of the function. A function is the constrained covariational relationship between two quantities where the value of one quantity uniquely determines the value of the second quantity (Thompson & Carlson, in press). In a study conducted that investigated undergraduate students' development of the function concept, undergraduate students were given the following question: "Compute $f(x+a)$ given $f(x)=3x^2+2x-4$" (Carlson 1998, p. 128). The mean response on this item was a 2.07/5 for a group of students who had recently completed College Algebra. The students who were able to correctly compute the answer justified their solution as "either a substitution of $x+a$ for *x*, or a procedure of adding *a* to every *x*" (Carlson 1998, p. 128). This suggests that students were not thinking about evaluating *f* at another point, nor were they thinking about $x+a$ as an input for *f*. However, the term *input* no longer applies when evaluating *f* at an expression, so I propose the use of the term *argument* in such situations. Figures 1 shows a traditional function definition and figure 2 shows the same function with a different representation of the argument.

$$\overbrace{\overbrace{\overbrace{f}^{\text{Name}}\ \overbrace{(t)}^{\text{Input}}}^{\text{Output}} = \overbrace{-9.8t^2+25t+80}^{\text{Rule of Assignment}}}^{\text{Function Definition}}$$

Figure 1 Conventions of function notation as depicted in *Pathways*

$$\overbrace{\overbrace{f}^{\text{Name}}\ \overbrace{(\underbrace{t}_{\text{input}} - 3)}^{\text{Argument}}}^{\text{Output}} = -9.8(t-3)^2+25(t-3)+80$$

Figure 2 Introduction of the argument to a function.

Once a function has been defined, the relationship between the quantity whose values are the output of the function and the quantity whose values are represented by the argument is fixed. By changing the representation of the argument, we can use the same function to relate a new input variable to the values of the output of the function. The purpose of this study was to develop a meaning for argument that would promote student's understandings of functions.

An Instructional Investigation

A hypothetical learning trajectory was created to build this meaning for argument with a class of 40 precalculus students in the *Pathways to Calculus* curriculum (Carlson, Oehrtman, and Moore, 2013). An investigation was designed involving Tommy throwing rocks off of a bridge into a lake. A didactic object (Thompson, was created using a computer program called *Graphing Calculator* (Avitzur, 2011). Students were given the following situation:

> 1. Tommy is standing on a bridge and throwing rocks off of the bridge into a lake. Suppose the function *f* models the relationship between the height above the lake (in feet) of any rock Tommy throws with respect to the time elapsed (in seconds) since Tommy threw that rock off the bridge.

As students progress through the task, they are asked to think about Tommy throwing a rock C that is thrown exactly 3 seconds after rock A is thrown. If t represents the time (in seconds) since rock A was thrown, $t-3$ represents the time (in seconds) since rock C was thrown. The height above the lake (in feet) of rock C can now be represented as $f(t-3)$, with $t-3$ now representing the argument to *f*. The figure below shows a screenshot of the didactic object.

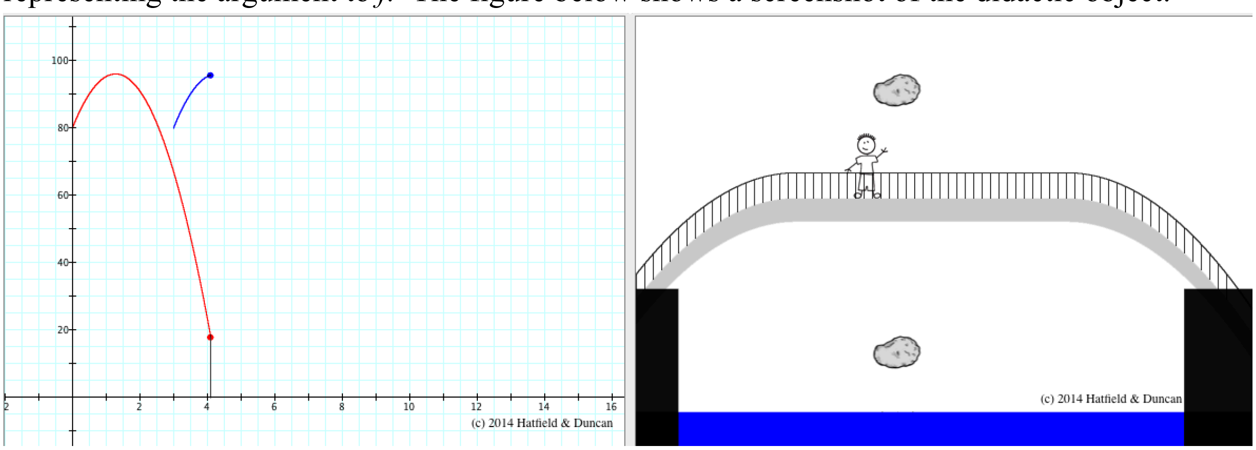


Figure 3 Tommy throws rock A.

Results and Future Research Implications

Students were assigned two online homework questions related to the task. One of the questions gave them $g(x+3)=2x+7$ and asked them to identify the argument of *g*, the input to *g*, and to evaluate $g(4)$. The mean scores on these two questions were 92% and 91%. This is an improvement from the study conducted by Carlson (1998). There are many future implications for the usefulness of being able to change the representation of the argument of a function such as with function composition, n-unit growth factors, and function transformations, to name a few. Further research is necessary to connect each of these ideas to this meaning of argument so that more students will be able to successfully build meanings for each of these concepts.

A preliminary categorization of what mathematics undergraduate students include on exam "crib sheets"

Antony Edwards
Swinburne University of Technology

Birgit Loch
Swinburne University of Technology

Many undergraduate examinations permit students to use a limited quantity of previously prepared notes: so-called "crib sheets", or "cheat-sheets". The majority of evidence from the literature suggests that students sitting such exams feel less anxious, and that they perform to a higher standard, although such results are idiosyncratic to discipline and course, and few are set in the context of undergraduate mathematics. Less is known about what content students choose to include on such a sheet, and how they interact with this material. This preliminary research report presents the first results from a three-year project investigating students' use of crib sheets in undergraduate mathematics exams. It explores the content and layout of crib sheets used by students for an end-of-semester calculus exam.

Key words: Examinations; Revision; Phenomenography; Calculus

Introduction

In mathematics exams at Swinburne University of Technology, Australia, undergraduate students may bring with them a single, double-sided, page of written notes. It is hoped that, in the process of creating these "crib sheets", students will revise effectively as they will have access to certain content of their choosing during the exam. There is also a belief that there will be a reduction in anxiety and the amount of rote learning of formulae and procedures required of students, who are then freed up to focus on deeper conceptual understanding of material.

Literature

When crib sheet exams are referred to in the literature, it is typically within the contexts of exam anxiety and exam performance. When considering the effect on exam anxiety, the majority of authors note that the use of crib sheets is effective in reducing anxiety (Butler & Crouch, 2011; Erbe, 2007; Janick, 1990; Weimer, 1989). Dickson and Miller's (2005) study with students in an undergraduate child development course found that students' prior beliefs about anxiety and subsequent reflections may not align: 80% believed that by using a crib sheet they would feel less anxious, but only 40% stated after the exam that they had done so.

In terms of exam performance, Dorsel and Cundiff (1979) noted that there is a link to the note-taking literature, in which authors such as Rickards and Friedman (1978) describe an 'external storage hypothesis' which implies that students will do better because they have access to more information during the exam, and an 'encoding hypothesis' that suggests the process of creating crib sheets allows students to organize their thinking. In line with these hypotheses, Gharib, Phillips, and Noelle (2012) found that psychology undergraduates scored significantly better on crib sheet exams than closed book, but also that they did better still on open book tests. Similar improvements were seen with Economics students (Wachsman, 2002) and students in a teaching course (Skidmore & Aagaard, 2004). In contrast, other authors have found no significant effect on performance (e.g. Dickson & Miller, 2005, and the statistics undergraduate students in Gharib et al.'s 2012 study). Such results do vary with the type of assessment, with improvements stronger for recall-based tests. Dickson and Bauer (2008) investigated the encoding hypothesis, and found that construction of crib sheets did not improve performance when students did not have access to their crib sheets. A meta-analysis of quantitative studies on crib sheets and open book exams by Lawin, Gorman, and Larwin (2013) found that overall, there is a "substantially higher" effect size for studies comparing student performance between crib sheet exams and closed-book exams, compared with those studies that had compared open-book and closed-book exams.

Less has been researched in the context of undergraduate mathematics. It is often argued that students consider mathematics to be primarily based on surface procedures (Crawford, Gordon, Nicholas, & Prosser, 1994; Frank, 1988; Garofalo, 1989). We might expect from the external storage hypothesis that crib sheets will boost performance in exams that require such procedural understanding. There is, however, growing evidence that crib sheets encourage reliance on procedural surface-level understanding of topics (Dickson & Bauer, 2008), dependence on the crib sheets (Funk & Dickson, 2011), and a "search" mentality when stuck on a problem (Burns, 2014). Whitley (1996) proposed a null effect: having more information is counter-acted by a belief from students that they need to understand it less well.

There is a need therefore to investigate crib sheet use in the context of undergraduate mathematics. We begin to do this here by first exploring the content on students' crib sheets. A limited amount of work has been done in this area outside of mathematics. Ludorf and Clark (2014) measured the quality of psychology undergraduates' crib sheets subjectively on scales measured 1-5 for *Overall quality*, *Verbal process information*, *Numeric process information*, *Density*, *Organization*, *Use of color*, and *Date of submission* (it is not clear from the paper how each scale was constructed, or how they compare). They found a positive correlation between the quality of a crib sheet, and performance in an exam. Content of the crib sheets produced by undergraduates on a programming course was encoded by de Raadt (2012). This took the form of binary indicators in the themes of *Layout* and *Content*, which were broken down into sub-themes: *Density*/*Organization*/*Ordering* and *Examples [of code]*/*Abstract representations [diagrams]*/*Sample Answers*/*Duplication*. Better exam performance was linked to those students whose crib sheets had a similar order to the course, who gave abstract representations, whereas indicators of poor performance were giving examples of code and sample answers. Our preliminary coding of our data from an undergraduate calculus class echoes de Raadt's broad themes, although inherent differences between the disciplines result in several differences at a more detailed level.

Methodology

This report presents preliminary results from the first of a four-stage research project. It addresses the questions "what do students choose to include on crib sheets?" and "how is this information presented?" Our data is crib sheets produced (and used) by students for a final unit examination in a first calculus course for non-mathematics majors.

Underlying our work is an assumption that a student's crib sheet represents that student's awareness of a course. More specifically, we believe crib sheets are a good indicator of the content students consider important for an exam and the material with which they feel least confident, both presented in a way that is intended to be helpful in an exam. We acknowledge that material already committed to memory may not be included on crib sheets, but we note that authors suggest that students aim to fill up their sheets completely (Erbe, 2007; Visco, Swaminathan, Zagumny, & Anthony, 2007). We have framed the four-stage project is within the interpretivist methodology of Phenomenography (Marton & Booth, 1997; Marton & Saljo, 2005), with an aim to explore students' use of crib sheets as a lens indicating their awareness of the course content. The results presented here from the first stage of the project are categories of description of the salient features of our students' crib sheets. We subscribe to the principle of Variation Theory (Runesson, 1999; Watson & Mason, 2006), in that we believe that if students are exposed to and become comfortable with many facets of crib sheet construction and content, they will be able to produce better, more effective crib sheets.

Our preliminary analysis is based on both authors independently open-coding salient properties of 30 crib sheets. The authors then discussed their codings, and constructed

Table 1
Dimensions of variation emerging from the preliminary coding exercise

Theme	Category of description	Types of difference
Layout-based	Density	Font size, Amount of white space, Location of white space
	Emphasis	Coloring, Boxing, Headings, Linearity, Box outs, Starring, Separation
	Sheet structure	Course Structure, Similarity structure, Neatness, Bullets, Orientation, Sub-division
Content-based	Examples	Worked solutions, Boundary examples, Sketches, Transferability of examples
	Representations	Brevity / Complexity, Calculator procedures, Fill in the gaps, Other languages
	Formulae	Listed, Grouped, In situ, Repetition of formulae sheet content
	Meta-content	Reminders, Messages, Thematic commentary, Arrows Indexing with color themes
	Correctness	Errors, Completeness

dimensions of variation, presented in Table 1. In the next stage of the project, we will examine all crib sheets from this class and also those from a different, more advanced course, in an attempt to describe the categories of variation as completely as possible, and to see if there are any differences between the groups. By the time this work is presented, we will have recorded a complete set of dimensions of variation for the crib sheets, and also recorded their occurrences and linkages between occurrences. Further stages of the project are planned, have received ethics clearance, and are described in more detail below.

Preliminary Results

Our preliminary themes and categories of description are listed and described in Table 1. We noted a distinction between the themes of content and layout, a distinction also made by de Raadt (2012). Due to the differences in subject and course, our categories of description were different to de Raadt's, and we note that our categories are rarely binary indicators.

In terms of layout, there were distinctions between density of text (i.e. font size), amount of white space (or gaps), and where white space was located. We also observed different techniques used to emphasize parts of their sheets: coloring, boxing, underlining. On some sheets these were present throughout, and on others only for key concepts (which is a link to the content-based categories of description). The structure of sheet layout also varied, with various methods to sub-divide space. In terms of content-based categories, there is a difference from de Raadt in that no students choose to include portions of code, but instead we saw many worked examples from lectures stated verbatim. The parts of these examples that were generalizable to the topic in question were seldom indicated and some contained copying errors. Different representations were used, with formulae statements being so ubiquitous that they were included as a sub-theme. A minority of students included content that was not directly from lectures and we labeled this meta-content.

Discussion

By taking the crib sheets as primary data, we have a snapshot of what students consider important to their exam performance. We do not know how the sheets were constructed, why particular layout and content were chosen, the relative importance of the content, or any links to anxiety or exam performance. This initial analysis has been necessary to address the gap in the literature of what students choose to include in crib sheets for undergraduate mathematics exams. It also will allow us to ground further parts of this project with an appropriate coding structure, rather than one taken from the literature of a different discipline area.

With this in mind, for the remainder of the first stage of the project, we will assume that students considered everything on their sheets important, and draw the inference that certain layout properties on the sheet such as boxing, underlining, highlighting and the use of color indicate emphasized importance. For instance, although worked examples were commonplace on many sheets, indicating students found them an important part of the course, they were rarely given a status of emphasis. By the time this work is presented we will have a more detailed description for each dimension of variation for two different levels of students, we will have investigated which mathematical topics are associated to which categories of description, and also considered the associations between categories themselves.

The next stage in the project will be to explore the links between crib sheet content and exam performance (including total and by-question scores, and displayed misconceptions). We will also conduct research interviews with students, to see if our interpretations of the data fit student views on the construction of crib sheets. Both these parts of the study will be completed in the second half of 2015. Our overall aim with the project is to formulate guidance to give to students before they produce crib sheets, and to conduct a randomized trial to see whether such guidance brings a performance gain in examinations (and thus, we hope, understanding of the course content). As Visco et al. (2007) noted, there is seldom a social aspect to construction of crib sheets, and we hope that encouraging a dialogue will aid construction and therefore performance. Such a trial will take place in 2016.

Questions to consider:

1. What is the relationship between content included on a crib sheet, the perceived importance of such content, and a students' level of confidence with that material?
2. Do crib sheets promote surface-level procedural understanding of a topic, or do they help students prioritize deeper understanding by relegating procedural content to the sheet?

References

Burns, K. (2014). Security Blanket or Crutch? Crib Card Usage Depends on Students' Abilities. *Teaching of Psychology, 41*(1), 66-68.

Butler, D., & Crouch, N. (2011). Student experience of Making and Using Cheat Sheets in Mathematical Exams. *Mathematics: Traditions and New Practices*, 131-141.

Crawford, K., Gordon, S., Nicholas, J., & Prosser, M. (1994). University Mathematics Students' Conceptions of Mathematics. *Studies in Higher Education, 23*(1), 87-94.

de Raadt, M. (2012). *Student Created Cheat-Sheets in Examinations: Impact on Student Outcomes.* Paper presented at the Proceedings of the 14th Australasian Coimputing Education Conference, Melbourne, Australia.

Dickson, K. L., & Bauer, J. (2008). Do Students Learn Course Material During Crib Sheet Construction? *Teaching of Psychology, 35*(2), 117-120.

Dickson, K. L., & Miller, M. (2005). Authorized Crib Cards Do Not Improve Exam Performance. *Teaching of Psychology, 32*(4), 230-233.

Dorsel, T. N., & Cundiff, G. W. (1979). The cheat-sheet: Efficient coding device or indispensable crutch? *Journal of Experimental Education, 48*(39-42).

Erbe, B. (2007). Reducing Test Anxiety While Increasing Learning: The Cheat Sheet. *College Teaching, 55*(3), 96-98.

Frank, M. (1988). Problem Solving and Mathematical Beliefs. *Arithmetic Teacher, 35*(5), 32-34.

Funk, S., & Dickson, K. L. (2011). Crib Card Use During Tests: Helpful or a Crutch? *Teaching of Psychology, 38*(2), 114-117.

Garofalo, J. (1989). Beliefs and Their Influence on Mathematical Performance. *Mathematics Teacher, 82*(7), 502-505.
Gharib, A., Phillips, W., & Noelle, M. (2012). Cheat Sheet or Open Book? A Comparison of the Effects of Exam Types on Performance, Retention and Anxiety. *Psychology Research, 2*(8), 469-478.
Janick, J. (1990). Crib Sheets. *Teaching Professor, 4*(6), 2.
Lawin, K., Gorman, J., & Larwin, D. (2013). Assessing the Impact of Testing Aids on Post-Secondary Student Performance: A Meta-Analytic Investigation. *Educational Psychology Review, 25*(3), 429-443.
Ludorf, M., & Clark, S. (2014). Help Sheet Content Predicts Test Performance. In W. Altman, L. Stein & J. Stowell (Eds.), *Essays from E-xcellence in Teaching* (Vol. XIII): Society for the Teaching of Psychology.
Marton, F., & Booth, S. (1997). *Learning and Awareness*. Hillsdale, NJ: Lawrence Erlbaum.
Marton, F., & Saljo, R. (2005). Approaches to Learning. In F. Marton, D. Hounsell & N. Entwistle (Eds.), *The Experience of Learning: Implications for Teaching and Studying Higher Education*. Edinburgh: University of Edinburgh, Centre for LEarning and Teaching.
Rickards, J., & Friedman, F. (1978). The Encoding Versus the External Storage Hypothesis in Note Taking. *Contemporary Educational Psychology, 3*, 136-143.
Runesson, U. (1999). *Teaching as constituting a space of variation*. Paper presented at the 8th EARLI conference, Goteborg, Sweden.
Skidmore, R., & Aagaard, L. (2004). The Relationship between Testing Condition and Student Test Scores. *Journal of Instructional Psychology, 31*(4), 304-313.
Visco, D., Swaminathan, S., Zagumny, L., & Anthony, H. (2007). *Interpreting Student-Constructed Study Guides: a Constructivist/Constructionist Perspective*. Paper presented at the 6th Annual Meeting, American Institute of Chemical Enginners.
Wachsman, Y. (2002). Should Cheat Sheets Be Used as Study Aids in Economics Tests? *Economics Bulletin, 1*(1), 1-11.
Watson, A., & Mason, J. (2006). Seeing an Exercise as a Single Mathematical Object: Using Variation Theory to Structure Sense-Making. *Mathematical Thinking and Learning, 8*(2), 91-111.
Weimer, M. (1989). Exams: Alternate ideas and approaches. *Teaching Professor, 3*(8), 3-4.
Whitley, B. E. J. (1996). Does `cheating' help? The effect of using authorized crib notes during examinations. *College Student Journal, 30*(4), 489.

The Structure, Content, and Feedback of Calculus I Homework at Doctoral Degree Granting Institutions and the Role of Homework in Students' Mathematical Success

In this study we investigate the relationship between the nature of Calculus I homework and student success. We examine these connections at five PhD granting institutions that were identified in a large US national study as having relatively successful Calculus I programs (compared to similar institutions) and we draw on both qualitative and quantitative from this study. Mixed method analyses point to a clear relationship between homework systems with varied structure, high feedback, and varied content emphases at more successful Calculus I programs where students persist onto Calculus II at higher rates and where students maintain more positive dispositions towards mathematics.

Keywords: Calculus, Homework, Student success, Mixed methods, Instructional triangle

Homework is an important part of how students develop fluency with the ideas and techniques in mathematics in general, and in calculus in particular. This is especially true at the undergraduate level, where the expectation is that students spend considerable time outside of class working on homework and studying course material. The time undergraduate students spend outside of class is an important venue to grapple with the material and interact with the content. In this study, we investigate the relationship between the nature of Calculus I homework and students' success in Calculus I at five PhD granting institutions in the US. The Calculus I programs at the five US institutions were selected as part of a large, national study at over 500 institutions, including Community Colleges, Bachelors-granting institutions, Masters-granting institutions, and PhD-granting institutions. These sites were selected because students at these institutions were more successful in Calculus I when compared to students at comparable institutions. This project enables us to ask the following questions that shed light on how homework is related to student success in Calculus I:

1. What characterizes the nature of homework at the selected PhD institutions with demonstrated successful Calculus I programs?
2. How does the nature of Calculus I homework relate to student success at PhD institutions?

To answer our first research question, we rely on the case study data from the five selected institutions to describe the nature of homework at each institution and more broadly. To answer our second research question, we draw on both the survey data to compare the nature of homework at selected institutions to non-selected institutions, and the student focus group interviews to examine students' accounts of the role of homework on their success.

Theoretical Background

We draw on Herbst and Chazan's (2012) elaboration of the instructional triangle (Cohen, Raudenbush, & Ball, 2003) to examine the relationships between students, their instructors, and the content. Herbst and Chazan's (2012) elaboration of the instructional triangle employs the concepts of didactical contract and milieu (Brousseau, 1997) to describe the interactions at play between teachers, students, and knowledge at stake through instruction, as shown in Figure 1. The knowledge at stake is used by Herbst and Chazan refers to the potential mathematical knowledge one might learn through instruction in all its forms. The major difference from the original instructional triangle is that the student component has been expanded into a subsystem in which students work on tasks maintained by the milieu. In this context, the milieu is "a counterpart environment that provides feedback on the actions of the students" (Herbst & Chazan, 2012, p. 607). Included within the milieu are the goals for students and the resources available to students while working on tasks (see Figure 1).

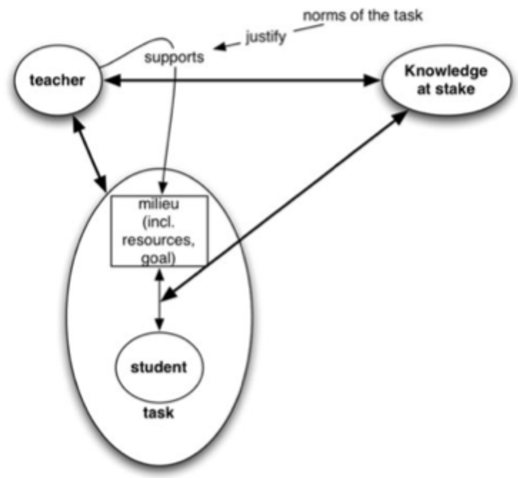


Figure 1. Instructional Triangle adapted by Herbst & Chazan (2012).

Following Herbst and Chazan (2012), we also use the construct of didactical contract (Brousseau, 1997) in which teachers, students, and knowledge at stake are bound to each other and the environment through implicit responsibilities. These implicit responsibilities are represented by the bidirectional arrows between each of the components in Figure 1. To satisfy the didactical contract teachers are responsible for students' development of the knowledge and students are responsible for partaking in the tasks needed to construct the knowledge.

Differences in the didactical contract exist between education at the K-12 level and education at the undergraduate level. In contrast to the K-12 level, undergraduate students are expected to put in considerably more hours in doing homework to learn course material. Therefore, studying the nature of homework at the undergraduate level can aid in the understanding of students' success.

Literature Review

Educators have investigated the role homework plays in student success for many years (Cartledge, & Sasser, 1981; Halcrow & Dunnigan, 2012; Morrel, 2007; White & Mesa, 2014). Student success entails both student learning and the confidence that one can achieve this learning (Dweck, 2008; Seymour & Hewitt, 1997). Research into the role homework has on student success has demonstrated the potential positive influence homework can have on student learning as well as student confidence (Halcrow & Dunnigan, 2012; Morrel, 2007; White & Mesa, 2014; Young et al., 2011). These studies also demonstrate that this is dependent on many aspects of the instructional environment and the interactions that take place between the teacher, student, and content within the instructional environment. Our study contributes to this literature by exploring the nature of homework assignments at selected institutions that were identified as having a successful Calculus I program. Further, we ground this exploration within the perspective of the instructional triangle, which allows us to examine relationships between aspects of homework as part of a system involving knowledge at stake, students, and the teacher.

Methods

To answer our research questions we conducted a mixed method analysis on data coming from a large, national study focused on successful calculus programs in the US, the Characteristics of Successful Programs in College Calculus (CSPCC) project. The CSPCC study consisted of two phases, the first of which was a survey given to Calculus I students and their instructors at the beginning and end of the term. The goal of this phase was to identify institutions with more successful Calculus I programs, as well as to learn more about the landscape of Calculus I in the US. The second phase of this study included explanatory case studies at five doctoral granting institutions selected for having more successful Calculus I programs as measured by increased student confidence, enjoyment, and interest in mathematics, Calculus I grade, and persistence onto Calculus II.

The five selected institutions varied in characteristics related to institution type, enrollment, the structure and class size of Calculus I sections, and the type of faculty employed to teach Calculus I. These five institutions included private, public, technical, small (less than 20,000 undergraduate population) and large (more than 20,000 undergraduate population) universities. The enrollment of undergraduate students ranged from approximately 3,000 to over 25,000. Also important to note was that the structure of the course and class size varied. Three of the institutions convened small class sizes of about 30 students, while the others had large lecture sections of about 200 students accompanied by smaller recitation sections.

A mixed methods approach was taken to analyze the quantitative survey data and the qualitative case study data (Creswell, 2002). End-of-term survey data from 2,023 students and 204 instructors from 60 PhD-granting universities were analyzed, as well as student focus group interviews and instructor interviews at the five case study PhD-granting universities. Of the 2,023 students, 597 came from a selected case study institution. Of the 204 instructors, 51 came from a selected institution. Descriptive analyses were conducted on both student and instructor responses to understand the nature of the homework at selected and non-selected institutions. Concurrently, inductive thematic analysis (Braun & Clarke, 2006) of the case study data was conducted to understand the nature of the homework assignments implemented at each of the case study sites.

Results

Research Question 1. The inductive thematic analysis of the case study data resulted in the identification of three salient features related to the nature of homework: structure, content, and feedback. We refer to these aspects together as a *homework system.* Figure 2 provides an overview of the homework systems at the five selected institutions.

	Structure			Content		Feedback	
	online	written	group	procedural	conceptual	collect/ graded	HW quiz
Public Technical Institute (PTI)	◌	◌	○	◌	◌	◌	○
Large Private University (LPU)	●	●	○	●	●	●	○
Private Technical University (PTU)	●	◌	○	●	◌	●	○
Large Public University 1 (LPU1)	●	●	●	●	●	●	○
Large Public University 2 (LPU2)	○	●	○	●	●	○	●

● Homework Component is uniform across all sections

◌ Homework Component is at instructors' discretion

○ Homework Component did not exist

Figure 2. Homework systems at selected institutions

At some of the institutions we saw that a part of the homework system was uniform across instructors, which is indicated in Figure 2 by a filled in circle. For example, Private Technical University (PTU) had an online procedural component that every Calculus I instructor assigned. We also saw that some but not all instructors at PTU assigned additional written conceptual homework, as indicated by the circle with several dots in it. We represent aspects of homework that we saw no evidence of existing with an empty circle.

As shown in Figure2, all institutions incorporated a combination of procedural and conceptual problem solving in their assignments, and all provided some form of feedback to the students. However, the structure of the different homework components, whether it be written, online, or how in which they gave feedback, varied among the five selected institutions. Using Brousseau's (1997) language, the homework system (structure, content, and feedback) functions as an important component of the milieu in which students interact with the mathematics. The homework system allows students to complete tasks, receive feedback, and avail themselves of certain resources to complete the tasks. For example, a homework system may create an environment where students can receive instant feedback by completing online homework and have access to resources such as hints provided by the online homework provider. Viewing the homework system as an important component of the milieu allows us to focus on the interactions between the homework system, the teacher, and the knowledge at stake for the course as well as the external environment that the interactions take place within. We attend to how the homework systems vary and the interactions that cause them to vary across the five selected institutions.

In terms of the instructional triangle (Herbst & Chazan, 2012), we view the *teacher* as anyone in a position to alter or create the homework system. This may include the course instructor, as well as recitation leader, or course coordinator. We view the *knowledge at stake* as the mathematical content covered within the course that is specified in the syllabus, textbook, or course objectives. In Herbst and Chazan's (2012) elaboration of the instructional triangle, they characterize the milieu as a "counterpart environment that provides feedback to the student" and includes the goals students are working towards and the resources with which the students are operating (p. 607). We argue that the homework system (structure, content, feedback) is an especially important component of the milieu within the undergraduate mathematical environment, necessitated by the shifted didactical contract.

Research Question 2: In the following we compare the responses from students or instructors at the selected institutions to those at the non-selected institutions along each of the three components of homework system: structure, content, and feedback.

Structure. Through the qualitative analyses, we attended to the mode of delivery of the homework as the key component of structure (online, written, and/or group). In the quantitative analyses, we also have reports of how often it was assigned and/or collected. As shown in Table 1, there were significant differences between student reports of the structure of homework at selected versus non-selected institutions. Compared to students at non-selected institutions, (a) students at successful institutions report that assignments were assigned and collected more frequently, (b) that group projects were assigned more frequently, and (c) that online homework was more likely to be used to complete homework assignments.

Table 1 Comparison of Structure of Assignments and Assessments

Student reports		**Non-Selected (n~1410)**	**Selected (n~590)**
How often was homework collected (either hard copy or online)?* *(1 = Never; 5 = Every class session)*	*Mean*	3.31	3.87
	Std. Dev.	1.32	1.34
How often did your instructor assign homework? ** *(1 = Never; 5 = Every class session)*	*Mean*	4.11	4.35
	Std. Dev.	0.98	0.92
Assignments completed outside of class time were submitted as a group project. *** *(1 = Not at all; 6 = Very often)*	*Mean*	1.34	2.97
	Std. Dev.	0.95	2.03
Assignments completed outside of class time were completed and graded online. *** *(1 = Not at all; 6 = Very often)*	*Mean*	3.56	4.81
	Std. Dev.	2.30	1.90

Note. * = $p \leq .10$, ** = $p \leq .05$, *** = $p \leq .001$; n varied slightly based on the question.

The combination of having multiple opportunities to interact with the material with supports available allows students to benefit from the extra time they are spending outside of class grappling with the homework content.

Content. The content of the homework can be thought of as a source of potential knowledge for students to develop. The balance between procedures and concepts is often an issue for instructors, both in terms of what is valued and what one has time for (Johnson, Ellis, Rasmussen, 2014). At all five of the selected institutions assigned both procedural and conceptual questions on their homework. Interestingly, we do not find the same balance at non-selected institutions. Table 2 shows that students at successful institutions reported having to explain their thinking more on homework problems and were assigned more word problems than students at less successful institutions. Table 2 also shows that instructors reported assigning more homework that focused on graphical interpretation, complex or unfamiliar word problems, and proof or justifications more than instructors at non-selected institutions.

Table 2 Comparison of Content of Assignments and Assessments			
Student reports		**Non-Selected (n~1410)**	**Selected (n~590)**
How frequently did your instructor require you to explain your thinking on your homework?*** *(1 = Not at all; 6 = Very often)*	*Mean*	3.01	3.62
	Std. Dev.	1.77	1.73
The assignments completed outside of class time required that I solve word problems *** *(1 = Not at all; 6 = Very often)*	*Mean*	4.59	5.01
	Std. Dev.	1.21	1.09
Instructor reports		**Non-Selected (n~158)**	**Selected (n~46)**
End-of-term: On a typical assignment, what percentage of the problems focused on:			
*skills and methods for carrying out computations (e.g., methods of determining derivatives and antiderivatives)?***	*Mean*	51.08%	39.57%
	Std. Dev.	19.21%	20.87%
*graphical interpretation of central ideas?****	*Mean*	21.44%	33.33%
	Std. Dev.	11.26%	20.67%
solving standard word problems?	*Mean*	23.65%	26.44%
	Std. Dev.	11.70%	16.26%
*solving complex or unfamiliar word problems?****	*Mean*	15.79%	28.22%
	Std. Dev.	11.65%	23.77%
*proofs or justifications?***	*Mean*	9.32%	14.42%
	Std. Dev.	8.38%	18.30%

Note. * = $p \leq .10$, ** = $p \leq .05$, *** = $p \leq .001$; n varied slightly based on the question.

Table 2 also shows that instructors at both selected and non-selected institutions report that they assigned homework involving standard word problems. Although successful institutions still assigned standard problems, they were complemented by complex or unfamiliar word problems more so than non-selected institutions.

Feedback. Feedback on homework emerged as a salient feature of the homework systems at selected institutions, and from a theoretical perspective the feedback given on the homework is a critical way the teacher can interact with the student, through the milieu of the homework system. Students were asked to report on a number of aspects of the nature of the feedback on their homework, both written and online. As shown in Table 3, students from selected institutions and non-selected institutions report that their assignments are more frequently completed and graded online at the same frequency, but students from selected institutions reported that they are returned with helpful feedback/comments. These results indicate that students from selected institutions receive more helpful feedback on both online and written homework than students at non-selected institutions.

Table 3 Student reports of the nature of homework feedback.

Assignments completed outside of class time were: (1 = Not at all; 6 = Very often)		**Non-Selected (n~1410)**	**Selected (n~590)**
Completed and graded online. ***	*Mean*	3.56	4.81
	Std. Dev.	2.30	1.90
Graded and returned to me.	*Mean*	4.25	4.25
	Std. Dev.	2.02	1.98
Returned with helpful feedback/comments. ***	*Mean*	2.70	3.26
	Std. Dev.	1.75	1.85

Note. * = $p \le .10$, ** = $p \le .05$, *** = $p \le .001$; n varied slightly based on the question.

Because undergraduate students are expected to spend more time constructing knowledge outside of class compared to during high school, the nature of the homework must support them in doing so. One key component of this appears to be providing responsive feedback to students as they grapple with the material outside of the classroom, with or without their peers.

Conclusion

Taken together, the above findings point to a significant shift in the didactical contract surrounding who is responsible for grappling with difficult material and where this takes place. The homework systems at selected institutions respond to this shift by providing multiple opportunities for students to interact with the content outside of class, provide feedback as a way for the teacher to interact with the student and the content while students work outside of class, and expect students to struggle with more complex content and explain their thinking related to this content. Perhaps these features are one reason why students at the selected institutions experienced an increase in their mathematical confidence, realizing that they are capable of succeeding with more difficult material.

These findings have both theoretical implications and practical implications. Our findings provide evidence for the applicability of the instructional triangle at the undergraduate level.. At the K-12 level, the majority of the interactions between student, teacher, and content occur *within* the classroom. However, at the undergraduate level, many of these interactions take place *outside* of the classroom. Thus, the homework system plays a heightened role in undergraduate mathematics because it acts as the milieu for these interactions to occur both inside and outside the classroom. Our findings also have direct implications for the classroom. As an undergraduate instructor, the homework system is an especially important resource for extending and furthering the learning that takes place during lecture. One must be purposeful about how to utilize online and/or written homework as a medium for students to practice skills and grapple with concepts, while providing feedback for the successful (and more enjoyable) development of both.

References

Blair, R., Kirkman, E.E., & Maxwell, J.W. (2012), *Statistical abstract of undergraduate programs in the mathematical sciences in the United States.* Conference Board of the Mathematical Sciences. American Mathematical Society, Providence, RI.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, 3(2), 77-101. doi:http://dx.doi.org/10.1191/1478088706qp063oa

Bressoud, D., Carlson, M., Mesa, V., & Rasmussen, C. (2013). The calculus student: Insights from the Mathematical Association of America national study. *International Journal of Mathematical Education in Science and Technology,* 44(5), 685-698. doi:10.1080/0020739X.2013.798874

Brousseau, G. (1997). Theory of didactical situations in mathematics: Didactique des mathématiques 1970–1990 (N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield, Eds. and Trans.). Dordrecht: Kluwer.

Cartledge, C. M., & Sasser, J. E. (1981). The effect of homework assignments on the mathematics achievement of college students in freshman algebra. Retrieved from http://www.johnsasser.com/pdf/article01.pdf

Chevallard, Y. (1991). La transposition didactique: Du savoir savant au savoir enseignée. Grenoble: La Pensée Sauvage.

Cohen, D., Raudenbush, S., & Ball, D. (2003). Resources, instruction, and research. Educational Evaluation and Policy Analysis, 25(2), 119–142.

Creswell, J. W. (2002). Educational research: Planning, conducting, and evaluating quantitative and qualitative approaches to research. Upper Saddle River, NJ: Merrill/Pearson Education

Dweck, C. S. (2008). *Mindsets and math/science achievement.* New York, NY: Carnegie Corp. of New York–Institute for Advanced Study Commission on Mathematics and Science Education.

Ellis, J., Kelton, M., & Rasmussen, C. (2014). Student perception of pedagogy and persistence in calculus. *ZDM.*

Enelke, N. (2007). A framework to describe the solution process for related rates problems in calculus. *RUME X Conference Proceedings.*

Halcrow, C., & Dunnigan G. (2012). Online homework in calculus I: Friend or foe?. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies,* 22(8), 664-682.

Herbst, P. , & Chazan, D. (2012). On the instructional triangle and sources of justification for actions in mathematics teaching. *ZDM*, *44*(5), 601-612.

Lenz, L. (2010). The Effect of a Web-Based Homework System on Student Outcomes in a First-Year Mathematics Course. *Journal of Computers in Mathematics and Science Teaching,* 29(3), 233-246. Chesapeake, VA: AACE.

Martin, T. (2000). Calculus students' ability to solve geometric related-rates problems. *Mathematics Education Research Journal*, 12(2), 74-91.

Morrel, J. (2007) Using problem sets in calculus, *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies,* 16(4), 376-384.

Piccolo, C., & Code, W. J. (2013) Assesment of of students' understanding of related rates problems. *Proceedings of the 16th Annual Conference on Research in Undergraduate Mathematics Education*; Denver, CO.

Rasmussen, C., & Ellis, J. (2013). Who is switching out of calculus and why? In Lindmeier, A. M. & Heinze, A. (Eds.). *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 4 (pp. 73-80). Kiel, Germany: PME.

Seymour, E. & Hewitt, N. M. (1997). *Talking about leaving: Why undergraduate leave the sciences*. Boulder, CO: Westview Press.

White,N., Mesa, V. (in press). Describing cognitive orientation of calculus I tasks across different types of coursework. *ZDM.*

Young, C. , Georgiopoulos, M. , Hagen, S. , Geiger, C. , Dagley-Falls, M. , et al. (2011). Improving student learning in calculus through applications. *International Journal of Mathematical Education in Science and Technology*, *42*(5), 591-604.

It's about Time: How Instructors and Students Experience Time Constraints in Calculus I

Jessica Ellis, Colorado State University
Estrella Johnson, Virginia Tech
Chris Rasmussen, San Diego State University

The goal of this research is to better understand the relationship between how quickly or deeply Calculus material is covered and how this is related to students' instructional experience and their persistence in a STEM major. To do so, we analyze data coming from a large national survey of Calculus I programs. Specifically, we first compare students' views of pacing to their instructor's views, resulting in four classifications of students. We then investigate various characteristics of these students, including: their institution type, their instructor type, their reported instruction, their mathematics beliefs, and their Calculus II intentions. Our findings suggest two important ways that pressure to cover material impacts Calculus I students and their instructors. First, when instructors feel pressure to cover material, student-centered teaching practices are often dropped. Second, feeling rushed to cover difficult material is a factor in losing STEM intending students.

Keywords: Calculus, Coverage, Pacing, Quantitative, Didactical Contract

The fact that calculus tends to be overstuffed with topics and taught in a manner that does not engage students is something that has been recognized by the broader mathematical community for decades. The calculus reform movement in the 1990's argued for a "lean and lively" approach to calculus. With the support of the National Science Foundation, the mathematical community developed a number of innovative approaches to calculus, including technology rich approaches to teaching, application rich content, a focus on students working collaboratively and on long-term projects, and an emphasis on a geometric perspective in addition to an analytic and numeric perspective on calculus. Evidence of lasting or systematic impact of these efforts has been minimal (Ganter, 2001; Haver, 1998).

The barriers that inhibit faculty from adopting leaner and livelier approaches to instruction are complex and involve the interplay of institutional, cultural, and cognitive factors. Student centered instructional approaches are often viewed as taking more time with less material being covered (Johnson, Caughman, Fredericks, & Lee, 2013). Faculty often cite concerns about coverage as reasons not to implement more student-centered instructional approaches (Christou et al., 2004; McDuffie & Graeber, 2003; Wagner, Speer, & Rossa, 2007). Research, however, continues to find that more active student instruction leads to deeper student understanding, longer retention of knowledge, more positive attitudes, and increased persistence in a STEM major (e.g., Freeman et al., 2014; Kogan & Laursen, 2013; Larsen, Johnson, & Bartlo, 2013; Rasmussen & Ellis, 2013; Rasmussen & Kwon, 2007). The previous overview points to a need to better understand the relationship between how quickly or deeply calculus material is covered and how this is related to students' instructional experience and their persistence in a STEM major.

We address this research need by analyzing data from a large national survey of Calculus I students and instructors. In our first stage of analysis, we examine the extent to which students and instructors report similar pressure regarding the speed at which material is covered in class. Specifically, our first research question is: *How do students' views of pacing compare to their instructor's views?* This comparison led to four classifications of students, which we leverage for the second research question: *For each of the four classifications, what are the student and* *instructor characteristics, including: their institution type, their instructor type, their reported instruction, their mathematics beliefs, and their Calculus II intentions?*

Theoretical Background

Embedded in this work are issues regarding the expectations of students and faculty. These expectations relate to who is responsible for learning, where learning occurs, and how much material is reasonable to cover. Theoretically, we see these types of expectations as part of the didactical contract (Brousseau, 1997). The notion of didactical contract refers to the set of reciprocal expectations and obligations between the instructor and the students, most of which are implicitly formed through patterns of interaction. For example, at the secondary school level students do not expect to have to cover large amounts of material on their own at home. Much of learning therefore occurs in class and students and their teacher are mutually responsible for learning. At the university level, however, these expectations and obligations may shift - the material covered increases, instructors tend to lecture more compared to secondary school teachers, and instructors expect students to learn more on their own at home. Students are often left feeling that their calculus courses are overstuffed and taught in an uninspiring and unresponsive manner (Seymour & Hewitt, 1997). It is precisely these aspects of the didactical contract that we aim to unpack at institutions with more successful calculus programs.

Methodology

To answer our research questions, we drew on survey data collected during the Characteristics of Successful Programs in College Calculus (CSPCC) project. CSPCC is a national study designed to investigate Calculus I. We have complete data (related to all questions that we address in these analyses) for 3,743 students that initially intended to take Calculus II, which we use as a proxy for being STEM intending. Surveys were sent to a stratified random sample of students and their instructors at the beginning and the end of Calculus I.

At the end of the term, both students and instructors were asked if they felt there was enough time for difficult ideas. Instructors were asked to respond to the prompt: When teaching my Calculus class, I had enough time during class to help students understand difficult ideas. Instructors were asked to provide a response ranging from 1 to 6 on a Likert scale, with 1 meaning “not at all” and 6 meaning “very often”. Students were asked to respond to the prompt: My calculus instructor allowed time for me to understand difficult ideas. Students were asked to provide a response ranging from 1 to 6 on a Likert scale with 1, meaning “strongly disagree,” and 6, meaning “strongly agree.” Instructor and student responses were linked, so we could match students’ responses to their instructor’s responses. Matched responses to these prompts were analyzed for our first research question.

To answer our second question, we investigated a number of aspects of students, including persistence, grade data, reported instructional experience, and reported beliefs. To examine their persistence, we used multiple questions across surveys to classify students into two categories: Persisters, and Switchers. Persisters are those students who initially intended to take more Calculus and did not change from this intention at the end of the term. Switchers, on the other hand, were those students that started Calculus I intending to take more calculus, but then by the end of the term changed their plans and opted not to continue with more calculus. We used instructors’ reports of students’ final grades as a measure of student achievement in the course.

To understand the relationship between students’ instructional experience and student and instructor reports of time to develop difficult ideas, we analyzed instructional practices as reported by students. Students were asked to report the frequency of 9 instructional activities, that we identify as instructor centered, including (a) instructor lecturing and (b) showing students how to work specific problems; student centered, including (c) having students give presentations; (d) having students work individually on problems or tasks; (e) asking students to explain their thinking; and (f) instructor asking questions; or interactive, including (g) having students work with one another; and (h) hold a whole-class discussion. Students were also asked to report how often their (i) prepared extra material to help students understand calculus concepts or procedures. Students were prompted to provide a response ranging from 1 to 6 on a Likert scale, with 1 meaning “not at all” and 6 meaning “very often”.

To investigate the relationship between students’ mathematical beliefs and student and instructor reports of time to develop difficult ideas, we used two sets of questions, coming from both student start of term and student end of term surveys. The first set of questions are worded identically on both surveys, and asked students to rate the level to which they agree with the statement “I am confident in my mathematical abilities,” with 1 meaning strongly disagree and 6 meaning strongly agree. Thus a positive change on this set of questions indicates that at the end of the term, the student gained mathematical confidence. The second sets of questions target the degree to which students agree that mathematics is about getting answers to specific problems. On both the start of term and the end of term survey, students were asked to rate the level to which they agree with the statement “Mathematics is about getting exact answers to specific problems,” with 1 meaning strongly disagree and 6 meaning strongly agree. Thus a positive change on this set of questions indicates that at the end of the term, the student agreed more reported that math is about getting exact answers (a more procedurally oriented perspective).

Results

To make the comparison between students’ views of pacing to their instructor’s views, we consider the four quadrants created by the two dimensions of student report and instructor reports. On one dimension, students either agree or disagree that there was enough time spent in class for them to understand difficult ideas. On the other dimension, instructors either agreed or disagreed to the same prompt. We computed a new value to indicate what quadrant the student would be in when graphing their response against their instructor’s response, as shown in Figure 1. Throughout this paper, we use this classification to demarcate students and to understand the calculus persistence and instructional experience of students in each quadrant.

Students in Quadrant I (QI) report having enough time to understand difficult ideas (answered 4-6) and their instructors agree (answered 4-6). Students in Quadrant II (QII) report not having enough time to understand difficult ideas (answered 1-3) but their instructors reported having enough time (answered 4-6). Students in Quadrant III (QIII) report not having enough time to understand difficult ideas (answered 1-3) and their instructors agree that there wasn’t enough time (answered 1-3). Students in Quadrant IV (QIV) report having enough time to understand difficult ideas (answered 4-6) though their instructors reported not having enough time (answered 1-3). From the lens of the didactical contract, students in QI and QIII share similar expectations as their instructors regarding the pace of the course. Students in QII and QIV have different expectations than their instructors regarding the pace of the course and different conceptions of the didactical contract within Calculus I.

As shown in Figure 1, almost 60% of students and their instructors agreed that there was enough time in class for them to understand difficult ideas, and around 6% of students and their instructors agreed that there was not enough time for them to understand difficult ideas. Nearly 15% of students felt that there was not enough time while their instructors thought there was, and around 20% of students felt there was enough time while their instructor thought there was not enough time.

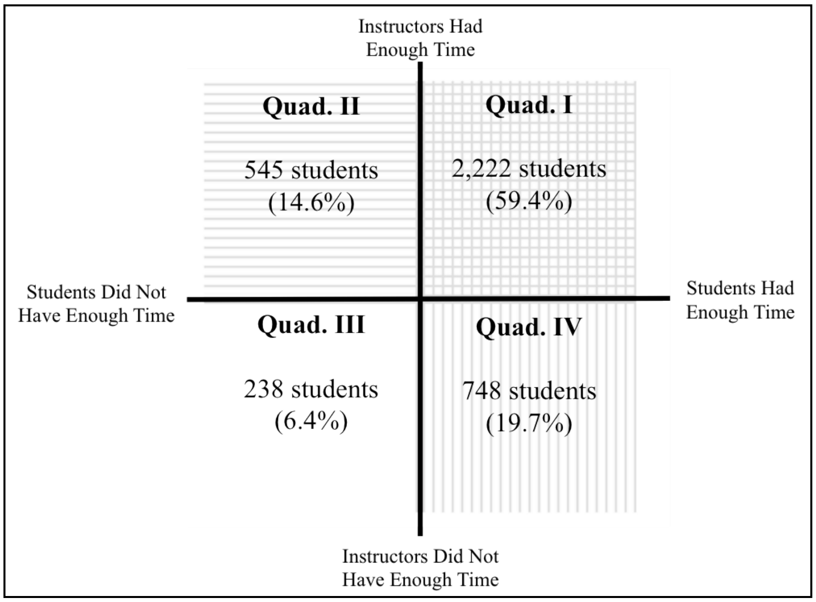


Figure 1. Classification of students based on agreement with instructor.

This data indicates that while the majority of students were in agreement with their instructors that there was enough class time to understand difficult ideas, there are many students that perceive the pacing of the class differently than their instructors, either as having more or less time than their instructors report. This finding allows us to examine various aspects of students from each of the four categories to gain a more holistic picture of their Calculus I experience. In the following sections, we present a summary of the multiple analyses conducted comparing students from each group along a number of dimensions to gain a profile of each group of students. As shown in Table 1, we compare: institution type; instructor type; reported instruction; and student success, specifically persistence onto Calculus II (as a proxy for STEM persistence), Calculus I grade, and mathematical beliefs.

Table 1. Comparisons across four types of students.

	QI	QII	QIII	QIV
Institution type (χ^2 (15, N = 3743) = 401.059, p < .001)				
CC	159	34	3	17
	7%	6%	1%	2%
BA	263	49	21	70
	12%	9%	9%	10%
MA	128	21	12	43
	6%	4%	5%	6%
Small PhD (<20k students)	533	169	132	328
	24%	31%	56%	44%
Large PhD (>20k students)	627	171	70	279
	28%	31%	29%	38%
Service Academy	512	101	0	1

	23%	19%	-	0%
Instructor Type (χ^2 (12, N = 3743) = 18.717, p < .001)				
Tenure track faculty	204	43	20	89
	9%	8%	8%	12%
Tenured faculty	516	129	120	288
	23%	24%	50%	39%
Other full time faculty	1205	296	60	264
	54%	54%	25%	36%
Other part time faculty	96	21	4	17
	4%	4%	2%	2%
Graduate teaching assistant	201	56	34	80
	9%	10%	14%	11%
Reported instruction				
How frequently did your instructor: (1=not at all; 6=very often)				
lecture?	4.8	4.9	5.5	5.4
	2215	545	238	738
	1.3	1.5	1.0	0.9
show how to work specific problems?	5.18	4.49	4.34	5.09
	2218	543	237	738
	0.9	1.3	1.5	1.0
have students give presentations?	2.1	1.5	1.2	1.6
	2207	544	238	737
	1.5	1.0	0.7	1.2
have students work individually on problems or tasks?	4.0	3.2	3.2	3.8
	2213	544	236	737
	1.6	1.7	1.7	1.7
ask students to explain their thinking?	4.3	2.9	2.3	3.7
	2215	545	237	737
	1.5	1.6	1.3	1.6
ask questions?	4.9	3.8	3.6	4.7
	2211	542	238	737
	1.0	1.3	1.3	1.1
have students work with one another?	3.7	2.8	2.2	2.9
	2211	545	237	737
	1.9	1.9	1.5	1.8
hold a whole-class discussion?	3.9	2.7	2.2	3.3
	2214	543	238	738
	1.8	1.7	1.5	1.8
prepare extra material to help students understand calculus concepts or procedures?	4.2	2.8	2.9	4.2
	2169	530	232	718
	1.4	1.5	1.5	1.4
Student Success				
Persisters	2005	469	191	645
(χ^2 (3, N = 3743) = 26.401, p < .001)	90%	86%	80%	87%
Switchers	217	76	47	93
	10%	14%	20%	13%
Expected grades (χ^2 (15, N = 752) = 67.250, p < .001)	160	12	9	79
A	160	12	9	79
	39%	15%	17%	39%
B	126	23	21	70
	30%	28%	40%	35%

C	86	20	13	43
	21%	24%	25%	21%
D	24	14	6	10
	6%	17%	12%	5%
F	18	11	2	1
	4%	13%	4%	1%
W	1	2	1	0
	0%	2%	2%	0%
Beliefs about math				
Change in confidence	-0.3	-0.9	-0.9	-0.3
(>0 means *more* confident)	1440	340	130	466
[$F(2375, 3) = 42.456$, $p < .001$]	1.0	1.3	1.2	1.0
Change in "Math is about getting exact answers to specific problems"	0.4	0.2	0.1	0.4
(>0 means *more* procedural)	1436	341	129	465
[$F(2370, 3) = 6.900$, $p = .016$]	1.4	1.5	1.5	1.4

QI: Students and their instructors reported having enough time

Compared to students in other quadrants, these students more often came from Community Colleges, Bachelors-granting institutions, and Service Academies and were more frequently taught by non-tenure-track full and part time faculty. These students reported relatively high frequencies of all instructional practices except lecture (which was reported at average frequency), indicating that there was a lot of diverse instruction occurring. This includes instructor-centered instruction, student-centered instruction, and interactive instruction. Additionally, students reported that their instructors frequently prepared extra material for them. These students switched their Calculus II intentions at the lowest levels (10% compared to the average 12%), received the highest percentage of A's, and average percentage of B's and C's, and F's, and relatively low percentage of D's and W's (Withdraws). Across all four quadrants, students' reported confidence in mathematics decreased over their Calculus I term. The students in QI reported average levels of a decrease in confidence. When asked to what degree they agreed that "Mathematics is about getting exact answers to specific problems," all students slightly increased in the amount they agreed, with average responses near 4, representing "slightly agree." Students in QI increased in the amount they agreed more than students in other quadrants.

QII: Students reported not having enough time but their instructors did

Compared to students in other quadrants, these students more often came from Community Colleges and large PhD granting institutions and were more frequently taught by non-tenure-track full-time and part-time faculty. These students reported relatively low-to-average frequencies of all instructional practices except having students explain their thinking and preparing extra materials, which were both reported at relatively low frequencies. These reports indicate that there wasn't a high frequency of any particular instructional activity occurring, but rather medium levels of many instructional activities. These students switched their Calculus II intentions at relatively high levels (14% compared to 12%), received the lowest percentage of A's compared to students in other quadrants, average percentage of B's and C's, and relatively high D's, F's, and W's. The students in QII reported relatively high levels of a decrease in confidence. These students also slightly increased in the amount they agreed, "Mathematics is about getting exact answers to specific problems."

QIII: Students and their instructors reported there was not enough time

Compared to students in other quadrants, these students more often came from small and large PhD granting institutions and were more frequently taught by tenured faculty and Graduate student Teaching Assistants (GTAs). These students reported relatively high frequency of lecture, relatively medium frequency of showing students how to solve problems, having students work individually and the instructor asking questions, and low levels of everything else. This indicates that these classes were largely dominated by instructor-centered instruction, with some individually oriented student-centered instruction, but low levels of interactive instruction. These students switched their Calculus II intentions at the highest levels of the four groups (20%), received a low percentage of A's compared to students in other quadrants, relatively high percentage of B's and D's, and relatively average C's, F's, and W's. The students in QIII reported relatively high levels of a decrease in confidence. These students had the least amount of change in the amount they agreed, "Mathematics is about getting exact answers to specific problems."

QIV: Students reported enough time but their instructors did not

Compared to students in other quadrants, these students more often came from small and large PhD granting institutions and were more frequently taught by tenure-track faculty. These students reported relatively high frequency of lecture, students working individually, the instructor asking questions, having a whole-class discussion, and the instructor preparing extra materials. All other instructional activities were reported at average frequencies. This indicates that there was a combination of instructor-centered instruction, student-centered instruction, and interactive instruction. These students switched their Calculus II intentions at relatively average levels (13%), received a high percentage of A's compared to students in other quadrants, relatively average percentage of B's and C's, and relatively low D's, F's, and W's. The students in QIV reported the relatively lowest levels of a decrease in confidence. These students had the most amount of change in the amount they agreed, "Mathematics is about getting exact answers to specific problems."

Discussion and Conclusions

These analyses paint four very different classroom images depending on if the instructor feels that there is enough time. For instance, in QI, where instructors agree that there was enough time to understand difficult ideas, there are higher than average reported frequencies of all instructional activities except for lecture. This is contrasted with QIII and IV, where instructors did not feel there was enough time. In QIII students and their instructors agree that there was not enough time in class to understand difficult ideas. In these classes, students reported higher than average levels of lecture and lower than average levels of all other practice. This environment appears very traditional and is consistent with the literature indicating that when there is a pressure for time, student-centered practices are sacrificed. Similarly, in QIV there were higher than average reported frequencies of showing students how to work specific problems, having students work individually on problems, lecture, and asking questions, and lower than average or average on the other practices. This suggests that when instructors felt some pressure to cover material, student-centered teaching practices were jettisoned.

In terms of students' perception of time, these findings indicate that students who switch their Calculus II intention are most likely to come from classes where the students do not feel there is not enough time to understand difficult ideas and where their instructors agree. In QIII, we saw 1) very traditional instructional practices, with high levels of lectures and low levels of any other instructional practice and, 2) especially large rates of students who changed their plans and opted not to continue with more calculus. These results are contrasted with results from QI, where students and their instructors both feel that they have enough time. In these classes, where there is a variety or traditional and student-centered instruction, students are more likely to continue with their intentions of taking further calculus courses.

Interestingly, in both QI and QIII, the students and their instructors seem to be in agreement on aspects of the didactical contract regarding pacing. In QI, students and their instructors agree that the course is moving at a reasonable pace, where there is enough time to understand difficult ideas. In QIII, students and their instructors agree that the course is moving at an unreasonable pace to support understanding. One way to understand the instructors in QIII is by considering the didactical contract they have with the broader mathematics and STEM departments. In this way, the instructor/student didactical contract may be related to an aspect of the instructor/department contract, one that pertains to the amount of material that needs to be covered in a given course. For instance, if an instructor has internalized an expectation that large amounts of material needs to be covered (an aspect of the instructor/department contract) then they may expect students to learn more material on their own (as aspect of the instructor/student contract). Given our findings, it appears that when both students and instructors internalize this pressure, we are most likely to lose STEM intending students.

References

Bressoud, D. (2009). Is the Sky Still Falling? *Notices of the AMS*, 56:2–7.

Brousseau, G. (1997). *Theory of didactic situations in mathematics* (N. Balacheff, Trans.). Dordrecht, Netherlands: Kluwer.

Carnevale, A. P., Smith, N., & Melton, M. (2011). STEM: Science, technology, engineering, mathematics. Georgetown University, Center on Education and the Workforce. Retrieved from http://www9.georgetown.edu/grad/gppi/hpi/cew/pdfs/stem-complete.pdf

Christou, C., Eliophotou-Menon, M., & Philippou, G. (2004). Teachers' concerns regarding the adoption of a new mathematics curriculum: An application of CBAM. *Educational Studies in Mathematics, 57*(2), 157-176.

Ganter, S. L. (2001). Changing calculus: A report on evaluation efforts and national impact from 1988–1998. *AMC*, *10*, 12.

Haver, W. E. (1998). *Calculus, Catalyzing a National Community for Reform: Awards 1987-1995*. Mathematical Association of America.

Johnson, E., Caughman, J., Fredericks, J., & Gibson, L. (2013). Implementing inquiry-oriented curriculum: From the mathematicians' perspective. *Journal of Mathematical Behavior. 32 (4).* 743 - 760

Larsen, S., Johnson, E., & Bartlo, J. (2013). Designing and scaling up an innovation in abstract algebra. *Journal of Mathematical Behavior 32 (4).* 693-711

McDuffie, A.R., & Graeber, A.O. (2003). Institutional norms and policies that influence college mathematics professors in the process of changing to reform-based practices. *School Science and Mathematics*, 103(7), 331-344.

President's Council of Advisors on Science and Technology (PCAST) (2012). *Engage to excel: Producing one million additional college graduates with Degrees in Science, Technology, Engineering, and Mathematics*. Washington, DC: The White House.

Rasmussen, C., & Ellis, J. (2013). Who is switching out of calculus and why? In Lindmeier, A. M. & Heinze, A. (Eds.). *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 4 (pp. 73-80). Kiel, Germany: PME.

Rasmussen, C., & Kwon, O. (2007). An inquiry oriented approach to undergraduate. *Journal of Mathematical Behavior, 26*, 189-194.

Seymour, E. (2006), Testimony offered by Elaine Seymour, Ph.D., University of Colorado at Boulder, to the Research Subcommittee of the Committee on Science of the U.S. House of Representatives Hearing on Undergraduate Science, Math and Engineering Education: What's Working? Wednesday, March 15, 2006

Wagner, J. F., Speer, N. M., & Rossa, B. (2007). Beyond mathematical content knowledge: A mathematician's knowledge needed for teaching an inquiry oriented differential equations course. *Journal of Mathematical Behavior*, 26, 247–266.

Wu, H. (1999). The joy of lecturing - With a critique of the romantic tradition of education writing. In S. G. Krantz (Ed.) *How to teach mathematics* (pp. 261-271). Providence, RI: American Mathematical Society.

Creating opportunities for students to address misconceptions: Student engagement with a task from a reform-oriented calculus curriculum

Sarah Enoch
Portland State University

Jennifer Noll
Portland State University

Research shows that students learn best when working with their peers as this creates opportunities for students to challenge one another's thinking, consequently building new knowledge. We will share analysis of three groups of calculus students engaging with an activity from a new reform-oriented calculus curriculum, which developed out of the philosophy that students working together promotes increased learning. Inspired by Gresalfi's engagement framework (2013), we developed an engagement coding scheme that differentiates between weak, procedural, and conceptual engagement. Our analysis focuses on the opportunities that emerged for students to address misconceptions. Findings show that how groups engaged with the same task varied significantly and, although opportunities to address misconceptions emerged in all groups, misconceptions were most effectively addressed in the group where conceptual engagement was prevalent. We discuss the extent to which norms, learning dispositions, and the activity itself played a role in the outcomes we observed.

Key words: Calculus, Engagement, Mathematical Tasks, Misconceptions

Introduction

Calculus has long been a gatekeeper for students entering STEM fields. Despite reform efforts to make calculus more accessible, high failure rates between 25%-40% persist (Bressoud et al, 2013; Burton, 1989). Even when students successfully pass calculus, research shows they often still have a poor grasp of many of the concepts and do not have the skills necessary for successful STEM careers (Selden, Selden, & Mason, 1994). As part of a larger NSF-funded research study that continues systematic inquiry into both curriculum development and research on student thinking, this study examines student learning of calculus content and the social aspects of the classroom curriculum. Our analysis focuses on groups of students working on a second derivative activity from a newly developed calculus curriculum, Process-Oriented Guided Inquiry Learning (POGIL). Our research question is: How does student engagement with the POGIL curriculum and with their peers in small groups afford opportunities for students to address misconceptions around the relationships between a function and its first and second derivative?

Background and Theoretical Frameworks

The research to date reveals that foundational calculus concepts are difficult for students and are often learned with significant misconceptions (Baker et al., 2000; Dreyfus & Eisenberg, 1981; Ferrini-Mundy & Graham, 1991; Tall & Vinner, 1981). Multiple studies have shown that students struggle with coordinating their understanding of first and second derivatives with that of the original function, particularly in a graphical context and that students often fail to recognize how the specific features of each of these graphs connect to one another (Berry & Nyman, 2003; Carlson et al, 2002; Christensen & Thompson, 2012; Orton, 1983).

We take the perspective that learning is a result of participation in a classroom community (see for example, Bowers, Cobb & MacClain, 1999; Greeno & Gresalfi, 2008; Gresalfi, 2009). Specifically, we focus on a framework of *affordances* for students to *engage* with mathematical ideas (Gresalfi, 2013; Gresalfi & Barab, 2011). Gresalfi and Barab (2011) use four types of engagement in their work: procedural, conceptual, consequential, and critical. They define procedural engagement as “using procedures accurately” and conceptual engagement as “understanding why an equation works the way it does” (p. 302). Consequential engagement “involves recognizing the usefulness and impact of disciplinary content” and critical engagement “involves questioning the appropriateness of using particular disciplinary procedures for attaining desired ends” (p. 302). They argue that the goal of curricular design and implementation is to foster consequential and critical engagement so that students use procedures and concepts as tools for investigating problems in meaningful ways. Their framework for engagement serves as a guiding lens for our analysis of the POGIL classroom observation data.

The Philosophy and Instructional Strategy of POGIL

POGIL was initially developed as a reform curriculum for chemistry. The POGIL curriculum is based upon the philosophy that students learn best when they are actively engaged with their peers and their instructor, discussing and exploring mathematical ideas. The POGIL activities are designed around a three-part learning cycle of *exploration* (in which students answer questions related to a model such as a graph or a problem situation that leads them towards an understanding of the concept to be learned), *concept formation*, and *application* (Hanson, 2006). The work presented here is part of a larger research project investigating the efficacy of the POGIL curriculum on student learning in calculus. The project is currently in its fourth year and data is still being collected.

Methods

The data used in this study was collected from two community colleges and one four-year University in the Pacific Northwest. The work presented here is based on video and audio recordings of classroom observations during an activity focused on the second derivative. Data from three groups are reported here – one from each school. During implementation of the POGIL activities, the classes were broken into groups of 3-4 students. Based upon a key strategy of the POGIL curriculum, each group member assumed a role such as recorder or spokesperson. While the roles from one classroom to the next were somewhat varied, there was always a group member whose responsibility was to read each question aloud. This role helped to ensure that the group members were all working on the same problem throughout the class time.

Our analysis focuses on the first half of an activity designed to introduce the concept of the second derivative. The activity is intended to guide students to recognize relationships between a function, its derivative, and its second derivative. The Model (see figure 1) used in the *exploration phase* of this activity is the graphs of a function and its first and second derivative. The activity prompts students to identify points and intervals on the graphs where the subsequent graphs were equal to zero and greater than or less than zero (respectively). As part of the concept formation phase of the activity (not shown in figure 1), which focuses specifically on $f''(x)$, the activity prompts students to describe what happens to the original function and the first derivative when $f''(x) = 0$, $f''(x) > 0$, and $f''(x) < 0$. The goal of the activity is to help students recognize that the original function is concave up when the second derivative is positive, concave down when the second derivative is negative, and has an inflection point where f ”(x) = 0. The activity is also intended to solidify students’ understanding that a function increasing or decreasing occurs when its derivative is positive or negative, respectively.

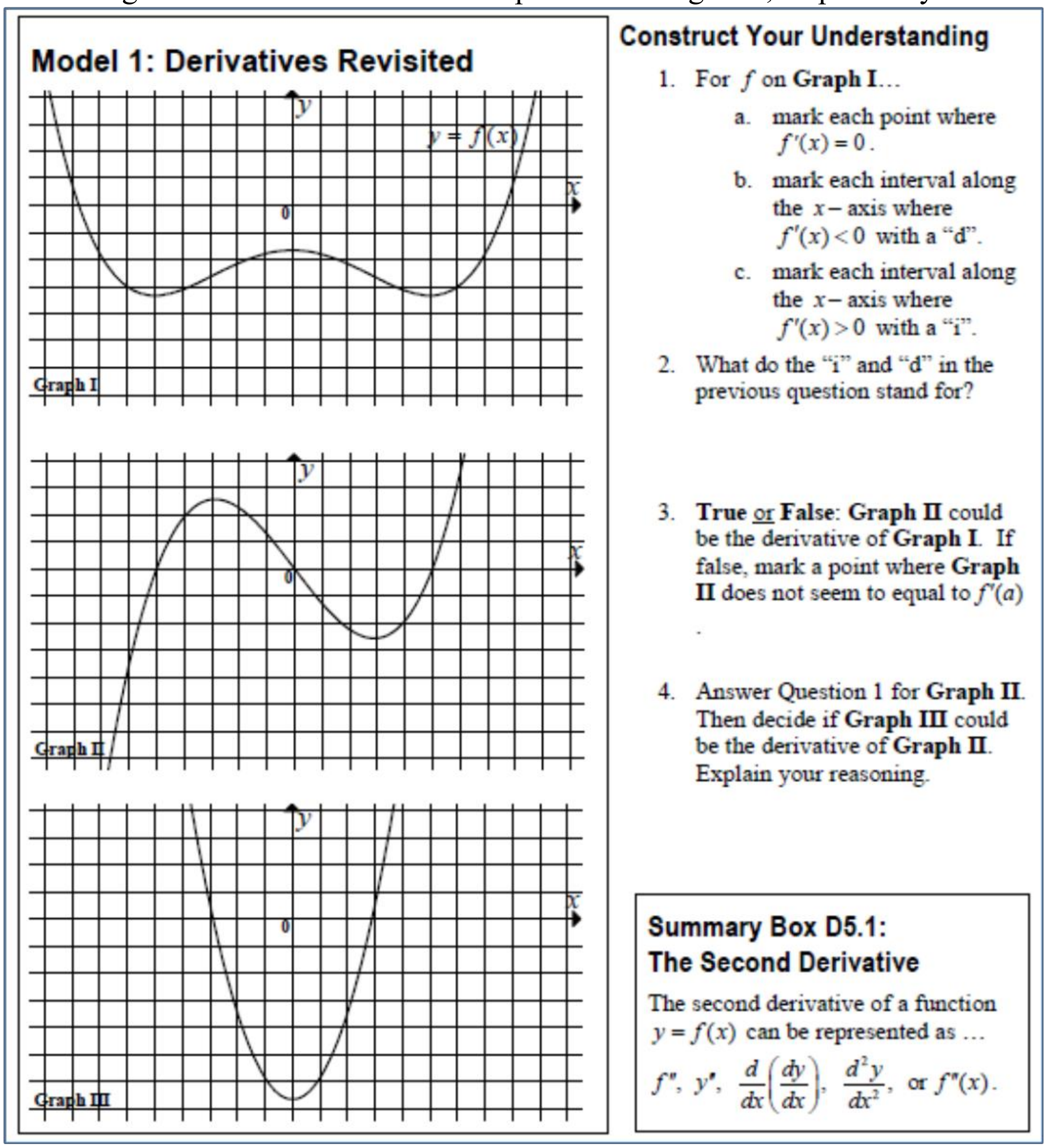


Figure 1. Beginning part of the second derivative activity.

Analysis

Building off of Gresalfi and Barab’s engagement framework (2011), we developed a coding scheme that broke down student engagement into weak, procedural, and conceptual engagement. Within each of these three levels, we further classified the type of activity the students were engaged in (see table 1). Note that, while consequential and critical engagement are not present in our coding scheme due to a general absence of these levels of engagement in the student discourse we observed, we made a note in our analyses when we felt that we observed these higher levels of engagement emerging from the discourse. In addition, we did not see strong opportunities in the task itself for consequential or critical engagement.

This coding scheme was used within each problem of the second derivative task. We found that, as the groups worked through a problem, the level of engagement often shifted. For example, a problem might open with W1 (Reading Task), followed by P1 (task interpretation) as the group attempted to make sure they understood what the problem was asking them to do. Then, some group members might share answers (W2) and this might be followed by a period of private think-time as the group members recorded their solutions (W4). We used these codes within our data to analyze the trends and developments of the groups' engagement with the task.

Table 1. Engagement Coding Scheme

Weak Engagement	
Reading task (W1)	Student reads task aloud
Sharing Answers (W2)	Students provide answers and the other group members accept it without further discussion.
Correcting errors (W3)	A student corrects another student's incorrect answer by telling the correct answer without an effort to address underlying procedures or concepts.
Private Think-Time (W4)	Students are working on their own without sharing their thinking with the other group members
Procedural Engagement	
Task Interpretation (P1)	Students discuss or explain what a problem is asking them to do.
Talking through Procedures (P2)	Using previously learned procedures to solve a task. Students are engaged in communicating the process.
Addressing Errors (P3)	Students are addressing errors by providing explanations that focus on the correct steps to be followed.
Using an incorrect procedure (P4)	Students are using a procedure incorrectly, that will lead to an incorrect solution.
Conceptual Engagement	
Reasoning through the task (C1)	The group does not have a clear solution or concept in mind, but is working towards making sense of the problem and generating possible reasons for their new thinking.
Conceptual Explanations (C2)	Providing conceptual explanations or justifications that are taken up by the group.
Addressing Misconceptions (C3)	Students help one another to work through a misconception by discussing the underlying concepts.
Non-viable conceptual engagement (C4)	The work either converges to a misconception, or does not converge to any concept.

Results and Discussion

Using our engagement codes to analyze the discourse that took place around the second derivative task within these three groups, we found that the levels of engagement for the three groups varied significantly. One group demonstrated frequent episodes of conceptual engagement, another episodes of procedural engagement, and the third episodes of weak engagement. We also found that, while these were the predominant levels of engagement for these groups, each group also demonstrated instances of higher levels of engagement, including some consequential engagement within the conceptual group.

We argue that the learning dispositions (Gresalfi, 2009) of the group members were influential in these levels of engagement. We observed that participants of the group with frequent episodes of conceptual engagement demonstrated a community in which each member was free to contribute as equals in the group regardless of prior knowledge or current level of understanding. In contrast, the procedural group demonstrated a dynamic in which there was a primary speaker and primary beneficiary. In this group, the primary speaker was committed to increasing understanding for his peers, but his explanations tended to be procedural in nature. In the weak engagement group, students were generally responsible for their own learning and engagement mostly took the form of vocalizing solutions without explanation. Despite the varying levels of engagement across the three groups, the POGIL activity afforded opportunities for students in all three groups to develop understanding of the relationship between a function and its first and second derivative, albeit the higher the level of engagement observed in the group, the deeper the level of understanding that was afforded.

Cappetta and Zollman (2013) propose four agents of change in the classroom as initiators of reflective thinking. These agents of change are the individual, their peers, the instructor, and the curriculum. The four agents of change framework is useful for interpreting our findings on students' engagement with the POGIL activity. The ways in which these students engaged at both higher and lower levels of thinking than what we felt the task afforded suggests that the individual and their peers played a significant role in the opportunities that were afforded for higher levels of engagement. A deeper investigation of the individual and peer groups is needed to better understand how instruction can capitalize on these agents of change in positive ways in the classroom. In addition, while some of these opportunities to develop understanding were afforded as a result of the POGIL curriculum and POGIL's teaching philosophies, we also recommend that the task could be modified to afford higher levels of engagement. Finally, more research needs to be conducted to help us to better understand the added aspect of the role of the teacher in affording higher levels of engagement.

Questions

During our presentation, we intend to pose the following questions to our audience: (1) Is our framework useful for capturing the varying levels of engagement of students working in small groups on mathematics tasks? (2) Our framework does not include levels of engagement higher than conceptual. Is it practical for us to include higher levels of engagement in our coding scheme? If so, what might that look like at the task level? At the student level? At the teacher level? (3) What changes might be made to the second derivative activity to increase the level of affordances for conceptual, consequential, or even critical engagement? (4) What is the role of the curriculum and the instructor in supporting both the individual and peer group to achieve higher levels of engagement?

References

Baker, B., Cooley, L., & Trigueros, M. (2000). A calculus graphing schema. *Journal for Research in Mathematics Education*, 31, 557-578.

Berry, J. & Nyman, M. (2003). Promoting students' graphical understanding of the calculus. *Journal of Mathematical Behavior*, 22, 481-497.

Bowers, J., Cobb, P., & McClain, K. (1999). The evolution of mathematical practices: A case study. *Cognition and Instruction*, 17, 25-66.

Bressoud, D; Carlson, M.; Mesa, V.; & Rasmussen, C. (2013). The calculus student: Insights from the Mathematical Association of America national study, *International Journal of Mathematical Education in Science and Technology*, 44, 685-698

Burton, M. (1989). The effect of prior calculus experience on “introductory” college calculus. *The American Mathematical Monthly, 96*, 350-354.

Carlson, M., Jacobs, S., Coe, E., Larsen, S. & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 33, 352-378.

Christensen, W. & Thompson, J. (2012). Investigating graphical representations of slope and derivative without a physics context. *Physical Review Special Topics – Physics Education Research*, 8, 023101.

Dreyfus, T. & Eisenberg, T. (1981). Intuitive functional concepts: A baseline study on intuitions. *Journal for Research in Mathematics Education*, 13, 360-380.

Ferrini-Mundy, J. & Graham, K. (1991). Research in calculus learning: Understanding of limits, derivatives, and integrals. *Paper presented at the Joint Mathematics Meetings*, San Francisco.

Greeno, J., & Gresalfi, M. (2008). Opportunities to learn in practice and identity. D. Pullin et al (eds.) *Assessment, equity, and opportunity to learn*, Cambridge, UK: Cambridge University Press. 170-199.

Gresalfi, M. (2009). Taking up opportunities to learn: Constructing dispositions in mathematics classrooms. *Journal of the Learning Sciences*, 18, 327-369.

Gresalfi, M. (2013). Technology in mathematics education: A discussion of affordances. *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Chicago, IL: University of Illinois at Chicago.

Gresalfi, M. & Barab, S. (2011). Learning for a reason: Supporting forms of engagement by designing tasks and orchestrating environments. *Theory Into Practice*, 50, 300-310.

Hanson, D. (2006). *Instructor's guide to Process-Oriented Guided-Inquiry Learning*. Lisle, IL: Pacific Crest.

Orton, A. (1983). Students' understanding of differentiation. *Educational Studies in Mathematics*, 14, 235-250.

Selden, J., Selden, A., & Mason, J. (1994). Even good calculus students can't solve nonroutine problems. In *Research issues in undergraduate mathematics learning*. MAA Notes, Mathematical Association of America: Washington, DC, Vol. 33, pp 19-26.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational studies in mathematics*, *12*, 151-169.

“What if we put this on the floor?”: Mathematical Play as a Mathematical Practice

Abstract: Mathematical play, as a mathematical practice, can be defined as the exploration of mathematical ideas in uninhibited and unconstrained ways, which could include engagement with physical devices, computer programs, imaginative acts, social interactions, and inscriptions. In this preliminary report I will discuss mathematical play and the ways in which it transpired in an undergraduate Foundation of Geometry course in which students engaged in activities to develop an understanding of physical, synthetic and analytic aspects of projective geometry. Additionally, I will discuss the ways in which mathematical play can be fostered through engagement in mathematically inspired art projects, as well as other artistic engagement.

Key Words: mathematical practices, mathematical play, arts integration, projective geometry

Introduction

Coinciding with the shift of researchers’ perspectives on learning from those that place sole emphasis on the cognitive aspects of learning to those that embrace the local and broader contexts in which learning occurs (Brown, Collins, & Duguid, 1989; Cobb & Yackel, 1996; Lave & Wenger, 1991), a greater emphasis is now being placed on mathematical practices – in the sense of practices of students in the mathematics classroom (e.g. Boaler, 1999; Cobb, Wood, Yackel, & McNeal, 1992; Stephan & Rasmussen, 2002), practices of mathematicians (e.g. Burton, 1999; Moschkovich, 2013), and practices of everyday people (e.g. Hoyles, Noss, & Pozzi, 2001; Lave, 1998; Nunes, Schliemann, & Carraher, 1993). This shift has resulted in an increased focus on the activities and practices in which students engage in the mathematics classroom. This is reflected in both the National Council of Teachers of Mathematics (NCTM) Principles and Standards document (NCTM, 2000) and by the introduction of the Eight Mathematical Practices into the Common Core State Standards Initiative documents (CCSSI, 2010). While the practices included in these standards documents are considered to reflect the authentic activity of professional mathematicians, they are generally geared toward mathematical performance, rather than mathematical appreciation and curiosity, which are essential characteristics of professional mathematicians as evidenced by aspects of their mathematical activity.

One activity that can promote mathematical appreciation and curiosity in students is that of *mathematical play*. In early childhood studies, play is considered a valuable activity in which students should engage (Piaget, 1951; Singer, Golinkoff, & Hirsh-Pasek, 2006), including as it relates to mathematical development (De Houlton, Ahmed, Williams, & Hill 2001 year; Ginsburg, 2006). However, students are not often engaged in play activities throughout their schooling, and in particular as they pertain to mathematics.

The notion of mathematical play is not new. De Houlton et al. (2001) define mathematical play as “that part of the process used to solve mathematical problems, which involves both experimentation and creativity to generate ideas, and using the formal rules of mathematics to follow any ideas to some sort of conclusion.” Within this characterization of mathematical play, the actor must follow through using mathematical rules to arrive at an answer to a problem. Engaging in play however does not always require a problem to be solved - yet problem solving can lead to mathematical play. Instead, the actor may simply be exploring the range of possibilities within a mathematical idea, and so she is not necessarily looking to arrive at a conclusion.

In these situations in which the actor is not attempting to solve a problem, mathematical play may purely be based on aesthetics, visual or otherwise, and can become a launching point for mathematical inquiry – in which case it may become a problem to solve. As such, in this study, I define *mathematical play* as exploring mathematical ideas in uninhibited and unconstrained ways, which could include engagement with physical devices, computer programs, imaginative acts, social interactions, and inscriptions. By uninhibited I mean the actor is freely exploring the mathematics at hand, without fear of judgment by others. By unconstrained I mean the purpose of the mathematical play may or may not be goal-driven, and the play may include the use of any tools at hand. Mathematical play often coincides with a mathematical disposition of curiosity, consistently asking "What if…" when considering mathematical ideas. For example, students using dynamic geometry software to explore triangles might ask themselves, "What if we doubled the length of each side of the triangle? Then what would happen (to the area, to the angles, etc.)?" Or similarly, "What if we add 1 to each side?"

Theoretical perspective

I approach this study from the perspective that all learning and knowing is situated (Brown, Collins, & Duguid, 1989; Greeno, 1998; Lave, 1988), and both socially and culturally mediated (Cobb & Yackel, 1996; Forman, 2003). Learners develop understanding through participation in cultural practices (Brown, Collins, & Duguid, 1989; Lave & Wenger, 1991). Furthermore, the particular modes of participation in a culture and context are mediated by the discourse of social interaction, cultural artifacts and tools, bodily engagement, and symbols. Consistent with this notion of learning and knowing, I align with the characterization of mathematical practices described by Moschkovich (2007), in which she describes mathematical practices as normative culturally and socially, as well as historically situated. In this framing, mathematical practices involve multiple resources, such as artifacts, tools, language, and other social aspects. As such, mathematical practices are embedded within the context in which they occur, and are constituted by the goals and meanings of discourse and purposeful activity (Moschkovich, 2007, 2013). Mathematical practices then can include discourse, behavior, and activity, in the context of the classroom community. These might include practices such as imagining, justifying, entertaining alternate possibilities, and ways of using mathematical tools. Mathematical play then, as a mathematical practice, can be composed of engagement with physical devices, computer programs, imaginative acts, social interactions, bodily engagement, and inscriptions.

Methodology

The setting for this study is a Foundations of Geometry course that took place during the fall semester of 2012 at a large Southwestern university. Participants included 16 out of 29 students enrolled in the course, and were primarily prospective secondary mathematics teachers. The focus of the course was projective geometry – a branch of geometry with roots in the formalization of the process by which artists can create a realistic drawing or painting of a three-dimensional object or scene. Through the use of physical devices and dynamic geometry software, students in this course worked in groups of three or four on novel problems designed to lead students to develop an understanding of physical, synthetic, and analytic aspects of projective geometry. The groups engaged in the activities and then reported back to the whole class on their ideas. As such, the mathematical ideas in the course were generally drawn out from the students, rather than explained by the instructor.

Students began their exploration of projective geometry ideas through the use of a physical device called the Alberti's Window, which consisted of a moveable eyepiece through which the students could view drawings or objects, and a 12x12-inch square piece of clear acrylic that stood perpendicular to a table (see *Figure 1*). Students sat facing the window, placed a drawing or object on the side of the window opposite themselves, looked through the eyepiece, and then traced on the window with a marker the drawing or object seen in front of them. Class discussions and further activities worked to extend the Alberti's Window activity to the geometric theory of projection, which included imagining the projection of points behind the eyepiece and between the eyepiece and the window.

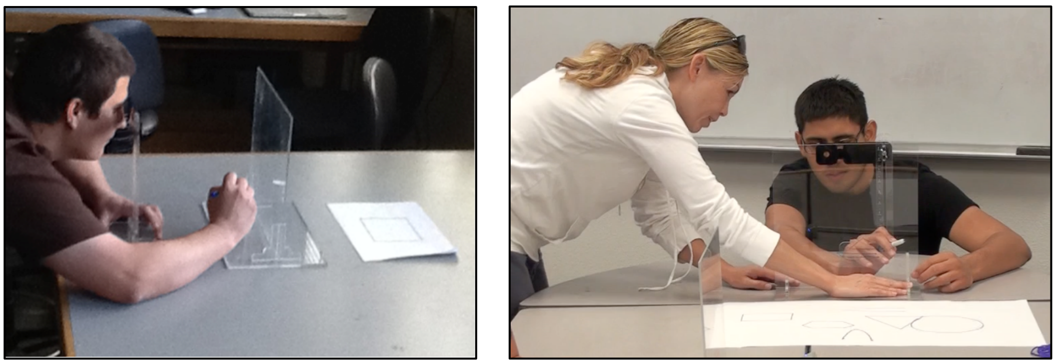


Figure 1: Students use the Alberti's window by looking through the eyepiece and tracing onto the window with a marker the object they see in front of them.

After exploration with the physical Alberti's Window, students were introduced to a Geometer's Sketchpad (GSP) (Jackiw, 1995) version of the Alberti's Window, which allowed them to further explore the geometry of linear projection. As a two-dimensional representation of the three-dimensional Alberti's Window situation, the GSP sketch shows an overlaying of both the horizontal plane (the tabletop) and the vertical plane (the window). This is obtained by rotating the horizontal plane 90 degrees about the intersection of the two planes. Since this GSP Alberti's Window represented two overlaying planes, objects in the sketch that were located on the horizontal plane (the tabletop) were colored green and objects located on the vertical plane (the window) were colored orange. The sketch contained a line that represented the intersection of the two planes, a line that represented the distance of the eyepiece from the window, and a line that represented the height of the eyepiece from the tabletop (see *Figure 2*).

In addition to exploring projective geometry through the use of the physical and GSP versions of the Alberti's Window, students analyzed various paintings and, as a culminating project, created their own painting with airbrushing techniques. Using the GSP version of Alberti's Window, students created an artistic design in the form of a GSP sketch fitting within a 10-inch x 13-inch frame. Students were required to use aspects of projective geometry discussed in the course to create their design, however not all aspect of the design needed to be projected objects. Stencils of each student's design were cut and students utilized the stencil in conjunction with an airbrush to paint their design in any way they desired. The intention of this project was not for students to demonstrate their understanding of projective geometry, but rather for the students to use the ideas of projective geometry for the sake of art creation. As such, the completed designs do not necessarily look like canonical images of projective geometry (see *Figure 3*).

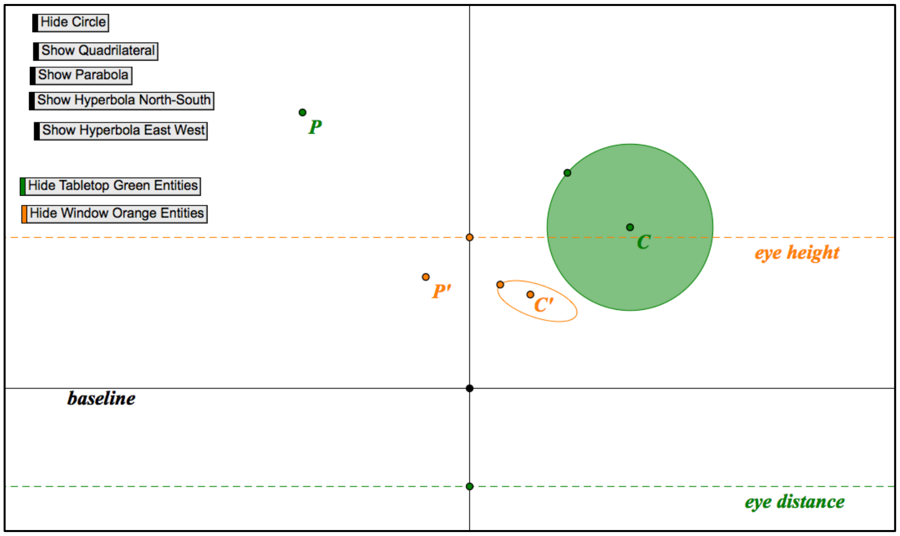


Figure 2. Screen shot of GSP version of Alberti's Window. The green entities are those on the tabletop and the orange entities are those on the window.

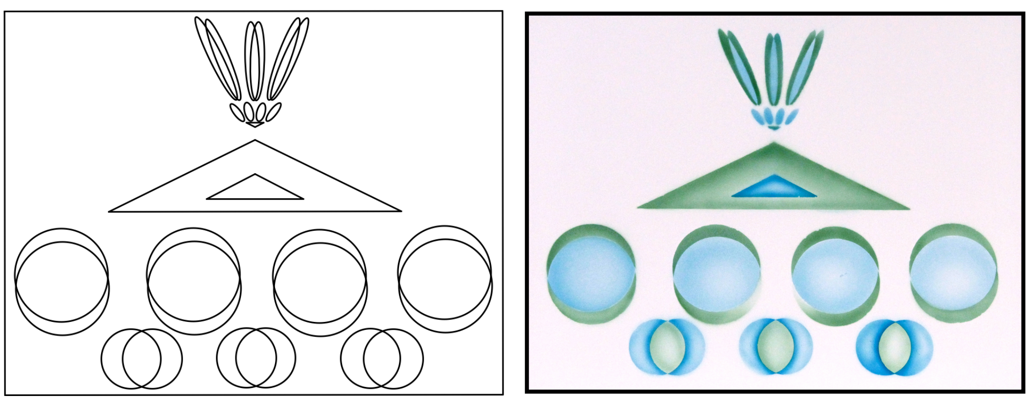


Figure 3. Left: The initial stencil design; *Right:* The completed painting.

This purpose of this research is two-fold. First, this research investigates the mathematical practices in which students engage during an activity-based Foundations of Geometry course with a focus on projective geometry. Second, this research explores the ways in which artistic engagement can enrich students learning experiences in this undergraduate geometry course. Analyses of classroom video data, individual student interviews, as well as students' written and video reflections, using a grounded approach to data analysis (Strauss & Corbin, 1990, 1994), form the basis for this preliminary report

Preliminary Results and Implications

Preliminary analyses suggest that students in this Foundations of Geometry course frequently engaged in mathematical play – both during classroom activities, as well as during the creation of and reflection on their mathematically inspired art projects. In particular, the design of the course, including the artistic engagement aspects, fostered mathematical play. In this course, mathematical play seemed to suggest multiple implications for students' development. Mathematical play appeared to help students develop their mathematical intuition and imagination, and served as a launching point for mathematical inquiry. This suggests that engagement in mathematical play, both inside and outside the classroom, has the potential to lead students to mathematical discoveries, or mathematical situations, which they otherwise may not encounter. As such, making space for mathematical play in the classroom should be considered.

Questions for audience:

1. What roles might mathematical play serve in developing mathematical understanding?
2. What are ways in which we might engage students in mathematical play at various levels of education?

References

Boaler, J. (1998). Open and closed mathematics: Student experiences and understandings. *Journal or Research in Mathematics Education, 29*, 41-62.

Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher*, 18, 32-41)

Burton, L. (1999). The practices of mathematicians: What do they tell us about coming to know mathematics? *Educational Studies in Mathematics, 37*(2), 121-143.

Cobb, P., Wood, T., Yackel, E., & McNeal, B. (1992). Characteristics of classroom mathematics traditions: An interactional analysis. *American Educational Research Journal, 29*(3), 573-604.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31*, 175-190.

Common Core State Standards Initiative (CCSSI). (2010). *The standards: Mathematics*. Retrieved from http://www.corestandards.org/the-standards/mathematics

De Holton, D., Ahmed, A., Williams, H., & Hill, C. (2001). On the importance of mathematical play. *International Journal of Mathematical Education in Science and Technology*, *32*(3), 401-415.

Forman, E. (2003). A sociocultural approach to mathematics reform: Speaking, inscribing, and doing mathematics within communities of practice. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 333-352). Reston, VA: National Council of Teachers of Mathematics.

Ginsburg, H. P. (2006). Mathematical play and playful mathematics: A guide for early education. *Singer et al., op. cit*, 145-165.

Greeno, J. (1998). The situativity of knowing, learning, and research. *American Psychologist, 53*(1), 5-26.

Hoyles, C., Noss, R., & Pozzi, S. (2001). Proportional reasoning in nursing practice. *Journal for Research in Mathematics Education, 32*, 4-27.

Jackiw, N. (1995). *The Geometer's Sketchpad*. Berkeley, CA: Key Curriculum Press.
Lave, J. E. (1988). *Cognition in practice*. Boston, MA: Cambridge.
Lave, J. E., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. New York, NY: Cambridge University Press.
Moschkovich, J. N. (2002). Chapter 1: An introduction to examining everyday and academic mathematical practices. In M. Brenner & J. Moschkovich (Eds.), *Everyday and Academic Mathematics: Implications for the classroom. Journal for Research in Mathematics Education*, Monograph Vol 11, 1-11.
Moschkovich, J. (2007). Examining mathematical discourse practices. *For the Learning of Mathematics, 27*(1), 24.
Moschkovich, J. N. (2013). Issues Regarding the Concept of Mathematical Practices. In *Proficiency and Beliefs in Learning and Teaching Mathematics* (pp. 257-275). SensePublishers.
National Council of Teachers of Mathematics (NCTM). (2000). *Principles and Standards for School Mathematics.* Reston, VA: Author.
Nunes, T., Schliemann, A., & Carraher, D. (1993). *Street mathematics and school mathematics.* Cambridge: Cambridge University Press.
Piaget, J. (1951). *Play, Dreams and Imitation in Childhood.* New York, NY: Newton.
Singer, D. G., Golinkoff, R. M., & Hirsh-Pasek, K. (Eds.). (2006). *Play= Learning: How play motivates and enhances children's cognitive and social-emotional growth*. Oxford University Press.
Stephan, M., & Rasmussen, C. (2002). Classroom mathematical practices in differential equations. *The Journal of Mathematical Behavior, 21*(4), 459-490.
Strauss, A., & Corbin, J. (1990). *Basics of Qualitative Research: Grounded Theory Procedures and Techniques*. Newsbury Park, California: Sage Publications.
Strauss, A., & Corbin, J. (1994) Grounded theory methodology: an overview. In N.K. Denzin & Y.S. Lincoln (Eds.), *Handbook of qualitative research*.

INSTRUCTIONAL SEQUENCE FOR MULTIDIGIT MULTIPLICATION IN BASE FIVE

In this poster I describe an instructional sequence for supporting the guided reinvention of multidigit multiplication in a preservice teacher content course. I use design heuristics from Realistic Mathematics Education (Gravemeijer, 1998) to design, test, and modify an instructional sequence leveraging the non-routine context of base five.

Key words: Multiplication, Alternate bases, Preservice teachers

My poster presents research on a design experiment for teaching multidigit multiplication in a preservice teacher content course. The design experiment resulted in an instructional sequence focused on an innovative way to approach the routine topic of multiplication of multidigit numbers using base five.

Problem of practice: Motivating preservice teachers

When working with preservice elementary teachers (PSTs), one of the biggest challenges is that PSTs tend to have procedural fluency with elementary mathematics but they may not have had the opportunity to explore the underlying mathematical concepts and structures. Research shows that because they can solve elementary math problems with memorized algorithms, they may not be motivated to learn mathematics at a deeper level ((Philipp et al., 2007; Thanheiser, Philipp, Fasteen, Strand, & Mills, 2013). This challenge has led me to leveraging alternate bases as a tool to help PSTs to build a conceptual understanding of place value structure and operations. The instructional sequence (see Figure 1) was built with the design principles of guided reinvention, modeling, and didactic phenomenology from Realistic Mathematics Education (Gravemeijer, 1998).

Results and Instructional sequence

My research focuses on the development of an instructional sequence which guides PSTs to reinvent a generalized algorithm for multidigit multiplication. The PSTs begin with problems that are small enough to be solved with repeated addition (see Tasks 1 and 2 in Figure 1). As the problems get larger (see Tasks 3 and 4), the context of area becomes both a model of multiplication and a tool for creating a general strategy for solving multidigit multiplication problems. For the 4th task in the sequence, the PSTs are asked to create a general strategy which would allow them to find the area of any base five rectangle.

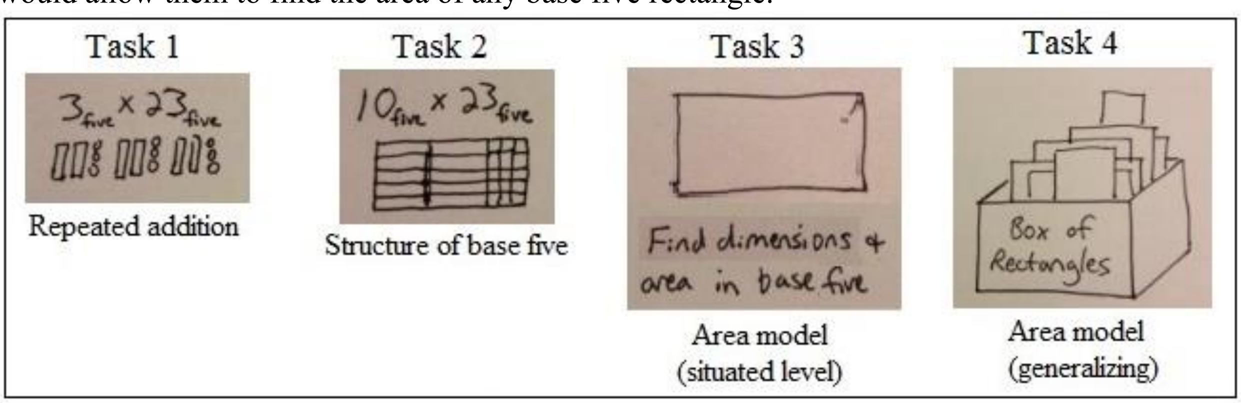


Figure 1. Instructional sequence for reinvention of multidigit multiplication.

The initial strategies generated by PSTs become the subject of further discussions and explanations as those strategies are shared, tested, and revised. As the PSTs engage in the construction of more and more efficient strategies, they are able to take ownership of their knowledge of multidigit multiplication and to reflect on their learning process. The reinvention process not only helps PSTs to build a conceptual understanding of the mathematics, but also helps them to reimagine what it looks like to teach mathematics as a sense making subject rather than a series of predetermined procedures.

References

Gravemeijer, K. (1998). Developmental research as a research method. In *Mathematics education as a research domain: A search for identity* (pp. 277–295). Springer. Retrieved from http://link.springer.com/chapter/10.1007/978-94-011-5470-3_18

Philipp, R. A., Ambrose, R., Lamb, L. L., Sowder, J. T., Schappelle, B. P., Sowder, L., … Chauvot, J. (2007). Effects of early field experiences on the mathematical content knowledge and beliefs of prospective elementary school teachers: An experimental study. *Journal for Research in Mathematics Education*, 438–476.

Thanheiser, E., Philipp, R. A., Fasteen, J., Strand, K., & Mills, B. (2013). Preservice-Teacher Interviews: A Tool for Motivating Mathematics Learning. *Mathematics Teacher Educator*, *1*(2), 137.

INTEGRATED MATHEMATICS AND SCIENCE KNOWLEDGE FOR TEACHING FRAMEWORK: KNOWLEDGE USED IN TEACHING APPLIED DERIVATIVE PROBLEMS

Shawn Firouzian
San Diego State University

Natasha Speer
University of Maine

Previous studies have indicated that effective teaching relies on teachers' knowledge of both student thinking and subject content. Very little is known about the integration (combination) of teachers' mathematical knowledge and science knowledge for teaching important topics like applied derivative problems. Using Ball and colleagues' framework for Mathematical Knowledge for Teaching (MKT), data were analyzed from interviews of eight calculus Graduate Teaching Assistants (GTAs) to examine the kind of knowledge used when talking about teaching applied derivative problems. Findings suggest that some of the domains of the existing MKT framework describe the kinds of knowledge GTAs draw on. However, not all elements of knowledge these GTAs used when discussing applied problems fit the MKT framework. Modifications to the framework are proposed to describe teachers' Integrated Mathematics and Science Knowledge for Teaching.

Keywords: Teacher Knowledge, Mathematical Knowledge for Teaching, Calculus, Applications in Calculus

Introduction

At elementary and secondary levels, the mathematics education and science education communities have independently witnessed that students' achievement is influenced by teachers' pedagogical content knowledge and subject matter knowledge of the science and mathematics they teach (Ball, Lubienski, & Mewborn, 2001; Carpenter & Fennema, 1988; Magnusson, Krajcik, & Borko, 1999). Researchers have documented that elementary and secondary school teachers with richer knowledge of typical student difficulties and strategies create richer learning opportunities for their students (e.g., Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). In this study, drawing on existing research findings, we analyzed the kind of knowledge teachers of undergraduate mathematics draw on when talking about applied calculus topics, in particular, applied derivative problems. Knowing more about the knowledge needed in this teaching situation can inform the design of professional development for instructors so they can better assist their students as they learn applications of calculus.

National Context and the Role of Calculus

The 2012 report of the United States President's Council of Advisors on Science and Technology (PCAST) issued an urgent need to produce one million additional college graduates with degrees in Science, Technology, Engineering, and Mathematics (STEM) fields. Concerns about student achievement in STEM have been prompted by assessments such as the National Assessment of Educational Progress (NAEP) and the Trends in International Mathematics and Science Study (TIMSS). The Congressional Research Service issued a report pointing out the areas of concern in STEM education including "U.S. student performance on international mathematics and science tests… global STEM education attainment, U.S. STEM teacher quality, and the U.S. STEM labor supply" (Gonzalez & Kuenzi, 2012, p. 12).

Calculus is considered a "gatekeeper" course for STEM fields (Moore, 2005). Several researchers have shown that students' success or failure in STEM majors and careers correlates with their performance in calculus (Bundy, LeBold, & Bjedov, 1998; Burton, 1989; Thomasian, 2011; Tyson, 2011). The importance of calculus can also be shown by its applications in different disciplines (Adler, 1997; Cullen, 1983; Goldstein, Lay, & Schneider, 2004). These applications not only provide students opportunities to develop their conceptual understanding of calculus but they also provide a context for deeper understanding of calculus ideas (Bressoud, 1991; Brooks & Brooks, 1993; Schwalbach and Dosemage, 2000; White & Mitchelmore, 1996).

Among topics covered in mainstream calculus classes, derivative and applications of the derivative are key ideas. Understanding these key ideas is fundamental for learning calculus. If students understand these important concepts, they are better prepared to learn future topics of calculus or calculus applications in other disciplines (National Council of Teachers of Mathematics, 2000).

Students' difficulties with applied derivative problems have been well documented (Çetin, 2009; Marrongelle, 2004; Moore & Carlson, 2012; White & Mitchelmore, 1996). They have issues with defining necessary variables for the unknown in the question, then they face difficulties constructing the necessary equation, and even if they get the equation correct, they have a hard time applying the necessary calculus. Finally, once they get the calculus correct, most of the time students cannot translate the answer back into the context of the question, using proper units, etc.

The Role of Interdisciplinary Curricula and the Need for Research on Teachers' Knowledge

In order to overcome the challenges discussed above, the remedy proposed by PCAST and supported by the American Mathematical Society, was a powerful "collaborative coalition" between different disciplines in order to achieve the PCAST goals (Jackson, 2012). In addition, during the last decades of the twentieth century, there was increased development of interdisciplinary curricula (Berlin, 1990). These deliberate efforts to offer interdisciplinary courses and programs are aiming "to broaden local knowledge of the specific fields with the integration of knowledge and applications of the other fields" (Ferrini-Mundy & Güçler, 2009, p. 64).

Teachers play a central role in students' success. At elementary and secondary levels, the mathematics education and science education communities have independently witnessed that students' achievement depends on teachers' pedagogical content knowledge and subject matter knowledge of the science and mathematics they teach (Ball, Lubienski, & Mewborn, 2001; Carpenter & Fennema, 1988; Davis & Smithey, 2009; Magnusson, Krajcik, & Borko, 1999; Shulman, 1986). Researchers have documented that elementary and secondary school teachers with richer knowledge of typical student difficulties and strategies create richer learning opportunities for their students (Carpenter, Fennema, Peterson, & Carey, 1988; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). It has been also been shown that a lack of adequate content knowledge can become a barrier to integrated approaches to instruction (Berlin & Lee, 2005; Stinson, Harkness, Meyer, & Stallworth, 2009).

It is contended that the findings at K-12 levels are "likely to be true at the collegiate level and that the practices of collegiate teachers are worthy and fruitful targets of research" (Speer, Smith, & Horvath, 2010). However, researchers have cautioned using models of K-12 teachers' mathematical knowledge for teaching to describe collegiate-level teachers' knowledge without also, simultaneously, examining how well the models actually fit that population of teachers (Speer, King, & Howell, 2014).

Research Questions

If one approach to improving STEM education includes the use of more interdisciplinary curricula and if teaching effectiveness is influenced by teachers' knowledge, then it stands to reason that the success of these interdisciplinary approaches will be influenced by what the mathematics education research community knows about the knowledge teachers use and need to use when using interdisciplinary tasks with their students. In addition, it will be valuable for the mathematics education community to examine the extent to which this kind of knowledge for teaching fits into existing models used to characterize knowledge needed for teaching given that these frameworks were developed primarily to describe knowledge needed for teaching non-applied content. To that end, in this work we carry out an exploratory project to begin to address the following research questions:

1. What kind of knowledge do instructors draw on when using applied derivative problems?
2. Can the existing model of mathematical knowledge for teaching describe instructors' knowledge for teaching applied derivative problems?
3. If not, what should we do? Can we adapt/modify the MKT framework to accommodate such knowledge?

We examined these questions by gathering data on knowledge used by college mathematics instructors as they discussed student thinking about applied derivative problems and trying to use an existing framework for MKT to characterize that knowledge. We identified knowledge that did and did not fit the definitions for knowledge categories in the MKT framework. We share our characterization of the knowledge that did not easily fit into the framework and share our thoughts on modifying or expanding the existing MKT framework in ways that would accommodate our findings.

Understanding more about the nature of knowledge teachers need to use applied derivative problems can inform the design of professional development for college teachers and this kind of exploration into the constraints and affordances of existing models for knowledge can strengthen the theory of mathematical knowledge for teaching.

Existing Model of Teachers' Mathematical Knowledge for Teaching

Although the specific boundaries and names of categories vary across publications, in this research we use one of the most widely-cited frameworks for characterizing knowledge used in teaching mathematics. Ball, Thames, & Phelps' (2008) model (see Figure 1) was produced by modifying the original categories defined by Shulman (1986) in order to more completely describe the knowledge teachers use in teaching.

Figure 1. Ball and colleagues' framework for Mathematical Knowledge for Teaching

<table>
<tr><td colspan="2">Subject Matter Knowledge (SMK)</td><td colspan="2">Pedagogical Content Knowledge (PCK)</td></tr>
<tr><td>Common Content Knowledge (CCK)</td><td rowspan="2">Specialized Content Knowledge (SCK)</td><td>Knowledge of Content and Student(KCS)</td><td rowspan="2">Knowledge of Content and Curriculum (KCC)</td></tr>
<tr><td>Knowledge at the Mathematical Horizon (KMH)</td><td>Knowledge of Content and Teaching (KCT)</td></tr>
</table>

As defined by Ball and colleagues, the mathematical knowledge known in common with others who know and use mathematics (in this case calculus) is called common content knowledge (CCK). For instance, if an instructor says "instantaneous rate of change" is a way of thinking about the derivative, this is the knowledge we would expect of a student in a calculus class. Therefore this knowledge is defined as part of instructor CCK.

Ball and colleagues defined Knowledge of Content and Students (KCS) as the kind of knowledge teachers draw on when they need to know and predict what subjects or concepts students likely find interesting and motivating or to know their level of difficulties with those topics. Examples of KCS in the context of calculus include a teacher knowing that concepts related to limit are especially challenging for students, computations that make use of the Chain Rule can be difficult for some students, and Related Rates problems can pose many, complex issues for students. Knowledge of Content and Curriculum (KCC) relates to knowledge of the curriculum being taught in other classes or other subject areas. For instance, KCC in the context of calculus includes a teacher knowing about the concepts covered in other science classes (chemistry, physics, biology, etc.) that are applicable to the mathematics.

The definition of Specialized Content Knowledge (SCK) provided by Ball and colleagues is the kind of knowledge used when looking for mathematical patterns in student errors, identifying and validating the mathematical correctness of student work, etc. For instance when "looking for patterns in student errors or in sizing up whether a nonstandard approach would work in general," or identifying the mathematical corrections of students' work, teachers do work that involves an "uncanny kind of unpacking of mathematics that is not needed – or even desirable– in settings other than teaching" (Ball et al., 2008, p. 400). For instance, calculus teachers utilize SCK when they examine (possibly unfamiliar-looking) written work on an exam to determine whether the method the student used to get the answer is mathematically valid.

Research Design

Participants and Setting

Participants were eight graduate student teaching assistants (GTAs), all enrolled in a mathematics masters degree program in a university in the northeastern United States. As pseudonyms, Greek letters are used. The GTAs all were either currently teaching differential (first semester) calculus or had taught it in the past. The amount of teaching experience they had ranged from less than one semester to more than four semesters.

Data Collection Methods

Semi-structured, task-based clinical interviews (Hunting, 1997) were conducted. The interview tasks were based on previous and ongoing work on students' difficulties and understanding of applied derivative problems (Ferrini-Mundy & Graham, 1994; Firouzian, 2013; White & Mitchelmore, 1996; Zandieh, 2000). Using these tasks, we designed an interview protocol (modeled after Frank & Speer, 2012) to target possible domains of knowledge teachers may use for teaching derivative and applied derivative problems. Figure 2 shows one of the tasks used in the interviews.

Cowboy Clint wants to build a dirt road from his ranch to the highway so that he can drive to the city in the shortest amount of time (Figure below). The perpendicular distance from the ranch to the highway is 3 km, and the city is located 9 km down the highway. Where should Clint join the dirt road to the highway if the speed limit is 80 kph on the dirt road and 100 kph on the highway?

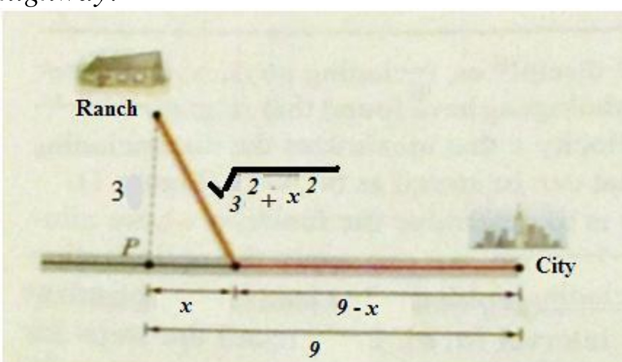


Figure 2. One of the tasks for which interviewees were asked to solve and discuss students' ideas and difficulties (adapted from Rogawski, 2011).

For each of several tasks presented during the interviews, GTAs were asked to solve the task, to discuss possible student difficulties with the task, and to examine samples of student work on the task that illustrated known difficulties students have with such tasks. Specific interview questions included:

- Why would someone who is not a mathematics major care about the concept of derivative? Can you come up with specific applied derivative questions in other disciplines or in a real-world situation? How would you teach or present the concepts to students?
- How would you solve the problems?
- What do you think the students' typical difficulties are with the above question?
- [Given sample student work, not similar to the difficulties they mentioned already, they were then asked:] Please explain what the students are thinking based on their answers.

Data Analysis Methods

Interviews were transcribed and GTAs' knowledge of students' thinking and difficulties about applied derivative problems were compared to that found in existing research. To analyze the kind of knowledge the interviewees drew on, first existing domains of teachers' knowledge for teaching (Figure 1) were used to categorize the knowledge. Grounded Theory (Strauss & Corbin, 1990) and the findings from existing research on students' difficulties and thinking about applied derivative problems were used to identify themes in the kinds of knowledge displayed by GTAs (both within categories that fit the MKT framework and among instances where the knowledge displayed was not easily categorized with the MKT framework).

Findings

We found that we could use the MKT framework to describe some of our data but there were several instances where the MKT categories did not seem to fit our data. In this section, first we discuss a few examples of when the MKT framework fit our data and then we discuss instances when we encountered difficulties utilizing the MKT framework to characterize the knowledge displayed by the GTAs. We provide fine-grained analysis of excerpts of interviews as support for our claims about the fit and lack-of-fit of our data to the MKT framework.

Examples of Our Data Fitting the MKT Framework

Based on our analysis, we found examples of knowledge displayed by the GTAs that fit the definitions of various MKT domains including the categories of common content knowledge, knowledge of content and students, and knowledge of content and curriculum.

Common Content Knowledge. GTA Delta was asked how he would explain to someone what the derivative of a function means. He talked about "instantaneous rate of change" as a way of thinking about the derivative and incorporated that into his explanation. This is a description of the derivative that is routinely found in textbooks and other curricular materials. Therefore, it fits the definition of Common Content Knowledge as being the mathematical knowledge known in common with others who know and use mathematics.

Knowledge of Content and Students. GTA Pi was asked to solve the optimization problem given in Figure 2. Once she was done, she was asked to describe potential difficulties students might have with this problem. She talked about students' difficulties with optimization problems as two problems: "one is setting up the time and one is finding the min[imum]. If we don't give the x they might have difficulty. They don't know how to set up the variable." This is an example of knowing what ideas are difficult for students and thus fits into the MKT framework category of Knowledge of Content and Students.

Knowledge of Content and Curriculum. GTA Theta was asked why someone who is not a mathematics major would care about the concept of the derivative. He replied:

> If they are economics or business [majors] they are going to talk about marginal rate at which profit or GDP…if they are in engineering they are going to talk about position and velocity. If they are in biology they are going to talk about population and growth, in chemistry they are going to talk about heating and cooling and in chemical reactions, in physics they are going to talk about position and velocity…

In this discussion GTA Theta is drawing on his knowledge of other subjects in the undergraduate curriculum to come up with applications of the derivative in fields outside of mathematics. The knowledge that Theta displayed of the curriculum being taught in other classes or other subject areas fits the definitions of Knowledge of Content and Curriculum as used in the MKT framework.

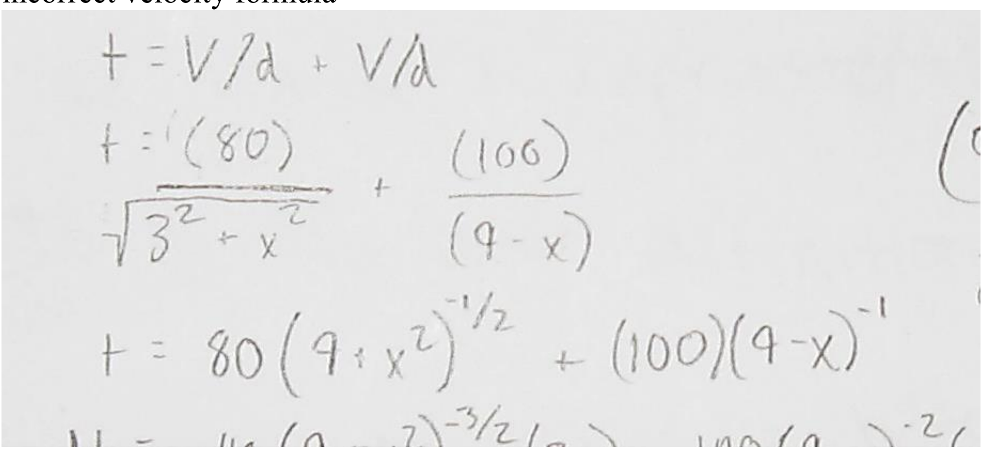
We also identified examples of GTAs' drawing on elements of Specialized Content Knowledge and Knowledge of Content and Teaching. However, there were also instances where our data did not appear to fit the MKT framework definitions for categories of knowledge. Several of these instances are discussed next.

Examples of When Our Data Did Not Fit the MKT Framework

There were multiple instances of knowledge displayed by the GTAs that did not clearly fit the definition of a category in the MKT framework. Here we focus in particular on instances where GTAs appeared to make use of knowledge of science content and/or knowledge of how students think about science. Although it seems quite natural that these kinds of knowledge would be used in the teaching of calculus when the problems involve the application of calculus ideas to science, nonetheless, the categories in the MKT framework seem not to be defined in ways that can accommodate these kinds of knowledge. Here will illustrate this claim with a few examples.

Example 1: Knowledge of Students' Difficulties with Physics and Knowledge of Physics Content. GTA Theta had just solved the optimization problem shown in Figure 2 and was asked what he thought students' typical difficulties might be with this problem. He talked about their difficulties with variables and setting up the equation. This knowledge fits well with the definition of Knowledge of Content and Students in the MKT framework. He was then asked to analyze a sample of student work (shown in Figure 3).

Figure 3. Sample student work where the student set up an incorrect equation by using an incorrect velocity formula



As we can see in the Figure 3, this student struggled with the speed formula and this led him/her to construct an incorrect equation. While GTA Theta was trying to make sense of the student's written work, the following conversation took place:

Theta: It is not as simple, they parameterize *x*, they also don't divide by the rate. They should have divided by the rate instead of multiplied. So instead of dividing they are still trying to minimize the distance when they should be trying to minimize the time…Yeah so that is the mistake. If they were trying to minimize the distance they did a great job.

Interviewer: So their difficulty is coming from their understanding of the distance, time and the velocity?

Theta: Yeah and that is just a set-up thing right? Like I said they should have had that process in mind where they set up a function, they differentiate the function and find the critical point.

Interviewer: But they setup the function correctly in a way.

Theta: They set up the function incorrectly for what they are looking for so they misread the model.

Interviewer: Correct but they had the wrong equation because they didn't know the math part of it or could it be with the physics?

Theta: The physical application. They didn't know that they were looking for time not distance.

GTA Theta figured out that the student was having a hard time setting up the equation. This can be characterized as knowledge of a general difficulty because it involves defining variables and constructing the relationship between them and therefore fits the MKT category of Knowledge of Content and Students. For these elements of knowledge, the MKT framework worked well to analyze and characterize the kind of knowledge this GTA displayed.

The existing framework, however, does not easily accommodate the kind of knowledge this GTA displayed when narrowing down the student's difficulty to a difficulty with the science concept of velocity or speed. The last few lines of the conversation above seem to show the GTA's knowledge of the physics formula applied in the problem. We hypothesize that GTA Theta drew on his Knowledge of Content and Curriculum in order to figure out that the student's knowledge of physics might be relevant to the problem. However, then GTA Theta states a *particular difficulty* with the ideas that could come from a student's under-developed understanding of the relationships among distance, time and velocity. Recognizing that this may be a factor in the students thinking seems to demand both knowledge of the specific physics content and knowledge of particular difficulties students may have with physics.

Example 2: Knowledge of Students' Difficulties with Physics and Knowledge of Physics Content. In another case, GTA Alpha was asked what she thought students' typical difficulties would be with an applied problem similar to that shown in Figure 2. The following conversation took place:

> *GTA Alpha:* The first step. Finding the function.
> *Interviewer*: Which part of it is difficult though?
> *GTA Alpha*: Maybe they have a problem finding the length?
> *Interviewer*: What do you mean finding? Like calculating?
> GTA Alpha: Yes. Calculating the length, like using [the] Pythagorean [Theorem].

The MKT framework fit this part of our data quite well. GTA Alpha drew on her Knowledge of Content and Students in order to list some typical difficulties such as using formulas for calculating values. What she said next about students' difficulties with the problem shown in Figure 2 could not be categorized as easily:

> Sometimes they [students] would be confused about which part the speed is. Sometimes they say this part, the speed is 100 [pointing to the dirt road] and that side is 80 [referring to the highway]. So from my experience this part is the most difficult part…a few of them would have [a] problem finding the T' but most of them can do the T' correctly if they have the function. They have this function, they would do the T' correctly. Another problem is when they set T' equal to zero they need to solve for x and that is another problem, solving the equation.

The first few lines of the quote above show the GTA's knowledge of students' difficulties with the formula for speed as it is being applied in the calculus problem. We propose that GTA Alpha displayed both knowledge of the particular physics in the problem and knowledge of how students think about such ideas. She recognized that the relevant physics concept was speed and then stated that students might have an especially difficult time determining how speed is represented in the problem. This kind of analysis seems to demand both knowledge of science content and knowledge of students' difficulties with science.

Example 3: Science Common Content Knowledge. In another instance, GTA Gamma was asked to discuss the underlying ideas and concepts students need to know in order to solve applied derivative problems successfully. She had provided an example of an applied derivative problem in physics so we asked her what she thought the underlying concepts and ideas are that students need to know related to those kinds of problems. The following conversation occurred:

> *GTA Gamma:* They need to know where the velocity is.
>
> *Interviewer:* Would that be a math thing or would that be a physics thing?
>
> *GTA Gamma:* It'd be a physics thing. For biology, it is not that they need to know the hard biology stuff. They just need to know that bacteria multiplies, creating more bacteria.

The MKT framework did not seem to easily fit these data. GTA Gamma talked about the science concepts underlying the applied derivative problem and none of the domains of the MKT framework can be easily used to characterize the kind of knowledge GTA Gamma displayed. We hypothesize that she drew on her Knowledge of Content and Curriculum in order to identify the applications of calculus in other sciences. She then drew on her common content knowledge in physics or biology in order to describe the specific concepts where the derivative can be applied. Similar responses were witnessed in other interviews as well.

Example 4: Horizon Content Knowledge in Science. GTA Alpha was asked why someone who is not a mathematics major would care about the concept of derivative or where she thought the concept of derivative might be used. GTA Alpha responded: "I always think biology has a lot of relationships with mathematics because like well, I don't think they need to use any math when they are undergraduate but when they want to go to graduate school, math becomes very important for them." The knowledge GTA Alpha displays in this statement does not fit easily into categories in the MKT framework. She spoke about the application of mathematics in other parts of undergraduate and graduate curricula, in biology in particular. The fact that she was making connection between calculus and other sciences shows that she drew on her Knowledge of Content and Curriculum. However, the MKT framework falls short in explaining her knowledge of the biology undergraduate curriculum. We hypothesize that she drew on a type of knowledge that might be characterized as "horizon content knowledge in the sciences."

Discussion, Conclusions and Implications

For the purposes of informing the design of professional development, and for theory-building efforts, it seems important that the mathematics education research community continue to examine and refine frameworks used to characterize teacher knowledge. In an effort to contribute to those goals, we analyzed the knowledge displayed by instructors while they discussed problems that are part of the typical calculus curriculum but differ from the type at play most commonly in such research. In particular, we found that when asked to engage in teaching-related tasks such as examining student work, our GTA participants displayed knowledge of science content as well as knowledge of particular ways that students think about science content.

Given the importance of interdisciplinary and applied problems in the STEM curriculum, it seems likely that the education community would benefit from having frameworks that characterized the knowledge needed in this kind of teaching. The findings raise the question of how best to proceed with theory development. One approach would be to augment the definitions of components of MKT so they also include the science-related knowledge relevant to the teaching of mathematics. Or we may need a different kind of framework to fully characterize this knowledge. Determining which approach is best-suited to the community's needs will require additional research and theory-testing. In the next section, we share some preliminary thoughts about theoretical constructs that may be useful as we and others work in this area.

Potential Connections to Other Theoretical Constructs

As we discussed in the findings, GTAs drew on multiple domains of their mathematical knowledge in order to discuss the applied derivative problems and students' difficulties with them. Sherin (2002) described how connections and negotiations between different aspects of teachers' knowledge enable them to deal with different components of teaching both in terms of knowledge and pedagogy. We witnessed similar connectivity and negotiation between GTAs' domains of MKT. As shown in some of the instances discussed above, GTAs displayed knowledge from multiple domains of MKT while working to make sense of student work.

Another theory-related idea that may turn out to be useful in this work is that of a "knowledge package." Ma (1999) defined *knowledge package* as a collection of conceptual and procedural ideas related to the understanding and teaching of a topic. Perhaps one approach to characterizing the knowledge displayed by these instructors would by defining a knowledge package that contains elements of MKT *and* knowledge of science or student thinking about science. For example, in GTA Alpha's responses to the interview questions, we saw evidence of her Knowledge of Content and Curriculum and perhaps that category could be reconceived as a knowledge package consisting of her knowledge of calculus application in other sciences and also her knowledge of undergraduate curriculum in biology. That would account for the elements of knowledge she appeared to draw on when doing this particular kind of teaching-related work in the context of applied calculus problems.

The existing research in science PCK includes knowledge of students' difficulties with science concepts (Magnusson, Krajcik, & Borko, 1999). We propose that teachers' knowledge for teaching can perhaps include elements of MKT and science knowledge for teaching while teaching applied problems. Several researchers and practitioners defined *integrated knowledge* as a blending of science and mathematics content knowledge such that the separate parts are not discernible (Czermiak, Weber, Sandman, & Ahern, 1999; Lederman and Niess, 1997). Perhaps it is the case that GTAs such GTA Theta, draw on their integrated knowledge of mathematics and sciences in describing students' difficulties with the applied problems or teaching those topics.

Directions for Future Research and Implications for Instruction

To explore the nature of science knowledge, instructors draw on when talking about teaching applied problem, additional studies are called for. One potentially useful approach for further exploration of theory would be to conduct interviews with science instructors as they consider problems that involve their particular science discipline. Then comparisons could be made of the knowledge they display with the knowledge displayed by the mathematics GTAs in our sample. Such comparisons and potential distinctions may help shed light into the nature of these kinds of integrated mathematical and science knowledge for teaching.

We suggest incorporating applied problems into "examining student work" activities for instructor professional development. Providing novice (or experienced) instructors with opportunities to consider the kinds of problems and student thinking they will encounter as their students tackle applied problems could create opportunities for them to learn how students think about such ideas, what common difficulties are and what productive strategies they are apt to witness in their students' work. It has also been shown that a lack of adequate content knowledge can become a barrier to integrated approaches to instruction (Berlin & Lee, 2005; Stinson & Harkness, 2009). Perhaps in professional development instructors could also have opportunities to engage with content-specific activities that would create opportunities for them to refresh or deepen their knowledge of the non-mathematics content they may need when teaching applied calculus problems. Such learning opportunities might enhance instructors' capacities to understand the curriculum and support their students learning

Taken together, efforts to further document and characterize the knowledge needed when utilizing applied problems in instruction can help improve student learning of both mathematics and other STEM disciplines. Such improvements may, in turn, improve both student enjoyment of and persistence in STEM majors and careers.

References

Adler, L. A. (1997). The role community colleges should play in job placement. In E. I. Farmer & C. B. Key (Eds.), *School-to-work systems: The role of community colleges in preparing students and facilitating transitions*. *New Directions for Community Colleges, no. 97* (pp. 41-48). San Francisco: Jossey-Bass.

Ball, D. L., Lubienski, S. T., & Mewborn, D. S. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. *Handbook of research on teaching*, *4*, 433-456.

Berlin, D. F. (1990). Science and mathematics integration: Current status and future directions. *School Science and Mathematics, 90(3),* 254-257.

Berlin, D. F., & Lee, H. (2005). Integrating science and mathematics education: Historical analysis. *School Science and Mathematics*, *105*(1), 15-24.

Brooks, J. G., & Brooks, M. G. (1993). In search of understanding: The case for constructivist classrooms. Alexandria, VA: ASCD.

Bressoud, D. M. (1991). *Second year calculus: from celestial mechanics to special relativity*. New York, NY: Springer-Verlag New York, Inc.

Budny, D., LeBold, W., & Bjedov, G. (1998). Assessment of the Impact of the Freshman Engineering Courses. *Journal of Engineering Education, 87*(4), 405-411.

Burton, M. B. (1989). The effect of prior calculus experience on "introductory" college calculus. *The American Mathematical Monthly, 96*(4), 350–354.

Carpenter, T. P., & Fennema, E. (1988). Research and cognitively guided instruction. *Integrating research on teaching and learning mathematics*, 2-19.

Carpenter, T. P., Fennema, E., Peterson, P. L., & Carey, D. A. (1988). Teachers' pedagogical content knowledge of students' problem solving in elementary arithmetic. *Journal for research in mathematics education*, 385-401.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C. P., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal, 26*(4), 499-531.

Çetin, N. (2009). The ability of students to comprehend the function-derivative relationship with regard to problems from their real life. *PRIMUS*, *19*(3), 232-244.

Gonzalez, H. B., & Kuenzi, J. J. (2012, August 1). *Science, technology, engineering, and mathematics (STEM) education: A primer*. Retrieved from http://fas.org/sgp/crs/misc/R42642.pdf

Cullen, M. R. (1983). *Mathematics for the Biosciences*. Boston, MA: Techbooks.

Czerniak, C., Weber, W. B., Sandman, A., & Ahern, J. (1999). A literature review of science and mathematics integration. School Science and Mathematics, 99(8), 421–430.

Davis, E. A., & Smithey, J. (2009). Beginning teachers moving toward effective elementary science teaching. *Science Education*, *93*(4), 745-770

Ferrini-Mundy, J., & Güçler, B. (2009). Discipline-based efforts to enhance undergraduate STEM education. *New Directions for Teaching and Learning*, (117), 55-67.

Firouzian, S. (2014). Correlations of students' ways of thinking about derivative to their success in solving applied problems. In S. Brown, S. Larsen, Marrongelle, K. M. Ohertman (Eds.), *Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education*. Denver, CO.: Available online.

Frank, B., & Speer, N. (2012). On the Job Learning: Instructors' Development of Knowledge for Teaching. *In preparation for submission to Cognition & Instruction*

Goldstein, L. J., Lay, D. C., & Schneider, D. I.(2004). *Calculus and its applications (10th ed.)*. Upper Saddle River, NJ: Pearson.

Hunting, R. P. (1997). Clinical interview methods in mathematics education research and practice. *The Journal of Mathematical Behavior*, *16*(2), 145-165.

Jackson, A. (2012). Presidential Report Draws Criticism from Mathematicians. *NOTICES OF THE AMS*, *59*(9).

Lederman, N. G., & Niess, M. L. (1997). The nature of science: Naturally? *School Science and Mathematics*, *97*(1), 1-2.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.

Magnusson, S., Krajcik, J., & Borko, H. (1999). Nature, sources, and development of pedagogical content knowledge for science teaching. In J. Gess-Newsome & N. Lederman (Eds.), E*xamining pedagogical content knowledge: The construct and its implications for science education* (pp. 95-132). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Marrongelle, K. A. (2004). How students use physics to reason about calculus tasks. *School Science and Mathematics*, *104*(6), 258-272.

Moore*, J. (2005). Undergraduate mathematics achievement in the emerging ethnic engineers programme. *International Journal of Mathematical Education in Science and Technology*, *36*(5), 529-537.

Moore, K. C., & Carlson, M. P. (2012). Students' images of problem contexts when solving applied problems. *The Journal of Mathematical Behavior*, *31*(1), 48-59

National Council of Teachers of Mathematics (NCTM). (2000). Principles and standards for school mathematics. Reston, VA: Author.

Rogawski, J. (2011). *Calculus: Early Transcendental*. New York, NY: Macmillan.

Schwalbach, E. M., & Dosemagen, D. M. (2000). Developing student understanding: Contextualizing calculus concepts. *School Science and Mathematics*, *100*(2), 90-98.

Sherin, M.G. (2002). When teaching becomes learning. *Cognition and Instruction, 20*(2), 119-150.

Speer, N. M., King, K. D., & Howell, H. (2014). Definitions of mathematical knowledge for teaching: using these constructs in research on secondary and college mathematics teachers. *Journal of Mathematics Teacher Education, 18*(2), 105-122.

Speer, N., Smith, J., & Horvath, A. (2010). Collegiate Mathematics Teaching. *The Journal of Mathematical Behavior, 29*(2), 99-114.

Stinson, K., Harkness, S. S., Meyer, H., & Stallworth, J. (2009). Mathematics and science integration: Models and characterizations. *School Science and Mathematics*, *109*(3), 153-161.

Strauss, A., & Corbin, J. M. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. Thousand Oaks, CA: Sage Publications, Inc.

Thomasian, J. (2011). Building a Science, Technology, Engineering, and Math Education Agenda: An Update of State Actions. *NGA Center for Best Practices*.

Tyson, W. (2011). Modeling engineering degree attainment using high school and college physics and calculus course taking and achievement. *Journal of Engineering Education*, *100*(4), 760-777.

White, P., & Mitchelmore, M. (1996). Conceptual knowledge in introductory calculus. *Journal for Research in Mathematics Education*, *27*(1), 79-95.

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. In E. Dubinsky, A. Schoenfeld, & J. Kaput (Eds.), *CBMS Issues in Mathematics: Research in Collegiate Mathematics Education, IV*(8), 103-127.

Students' conceptions of rational functions

Nicholas Fortune
North Carolina State University

Derek Williams
North Carolina State University

Abstract

Mathematics education research on dynamic technologies incorporated into learning environments indicates that they possess the ability to enrich students' mathematical conceptual understanding. This study explores how three community college students conceptualize rational functions by classifying their mathematical thinking according to the APOS theory. This study highlighted students' conceptualization of rational functions as actions (required sequences of observable external steps), processes (sequences with no need to externalize), and objects (thinking about rational functions as a single object or entity).

Keywords: Rational functions; APOS; conceptions

Purpose

A large percent of students at the university level begin their mathematics education with pre-calculus, a course that includes the study of complex functions. There is a lack of research about how students come to understand those complex functions; rational functions in particular. In this poster, the authors use dynamic software as a tool to help support and classify community college students' conceptions of rational functions as actions, processes, and objects according to the APOS theory (see Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dubinsky & Harel, 1992; Dubinsky, 1991).

Theoretical Framework

It is anticipated that students will display various stages of mathematical understanding of the construction and composition of rational functions. APOS theory, as defined by Dubinsky and colleagues, is appropriate to help researchers understand the students' conception development in the form of action, process, objects, or schema (see Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dubinsky & Harel, 1992; Dubinsky, 1991). In this case, the authors have found that APOS theory may define students' conceptualizations of rational functions as actions (required sequences of observable external steps), processes (sequences with no need to externalize), objects (internalizing rational functions as a single object or entity), or schema (full understanding and containment of all objectified forms of rational functions).

Methodology

This mini-study is part of a larger unpublished study where one of the authors conducted research with three students from a local community college, two male and one female; all had little background in rational functions prior to the interviews. In that research, students were first introduced to *GeoGebra* (Hohenwarter, 2002) files and were provided information on how to use the software for their explorations. Within each exploration, students made conjectures about different parameters of rational functions. To end each session, students participated in an assessment game where they could test their knowledge obtained while investigating. Following the sessions students were given a summative post-test by the interviewer. Further, some items on the post-test were used as tasks in clinical interviews. The present study focuses on responses from one question on the post-test and the corresponding portion of the interviews. The particular question is as follows:

> Give an example of a rational function that has a hole at $x = -2$, vertical asymptote at $x = 3$, and has zeros at $x = 1, 2, -5$. Explain how you know that these conditions have been met.

This question was chosen for this mini-study for a variety of reasons. First, this is a summative question that includes topics from all of the sessions about the construction and composition of rational functions. Additionally, this is an open-ended question with a justification component and thus has the ability to illustrate students' conceptions of rational functions as actions, processes, or objects.

Results and Discussion

Results from this study indicate that not all students conceptualize rational functions the same. Each of the three student's response to the interview question was coded differently as showing an action, process, or object conception of rational functions. The results presented will include examples of evidence discovered from this study, a brief summary of students' conceptions of rational functions, and a claim for the need of a larger study of this nature to future develop implications for teaching rational functions at the collegiate level.

References

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics*, *23*(3), 247–285.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinkning* (pp. 95–123). Dordrecht, The Netherlands: Kluwer.

Dubinsky, E., & Harel, G. (1992). The nature of the process conception of function. In E. Dubinsky & G. Harel (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 85–106). Washington DC: Mathematical Association of America.

Hohenwarter, M. (2002). *GeoGebra: A software system for dynamic geometry and algebra in the plain.* University of Salzburg.

Public versus private mathematical activity as evaluated through the lens of examples

Tim Fukawa-Connelly
Drexel University

Lecture has often been critiqued as obscuring the mathematical habits such as sense-making about abstract statements about mathematical concepts, the creation of conjectures about those concepts, and to the processes of proof-writing and exhibition of counter-examples. In all of these actions, a mathematician or student should be able to draw upon a rich store of examples in order to make meaningful progress and researchers have argued that it is via examples that they are best able to engage in such processes. Inquiry-based classes have, as part of their promise, making more visible the mathematical promises that lecture obscures behind the polished formalism. This preliminary report explores whether and how students in an inquiry-based abstract algebra class engage in public example-based reasoning as a means to explore public versus private mathematizing.

Keywords: example-based reasoning, public vs. private mathematizing, opportunity to learn, mathematical processes

Many researchers argue that lecture-based mathematics instruction is intimidating and misleading to students about the nature of mathematics, especially in proof-based courses (Cuoco, 2001; Thurston, 1986). Some contend that it hides much of the process used in mathematical thinking and makes it difficult for students to develop an appreciation for the discipline (Dreyfus, 1991). One goal of ongoing efforts in mathematics education has been to develop interventions that change undergraduate teaching practices, and, there are a growing number of inquiry-based classes across the undergraduate spectrum. Yet, only a few such courses have been described in the literature (c.f., Cook, 2014; Larsen, 2013; Dubinksky & Leron, 1997), and, those descriptions are more likely to focus on how students come to understand a particular piece of content than how they experience mathematical processes. Larsen and Zandieh's (2008) piece is a notable exception, although, it was describing the results of a teaching experiment with only a pair of students, based on available literature it appears that students in Larsen's (2013) curriculum have a similar experience creating the definition of group.

The pedagogical importance of examples

Examples are believed to be very important in developing conceptual understanding of mathematical ideas (Mason & Watson, 2008; Vinner, 1991). Examples give insight into mathematical definitions, theorems and proofs, and can be used to create them, as well (Cuoco, Goldenberg and Mark, 1996; Lakatos, 1976). Several studies have focused on student exemplification and the use of examples to learn about concepts and proving (Alcock & Inglis, 2008; Dahlberg & Housman, 1997; Mason & Watson, 2008). As yet, there are no studies of instructors' teaching with examples in undergraduate proof-based mathematics courses. Studying teaching is by nature a difficult process and little empirical research has described and analyzed the practices of teachers of mathematics at the undergraduate level despite repeated calls for this type of study (Harel & Sowder, 2007; Harel & Fuller, 2009, Speer, et al. 2010). As a result, this study addresses the following questions:

- How do students in an IBL class engage in public mathematical-reasoning? Here, public is taken to be whole-class discussion, while I call small-group-work semi-private.

- How do students reason with and about examples as tools for exploring and making evident mathematical processes?

The theoretical orientation for this study is a version of social constructivism referred to as the emergent perspective (Cobb & Bauersfeld, 1995; Cobb & Yackel, 1996). Within this research tradition, the study draws upon ideas primarily from the social perspective. "The social perspective indicated is concerned with ways of acting, reasoning, and arguing that are normative in a classroom community" (Cobb, Stephan, McClain, & Gravemeijer, 2001, p. 118). Here, an individual's reasoning is understood as the action of participating in the classroom normative activities. When the classroom is the focus of study, the norms and processes are understood to be emergent and under negotiation by the students and teacher throughout the course of a semester rather than understanding the students being inducted into the existing community of mathematicians (Cobb et al., 2001). Sociomathematical norms refer to classroom practices that are specific to the discipline of mathematics, such as what constitutes a proof or a good explanation (Rasmussen & Stephan, 2008; Stephan et al., 2003; Yackel & Cobb, 1996). Yackel and Cobb (1996) claimed that students develop beliefs and values, specific to mathematics. Further, the negotiation of some sociomathematical norms may lead students to act more autonomously when doing mathematics.

Methodology

The setting was a first semester abstract algebra course at a PhD granting university in the US. The instructor had previously taught abstract algebra and was committed to a non-traditional pedagogy that he alternatively described as inquiry-based learning course. The professor had collected some notes from other abstract algebra classes and had substantially modified them, these formed the basis of the students' daily activities and consisted of expository text as well as exercises and problems for students to work. On a daily basis the students were seated in groups of four and in every class meeting there was time that they worked on mathematical tasks, presented at the board, and the professor lead whole-class discussions. The course met for 75 minutes three days per week and students were responsible for daily homework, only some of which was to be submitted for a grade. All class meetings were video recoded by a graduate student in mathematics education who took notes about the class. In the first few days, some small-group data was lost due to issues with a microphone.

Transana was used to code all incidents where an example or non-example was shown, constructed or analyzed in class. We created an example log (including non-examples), similar to Stephan and Rasmussen's (2008) argument log, characterizing each in four columns.

- Column 1: each example or non-example of the particular construct (in this case, an algebraic group).
- Column 2: counts the number of class meetings since the formal definition of a group (a written homework assignment was coded as occurring on the day that it was assigned).
- Column 3: description of the qualities of the example or non-example. In the case of examples, the third column described any additional qualities that the example possessed from a list that would be known to first semester algebra students by the midpoint of the semester (e.g., being a commutative group, a finite group, or a cyclic group). In the case of non-examples we described any properties of the construct that were missing as well as additional properties that the non-example possessed from the list above.
- Column 4: description of the manner in which the example or non-example was made part of the classroom discourse and what the students were doing with it.

Because the focus of this investigation is how students engage in public mathematical reasoning, rather than simply seeing the results of such reasoning, and how they use examples to do so, in my analysis I first focus on classifying how the examples were included in the class; describing whether students were analyzing them or using them as tools for reasoning. When students were using analyzing them, I described what they were attempting to come to understand (what concept they were analyzing them for). When they were using examples as tools for mathematical reasoning, I coded what the students were attempting to do (or, asked to do). Finally, for each use of examples, I described them as semi-private or public, and, then aggregated across the examples of public mathematizing and described the range of activities. I draw some tentative conclusions about the types of mathematical activities most likely to be engaged in publically and, based on that, make some tentative conclusions about the opportunity to learn about mathematical activities that the students had. I argue that the regularity of practice constitutes evidence of sociomathematical norms of practice, and, thus, what is appropriate mathematical practice to make public.

Preliminary Data and Results

The first finding was that most examples discussed by the class were those assigned by the professor (via the class notes), and, generally, students were analyzing the examples to determine whether they were examples of a particular construct. This conforms with Fukawa-Connelly and Newton's (2014) finding that almost all of the examples students encountered in a traditional class were used as examples of a concept or were to be tested to verify that they were. That is, most public and semi-private mathematizing via examples was simply about students developing understanding, such as the appropriate limits of variation (Fukawa-Connelly & Newton, 2014), of a concept that had already been defined, and, most such examples were provided by the notes. There were a number of instances in which the students suggested their own candidates for examples of a concept. In all, this use of examples seems to not address the critiques of lecture in a meaningful way as it already appears to be common.

The students also used examples to make observations about regularities and propose new ideas. There were three primary cases of this; subgroups, isomorphism, and ideals. In the case of subgroups, the students had been analyzing the group D_3, and, in doing so, had written the operation table on the board. One student, during whole-class discussion, said, "that looks like a group inside the group." The professor asked if other students noticed the same thing, and, if the same phenomenon appeared in other Cayley tables of the dihedral groups. That is, the students appeared to publically move from reasoning about the example (that it, it was a model-of a particular idea) to using D_3 as a model-for reasoning generally about mathematics, and, doing so in a way that they were reinterpreting a known concept in a new setting, thus, they could be seen as operating in the *general phase* of the reinvention process (Gravemeijer, 1998).

There is some, emerging evidence, of students using examples in support of proof-writing activities. In some cases, these were prompted by the professor's suggestion when he found the students struggling with an abstract proof, he suggested working with an example, and, the students to whom he suggested that did so attempt it. The original coding did not always make clear how individual groups chose to begin working with an example though, and, there are a few points in the data that need to be re-coded, moreover, there is one class period in which the original coding of Column 4 is unclear about whether students included example-based reasoning in their presentation to the class. In no instances, with one day left to re-code, were students observed to publically engage in mathematical exploration or example-based reasoning, instead, when presenting proofs, they would present their attempt at an analytic proof. That is, it appears that all uses of examples in support of proof-writing were semi-private. More coding is needed to explore whether examples are used as counter-examples for any assertions in student presentations, thus far, none have been found. Which, again suggests that only polished proofs were presented, and mathematical reasoning was not publically displayed. While there is value in semi-private exploration, I argue that whole-class exploration and demonstration of in-progress mathematical reasoning has a different value in that it would communicate a different understanding of what it means to do mathematics; mathematics is about reasoning as opposed to being about results, meaning some of the promise of IBL was not realized.

Questions for Discussion

1) While we believe this a helpful way of thinking about making mathematical processes visible, we wonder if it is too narrow of a lens?
2) Similarly, is it too time-intensive to be useable in a meaningful way?
3) Besides glaringly obvious teaching suggestions like, "show more incomplete and incorrect reasoning and focus on the process" what potential does this have for affecting instruction of either lecture or Inquiry based teaching? Further research?
4) What more should we I doing? Look beyond examples?

References

Alcock, L. & Inglis, M. (2008). Doctoral students' use of examples in evaluating and proving conjectures. *Educational Studies in Mathematics, 69*, 111-129.

Cobb, P., & Bauersfeld, H. (1995). Introduction: The coordination of psychological and sociological perspectives. In P. Cobb, & H. Bauersfeld (Eds.), Emergence of mathematical meaning: Interaction in classroom cultures (pp. 1–16). Hillsdale, NJ: Lawrence Erlbaum Associates.

Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2001). Participating in classroom mathematical practices. *Journal of the Learning Sciences, 10*(1–2), 113–163.

Cook, J. P. (2014). The emergence of algebraic structure: students come to understand units and zero-divisors. *International Journal of Mathematical Education in Science and Technology*, *45*(3), 349-359.

Cuoco, A. (2001). "Mathematics for Teaching." *Notices of the American Mathematical Society, 48*(2): 168-174.

Cuoco, A., Goldenberg, E.P., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. *Journal of Mathematical Behavior, 15*, 375-402.

Dahlberg, R. P., & Housman, D. L. (1997). Facilitating learning events through example generation. *Educational Studies in Mathematics, 33*, 283–299. doi:10.1023/A:1002999415887.

Dreyfus, T. (1991). On the status of visual reasoning in mathematics and mathematics education. In: Furinghetti, F. (Ed.), *Proceedings of the 15th Conference of the International Group for the Psychology of Mathematics Education,* Assisi - vol.1, 33-48.

Dubinsky, E., & Leron, U. (1994). *Learning abstract algebra with ISTEL*. New York: Springer-Verlag.

Fukawa-Connelly, T. P., & Newton, C. (2014). Analyzing the teaching of advanced mathematics courses via the enacted example space. *Educational Studies in Mathematics*, 1-27.

Gravemeijer, K. (1998). Developmental research as a research method. In*Mathematics education as a research domain: A search for identity* (pp. 277-295). Springer Netherlands.
Harel, G., & Fuller, E. (2009). Contributions toward perspectives on learning and teaching proof. In D. Stylianou, M. Blanton, & E. Knuth (Eds.), *Teaching and learning proof across the grades: A K-16 perspective* (pp. 355-370). New York, NY: Routledge.
Harel, G., & Sowder, L. (2007). Towards a comprehensive perspective on proof. In F. Lester (Ed.), *Second handbook of research on mathematical teaching and learning* (pp. 805-842). Washington, DC: NCTM.
Lakatos, I. (Ed.). (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge university press.
Larsen, S. P. (2013). A local instructional theory for the guided reinvention of the group and isomorphism concepts. *The Journal of Mathematical Behavior*,*32*(4), 712-725.
Larsen, S., & Zandieh, M. (2008). Proofs and refutations in the undergraduate mathematics classroom. *Educational Studies in Mathematics*, *67*(3), 205-216.
Mason, J. & Watson, A. (2008). Mathematics as a Constructive Activity: exploiting dimensions of possible variation. In M. Carlson & C. Rasmussen (Eds.) Making the Connection: Research and Practice in Undergraduate Mathematics, Washington: MAA. p189-202.
Rasmussen, C. & Stephan, M. (2008). A methodology for documenting collective activity. In A. E. Kelly & R. Lesh (Eds.). Handbook of design research methods in education: Innovations in science, technology, engineering, and mathematics learning and teaching (Pp 195-215). Mawah, NJ: Erlbaum.
Speer, N. M., Smith III, J. P., & Horvath, A. (2010). Collegiate mathematics teaching: An unexamined practice. *The Journal of Mathematical Behavior*,*29*(2), 99-114.
Stephan, M., Cobb, P., & Gravemeijer, K. (2003). Coordinating social and individual analyses. In M. Stephan, J. Bowers, P. Cobb, & K. Gravemeijer (Eds.), Supporting students' development of measuring conceptions: Analyzing students' learning in social context (pp. 67–102). Reston, VA: National Council of Teachers of Mathematics.
Thurston, W.P. (1986). On proof and progress in mathematics. In T. Tymozko (Ed.), *New directions in the philosophy of mathematics*. (2nd ed.). Princeton: Princeton University.
Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25-41). Dordecht: Kluwer.
Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. Journal for Research in Mathematics Education, 27, 458–477.

Opportunity to learn the concept of group in a first class meeting on abstract algebra

Tim Fukawa-Connelly
Drexel University

This paper is a case study of the teaching of an undergraduate abstract algebra course; in particular, it examines the opportunity that the students had during the first class meeting to learn about the concept of a mathematical group and how much intellectual responsibility for the definition they were given. The paper offers a description of the classroom teaching and discussion an inquiry-oriented abstract algebra course. It focuses on the mathematical tasks that the professor used to introduce the concept of a group, the student solutions and classroom presentation of those tasks, and the professor's activities. It analyzes the class in terms of providing students the opportunity to learn the concept of a mathematical group and about the mathematical process of defining. It also analyzes the professor's activities in terms of devolving intellectual responsibility for writing the definition of a group. Finally, the analysis suggests ways that the professor's actions failed to devolve mathematical authority and limited opportunities to learn.

Keywords: abstract algebra, opportunity to learn, inquiry-based teaching

The abstract algebra course is an important point in the education of undergraduate mathematics majors and secondary mathematics teachers (Committee on the Undergraduate Program in Mathematics (CUPM), 1971; Mathematical Association of America (MAA), 1990; Conference Board of the Mathematical Sciences (CBMS), 2001). In its current incarnation, the abstract algebra course should help students develop mastery of the content of groups, rings and fields. Many researchers would argue that there is another (often implicit) content-related learning goal—that the students should be improving their algebraic thinking skills such as capturing patterns with symbols and (Cuoco, Goldenberg, & Mark, 1996; Smith, 2003). Mathematics education has shown that definitions (and thus, the idea of defining) is particularly critical for student's mathematical proficiency (CITES). Zazlavsky and Shir (2005) have described the concept of a mathematical definition via roles and features. They described the features as the list of qualities that a definition typically fulfills, including that it is non-contradicting, unambiguous, invariant under changing representations (for example, from symbols to words), non-circular, and, they are commonly asked to be minimal. They described the roles of definition in mathematics as including that they are to convey the characterizing properties and introduce the concept (Mariotti & Fischbein, 1997; Pimm, 1993), including "capturing the essence of a concept" (p. 317). Mathematical definitions establish the basis for proving and other problem-solving activities (Moore, 1994; Weber, 2002), and establish a community-wide agreement how to specific a concept, and thereby allow for more efficacious mathematical communication. Thus, the means by which a prospective course of instruction should be judged is via the success in conveying (or, at least giving students the opportunity to learn) the formal definition, the essence of the concept, and the ability to then operate on and with, and communicate about the concept, these are all known to be difficult for students.

Research questions:
This preliminary report will investigate the OtL about (a) groups, and (b) mathematical processes that students in an IBL abstract algebra class had.

The Opportunity to Learn

We align ourselves with Hiebert and Grouws' (2007) claim that, at minimum, the opportunity to learn requires both time on task and topic coverage. By topic coverage, we mean the mathematical content that appears in the lecture (as interpreted by expert observers of the class). In the account that follows, we refer to diagrams and explanations as "illustrations," because we believe they illustrate particular aspects of the concept of equivalence class. In sum, we describe a collection of illustrations of the concept of equivalence class as well as our post-hoc interpretation of how each of the illustrations can be understood to indicate particular aspects of the concept of equivalence class, the meaning of this concept, and the mathematical content that is available for students to learn about this concept. Following Gresalfi, Barnes, and Cross (2011), this study takes the position that the opportunity to learn is best understood as "the interrelations between the affordances of the designed learning environment" (p. 2) and whether and how those affordances are acted upon (Barab et al., 1999; Shaw, Effken, Fajen, Garrett, & Morris, 1997). Affordances are the set of actions (including mental ones) that are made possible by a particular aspect of the class (Gresalfi, Barnes, & Cross, 2011). As Gresfaldi (2009) noted, it is impossible to describe student learning without describing the presentation of content students experience, and how teacher actions make it accessible. Coupled with the notion of content being 'present' is also the notion of intellectual responsibility.

Methods

The setting was a first semester abstract algebra course at a PhD granting university in the US. The instructor had previously taught abstract algebra and was committed to a non-traditional pedagogy that he alternatively described as inquiry-based learning course. On a daily basis the students were seated in groups of four and in every class meeting there was time that they worked on mathematical tasks, presented at the board, and the professor lead whole-class discussions. The course met for 75 minutes three days per week and students were responsible for daily homework, only some of which was to be submitted for a grade. The particular course meeting presented here is about the definition and examples of a group.

Every class meeting was video recorded and a graduate student in mathematics education observed and took notes about each class meeting. I transcribed the recorded classroom dialog on the day that the students developed the definition of group. I differentiated between whole-class and small-group discussions, and noted everything written on the board that was linked, in appropriate places, to the transcript. The professor both participated in interviews throughout the semester and wrote notes describing his thinking related to the content, pedagogy and goals for the class. I note that in the first few classes we attempted to record some student work with a desktop microphone that we later learned did not work, and, thus, there are a few missing pieces in the transcript.

As a first pass through the data, I attempted to interpret what content could be learned through the lecture from by a mathematically enculturated individual, in this case, the author who has a masters degree in mathematics and researches the teaching and learning of abstract algebra. I first watched video of the class, then, created the transcript, and, finally, read the transcript. My first goal in coding was to indicate any places where I thought the whole-class discussion was conveying ideas about mathematics, then, described what idea or ideas were being conveyed and how. Additionally, to confirm his observations, the author asked two other mathematics education researchers with a masters degree in mathematics to read the transcript and describe the content that students have the opportunity to learn. I subsequently coded, as appropriate, any of the roles of definition (Zazlavsky & Shir, 2005) that were covered and how it was conveyed. The professors' claims were coded similar to the class, focusing on the roles of definition. Generally, the professor's claims aligned with the points in the subsequent analysis.

Data and analysis

Mathematical prompts--The definition of a group

The class in which the students had the first opportunity to learn the definition of group began with the instructor noting that the students were to have solved some equations, listed in the notes, overnight. The students solved the equations in the noted systems:

Set: Integers, Operation: Addition $3 + x = 7$	*Set: 2x2 invertible Matrices, Operation: Multiplication* $A \cdot x = B$	*Set: real numbers, Operation: multiplication* $\pi x = 8$

The next prompt was:

> As we've seen with the simple equations above, there is interesting work to be done when we focus on one set and operation at a time. For right now, let's continue in that way; simple structures (S, *) consisting of a set of objects on which one binary operation is defined. Based on your work above, let's operate on elements in the set S using the operation *.
>
> **Carefully Solve: a*x=b**

The notes also asked, "What properties did you use? Could you have used fewer?"

The class period described here is the third meeting of the class and began with students solving linear equations that the instructor provided. In subsequent sections I explore the ways that the professor and students interacted and how the students engaged with the mathematical content as a means of describing their opportunity to learn and intellectual responsibility for the definition of a mathematical group. Throughout the class, especially during discussions the professor acted as an arbiter of the mathematical norms of the class. For example, to open discussion about the third equation the professor prompted:

> T: So, take a look at pi-x = 8 there, Dan claims that he used the multiplicative inverse and the balance beam principle but no other properties. Are you willing to believe him?
> S: I think he used associativity
> B: you re-arranged the parenthesis so that you're multiplying pi-inverse times pi on both sides…
> D: Is that necessary?
> T: Well, looking at what you've got, there's two things in a parenthesis…

Again, the professor here acted as an arbiter of notation and whether or not rules were followed. He first stated the properties that the student claimed he used, asked for comments, and, when S2 claimed that S1 used the associative property, the professor indicated how S1's notation used the associative property. There were at least 8 instances in which I described the professor's role in the interaction as promoting or moderating the mathematical norms. In describing the professor as adjudicating mathematical norms, it is important to note that the norms that he promoted and enforced were about the whether the students needed to list all used properties, whether they were using properties listed in the notes, and whether they were in agreement about the terms.

The professor presented the definition of a group

Following the discussion of the fourth equation, the professor immediately moved to defining a group. He said:

> So, looking across these four different equations, when we tried to solve equations, in each case, we needed closure, associativity, an addit, an inverse property of whatever it was and an identity. Some interesting things going on… We didn't need commutativity. You have to be a little bit careful, with your matrices because they're, in fact, not commutative. You can't claim that A^-1B is the same as BA^-1. So, here's a cool thing. In mathematics, you get to make up definitions of new things, if you run into something that you're going to want to keep track of you can make up a definition that says, here's this new cool idea.

In these comments, which are similar, to the final prompt in the notes, the professor makes available a number of opportunities to learn about mathematical processes and the role of definitions. First, he explained that they were to look across the solutions of the four equations, meaning that they were to look for repeated reasoning, and, served to remind them, again, that their goal had been to solve equations, which, in this case, can be understood as what the professor's words and actions conveyed, and, the professor specifically indicated that this was his goal for student understanding. The professor then reminded the students of the four properties that they had used, and, that they did not use the commutative property, and could be certain of this because matrices are not commutative. In terms of the process of defining, the professor expressed the notion that mathematical definitions are created by people to respond to some intellectual need; that there is an idea that they want to use again.

The professor continued by explaining both how they would create the definition, and, that the names of concepts are, essentially arbitrary:

> It turns out that we're going to take your 4 properties and we're going to turn them into a definition. And, here's what we're going to do… It's not very complicated. We're going to say, given a set S, and an operation on the elements of that set, and since I want to be able to talk about all kinds of operations I'm going to call it star, right. If the set and operation have the following properties… It's all we've done right? We've take a bunch of sets and a bunch of operations, we solved some equations and said, "what properties do I want?" Well, what properties do I want? Where do we start?

Here, the professor has transitioned to describing the process by which he will write the definition of the new concept. The professor described the goal was to write a definition that was applicable to "a bunch of cases" which we believe is reasonably interpreted as all cases, which the professor reinforces by saying, "given a set S, and an operation on the elements of that set." He then asked, what properties a set and operation require, in order to solve an equation.

The students took his question as an opportunity to suggest the needed properties, the professor used this as an opportunity to describe the logical relationship between the properties:

> S: closure!
>
> T: Closure. 2! What order should I go in? Order might matter? What do you want next?
>
> S: inverses?
>
> T: Could have inverse next, could have associativity next, or could have identity next. Question, "can you have inverses without having an identity?"
>
> Ss: no
>
> T: so, we're going to put inverses below identity in our hierarchy. And now everybody's thinking, "huh?" So, let's put identity, inverses, and, inverses for all? For some? For one?
>
> S: for all
>
> T: right, because if I'm going to solve equations I need inverses for all. And, I need associativity. … Quick pause, this is not quite in order, but, claim: you can't have inverses without an identity. I'm going to write it down, I'm going to leave it there, and let it hang out for a little while. If S-star obeys these rules…

The students know the four properties that they have been discussing, thus, listing them can serve to reinforce them and help students associated them with the concept. By asking and then answering the question, "can you have inverses without an identity?" I assert that the professor offered students the opportunity to learn that there is a logically necessary relationship between the existence of identity and inverses. Moreover, the professor and a student have both asserted the importance of all elements having an inverse. Finally, the professor made claims about the conventions of naming in mathematics:

> Well, here's the cool thing about writing a new definition, it turns out that you can call it whatever the heck you want to call it. If you want to call it a bob, you could, if you're the first one to make it up. Some people, in fact, name things after themselves, and it turns out that there's a goodly quantity of stars and planets named after people. But, it turns out that people have

> thought this up before, and, so while in this class we could call it a Megan, we're going to call this a group. G-R-O-U-P.

Here, the professor, made an explicit statement about the naming conventions in mathematics, and, via his claim that it is possible to name a structure after one of the students in the class, he conveyed that names of structures might be arbitrary. Through the entirety of his presentation about the definition of a group the professor conveyed that the essence of the concept of a group is that it's a system in which is possible to solve equations, that name of the concept is a group (allowing for communication), and the name of each of the four relevant properties. Thus, at this point of the lesson, while the students had the opportunity to learn many aspects of the concept of group, the professor did not here give a formal statement of any of the properties here. At best, the students could refer back to the original statement of the properties in their notes, but, while given in a general form, they are written in the context of "numbers" and thus the students do not have an appropriate formal mathematical statement for the properties of identity or inverse, and, even closure and associativity require changing the operations and the set from which the elements are drawn.

Summary of Opportunity to Learn

By the end of the first day of class the students had the opportunity to learn that the essence of the concept is a system in which it is possible to solve equations. They saw, via solving the equations, the necessity of each of the four characteristic properties, and, via their work, that those were the only needed properties. Thus, they had the opportunity to learn that the four properties are both minimal and completely specify the essence of concept. Moreover, the students, in their discussion, explored issues of abstraction and generality with the properties, terminology and notation. For example, through the discussion of the solution to a*x=b, the students appeared to recognize a need for a new term, rather than 'multiply' to describe the action of an abstract operation, and the need for new notation, rather than "1" to denote the identity element when working with an abstract set and operation. Moreover, the students had the opportunity to learn the statement of the definition of a group from the notes, including the symbolic version, and, it had all of the essential characteristics of a definition. I argue that the students had the opportunity to establish a community-wide agreement on how to specify the concept, especially that a group does not need to be commutative.

Intellectual responsibility

In terms of the intellectual responsibility for the work of defining, the professor had significant responsibility and devolved much less to the students than in Larsen's (2013) curriculum but more than in Leron and Dubinsky (1997) or the traditional curriculum described in Fukawa-Connelly and Newton (2014). In particular, the instructor established that the essence of the concept of a group is a system in which it is possible to solve equations. The students did solve, listing the properties the minimal set of properties needed to solve, and, collectively agreeing on the set of properties used to solve each of the four equations. The professor then indicated that they would write a definition that captured this idea, but, he is the one that wrote the idea, and, presented the students with a formal statement of the definition that he had pre-written in a second handout. Thus, the students' work formed the basis for the professor's writing of the definition, and therefor I suggest that they had intellectual responsibility for the content of the definition.

Discussion

The first conclusion of this study is that the professor's notes, planning, and classroom activities made available significant opportunity to learn about the definition of a group and the example space for a group, and finally the process of writing a mathematical definition. The second conclusion relates to the intellectual responsibility that the students and the professor had for developing the definition of a mathematical group. While the students had more intellectual responsibility than in a lecture class, and when compared with the responsibility offered by the Dubinsky and Leron (1994) curriculum, it was less than that of Larsen's (2013) curriculum. These observations illustrate how well-intentioned instruction, even instruction that includes a lot of student participation, and intellectual responsibility, might provide students with less responsibility than is reflected by the text/notes. While the professor in this study wrote notes that allowed for significant responsibility, and, the amount of student argumentation and discussion is productive and valuable, some of his choices. However, the in-class questions were also limited, because they did not provide the students with opportunities to practice some important aspects of writing and revising a definition.

Guzzetti, B. J., Snyder, T. E., Glass, G. V., & Gamas, W. S. (1993). Promoting conceptual change in science: A comparative meta-analysis of instructional interventions from reading education and science education.*Reading Research Quarterly*, 117-159.

Studying students' preferences and performance in a cooperative mathematics classroom

Sayonita Ghosh Hajra
University of Utah

Natalie L.F. Hobson
University of Georgia

In this study, we discuss our experience with cooperative learning in a mathematics content course. Twenty undergraduate students from a southern public university participated in this study. The instructional method used in the classroom was cooperative. We rely on previous research and literature to guide the implementation of cooperative learning in the class. The goal of our study is to investigate the relationship between students' preferences and performance in a cooperative learning setting. We collected data through assessments, surveys, and observations. Results show no significant difference in the comparison of students' preferences and performance. Based on this study, we provide suggestions in teaching mathematics content courses for prospective teachers in a cooperative learning setting.

Keywords: cooperative learning, performance, prospective elementary teachers

Many mathematics education researches emphasize the importance of mathematical reasoning in learning mathematics (Kramarski and Mevarech, 2003; National Council of Teachers of Mathematics, 1989, 2000). According to the Common Core State Standards (Common Core State Standards Initiative, 2010), mathematical understanding is the ability to justify why a particular statement is true. It is important for students to not only construct viable arguments but also critique the reasoning of others (Common Core State Standards Initiative, 2010). These components of learning are also elements of cooperative learning, which suggests students critique and learn from each other (Evans, Gatewood and Green, 1993; Sheehy, 2004).

The basis of a cooperative classroom is group success. It is a form of instruction where students work in groups, share common goals, and are accountable for their actions (Johnson & Johnson, 1994a). Various studies (Slavin, 1996; Webb, 1989; Webb & Farivar, 1994) suggest multiple benefits of cooperative instruction. According to Slavin (1996), peer collaboration can encourage creative thinking and helps generate new ideas. Many researches suggest that when instruction shifts from individual to group individual learning is enhanced (Marzano, Pickering, & Pollock, 2001; Slavin, 1996).

According to Johnson and Johnson, the following are not features of cooperative learning: students sitting side by side at a table and talking while completing their own individual assignments, one large assignment given to a group of students where only one student contributes to the assignment and others take credit, students helping each other on individual assignments after finishing their own work (1994a, 1994b). Cooperative learning does involve students working together to complete a task in groups. Many researchers focus and identify different forms of cooperative learning. In our study, we focus on the learning together form of cooperative learning (Johnson and Johnson, 1975). In the learning together form, all students work together on the same task sharing a common goal. According to Johnson and Johnson (1994b), characteristics of cooperative learning are group interdependence, individual accountability, group processing, and face-to-face interaction.

In this study, our goal is to understand how students' preferences and performance are related in a cooperative mathematics classroom. Students come in the classroom with their preferences—some like to work in groups and some do not like to work in groups. We study and explore how students' preferences affect students' individual performance in a cooperative learning setting. In this exploration, we also learn about students' opinions about group work after a cooperative mathematics experience. This gives us insight for future cooperative learning implementation.

Methodology

We collected three forms of data from study participants throughout the length of this study. These forms of data include a preliminary survey of students' initial preferences for working in a group, students' work on group assignments and corresponding individual assignments (see Appendix A), and students' responses on an end-of-course survey (see Appendix B).

At the time of this study one of the researchers was the instructor of an arithmetic content course for future elementary teachers at a public university in the southeastern United States. Study participants consisted of the twenty students enrolled in this course. The class consisted of females in their third year of undergraduate study. The textbook for the course was Beckmann (2011a, 2011b). These participants had all previously completed similar content courses for prospective elementary teachers in geometry, number, algebra, and statistics at the same university with different instructors from this same textbook.

In order to make clear our focus of analysis we define two terms we use throughout our investigations.

Preference refers to a student's liking of group work. We classify preferences as like, like with a choice, and dislike. Like with a choice refers to a student's liking of group work if allowed to choose her own group.

Performance refers to a student's scores on group and individual activities based on a rubric discussed in the following.

We structured each class to incorporate a group activity and an individual activity related to the group activity. Each class period was 50 minutes in length. Each class was structured with 10 minutes of lecture, 20 minutes of group activity, 10 minutes of individual activity, and 10 minutes of class discussion. Students worked in assigned groups of three. Groups were randomly reassigned each week with the condition of no two students remaining in the same group. Each group worked on one activity sheet to encourage cooperation between members.

The activities covered the following content sections: adding and subtracting fractions, commutative property of multiplication, associative property of multiplication, distributive property of multiplication over addition, mental math, fraction multiplication, and the partial product algorithm.

Before implementing cooperative learning in the classroom, the researchers gave students a preliminary survey asking for their preferences of working in groups. At the completion of the course, an end-of-course survey asked open-ended questions to students soliciting as many responses as possible to help us understand their opinions of and experiences in the group and individual settings.

Data Analysis

In the preliminary survey students generated and self identified with one of the following three categories coded as follows, like group work (L), like group work if given a choice of group members (C), and dislike working in a group (D).

Researchers determined student performance on group and individual assignments using a general rubric (see Table 1). With this general framework, researchers created a specific rubric for each activity with specific details in order to determine the sufficiency level of responses. To ensure reliability of data analysis on performance, one of the two researchers wrote a rubric for each activity and assigned each student a performance number from this rubric, the other researcher then used this same rubric to give each student a performance rating also. Researchers met to discuss any discrepancies in the scoring and legitimacy of the rubric. For group assignments, the researchers assigned the same performance score to each student in the group.

Table 1: General grading rubric.

Score	Meaning	Description
3	Sufficient with detail	Correct, logical argument with extra detail (typically includes extra diagrams and images to support argument)
2	Sufficient	Correct, logical argument
1	Almost sufficient	Correct argument but lacks some justification
0	Not sufficient	Inaccurate or flawed argument

Researchers coded students with the following three data points: initial preferences, average group performance, and average individual performance. Researchers used t-test to quantitatively analyze the data to compare preferences with performance.

The non-teacher researcher collected the end-of-course survey forms and only after turning in the grades, the teacher researcher was able to see the survey. Each researcher individually read all surveys and made a rubric to help categorize student responses after reading through what students had written. Researchers met to organize the rubric and layout a framework to understand students' experiences in the classroom.

Results

We collected data from the six topic sections of the course as discussed above. We use two sample t-test assuming unequal variances to analyze our data.

Quantitative Analysis

Among the 20 students, students self-identified their group preference as follows, 6 like, 8 like with a choice, and 6 dislike. The group activity performance means for the like, like with a choice, and dislike students are 61.23, 59.10, and 56.44 respectively. We computed the observed p-value of the t-test for the categories of dislike and like with a choice, dislike and like, and like with a choice and like. Respectively these values are 0.42, 0.129, and 0.407; each of which is greater than the level of significance, 0.05.

Hence, we saw no significant difference in performance of group work between students with different group preference. In a similar manner, we performed the two sample t-test to compare the *individual activity performance* between students with the three different preferences. We saw no significant difference in the individual performance across students with different preferences.

Student Opinions of the Cooperative Learning Experience

The end-of-course survey solicited a variety of student responses on beliefs and opinions while participating in a cooperative learning classroom. These responses provide a categorical description of possible student beliefs about cooperative learning in mathematics. This helps to provide a framework to understand and gain awareness of student experiences in the cooperative learning environment and helps us learn how to improve this setting for students.

The following (Table 2 and Table 3) gives an organization of responses and reflections on group and individual activities gathered from the end-of-course survey.

Table 2: Opinions of group activity experience.

Overall Group Experience	Reason
• Beneficial for long term and increased understanding	• Metacognitive aspect (i.e., why, what, and how) • Multiple ideas or approaches from group • Verbal expression of ideas • Clarify misconceptions
• Beneficial for completing activity	• Immediate input from other students
• Not beneficial for completing activities	• Contradictory group input • Multiple approaches distracting • Topic too elementary for group discussion • Dependence on others
• Dislike	• Unnecessary and excessive • Switching groups too often
• Like	• Social interaction • New way of learning (because beneficial)

Table 3: Opinions of individual activity experience.

Overall Individual Activity Experience	Reasons
• Dislike	• Redundant, repetitive problems • Feels like memorization • Group activity made student feel unconfident
• Like	• Self-reflection and self-monitoring (e.g., what is done and what needs to be done) • More practice • Group activity made student feel confident • Assessment of knowledge • Fewer questions help to focus

Conclusion

From our study, we find that a student's preference of working in groups does not necessarily determine the student's performance in the group setting. Preferences may vary from student to student and performance relies on the understanding of the task. This understanding of the task may or may not develop in students through group activities regardless of the their initial preferences for group activities.

Discussion and Future Implementations

The results of this study provide several suggestions to help teachers teaching math content courses more successfully implement cooperative learning in their classrooms and improve the cooperative learning experience for their students. Specifically the student responses in the end-of-course survey and our observations in the classroom provide several reasons why students feel the group activities implemented by the teacher researcher were not beneficial. As teachers, we can make aims to address such factors in order to help students feel the group activity is beneficial and to help students get a fulfilling experience from these activities. We discuss these suggestions below.

Teacher's interaction during cooperative learning activity- In our study we observed situations in which a few of the students in a group convinced other students of inaccurate reasoning. As a teacher implementing cooperative learning it is important that the teacher maintain a very active role in the classroom by listening to students' discussion and provide direction when students collectively engage in inaccurate reasoning or problem solving (Davidson, 1990). It is also important for teachers to help students who feel unheard in a group speak out and share ideas, especially when those ideas are beneficial for the entire group. Teachers must play an active role of engaging in small group discussions by listening to the group's discussion and helping the group focus on insightful approaches.

Suitable tasks- In a group activity, the cognitive demand of the task is very important. When there is not enough content for the students to discuss, students are not able to engage with each other and interact. If a concept does not require extensive investigation of the topic, the task might best be implemented during an individual activity rather than a group activity.

Suitable questions- Additionally, our experience found that group tasks with direct questions of why and how provide more opportunity for students to discuss. For example, during group activities involving addition and multiplication of whole numbers, when we asked students to compute a solution, students quickly completed the assignment and did not have discussions. However, when we asked students to explain properties of an operation used in computation (e.g., associative or commutative) students engaged in discussion with each other in order to check the validity of each step involved in the calculation. With this type of question students had to discuss and debate strategies and reasonings. This encouraged group interaction and collaboration.

In finale, we offer the above as suggestions for teachers implementing cooperative learning in a mathematics content course for pre-service elementary teachers. We also want to emphasize that teachers should not be discouraged if their students at first do not like working in groups. As our study shows, students generate many positive beliefs about group work after they have encountered a cooperative learning experience in a mathematics course.

References

Beckmann, S. (2011a). *Mathematics for elementary teachers with activity manual*. Boston, MA: Pearson.

Beckmann, S. (2011b). *Mathematics for elementary teachers activity manual*. Boston, MA: Pearson.

Common Core State Standards Initiative. (2010). *The common core state standards for mathematics*. Washington, D.C.: Author.

Davidson, N. (1990). Small-group cooperative learning in mathematics. In T. J. Cooney & C. R. Hirsch (Eds.), *Teaching and Learning in Mathematics in the 1990s, 1990 Yearbook of the National Council of Teachers of Mathematics (NCTM)* (pp. 52–61). Reston, VA: NCTM.

Evans, P., Gatewood, T., & Green, G. (1993). Cooperative learning: Passing fad or long-term promise? *Middle School Journal 42* (3), 3–7.

Johnson, D.W., & Johnson, R.T. (1975). *Learning together and alone*. Englewood Cliffs, NJ: Prentice Hall.

Johnson, D. W., & Johnson, R. T. (1994a). *Cooperative learning in the classroom*. Alexandria, VA: Association for Supervision and Curriculum Development.

Johnson, D. W., & Johnson, R. T. (1994b). An overview of cooperative learning [Electronic

version]. In J. Thousand, A. Villa and A. Nevin (Eds), *Creativity and collaborative learning*. Baltimore: Brookes. Retrieved November 4, 2001, from: http://www.co-operation.org.
Kramarski, B. & Mevarech, Z. (2003). Enhancing mathematical reasoning in the classroom: The effects of cooperative learning and metacognitive training. *American Educational Research Journal, 40*: 281–310.
Marzano, R. J., Pickering, D. J., & Pollock. (2001). *Classroom instruction that works: Research-based strategies for increasing student achievement*. Alexandria, VA: Association for Supervision and Curriculum Development.
National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. New York: Routledge & Kegan Paul.
National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
Sheehy, L. A. (2004). *Using student voice to deconstruct cooperative, mathematical problem solving* (Unpublished doctoral dissertation). The University of Georgia, Athens, GA.
Slavin, R. E. (1996). Research on cooperative learning and achievement: What we know, what we need to know. *Contemporary Educational Psychology, 21,* 43–69.
Webb, N. (1989). Peer interaction and learning in small groups. *International Journal of Educational Research, 13,* 21–40.
Webb, N., & Farivar, S. (1994). Promoting helping behavior in cooperative small groups in middle school mathematics. *American Educational Research Journal, 31,* 369–396.

Appendix A

Example of a group activity and an individual activity

Group Work:
Student 1.
Student 2.
Student 3.

1. Use the array to help you explain why 3X5 and 5X3 must be equal. In other words, explain why 3X5 = 5X3 (Beckmann, 2011b, p. 71)

   *****
   *****
   *****
   Solution:

2. Why does the commutative property hold for numbers other than 3 and 5? In other words, why is it true that AxB = BxA, no matter what counting numbers A and B are? To answer this question you may need to modify your explanation for why 3X5 is equal to 5X3 so that it is a general conceptual explanation, that is, one that doesn't refer to the number 15 but refers only to 3 and 5 and to the underlying array (Beckmann, 2011b, p. 71)
   Solution:

Individual work:
Student Name:

Problem 1: Here is Amy's explanation for why the commutative property of multiplication is true for counting numbers:

> Whenever I take two counting numbers and multiply them, I always get the same answer as when I multiply them in the reverse order. For example, 6 X 8 = 48, 8 X 6 = 48; 9 X 12 = 108, 12 X 9 = 108, 3 X 15 = 45, 15 X 3 = 45.
>
> It always works that way; no matter which numbers you multiply, you will get the same answer either way you multiply them.

Explain why Amy's discussion might not convince a skeptic that the commutative property should always be true for any pair of counting numbers. Then explain why the commutative property of multiplication is valid in another way (Beckmann, 2011a, p. 159).
Solution:

Appendix B

End-of-course Survey

Please fill out the survey form clearly.
Student Name:

1. Describe your overall experience working in groups this semester in the following situations:
    a. Group Activities
    b. Final Project
2. Describe your overall experience working on the individual activities this semester.
3. Do you feel comfortable working in a group? (Yes/ No)
4. Did working with a group help you to learn multiple ways to solve problems? (Yes/ No)
   Explain question 4. How did the group help/ not help?
5. Would it have been beneficial for you if you had worked alone? Explain.
6. Has your confidence in explaining math in writing and/or orally increased? (Yes/ no).
    If yes, what percentages do you attribute to the following class components for your achievement of confidence increase (total should be 100%):
        Group activities:
        Individual assignments:
        Class discussions:
        Homework:
        Peer review:
    If no, explain why not.
7. What was your favorite class activity this semester? (Class activities were: group assignments, paper ball activity, wall activity, reading pop quiz, etc.). Why?
8. What class component helped most with your learning of mathematics?

A mathematician's experience flipping a large lecture calculus course

Erin Glover
Oregon State University

There is a growing body of literature that suggests students do better in classes that implement active-learning strategies (e.g., group work, problem solving). Flipping classroom instruction is becoming a popular innovation to support active learning in the classroom. This preliminary report highlights one instructor's experiences when implementing ambitious teaching practices in a flipped large lecture Calculus I course. Data collection consists of email correspondence, open-ended interviews, and weekly questionnaires. Particular attention will be made to address how the instructors' professional identity impacted instruction throughout the term.

Keywords*: flipped classroom, large lecture, ambitious teaching, calculus, professional identity*

The large-scale national study funded by the MAA and the NSF, *Characteristics of Successful Programs in College Calculus* (CSPCC) sought to describe the demographics of Calculus students and the characteristics of successful programs (Bressoud, Carlson, Mesa, & Rasmussen, 2013). Results from the study are still rolling in, but the overarching message is that factors contributing to a successful program are complex and Calculus instruction matters. The CSPCC student survey results found that students with instructors who engaged in ambitious teaching practices had a positive impact on students' retention in the Calculus sequence (Rasmussen, Ellis, & Bressoud, 2013), but that it also had small negative impact in students' confidence, enjoyment, and interest in mathematics (Sonnert, et al., 2013). Ambitious teaching practices include things like group work, group discussion, requiring students to explain their thinking, student presentations, challenging problems on exams, etc. In response to these mixed survey results, Larsen, Glover, & Melhuish (2014) provided a brief review of the literature related to ambitious teaching and a discussion of two substantial cases of ambitious teaching observed during the case study phase of the CSPCC project. They concluded that engaging in ambitious teaching is a complex activity that requires institutional support and a significant commitment on the part of the instructors. My study was very much inspired by the results from the CSPCC project. I encountered a professor faced with a 270-person large lecture Calculus I teaching assignment who was passionately (and very openly) opposed to large lecture instruction. Instead of teaching a traditional large lecture, he chose to implement various ambitious teaching practices supported by flipping his classroom. This preliminary report will describe the instructor's experience, and more specifically, the relationship between his instruction and how he viewed himself as a teacher.

Relation to research literature

A wealth of research in STEM indicates that students do better in classes that employ active-learning strategies (e.g., group problem solving, student presentations, clicker surveys). The physics discipline has sought to "transform" their large lectures by implementing researched-based instructional changes. One such study in a calculus-based physics course found that combining interactive engagement methods in lecture, tutorials, and homework had a positive effect on conceptual and attitudinal development" (Pollock, 2006, p. 142). A meta-analysis of 225 STEM studies found that instruction that included at least some active-learning strategies (e.g., group problem solving, worksheets, tutorials, workshop) increased student performance on examinations and concept inventories

(Freeman, et al., p. 3). A popular innovation in the mathematics classroom to make time for these types of strategies is known as "flipping" (or inverting). This was described by Lage, Platt, and Tregalia (2000) as "events that have traditionally taken place *inside* the classroom takes place *outside* the classroom and vise versa." Numerous studies have appeared in RUME proceedings in recent years, comparing various effects of instructional innovation including flipping instruction (Bagley, 2013, 2014; Bowers & Zaskis, 2012; Wasserman, 2013). Larsen, et al. (2014) presented a case study that was a robust example ambitious teaching. The calculus program's success was facilitated by the combination of flipping, group work, and technologically supported innovations (e.g., use of iPads and clickers). This report adds to both the body of literature on large lecture instruction, flipped instruction, and literature on the implementation of ambitious teaching practices. This will also add to literature about the connection between instructor professional identity and instruction.

Theoretical perspective

The analysis for this report will draw from the work of Remillard (2005) that suggests how teachers interact with novel curricular materials is directly influenced by their professional identity (p. 237). Spillane (2000) defined identity as "an individual's way of understanding and being in the world, in this case the world of work. Although identity includes what one knows and believes, it also encompasses dispositions, interests, sense of efficacy, locus of control, and orientations toward work and change" (p. 308). Initial interviews with the large lecture instructor indicated that his personal beliefs about himself as a teacher did not align with the teaching assignment he was given. At the outset of this project his professional identity can already be seen in shaping instruction, which is evidenced by his decision to flip the course. Given the proven utility of identity in explaining teachers experiences with curricular materials, I believe it is likely that the way instructors experience other teaching challenges (such as an extremely large class size) can be understood in terms of the instructor's professional identity.

Research methodology

This preliminary report is part of a larger study that will examine all of the participants' experiences in one flipped large lecture Calculus I course at a large, public university. This report will focus on the large lecture instructor, Mark[1]. Data collected will consist of audio-recorded open-ended interviews, email correspondence, and weekly online questionnaires. Initial interviews revealed that the Mark's professional identity impacted how he dealt with his teaching assignment. The weekly online questionnaires are intended to track Mark's instruction throughout the term and how it related to his professional identity. A subset of weekly questionnaires items include questions like (1) "What is something you did in class this week that is consistent with how you view yourself as a teacher?" and (2) What is something that happened that was inconsistent with the way you view yourself a teacher? Future interviews with Mark will focus in how his instruction changes and what promoted him to make the changes and whether or not these changes align with how he views himself as an instructor. All interviews will be analyzed and relevant sections transcribed. The analysis will look for specific examples of Mark's professional identity that relate to his instruction. For example, broadly, his decision to flip the course could be explained by the fact that he identifies himself as a mathematician who supports mathematics education and engages in student-centered learning in his own classrooms. As the course unfolds, future interviews will be analyzed to find other examples of Mark's professional identity impacting his instruction.

A second part of this study focuses on the students in the large lecture course. The students will complete both the pre and post student surveys used in the CSPCC project. These surveys will highlight the demographics of the students, but also reveals various aspects of their experience in their Calculus course. In addition to the CSPCC survey data, students answer Clicker survey question related to the instructional innovation (e.g., group work, use of clickers, flipping) at various times throughout the term. This preliminary report will be limited to just the experiences of the lecture instructor.

Preliminary Results

The large lecture instructor, Mark, is a professor with experience teaching calculus. He is a mathematician and serves as the calculus coordinator. Mark also served as department chair in the past and now serves as a consultant on math education grants, serves on dissertation and exam committees for both the math education PhD programs and Math PhD programs.

[1] Pseudonym.

This is the first time Mark has tried to flip one of his courses, but was already quite familiar with the efforts of another instructor who did so. He wanted to flip his instruction to "minimize what we all know will be the negative impacts on student success and student learning."

To support his course set-up, Mark worked hard to coordinate resources because he suspected there was going to be little support from the department to implement his flipped large lecture. He responded to his teaching assignment by reaching out to current Calculus instructors to recruit high-quality undergraduate "peer mentors" (TA) to volunteer in his 270-person large lecture because the "experience one would get in this environment will be helpful on [their] resume."

Mark also selected his recitation leaders; one had experience teaching calculus, the other who recently mentored with the Mark in Calculus, and the third was a new graduate teaching assistant (GTA). Mark intended to give extra support to the new GTA by placing a dedicated undergraduate TA in their recitation sections. He required recitation leaders to attend lecture "to engage students" and support the undergraduate peer mentors. Early communication with colleagues resulted in "a lot of people that [were] interested." This included a graduate student that was interested in "spending time in my class to get some experience" which would help in "applying for a [G]TA to push through to get his degree."

Simultaneously, he began to scour math education literature related to large lecture instruction, flipping, and other innovations. The research literature helped make his decision to flip his course, but it also helped frame the request for support he submitted to the math department. This contained references to the literature to justify these requests which included (1) recitation sections capped at 20, (2) Meeting times twice (to support longer meeting times for the large lecture) (3) undergraduate peer mentors, and (4) a one-time $3000 grant to support course design and set-up. Mark reported, "I got nothing." The math department did give Mark a course that met twice a week, but was told his lecture would be 50 minutes shorter to make up for the time students spend in smaller recitation sections. He decided, "If students want to stay late for the last hour, they can. [The Hall] is open. They can work on problems if they want. That sort of gives me extra time I can use, although I'm not supposed to be there." Mark does see the value of the GTA-led recitation sections attached to his course saying, "My theory is that those should be places to build community around the subject of calculus."

Discussion and further research

My preliminary analysis is focused on what Mark did to support getting his course up and running. Future analysis will include Mark's weekly online questionnaires and his final reflection of the course in addition to further interviews. Data from these interviews will detail whether Mark believes his ambitious teaching successfully addressed his concerns with large lecture instruction and how these might be connected to his professional identity. I suspect that his professional identity will have various impacts on his ambitious teaching as the term progresses. This might be seen in the level of group work, student presentations, or the use of clicker responses to balance the instructional goals any given week. It will be interesting to hear about how Mark views his instructional moves and to what level they aligned with how he views himself as a teacher.

Questions for The Audience

1. Is there other literature that might help to inform this research?
2. Are their other theoretical constructs that might help in making sense of the instructor's experience?
3. In this case, Mark was asked to do something he did not want to do (teach a very large section of calculus). How as a researcher can I best address this aspect of the study?

References

Bagley, S. (2013). A comparison of four pedagogical strategies in calculus. In *Poster presented at the 35th Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education, Chicago, IL*.

Bagley, S. (2014). Best Practices for the Inverted Classroom (Under Review).

Bowers, J., & Zazkis, D. (2012). Do students flip over the "flipped classroom" model for learning college calculus? In L. R. Van Zoest, J.-J. Lo, & Kratky, J. L. (Eds.), *Proceedings of the 34th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (849-852). Kalamazoo, MI: Western Michigan University.

Bressoud, D., Carlson, M., Mesa, V., & Rasmussen, C. (2013). The calculus student: Insights from the Mathematical Association of America national study. *International Journal of Mathematical Education in Science and Technology*, *44*(5), 685-698. doi:10.1080/0020739X.2013.798874.

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences*, 201319030.

Lage, M. J., Platt, G. J., & Treglia, M. (2000). Inverting the classroom: A gateway to creating an inclusive learning environment. *The Journal of Economic Education*, *31*(1), 30. doi:10.2307/1183338

Larsen, S., Glover, E., Melhuish, K. (2014). *Beyond Good Teaching: The Benefits and Challenges of Implementing Ambitious Pedagogy.* (Submitted.)

Pollock, S. J. (2006, February). Transferring transformations: learning gains, student attitudes, and the impacts of multiple instructors in large lecture courses. In *2005 Physics Education Research Conference* (Vol. 818, pp. 141-144).

Rasmussen, C., Ellis, J., Bressoud, D. (2013). Who Are the Students that Switch out of Calculus and Why? Under Review.

Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research*, *75*(2), 211-246.

Sonnert, G., Salder, P., Sadler, S., Bressoud, D. (2014). *The Impact of Instructor Pedagogy on College Calculus Students' Attitude Toward Mathematics.* Submitted to International Journal of Mathematics Education for Science and Technology.

Spillane, J. P. (2000). A fifth-grade teacher's reconstruction of mathematics and literacy teaching: Exploring interactions among identity, learning, and subject matter. *Elementary School Journal*, 100(4), 307–330.

Wasserman, N. H., Norris, S., & Carr, T. (2013). Comparing a "flipped" instructional model in an undergraduate Calculus III course. In *Conference on Research in Undergraduate Mathematics Education, Denver, CO*.

Understanding participants' experiences in a flipped large lecture calculus course

Erin Glover
Oregon State University

There is a growing body of literature that suggests students do better in classes that implement active-learning strategies (e.g., group work, problem solving). Flipping classroom instruction is becoming a popular innovation to support active learning in the classroom. The study reports on one large lecture calculus course where instruction was flipped in order to employ active-learning strategies. This poster will highlight the experiences of the large lecture instructor, recitation leaders, and student focus groups.

Keywords: *flipped classroom, inverted classroom, instructional models, calculus*

Introduction and Relation of this Work to the Research Literature

A wealth of research in STEM indicates that students do better in classes that employ active-learning strategies (e.g., group problem solving, student presentations, clicker surveys). A popular innovation in the mathematics classroom to make time for these types of strategies is known as "flipping" or "inverting" (Lage, Platt, & Tregalia, 2000). This study details the experiences when a mathematics professor flipped his instruction to employ active-learning strategies into his large lecture calculus course. Numerous studies have appeared in RUME proceedings in recent years, comparing various effects of instructional innovation including flipping instruction (Bagley, 2014; Bagley 2013; Bowers & Zaskis, 2012; Wasserman, 2013). This goal of this presentation is to contribute to the body of literature on specific instructional models, like large lecture and flipped classrooms, and the literature on ambitious teaching practices.

Theoretical Perspective and Research Methodology

This poster presentation will highlight experiences of the various participants in a flipped large lecture Calculus I course. Data collected consists of audio-recorded semi-structured interviews with the department chair, large lecture instructor, recitation leaders, and student focus groups. The large lecture instructor also participated in additional weekly online reflection surveys intended to track his experiences in greater detail. Classroom observations were conducted in both the large lecture and recitation sections throughout the term. Students in the course also participated in a pre and post survey to identify student demographics and their success in the course. This survey was used in the MAA's national NSF-funded grant *Characteristics of Successful Program in College Calculus* (Bressoud, Carlson, Mesa, & Rasmussen, 2013). A grounded theory approach is being used to identify themes that emerge from the interviews, surveys, and classroom observations (Charmaz & Belgrave, 2002; Corbin & Strauss 2008).

Preliminary Results and Future Research

Preliminary results indicate mixed feelings about the flipped large lecture/recitation (LLR) format from the participants' perspectives. The large lecture instructor consistently believed that there was room for improvement during the course regarding instruction, organization of the course, and using technology-based innovations effectively. There were varied opinions about the effectiveness of technology-related instruction (e.g., clickers, online videos) that were intended to engage students inside out outside of class. Student focus group interviews suggested that, despite the amount of group work in the LLR, students felt disconnected from other students, in addition to their instructor and recitation leader. Students reported they believed in the value of group work, but wanted to have more lecture throughout the course. Recitation leaders reported being cognizant of the students' displeasure of the format of the course and attempted to modify their instruction in response to this. Future analysis will describe in greater detail what the participants believed to be successful and what parts of the course needed improvement.

References

Bagley, S. (2013). A comparison of four pedagogical strategies in calculus. In *Poster presented at the 35th Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education, Chicago, IL*.

Bagley, S. (2014) Best Practices for the Inverted Classroom (Under Review).

Bowers, J., & Zazkis, D. (2012). Do students flip over the "flipped classroom" model for learning college calculus? In L. R. Van Zoest, J.-J. Lo, & Kratky, J. L. (Eds.), *Proceedings of the 34th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (849-852). Kalamazoo, MI: Western Michigan University.

Bressoud, D., Carlson, M., Mesa, V., & Rasmussen, C. (2013). The calculus student: Insights from the Mathematical Association of America national study. *International Journal of Mathematical Education in Science and Technology*, *44*(5), 685-698.

Charmaz, K., & Belgrave, L. (2002). Qualitative interviewing and grounded theory analysis. *The SAGE handbook of interview research: The complexity of the craft*, *2*.

Corbin, J., & Strauss, A. (Eds.). (2008). *Basics of qualitative research: Techniques and procedures for developing grounded theory*. Sage.

Freeman, S., Eddy, S., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences*, 201319030.

Lage, M. J., Platt, G. J., & Treglia, M. (2000). Inverting the classroom: A gateway to creating an inclusive learning environment. *The Journal of Economic Education*, *31*(1), 30. doi:10.2307/1183338

Wasserman, N. H., Norris, S., & Carr, T. (2013). Comparing a "flipped" instructional model in an undergraduate Calculus III course. In *Conference on Research in Undergraduate Mathematics Education, Denver, CO*.

Marginalizing, centralizing, and homogenizing: An examination of inductive-extending generalizing among preservice secondary educators

Duane Graysay
The Pennsylvania State University

Policy and standards documents from recent decades emphasize the importance of attending to students' development of ways of engaging in mathematical activities such as generalizing. Therefore, we need to be able to describe proficient generalizing and ways of explaining and predicting the strategies that individuals use to generalize. However, existing research has largely focused on examining generalizing at the K-12 level on a specific type of task. A broader understanding of mathematical generalizing is important for developing potential developmental trajectories. This report presents tentative findings from the preliminary analysis of data collected as part of the author's dissertation research on generalizing among university students.

Key words: Preservice Secondary Teachers, Generalizing, Mathematical Reasoning

For at least the past several decades, standards and policy documents have been arguing that there is a fundamental limitation in the epistomological position that knowing mathematics includes only distinct concepts and procedures (NCTM, 1989, 2000; NGA/CCSSO, 2010; NRC, 2001). These documents align with suggestions that a more sophisticated position is to also consider the ways that individuals engage in problem solving (Schoenfeld, 1989); their mathematical habits of mind (Cuoco, Goldenberg, & Mark, 1996); the mathematical practices in which they engage (NGA/CCSSO, 2010); their strategic competence, adaptive reasoning, and productive disposition (NRC, 2001); or the ways that they engage in mathematical processes (NCTM, 2000). Common across these notions is that knowing mathematics includes the ways that one engages in mathematical activity.

One type of activity that is essential to mathematics -- though not uniquely mathematical-- is the process of generalizing. To understand the importance of generalizing, one must only consider the typical conceptual and procedural content of school mathematics, which often consists of generalized procedures for responding to tasks (e.g., an algorithm for multidigit addition, or a procedure for identifying the derivative of a function based on the equation that describes the function) or of generalized claims about categories of objects (e.g., the Pythagorean Theorem or the Intermediate Value Theorem). Because of the central nature of generalizations in mathematics, it follows that the process of creating generalizations "lies at the very heart of mathematics" (Mason, 2008, p. 63). Consequently, if a robust mathematics education is one in which participants have opportunities to develop more sophisticated ways of engaging in mathematical activity, then such an education must be one in which participants have opportunities to develop as generalizers.

One hypothesized mechanism for the development of generalizing is Vygotsky's (1978) *zone of proximal development*, which suggests that a potential is created when a learner is engaged as an active collaborator in generalizing activity that is more sophisticated than the generalizing that he or she can engage in alone. Such activity would typically be led by the collaborators who are able to engage in the most sophisticated types of generalizing. It is the goal of college and university teacher education programs to prepare preservice teachers to shoulder the role of the more proficient members of the classroom community and to use their own mathematical knowledge to lead their students' mathematical development. It follows that the knowledge of mathematics teachers should include understanding of the generalizing process so that those mathematics teachers can create developmental trajectories that will help students develop into more capable and sophisticated generalizers. This raises a need for research to understand the nature of the generalizing process, particularly among mathematically proficient students.

Unfortunately, there is little empirical research on the mathematical generalizing of students beyond early high school, and what little we know of generalizing among school students (i.e., students in K-12) has been gathered through research that relies heavily on a fairly narrow range of types of generalizing tasks. One strategy that has been used to study generalizing has been through tasks that present learners with a finite collection of geometric images, numbered in sequence (e.g., the 'Christmas Trees' problem, Stacey, 1989; see Figure 1).

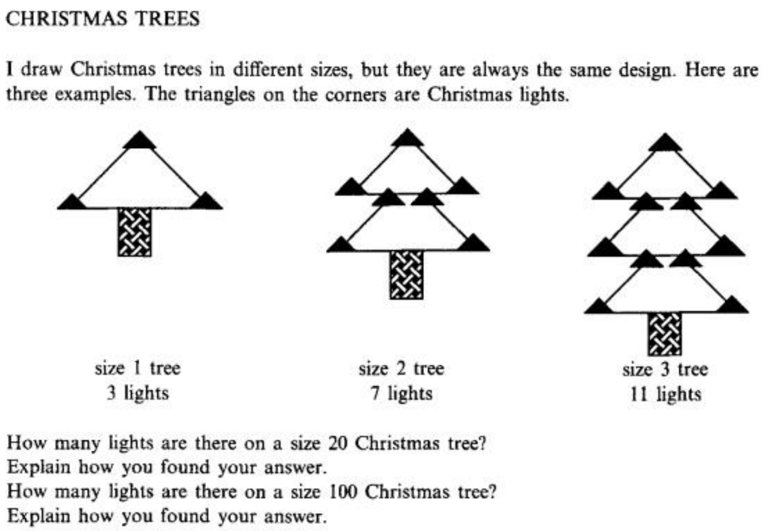


Figure 1: The Christmas Trees problem, Stacey, 1989

Such tasks then ask the learner to make predictions regarding some quantifiable attribute of an image that would appear later in the sequence. To accomplish this task, learners must inductively characterize the given collection and then extend that characterization in order to predict the appearance and attributes of the projected example. Although such inductive-extending tasks have allowed researchers to identify varied strategies that learners use (Stacey, 1989) and to hypothesize the conditions under which such strategies appear (Lannin, Barker, & Townsend, 2006), such findings are necessarily limited to describing and predicting generalizing in similar inductive-extending tasks.

In order to understand the nature of generalizing in its role as an essential mathematical activity, it is necessary to expand the existing body of research to examine the process in a broader range of types of generalizing tasks and among a more mathematically knowledgeable set of participants. The broad questions that frame this study are:

1. What are the strategies that individuals use to respond to generalizing tasks?
2. What explains or predicts the ways that individuals respond to a generalizing task?

Method

To broaden the scope of the empirical research on generalizing, I began by defining generalizing as activity that associates a set of mathematical objects with an ostensible generic claim. I then designed a collection of generalizing tasks that avoided presenting instances as a sequence or that would engage participants in other forms of generalizing activity. The first set of tasks were *inductive-extending* tasks, in which an initial set of examples was presented to a participant without numerical labels, and with the instances drawn from an envelope at random by the participant. The second set of tasks were *populating* tasks, in which participants were provided with a mathematical claim and were asked to identify a collection of examples for which the claim would be true. The third set of tasks, *reconstructing* tasks, presented a generalization about a set of mathematical objects (e.g., any product of four consecutive whole numbers is divisible by 12), then asked the participant to rewrite the claim so that it would be true for a superset of the domain (e.g., for products of three or more consecutive whole numbers).

To further broaden the scope of empirical research, I recruited participants from among preservice secondary mathematics teachers at a large mid-Atlantic University. Although much has been learned about the generalizing of elementary, middle, and early high school students, much less is known about the generalizing strategies of this target population.

I conducted one-on-one task based interviews with five different participants from the target population. After transcribing each interview, I identified distinct data by looking for shifts in the participant's activity. Each shift signified the end of one datum and the beginning of another. In particular, I looked for evidence of 1) a shift in the type of inscriptions that were used to represent examples, 2) changes in the specific examples that were being considered, 3) changes in the attributes that were being examined, or 4) a shift from one type of activity to another (e.g., from comparing two examples to each other to contrasting those examples against each other).

After parsing each transcript, I then coded each datum using an open coding process and began to use those codes to develop 1) descriptions of the attributes of examples that seemed salient to participants, 2) characterizations of the activities that constituted generalizing for the participant within the context of the task, and 3) hypotheses that might explain the characteristics of generalizing and their relationships to the salient attributes of examples.

Findings

Although this research is still in its earliest stages of analysis, initial findings seem to cluster around three key points. First, participants tended to focus on inscription-level relationships within and among the examples that were presented to them. Second, one of the strategies that participants employed was to engage in activities that implied an assignment of status to each example as a representative of the entire collection. These activities resulted in a *centralizing* of some examples as a representative subset of the collection, while *marginalizing* other examples based on a perception of the marginalized examples as dissimilar to the centralized set. For example, when describing the collection shown in Figure 2, Alison characterized the collection based on attributes that she associated with Examples 2, 3, and 4, namely that the figures are all blue in color and that for each Example "you can outline the overall figure with a triangle" (Alison, Interview 1). She also marginalized Example 6 because "you kind of have to use your imagination to see a bigger triangle there" and Example 1 because it was "not a triangle".

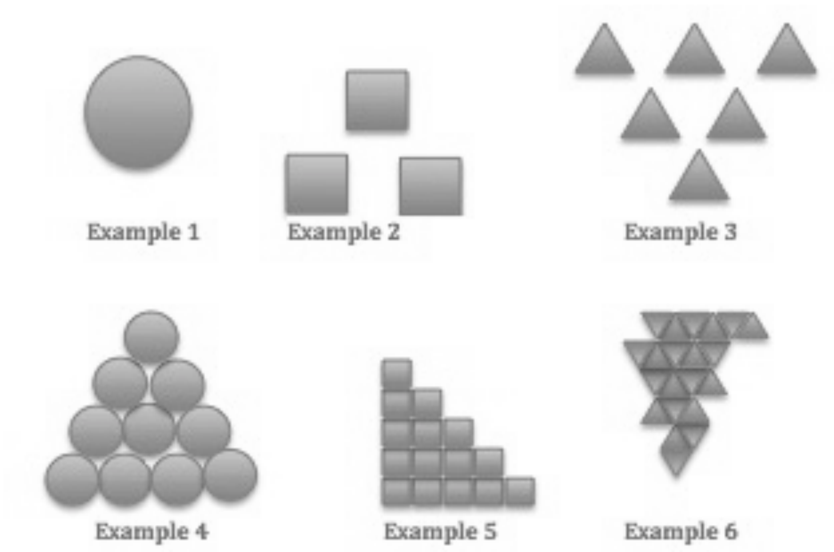


Figure 2: The collection for Alison's inductive-extending task.

A possible explanation for the marginalizing and centralizing phenomena is a desire on the part of the participant to establish homogeneity among the elements of the collection. For instance, while examining the examples from Figure 2, Alison rotated Example 5 to the position shown in Figure 3 because the rotation "gives the illusion of a . . . an equilateral triangle, even though [the orientation shown in Figure 2] has the same . . . organization of blocks, they're just shifted to be a right triangle".

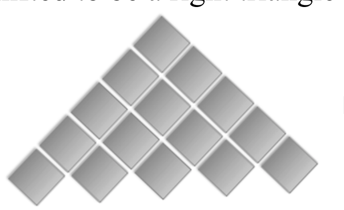


Figure 3: Alison's rotation of Example 5.

Alison's actions and statements suggest that the arrangement of geometric elements is, for her, a powerful attribute of the Examples and that her preference would be to establish homogeneity with respect to those arrangements. This preference for homogeneity may explain why Example 1 and Example 6 are relegated to a marginal status compared to the central collection consisting of Examples 2, 3, 4, and 5.

Questions for Discussion

1. Are there suitable theoretical frameworks that would help to make sense of participants' actions when engaged in populating or reconstructing tasks?
2. What suggestions might the audience have for focusing the coding scheme during subsequent phases of analysis? The open coding has yielded a proliferation of codes, and I am struggling to condense them.

References

Bishop, A. J. (1991). *Mathematics education library: Vol. 6. Mathematical enculturation: A cultural perspective on mathematics education.* Dordrecht, The Netherlands: Kluwer.

Cuoco, A., Goldenberg, E. P., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. *Journal of Mathematical Behavior, 15,* 375–402.

Cuoco, A., Goldenberg, E. P., & Mark, J. (2010). Organizing a curriculum around mathematical habits of mind. *Mathematics Teacher, 103,* 682–689. www.nctm.org/publications/mt.aspx

Harel, G., & Tall, D. (1991). The general, the abstract, and the generic in advanced mathematics. *For the Learning of Mathematics, 11*(1), 38-42.

Lannin, J. K. (2005). Generalization and justification: The challenge of introducing algebraic reasoning through patterning activities. *Mathematical Thinking and Learning, 7,* 231–258. doi:10.1207/s15327833mtl0703_3

Lannin, J., Barker, D., & Townsend, B. (2006). Algebraic generalisation strategies: Factors influencing student strategy selection. *Mathematics Education Research Journal, 18*(3), 3–28. doi:10.1007/BF03217440

Mason, J. (2008). Making use of children's powers to produce algebraic thinking. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 57-94). New York, NY: Erlbaum.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

National Governor's Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Washington, D.C.: Authors. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

National Research Council. (2001). *Adding it up* (J. Kilpatrick, J. Swafford, & B. Findell, Eds.). Washington, D.C.: Author.

Stacey, K. (1989). Finding and using patterns in linear generalising problems. *Educational Studies in Mathematics, 20,* 147–164. doi:10.1007/BF00579460

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes* (M. Cole, V. John-Steiner, S. Scribner, & E. Souberman, Eds.). Cambridge, MA: Harvard University Press.

Effects of engaging students in the practices of mathematics on their concept of mathematics

Duane Graysay
Pennsylvania State University

Shahrzad Jamshidi
Pennsylvania State University

Monica Smith Karunakaran
Pennsylvania State University

Abstract

Two important parts of mathematical proficiency are the individual's understanding of mathematics and his or her self-efficacy beliefs with respect to mathematics. Understanding the nature of mathematics should include recognizing mathematics as a field of inquiry. However, it is not clear how changes in a student's understanding of the nature of mathematics might affect the ways that the student perceives his or her mathematical abilities. We designed a 5-week course around inquiry projects in an attempt to promote a more robust understanding of the nature of mathematics. Using surveys and interviews, we gathered information about students' perceptions of mathematics and about their own mathematical abilities. Our exploratory research suggests that engaging in an inquiry-based course experience helped students to recognize the roles of collaboration and communication in mathematics, but may also have led them to perceive mathematics as inherently more difficult and themselves as less able to communicate mathematically.

Rationale

A goal of mathematics instruction is to help students become mathematically proficient. Two components of proficiency are (a) ideas about *what it means to do mathematics* (which helps one to recognize situations in which mathematical knowledge can be used) and (b) *mathematical self-efficacy* beliefs. Students with low mathematical self-efficacy beliefs are less likely to persist when faced with challenges, and may be less likely to choose STEM-oriented careers (cf. *strategic competence* and *productive disposition,* National Research Council, 2001).

Typical approaches to teaching lower-level mathematics classes involve demonstrating an application of some mathematical fact or procedure to an example, accompanied by students' practice with similar examples. However, it is also important for students to experience mathematics as a field of inquiry**,** in which they learn to think and communicate mathematically by working collaboratively on novel problems (e.g., Goos, 2004). We consider how such an approach might affect both their understanding of mathematics and their perceptions of themselves. We designed and implemented this exploratory project to investigate two questions:

How do exploratory projects designed to loosely mimic mathematical research practices impact students'

- understandings of mathematics, including mathematical research?
- perceptions of themselves as mathematically able?

Method

We developed a 5-week class for pre-calculus high school students that consisted exclusively of projects designed to engage students in the kinds of practices that mathematicians use to solve problems. The goal was for students to learn that doing mathematics includes making sense of problems, posing partial solutions for public critique, collaborating with others to work toward a solution, and formalizing results in the form of a viable argument (cf. *Common Core State Standards for Mathematical Practices*, NGA Center & CCSSO, 2010). We used a survey instrument to examine students' beliefs both before and after the course on (a) the discipline itself, (b) learning mathematics, and (c) their personal abilities to learn the subject. We also interviewed students about their course experiences.

Results

The course experience seemed to correspond with moderate, productive changes in beliefs about the nature of mathematics and about the roles of collaboration in the doing of mathematics. Some students also seemed to develop more detailed descriptions of the work of mathematicians, and appeared to change their attribution of success from *knowledge of mathematical facts and procedures* to *ways of thinking about mathematics*. We also detected negative changes in students' perceptions of the difficulty of mathematics for them and of their own abilities to explain their mathematical thinking to others.

References

Goos, M. (2004). Learning mathematics in a classroom community of inquiry. *Journal for Research in Mathematics Education, 35,* 258-291. doi:10.2307/30034810

National Governor's Association Center for Best Practices, & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Washington, DC: Authors.

National Research Council. (2001). *Adding it up: Helping children learn mathematics* (J. Kilpatrick, J. Swafford, & B. Findell, Eds.). Washington, DC: National Academy Press.

A comparison of self-inquiry in the context of mathematical problem solving

Todd A Grundmeier
Cal Poly, San Luis Obispo

Dylan Retsek
Cal Poly, San Luis Obispo

Dara Stepanek
Central Coast New Tech High School

Self-inquiry is the process of posing questions to oneself while solving a problem. The self-inquiry of thirteen undergraduate mathematics students and one mathematics professor was explored. Student self-inquiry was explored via structured interviews requiring the solution of both mathematical and non-mathematical problems. The professor's self-inquiry was explored through self-reporting of questions asked in an advanced problem-solving context. Using transcripts of the student interviews, a coding scheme for questions posed was developed and all questions were coded. Data analysis of the posed questions suggests that the "good" mathematics students focus more questions on legitimizing their work and fewer questions on specification of the problem-solving task. Data analysis of the professor's self-inquiry is ongoing and will be compared and contrasted to that of the undergraduates.

Key-Words: Problem Solving, Proof, Self-Inquiry, Logic, Questioning

Experts in mathematics do things differently than the masses. It therefore makes sense to rigorously study exactly what characterizes expert mathematical thought, ultimately aiming to transfer this understanding to better educate undergraduates in mathematics. Indeed, much recent work in undergraduate mathematics education has explored this very idea. From how experts read proofs (Inglis & Alcock, 2012) and vet the work of their peers (Inglis, Meija-Ramos, Weber & Alcock, 2013) to how they make conjectures (Belnap & Parrott, 2013) and use metaphors/perceptuo-motor activity (Soto-Johnson, H., Oehrtman, M., Noblet, K., Roberson, L., & Rozner, S., 2012), a clearer picture of expert mathematical practices is beginning to emerge. In this preliminary report the authors begin to add expert "self-inquiry" to the canon.

Many teachers refuse to simply answer a student's question; instead, these teachers insist on responding to the student's misconceptions with other related questions that the student can answer, slowly scaffolding the student's responses until the student has answered (knowingly or unknowingly) their own question. This method, when done correctly, allows the student to recollect related knowledge, receive a confidence boost in their own knowledge of the subject, and receive a lesson in problem-solving strategies that could be utilized to solve future problems.

This method of answering questions with other questions seems to work extremely well for student ownership of material, but the question remains as to why students don't ask themselves some or all of these leading questions. Since the student is capable of answering the posed questions that lead them to the solution, what is stopping the student from posing these questions themselves? Is effective self-inquiry a mark of a "good" student? What types of questions do these "good" students ask themselves while problem solving? More importantly, how can we foster pedagogical knowledge from these "good" students' questions so that teachers can guide all students toward productive self-inquiry?

A detailed initial exploration motivated by these questions (Grundmeier, Retsek & Stepanek, 2013) suggests marked differences between the questioning profiles of "strong", "average" and "weak" students. In order to shed further light on these questions, the authors designed and undertook a similar data collection process wherein a single "expert" recorded his own self-inquiry over an extended period of study on an advanced mathematical topic.

The overarching goal of this study is to compare and contrast these expert questions to those of undergraduate students (Grundmeier et al., 2013). Following an identical coding scheme, ongoing analysis of expert questions posed during the problem solving process will shed further light on what makes for "good" self-inquiry, will test the adequacy and completeness of earlier question coding schemes, and will potentially generate further data collection tools. The coding scheme is represented in the "question tree" below and will be explained in detail during the presentation.

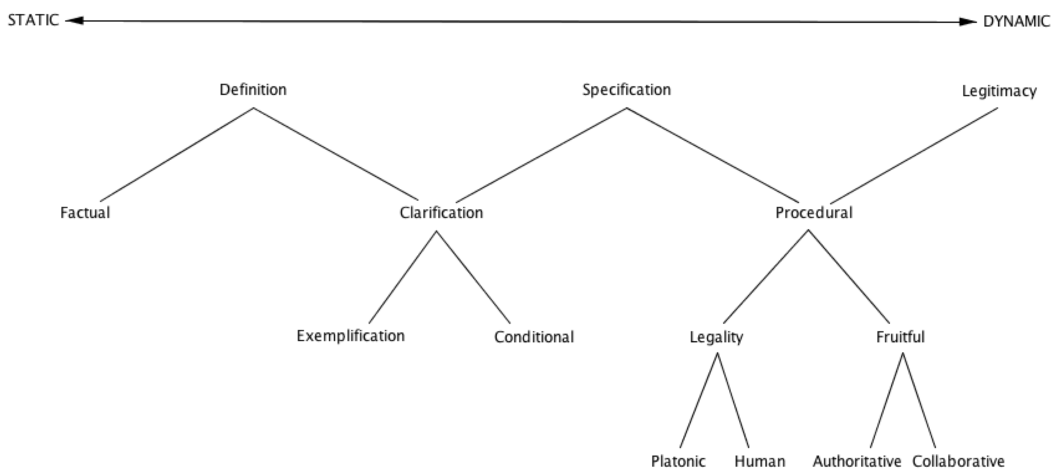


Figure 1: The question tree

While related research has been conducted in secondary education and reading comprehension (Kramarski & Dudai, 2009; King, 1989) and in general mathematical thinking (Schoenfeld, 1992), it seems that the self-inquiry of undergraduates in the process of mathematical problem solving, particularly in comparison to analogous expert self-inquiry, has not been explored. Therefore another goal of this project is to continue this line of inquiry and add to the current mathematics education research related to problem solving.

Further Data Collection

As the main activity in a quarter long sabbatical the participant worked through the majority of the first two chapters of the text *Real Analysis: Modern Techniques and Their Applications* (Folland, 1984). This text was chosen because it is frequently used in graduate course work and would allow for a faculty mentor just in case mathematical questions needed to be referred to a colleague.

The participants' typical plan for working on the material was to carefully read each section of the text while noting questions that arose. He then attempted the problems that had been assigned in a recent Real Analysis course. While attempting to solve each problem the participant would document all questions that arose as well as the time between each question. Many problems required multiple attempts before a solution or proof was reached and questions during each attempt were recorded separately. For example, in organizing the data many headings such as “Section 2.3, Problem #22 attempt #2” appear. The choice to record these questions separately was made for a number of reasons.

First there was often a significant amount of time between attempts, as the participant might have tried another problem in between or needed to sleep on the strategy he was using. Second, the participant assumed there would be overlap between the questions asked which might be important to analyze and discuss. Finally, it may be interesting to determine if the types of questions asked were different after some time subconsciously considering the problem. Working through this process for the first two chapters of the text led to the collection of 404 questions.

Further Analysis

The newly generated data will be analyzed using the coding scheme developed and used by the authors (Grundmeier et al., 2013). The goal of the analysis will be to compare the professor’s questioning to that of our previous research participants, to determine if our coding scheme needs to be refined, to create a detailed description of the participant as a problem solver, and to generate further data collection tools.

Questions for the Audience

1. Are there other potentially fruitful ways to analyze this data set and/or make comparisons of self-inquiry?
2. What other data collection tools or research design options would help explore self-inquiry?
3. Is the refinement of the coding scheme a worthwhile research task?

References

Belnap, J. & Parrott, A. (2013). *Understanding Mathematical Conjecturing*. In S. Brown, G. Karakok, K. H. Roh, and M. Oehrtman, *Proceedings of the 16th Annual Conference on Research in Undergraduate Mathematics Education*, 2013, Denver, Colorado.

Folland, Gerald B. (1984). *Real Analysis: Modern Techniques and Their Applications*, New York: John Wiley & Sons.

Grundmeier, T.A., Retsek, D. and Stepanek D. (2013). A Foray Into Describing Mathematics Majors’ Self-Inquiry During Problem Solving. In S. Brown, G. Karakok, K. H. Roh, and M. Oehrtman, *Proceedings of the 16th Annual Conference on Research in Undergraduate Mathematics Education*, 2013, Denver, Colorado.

Inglis, M. & Alcock, L. (2012). Expert and novice approaches to reading mathematical proofs. *Journal for Research in Mathematics Education*, 43, 358-390.

Inglis, M., Mejia-Ramos, J.P., Weber, K., & Alcock, L. (2013). On mathematicians’ different standards when evaluating elementary proofs. *Topics in Cognitive Science, 5*(2), pgs. 270-282.

Kramarski, B. and Dudai, V. (2009). Group-metacognitive support for online inquiry in mathematics with differentiated self-questioning. *Journal of Educational Computing Research, 40*(4), pgs. 377-404.

King, A. (1989). Effects of self-questioning training on college students' comprehension of lectures. *Contemporary Educational Psychology, 14*(4), pgs. 366-381.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), Handbook for Research on Mathematics Teaching and Learning (pp. 334-370). New York: MacMillan.

Soto-Johnson, H., Oehrtman, M., Noblet, K., Roberson, L., & Rozner, S. (2012). *Experts' Reification of Complex Variables Concepts: The Role of Metaphor*. In S. Brown, S. Larsen, K. Marrongelle, and M. Oehrtman *Proceedings of the 15th Annual Conference on Research in Undergraduate Mathematics Education*, 2012, Portland, Oregon.

A discursive approach to support teachers' development of student thinking about functions

Beste Güçler
University of Massachusetts Dartmouth

Heather Trahan-Martins
University of Massachusetts Dartmouth

Introduction

Function is a critical concept for mathematics and related branches of sciences. Despite its centrality in K-12 and undergraduate mathematics, it is one of the most challenging topics to master in school learning due to the various notions associated with the concept (Eisenberg, 1991). Existing research highlights many challenges students face as they learn about functions. Researchers argue that students often associate function with a single algebraic expression or rule (Vinner & Dreyfus, 1989) and have difficulties with piecewise functions (Sfard, 1992). Students also struggle with conceptualizing function both as a process and a mathematical object (Gray & Tall, 1994; Sfard, 1992) and moving flexibly across graphical, algebraic, tabular, and verbal representations of functions (Monk, 1994; Sierpinska, 1992; Tall, 1996). They have difficulties interpreting dynamic representations and information demonstrating the relationship between two variables (Carlson, 1998; Kaput, 1992; Monk, 1994). Given the multiplicity of notions and complexities inherent in the learning of functions, it is important for teachers to address their students' difficulties in the classrooms. In order to this, teachers need to understand the content and the challenges it presents to students.

Shulman (1986) identified pedagogical content knowledge (PCK) as a combination of content knowledge and pedagogy that is distinct to the teaching profession. Ball, Thames, and Phelps (2008) built on Shulman's work and provided two distinct categories of PCK: (a) awareness of what makes mathematical topics difficult for students to learn, and (b) knowledge of strategies for helping students overcome their difficulties. They pose an important question "whether and how different approaches to teacher development have different effects on particular aspects of teachers' pedagogical content knowledge" (Ball, Thames, & Phelps, 2008, p. 405). The aim of this study is to present one such approach that uses a discursive lens in the context of a post-secondary classroom. The study addresses the following question: How does an instructional approach that promotes prospective and in-service high school teachers to reflect on their own discourses help them develop their thinking about student learning about functions?

Theoretical Framework

This work has theoretical assumptions based on Sfard's (2008) communicational approach to cognition, which underlines the close relationship between thinking and communication. In this theory, thinking is conceptualized as an individual form of communication and mathematics is considered as a discourse that can be characterized by its word use, visual mediators, routines, and endorsed narratives. *Word use* refers to the ways in which participants use words in their mathematical discourse. *Visual mediators* refer to the visible objects created and operated upon for the sake of mathematical communication. *Routines* are the collection of meta-level rules characterizing the repetitive patterns in participants' discourses. *Endorsed narratives* refer to the set of utterances describing mathematical objects and their relationships that the participants of the discourse consider as true. Sfard (2008) views learning as becoming a more fluent participant of mathematical discourse as participants change their previously existing discourses.

There are various reasons why we used a discursive approach to help teachers develop their thinking of student learning on functions. First, it is important for teachers to elicit their students' discourses in the classroom to be able to identify their difficulties and address them. Second, teachers need to be aware of and attend to their own discourses to enhance classroom communication (Güçler, 2013, 2014). This requires that they explicitly discuss the different ways they use mathematical words, visual mediators, routines, and endorsed narratives about a concept for students to see how a concept can be viewed differently depending on the context. Third, in a content course or a professional development setting, it is critical to elicit teachers' own discourses on mathematical concepts rather than directly exposing them to student difficulties about the concepts. Before responding to their students' discourses, teachers need to be aware of their own mathematical discourses. These elements constitute the main principles of the teaching experiment we designed for this work.

Methodology

The main focus of the teaching experiment was to make teachers' discourses on function explicit topics of reflection to help them realize their own thinking about the concept to then help them develop their thinking about student learning of the concept. In this work, teacher development was evidenced by the changes in teachers' discourses in terms of how they talked about the student difficulties about function and the teaching strategies that can be used to address those difficulties. Consistent with the theoretical assumptions of this work, we aimed to elicit the student difficulties about functions through the teachers' own discourses. In other words, we initially considered the teachers as learners to examine how they thought about functions. We then kept a record of all the difficulties the teachers faced when working on function-related problems and made those instances explicit topics of discussion in the classroom for teachers to reflect on their own discourses. Through these experiences, the teachers explicitly talked about how they used words, visual mediators, and routines to endorse particular narratives about functions. It was only after these experiences that the teachers were asked to think about possible student difficulties and how they would address those in their classrooms. The activities used in the classroom were developed in light of the research on student difficulties to enhance teachers' thinking about different conceptualizations and representations of functions. The goal was to use those activities to bring forth and reflect on teachers' discourses to support the development of their thinking on student learning about the concept.

The participants were one pre-service and seven in-service high school teachers, hereon referred to as teachers, taking a content course on calculus to meet the criteria for their initial or professional licensure program. Except for the pre-service teacher, who had no prior teaching experience, the participants' experiences ranged between 4 to 12 years. The researcher was the instructor of the course. Two graduate students assisted the researcher during data collection and analysis. Although the larger study explored teachers' discourses on all calculus concepts over the course of 13 weeks, this paper focuses on functions. The study followed a teaching experiment methodology as outlined by Steffe and Thompson (2000).

For functions, the data consisted of an initial survey given to the teachers, three video-taped classroom sessions (lasting 2.5 hours each), weekly homework assignments, weekly journal reflections collected from the teachers, and audio-taped interview sessions conducted at the end of the semester. The interview sessions and classroom discussions during which the teachers talked about functions were transcribed. The transcripts were independently and then collectively examined to identify how the teachers' discourses on student thinking changed during the course in relation to their reflections on their own discourses on functions.

Results

In the initial survey, which was administered at the beginning of the first lesson on functions, the teachers were asked to define what a function is. Their definitions suggested that the majority of the teachers viewed function as a process that takes any given input and produces one unique output. Two teachers endorsed mathematically incorrect narratives about function (e.g., it has to be continuous or one-to-one) and four teachers used phrases they associated with function (e.g., "for every input, there is only one output", "passes vertical line test") rather than defining what a function is. Two teachers considered function as the same thing as one of its representations by defining it as a graph or algebraic rule. These responses indicated that some of the student difficulties identified by the literature were present in the teachers' discourses at the beginning of instruction. In the survey, the teachers were also asked to list some student difficulties about functions that they experienced or anticipated. Table 1 shows the written responses they provided for this question (the names used in the study are pseudonyms).

Table 1: Survey responses to experienced/anticipated student difficulties with functions

Teachers	Survey Responses
Carrie	[1] Students have difficulties with trig functions and inverse trig functions because they are less familiar
Fred	[2] Inputs of non-integer values and how to conceptualize behavior of functions without reaching for graphing calculator
Lea	[3] Understanding of dependent variable vs. independent variable; graphs of different functions; evaluating functions
Martin	[4] Displaying and modeling
Milo	[5] ?
Ron	[6] Student interchange the input and outputs
Sally	[7] One to one and onto
Steve	[8] The closeness to an equation/what is really different about the two

The responses in Table 1 demonstrate that, when thinking about student difficulties, the teachers often talked about particular actions to be performed on functions such as evaluating [2-3], interchanging the input and output [6], and displaying and modeling [4] without explaining why those would be challenging for students. One teacher could not think of any student challenges as indicated by the question mark he provided in his response [5]. Some teachers mentioned students' lack of familiarity with particular types of functions [1] and their problems with various characteristics of functions such as one-to-one and onto [7]. Despite the significant years of teaching experience of these participants, only a few responses touched on some of the conceptual challenges with functions as indicated by the literature such as struggles with conceptualizing the behavior of a function [2], lack of understanding of independent and dependent variable [3], and confusion about an equation and a function [8].

The activities in the classroom gave teachers ample opportunities to think about their discourses on function and how the narratives they endorsed about the concept were regulated by their word use, routines, and visual mediators. For example, there were activities during which the teachers reflected on their own definitions of function as well as the different definitions formulated by mathematicians over the course of the historical development. These experiences enhanced teachers' awareness of the elements of their discourses on function and how different views of the concept are based on different utilizations of those elements. Through the classroom discussions, the teachers recognized that their initial endorsed narratives about a function as consisting of a process or graph were based on the routine of using the metaphor of continuous motion, which was supported by their use of graphs as visual mediators. They were able to juxtapose these elements of their discourses with other endorsed narratives about function such as the Bourbaki definition, which views function as consisting of a set of ordered pairs through the routine of using the metaphor of discreteness, which was supported by the use of algebraic representations as visual mediators. Similar findings were observed when teachers worked on activities that were focused on the representations of functions. The foci when discussing the various representations of functions were to examine whether a function is the same thing as its representation (e.g., is a function a graph/an algebraic rule?) and to think about the similarities and differences across multiple representations of functions. The teachers had opportunities to think about the verbal, tabular, graphical, and algebraic representations of functions by modeling real-life problems and using dynamic geometry software. During these activities, the teachers had to work with atypical and piecewise functions and realized that they often considered functions as continuous, linear, and consisting of a single rule. They compared and contrasted the representations based on a static view of functions (e.g., tabular representations as a set of ordered pairs) with those that are based on a dynamic view (e.g., graphs). All these instances helped the teachers (a) recognize their own difficulties about functions, (b) reflect on their own discourses on function, and (c) recognize the dynamic relationship among the different elements of their discourses and the importance of explication of those elements to enhance mathematical communication in the classrooms.

Teachers' weekly journals and responses to the interviews revealed that, they were able to identify many of the conceptual challenges associated with function at the end of their instruction. Some of their responses are presented in Table 2.

Table 2: Responses to anticipated student difficulties with functions

Teacher	Response	Source
Carrie	[9] The idea of function as a single rule but it can be multiple rules. Piecewise functions would pose a big issue [for students] as well.	Interview
Fred	[10] Linear functions cloud student perceptions about all functions.	Journal entry
Lea	[11] They would struggle with seeing functions as coming in all shapes and forms and to recognize each of them as being about a function. Also looking at function as a process and not just as a number coming out as the output. Translations from the verbal to the graphical and algebraic representations would be useful.	Interview
Martin	[12] Students have a tendency to assume continuity despite not being told or shown that a function is continuous. Students seem to get fixated on the notion of a single rule.	Journal entry
Milo	[13] The elaborate [formal] definition of function wouldn't mean anything to a lot of students unless it fits to their existing definitions.	Interview
Ron	[14] Students have issues with functions with split domains because they have a hard time with the concept that a function can change in pattern. They have difficulties with constant functions...and can't seem to understand a function doesn't necessarily have to change.	Journal entry
Sally	[15] Students consider a function just the equation or graph or table.	Journal entry
Steve	[16] Just looking at the procedures, like the vertical line test, without understanding the definition of the concept.	Interview

In Table 2, the teachers talked about the conceptual challenges with functions and the difficulties they identified were consistent with those in the literature, which contrasted sharply with the responses they provided in Table 1. Table 2 indicates that, at the end of their instruction on functions, the teachers recognized the student difficulties related to piecewise functions and the view of function as a single rule ([9-10], [12], [14]); translations across different representations [11]; associating function with a given representation [15]; and the role of the definition ([13], [16]) when thinking about functions. They also highlighted student challenges regarding change and variation [14]; the use of the metaphor of continuous motion [12]; and the views of function as a process (e.g., mapping) and a mathematical object (e.g., a number) obtained at the result of that process [11]. The interview responses suggested that the teachers were able to provide explanations when asked to elaborate why those difficulties occurred. When asked how they knew about those difficulties, they often referred to their own struggles since many of them demonstrated the difficulties in Table 2 in the classroom.

The responses in the weekly journals and interviews also indicated that many of the teachers considered the activities and discussions they engaged in the classroom as resources they would use to address their students' difficulties. For example, Ron said he would use the classroom activity in which they examined each other's definitions of functions in his classroom. Milo noted that he benefited from the examination of the historical development of function and wanted to use it as a resource to enhance his students' thinking about definitions of functions. Sally considered working on different representations in the classroom as a useful way to teach functions to her students. Fred mentioned that he would now ask his students to keep journals.

Discussion

The results of the study indicate that making teachers' discourses on function explicit topics of reflection in the classroom supported the development of their thinking about student learning. It also helped them think about possible teaching strategies for helping their students. These two aspects of the teachers' development were in alignment with Ball, Thames, and Phelps' (2008) notion of PCK that includes awareness of what makes mathematical topics challenging for students and strategies to address those. In addition, the pedagogical approach used in the study helped teachers realize their own struggles with the function concept, which is critical for enhancing their content knowledge. These findings indicate that teachers' deep exploration of their own mathematical discourses can be a useful approach for them to think about their own knowledge, practice, and students.

The teachers' consideration of the activities and discussions they engaged in the classroom as useful resources for their own students suggests that they recognized the importance of discursive transparency in enhancing mathematical communication. For such transparency, the teachers do not only need to elicit their students' mathematical discourses in the classroom but also explicitly attend to the elements of their own discourses. Otherwise, those elements could remain implicit in the classroom, leading to miscommunication. Indeed, the teachers in the study often mentioned that they never learned about the connections among the different elements of the discourse on functions in their prior education. Such tacitness was possibly one of the reasons why, despite their experience, they showed difficulties with various aspects of the concept.

The discursive approach used in this study is certainly not the only one that can help teachers enhance their thinking about student learning. On the other hand, rather than treating teachers' knowledge as static entities and focusing on the teachers' limitations (e.g, lack of knowledge, insufficiency of cognitive schema), it characterizes teacher learning as a contextual, dynamic, and ongoing process of developing particular types of discourses to enhance participation in mathematical discourse.

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching what makes it special? *Journal of Teacher Education*, *59*(5), 389-407.

Carlson, M. P. (1998). A cross-sectional investigation of the function concept. In A. H. Schoenfeld, J. Kaput & E. Dubinsky (Eds.), *Research in Collegiate Mathematics Education. III* (pp. 114-162), CBMS Issues in Mathematics Education Vol. 7. Providence, Rhode Island: American Mathematical Society.

Eisenberg, T. (1991). Functions and associated learning difficulties. In D. O. Tall (Ed.), *Advanced mathematical thinking* (pp. 140-152). Dordrecht, The Netherlands: Kluwer.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education*, *26*, 115-141.

Güçler, B. (2013). Examining the discourse on the limit concept in a beginning-level calculus classroom. *Educational Studies in Mathematics*, *82*(3), 439-453.

Güçler, B. (2014). The role of symbols in mathematical communication: The case of the limit notation. *Research in Mathematics Education*. Advanced online publication. doi: 10.1080/14794802.2014.919872

Kaput, J. (1992). Patterns in students' formalization of quantitative patterns. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy*, MAA Notes, *Vol.* 25 (pp. 290-318). Washington DC: Mathematical Association of America.

Monk, G. S. (1994). Students' understanding of functions in calculus courses. *Humanistic Mathematics Network Journal*, *9*, 21-27.

Sfard, A. (1992). Operational origin of mathematical objects and the quandary of reification – the case of function. In E. Dubinsky & G. Harel (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 59–84). Washington, DC: Mathematical Association of America.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses and mathematizing*. New York: Cambridge University Press.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.

Sierpinska, A. (1992). Theoretical perspectives for development of the function concept. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy*, MAA Notes, *Vol.* 25 (pp.23-58). Washington DC: Mathematical Association of America.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research Design in Mathematics and Science Education* (pp. 267-307). Hillsdale, NJ: Erlbaum.

Tall, D. (1996). Functions and calculus. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick and C. Laborde (Eds.), *International Handbook of Mathematics Education*, (pp. 289–325) Kluwer Academic Publishers, Dordrecht.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, 356-366.

Students’ understanding of concavity and inflection points in real-world contexts: Graphical, symbolic, verbal, and physical representations

Michael Gundlach
Brigham Young University

Steven R. Jones
Brigham Young University

Much of what we know about student understanding of concavity and inflection points comes incidentally from studies looking at the calculus activity of sketching the graphs of functions. However, since concavity and inflection points can be useful in conveying information in disciplines like physics, engineering, biology, and economics, it seems important to study how students understand these two concepts in these contexts. This study attempts to provide insight into this area.

Key words: calculus, concavity, inflection point, applications, representations

Concavity and inflection points are important characteristics of a function’s “behavior,” and are key elements in the calculus activity of sketching the graphs of functions (see Baker, Cooley, & Trigueros, 2000). In fact, most of the insight on student understanding and usage of concavity and inflection points comes incidentally from research on this particular activity (e.g., Baker et al., 2000; Berry & Nyman, 2003; Carlson, 1998). However, these two concepts have much potential usefulness in disciplines outside of mathematics, such as in physics, economics, biology, and engineering. It is important to know how science-bound calculus students might make meaning out of concavity and inflection points in these contexts.

Carlson, Jacobs, Coe, Larsen, and Hsu (2002) discussed inflection points in real-world contexts by examining how students were able to model the height of water in a bottle over time and by having students reconstruct a temperature graph based on its rate of change graph. They noted that several students had difficulty conceptualizing a changing rate of change. The students who were more successful were able to use “covariational reasoning” in order to track these changes. Since it is important to connect mathematics education to real-world contexts (President's Council of Advisors on Science and Technology, 2012), we propose to build on this work by examining students’ understanding of concavity and inflection points in a wider variety of applications.

Four representations for concavity and inflection points

Some studies have documented difficulties students have in working with concavity and inflection points (e.g., Baker et al., 2000; Carlson et al., 2002; Tsamir & Ovodenko, 2013), but each study has typically focused on only certain representations. For example, Tsamir and Ovodenko (2013) focused mostly on the symbolic and visual representations and noted that students struggled in applying logic to statements about inflection points. Carlson et al. (2002), on the other hand, focused on covariational reasoning and showed that is quite useful for making sense of concavity and inflection points. In order to talk about a variety of representations at once, and to have a common language for these various representations, we employ the names of four representations used in Zandieh (2000), which she laid out to discuss students’ understanding of the derivative. In particular, we use the categories of *symbolic*, *graphical*, *verbal*, and *physical* representations for concavity and inflection points.

Graphical: In this representation, concavity and inflection points are considered in a purely visual manner, with the salient feature being the “shape” of a graph. This might be communicated through the use of familiar images, such as bowls, smiley faces, or parabolas. An inflection point may be seen *graphically* as the linking of two such images.

Symbolic: Concavity and inflection points can both be represented abstractly through mathematical symbols. A student may consider whether the second derivative is positive or negative, $f''(x) > 0$ or $f''(x) < 0$, to decide whether a function is said to be concave up or down. A change in the *sign* of the second derivative may indicate an inflection point.

Verbal: In a way similar to Zandieh's derivative framework (2000), we use *verbal* to indicate the interpretation of concavity as the "rate at which a rate of change changes." This is essentially the same as Carlson et al.'s (2002) covariational reasoning, in which a student discusses a changing rate of change. By extension, the inflection point is the *switching* of this rate of change in the rate of change.

Physical: It is possible to think about concavity in terms of familiar physical phenomena, such as acceleration, or something akin to force. Note that while this category is labeled "physical," it does not necessarily mean that any time a student discusses concavity or inflection points in a real-world context that they are working within this representation. Rather, it requires the conception and usage of some well-known, intuitive real-world phenomenon that captures, for the student, the essence of the concavity or inflection point.

In this paper, these four categories of representation are meant to capture different possible aspects of understanding concavity and inflection points, though they might not necessarily capture them all. Also, these categories are not mutually exclusive; it is quite possible (and even likely) that a student would draw on multiple representations in order to make sense of concavity.

Methods

For this study, a group of eight, randomly chosen students from a large-lecture, first-semester calculus course at a major university in the United States were asked to participate in three 45-minute task-based interviews (Goldin, 1997) regarding calculus concepts. One of the interviews, which centered on concavity and inflection points, is used for this study. To focus on how students made sense of these two concepts in real-world contexts, the first four interview items were based on physics, economics, and biology examples. For the first item, we wished to elaborate on the temperature task given in Carlson et al. (2002), by providing an open ended prompt to the students that "there is an inflection point in air temperature." By doing so, it takes the focus away from the activity of graphing and places it on students' informal understanding. The second item asked the students to consider an inflection point in housing prices during the economic recession following the year 2008. The third item asked the student to describe the concavity of a person's height over their life time. And the fourth item asked the students to describe the expansion of the universe based on information given symbolically about the second derivative of a function modeling the universe's size. The fifth item was a traditional calculus graphing problem, given to the students in order to have a baseline reading of how they understood concavity and inflection points in a pure mathematics context. We analyzed the data by coding student statements based on the four different representations listed above. We then examined each of the four categories for patterns in how the students drew on them to make sense of the real-world contexts.

Results

In the following subsections, we describe the students' usage of each of the four representations. We also provide results on typical difficulties that arose with certain representations.

Graphing: All eight students, not surprisingly, drew heavily on the graphical representation and were often able to use it to effectively discuss concavity and inflection points. For example, three of the students used the image of an upward-facing bowl or an upside-down bowl to determine whether a graph was concave up or concave down (cf. Baker et al., 2000). We label these visual cues "*image pairs*," since they always consisted of complementary images for concave up and concave down. Other image pairs the students used in determining concavity included smiles and frowns, hills and valleys, a “U” and an “n,” and upward and downward facing parabolas. Note that even though a parabola potentially has other layers of meaning, the students in this study seemed to use “parabola” mostly for its “U-shape.”

The students located inflection points often using the same visual cues as concavity. They typically looked for places where the complementary image pairs were “linked” together.

> *Ryan*: I’d look and try to figure out where, like, there are two parabolas coming together, so like I see this thing right here, and I see another bigger one right there, and I look, like, for where they intersect.

While the graphical representation of concavity and inflection points was a comfortable context for the students to work in overall, the reliance on the familiar images of bowls, hills, smileys, and parabolas did occasionally lead students toward incorrect interpretations regarding the nature of concavity and inflection points. First, four of the eight students stated at least at one point in the interview that if a graph was concave up it would have to have a minimum and if it was concave down it would have to have a maximum. For example, while discussing an inflection point in the temperature, Doug said the following:

> *Interviewer*: So, in your opinion, would an inflection point always be associated with a max and a min on either side, or can you have an inflection point that doesn’t necessarily have maxes and mins on either side?
>
> *Doug*: I think in my opinion it would, it would be, it would need to be based on, like, a maximum or a minimum. At least, just to make sense in my mind, so that you can know something is concave up or concave down, because those always have a maximum or a minimum.

This seems related to the fact that six of the eight students claimed at least once during the interview that an inflection point must be where the slope was the steepest, which is only true for certain types of inflection points. When an inflection did not occur at this location, the students were often more hesitant. As an example, Kylie generally seemed less confident about inflection points that were on the *least* steep part of the graph, often calling them less “defined” than inflection points that occurred on the steepest part of the graph. She said the following while thinking through the expansion of the universe over time.

> *Kylie*: If the concavity’s down, but it’s still increasing, then it’s going to be… [draws left half of the graph in Figure 1]. Now concavity’s changed to up, so the curve will look something like this [draws right half of the graph]. And this is an inflection point, not a very defined one [puts a dot on the graph].

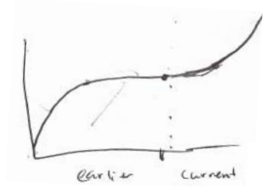


Figure 1: Kylie’s “not very defined” inflection point

Next, this reliance on familiar, curved shapes to depict concavity led some students to make assertions in conflict with common sense for certain real-world contexts. For example, Doug was using the idea of concavity to sketch a graph of a person’s height over their lifetime, and the “hill” image caused him to think that any concave down part of the graph would necessary have to decrease from a maximum before switching to concave up.

> *Doug*: I guess, in my mind, everybody either like, increases over time and then they reach like a certain peak in their life, and then they go back down. But I guess there are like, certain people where maybe they, maybe they reach a growth spurt in high school, shrink a little bit in college, maybe a little bit, maybe grow back in their twenties or so. And I guess maybe that would change the concavity of it.

Doug's "hill" shape for concavity was strong enough that he forced reality to fit his picture of a concave down portion of a graph. This created the strange idea that people must shrink at various times in life in order to accommodate the concavity inherent in his mathematical model. Six of the eight students made some sort of unrealistic statement about a physical context in order to preserve their strong, familiar "U-shaped" images of concavity. This resonates with other studies that have discussed the potentially problematic nature of visual representations in student thinking (e.g., Aspinwall, Shaw, & Presmeg, 1997).

In general, and not surprisingly, the graphical representation for concavity and inflection points was the most prominent in student thinking. Some students, when asked to think about inflection points using alternate representations, insisted that it would be best to just find a way to turn it into a graph and find the inflection points that way. One student adamantly exclaimed, "Why wouldn't you make it into a graph?" Overall, while the visual representation can be powerfully useful in student thinking, it appears that, without care, it can also lead toward difficulties as well.

Symbolic: The students in this study were largely comfortable working within the symbolic representation. Six of the eight students had no difficulty in determining concavity based solely on the sign of the second derivative and using the second derivative to make correct interpretations. However, for one student, the sign of the second derivative seemed automatically associated with either maximums and minimums, or the fastest or slowest growth—essentially bypassing concavity altogether.

> *Nathan*: The second derivative is less than zero. That indicates that there's going to be a maximum. And then, for, it's greater than zero, then, it indicates there's going to be a minimum.

Nathan seems to have internalized the results of the "second derivative test" as becoming the actual *meaning* of the second derivative. This potential conflation of meaning suggests the need for students to carefully construct what the second derivative test is and why it works. Yet, despite these occasional difficulties, the students were generally successful in working within the symbolic representation.

Verbal: In moving from the graphical and symbolic representations, we see a critical shift in the students' ability to work within the verbal representation, in that they had difficulty in accurately describing the rate at which the rate of change changes (cf. Carlson et al., 2002). This difficulty became pronounced in the various real-world scenarios. In particular, students seemed much more prone to confuse inflection points with maximums and minimums in the real-world contexts than in the pure mathematics graphing item. For example, Gavin struggled to describe what an inflection point meant, saying at first it was a point where the graph "stops increasing at such a rate, and starts decreasing at such a rate." In fact, when asked about an inflection point in the context of temperature, five of the eight students stated it would be a point where the temperature reached either a maximum or minimum.

> *Ryan*: That is, that'll be where it, it changes from like, getting hot to getting cold. So if you're looking at, if you're looking at, like, the temperature, and it's getting hotter, hotter, hotter [draws the left half of a "parabola" shape, see Figure 2], it's not getting as hot anymore, and then it'll start getting colder [draws the right half], this would be the inflection point [puts a dot at the maximum, see Figure 2].

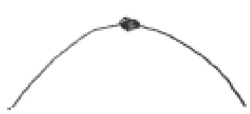


Figure 2: Ryan's "inflection point" at the maximum

In another example, students were given an item describing the beginning of a sharp decline in housing prices in 2008 and were told that an economist claimed there was an inflection point in housing prices in 2009. When asked to interpret what that inflection point in housing prices would mean, three of the eight students claimed that it would be a minimum.

> *Interviewer*: If you were living in 2009, what would tip you off that, okay, an inflection point just happened in the housing prices right there?
> *Nathan*: Um, realize the housing prices were up. That they would start to rise. The value of your house has started to rise.

In general, this tendency to conflate inflection points with maximums and minimums was a recurring theme across the interview items involving real-world contexts.

Physical: Little research has been done on how students draw on a physical representation of concavity and inflection points, so we relate some of the ways in which our students did so. Six of the eight students drew on some sort of physical representation during the interview, which was done most commonly by rendering concavity as the acceleration of the quantity in question. This includes the "acceleration" of temperature, the "slowing down" of a person's height, or the "speeding up" of the universe's expansion. Occasionally, a student would employ a distinctive physical metaphor to think about concavity, such as cold and warm winds (for the temperature item) or emotional optimism (for the economics item).

> *Interviewer*: If you were to describe [the inflection point], how would, what would sort of be your everyday language way of saying what's going on?
> *Ryan*: I would say that the future is optimistic, I guess. Like, things have been bad, but now we think things are going to turn around and start getting better.

Yet, like the verbal representation, there was increased difficulty for the students when drawing on the physical representation. In fact, once again, much like with the other three representations, the students at times mixed up inflection points with maximums and minimums when working within the physical representation.

> *Interviewer*: What would you be experiencing as far as temperature goes at that inflection point?
> *Camille*: It would be suddenly feeling a cold breeze and suddenly feeling, like, a warm breeze.
> *Interviewer*: Okay, so it would switch kind of from cold to hot, sort of a thing?
> *Camille*: Yeah.

Overall, we observed that the physical representation was also difficult for students to think and reason with. In fact, it did not seem to be a major driver in any of the student's thinking, but was mostly a way of embellishing a previously used graphical or symbolic representation. And while the acceleration notion was effectively drawn on at times, it appears that the students had not had the opportunity to create strong physically-based meanings for concavity and inflection points.

Discussion

The students in this study were able to draw on a variety of representations to think about concavity and inflection points in real-world contexts. Many students often drew on more than one at a time. Yet, one striking theme that appeared throughout all contexts and representations was that students tended to conflate concavity and inflection points with maximums and minimums when considering real-world scenarios. Students often thought of maximum temperatures or minimum housing prices when explaining inflection points. This result adds another piece of evidence in support of the need for students to develop strong covariational reasoning (Carlson et al., 2002). Not having this reasoning seemed to lead students toward mixing up what the "change" was that they were looking at. That is, it seems possible that they knew concavity and inflection points dealt with a "change" of some sort, but without being able to think through a change in how the rate of change changes, these students may have confused "how the rate of change changes" with simply "how the rate changes." If this is the case, it would explain why students tended to think of an inflection point as merely implying a switch from increasing to decreasing, or vice versa, instead of implying a change in how the *rate of change* is changing.

In addition, the visual representation had a peculiar difficulty associated with it, in that it seemed to lead students to rely heavily on a full "bowl" shape, that must both decrease and increase. This led some students to make illogical conclusions regarding the real-world contexts they were discussing. It may be that the students were *over-relying* on this representation, since it is what is most commonly used during instruction (see Baker et al., 2000). In either case, it appears that relying too much on a specific visual image may impact how students think about these important concepts (cf. Tsamir & Ovodenko, 2013).

Overall, these students did not seem equipped to apply their knowledge of concavity and inflection points to real-world contexts. In order to support students' usage of their calculus-based knowledge in these contexts, our results suggest, along with Carlson et al. (2002), that we may need to increase classroom attention given to concavity and inflection points *outside* of the classical graph-sketching activity. In addition, we propose that including contexts outside of mathematics, such as physics, economics, or biology, may help students construct covariational reasoning in a way that enables them to think through these types of scenarios using the verbal or physical representations more successfully. In doing so, we believe students will be better able to use their knowledge of concavity and inflection points to understand the real-world phenomena they encounter outside of the calculus classroom.

References

Aspinwall, L., Shaw, K. L., & Presmeg, N. C. (1997). Uncontrollable mental imagery: Graphical connections between a function and its derivative. *Educational Studies in Mathematics, 33*(3), 301-317.

Baker, B., Cooley, L., & Trigueros, M. (2000). A calculus graphing schema. *Journal for Research in Mathematics Education, 31*(5), 557-578.

Berry, J. S., & Nyman, M. A. (2003). Promoting students' graphical understanding of the calculus. *The Journal of Mathematical Behavior, 22*, 481-497.

Carlson, M. (1998). A cross-sectional investigation of the development of the function concept. In A. Schoenfeld, J. Kaput & E. Dubinksy (Eds.), *Research in collegiate mathematics education III* (pp. 114-162). Providence, RI: American Mathematical Society.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-378.

Goldin, G. A. (1997). Observing mathematical problem solving through task-based interviews. *Journal for Research in Mathematics Education. Monograph, 9*, 40-62.

President's Council of Advisors on Science and Technology. (2012). *Engage to excel: Producing one million additional college graduates with degrees in science, technology, engineering, and mathematics.* Washington, DC: The White House.

Tsamir, P., & Ovodenko, R. (2013). University students' grasp of inflection points. *Educational Studies in Mathematics, 83*(3), 409-427.

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. In E. Dubinksy, A. Schoenfeld & J. Kaput (Eds.), *Research in collegiate mathematics education IV* (pp. 103-127). Providence, RI: American Mathematical Society.

Undergraduate students' understandings of functions and key calculus concepts

CAROLINE J. HAGEN
TUFTS UNIVERSITY

1. Problem Statement

The transition into university-level mathematics is a critical juncture in the education of future science, technology, engineering, and mathematics (STEM) professionals. As student retention in STEM _fields remains far too low, particularly amongst underrepresented groups such as women and people of color (PCAST, 2012), there is a need to improve student learning outcomes in the courses that often push students out of STEM _fields, particularly introductory calculus. That even high-performing calculus students demonstrate weak understandings of key calculus concepts (e.g., Carlson, 1998; Selden et al., 2000) further suggests that work needs to be done to foster more robust learning in undergraduate calculus. In addition, functions are fundamental objects of study in mathematics, and research has shown that strong understandings of functions support many kinds of mathematics learning. To date, not enough is known about students' knowledge of mathematical functions and how this may influence their learning of key concepts of introductory calculus such as limits and rates of change. This study helps address this research gap by examining beginning calculus students' ideas about functions, as well as their ideas about limit and rate of change. We will share selected examples of students' work to illustrate student thinking and will discuss the implications of the study results on teaching calculus at the undergraduate level.

2. Theoretical Framework

A widely accepted idea is that students should understand the conceptual basis of mathematics, rather than only knowing procedures (Hiebert & Carpenter, 1992). In other words, computational proficiency, while valuable, is not on its own a suficient goal of mathematics education. Furthermore, computational proficiency often does not mean that students actually understand the mathematical concepts that underlie the procedures they use (e.g., Wearne & Hiebert, 1988), nor does it mean that students are able to determine in which novel situations procedures can be appropriately used (e.g., Lave, 1988; Stigler & Baranes, 1988). Carlson (1998) and Selden et al. (2000) found that even high-performing undergraduate calculus students possess weak understandings of the key ideas of calculus such as limits and derivatives. However, calculus curricula in the United States (US) have not changed substantially since the 1950s (Bressoud et al., 2013). The focus of calculus instruction, by and large, continues to be on developing procedural proficiency in computing limits, derivatives, and integrals, rather than fostering conceptual understanding of these ideas.

This study draws specifically on existing work that provides a foundation for understanding student reasoning and the development of learning about functions (e.g., Carlson, 1998);

student performance in and perceptions of introductory calculus courses (e.g., Bressoud et al., 2013); and development of student understanding of the structure of mathematics (e.g., Richland, Stigler, & Holyoak, 2012; Vergnaud, 1996). Core ideas from these frameworks are that students hold varied and dynamic conceptions of mathematical phenomena (including functions, limits, and rates of change) rather than unitary, static conceptions of such phenomena; that the particular conception of a mathematical phenomenon that students employ when solving a problem is likely dependent on the context of the problem (e.g., Wagner, 2006); and that the view of functions a student takes will likely have an impact on how they reason about calculus problems that deal with functions, such as finding limits and derivatives. These frameworks suggest that student learning of calculus concepts may be strengthened by fostering strong and flexible conceptions of mathematical functions.

3. Methodology

The primary research question of this study is: How do beginning students' abilities to work with functions interact with their abilities to work with basic notions of limits and rates of change? We report on first-semester undergraduate calculus students' responses to: (a) a written function assessment that asked them to reason about functions presented in varied contexts and representations, (b) a question about rates of change on a homework assignment, and (c) a final exam question focused on limits. In order to examine students' ideas about functions, we administered a 13-item written assessment to 23 of the 28 undergraduate students enrolled in an introductory calculus course at a research university in the northeastern US. This assessment was offered to all 28 enrolled students at the beginning of the semester and asked students to reason about functions across different representations (e.g., algebraic formulae, graphs, natural language) and with different function types (e.g., linear, polynomial, periodic). In order to examine students' ideas about key calculus concepts, we analyzed responses to selected homework and final exam questions from all 28 students enrolled in the course. The limit question from the final exam (Figure 1) required students not only to compute a numerical response, but also to draw a graphical representation. The rate of change homework problem (Figure 2) required students to reason about rate of change of height of water in a bottle in the absence of an explicit formula.

(6) (12 pts) For each of the following, compute $\lim_{x\to a} f(x)$ and draw a picture of the function $y = f(x)$ near the location of the limit:

$(a) \lim_{x\to 2} \frac{x^2-4}{x-2}$ $\quad$ $(b) \lim_{x\to\infty} \frac{3x^2-4x+1}{4x^2-3}$ $\quad$ $(c) \lim_{x\to 3} \frac{1}{(x-3)^2}$

Figure 1. Final exam limit question.

4. Results

Table 1 shows the distribution of scores on the function assessment from all 23 participating students. Of the 13 items on this assessment, the mean number of items answered correctly was 7.6, which is just under 59% of questions correct. The highest score was 12 (92% of

Suppose that water is being poured into a spherical bottle. Sketch a graph of the *height* of the water in the bottle as the *amount* of water that has been poured into the bottle changes.

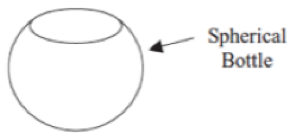


Figure 2. Rate of change homework problem (adapted from Carlson et al., 2010).

questions correct) and the lowest was 5 (38% of questions correct). These data suggest that students arrived to this introductory calculus course with varied understandings of functions. The significant numbers of students (39% of respondents) who answered fewer than six questions correctly (less than half of the questions on the assessment) suggests that the function concept is challenging for many students, and that instructors should not assume that students arrive to undergraduate calculus courses with strong understandings of functions.

Table 1. Distribution of total scores on written function assessment.

Score (out of 13 possible points)*	Number of students receiving this score ($n = 23$)
≤ 4	0 (0%)
5	5 (22%)
6	4 (17%)
7	3 (13%)
8	3 (13%)
9	2 (9%)
10	4 (17%)
11	1 (4%)
12	1 (4%)
13	0 (0%)

* One point earned for each correct response to the 13 questions of the assessment.

Examining student responses to the limit question on the final exam (see Figure 1) gives us an opportunity to compare students' computational skills with their interpretation of what limits mean in terms of a function's behavior. In this question students were asked to compute the limit and to draw a picture of the function. Table 2 shows the distribution of correct responses to the computational and drawing components of this question.

It is striking that while the majority of students were able to compute the correct limit ($> 72\%$ for each part), relatively few students were able to draw correct approximate behavior of the function near that limit, especially for parts (a) (25% of students) and (c) (32% of students). This suggests that students may be memorizing procedures for calculating limits without understanding what the results of the procedures mean for the behavior of the function. Figure 3 shows examples of typical incorrect student drawings. The typical incorrect drawing for part (a) that includes 2 in the domain of the function suggests that many students are not paying attention to the difference between the limit and the function

Table 2. Correct responses on final exam question about limits.

Question part	Type of limit	Students who computed correct response ($n = 28$)	Students who drew correct behavior ($n = 28$)
(a)	removable discontinuity	20 (72%)	7 (25%)
(b)	horizontal asymptote	22 (78%)	18 (64%)
(c)	vertical asymptote	21 (75%)	9 (32%)
all parts		20 (72%)	4 (13%)

value itself. The typical incorrect response for part (c) indicates that many students may interpret the limit not existing as the function itself not existing, or as impossible to draw.

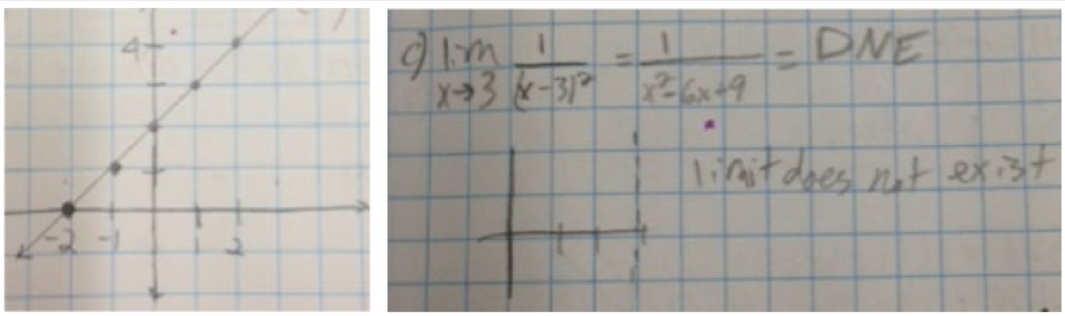


Figure 3. Typical student responses to limit question (part (a) left and part (c) right).

Last, examining students' responses to the bottle problem (see Figure 2) gives us insight into their understandings of rates of change. This problem does not provide any explicit formula for calculating height or volume and instead asks students to sketch the general shape of the graph. This requires students to think conceptually about what is going on in the problem context, as there are no numbers or formulae to manipulate in order to arrive at an answer.

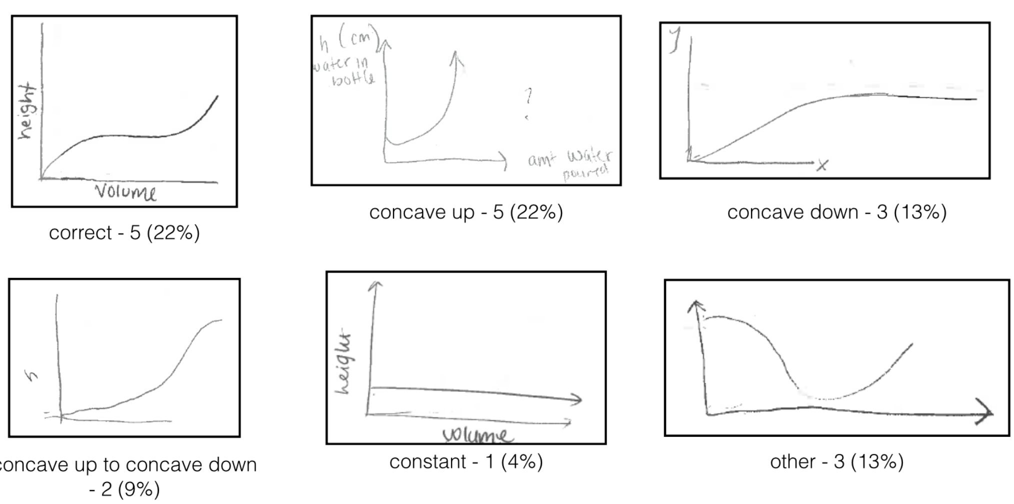


Figure 4. Examples of student responses to the bottle problem in each category.

Only five students (22%) drew graphs that could be considered correct for this problem (an increasing concave down curve changing to concave up). The remaining students gave a variety of responses, including graphs that were exclusively concave up, exclusively concave down, concave down changing to concave up, constant, and unclassifiable (see Figure 4 for examples of student responses). The explanations given for the responses also varied. As an example, students who drew increasing graphs that were exclusively concave up or concave down often provided explanations such as "as the volume goes up the height goes up too" without considering how this relationship varies with the shape of the bottle.

We have conducted a preliminary analysis comparing students' scores on the function assessment with their responses on the limits question and the bottle problem (both described above). The five students who answered the bottle question correctly also scored highly on the function assessment; all scored 10 or above out of 13 possible points. These students also did well on the limit question; each of them correctly computed all three limits and also drew correct behavior for the functions for at least two of the three parts of the question. Furthermore, of the six top-scoring students on the function assessment (scores of 10 or above), five of them also correctly completed the bottle problem (one left that question blank), and of the five of these six students who took the final exam (one top-scoring student on the function assessment later dropped the course), all five correctly computed all limits and drew correct function behavior for at least two of the three parts of the final exam problem. This suggests a relationship between students' understandings of functions and their abilities to learn new calculus concepts such as limits and rate of change.

5. Implications

Students' difficulties with learning the key ideas of introductory calculus have been well-documented (e.g., Carlson, 1998; Selden et al., 2000) and affect students from high school through graduate school. Our results support these findings as the data show that students may learn computational skills in calculus yet still struggle to develop understandings of the key concepts that underlie those computations. In addition, these data suggest that developing a strong concept of mathematical functions has the potential to enhance student learning in calculus. More research is needed in this area, but this study provides a starting point for researchers and instructors to engage with students' ideas about functions, limits, and rates of change and to further explore ways in which students' varied ideas can be leveraged to support learning of core calculus concepts in undergraduate classrooms.

6. Questions and Audience Discussion

The main questions to be posed for audience feedback and discussion in order to help further this line of research are:

- What ideas do students have about functions that might be either productive or problematic for their learning of key calculus concepts?
- What kinds of measures can instructors use to gauge students' understanding of key calculus concepts, as opposed to their ability to apply procedures accurately?

- How important is it for students to develop strong understandings of functions *before* understanding key calculus concepts? Could it be more productive to help them develop understandings of functions and key calculus concepts concurrently rather than one before the other?

7. References

Bressoud, D. M., Carlson, M. P., Mesa, V. & Rasmussen, C. (2013). The calculus student: insights from the Mathematical Association of America national study, *International Journal of Mathematical Education in Science and Technology*, 44(5), 685-698.

Carlson, M. P. (1998). A cross-sectional investigation of the development of the function concept. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education* (vol. III, pp. 114-162). Providence, RI: American Mathematical Society.

Carlson, M. P., Oehrtman, M., & Engelke, N. (2010). The Precalculus Concept Assessment: A tool for assessing students' reasoning abilities and understandings, *Cognition and Instruction*, 28(2), 113-145.

Dubinsky, E., & Harel, G. (1992). The nature of the process conception of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy.* MAA Notes (vol. 25 pp. 85-106). Washington, D.C.: Mathematical Association of America.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-97). New York, NY: McMillan.

Lave, J. (1988). *Cognition in practice.* New York, NY: Cambridge University Press.

President's Council of Advisors on Science and Technology (PCAST). (2012). Engage to excel: Producing one million additional college graduates with degrees in science, technology, engineering, and mathematics. Report to the President. Retrieved on June 1, 2014.

Richland, L. E., Stigler, J. W., & Holyoak, K. J. (2012). Teaching the conceptual structure of mathematics, *Educational Psychologist*, 47(3), 189-203.

Selden, A., Selden, J., Hauk, S., & Mason, A. (2000). Why can't calculus students access their knowledge to solve non-routine problems? In: J. Kaput, A. Schoenfeld, & E. Dubinsky, (Eds.) *Research in collegiate mathematics education* (vol. 4, pp. 128-145). Washington, DC: American Mathematical Society.

Stigler, J. W., & Baranes, R. (1998). Culture and mathematics learning. In *Review of research in education* (vol. XV). Washington, DC: American Educational Research Association.

Vergnaud, G. (1996). The theory of conceptual fields. In L. Steffe, P. Nesher, P. Cobb, G. Golding, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 219-239). Hillsdale, NJ: Lawrence Erlbaum Associates.

Wagner, J. (2006). Transfer in pieces. *Cognition and Instruction*, 24:1, pp. 1-71.

Wearne, D., & Hiebert, J. (1988). Constructing and using meaning for mathematical symbols: The case of decimal fractions. In J. Hiebert and M. Behr (Eds.), *Number concepts and operations in the middle grades* (vol. 2, pp. 220-235). Mahwah, NJ: Lawrence Erlbaum Associates.

Linear algebra in the three worlds of mathematical thinking: The effect of permuting worlds on students' performance

John Hannah — Canterbury University
Sepideh Stewart — Oklahoma University
Michael O.J. Thomas — Auckland University

Linear algebra is a required course for STEM majors. Many undergraduate students struggle with the sudden exposure to abstraction which is almost an unavoidable feature of the course. Although research on students' conceptual understanding of linear algebra is going forward, no research has focused on how students react to the order in which the concepts are taught. In this study, we use Tall's three-world model of mathematical thinking to examine students' performance in a first year linear algebra course. The study examined two sections of a linear algebra course simultaneously while the instructor changed the order in which she taught the concepts in each class. The result of this investigation so far suggests no significant difference on students' performance.

Keywords: Reflections on Teaching, Linear Algebra, Three Worlds Model of Mathematical Thinking, Contingent teaching, Clickers

Background

Many STEM students take an introduction to linear algebra course after completing their calculus sequence requirement, hence by the time they study linear algebra they have some familiarity with university level mathematics. Despite this, many still struggle with grasping the more theoretical aspects of linear algebra. Research on students' conceptual difficulties with linear algebra first made an appearance in the 90's and early 2000's (e.g. Dorier, 1990; Carlson, 1997; Dorier & Sierpinska, 2001). Over the past decade, research on linear algebra has concentrated on the nature of these difficulties and students' thought processes (e.g. Stewart & Thomas, 2009; Hanah, Stewart, & Thomas, 2013; Wawro, Sweeney, & Rabin, 2011; Wawro, Zandieh, Sweeney, Larson, & Rasmussen, 2011).

In this study we applied Tall's (2004) three worlds of mathematical thinking to examine whether the order of the manner of presentation of linear algebra concepts had an effect on students' performance in the course. In his theory, Tall introduced a framework based on three worlds of mathematical thinking: the *conceptual embodiment*, *operational symbolism* and *axiomatic formalism*. The world of conceptual embodiment is based on "our operation as biological creatures, with gestures that convey meaning, perception of objects that recognise properties and patterns...and other forms of figures and diagrams" (Tall, 2010, p. 22). Embodiment can also be perceived as the construction of complex ideas from sensory experiences, giving body to an abstract idea. The world of operational symbolism is the world of practising sequences of actions that can be achieved effortlessly and accurately. The world of axiomatic formalism "builds from lists of axioms expressed formally through sequences of theorems proved deductively with the intention of building a coherent formal knowledge structure" (p. 22). Tall (2013) suggested that: "Formal mathematics is more powerful than the mathematics of embodiment and symbolism, which are constrained by the context in which the mathematics is used" (p. 138).

Tall observes that although learners often meet embodiment first, followed by symbolism and formalism, “when all three possibilities are available at university level, the framework says nothing about the sequence in which teaching should occur” (Tall, 2010, p. 22).

Method

This research project describes a case study in which a university mathematics instructor (second named author) examines Tall’s three-world model. The first phase of the project was conducted in the Fall of 2013 at The University of Oklahoma. The instructor was teaching two sections (004 & 001) of an introduction to linear algebra course.

To investigate whether the order (ie Embodied (E), Symbolic (S) and Formal (F)) in which the material is presented has an impact on students’ learning and attitudes, each of the following possible combinations of teaching various concepts was used: ESF; EFS; SEF; SFE; FSE; FES as well as the two common ways of teaching with no embodied exposure at all: FS and SF. The aim was to try to establish whether the order of presentation influences understanding of a particular concept. For example, concept A was taught in the morning to class (004) in the ESF order, whereas in the afternoon class (001) was taught in the FSE order (see Figure 1). To expose students to as many orders as possible, concept B was taught using SFE in the morning and SEF in the afternoon section, and so on. Hence, each concept was taught in all three worlds of embodied, symbolic and formal mathematical thinking to each class, but in different orders. To try to gain some measure of students’ understanding the instructor incorporated clicker quiz questions into the presentation of the teaching material. The design of suitable quizzes, posed at the right moment, was a crucial part of the project. The students were also given clicker opinion questions throughout the lecture regarding their preference of the order of presentation, to gauge the reaction of the class and make sure everyone was following. These included questions such as: How would you like to be taught this particular concept? (a) by a definition, (b) an example, (c) a picture. Now that you have seen the examples, what would you prefer to see next? Students were also asked a number of True/False opinion questions regarding their understanding. (e.g. I fully understand this theorem. T/F). Data was collected from the student clicker quizzes to try to establish the effect of a particular order on student attitudes and learning. It was noticeable that this approach changed the class atmosphere and it appeared that students were more involved and engaged, started to respond better and embraced the lecture style.

Other forms of data gathering occurred through the instructor’s daily journals for each lecture, homework assignments, tests, final examination questions and student interviews. Of the 82 students in the classes 68 (41 from 001 and 27 from 004) gave consent for their data and course material to be used. In addition, during the final two weeks of the course, 10 (7 from 001 and 3 from 004) student volunteers from both sections were given semi-structured interviews by a colleague, using questions such as: Did you notice any difference in the way Dr. Stewart taught different concepts in her lessons this semester? If so, in what way were they different? If not, was her approach in teaching concepts always the same? If you prefer teaching to start with one particular approach, which one would it be? Can you explain why you prefer this approach? Do you think that step should always come first (second, third), or are there situations where you would prefer a different order? Which type of thinking do you prefer, or feel most comfortable with: embodied, symbolic or formal? Do you think any of these types of thinking is more important than the others in mathematics? If so, which one? What do you think about clicker questions (quizzes and opinion)?

The research question for this part of the study is as follows: Is there any influence of order of presentation on overall student performance?

Math3333-004 9:30 class

Worlds Concepts	ESF	EFS	SEF	SFE	FSE	FES	FS	SF
Subspace					*			
Linear Combination	*							
Span						*		
Linearly Independent/dependent		*						
Basis				*				
Column space							*	
Null space								*
Transformation								
Kernel and range					*			
Eigenvalues & eigenvectors	*							
Least square								

Math3333-001 12:30 class

Worlds Concepts	ESF	EFS	SEF	SFE	FSE	FES	FS	SF
Subspace	*							
Lin Combination					*			
Span	*							
Linearly Independent/dependent						*		
Basis						*		
Column space							*	
Null space								*
Transformation								
Kernel and range				*				
Eigenvalues & eigenvectors	*							
Least square								

Figure 1: Linear Algebra concepts and the order in which they were presented in each section.

Results and Discussion

The initial results from students' interviews showed that their preferred order included having the symbolic exposure to linear algebra first (Hannah, Stewart &Thomas, 2014). As the analysis of the rest of the data (homework questions, midterm tests and the final examination) continued, it was decided to try to ascertain whether there is a preferential order for a particular concept, and thus the following strategy was undertaken.

The argument: by taking each group as having Embodied (E) before Formal (F) for two concepts and Formal before Embodied for two concepts we searched for some evidence of which approach might be preferential, namely E followed by F or F followed by E. We compared the results as follows: *Embodied first*: Subspace 001; Span 001; Linear combination 004; Linear independence 004, versus *Formal first*: Subspace 004; Span 004; Linear combination 001; Linear independence 001, using questions from two homework assignments and the final exam (see Figure 2). The analysis of the data from question 1, in Assignment 5 revealed that a majority of students from both sections scored 100%, making it difficult to distinguish any possible effects of the order in which the concepts were taught. It is noteworthy to mention that the instructor asked her grader to give full points (10 points) to any concept maps that students drew in Part c and not to punish them for any possible misconceptions.

Assignment 5 Question 1:

1. [**30 marks**]

Consider the following concepts: Subspace, Linear combination, Span, Linear independence and dependence, Basis. Your task is to:

(a) Define each concept.

(b) Show a geometrical representation of each concept (draw a picture).

(c) On a separate page, carefully design a concept map relating the above concepts wherever possible.

Assignment 6 Question 1:

1. [**15 marks**] Consider the vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}$$

On a separate page write a short paragraph about $\mathbf{u}$,$\mathbf{v}$ and $\mathbf{w}$. Your paragraph should be about 70-80 words long and should contain the following linear algebra terms:
basis, dimension, subspace, linear combination, linear independence, linear dependence and span.

Exam Question 3(c):

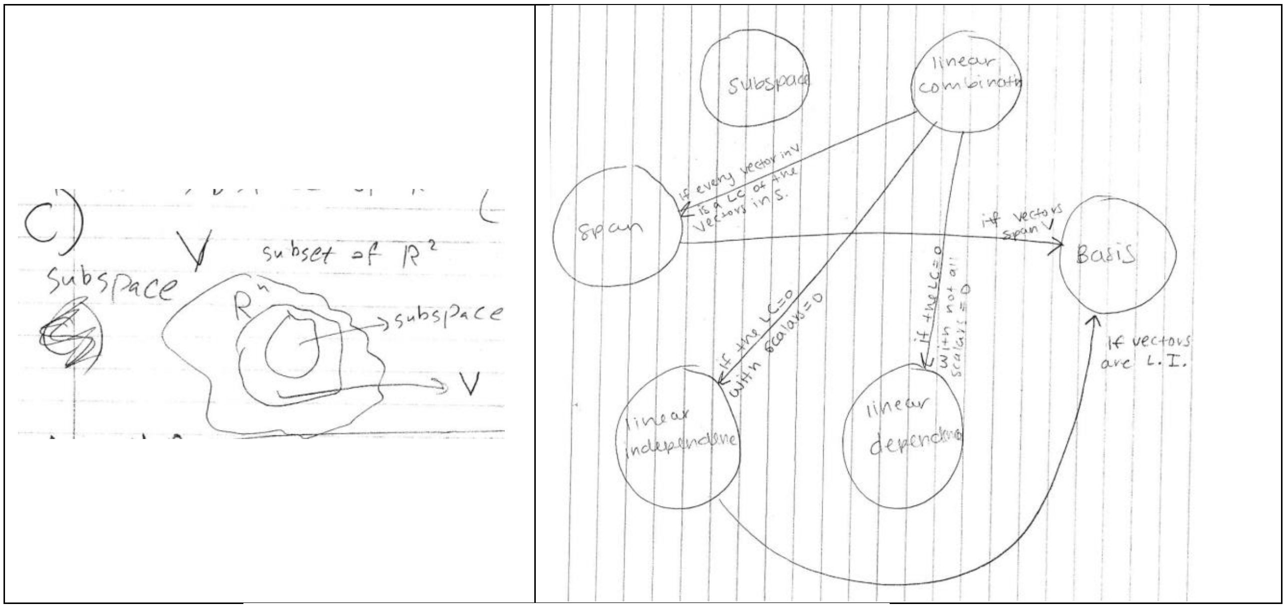
(c) [**5 marks**] Describe the following set of vectors in terms of span, linearly independent, linearly dependent, basis and subspace. No calculations is required.

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \qquad \text{where } \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

Figure 2. A selection of questions given to both sections during the course.

Further investigation of each concept and students' performances in Question 1 from Assignment 5 showed that, 53% of students from section 004 copied the definitions form the text book, in comparison to only 10% in section 001. It was noted that, 53% of students from section 004 did not mention the properties of a subspace and only described the term as a subset of a vector space, whereas, 90% of students from section 001 gave a more compelling description of the term in their own words. Most students in both sections were able to draw pictures to illustrate each term well. These pictures were either from their notes (given to them in lectures) or from the text book. The results showed that at least two students (one from each section) drew different pictures of the concept subspace than the rest (see figure 3(a)). Students' concept maps (Question 1, part c) gave more insight into how students were thinking about the concepts and connecting the ideas together, with some drawing more well thought-out maps than others. A number of students from both sections raised their concern with regard to where they should place the concept of subspace (Question 1(c)) and a few did not connect this concept with any other concepts but left it on its own (see Figure 3(b)).

Question1 from Assignment 6 was considered as a non-routine problem and most students from both sections struggled to express their understanding of the concepts well. The box plot of their grades (see Figure 4) shows that students in section 004 performed slightly better than students in section 001.



(a) (b)

Figure 3: An illustration of a subspace (student 3, 004) and a concept map (student 32, 004).

However, when a similar question was posed in the final examination (Question 3(c)), students' scores in both sections, increased significantly.

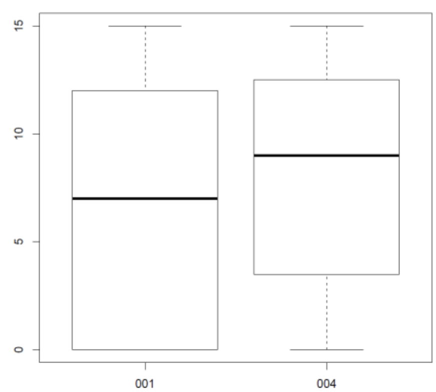


Figure 4. Students' grades for question 1 in Assignment 6.

Final Remarks

The overall performance of students (the final grades) showed no significant difference between students' performances in section 001 (79.19%) and their counterparts in section 004 (77.45%). The more comprehensive qualitative analysis of each question mentioned in this study is beyond the scope of this proposal, however, the authors are in the process of making it available in the near future. As the search for finding a recommended order of teaching linear algebra concepts continues, our future research will examine a fewer number of permutations of the worlds (Embodied, Symbolic and Formal) with more frequency of their occurrence, to narrow down the feasible tracking processes and to make the comparison between the groups more manageable.

References

Carlson, D. (1997). Teaching linear algebra: Must the fog always roll in? In D. Carlson, C. R. Johnson, D. C. Lay, A. D. Porter, A. Watkins & W. Watkins (Eds.), *Resources for Teaching Linear Algebra, MAA Notes* (Vol. 42, pp. 39-51). Washington: Mathematical Association of America.

Dorier, J. L. (1990). Continuous analysis of one year of science students' work, in linear algebra, in first year of French university. *Proceedings of the 14th Annual Conference for the Psychology of Mathematics Education*, Oaxtepex, Mexico, II, 35-42.

Dorier, J. L., & Sierpinska, A. (2001). Research into the teaching and learning of linear algebra. In D. Holton, M. Artigue, U. Krichgraber, J. Hillel, M. Niss & A. Schoenfeld (Eds.), *The Teaching and Learning of Mathematics at University Level: An ICMI Study* (pp. 255-273). Dordrecht, Netherlands: Kluwer Academic Publishers.

Hannah, J., Stewart, S., & Thomas, M. O. J. (2013). Emphasizing language and visualization in teaching linear algebra. *International Journal of Mathematical Education in Science and Technology*, *44*(4), 475-489. DOI: 10.1080/0020739X.2012.756545.

Hannah, J., Stewart, S., & Thomas, M. O. J. (2014). Teaching linear algebra in the embodied, symbolic and formal worlds of mathematical thinking: Is there a preferred order? In Oesterle, S., Liljedahl, P., Nicol, C., & Allan, D. (Eds.) *Proceedings of the joint meeting of PME38 and PME-NA 36*, Vol. 3, pp. 241-248. Vancouver, Canada: PME.

Stewart, S., Thomas, M.O.J. (2009). A framework for mathematical thinking: the case of linear algebra. *International Journal of Mathematical Education in Science and Technology*, *40*(7), 951-961.

Tall, D. O. (2004). Building theories: The three worlds of mathematics. *For the Learning of Mathematics*, *24*(1): 29-32.

Tall, D. O. (2010). Perceptions, operations, and proof in undergraduate mathematics. *Community for Undergraduate Learning in the Mathematical Sciences (CULMS) Newsletter*, *2*, 21-28.

Tall, D. O. (2013). *How humans learn to think mathematically: Exploring the three worlds of mathematics*, Cambridge University Press.

Wawro, M., Sweeney, G., & Rabin, J. (2011). Subspace in linear algebra: Investigating students' concept images and interactions with the formal definition. *Educational Studies in Mathematics*. doi: 10.1007/s10649-011-9307-4

Wawro, M., Zandieh, M., Sweeney, G., Larson, C., & Rasmussen, C. (2011). *Using the emergent model heuristic to describe the evolution of student reasoning regarding span and linear independence*. Paper presented at the 14th Conference on Research in Undergraduate Mathematics Education, Portland, OR.

Building Student Communities Through Academic Supports

Kady Hanson
San Diego State University

Estrella Johnson
Virginia Tech

In this report, we draw on the Characteristics of Successful Programs in College Calculus (CSPCC) survey and interview data in order to better understand the student supports available for Calculus I students. We began by investigating possible differences in the availability of supports offered by institutions that were identified as having a successful Calculus I program. This quantitative analysis found no statistical differences in regards to tutoring centers at institutions that were identified as successful and tutoring centers at the other institutions in our study. We then analyzed student interview data in order to investigate how the supports offered at these institutions contribute to student success. This qualitative analysis identified several ways in which academic support systems (and tutoring centers in particular) can facilitate social and academic integration for Calculus I students.

Key Words: Calculus, Student Support, Communities, Social integration

Nationally there is tremendous need to retain more post secondary STEM intending students. As reported by the 2012 President's Council of Advisors on Science and Technology, current approaches to educating STEM majors is insufficient to meet the demands of the workforce (PCAST, 2012). However, this shortage can be significantly alleviated with just a 10% increase in the retention of STEM intending students (Carnevale, Smith, & Melton, 2011; PCAST 2012). Retaining more STEM intending students, however, has been and continues to be problematic and the subject of much scholarly inquiry, including the work carried out by the Characteristics of Successful Programs in College Calculus (CPSCC) project.

The CSPCC project is a large empirical study designed to investigate Calculus I across the United States. The primary focus of the CSPCC project is to identify factors that contribute to student success in Calculus I and to better understand how such factors actually contribute to student success. In this report, we draw on the CSPCC survey and interview data in order to better understand the different avenues of student support that are available for Calculus I students. He we will focus specifically on mathematics tutoring available to Calculus I students. We begin by investigating possible differences in tutoring services offered by institutions that were identified as having a successful Calculus I program and those offered at the rest of the institutions in our study. We then investigating how tutoring services offered at these institutions may contribute to student success. In particular, we will investigate the ways in which tutoring centers can facilitate social and academic integration for Calculus I students.

Theoretical Background

Higher education research has found that students are more likely to persist in college when they are integrated into both a social and an academic community (Tinto, 1997). The extent to which a student has integrated into a social community can be understood in terms of the richness in peer-to-peer interactions and faculty-student interactions, whereas indications that a student is integrated into an academic community include academic progress, satisfaction with intended major, and a clear understanding of the academic expectations at their institution (Kuh et al., 2006). Such integration is particularly important in the first year of college when attrition is more likely to happen, as “nearly half of all leavers depart before the start of the second year” (Tinto, 1997, p. 167). Analogously, one would expect that students are more likely to persist in STEM related fields when they have strong academic and social connections in the STEM community. Further, because students’ first-years experience is to important to persistence on the whole, it follows that social and academic integration within students first-year STEM courses, such as Calculus I, is of particular importance for STEM intending students. Understanding how student support programs at successful institutions foster social and academic integration into Calculus I student communities is the focus of this work.

Data and Methods

In order to answer our research questions, we draw on data collected in the two phases of the CSPCC project. The first phase of the CSPCC study involved surveys sent to a stratified random sample of institutions. Course coordinators, instructors, and students were asked to complete surveys at the beginning of Calculus I. Instructor and students were also asked to complete a survey at the end of Calculus I. These surveys were designed to gain an overview of the various calculus programs nationwide, and to determine which institutions had more successful calculus programs. Here we define success based on a number of student variables, including: persistence in the calculus sequence; high pass rates; and positive affective changes, including enjoyment of math, confidence in mathematical ability, and interest to continue studying math. Based on these student variables, 18 institutions were selected for follow-up site visits. In this second phase of this project, we conducted three-day site visits at each of the 18 selected institutions. During these site visits members of the CSPCC team: interviewed students, instructors, and administrators; observed classes; and collected exams, course materials, and homework.

In order to identify possible differences in the tutoring supports at selected and non-selected institutions, we analyzed course-coordinator and end-of-term student surveys. Specifically, we analyzed survey questions related to the availability of mathematics tutoring centers and how frequently students visited tutoring centers. Then, in order to determine how tutoring centers were contributing to student success, we analyzed transcripts from student focus groups. Here our focus was on social integrations. Specifically, we identified instances in which students described ways in which the available support programs facilitated either student-to-student or student-to-faculty interactions.

Results

We will begin with some descriptive and comparative statistics about tutoring centers. Both course coordinators and students responded to survey questions regarding the availability and frequency of use of tutoring centers. We then present two examples from our selected institutions that highlight ways in which tutoring centers can facilitate social integration.

Tutoring Centers – By the Numbers

1) Selected institutions were not more likely to have a tutoring center than non-selected institutions. Course coordinators were asked: “Does your department, college, or university operate a mathematics tutoring center available to Calculus I students?” Course coordinators from 13 of the 18 selected institutions and 105 of the 150 not-selected institutions answered this question. All 13 of selected institutions that answered

this question (100%) have a tutoring center that offered Calculus I assistance, compared to 102 of the 105 non-selected institutions that answered this question (97.1%) (p=.537).

Table 1 Prevalence of tutoring centers among selected and non-selected institutions

Does your department, college, or university operate a mathematics tutoring center available to Calculus I students?

	# of institutions that answered the questions	# of institutions that said "Yes"	% of institutions with a Tutoring Center
Selected	13	13	100%
Not Selected	105	102	97.1%

2) At institutions where course coordinators indicated that they have tutoring centers, there is no difference in regards to the tutoring center staff at selected and non-selected institutions. The course coordinator survey provided 5 classifications for tutoring: tutoring by undergraduate students, tutoring by graduate students, tutoring by paraprofessional staff, tutoring by part-time math faculty, and tutoring by full-time math faculty. None of the intra-category comparisons were signification at the .1 level. The most common tutoring center staffing was by undergraduate students, with graduate students the next most prevalent.

Table 2 Tutoring Center Staff

What services are available to students in the Tutoring Center?

	tutoring by undergraduate students	tutoring by graduate students	tutoring by para-professional staff	tutoring by part-time math faculty	tutoring by full time math faculty
Selected	12	4	2	1	2
	92.3%	30.8%	15.4%	7.7%	15.4%
Not	91	35	28	23	31
Selected	86.7%	33.3%	26.7%	21.9%	29.5%

3) At institutions where course coordinators indicated that they have tutoring centers, course coordinators at selected and non-selected institutes report similar student usage rates. When asked to respond to the statement "Students in Calculus I take advantage of the tutoring center", 10 of the 13 (77%) course coordinators either 'agreed" or "strongly agreed" with this statement, where 86 of the 102 (85%) courses coordinators from the

non-selected intuitions "agreed" or "strongly agreed". These differences were not statically significant at the .1 level.

Table 3 Student Tutoring Center Usage as Reported by Course Coordinators

	Students in Calculus I take advantage of the tutoring center (Strongly agree or agree)
Selected	10 77%
Not Selected	86 85%

4) At institutions where course coordinators indicated that they have tutoring centers, student survey responses about tutoring center usage agreed with course coordinator responses and showed no differences between selected and non-selected institutions. At the end of the term students were asked "How often do you visit a tutor to assist with this course: Never, Once a month, A few times a month, Once a week, More than once a week?" (see Table 3). Again, none of the intra-category comparisons were significant to the .1 level.

Table 4 Student reported frequency of tutoring

	How often do you visit a tutor to assist with this course?				
	never	once/ month	a few times/ month	once/ week	> once/ week
Selected	762 61.4%	131 10.6%	145 11.7%	107 8.6%	96 7.7%
Not selected	2367 57.8%	487 11.9%	498 12.2%	399 9.7%	342 8.4%

When we translated these responses into a 1-5 scale (with 1 being "Never" and 5 being "More than once a week") and averaged the numerical values. We found that the average responses from students at selected institutions was 1.91 and the average response from students at non-selected institutions was 1.99 (Table 4). So, on average, students at both selected and non-selected institutions are visiting tutoring centers about once a month.

Table 5 Student reported average tutor frequency

		How often do you visit a tutor to assist with this course?
Selected	Mean	1.91
	N	1241
	(Std. Dev.)	(1.328)

	Mean	1.99
Not selected	N	4093
	(Std. Dev.)	(1.357)

In summary, we found no statistical differences in the number of institutions providing tutoring centers, the composition of tutor center personnel, or in the reported frequency of students visiting the tutoring centers. We now take a deeper look at how tutoring centers at our selected institutions went beyond merely offering mathematics assistance and were able to actually contribute to the development of a student community for Calculus I students.

Two examples of support systems

Here we present two examples of tutoring centers at successful institutions that were able to foster social integration through providing opportunities for student-to-student interaction.

Large Private PhD-Granting University (LPPU): At LPPU the math lab is divided by courses, therefore all students in Calculus I gather together in the same location to get tutoring assistance on homework and other assignments. If students are interested in drop in tutoring, they put their name on a sign-up list with the exact section and problem number with which they need help. The tutors then go to that list, call out the next name, and help them with their problem. Students often reported using the tutoring center as a place to do homework, even if they were not there specifically for tutoring services. As one student described:

> *I can walk into the Math Lab, and I can sit down with someone from a different class, and I can do my homework with them. And that really to me is really beneficial because it's like, 'I got stuck on this problem,' there's nobody from my class, I have a class of 250. Every time I go there's maybe 3 other people there, but then there's also going to be 4 people from Barret's class, Franz's class and things like that.* (Student, LPPU)

In this way, the math center was able to create a working environment where students could go to get tutoring or just to meet other students in Calculus I that may be in different sections. Many students at LPPU explained that the tutoring center led them to start working with other students whether it was on homework, studying, or just practicing concepts. These students were able to find a group that was similar to them and had the same goals and motivation. These connections allowed students at LPPU to have an outlet to get guidance, discuss frustration, and to simply make friends that they can go through college with.

Urban Public Associates-Granting University (UPAU): The math lab at UPAU is equipped with two large white boards, with about 50 to 75 chairs located throughout the room and can serve anywhere from 50 to 100 students per day. Students have two options for tutoring at the UPAU math lab: one-on-one tutoring and group review. If they want one-on-one tutoring the students must raise their hand and wait for a tutor to come and help them. Depending on how busy the center is, the tutors can help with multiple problems at one time, or only assist with one before moving onto the next student.

In the group review, the groups of 3 or more students can make a reservation to have a tutor assist them with a review of a topic or an assignment.

> *Also in the math lab if you have three or more people and you can have someone there tutoring you in the study group as you go through on say a mini grade or problems from the book. It's extremely helpful because you have instead of going to math lab and having a tutor come up to you if you raise your hand, you get personal attention for about an hour or so.* (Student, UPAU)

By offering dedicated tutor support to groups of students, the math lab provides motivation for student to form groups. These groups contribute to strong student communities, as these students work on assignments together, study for exams together, and build relationships where they are not afraid to ask questions and admit they need help.

Discussion and Conclusions

When we only analyzed the CSPCC survey data, we were unable to identify any differences in selected and non-selected institutions. This made it impossible to make any claims about how tutoring centers contributed to student success. However, by analyzing the student focus group interviews, we were able to identify an important component of tutoring centers at successful institutions – fostering a student community for Calculus I students. By identifying these specific components and policies of tutoring centers at selected institutions, this work helps provides examples for how student support services can facilitate social integration. At LPPU, this student community was facilitated by common homework assignments, space for informal study groups, and tutors that only answered one question at a time – making it more time efficient to ask a student sitting next to you. At UPAU, the student community was fostered by the tutoring center by providing a common area for students to work together and rewarding students to work form groups by providing extended tutoring services to groups of three or more.

.

References

Carnevale, A. P., Smith, N., & Melton, M. (2011). STEM: Science, technology, engineering, mathematics. Georgetown University, Center on Education and the Workforce. Retrieved from http://www9.georgetown.edu/grad/gppi/hpi/cew/pdfs/stem-complete.pdf

Kuh, G., Kinzie, J., Buckley, J., & Hayek, J. (2006). What matters to student success: A review of the literature (Executive summary). Commissioned report for the National Symposium on Postsecondary Student Success.

President's Council of Advisors on Science and Technology (PCAST) (2012). *Engage to excel: Producing one million additional college graduates with Degrees in Science, Technology, Engineering, and Mathematics*. Washington, DC: The White House.

Tinto, V. (1997). Colleges as communities: Taking research on student persistence seriously. *The review of higher education*, *21*(2), 167-177.

Examining proficiency with operations on irrational numbers

Sarah Hanusch
Texas State University

Sonalee Bhattacharyya
Texas State University

Introduction

Undergraduate students need proficient knowledge of the real numbers, because the real numbers form the foundation for more advanced mathematical concepts. To achieve numerical literacy, students must have some proficiency in the real number system (Fischbein, Jahiam, & Cohen, 1995; Guven, Cekmez, & Karatas, 2011). Studies have shown that the set of irrational numbers is difficult to grasp, because of challenges with the definitions of rational and irrational numbers (Fischbein et al., 1995), with the connection between irrational numbers and limits (Peled & Hershkovitz, 1999), and with moving between multiple representations of irrational numbers (Arcavi, Bruckheimer, & Ben-Zvi, 1987; Sirotic & Zazkis, 2007).

In this study, we sought to see how students approached problems involving operations on irrational numbers. We chose developmental mathematics students at a large university for the sample because rational and irrational numbers are included in the curriculum for the course. Few studies have focused on the mathematical knowledge of students at the developmental (non-credit bearing) level (Givven, Stigler, & Thompson, 2011; Grubb & Cox, 2005; Stigler, Givven, & Thompson, 2010). However, Stigler et al. (2010) and Givven et al. (2011) collected data from community college developmental mathematics students to conclude that most students at the developmental level suffer from "conceptual atrophy," meaning the students are unable to connect "basic intuitive ideas about mathematics" (Stigler et al., 2010, p.15) with procedures and concepts.

In this study, we consider an individual to be mathematically proficient if they demonstrate procedural fluency, conceptual understanding, adaptive reasoning, strategic competence and productive disposition, as described in Adding It Up (Kilpatrick, Swafford, & Findell, 2001). These five are titled the strands of proficiency, and they "are interwoven and interdependent in the development of proficiency in mathematics" (Kilpatrick et al., 2001, p. 137). The interconnectedness of these strands emphasizes the importance of making connections between mathematical topics and skills.

Our research questions for this study are:

- Within each strand of proficiency, what are the developmental mathematics student's ideas relating to the closure of operations on the irrational numbers?
- In which strands of proficiency do the students demonstrate strengths and weaknesses with irrational numbers?
- Do developmental mathematics students demonstrate overall proficiency regarding the closure of operations on the irrational numbers?

Method

The sample in this study is developmental mathematics students at a large university in the southwest of the United States. This university offers two mathematics courses at the developmental level, meaning non-credit bearing courses, which serve as prerequisites for entry level courses, such as College Algebra. Students placed into level one must successfully pass level one and level two before enrolling in credit bearing courses.

For this project, the population is the students enrolled in the level one course. This course is organized with large lecture sections and smaller laboratory sections. Each week, students spend one hour in a large lecture taught by the instructor of record, and three hours in smaller laboratory sections, taught by graduate assistants. A majority of the instruction occurs in the laboratory sections with a standardized curriculum among all sections. The weekly lecture is intended to review, expand, and elaborate on the topics discussed in lab. Although most assignments were assigned and collected in the laboratory, this semester the lecturer assigned weekly homework assignments that connected directly to the lecture.

During the semester in which data was collected, 77 students were enrolled in all sections of the level one course. The data collected was a portion of one lecture homework assignment. After the students turned the homework into their instructor, photocopies of the students' responses and the consent form were given to the researchers for analysis. Only 31 students provided consent for their homework to be analyzed. This study includes responses from two questions:

1. What can you conclude about the sum of two irrational numbers? Is it always irrational? Always rational? Sometime irrational and sometimes rational? Explain your reasoning.
2. What can you conclude about the product of two irrational numbers? Is it always irrational? Always rational? Sometime irrational and sometimes rational? Explain your reasoning.

The written homework assignments were analyzed for evidence of the five strands of proficiency outlined in *Adding it Up* (2001). An open coding scheme was developed within the lens of each strand of proficiency, in the style of Glaser and Strauss (1967). Within each strand, four categories were established, typically strong, moderate, weak and no evidence. A more detailed description of each category is found in the results section.

Results

In the results that follow in the following sections, all counts are per problem. This means that there are 62 responses, two for each student. We made this decision because in several instances an individual student performed differently on the two questions. All names included are pseudonyms that reflect gender.

Procedural Fluency. To analyze procedural fluency we considered three criteria: 1) the response included the correct operation, 2) the response included correct computations, and 3) the response correctly identified numbers as rational or irrational. A student demonstrated strong procedural fluency if they satisfied all three criteria, moderate if they failed one criterion and weak if they failed two or more. Some responses did not include any procedures, and these were coded as no evidence.

Twenty-two of the responses exhibited strong procedural fluency, 10 exhibited moderate, 11 exhibited weak and 19 provided no evidence. Only two of the responses contained incorrect computations, and both of those responses came from the same student. One instance of a student whose response did not reference a specific operation is Olivia, who claims the sum of two irrational numbers "can be both, because an irrational number can be turned into a fraction which would be considered rational." Although Olivia does reference division, her explanation does not reference the relevant operation of addition.

Conceptual Understanding. To analyze conceptual understanding we considered either the definitions provided for rational and irrational numbers, or the examples provided.

Responses were classified into four categories: *correct, small error, significant error* and *no evidence*. Twenty-four responses were classified as correct, meaning the student provided either a correct definition for rational or irrational numbers, or the students classified all numbers generated correctly as rational or irrational.

Seven responses were nearly correct, but had *small errors*. One instance in this category is Nathan who wrote "irrational # is any number that isn't rational," but then claims that zero is both a rational and irrational number. The other responses in this category misclassified the square roots of perfect squares as irrational numbers. In some instances, the students believed that the rationality depends on the representation of the number, i.e. $\sqrt{4}$ is irrational, but 2 is rational; see Carl's response in figure 1.

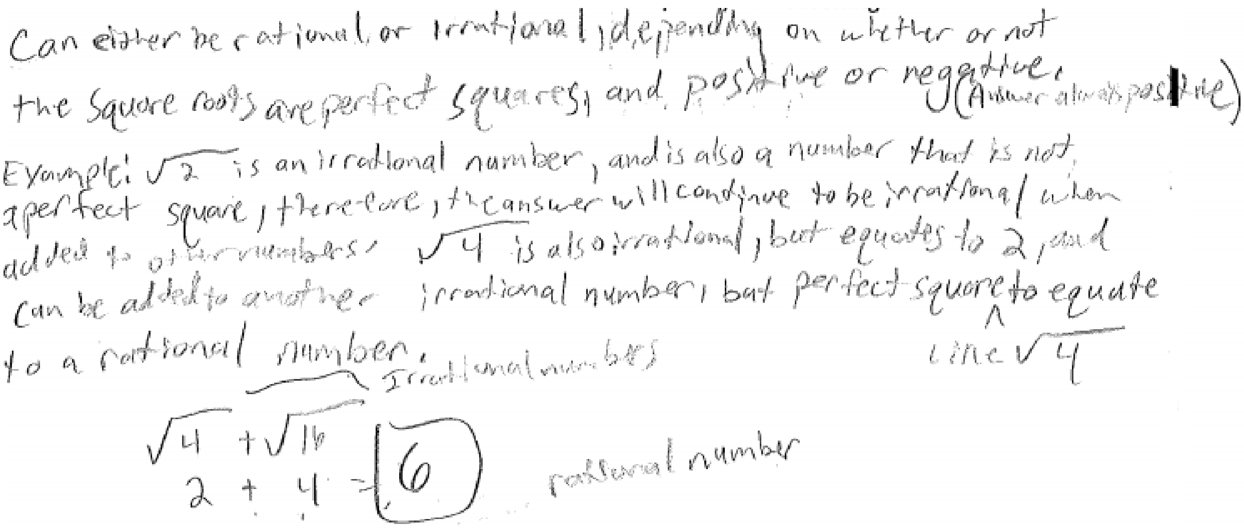


Figure 1 Carl's response to the sum problems where he classifies $\sqrt{\mathbf{4}}$ as irrational, and **2** as rational.

Ten responses were classified as having a *significant error*. In six of these responses, the student classified rational numbers as irrational numbers, and integers as rational numbers. In four other responses, the students indicated that all fractions are rational, not just those with integers. For instance, Andy claimed that "if we put 1 under any number, it would be rational." These students have misconceptions about rational numbers and fractions.

The final category, *no evidence*, had 21 responses. These responses did not provide sufficient information to infer the students' understanding of the concepts. For instance, Terese simply wrote "The sum can be irrational or rational" but offered no explanation or indication of her conceptual understanding. Some of these students reached the correct conclusions to the problems, but many reached incorrect conclusions.

Strategic Competence. In this category, the students were classified by the strategy the used to approach the problem. The first category, called *correct with examples*, a response had to indicate that both rational and irrational sums are possible, and they had to attempt to provide at least one example to support each of those situations. These examples were not always correctly identified. The second category, called *incomplete with examples*, a response included examples to justify their responses, but only included a rational or irrational example. The third category, called *properties*, the students attempted to use properties of irrational numbers to justify their claims. The final category is *no evidence* where the response includes no justification, or did not fit any other category.

Sixteen responses included the correct strategy, although not all of these identified the rational and irrational numbers correctly. Eighteen responses were classified as *incomplete with examples*. An incomplete strategy took one of two forms: either the participants claimed that the sum or product was always rational or always irrational and only provided examples that supported their claim, or the participants claimed that the sum or product could be rational or irrational but only provided examples to show one of the cases. Eleven responses attempted to use *properties*. These responses were split between claiming the sum or product is always irrational and claiming the sum or product could be both. None of these arguments produced valid results. Of the remaining 17 responses in the *no evidence* category, one response was particularly unusual. Heather attempted to use the variable x to create a general argument. She frequently used x^2, and while we suspect she meant to use $\sqrt{x}$ there is no evidence to support that supposition. However, she did recognize that a number plus its negative is zero, and then attempted to use that to help her argument. She claimed “If two numbers sum to a rational number, both must be rational or both must be irrational," but she did not justify this claim any further.

Adaptive Reasoning. To analyze the strand of adaptive reasoning, we initially separated the responses into three categories. The categories were students who made a claim and provided *complete justification*, students who made a claim and had *incomplete justification*, and students who made a claim, but provided *no justification*, or students who did not make any claim at all.

Sixteen responses made a claim and provide a *complete justification*. Not every student in this category used correct definitions of rational and irrational, but they did include an example labeled as rational and another labeled as irrational as justification for their conclusion.

The *incomplete justification* group contained 24 responses, and these responses were characterized by an attempt to justify a claim, but their reasoning erred at some point. For instance, Victor made a claim that the sum of two irrational numbers is always irrational, but his justification was just a few examples. We infer that Victor's reasoning was limited by his small example space. It is unclear whether or not Victor recognized that he could not prove a statement with examples.

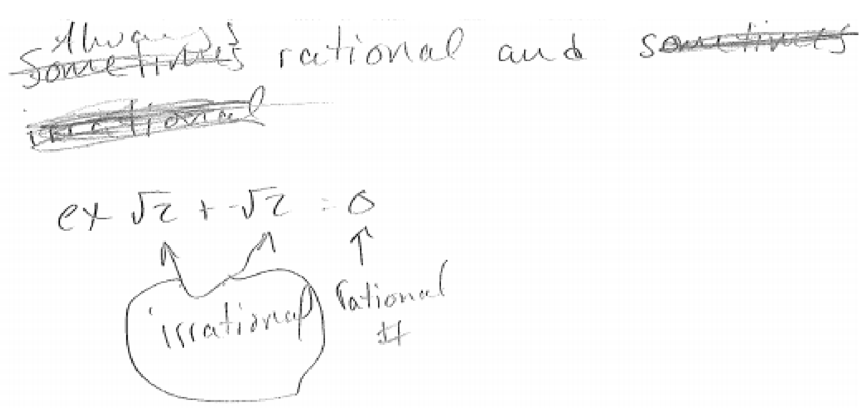


Figure 2 Betsy changed her claim after working examples

Two students had the correct claim, but only justified for the rational case of both questions. It seems possible that these students assumed it was obvious that the sum or product of two irrationals could be irrational, but we have no evidence to support this supposition.

Betsy is the only student to indicate indecision in a response, see Figure 2. She initially claimed that that the sum of two irrational numbers is sometimes rational and sometimes irrational, but then crossed out her response and changed it to say always rational. It seems that she was only able to produce a rational example, and then made a new claim.

In the final category, most of the students either left a blank page or only included a phrase indicating their claim. A few attempted some justification, but these showed severe limitations in reasoning ability.

Productive Disposition. The data collected in this study is inadequate to provide deep insight into the productive disposition of the students in the sample. However, a few insights can be gleaned from this data. One fact that may indicate low productive disposition among this group of students is the fact that fewer than 50% of the students enrolled in the course participated in this study. While that number is not unusual in research studies, in this study data collection came from a single homework assignment. Anecdotally, we know that many students enrolled in this course do not participate in the lectures, and as such we suspect that many of the students did not turn in the assignment.

A few additional responses also indicated a low productive disposition. One student, Peter, turned in a completely blank paper. Another student, Rachel, answered the question, but said "I'm not sure why, or I can't give you reasoning. Without really studying notes or referring back, it's just what I think." It seems that either she felt that she was not allowed to use references, or she simply chose not to make the effort. She also included a frown face figure on the second question, indicating discontentment with the finished product.

Overall. Looking at the various responses as a whole we observed that 42 of the 62 responses stated the correct claim of *sometimes rational and sometimes irrational*, see Table 1. Of the 31 students, 19 made the correct claim about sums of irrational numbers and 23 made the claim about products. Some of the responses were difficult to categorize because the students wrote statements such as "the sum can be rational." However, unless a response explicitly mentioned the sum or product can be irrational, then we classified the response as always rational. The analogous protocol was used for always irrational responses.

Although the majority of the claims were correct, many of the justifications were not. The most frequent justification scheme, 27% of the responses, included both rational and irrational examples. Tied at the same level were the responses that made a claim but did not provide any justification for their response. Attempting to use properties was the justification scheme used in 21% of the responses. This scheme is admirable because it could have led to sophisticated arguments, but unfortunately nearly all of the properties that the students used were false statements. The remaining responses provided examples that were either just rational, or just irrational.

Table 1 A table of the claims made on the responses and the nature of the justifications

Claim				Justification		
Rational or Irrational	42	68%		Two Examples	17	27%
Always Rational	8	13%		Rational Example	9	14%
Always Irrational	9	14%		Irrational Example	6	10%
				Properties or Other	13	21%
No Claim	3	5%		No Justification	17	27%

Discussion

The responses of several students indicate that they have insufficient examples of irrational numbers to be able to answer the questions posed. Leslie, George and David were the most egregious instances of insufficient example spaces since these students clearly could not correctly distinguish the definitions of rational and irrational numbers. Other students demonstrated subtler problems with their example space. For instance, Carl, Jill and Walter struggled with the classification of the square roots of perfect squares.

Relating to the examples spaces of these students, nearly every student chose square roots of small positive integers for the irrational number examples. The only other irrational number chosen was π. This is consistent with the work of Sirotic and Zazkis (2007b), where they argue that a limited example space can lead to misconceptions of irrational numbers. Although some of the students mentioned a decimal representation of irrational numbers, none of the students chose irrational numbers in this representation. A few expressed the sum or product in such a representation, although this seems to be a reaction from using a calculator to compute. More students made the correct claim about the products than the sums, and we attribute this to the choice of radicals for the examples. The procedures for multiplying radicals are significantly easier than adding radicals, and may have contributed to this result.

We support some changes to the instruction given to students on irrational numbers during grade school, and in developmental mathematics courses. Specifically, we support instruction that connects multiple representations of irrational numbers, including decimal expansions, radicals, other constants and placement on the number line. Providing students questions that involve reasoning, such as "find an irrational number that lies between $\frac{3}{2}$ and $\frac{5}{3}$ on the number line," can promote reasoning and translating between representations. Such a question also open a discussion about the abundance of irrational numbers without directly addressing the concepts related to the cardinality of sets. Clear understanding of representations of irrational numbers may aid students as they progress through algebra, especially with regards to identifying the x-intercepts of polynomials with irrational roots.

References

Arcavi, A., Bruckheimer, M., & Ben-Zvi, R. (1987). History of mathematics for teachers: The case of irrational numbers'. *For the Learning of Mathematics, 7*, 18-23.

Fischbein, E., Jahiam, R., & Cohen, D. (1995). The concept of irrational numbers in high-school students and prospective teachers. *Educational Studies in Mathematics,29*, 29-44.

Givven, K. B., Stigler, J. W., & Thompson, B. J. (2011). What community college developmental mathematics students understand about mathematics, part 2: The interviews. *MathAMATYC Educator, 2,* 4-18.

Grubb, N. W., & Cox, R. D. (2005). Pedagogical alignment and curricular consistency: The challenges for developmental education. *New Directions for Community College*, 93-103.

Guven, B., Cekmez, E., & Karatas, I. (2011). Examining preservice elementary mathematics teachers' understandings about irrational numbers. *PRIMUS, 21,* 401-416.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. National Academy Press.

Peled, I., & Hershkovitz, S. (1999). Difficulties in knowledge integration: revisiting Zeno's paradox with irrational numbers. *International Journal of Mathematical Education in Science and Technology, 30*, 39-46.

Stigler, J. W., Givvin, K. B., & Thompson, B. J. (2010). What community college developmental mathematics students understand about mathematics. *MathAMATYC Educator, 1*, 4-16.

A study of mathematical behaviors

Nadia Hardy
Concordia University, Montreal, Canada

In this poster presentation we bring together different characterizations of mathematical thinking, doing and behaving that researchers have brought forward over the last three decades, to compose, in a way of speaking, a collage *of what we came to call* mathematical behaviors*. In previous work, we have combined some of these characterizations to identify opportunities to engage in mathematical behaviors that students may encounter in undergraduate courses, and to design and analyze tasks that may foster the development of such behaviors. The poster format allows us to play with a pictorial representation that reveals the relations and complementarities between the 'images' of the* collage*. We hope to discuss (a) these relations and complementarities, (b) strategies to foster the development of mathematical behaviors in undergraduate mathematics students, and (c) methods that may allow us, as researchers or as teachers, to compose accounts of students' mathematical behaviors.*

Key words: Mathematical thinking, Mathematical behaviors, Undergraduate mathematics

In a 1972 paper, Seymour Papert begins by claiming that being a mathematician is no more definable as 'knowing' a set of mathematical facts than being a poet is definable as knowing a set of linguistic facts. (p. 249) Likely at the time of publication, and certainly today, most mathematicians will agree to this statement. Knowing *about* mathematics may be necessary but does not seem to be sufficient to *do* mathematics (as a mathematician does). Papert goes on to advocate for research to identify and name the concepts that enable mathematicians to think mathematically and to discuss their mathematical way of thinking in an articulate way. What is, the author asks, that something, other and more general than the specific content of particular mathematical topics, that one learns in becoming a mathematician? (p. 250) Since then, many researchers in mathematics education have characterized, in different ways and from different perspectives, certain aspects of *mathematical thinking* and *doing*, and have discussed and reflected on strategies to teach students of all levels to *think mathematically* and to *do mathematics*.

In this poster presentation we bring together many of these characterizations to compose, in a way of speaking, a *collage* that provides an illustration of what we came to call *mathematical behaviors*. In particular, we consider the work of Schoenfeld (e.g., 1987, 1989), Sierpinska (2002), Mason (e.g., 2000; see also Mason et al., 1982), Cuoco et al. (1996; see also Lim and Selden, 2009), Selden and Selden (e.g., 2005, 2013), and Burton (e.g., 1999, 2001, 2004).
In previous research, we have brought together some of these characterizations to identify opportunities that students may encounter in undergraduate mathematics courses to engage in mathematical behaviors, and to design and analyze tasks that can foster the development of such behaviors. (Hardy & Challita, 2012; Hardy et al., 2013) The poster format provides us with the opportunity to consider all these characterizations at once, and to actually play with a pictorial representation of the *collage* we have imagined.

Our interest in mathematical behaviors stems from (a) the work of mathematics educators who have shown several instances (and of different natures) of students unfamiliarity with mathematical thinking, doing, and behaving; and from (b) conversations with colleagues from mathematics departments who refer to (or perceive) a deteriorated profile of graduating mathematics students – students arrive to graduate programs (masters, doctoral and even post-doctoral), it is often claimed, without the ability to *do* mathematics. Many colleagues, and even students, blame this on the lack of opportunities to *engage in mathematical behaviors* during undergraduate (and graduate) studies. Furthermore, some anecdotal data may suggest that graduating students do not choose to pursue graduate degrees or professional careers in mathematics out of disappointment of what they end up believing doing mathematics is all about. Our ultimate goal in inquiring into mathematical behaviors is to devise strategies to foster such behaviors in undergraduate mathematics students – to provide them with opportunities to experience doing mathematics, behaving mathematically.

Presenting this poster has two main goals: one, to discuss the pertinence of the *collage* and its 'images,' and the way(s) in which these relate to, and complement, one another; and two, to inquire and reflect on methods that can help us, as researchers or as teachers, to report on students' mathematical behaviors.

References

Burton, L. (1999). Why is intuition so important to mathematicians but missing from mathematics education? *For the Learning of Mathematics*, 19(3), 27-32.

Burton, L. (2001). Research mathematicians as learners – and what mathematics education can learn from them, *British Educational Research Journal,* 27(5), 592-599.

Burton, L. (2004). *Mathematicians as inquirers: Learning about learning mathematics*. Springer: Netherlands.

Cuoco, A., Goldenberg, E. P., & Mark, J. (1996). Habits of mind: An organizing principle for a mathematics curriculum. *Journal of Mathematical Behavior,* 15(4), 375-402.

Hardy, N., Beddard, C., & Boileau, N. (2013). Teaching first-year undergraduates to think mathematically. *Proceedings of the Psychology of Mathematics Education North American Chapter* (PME-NA 35). Chicago, Illinois. November 14-17, 2013.

Hardy, N., & Challita, D. (2012). Students' perceptions of the role of theory and examples in college level mathematics. *Proceedings of the Psychology of Mathematics Education North American Chapter* (PME-NA 34). Kalamazoo, Michigan, November 1-4, 2012.

Lim, K., & Selden, A. (2009). Mathematical habits of mind. In S. L. Swars, D. W. Stinson, & S. Lemons-Smith (Eds.), *Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1576-1583). Atlanta, GA: Georgia State University.

Mason, J. (2000). Asking mathematical questions mathematically. *International Journal of Mathematical Education in Science and Technology*, 31(1), 97-111.

Mason, J. Burton L., & Stacey K. (1982). *Thinking Mathematically*. Addison Wesley, London.

Papert, S. (1972). Teaching children to be mathematicians versus teaching about mathematics. *International Journal of Mathematical Education in Science and Technology*, 3(3), 249-262.

Schoenfeld, A. H. (1987). What's all the fuss about metacognition? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 189-215). Hillsdale, NJ: Lawrence Erlbaum Associates.

Schoenfeld, A. (1989). Exploration of students' mathematical beliefs and behavior, *Journal of Research in Mathematics Education*, 20, 338-355.

Selden, A., & Selden, J. (2005). Perspectives on advanced mathematical thinking. *Mathematical Thinking and Learning,* 7, 1-13.

Selden, A., & Selden, J. (2013). Persistence and self-efficacy in proving, *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 304-307). Chicago, IL: University of Illinois at Chicago.

Sierpinska, A., Nnadozie, A., & Oktac, A. (2002). *A study of relationships between theoretical thinking and high achievement in Linear Algebra*. Concordia University. Unpublished Manuscript.

Elementary mathematics pre-service teachers' consequential transitions from formal to early algebra

Charles Hohensee
University of Delaware

Siobahn Young
University of Delaware

Mathematics educators have long advocated for early algebra to be introduced into the elementary grades. However, little research is currently available to inform teacher preparation programs about the work of preparing undergraduate pre-service teachers for teaching early algebra. The research reported here examines the experiences of undergraduate pre-service teachers as they make consequential transitions from formal to early algebra. Preliminary results suggest that making this transition is far from trivial for undergraduates and that, to varying degrees, they face four kinds of conceptual challenges.

Key words: early algebra, consequential transitions, pre-service teachers, elementary education

Although the Common Core State Standards in Mathematics (CCSS-M, 2010) were intended for K-12 mathematics education, the standards impact undergraduate mathematics education. The specific impact addressed by the research reported here is the impact on the preparation of undergraduate pre-service teachers (PSTs). In particular, we examined the preparation of undergraduate elementary PSTs to teach early algebra. By *early algebra*, we mean an exploration of algebraic ideas that does not involve the symbols and equations that are normally associated with formal high school algebra (Carraher, Schliemann, & Schwartz, 2008). In other words, early algebra involves algebraic reasoning that elementary students would be able to understand.

The CCSS-M emphasizes algebraic reasoning throughout the elementary grades. The goal is for students to "begin developing an algebraic perspective many years before they will use formal algebraic symbols and methods" (Common Core Standards Writing Team, 2011, p. 13). This emphasis in the standards is consistent with the call made for early algebra by many mathematics education scholars over at least the last two decades (e.g., Cai, Ng, & Moyer, 2011; Carpenter & Levi, 2000; Carraher, Schliemann, Brizuela, & Earnest, 2006; Kaput & Blanton, 2000; Kieran, 1992; Nathan & Koellner, 2007; Radford, 2012).

Needless to say, those charged with teaching early algebra will be elementary teachers. Thus, if teacher preparation programs are going to prepare elementary PSTs to address the CCSS-M in their future classrooms, then part of this work must involve preparing undergraduate PSTs to teach early algebra. Undergraduates, who typically come to teacher preparation programs with high school algebra experience, may not have had (or do not remember) the kinds of early algebra experiences that they will soon be expected to engage their future students in. This sets up the unusual situation in which undergraduates have significant experiences with more-advanced mathematical ideas (i.e., formal algebra) and are then asked to learn less-advanced (but important) mathematical ideas (i.e., early algebra).

Currently, the research on preparing PSTs to teach early algebra is limited (for an exception, see Stevens, 2008). However, what we have observed in the two semesters since beginning to study the teaching of early algebra to undergraduates is that this topic elicits *greater-than-normal resistance*. For example, PSTs often don't understand why they need to learn about early algebra because they think it will be too difficult for children. We have thus come to the conclusion that making the move from formal to early algebra is far from trivial for undergraduates, and that it may be similar to what Beach (1999) calls a *consequential transition.*

Consequential transitions are defined as "the "developmental change in the relation between an individual and one or more social activities" (Beach, 1999, p. 114). Beach provides a number of illustrative examples, some of which reminded us of undergraduates preparing to teach early algebra. In one example, Beach describes the consequential transition that machinists went through when the trade moved from manual to computerized machining. Machinists with years of experience physically manipulating machines were suddenly faced with the challenge of switching to working with program codes. In particular, "[t]he shift in objects is difficult for highly skilled tool and die makers who may have spent 20 years on mechanical machines prior to learning computerized machining" (Beach, 1999, p. 123). Some machinists found computerized machining too great a departure from what they had been accustomed to doing and, as a result, accepted lower-tier jobs that allowed them to continue mechanical machining.

The machinist example reminded us of undergraduate PSTs learning about early algebra. Like the machinists, undergraduates, who have prior knowledge about how to solve algebra problems using their knowledge of formal high school algebra, are being asked as PSTs to learn a new way to solve the same problems. Also, like the machinists, undergraduates tend to meet the transition with resistance. These similarities suggest that, similar to how Beach investigated machinists' experiences, it could be illuminating to investigate undergraduates' experiences as they learn about early algebra. Our ongoing research into the preparation of undergraduates to teach early algebra addresses the following two research questions: (a) What conceptual challenges do undergraduates encounter when learning about early algebra? (b) How do the relations between undergraduate PSTs and early algebra ideas and activities develop over time?

Methods

Participants and Setting

Undergraduates enrolled in the third course in a sequence of three mathematics content courses that were required for an elementary and middle school teacher preparation program at a large mid-Atlantic American University were recruited to participate. The content course focused primarily on early algebra (i.e. 20 of 28 lessons were on early algebra). In particular, this course focused on "activities [that] can be engaged in without using the letter-symbolic, [but that] can be further elaborated at any time so as to encompass the letter-symbolic" (Kieran, 2004, p. 148, brackets added). This involved representing algebraic ideas with pictorial diagrams and story problems instead of with literal symbols (Cai et al., 2011; Carraher et al., 2008; Koedinger, & Nathan, 2004; Watanabe, 2011) and solving problems with quantitative reasoning instead of with symbolic manipulation of equations (Ellis, 2011; Kaput, 1995; Smith & Thompson, 2008; van Reeuwijk, 1995). Additionally, the early algebra portion of the course focused on three mathematical themes: (a) algebra as generalized arithmetic (Kaput, 1995; Nathan & Koedinger, 2000), (b) algebra from a functions approach (Chazan, 1996; Ellis, 2011; Kieran, Boileau, & Garançon, 1996), and (c) algebra and meanings of the equal sign (Knuth, Alabali, McNeil, Weinberg, & Stephens, 2005; McNeil et al., 2006).

Data Collection

Data collection thus far has consisted of interviewing 13 undergraduate participants in groups of two and three at the end of the early algebra component of the course to discuss early algebra problems. These interviews addressed the first research question. The interviews were semi-structured (Bernard, 1988) and had four parts. Participants were asked to (a) draw an informal diagram for a given story problem that involved an unknown; (b) explain a given functional relationship between two variables in a way that fourth graders could understand; (c) create a story problem for a given equation in one variable (where the variable was present on both sides of the equal sign) and solve the problem in a way that did not involve symbolic manipulation; and (d) describe their experiences with the early algebra portion of the course. During the interviews, which were video recorded, the interviewer provided no feedback on correctness of responses. To address the second research question, additional interviews with PSTs before, during and after the early algebra unit, which examine the evolving nature of the relationship between the undergraduates and early algebra are currently being conducted.

Data Analysis

The video recordings of the interviews were transcribed and a descriptive account, with minimal inference, was created. To answer the first research question, the interviews were compared to each other to see if themes emerged in terms of conceptual challenges that the undergraduates were experiencing. Based on identified themes, a coding scheme was developed, using a priori codes found in the literature. Additional codes grounded in the data were added as needed (Strauss, 1987). *Axial* and *selective coding* was used to identify relationships between codes and to develop hierarchies among codes (Strauss, 1987). The *constant comparison method* was used to further refine codes and theory (Strauss & Corbin, 1994). Intercoder reliability was examined to assess the reliability and validity of the codes and 90% agreement was achieved. To answer the second research question, additional qualitative analysis will be conducted on the interviews currently being conducted.

Results

Analysis thus far has revealed that undergraduates who explore early algebra encounter four kinds of conceptual challenges. First, 11 of 13 undergraduate PSTs *found it challenging to identify relationships in story problems that contained functional relationships*. In particular, they were either unsure how many variables were related or which two variables were related. Second, all 13 PSTs *found it challenging to distinguish between unknowns and variables*. Specifically, they referred to variables as unknowns and vice versa, they referred to variables as quantities that needed to be solved for, and they created story problems that involved quantities that did not typically vary. Third, all 13 PSTs *found it challenging to bracket their understanding of formal algebra*. For example, PSTs were unable to draw informal diagrams about story problems without first writing a symbolic expression and/or were unable to explain solving an equation without referring to symbolic manipulation. Fourth, 5 of 13 PSTs *found it challenging to draw informal diagrams of equations that involved subtraction of a constant from an unknown*. For example, some PSTs were unsure if they should represent (x-2) as two less than some underspecified quantity (correct) or if they should add two onto some underspecified quantity and then remove two (incorrect). Analyses of the data currently being collected that examines the evolving relations between PSTs and early algebra will also be reported.

Discussion

Preliminary results from this study suggest that for undergraduates training to be elementary teachers, the transition from formal to early algebra can be challenging. Furthermore, it may be that the challenges that PSTs experience with early algebra prove to be related to the resistance that they tend to exhibit toward early algebra. It is our goal that the additional analysis currently being conducted will help to answer this question.

Some of the identified challenges likely indicate weaknesses in the PSTs' understanding of algebra in general (e.g., the confusion between unknowns and variables), which then impacted their transition to early algebra. Other challenges were likely unique to transitioning to early algebra (e.g., bracketing one's understanding of formal algebra). By gaining a better understanding about the kinds of challenges undergraduates face, as well as more generally about their experiences with transitioning to early algebra, research could inform teacher preparation programs on how to better prepare undergraduate PSTs for teaching early algebra effectively.

References

Beach, K. (1999). Consequential Transitions: A sociocultural expedition beyond transfer in education. In A. Iran-Nejad & P. D. Pearson (Eds.), *Review of Research in Education* (Vol. 24, pp. 101-139). Washington, DC: American Educational Research Association.

Bernard, H. E. (1988). *Research methods in cultural anthropology.* Beverly Hills: Sage.

Cai, J., Ng, S. F., & Moyer, J. C. (2011). Developing students' algebraic thinking in earlier grades: Lessons from China and Singapore. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 25-41). Heidelberg, Germany: Springer.

Carpenter, T. P., & Levi, L. (2000). *Developing conceptions of algebraic reasoning in the primary grades* (Report No. 00-2). Madison, WI: National Center for Improving Student Learning and Achievement in Mathematics and Science.

Carraher, D. W., Schliemann, A. D., & Brizuela, B. M. (2001). Can young students operate on unknowns? In *Proceedings of the XXV Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 130-140). Utrecht, The Netherlands: PME.

Carraher, D. W., Schliemann, A. D., Brizuela, B. M., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics Education, 37*(2), 87-115.

Carraher, D. W., Schliemann, A. D., & Schwartz, J. L. (2008). Early algebra is not the same as algebra early. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 235-272). Mahwah, NJ: Erlbaum.

Chazan, D. (1996). Algebra for all students? *Journal of Mathematical Behavior, 15*(4), 455-477.

Common Core State Standards Initiative (2010). *Common Core State Standards in Mathematics*. Retrieved from: http://www.corestandards.org/assets/CCSSI_Math%20 Standards.pdf

Common Core Standards Writing Team (2011). *K, counting and cardinality; K–5, operations and algebraic thinking. Progression for the Common Core State Standards in Mathematics* (Draft). Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

Ellis, A. B. (2011). Algebra in the middle school: Developing functional relationships through quantitative reasoning. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 215-238). Heidelberg, Germany: Springer.

Kaput, J. J. (1995). Long-term algebra reform: Democratizing access to big ideas. In C. Lacampagne, W. Blair, & J. Kaput (Eds.), *The algebra colloquium,* (Vol. 1, pp. 1-44). Washington, DC: US Department of Education.

Kaput, J. J., & Blanton, M. L. (2000). *Algebraic reasoning in the context of elementary mathematics: Making it implementable on a massive scale.* Dartmouth, MA: National Center for Improving Student Learning and Achievement in Mathematics and Science.

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390-419). New York: Macmillan.

Kieran, C., Boileau, A., & Garançon, M. (1996). Introducing algebra by means of a technology-supported functional approach. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 257-294). Dordrecht, The Netherlands: Kluwer.

Knuth, E. J., Alabali, M. W., McNeil, N. M., Weinberg, A., & Stephens, A. C. (2005). Middle school students' understanding of core algebraic concepts: Equivalence and variable. *ZDM, 37*(1), 68-76.
Koedinger, K. R., & Nathan, M. (2004). The real story behind story problems: Effects of representations on quantitative reasoning. *Journal of the Learning Sciences, 13*(2) 129-164.
McNeil, N. M., Grandau, L., Knuth, E. J., Alibali, M. W., Stephens, A. C., Hattikudur, S., & Krill, D. E. (2006). Middle-school students' understanding of the equal sign: The books they read can't help. *Cognition and Instruction, 24*(3), 367-385.
Nathan, M. J., & Koedinger, K. R. (2000). Teachers' and researchers' beliefs about the development of algebraic reasoning. *Journal for Research in Mathematics Education, 31*(2), 168-190.
Radford, L. (2012). *Early algebraic thinking, epistemological, semiotic, and developmental issues*. Regular lecture presented at the 12th International Congress on Mathematical Education, Seoul, Korea. Retrieved at April, 7, 2014, from http://www.icme12.org/upload/submission/1942F.pdf
Smith, J., & Thompson, P. W. (2008). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 95-132). New York: Erlbaum.
Stephens, A. C. (2008). What "counts" as algebra in the eyes of preservice elementary teachers? *Journal of Mathematical Behavior 27,* 33-47.
Strauss, A. L. (1987). *Qualitative analysis for social scientists.* Cambridge University Press.
Strauss, A., & Corbin, J. (1994). Grounded theory methodology: An overview. In N. K. Denzin and Y. S. Lincoln (Eds.), *Handbook of Qualitative Research* (pp. 273-285). Thousand Oaks: Sage Publications.
van Reeuwijk, M. (1995, April). *The role of realistic situations in developing tools for solving systems of equations.* Paper presented at the annual meeting of the American Educational Research Association, San Francisco, CA.
Watanabe, T. (2011). Shiki: A critical foundation for school algebra in Japanese elementary school mathematics. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 215-238). Heidelberg, Germany: Springer.

The role of examples in understanding quotient groups

Carolyn McCaffrey James
Portland State University

This poster investigates two students' use of examples in the transition to advanced mathematical thinking in the context of a teaching experiment focusing on quotient groups. Based upon the theoretical perspective of Edwards, Dubinsky, and McDonald (2008), this poster extends the construct of imperfect models to include instances of generic examples. This study found that students leveraged examples to construct conjectures, form justifications, clarify misconceptions, and provide analogies for further reasoning. However, attending to the specific features of particular examples also led to over-generalizations and invalid reasoning.

Literature and Theoretical Framework

Imperfect models are a means of representing some, but not all, of the characteristics of an abstract mathematical object (Edwards, Dubinsky and McDonald 2005). This poster extends the construct of imperfect models to include reasoning based in generic examples (Mason & Primm, 1984). Any specific example is necessarily imperfect: it is only a single representative of an entire class of objects. However, the specific example can provide a means for describing abstract mathematical properties that transcend the limitations of the specific case. This poster describes how specific examples can be leveraged to help students transition to advanced mathematical reasoning within the context of reinvention of cosets and quotient groups.

Background and Methodology

Data for this analysis is drawn from a larger design experiment whose goal was to create an instructional approach for supporting the guided reinvention of the quotient group concept. The instructional intervention used in this study follows the philosophy of guided reinvention as described by Freudenthal (1991): students create mathematics through authentic interaction with challenging problems. Two student participants participated in ten 60 to 90 minute videotaped sessions. In the first stage of analysis a team of three researchers independently identified instances in which examples were used. Next, the researchers compared instances and identified themes. Finally, the first author returned to the video data and triangulated content log analysis with video, video transcript, and student work to find evidence for or against these emergent themes.

Results

We identified four primary ways that the students used examples: generic examples as generative representatives, generic examples as justifications, examples clarifying misconceptions, and example-based over-generalizations. Examples served as generative representatives when students reasoned generally about a specific example to help create new insight or understanding. Similarly, students used generic examples to justify claims. The distinction between these lies in the use of generic examples: to inform or to prove. Examples and counterexamples were also used to clarify student misconceptions. Finally, we found that examples occasionally hindered student progress: students constructed faulty arguments based on an invalid generalization of a particular feature of a specific example.

References

Edwards, B., Dubinsky, E., & McDonald, K. (2005). Advanced mathematical thinking. *Mathematical Thinking and Learning*, *7*(1), 15–25.

Freudenthal, H. (1991). *Revisiting mathematics education: The China lectures*. Dordrecht: Kluwer.

Mason, J. & Pimm, D. (1984). Generic Examples: Seeing the General in the Particular. *Educational Studies in Mathematics* 15(3), 277-290.

Commognitive conflicts in the discourse of continuous functions

Gaya Jayakody
Simon Fraser University

This paper reports on two commognitive conflicts that were identified in a larger research study conducted on university first year students' discourse on continuous functions. The study looks at continuity related aspects under a participationist, discursive lens in contrast to previous studies on continuity that have used cognitive theories that hold an acquisitionist view on learning. The study adopts Sfard's commognitive framework to analyze data. Among findings are different ways in which students use the word 'domain' and how they struggle with inconsistent 'realizations' arising from different definitions presented in text books.

Key words: Calculus, Continuous functions, Commognitive conflict, Realizations, Discourse

Introduction

Continuous functions has been a topic of interest for a number of researchers in Undergraduate Mathematics Education who have focussed on Calculus. Even though small in number, these studies have revealed some important findings around the understanding and learning of continuous functions by both students and teachers, a couple of which will be looked at in the forthcoming sections. However, they have not been able to move much forward past findings such as most common concept images of continuous functions. One reason for this stagnant nature might be the reliance of all these studies on theoretical frameworks that assumes an acquisitionist stance on concept formation. A few such most commonly used frameworks were Tall and Vinner's 'concept image and concept definition' and Skemp's 'theory of knowledge acquisition' along with his 'two types of understandings- instrumental and relational'. The current study strives to take a new look at the issues around the learning of continuous functions through a discursive lens. While the larger study uses Sfard's commognitive framework to analyse university first year students' discourse on continuous functions at large, this paper aims to report on a particular aspect of this discourse. This particular part of the study was guided by the following research question.

What *commognitive conflicts* arise within university first year students' discourse on continuous functions?

Literature

This section reviews the literature on continuous functions very briefly. Due to restriction of space, I only aim to point out the 'common images' of continuous functions held by both students and teachers as reported in previous studies. Many studies have found that a common reason thought by students for a function to be continuous was its graph being in one piece (Vinner, 1987; Tall & Vinner1981; Mastorides & Zachariages, 2004). The function being defined at a point and limit existing at a point were another two common conceptions found to be held by students to validate its continuity (Vinner, 1987; Bezuidenhout, 2010; Tall & Vinner, 1981). In addition to these cognitive accounts, Jayakody (2014) reports on conflicts between different ways in which students communicate about continuous functions that has connections to what is presented in this paper.

A diversion: A discursive theoretical framework

Deviating from the commonly used cognitive based frameworks on studies on continuity, this study adopts Sfard's communicational approach to cognition to analyse data. While acquisitionists view communication as a mere window to 'thinking' and 'mental schemas', Sfard views communication as tantamount to thinking. The main premise in this framework is that thinking is an individualized version of interpersonal communication. The theory views Mathematics as an autopoietic system that creates objects of its study. Mathematical objects are hence discursive objects. Three further constructs are primarily used in this paper. They are *signifiers*, *realizations* and *commognitive conflicts*. A *signifier* is a string of words or symbols that function as a primary object in the mathematical discourse. A *realization* of a signifier is a "perceptually accessible thing so that every endorsed narrative[1] about the signifier can be translated according to well defined rules into an endorsed narrative about the realization" (Sfard, 2008, pg.154). In other words, a realization is a possible way of communicating about a signifier. When all the realizations of a signifier are organized in a hierarchical manner, it is called a *realization tree*. A *commognitive conflict* is when two narratives originate in incommensurable discourses (discourses that differ in their use of words etc.) which can be interpersonal or intrapersonal.

Design of Study

The larger study consists of administration of a questionnaire to 54 students and interviews with 5 selected students from them. The students who took part in the study were first year undergraduate students who were taking a Calculus 1 course in a large university in Western Canada. A few days after the students learned the topic of continuity in their course, they were given a questionnaire in which they had to first *describe* and then *define* what a continuous function is and then to choose a set of functions given both in graphical and formula form as continuous or not giving reasons for their choices (See Appendix). Five students were then selected for interviews (semi-structured) based on their questionnaire responses. The interviews were transcribed and then realization trees were constructed for each of the five students for a continuous function and different features and patterns in their discourse were observed.

In addition, as part of the study, various mathematical resources in introductory level Calculus and Analysis were examined to identify different ways in which continuity related concepts are defined. It was found that there are two definitions used for 'continuity at a point' and also for a 'continuous function'. Only two labels are used DeCon1 and DeCon2 since the four definitions could be grouped into just two that were consistent. These four definitions are given in Table 1.

	Continuity at a point	Continuous function
DeCon1	A function f is said to be continuous at c if, 1. $f(x)$ is defined at $x = c$	A function is continuous if it is

[1] In mathematics, endorsed narratives are those that constitute mathematical theories.

	2. $\lim_{x \to c} f(x)$ exists. 3. $\lim_{x \to c} f(x)$ is equal to $f(c)$ f is discontinuous if any of the above conditions are not satisfied. (e.g., Tan, Menz & Ashlock 2011)	continuous on all real numbers. (e.g., Anton, 1995)
DeCon2	A function f is said to be continuous at $x = c$ in its domain if, $\lim_{x \to c} f(x) = f(c)$ And f is discontinuous at $x = c$ in its domain if, $\lim_{x \to c} f(x) \neq f(c)$ (e.g., Stahl, 2011)	A function is continuous if it is continuous on its domain. (e.g., Strang, 1991)

Table 1: Different definitions for continuity related concepts

It is noted that if DeCon1 for a continuity at a point and DeCon2 for a continuous function is applied together, it can result in seemingly inconsistent narratives about the continuity of a function. For instance, a function that is not defined at a point has a point of discontinuity at that point according to DeCon1 (for continuity at a point) but is a continuous function according to DeCon2 (for a continuous function).

The students in the study had only learnt the definition for continuity of a function at a point. And this definition was DeCon1.

Results

The interview data revealed two types of commognitive conflicts in students' discourse on continuous functions. I draw on the interviews with the students Jane, Chirag and Geet to illustrate these commognitive conflicts. The first step of analysis was to construct realization trees for the signifier 'continuous function' for the five students that I conducted interviews with. In order to keep the length of the paper within limit, I do not present these realization trees here but refer to the realizations in the discussion on commognitive conflicts. Realizations of a signifier in the same level were labelled with italicized bold-faced capital letters starting with ***A***. When these are further realized down to another level, numbering was started with ***A1*** and continued as ***A2***, ***A3*** and so on. In the following illustrations, some of these realizations will be referred to.

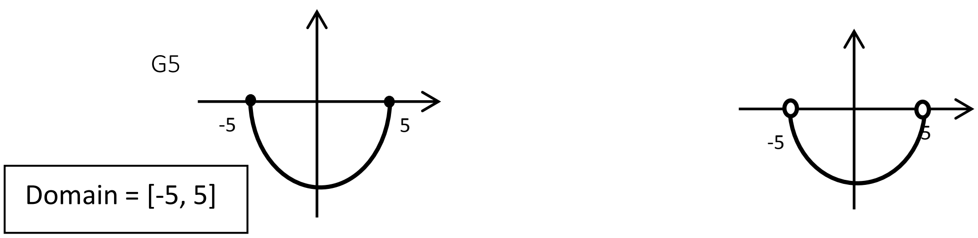
1. Commognitive conflict between the discourses of interviewer and interviewee on the same signifier 'domain'

Certain ways Jane and Chirag talked about 'domain' indicated that there was a commognitive conflict between my discourse and their discourse on the signifier 'domain'. We appeared to mean different things when we used the signifier 'domain' in certain parts of the conversation. By analyzing these utterances, this commognitive conflict could be suggested to have arisen due to the lack of the following signifier, realization pair, regarding domain.

There is a ‘hole’ in the graph (A point at which the function is not defined)	$\longleftrightarrow$	The corresponding x is not in the domain of the function

Below are two different utterances by Jane and Chirag that exemplify their use of ‘domain’.

Example 1- G5 was the fifth function represented in graphical form in the questionnaire (See Figure 1a). I drew a slight variation of G5 during the interview with Jane (See Figure 1b).



a) Graph G5 in questionnaire b) Graph I drew similar to G5

Figure 1

Referring to the graph that I drew during the interview, Jane said the following:

Jane: *If I was given this (pointing to Figure 1b) but the domain I was given that (*pointing to [-5, 5] in Figure 1a*), then that is not, it’s not continuous. But if I was given that (*pointing to (-5, 5) that I had written before*), I would assume that it was continuous*

Example 2 - Chirag referring to the sixth graph in the questionnaire, G6, (See Figure 2) said the following:

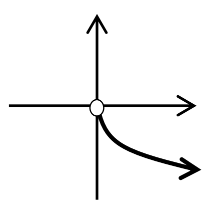


Figure 2: G6

Chirag: *It’s continuous if x is greater than 0, but it’s not continuous if 0 is included in the domain, yeah*

Also Jane, Chirag and Geet failed to identify the domain of some functions where they did not identify a ‘hole’ in the graph (a point at which the function is not defined) as a point that is excluded from the domain of the function. Below are three utterances in which each of the students expressed the domain of the function represented by the second graph G2 (See Figure 3) as all real numbers. These examples show how these three students’ way of talking about ‘domain’ is different from how the researcher (and the reader) would talk about it.

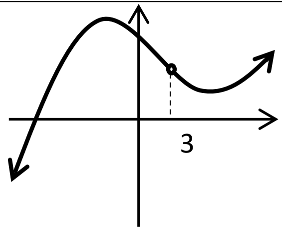


Figure 3: G2

Jane: *All real numbers (Responding to what the domain of G2 is)*
Chirag: *We don't know, infinity, negative infinity to infinity? (Referring to the domain of G2)*
Geet: *Because it is not continuous at this point (pointing to 3), this whole thing is the domain (Referring to G2)*

2. Commognitive conflict between the discourses arising from different realizations for the same individual

In addition to the commognitive conflict that occurred between the discourses of the participants and mine, there was another kind of conflict that was observed in Jane, Chirag and Geet's interviews. This conflict occurred *within* their own discourse and the conflict was *between* two realizations that they were using. For instance, Jane faced a tension between her realizations ***C*** and ***D*** (A detailed account of this can be found in Jayakody (2014)). Similar tensions were observed in Chirag's discourse between his realizations ***B*** and ***C***. And also Geet showed a tension when he was using his realizations ***B*** and ***A1***. One interpretative elaboration is presented in this paper to illustrate the tension created. These conflicts seemed to arise out of the discordance between the two definitions DeCon1 and DeCon2 that the realizations were based on. The realizations between which Jane, Chirag and Geet had tensions and their accordance with the two definitions DeCon1 and DeCon2 are given in Table 2 below.

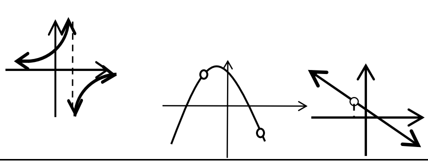
	Realizations in accordance with DeCon1	Realizations in accordance with DeCon2
Jane	***D*** - Functions that look like these with holes and asymptotes are excluded from continuous functions 	***C*** – For every point in its domain, $f(a)$ is defined and $\lim_{x \to a} f(x) = f(a)$
Chirag	***B*** - No discontinuity according to DeCon1	***C*** - A function that is continuous on its domain
Geet	***A1*** - A function that is defined, that doesn't have a hole, break or a jump	***B*** - Continuous on its domain

Table 2: Consistency of realizations that created tension, with DeCon1 and DeCon2

Table 3 gives an interpretative elaboration (a text that, utterance by utterance, elaborates on the text produced by the speakers) that presents and elaborates the tension created due to the conflict between Chirag's realizations ***B*** and ***C***. The episode refers to the graph G2 (See Figure 3) in the questionnaire. 'N' indicates the interviewer (the author) and 'C' indicates Chirag.

Uttera-nce no.	Who said	What is said	What is done	Interpretative elaboration
C501	N	Then what about this one?	Points to G2	I invite Chirag to reconsider the continuity of G2
C502	C	It's continuous except for, it's a continuous function except that point *x* equals 3		In the questionnaire he has classified G2 as 'not continuous', but now identifies as 'continuous except at 3'
C503	N	Oh ok, what does that mean? It's a continuous function except at point 3 means?		I want to see whether he can classify it as a 'continuous function' with a discontinuity at 3.
C504	C	There's a point of discontinuity at 3		Chirag rightly identifies 3 as a point of discontinuity according to realization ***C***.
C507	N	But the function is continuous?		
C508	C	Yeah, but I don't know, yeah because we put that so		Chirag accepts that G2 is continuous but his 'I don't know' reflects his uncertainty about it
C509	N	Yeah		
C510	C	So we don't really care what happens at 3. That's what, that's what I'm basing my thesis, do we really don't care what happens at that *x* value and I don't think we do so it's continuous throughout		He reasons to himself why G2 is continuous. He is using realization ***B***. He is explicit about "the thesis he's basing his conclusion on".
C513	N	Does that bother you?		He looks bothered and I ask him about it.
C514	C	Yeah	Laughs	He admits that it bothers him.
C515	N	It does?		
C516	C		Laughs	
C517	N	There is a discontinuity but it's a continuous function		I put forth the reason for the botheration: the two results from ***C*** and ***B*** explicitly;
C518	C	Yeah but it's a continuous function		And Chirag agrees

Table 3: Interpretative elaboration for Chirag's utterances from [C501] to [C518] that elaborates a tension between the realizations ***B*** and ***C*** for continuity

Table 3 shows how Chirag faces tension having to accept that a function can be called as continuous when it has a discontinuity at a point. Note that a transcript lacks many components of communication and only shows the verbal utterances and "does it bother you?" in [C513] was deliberately asked to have a written record of Chirag's tension expressed through facial expressions. Jane and Geet showed similar tensions throughout their interviews whenever they were trying to use the realizations that were based on DeCon1 and DeCon2.

Discussion

The paper illustrates two commognitive conflicts in the discourse of continuous functions; one interpersonal and one intrapersonal. The *interpersonal* commognitive conflict of using the term 'domain' has not been identified in earlier studies under cognitive lenses in cognitive terms. This conflict bears information about how students might be thinking of functions and domain of functions in general and hence has implications for teaching.

The second conflict which is *intrapersonal,* is the first to be recorded of the kind. In addition to being *intrapersonal,* this conflict highlights its origin. The conflict is between two realizations for the same signifier for the *same* individual. I have also attempted to frame these conflicting realizations as arising from nothing but the inconsistent definitions used for continuity; a concern that is seen to be present in textbooks, mathematical websites, or even arguably within classroom instruction and discourse.

References

Anton, H. (1995). Calculus with analytic geometry (5th ed.). New York, NY: Wiley and Sons, Inc.

Bezuidenhout, J. (2001). Limits and continuity: Some conceptions of first-year students. *International Journal of Mathematical Education in Science and Technology*, *32*, (4), 487-500. doi: 10.1080/00207390010022590

Jayakody, (2014). A case study of conflicting realizations of continuity. *Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 3). Vancouver, Canada: PME.

Mastorides, E., Zachariades, T. (2004). Secondary mathematics teachers' knowledge concerning the concept of limit and continuity. *Proceedings PME* 28, Vol. 4, 481-488

Sfard, A. (2008). Thinking as communicating: Human development, the growth of discourses, and mathematizing. New York: Cambridge University Press.

Stahl, S. (2011). Real Analysis: A Historical Approach (2nd ed.). New Jersey: Wiley.

Strang, G. (1991). Calculus. USA: Wellesley-Cambridge Press.

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151-169.

Tan. S., Menz, P., Ashlock, D. (2011), Applied Calculus for the managerial, life, and social sciences. Nelson Education Ltd.

Vinner, S. (1987), Continuous functions-images and reasoning in College students: Proceeding PME 11, II, Montreal, 177-183.

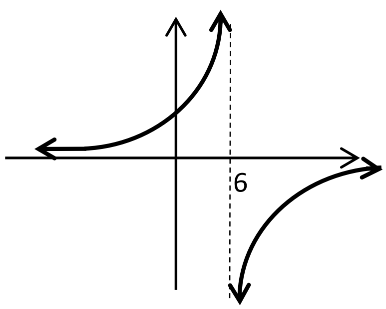
Appendix

<u>Questionnaire</u>

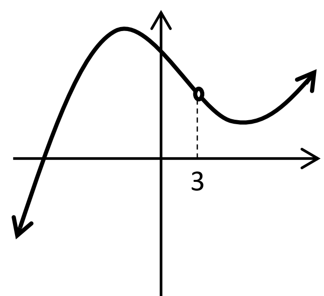
Answer all questions.

1. What do you understand by a ‘continuous function’?
2. Give the **definition** of :
   a) ‘continuity of a function at a point
   b) ‘continuous function’
3. Below are the graphs of six functions. Determine whether the functions are continuous or not. **Give reasons for your answer**. If any, state the point/points of discontinuity.

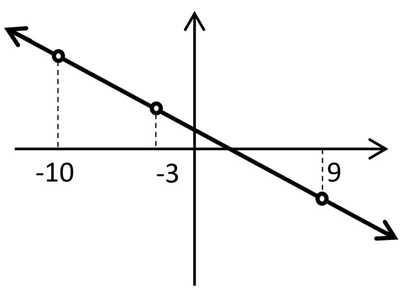
a)



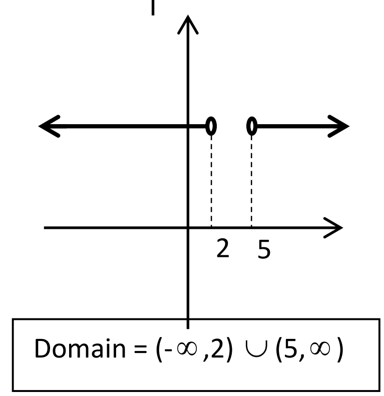
b)



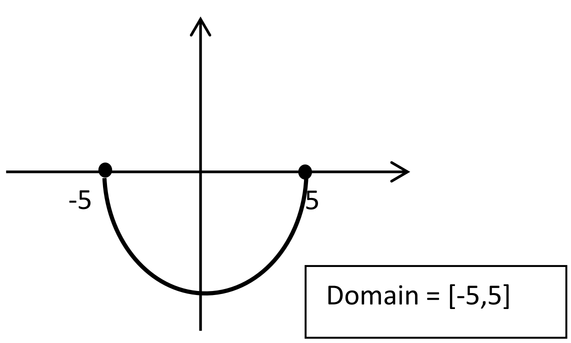
c)



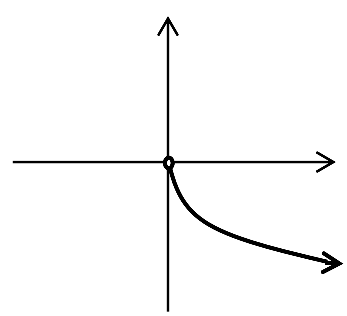
d)



e)



f)



4. Below are six functions given by their formulae. Determine whether the functions are continuous or not. **Give reasons for your answer**. If any, state the point/points of discontinuity.

a) $f(x)=\dfrac{1}{x}$

b) $f(x)=\dfrac{x^2(x-5)}{(x-5)}$

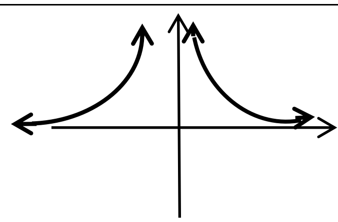
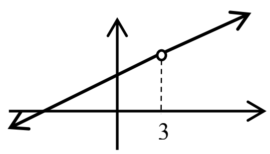
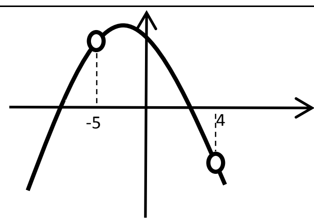
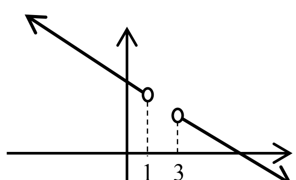
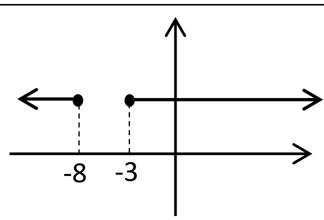
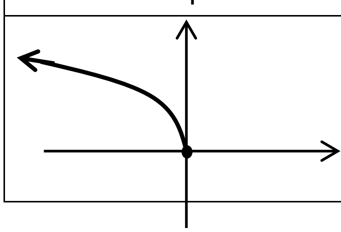
c) $f(x)=\dfrac{(x+2)(x-3)(x+1)}{(x-3)(x+1)}$

d) $f(x)=7 \quad ; \; x\in(-\infty,10]\cup[15,\infty)$

e) $f(x)=x^2 \quad ; \; x\in(-10,10)$

f) $f(x)=\sqrt{x} \; ; \; x\geq 0$

5. Place a tick in the correct box.

Function	Is a continuous function	Is not a continuous function
		
		
		
		
		
		

Towards a Measure of Inquiry-Oriented Teaching

Estrella Johnson
Virginia Tech

Abstract: Over the last decade, undergraduate mathematics researchers and curriculum developers have generated inquiry-oriented curriculum materials for courses from calculus through abstract algebra. These materials present a number of challenges for implementation, and as such, an instructional quality measure becomes a necessary requirement for making sense of student learning in these classrooms. The work here represents an initial attempt to define and map the domains of inquiry-oriented teaching. Specifically, classroom video data will be analyzed using a conjectured list of critical components of inquiry-oriented teaching. This analysis will be used to 1) refine/test the initial characterization of inquiry-oriented teaching, and 2) generate examples of how these critical components are actualized during instruction.

Key Words: Instructional Measure, Teaching, Inquiry-oriented

Researchers and curriculum developers have responded to the call for instructional improvements in undergraduate STEM education by developing numerous student-centered curricular innovations. In practice, student-centered instruction has been shown to support conceptual learning gains (e.g., Kogan & Laursen, 2013; Kwon, Rasmussen, & Allen, 2005; Larsen, Johnson, & Bartlo, 2013), diminish the achievement gap (e.g., Kogan & Laursen, 2013; Riordan & Noyce, 2001; Tarr et al., 2008), and improve STEM retention rates (e.g., Hutcheson, Pampaka, & Williams 2011; Rasmussen, Ellis, & Bressoud, 2013; Seymour & Hewitt 1997). While the research and development of student-centered curricular materials and the research showing positive student results are important, they miss a key component of undergraduate education – teacher practice. While undergraduate mathematics education researchers have produced only a small amount of research on the teaching practices of mathematicians (Speer, Smith, & Horvath, 2010), research at the K-12 level has provided significant evidence that instructional quality is a primary indicator of student achievement (e.g., Nye, Konstantopoulos, & Hedges, 2004; Ottmar, Rimm-Kaufman, Larsen, & Merritt, 2011). The lack of research into mathematicians' teaching practices and the K-12 research indicating the importance of instructional quality implies a gap in research regarding instructional quality at the university level.

Within the undergraduate mathematics community, the last decade has seen a sharp rise in inquiry-oriented, research based, instructional innovations. Inquiry-oriented teaching is a student-centered pedagogy being used in mathematics classes from calculus through abstract algebra. The limited research that does exist on mathematicians teaching practices has shown that these inquiry-oriented curricular materials present a number of challenges for implementation. Such challenges include: developing an understanding of student thinking, planning for and leading whole class discussions, and building on students' solution strategies and contributions (Johnson & Larsen, 2012; Rasmussen & Marrongelle, 2006; Speer & Wagner, 2009; Wagner, Speer, & Rossa, 2007). Given these challenges with the implementation of inquiry-oriented instructional materials, the need for a measure of instructional quality becomes a necessary requirement for making sense of student learning in these classrooms.

Before such an instrument can be developed and used as a measure of inquiry-oriented teaching, the concept of "inquiry-oriented teaching" first needs to be operationalized in a way that can be observed, measured, and analyzed. The work here represents an initial attempt to define and map the domains of inquiry-oriented teaching. Specifically, classroom video data will be analyzed using a conjectured list of critical components of inquiry-oriented teaching. This analysis will be used to 1) refine/test the initial characterization of inquiry-oriented teaching, and 2) generate examples of how these critical components are actualized during instruction.

Theoretical Perspective on Inquiry-Oriented Teaching

This study takes place in the classrooms of mathematicians implementing the Inquiry-Oriented Linear Algebra (IOLA) curriculum. The design of the IOLA curriculum was guided by the instructional design theory of Realistic Mathematics Education (RME) design heuristic. In addition to informing the development of the IOLA curriculum, RME also heavily influences how the curriculum materials are *intended* to be implemented. As described by Wawro et al. (2012), the IOLA curriculum was developed to be consistent with the idea that mathematics is a human activity, as opposed to a collection of predetermined truths (Freudenthal, 1991). This perspective can have significant implications for the structure of a course utilizing the IOLA materials, and ideally induces a classroom environment in which 1) students "learn new mathematics through *inquiry* by engaging in mathematical discussions, posing and following up on conjectures, explaining and justifying their thinking, and solving novel problems", and 2) "teachers routinely *inquire* into their students' mathematical thinking and reasoning" (Rasmussen & Kwon, 2007, p. 190). This duality between student inquiry into the mathematics and teacher inquiry into student thinking helps to ensure that "the classroom participants (teachers and students) lay down a mathematical path as they go, rather than follow a well-trodden trajectory" (Yackel, Stephan, Rasmussen & Underwood, 2003). Therefore, the goal of instructor inquiry into student thinking goes beyond merely assessing students, and seeks to reveal students' intuitive and informal ways of reasoning, especially those that can serve as building blocks for more formal or conventional ways of reasoning. The instructor's role is to guide and direct the mathematical activity of the students by listening to students and using their own reasoning to support the development of new conceptions.

Data and Methods

In order to investigate the different dimensions that comprise inquiry-oriented teaching, preliminary analysis was carried out on classroom video data of an instructors as she implemented an inquiry-oriented linear algebra (IOLA) unit in a freshman/sophomore level, introductory linear algebra course. Dr. Roberts, teaches at a small private university. This was her first implementation of the IOLA materials and she had 20 students. As Dr. Roberts implemented the IOLA instructional units, classroom video data was collected[1]. The camera was focused on the instructor, and both the instructor and the students are audible when the teachers pose tasks, lectures, and hold whole class discussions. These whole-class segments are the primary data source for this analysis.

In order to begin to map the domains of inquiry-oriented teaching, the teachers' activity and

---

[1] This data was collected as part of the NSF funded grant, *Collaborative Research: Developing Inquiry-Oriented Instructional Materials for Linear Algebra* (DUE-1245673, 1245796, and 1246083), M. Wawro (PI), M. Zandieh and C. Rasmussen (co-PIs).

practices were analyzed during these whole-class segments. An analytic framework informed by the research literature guided this analysis. Initially, by generalizing from the RME literature and from empirical studies of instructors implementing inquiry-oriented curricula materials, six critical components of inquiry-oriented teaching were generated:

1. Teachers engage students in challenging tasks that encourage development of important mathematical ideas.
2. Teachers actively inquire into students' thinking.
3. Teachers use student contributions to advance the mathematical agenda of the class.
4. Teachers support students to engage in one another's thinking.
5. Students, instructors, and materials work together to build coherent understanding of important mathematical ideas.
6. Instructors introduce a minimal amount of language and notation *prior* to students' engagement with a task; language and notation is introduced only when a need for it has been established.

The analysis of Dr. Roberts' classroom video data focuses on if/how these components appeared in her implementation of the IOLA materials. This analysis will serve to test the conjecture that these components are in fact important aspects of IO teaching and to generate examples of how these components may (or may not) be actualized during instruction.

Results of Preliminary Analysis

An initial analysis showed some evidence of all six of our conjectured critical components. Below is segment of transcript from Dr. Roberts' class in which a student is describing his group's work on the first task of the IOLA materials to the rest of the class (figure 1 shows the task and figure 2 shows the solution this group presented to the class).

You are a young traveler, leaving home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the *hover board's* movement by the vector $\begin{bmatrix}3\\1\end{bmatrix}$.
By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 miles East and 1 mile North of its starting location.

We denote the restriction on the *magic carpet's* movement by the vector $\begin{bmatrix}1\\2\end{bmatrix}$.
By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 mile East and 2 miles North of its starting location.

SCENARIO ONE: THE MAIDEN VOYAGE

Your Uncle Cramer suggests that your first adventure should be to go visit the wise man, Old Man Gauss. Uncle Cramer tells you that Old Man Gauss lives in a cabin that is 107 miles East and 64 miles North of your home.

TASK:
Investigate whether or not you can use the hover board and the magic carpet to get to Gauss's cabin.
If so, how? If it is not possible to get to the cabin with these modes of transportation, why is that the case?

As a group, state and explain your answer(s) on the group whiteboard. Use the vector notation for each mode of transportation as part of your explanation and use a diagram or graphic to help illustrate your point(s).

Figure 1. Task 1 of IOLA Materials

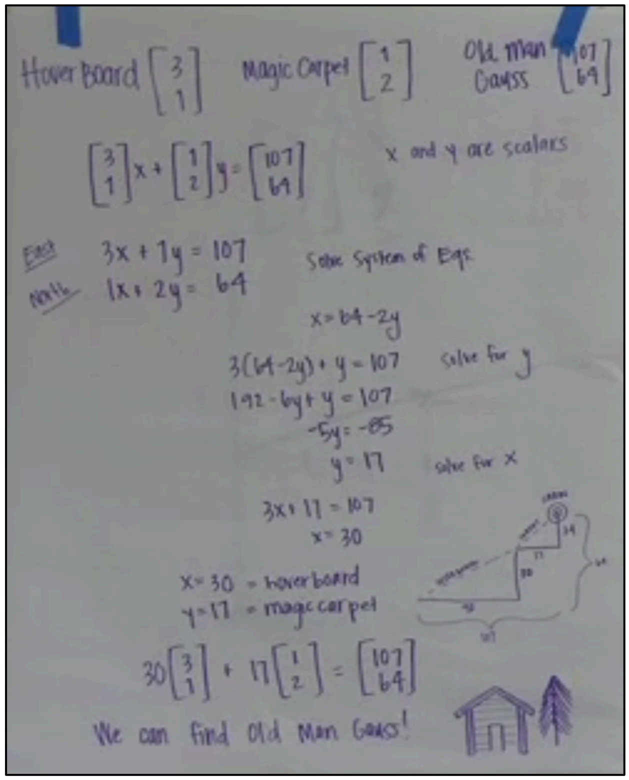


Figure 2. Group 1's Work on Task 1

Group 1 Representative: So with the hover board, we had to figure out how many hours he would go on the hover board, and then how many hours he would go on the magic carpet to get to the eventual 107 over 64.

Instructor: Good, pause. What represents the number of hours he spent on the hover board?

Group 1 Representative: That is the x

Instructor: Ah [points to the x on the paper] got it. OK. OK so I spend x hours here and y hours there, and then I need to…

Group 1 Representative: Get there eventually.

Instructor: end up here, OK. We on the same page about that? I saw this in a lot of places, so yeah, OK great, keep going.

Group 1 Representative: So I wrote that they are scalars, so then I just um…made two different equations for the east and the north. And so, that's what those are. And then I just solved for the x and the y, which were specifically the hover board and the magic carpet.

Instructor: Uhh…they're not the hover board and magic carpet themselves…

Group 1 Representative: Er, the hours.

Instructor: OK, terrific. Yeah, very good. Very good. So, excellent start. … this is called vector equation because it's an equation involving these vectors. Then we break it up into what's called a system of linear equations; they are linear because the highest power in any of our variables is one, and we're just putting scalars in front of them. So going from the vector equation to the system of equations…why are we allowed to do this? Like what suggests – yeah, Chris.

Chris: 'Cause uh, both vectors have like x components and y components – you just break it up…and make an equation of all x components going together and all the y components.

The students' work on this task, and Dr. Roberts' decision to have students present their work to the class, reflects component 1 and 4, respectively. Further, Dr. Roberts' question "So going from the vector equation to the system of equations…why are we allowed to do this?" can be seen as evidence of components 2, 3, and 5. Finally, Dr. Roberts choosing to introduce the term "vector equation" in reference to student work reflects component 6.

Future analysis will continue to investigate the extent to which these components are present in Dr. Roberts' implementation of the IOLA materials. Analysis will also be carried out on other instructors' implementations of the IOLA materials. This will allow for further refinements of the components and additional documentation of the various ways these components are enacted. Finally, analysis will turn to indications of how these components actually influence the student learning.

Questions for Discussion

1) How do these components of IO teaching overlap/related to other student-centered instructional approaches (e.g., Inquiry Based Learning)?
2) Are there aspects of inquiry-oriented, student-centered, or lecture that should have been included? For instance, is "mathematical correctness" or "formality of definitions" something that should be included?
3) Ideally, the measurement that is the ultimate goal of this work would be sensitive enough to differentiate between instruction in a "Modified Moore Method" course and an IO course. Do you expect there to be difference in these two pedagogical approaches given the list of critical components?

References

Freudenthal. H. *(1991). Revisiting mathematics education.* Dordrecht, The Netherlands: Kluwer Academic.

Hutcheson, G. D., Pampaka, M., & Williams, J. (2011). Enrolment, achievement and retention on 'traditional' and 'Use of Mathematics' pre-university courses. *Research in Mathematics Education*, *13*(2), 147-168.

Kogan, M., & Laursen, S. L. (2013). Assessing long-term effects of inquiry-based learning: A case study from college mathematics. *Innovative higher education*, 1-17.

Kwon, O. N., Rasmussen, C., & Allen, K. (2005). Students' retention of mathematical knowledge and skills in differential equations. *School Science and Mathematics*, *105*(5), 227-239.

Larsen, S., Johnson, E., & Bartlo, J. (2013). Designing and scaling up an innovation in abstract algebra. *The Journal of Mathematical Behavior*.

Ottmar, E. R., Rimm-Kaufman, S. E., Larsen, R., & Merritt, E. G. (2011). Relations between Mathematical Knowledge for Teaching, Mathematics Instructional Quality, and Student Achievement in the Context of the" Responsive Classroom (RC)" Approach. *Society for Research on Educational Effectiveness*.

Nye, B., Konstantopoulos, S., & Hedges, L. V. (2004). How large are teacher effects?. *Educational evaluation and policy analysis*, *26*(3), 237-257.

Rasmussen, C., Ellis, J., & Bressoud, D. (2013) Who is switching out of STEM and why? Manuscript under review, Journal for Research in Mathematics Education.

Rasmussen, C. & Kwon, O. (2007). An inquiry oriented approach to undergraduate mathematics. *Journal of Mathematical Behavior*. 26: 189–194.

Rasmussen, C., & Marrongelle, K. (2006). Pedagogical Content Tools: Integrating Student Reasoning and Mathematics in Instruction *Journal for Research in Mathematics Education, 37*(5), 388-420.

Riordan, J. E., & Noyce, P. E. (2001). The impact of two standards-based mathematics curricula on student achievement in Massachusetts. *Journal for Research in Mathematics Education*, 368-398.

Seymour, E., & Hewitt, N. M. (1997). *Talking about leaving: Why undergraduates leave the sciences* (pp. 115-116). Boulder, CO: Westview Press.

Speer, N. M., Smith, J. P., III, & Horvath, A. (2010). Collegiate mathematics teaching: An unexamined practice. *Journal of Mathematical Behavior*, 29, 99–114.

Speer, N. M., & Wagner, J. F. (2009). Knowledge Needed by a Teacher to Provide Analytic Scaffolding During Undergraduate Mathematics Classroom Discussions. *Journal for Research in Mathematics Education, 40*(5), 530-562.

Tarr, J. E., Reys, R. E., Reys, B. J., Chávez, Ó., Shih, J., & Osterlind, S. J. (2008). The impact of middle-grades mathematics curricula and the classroom learning environment on student achievement. *Journal for Research in Mathematics Education*, 247-280.

Wagner, J. F., Speer, N. M., & Rossa, B. (2007). Beyond mathematical content knowledge: A mathematician's knowledge needed for teaching an inquiry-oriented differential equations course. *The Journal of Mathematical Behavior*,*26*(3), 247-266.

Wawro, M., Rasmussen, C., Zandieh, M., Sweeney, G. F., & Larson, C. (2012). An inquiry-oriented approach to span and linear independence: The case of the magic carpet ride sequence. *PRIMUS*, *22*(8), 577-599.

Yackel, E., Stephan, M., Rasmusen, C., & Underwood, D. (2003). Didactising: Continuing the work of Leen Streefland. *Educational Studies in Mathematics 54*, 101-126.

Undergraduate students' experiences in a developmental mathematics classroom[1]

Durrell A. Jones
Michigan State University

Beth Herbel-Eisenmann
Michigan State University

Understanding students' perspectives about their mathematics classroom experience is important for supporting students' learning. As part of a larger study, 15 students enrolled in enrichment sections of a developmental mathematics course were interviewed to explore students' experiences and dispositions. We highlight areas that students described as helpful or not and the ways in which students' framed themselves as having a "productive disposition" (National Research Council (NRC), 2001). Preliminary findings suggest students found activities most helpful when they thought that the activity might help them be successful on other activities and that they described themselves as having many characteristics of a productive disposition. A detailed analysis of these descriptions, however, uncovered narrow interpretations of "understanding" and "making sense" in mathematics.

Key words: Productive Disposition, Developmental Mathematics, Students' Experiences

Background

Many high school students in the U.S. continue to graduate lacking skills necessary to take credit-granting courses required for a college degree (Attewell, Lavin, Domina, & Levey, 2006). Instead, these students are taking non-credit bearing developmental courses (Bahr, 2010). As many as one-third of post-secondary students require some sort of developmental[2] course (Moore, 2003). Further, students are failing at alarming rates in these developmental courses—and the trend is worse in mathematics. As Bahr (2010) pointed out, "less than one student in four (24.6%) completed a college-level math course successfully within six years of first enrollment." (p. 220). To try to reduce this trend, some universities provide additional supports like enrichment courses, which is the focus of this paper.

Even though this dilemma exists, there is literature mainly at the secondary level suggesting that focusing on students' experiences in mathematics may provide some insight into this issue (Martin, 2007). There is little research that points to students' perspectives on these develomental courses (e.g., Grubb, 2001; Larnell, 2011). Yet, Waxman (1989) argued that researchers must recognize student perspectives on classroom instruction and learning environments because students ultimately respond to what they find to be important. Thus, we contribute to this literature by exploring students' perspectives on their experiences in university enrichment courses. By doing so, we can identify key elements students suggest contribute to the way they see their world and the world of mathematics. Such findings have implications for course design and pedagogy in order to meet the needs of students and potentially affect their success rate in developmental mathematics and enrichment courses.

Although there are different ways researchers might frame students' perspectives, our focus was to investigate what students found helpful and to account for how they describe their dispositions. In particular, we focus on whether students' responses to our interview questions suggested characteristics of a "productive disposition" (ProDisp) or "the tendency to see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in learning mathematics pays off, and to see oneself as an effective learner and

[1] This material is based upon work supported by the National Science Foundation under Grant No. (NSF DUE-1245402). Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF.

[2] Although some authors call these "remedial" mathematics courses, we use the term "developmental" mathematics.

doer of mathematics" (NRC, 2001, p. 131 - 133). By probing to see whether students align themselves with characteristics of a ProDisp, we can see whether they think they have a ProDisp, a perspective that has been shown to affect success in learning.

Research Questions

As part of a larger study, we more broadly seek to explore the answer to the question: What do students' descriptions of their enrichment class tell us about their experiences in that class? Here we concentrate on the following related sub-questions: RQ1) What do students describe as most and least helpful aspects of the course? RQ2) In what ways do their descriptions frame themselves as having a productive disposition? and RQ3) What differences exist in the accounts provided by students from an intervention enrichment section as opposed to students from non-intervention sections?

Research Method

This qualitative research project features an analysis of semi-structured interviews with intervention and non-intervention students in order to address our research questions. Our interpretivist analysis used open coding (Esterberg, 2002), in the sense that we attended to patterns or themes across the set of interviews. We report on in-progress analyses, focusing on three of the questions students were asked in the interviews.

Context

Intermediate Algebra is a 3-credit online developmental mathematics course offered at a large university in the Midwest U.S. Students are placed into this course based on placement exams and students with the lowest scores are asked to enroll in the face-to-face "enrichment section" that meets twice a week for 2 hours each. The trend has been that 30% of the students in Intermediate Algebra do not complete the course or get a D or F. Over half the students in the enrichment section identify as African American. This paper focuses on one section of this enrichment course (the "intervention" section), drawing on a larger NSF-funded study in which prospective secondary mathematics teachers (PSTs) use research based teaching methods and innovative materials to teach this course. The larger study investigates the learning and experiences of both PSTs and students in the enrichment course.

Data Collection

In December 2013 and January 2014, we conducted 45-minute interviews with 15 students from the intervention (n = 7; 2 male and 5 female) and non-intervention (n = 8; 1 male and 7 female) sections of the enrichment course. A semi-structured interview protocol was designed to inquire about student's background, learning disposition, and overall experiences in the enrichment course in order to understand students' experiences and perspectives on different versions of the course. This particular analysis focuses on three of the seven total questions from the interview. We chose these questions to begin our analysis because we thought they were the most salient for improving the course; they offered the greatest insight and immediate applicability of findings into the design, planning, and implementation of the enrichment class for the fall 2014 semester.

Analysis

All interviews were transcribed and then students' responses were entered into separate excel files to compile all responses to each question in order to analyze them for major themes. To answer RQ1, we used open coding to identify themes and categories in the text, which led us to consider supports as structures in the classroom as described by Doyle & Carter (1984). Like these authors, we use the term "activity" to refer to "how groups of students are organized for working, e.g., seatwork, small group discussions, lectures, etc." (p. 131-132). This category includes, for example, working on worksheets, taking attendance quizzes, and reviewing for exams. The enrichment students' interviews helped us to understand what they found helpful or unhelpful in supporting their learning. The first author created a set of themes and wrote an analytic memo, which was discussed by both authors and revised based on further examination of the data and a search for discrepant events.

To analyze student responses to questions that focused on ProDisp, we created a 'scale of alignment' to capture the many characteristics of ProDisp as students described having them or not. For example, when we asked them "Is mathematics something that makes sense to you? In what ways? (probe for: makes sense in and of itself; makes sense in relationship to real life)?" if they said it made sense and explained how it made sense to them, we coded them as having alignment with "mathematics makes sense". If students disagreed or seemed unsure about their stance related to a particular characteristic of ProDisp, their response was coded as "not having alignment".

We categorized students as having "high alignment" if they responded positively to all four aspects of ProDisp, "medium alignment" if they responded positively to 2-3 aspects and "low alignment" if they responded to 0-1 aspects. The first author then grouped the students' responses based on level of alignment and we used open coding for each group to better understand the ways in which students explained that aspect of ProDisp. For example, we looked carefully at how students said mathematics "made sense" and saw that most students said it was because they said they could follow instructors' procedures easily or because mathematics was something they used in life. Additionally, we looked for themes within and across the intervention section interviews and the non-intervention sections. For instance, the themes that arose out of looking across the responses of how students viewed mathematics in terms of it being "useful & worthwhile" were: 1) Career Math (Math that is needed for careers), 2) World Math (Math that is needed to function in the everyday world outside of work), & 3) Classroom Math (Math that exists in traditional classrooms).

Current Findings

Because analysis of data for each research question offered slightly different aspects of students' perspectives, we share our findings about each. Related to what students found helpful (RQ1), we found that students described the in-class activities as most helpful for supporting their learning (see Table 1). The table below showed us the extent to which students found the different aspects of the course helpful. Knowing that most of the students, regardless which section they were in, responded favorably to "Class Activities" suggested exploring further how they experience the Intermediate Algebra class, which was delivered online. Additionally, none of the non-intervention students mentioned classroom routines as being helpful and, although some found this aspect helpful in the intervention section, more students said it was not helpful. This is also an area that offers some room to investigate as we look at the pedagogy in the classrooms. Finally, we briefly note that the same percentage of students found the instructors helpful, about half the intervention students versus non-intervention said the instructors were not helpful.

Table 1
Enrichment Students' Descriptions of Helpful/Unhelpful

Categories	**Helpful**		**Unhelpful**	
	Intervention (I)	Non-Intervention (NI)	Intervention (I)	Non-Intervention (NI)
Class Structure	7%	20%	13%	13%
Class Routine	13%	0%	33%	13%

Class Activities	27%	40%	20%	33%
External Resources	7%	13%	0%	13%
Characteristics of Teacher/Instructor	20%	20%	13%	20%
ALEKs	0%	7%	0%	13%

The findings (see Table 2) related to questions about ProDisp suggest that students in general leave the enrichment course with medium alignment to a ProDisp; almost all students aligned themselves with two or more aspects (6 of 8 non-intervention & 5 of 7 intervention). We were able to uncover further nuances from students' responses related to ProDisp. For example, we noticed that non-intervention students responded to the "Steady Effort Pays Off" question about how one might improve in mathematics because they had observed *someone else* improving in mathematics. In contrast, the intervention section students were specific about how they themselves could improve and gave responses that connected to their own intrinsic motivation as the means of improvement.

Table 2

Enrichment Students' Alignment to Productive Disposition

Alignment Scale (PDF)	Number of Dimensions Present	% of Alignment (Intervention)	% of Alignment (Non-Intervention)
Low	0 – 1	7%	0%
Medium	2 – 3	43%	36%
High	4	7%	7%

This research will continue with an analysis of remaining responses from the interview. We also plan to look at additional data sources like classroom observations in order to understand why students responded the ways they did. Continuing the research will allow for greater insights about how students describe their experiences and an ability to compare and contrast views of students to look for further trends, especially in relationship to current literature on secondary students' experiences and ProDisp.

Presentation & Audience Engagement

In our presentation, we will share common areas that students recognized as helpful or not and how the students described characteristics of ProDisp. We will allocate 10-20 minutes for questions and feedback. Questions we will pose include: 1) In addition to surveys and interviews, what other approaches have you used to address students' perspectives on developmental mathematics classes? 2) What aspects of less traditional pedagogy and tasks might be contributing to these findings and how might we better probe for those aspects of students' experiences? and 3) (from a practical standpoint) How might we use what we have learned to improve the course? For example, if the less conventional pedagogy is contributing to these findings, how might we mitigate students' experiences to help them understand why we are using these strategies?

References

Attewell, P. A., Lavin, D. E., Domina, T., & Levey, T. (2006). New evidence on college remediation. *The Journal of Higher Education*, 77(5), 886–924.

Bahr, P. R. (2010). Preparing the underprepared: An analysis of racial disparities in postsecondary mathematics remediation. *The Journal of Higher Education*, 81(2), 220.

Doyle, W., & Carter, K. (1984). Academic tasks in classrooms. *Curriculum Inquiry*, 14(2), 129–149.

Esterberg, K. G. (2002). Interviews. In K. G. Esterberg, *Qualitative methods in social research*. (pp. 83-114). Boston, MA: McGraw-Hill.

Grubb, W. N. (2001). *From black box to pandora'a box: evaluation remedial/developmental education*. New York, NY..

Larnell, G. V. (2011). *More than just skill: mathematics identities, socialization, and remediation among african american undergraduates*.(unpublished doctoral dissertation) Michigan State University, East Lansing, MI.

Martin, D. B. (2007). Beyond missionaries or cannibals: Who should teach mathematics to African American children? *The High School Journal, 91*(1), 6–28.

Moore, A. (2003). Shape up or ship out? *The Health Service Journal, 113*(5859), 12–3. Retrieved from http://www.ncbi.nlm.nih.gov/pubmed/22956564

National Research Council. (2001). *Adding it up: Helping children learn mathematics*. J. Kilpatrick, J. Swafford, and B. Findell (Eds.). Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press.

Waxman, H. (1989). Urban Black and Hispanic elementary school students' perceptions of classroom instruction. *Journal of Research and Development in Education*, *22*(2), 57-61.

Promoting students' construction and activation of the multiplicatively-based summation conception of the definite integral

Steven R. Jones
Brigham Young University

Prior research has shown how the multiplicatively-based summation conception (MBS) is important for making sense of definite integral expressions in science contexts. This study attempts to accomplish two goals. First, it describes introductory lessons on integration from two veteran calculus teachers as a way to possibly explain why so few students draw on the MBS conception when making sense of definite integrals. Second, it reports the results from a design experiment intended on promoting not only the construction of the MBS conception, but its priming for activation when students see and interpret definite integrals expressions.

Key words: calculus, definite integral, summation, accumulation, design experiment

The literature from undergraduate mathematics education and science education demonstrates that students are struggling to draw on and adequately apply their knowledge of integration to calculus-based coursework (Bajracharya & Thompson, 2014; Christensen & Thompson, 2010; Pollock, Thompson, & Mountcastle, 2007; Wemyss, Bajracharya, Thompson, & Wagner, 2011). In light of this difficulty, Jones (2014) subsequently analyzed three common conceptualizations of the definite integral in terms of how productive (i.e. useful or helpful) each conceptualization is for *making sense* of integral expressions and formulas. The results revealed that the familiar "area under a curve" and "anti-derivative" conceptions of the integral seem *less productive* for making sense of applied-science definite integrals, since they did not help the students see the relationships between the physical quantities involved in the integral expressions. By contrast, the "multiplicatively-based summation conception" (MBS conception), grounded in the notion of Riemann sums, seems *highly productive*, since it helped students understand *what* a given definite integral calculates and, more importantly, *why*. The MBS conception is defined as one that incorporates both the multiplicative relationship between the integrand and the differential and the summation or accumulation of the resulting quantity throughout the domain.

In response to these results, Jones (*under review*) later investigated, through a sample of 150 calculus students, how common the (a) area under a curve, (b) anti-derivative, and (c) MBS conceptions of the definite integral are in a general calculus student population. Unfortunately, the results showed that large percentages of students primarily interpreted the definite integral as an area under a curve (87.3%) or an anti-derivative (40.0%). Very few students (6.7%) interpreted the basic meaning of the definite integral through the MBS lens. While these results do not imply that students do not cognitively *possess* the MBS conception, they do demonstrate that students might not *draw on* the useful MBS conception in making sense of definite integrals.

While several educators are looking into how students learn and think about Riemann sums (e.g., Bressoud, 2009; Sealey, 2014; Sealey & Engelke, 2012), it is possible that students separate the Riemann sum and the definite integral as related-but-distinct notions (Jones, 2013). Thus, it is important to study whether instruction regarding Riemann sums actually helps students not only construct the MBS conception, but to have it cognitively primed for activation when students attempt to make sense of integral expressions. This study seeks to shed light on how introductory lessons on integration support, or not, the activation of the MBS conception in making sense of definite integrals.

Focusing Framework

In order to investigate the connections between introductory instruction on integration and how students construct and draw on their knowledge of integrals, I use two elements of the "focusing framework" as an analytic lens (Lobato, Rhodehamel, & Hohensee, 2012). First, *centers of focus* are features in the classroom that students may attend to, such as patterns, properties, or regularities. Second, these centers of focus are created through *focusing interactions*, which are discursive practices including speech, gestures, or visuals. By creating centers of focus, instructors directly influence what students may pay attention to or what they may perceive as useful or important as they construct their knowledge.

Analyzing a lesson for the focusing interactions that create certain centers of focus allows for the opportunity to uncover what students might *notice* during that lesson, defined as "the selection of certain information in the presence of competing sources of information" (Lobato et al., 2012, p.438). Examining what students might *notice* during introductory lessons on integration may help explain the conceptions students choose to draw on when thinking about integrals. This framework is also used to assist in evaluating experimental lessons designed at helping students create and draw on the important ideas that make up the MBS conception.

Methods

This study contains two main parts. In the first, two veteran calculus teachers at a large, four-year university had their first five lessons on integration observed in order to analyze the lessons through the focusing framework. Each fifteen-second segment of the lesson was coded as to whether the discussion was related to "area," "anti-derivatives," or "Riemann sums" (potentially including more than one). Each lesson segment dealing with Riemann sums was then investigated for the focusing interactions present in that segment. These focusing interactions were then aggregated across each lesson in order to determine what centers of focus, and consequently what student noticing, were possibly occurring.

In addition to the observations, at the end of the semester a survey was administered to a random selection of these two instructors' students (n = 55). Interviews were also conducted with six of the instructors' students, three from each instructor. The survey used in this study has been used previously (Jones, *under review*), and the reader is referred to that study for more detail regarding the survey questions. For brevity in this paper, only two of the survey questions are discussed. The first asked the students to explain the meaning of $\int_a^b f(x)\,dx$ in as many ways as they could think of, and the second asked them to describe and explain the physics equation $F = \int_S P\,dA$ (they were told in the survey item that force = pressure × area). The interviews focused on the students' understanding of the definite integral and asked the students to discuss pure mathematics and applied physics integrals using expressions similar to those used on the survey.

In the second part of this study, a design experiment (see Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) was conducted regarding a set of introductory lessons on integration with the goal of improving students' creation *and* usage of the MBS conception. The lessons created for this experiment were modified and improved over several semesters of instruction and the last iteration was given to two separate classes taught by the author. The students in the experimental classes were given the same survey at the end of the semester as the students in the observed courses (n = 41). Four students from the experimental classes were interviewed using the same interview protocol as given to the other instructors' students.

The first experimental lesson centered on a series of activities dealing with water leaking out of a pipe, first at a constant rate and then at a non-constant rate (see Jones, 2013/14). The students were led through the construction of a Riemann sum for each step in the activity, including the eventual limit of Riemann sums, which was defined as the definite integral.

That is, “area under a curve” was explicitly *not* used as the primary motivator for the definite integral. The area under a curve was discussed later as an *application* of Riemann sums, in an attempt to place the central meaning of definite integrals within the Riemann sum, as opposed to the area under a curve as is usually done (see, for example, Stewart, 2012).

Results from the Observed Instructors and Their Students

The lessons from the two instructors contained a great deal of time spent on Riemann sums. In fact, one instructor spent 85.9% of the mathematical discussion in the first lesson and 65.7% of the mathematical discussion in the second lesson explicitly on Riemann sums. Similarly the other instructor spent 96.2% of the first lesson and 39.0% of the second lesson explicitly on Riemann sums. Given that both instructors focused so heavily on Riemann sums, it might be reasonable to expect that their students would show a greater tendency to draw on the MBS conception than was found in the general population.

Unfortunately, it appears that the students from the two observed instructors also had a tendency to rely on the area under a curve and anti-derivative conceptions in their responses to the survey items, and to draw less on the MBS conception of the integral. The number of responses that fit into each category is shown in Table 1, with all blank responses or responses that did not fit into the three categories listed under “None/Other.” Note that students were encouraged to describe the integrals in the survey in as many ways as possible, meaning that the frequencies add up to more than the sample size.

	Area	Anti-derivative	MBS	None/Other
$\int_a^b f(x)\,dx$	47 (85.5%)	14 (25.5%)	8 (14.5%)	7 (4.7%)
$F = \int_S P\,dA$	14 (25.5%)	7 (12.7%)	11 (20.0%)	31 (56.4%)

Table 1: Frequencies of responses (n = 55) based on the (a) area under a curve, (b) anti-derivative, (c) multiplicatively-based summation, or (d) “other/none” conceptions

Only one-fifth of the students from these two instructors used the MBS conception to make sense of the physics integral, $F = \int_S P\,dA$. This is problematic in light of the fact that the MBS conception is important for making sense of physics and engineering integrals (Jones, 2014). The student interviews support the results from the survey and show that although the students all demonstrated cognitively possessing the Riemann sum conception, they did not see it as important in making sense of definite integrals.

S1: I’m sure, back in the day, before they had integrals [the student used “integral” here to mean “anti-derivative”] that Riemann sums were probably big. And that would just take a lot of time. Like I remember doing one problem that took like twenty minutes. It was terrible. Whereas doing integrals takes like, if the equation is easy, only a couple of minutes or less.

This example is typical of how these students discussed the Riemann sum in the interviews. The student described it as essentially an “old-fashioned” method for calculating area under a curve, and indicated that anti-derivatives can replace Riemann sums in accomplishing this goal. In fact, this student used the word “integral” to mean, literally, an “anti-derivative.” In this way, it can be seen that the area under the curve and the anti-derivative notions occupy the primary meanings for this student as to what an integral actually is. Many of these students said similar things, such as, “I feel like this [the Riemann sum] is the original way people did it and then they, like, found the shortcut.” Despite cognitively possessing the concept of the Riemann sum, and being able to discuss it when prompted, it is clear that the students did not conceive of Riemann sums as scaffolding the underlying meaning of the definite integral. The Riemann sum is merely a calculational tool for getting at the *real* problem—calculating area under a curve. In this way, it is not surprising that only a relatively small percentage of students drew on the MBS conception to explain the meaning of the integral expressions in the survey items or interviews.

When the observed lessons were analyzed through the focusing framework, certain features of the lessons stood out that might cue students into this type of thinking. In particular, I describe three specific focusing interactions that may have played a role in leading students toward this thinking. While these focusing interactions are evidenced in many places through the lessons, only single examples of them are provided here, for brevity. The following excerpts are typical of how both instructors discussed definite integrals during the introductory lessons. First, both instructors emphasized the area under a curve as the primary meaning for integrals. Additionally, despite spending a great deal of time discussing Riemann sums, they both couched Riemann sums as essentially just a calculational technique for the "real" task of finding area under a curve.

> I1: There are two main ideas in calculus… The integral can be thought of as area under a curve… We want to estimate that area, it's hard to find that exact area, it's hard to do that. So how can we estimate it by fitting in rectangles here? So think about this, if I broke that up into a set of rectangles, I could approximate the area there.

Second, both instructors emphasized that integrals are, in essence, simply the reverse process of the derivative.

> I2: If you start with a function that gives you the amount of something and you take the derivative, that gives you the rate of change. If you take the integral of the rate of change, it takes you back to what you started with. So, derivatives and integrals are going to undo each other.

Third, both instructors used the Fundamental Theorem of Calculus as a way of portraying the Riemann sum as an "outdated" technique that could be replaced with the more efficient anti-derivative technique.

> I1: This makes integrals so much easier. Remember when you were doing, a few class periods ago, and you were doing those Riemann sums? OK, and how much you, let me say the word, *disliked* them? OK, they were a pain to do?... This allows you to just *bypass* all that. It makes it so much easier… If you did all that really long stuff, the Riemann sum, and spent a half an hour on the problem… And if you do [the anti-derivative] all correctly, you'd get the same answer.

Through these focusing interactions, the instructors have, perhaps inadvertently, set up students to *notice* the area and anti-derivative conceptions of the definite integral as being the most important and most useful. On the other hand, they may lead students to believe that Riemann sums are just a time consuming, even "painful" way to calculate area—a method that can and should be cognitively "demoted" once the anti-derivative method is established. While I acknowledge that other factors may be at play that would lead students to "demote" the Riemann sum, I claim that these focusing interactions do play a role.

Results from the Design Experiment

In contrast to these students, the students from the design experiment were much more likely to draw on the MBS conception to make sense of the integrals in the survey items (see Table 2). Here we can see that a *majority* of students drew on the MBS conception to make sense of both the mathematics and physics integral expressions, *in addition* to drawing on the area under a curve conception.

	Area	Anti-derivative	MBS	None/Other
$\int_a^b f(x)\,dx$	37 (90.2%)	12 (29.3%)	24 (58.5%)	10 (24.4%)
$F = \int_S P\,dA$	5 (12.2%)	1 (2.4%)	24 (58.5%)	15 (36.6%)

Table 2: Frequencies of responses (n = 41) based on the (a) area under a curve, (b) anti-derivative, (c) multiplicatively-based summation, or (d) "other/none" conceptions

The student interviews reinforce these results to show that the students had a better idea of the multiplicative relationship between the integrand and the differential, and the summation or accumulation of the resulting quantity.

> S2: Yeah, you could split it up into tiny areas… so you find this, the force for that, for the small rectangle by multiplying whatever that small area is by the pressure in that small area… So, since the pressure is different throughout the balloon, or whatever it is, you can't just do the balloon as a whole to find the force, you need to split it into the small increments… You can split it up into small increments and then just add them all up to get the total force.

As similarly documented in Jones (2014), the MBS conception allowed this student to reason through this physics integral, preserving the scientific relationships between pressure, surface area, and force. It helped her know *what* the integral was calculating and *why* it was calculating it. In this way, she was able to understand the integral expression. This example is typical of the ways in which the students from the design experiment discussed the definite integral, both in pure mathematics contexts and in applied physics contexts.

These results should not be too surprising given that the design experiment was purposefully crafted to achieve this outcome, by helping students *notice* two ideas: (a) the multiplicative relationship between the integrand and the differential and (b) the summation/accumulation of the resulting quantity throughout the domain (see also Thompson & Silverman, 2008). Yet, the focusing framework can help us identify *why* the students *noticed* the Riemann sum as a more central, inherent meaning for integrals. Space does not permit a full description of all centers of focus, but the following examples are typical of the types of centers of focus regarding the Riemann sum that were created in these lessons.

The introductory activity from the first lesson used water leaking out of a pipe over time (see Figure 1) to initiate students into the notion of Riemann sums. The students used the fact that the rate of leakage multiplied by time produced an *amount* of water. Then, depending on the rate of leakage, the amount of water *accumulated* faster or slower. The final amount of water depended on the time over which the water was accumulating (i.e. the limits of integration).

Water leakage recorded every 30 seconds:

time (*min*):	0	0.5	1	1.5	2	2.5	3	3.5	4
Rate (*L/min*):	0	5	9	12	11	10	8	5	2

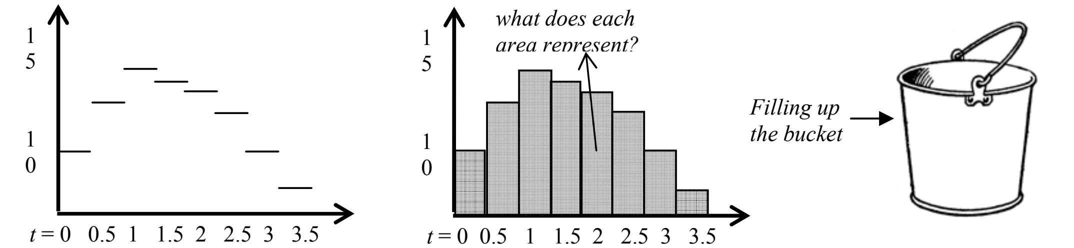
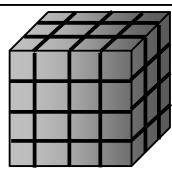


Figure 1: Center of focus regarding the multiplicative relationship between rate and time, as well as the accumulation of a total amount of water

Similarly, other centers of focus were created to promote the meaning of the differential–and the multiplicative relationship between it and the integrand–*without* depending on a "rate" type problem, which can be conflated with derivatives. The students were shown a box with varying density and asked how they could estimate the mass of the box (they were told that density × volume = mass). The students worked through the idea of breaking the box into smaller pieces (ΔV) and finding the mass of each to come up with an estimate (see Figure 2).



Divide the box into small pieces and take the density at a point in each piece:
density × ΔV = *mass of one piece.*

$$\sum density \times \Delta V \rightarrow \int_{box} D\, dV$$

Figure 2: Center of focus regarding the differential and its multiplicative relationship with the integrand

Throughout these activities, the definite integral was defined via the multiplicatively-based summation perspective. Furthermore, the summation notation for the limit of Riemann sums, $\lim\limits_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x_k$, was explicitly connected to the notation for definite integrals, $\int_a^b f(x)\,dx$, on many occasions during the lesson (see Figure 3).

As Δx gets small $\rightarrow dx$

$$\underbrace{\lim_{n\to\infty} \sum_{k=1}^{n}}_{\downarrow\ \int_a^b} \underbrace{f(x_k)\cdot \Delta x_k}_{\downarrow\ f(x)\cdot dx}$$

lim$\sum$, as n goes to $\infty \rightarrow \int$

Figure 3: Center of focus regarding summation notation and definite integral notation

Conclusion

Through these types of activities, the experimental lessons appear to have helped the students develop a strong connection between the ideas present in the Riemann sum and the symbols of the definite integral expression. In particular, the students showed a much stronger tendency to draw on the multiplicative relationship between the integrand and differential to make sense of both the mathematics and physics integral expressions. The students also used the summation/accumulation notion to conceive of capturing the total amount of the resultant quantity. The experimental lessons were far from perfect, considering that around 40% of the students *still* did not draw on the MBS conception in the survey. However, they demonstrate a significant step in the right direction, and constitute lesson units that could be easily employed in any first-semester calculus classroom.

References

Bajracharya, R., & Thompson, J. (2014). Student understanding of the Fundamental Theorem of Calculus at the mathematics-physics interface. In T. Fukawa-Connelly, G. Karakok, K. Keene & M. Zandieh (Eds.), *Proceedings of the 17th special interest group of the Mathematical Association of America on research in undergraduate mathematics education*. Denver, CO.

Bressoud, D. M. (2009). Restore the integral to the fundamental theorem of calculus. *Launchings: Mathematical Association of America.* Retrieved September 10, 2014, from http://www.maa.org/external_archive/columns/launchings/launchings_05_09.html

Christensen, W. M., & Thompson, J. R. (2010). Investigating student understanding of physics concepts and the underlying calculus concepts in thermodynamics. *Proceedings of the 13th special interest group of the Mathematical Association of America on research in undergraduate mathematics education*. Raleigh, NC.

Cobb, P., Confrey, J., diSessa, A. A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(1), 9-13.

Jones, S. R. (2013). Understanding the integral: Students' symbolic forms. *The Journal of Mathematical Behavior, 32*(2), 122-141.

Jones, S. R. (2013/14). Adding it all up: Reconceiving the introduction to the integral. *Mathematics Teacher, 107*(5), 372-377.

Jones, S. R. (2014). Three conceptualizations of the definite integral in mathematics and physics contexts. In T. Fukawa-Connelly, G. Karakok, K. Keene & M. Zandieh (Eds.), *Proceedings of the 17th special interest group of the Mathematical Association of America on research in undergraduate mathematics education*. Denver, CO.

Jones, S. R. (*under review*). The prevalence of area-under-a-curve and anti-derivative conceptions over Riemann-sum based conceptions in students' explanations of definite integrals.

Lobato, J., Rhodehamel, B., & Hohensee, C. (2012). "Noticing" as an alternative transfer of learning process. *The Journal of the Learning Sciences, 21*(3), 433-482.

Pollock, E. B., Thompson, J. R., & Mountcastle, D. B. (2007). Student understanding of the physics and mathematics of process variables in PV diagrams. *AIP Conference Proceedings, 951*, 168-171.

Sealey, V. (2014). A framework for characterizing student understanding of Riemann sums and definite integrals. *The Journal of Mathematical Behavior, 33*(1), 230-245.

Sealey, V., & Engelke, N. (2012). The great gorilla jump: An introduction to Riemann sums and definite integrals. *MathAMATYC Educator, 3*(3), 18-22.

Stewart, J. (2012). *Calculus: Early transcendentals* (7th ed.). Belmont, CA: Brooks/Cole.

Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in calculus. In M. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 117-131). Washington, DC: Mathematical Association of America.

Wemyss, T., Bajracharya, R., Thompson, J., & Wagner, J. (2011). Student understanding of integration in the context and notation of thermodynamics: Concepts, representations, and transfer. *Proceedings of the 14th special interest group of the Mathematical Association of America on research in undergraduate mathematics education*. Portland, OR.

Students' generalizations of single-variable conceptions of the definite integral to multivariate conceptions

Steven R. Jones
Brigham Young University

Allison Dorko
Oregon State University

Prior research has documented several conceptualizations students have regarding the definite integral, though the conceptualizations are largely based off of single-variable integral expressions. No research to date has documented how students' understanding of integration becomes generalized for multivariate contexts. This paper describes six conceptualizations of multivariate definite integrals and how they connect to students' prior conceptions of single-variable definite integrals.

Key words: calculus, definite integral, multivariate integration, generalization

The undergraduate calculus series is a rich place for research in mathematics education. It takes core concepts, such as limits, derivatives, and integrals, and extends them from the single-variable context to the multivariate context to the more abstract context of real and complex analysis. This creates many instances in which students construct knowledge in first-semester calculus that is then increasingly generalized in subsequent courses.

While the core concepts of limits, derivatives, and integrals are each worth studying, the definite integral is a particularly useful construct that is used frequently in pure mathematics (Brown & Churchill, 2008), physics (Serway & Jewett, 2008), engineering (Hibbeler, 2012), and other sciences (Salvatore, 2008). Since many applications in these areas of study deal with multivariate integration, it is important to know how students generalize their knowledge of single-variable integration to multivariate integration. Some studies have looked at students' generalized knowledge of function (Kabael, 2011; Martinez-Planell & Trigueros-Gaisman, 2013; Weber & Thompson, 2014), domain and range (Dorko & Weber, 2014; Martinez-Planell & Trigueros-Gaisman, 2012), and derivative (Martinez-Planell, Trigueros-Gaisman, & McGee, 2014; Yerushalmy, 1997), but as far as the authors are aware there is no research that has been done regarding students' conceptions of multivariate definite integrals or how that knowledge is generalized from the single-variable context to the multivariate context. This study seeks to document and describe students' conceptions of multivariate definite integrals and how they are related, or not, to conceptions of single-variable definite integrals.

Analytic Lenses

In this study, we begin with a set of previously documented student conceptualizations of the definite integral in the single-variable context (Jones, 2013). Note that while some interview items from that study involve multivariate contexts, the documented conceptualizations are largely based on the single-variable definite integral structure. Three main conceptualizations of single-variable definite integrals are used in this study to organize our coding and frame our analysis of the student data: (1) perimeter and area, (2) function matching, and (3) adding up pieces. We focus on these three because they deal explicitly with the entire definite integral expression and this study is intended to capture large-grained ways in which students conceptualize the definite integral as a whole. A brief description of each of these conceptualizations is provided here.

Perimeter and area: This conceptualization interprets each "box" in the integral expression, $\int_{[]}^{[]}[]\,d[]$, as being one part of the perimeter of a shape in the (x-y) plane. The integrand forms a (usually curvy) top, the limits of integration create vertical lines for the sides, and the differential dictates the variable that resides on the horizontal axis, which forms the bottom of the shape. The integral is the area of this newly created shape, which is taken as a fixed, undivided whole.

Function matching: This conceptualization interprets the integrand as having come from some other "original function." The original function became the integrand via a derivative, and the differential dictates the variable with respect to which the derivative was taken. The integral symbol is conceived of as an implicit instruction to find this original function. The limits of integration are values that are to be input into the original function.

Adding up pieces: This conceptualization contains the idea of the domain being broken into infinitely-many, infinitesimally-small sections. Within a single piece, called a *representative rectangle*, the quantities represented by the integrand and the differential are multiplied to capture a tiny amount of the resulting quantity. This resulting quantity is then added up (or accumulated) throughout all the sections in the domain to get the total amount.

These provide a lens for investigating how student conceptions of multivariate integrals may be generalized from them. In our analysis we adopt the framework of generalization described in Harel and Tall (1991). In particular, Harel and Tall describe (a) *expansive generalization*, as one where the applicability of an already existing schema is simply broadened, (b) *reconstructive generalization*, in which the existing schema is altered in order to broaden its applicability, and (c) *disjunctive generalization*, where a new schema is created for the new context, which remains disjoint from the previously existing schema. Using this framework enables us to explore the nature of the students' generalizations of the definite integral by comparing the single-variable conceptions with the multivariate conceptions.

Methods

In order to capture how students think about and discuss multivariate definite integrals, we conducted two 45-minute, task-based interviews (see Goldin, 1997) with 12 students who were at various stages of their mathematical study, ranging from several students who had recently taken multivariate calculus to two graduate students. In the first interview, the students were given two tasks that asked them to discuss single-variable integrals. For the remainder of the two interviews, the students were given items that showed multivariate integral expressions, like $\int_c^d \int_a^b g(x,y)\,dxdy$ or $\int_{a_n}^{b_n} ... \int_{a_1}^{b_1} g(x_1,...,x_n)\,dx_1...dx_n$, and asked them to discuss the meaning of the expressions, or how they could be used in applied situations.

Since we already had baseline categories that could inform the coding and analysis, we drew on the "analytic-inductive" method (Knuth, 2002) to code the data. This method is comprised of "researcher-generated" categories, including extrapolations of the three conceptualizations listed above, and "data-grounded" categories, which come directly from the data. Based on the researcher-generated categories, we identified places in the data where students discussed ideas related to the three conceptualizations of single-variable integrals, such as the boundary of a shape, area or volume, anti-derivatives, rectangles, and multiplying quantities together. While reviewing the data with these conceptualizations in mind, we also looked for places that required the creation of new data-grounded codes. We then made a final pass through the data to ensure we had identified all places where the data fit into either the researcher-generated or data-grounded categories. Once the data were coded, each category was analyzed for the underlying student conceptualization.

Results

In total we identified six categories that represent student conceptualizations of the definite integral in multivariate contexts. While we in no way claim that these categories are an exhaustive list of all possible student conceptualizations, they do seem to capture most of the ways in which these 12 students thought about multivariate integration. Four of the conceptualizations, which we name (1) *boundary and volume*, (2) *adding up slices*, (3) *infinite strip*, and (4) *abstract space underneath*, appear to be generalizations of the perimeter and area conceptualization. Boundary and volume and infinite strip are *reconstructive* generalizations, while adding up slices and abstract space underneath are *expansive* generalizations. The fifth conceptualization, which we name (5) *building an expression of inputs*, can be considered an *expansive* generalization of function mapping. Lastly, the sixth conceptualization, which we continue to call (6) *adding up pieces* is a *reconstructive* generalization of the previously documented adding up pieces.

1. Boundary and volume: The boundary and volume conceptualization is largely a higher-dimensional extension of perimeter and area, though there are some important additional conceptual components. Consider student C's description of the meaning of $\int_c^d \int_a^b g(x,y)\,dx\,dy$ (see Figure 1).

> *C*: We make a rectangle, in the *x*-direction it will be from *a* to *b* and then in the *y*-direction it will be from *c* to *d*. And we find the volume between the *x-y* plane, this *g*(*x,y*), and the planes that are formed by our edges… [Draws a two-dimensional graph for *g*.] We'll call the top of this *g*(*x,y*)… Here's *a*, here's *b*, here's *c*, here's *d* [marks them off on the *x*- and *y*-axes and creates straight lines extending them into the *x-y* plane]. We're finding the volume in here [makes lots of dots underneath the graph].
>
> …
>
> *C*: We have the analogy of, we could take the volume of water. So, between the *x-y* plane and the surface [i.e. the graph of *g*]. Fill that entirely with water within the rectangle that we described by our bounds, *a*, *b*, *c*, and *d*.

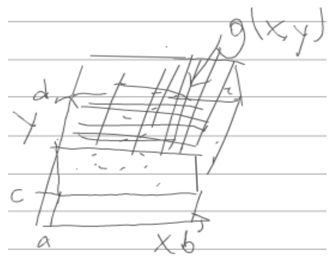


Figure 1: Volume under the graph of *g*, bounded by vertical planes

In this description we can see that, first, the (constant) limits of integration are thought of as marking off straight lines in the *x-y* plane, which intersect each other to create a region that constitutes the "bottom" of a three-dimensional shape. The variables represented by the differentials name the type of plane (i.e., *x-y*) that exists for the bottom. Second, vertical planes are conceived of as arising from this perimeter to create the lateral sides of the shape. These vertical planes make up an additional layer of meaning placed on the integral that goes beyond just the limits of integration, since there is no single symbol in the integral expression that seems to correspond with these vertical planes. Contrast this with perimeter and area, in which the vertical lines that create the sides of the shape are directly represented by the limits of integration. Lastly, the two-dimensional graph of *g*(*x,y*) forms the "lid" to this shape. As with perimeter and area, the volume seems to be considered a fixed, undivided whole. Student C used the analogy of the shape being a tank that holds a certain volume of water.

2. Adding up slices: This conceptualization is a direct extension of the area under a curve notion. For example, student G had been drawing on the area under a curve idea when explaining single-variable integrals and then extended this thinking to his discussion regarding $\int_c^d \int_a^b g(x,y)\,dx\,dy$. Note that he refers to *g*(*x,y*) by the symbol *z*.

G: We're coming up with a slice at each *y*-value… So perhaps first we would integrate with respect to *x*, integrate this function, *z* equals whatever, with respect to *x*, which we would do the same way we did with the single variable [i.e. find the area under the curve]. But then we'd get kind of a slice of the whole volume. And we'd have to do that process for each slice, for each *y*-place [i.e. *y*-value]. And then we'd sum up all those slices.

This conceptualization takes the familiar area under a curve notion and repeats it over and over (infinitely many times) at every single *y*-value to create the object represented by the integral. By taking infinitely many "sheets" of area and pasting them together, the volume under the graph is constructed. While this may appear to have surface features in common with adding up pieces, it seems mostly to be an *expansive* generalization of the area conception. Since it is focused on two-dimensional area and three-dimensional volume, it is unclear whether this specific conception has a higher dimensional analog or not.

3. Infinite strip: This conceptualization also deals with boundary and volume ideas, yet it is distinct from adding up slices in how the volume is perceived to be constructed, leading to an important cognitive difference. Student D was explaining how the integral $\int_c^d \int_a^b g(x,y)\,dx\,dy$ produced volume when she described the following (see Figure 2),

D: So, some things I'm trying to think about are whether we're just finding [volume] under the surface, for everything that would fall in between this *a* and *b* [Draws extended straight lines through $x = a$ and $x = b$], like, for no matter, for all *y*-values [sweeps hands from her graph to off the table], like everywhere.
Interviewer: So kind of like on forever in both directions [sweeps hands to both sides]?
D: Right. So in that case we'd be finding the… volume for this whole section [places fingers on $x = a$ and $x = b$ and then slides them away in opposite directions]. And then when we do the next integral, we're cutting it off here [places straightened hands on the lines representing $y = c$ and $y = d$] and finding the volume under just this section here [shades in the rectangle bounded by $x = a$, $x = b$, $y = c$, and $y = d$].

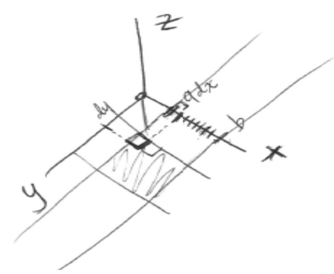


Figure 2: An infinite strip after integrating with respect to *x*

This student broke down the integral in an attempt to describe what was happening at each step. After completing the integral with respect to only one of the variables (in this case, *x*), she invoked the boundary and volume conception to conceive of an infinitely long strip between the straight lines $x = a$ and $x = b$. In this way, if one stops the integration process part-way through, an interesting abstract picture emerges of an infinite volume under the graph that extends forever in both directions. It is only after the second step of the integration process occurs that the infinitely long section is "cut off" at the other boundary lines, $y = a$ and $y = b$, to make the regular, finite shape with its accompanying volume.

4. Abstract space underneath: Many of the students, when encountering the higher dimensional integral, $\int_{a_n}^{b_n} \ldots \int_{a_1}^{b_1} g(x_1,\ldots,x_n)\,dx_1 \ldots dx_n$, continued to draw on the "area underneath a curve" or "volume underneath a curve" idea, as exemplified by student F.

F: So, you're finding some kind of curve, up through the *n*-th variable, and you're finding the area, or the volume, or the density, or something in space that we can't even visualize, underneath that [the graph of *g*], on the boundaries between *a* and *b* from the sub-1 [i.e. a_1 and b_1] to the sub-*n* [i.e. a_n and b_n].

…
F: [The integral] is going to represent, everything beneath, or all those variables integrated beneath that [the graph of *g*]… They stretch up in this weird *n*-space that you can't actually draw!

The central aspect of this conceptualization is that, even in higher dimensions, the notion of area or volume *beneath* a graph may remain a key part of students' conceptualizations. That is, there is a curve somewhere "we can't even visualize" that has something analogous to a higher-dimensional "volume" underneath it.

5. Building an expression of inputs: As with single-variable integrals, many students interpreted the underlying meaning of multivariate integrals as being a process of anti-differentiation, yet generalized by being repeated over and over. Unlike single-variable integrals, where the limits of integration are simply inserted at the end, inputs in the multivariate case are inserted *along the way*. In fact, some of the students seemed to be thinking of the multivariate integral as constructing an entire *expression of inputs*.

D: So, we'd get a function, maybe I should say, take the anti-derivative in terms of x. I don't remember how to write that in notation. Anti-derivative of b, y minus our anti-derivative in terms of x, of a, y [writes $\int_c^d G(b,y)-G(a,y)\,dy$]. Then we take the anti-derivative of our function now in terms of y, whatever that would be [laughs and writes a square capital G, "[G]"]… So this would be [G](b,d) minus [G](b,c), whatever this one is, [G](a,d) minus [G](a,c) [writes out: "[G](b,d) – [G](b,c) – [[G](a,d) – [G](a,c)]"].

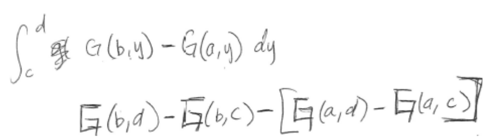


Figure 3: The definite integral as building an expression of inputs

We contend that this conceptualization includes both an *object* and a *process* (see Sfard, 1991). While the process is the repeated anti-differentiation procedure, the object is the outcome of a newly constructed expression which has translated parts of the integral expression (the function g and the limits of integration a, b, c, and d) into a new formulation. An interesting note should be made here that, unlike the function matching conceptualization for single-variable integrals, the students in this study seemed to have dropped the notion of an "original function." That is, the integrand is no longer thought of as having come from somewhere else. Instead, the integral seems to deal more with taking a *self-existent* integrand function and translating it into a new expression with inputs inserted into it.

6. Adding up pieces: The final conceptualization of multivariate integration we discuss in this paper is essentially identical to the conceptualization described in Jones (2013), wherein infinitely many pieces are added together over the domain. Thus, the interested reader is referred there for a more detailed account of it and in this paper we only discuss a single key feature in the generalization of this conception. In Jones, the students were described as typically reasoning within a single piece, called a *representative rectangle*, in which the student identifies the relationship between the integrand and the differential and determines the resultant quantity. In the single-variable context, the use of the word *rectangle* is appropriate, since the differential can be thought of as small segments along the horizontal axis, and the multiplication between the integrand and the differential can be visually depicted as rectangles underneath the graph of the function. Yet, in the multivariate case, the pieces are no longer represented by *rectangles*, but rather by prisms, cubes, discs, or other three-dimensional (or, abstractly, higher-dimensional) shapes (see Figure 4).

C: We could take rectangles of width dx, depth dy, and height $g(x,y)$ [draws a thin prism]… Yeah, rectangular prism. And then add all those together in this rectangle from a to b, c to d [draws a rectangle to represent the domain]. So we have a bunch of these *dx* by *dy* rectangles here [draws little squares inside the domain], and we're measuring the rectangular prisms underneath the surface.

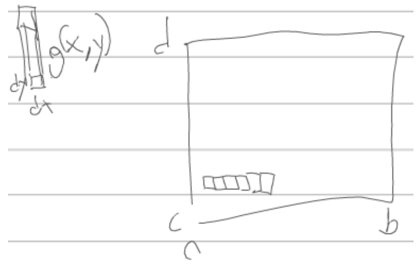


Figure 4. Adding up pieces, with a *representative rectangular prism*

This student clearly demonstrated the usage of a single *representative prism* in order to frame his understanding of the quantities *dx*, *dy*, and *g*(*x,y*), and how they relate to each other. Once their relationship was established, the student thought of this prism as being representative of all other prisms throughout the domain. That is, only a single prism is needed to make sense of the integration process, and the final step is to "add up" (or accumulate) the infinitely-many prisms over the domain to determine the total amount. In this way, we wish to rename the *representative rectangle* a "*representative piece*" in order to for it to apply to any definite integral context.

Conclusion

In this paper, we have documented several generalized student conceptualizations regarding multivariate integration. These conceptualizations are largely based off of the students' single-variable conceptions, yet many important additional features are included in the students' restructuring of their integration knowledge. Some conceptualizations, such as the boundary and volume, infinite strip, and adding up pieces are *reconstructive* generalizations in that the students have reorganized their knowledge in order to broaden it to the multivariate context. On the other hand, abstract space, adding up slices, and building an expression of inputs seem to be *expansive* generalizations in that the students simply extend their prior knowledge to the multivariate context. Interestingly, none of the conceptualizations documented in this paper seems to be *disjunctive*. That is, the students' conceptualizations all seem rooted in and connected to prior understandings.

References

Brown, J., & Churchill, R. (2008). *Complex variables and applications* (8th ed.). New York, NY: McGraw-Hill.

Dorko, A., & Weber, E. (2014). Generalising calculus ideas from two dimensions to three: How multivariate calculus students think about domain and range. *Research in Mathematics Education*.

Goldin, G. A. (1997). Observing mathematical problem solving through task-based interviews. *Journal for Research in Mathematics Education. Monograph, 9*, 40-62.

Harel, G., & Tall, D. (1991). The general, the abstract, and the generic in advanced mathematics. *For the Learning of Mathematics, 11*(1), 38-42.

Hibbeler, R. C. (2012). *Engineering mechanics: Statics* (13th ed.). Upper Saddle River, NJ: Pearson Prentice Hall.

Jones, S. R. (2013). Understanding the integral: Students' symbolic forms. *The Journal of Mathematical Behavior, 32*(2), 122-141.

Kabael, T. U. (2011). Generalizing single variable functions to two-variable functions, function machine and APOS. *Educational Sciences: Theory and Practice, 11*(1), 484-499.
Knuth, E. J. (2002). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education, 33*(5), 379-405.
Martinez-Planell, R., & Trigueros-Gaisman, M. (2012). Students' understanding of the general notion of a function of two variables. *Educational Studies in Mathematics, 81*(3), 365-384.
Martinez-Planell, R., & Trigueros-Gaisman, M. (2013). Graphs of functions of two variables: Results from the design of instruction. *International Journal of Mathematics Education in Science and Technology, 44*(5), 663-672.
Martinez-Planell, R., Trigueros-Gaisman, M., & McGee, D. (2014). On students' understanding of partial derivatives and tangent planes. In S. Oesterle, C. Nicol, P. Liljedahl & D. Allan (Eds.), *Proceedings of the 38th annual meeting of the International Group for the Psychology of Mathematics Education*. Vancouver, Canada: PME.
Salvatore, D. (2008). *Microeconomics: Theory and applications* (5th ed.). Oxford, UK: Oxford University Press.
Serway, R. A., & Jewett, J. W. (2008). *Physics for scientists and engineers* (7th ed.). Belmont, CA: Thomson Learning.
Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1-36.
Weber, E., & Thompson, P. W. (2014). Students' images of two-variable functions and their graphs. *Educational Studies in Mathematics, 87*(1), 67-85.
Yerushalmy, M. (1997). Designing representations: Reasoning about functions of two variables. *Journal for Research in Mathematics Education, 28*(4), 431-466.

Bundles and associated intentions of expert and novice provers: The search for and use of counterexamples

Shiv Smith Karunakaran
Washington State University

The argument for the importance of proving and of proof in the teaching and learning of mathematics has been repeatedly made by mathematics education researchers and by policy documents. There is also considerable research examining the existence of a gap in the proving and proof-constructing abilities of "novices" and "experts" in mathematics. However, considerably less research examines the nature of what constitutes expertise in proving mathematical statements, specifically with regard to the use of the individuals' mathematical knowledge. This study uses grounded theory methods to examine "expert" and "novice" mathematicians in the process of proving mathematical statements. The result reported here focuses on the differences in the use of and search for counterexamples by the two populations. More specifically expert provers seem to value the unsuccessful searches for counterexamples, as well as the finding of valid counterexamples.

Key words: Proving and proof, Counterexamples, Expert and novice mathematicians

The importance of proof has long been emphasized by both mathematics teacher education organizations and groups formed by professional mathematics organizations (e.g., NCTM, 2000; CBMS, 2001). Moreover, the process of proving is indispensable to the act of doing mathematics. At the higher academic levels (graduate and professional mathematics), proving can be considered as the way in which the truth of a claim is established or realized (Hanna, 2000) and a way in which new knowledge may be created. However, the research on proof has routinely focused more on the production and evaluation of a finished and valid proof and less on the purposes or intentions behind the provers' use of their demonstrated mathematical knowledge during the process of proving. This report focuses on the process of proving, particularly the use of and search for counterexamples. The finding presented in this report is part of a larger study (Karunakaran, 2014) that examined how and why provers used the mathematical knowledge they called on in the service of proving a mathematical statement. The primary research question guiding the larger study was: In what ways are expert and novice provers of mathematics similar and different in their use of the mathematical knowledge they call on during the process of proving a mathematical statement?

Research has demonstrated that a gap may exist between a novice's understanding of the proving process and an expert's understanding of the proving process. Raman (2002) posits that a mere presentation of the statement of a theorem and the subsequent presentation of the proof of the theorem may not engender students' understanding of the proving process. Also, Chin and Tall (2002) have stated that in mathematical textbooks "we can simply see the process of a mathematical proof [sic] as the development of a sequence of statements using only definitions and preceding results, such as deductions, axioms, or theorems" (p. 213). In fact, one mathematics professor quoted by Ayalon and Even (2008) expressed the view that a student "thinks about something, he draws a conclusion, which brings him to the next thing ... Logic is the procedural, algorithmic structure of things" (p. 240). This "systematic, step–by–step manner" (Ayalon & Even, 2008) expressed in final written proofs does not reveal what led the mathematicians to produce these particular arguments or "steps." This may, in turn, promote the memorization of the proofs by the students without understanding the proving process. Students may look at proof as a finished product generated by someone other than himself or herself and not as a constructed argument that the students themselves could produce.

In fact, Romberg (1992) suggested that mathematics should no longer be thought of as a finished product, but as a "process of inquiry and coming to know, a continually expanding field of human creation and invention" (p. 751). This revised view of mathematics focuses on the perspective that to know mathematics is to do mathematics. This, in turn, calls for more focus on mathematical work and activity of practicing mathematicians, and the mathematical activity of these mathematicians could subsequently serve as a model for students' mathematical activity. Lampert (1990), Schoenfeld (1992), and Stylianou (2002) have called for one of the goals of mathematics instruction to better understand the practices of mathematicians. Harel and Sowder (2007) have unambiguously claimed that the one of the goals of mathematics instruction should be to "help students gradually develop an understanding of proof that is consistent with that shared and practiced in contemporary mathematics" (p. 807). Furthermore, Weber and Mejia–Ramos (2011) have also listed as one of the goals of mathematics instruction to be for students to behave more like mathematicians in proof–related activities or tasks. To work towards these goals, researchers (Blum & Kirsch, 1991; Weber, 2001) have emphasized how teachers can help students better learn from and understand a mathematician's work with proof by making the act or process of proving clearer to the students. The present study (a part of which is reported here) aimed to contribute new knowledge about the similarities and differences between the observed use of mathematical knowledge by expert provers and novice provers while proving a mathematical statement.

Theoretical Perspective

Gaining expertise in the act of doing mathematics, and thus in the process of proving mathematical statements (which is a subset of doing mathematics), involves the use of the existing knowledge that the individual has accumulated. It is not merely the fact that individuals have assimilated this wealth of knowledge, but also how they call upon the various facets and parts of this knowledge that allows them to demonstrate expertise in the act of proving mathematical statements. To this end, it seems useful to examine portions of the mathematical knowledge that can be inferred as an individual proves a mathematical statement. These inferences about mathematical knowledge can be drawn from observations of the properties, objects, procedures, definitions, theorems, and so on, that an individual brings to bear in the service of proving a statement. The parts of mathematical knowledge that are called on during the process of proving is referred to as the *resources* to which the individual accesses during the act of doing mathematics. It is further posited that an individual's mathematical knowledge is comprised of a connected network of such resources. This view of mathematical knowledge and of doing mathematics as using a network of relations is not novel within mathematics education research. Hiebert and Lefevre (1986) view mathematical knowledge (more specifically conceptual mathematical knowledge) as a "web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information" (p. 3).

Once the individual is observed calling on one or more resources, then he or she can be reasonably observed acting on these resources. That is, he or she may use the resources to perform certain *actions* such as asking a question based on the resource(s), constructing an example, searching for a counterexample, and using a form of reasoning. The terms "actions" and "resources" are adapted and expanded from the work of Wilkerson–Jerde and Wilensky (2011). Those researchers described how mathematicians use different *resources of mathematical understanding* and *acts of mathematical understanding* in order to read and understand a published, but unfamiliar mathematical paper about knot theory. Wilkerson–

Jerde and Wilensky do not offer any definition of resources of mathematical understanding other than to equate them to “specific knowledge” (2011, p. 22) that the mathematicians used in order to understand the unfamiliar mathematics present in the paper. Although Wilkerson–Jerde and Wilensky only focus on three acts of mathematical understanding (namely *question*, *resolution*, and *explanation*), the concept of acts of mathematical understanding has been expanded for the purposes of the present study to include a wider range of actions that the prover may utilize in the process of proving.

However, simply identifying the resources and the actions an individual uses in the process of proving does not give sufficient insight into the individuals’ rationale underlying their use of the resources and actions. Skemp’s construct of relational understanding of mathematics (1976) involves knowing not just what to do when doing mathematics, but also why to do it. The assumption both Skemp and the authors of the present study make is that an individual makes intentional decisions to use what they know and how to use what they know. Skovsmose (2005) differentiates the notions of action and “blind activity” using this same assumption. Blind activity is characterized by automatic behavior and it assumes that there is no true rationale behind what an individual is doing. In contrast, an action presupposes some degree of choice and as such assumes that the individual has a purposeful intention behind performing said action. For the present study, the authors adopt Skovsmose’s notion that any action or set of actions identified during the process of proving a statement is associated with intention(s). That is, one cannot truly describe the actions of an individual without considering the intention behind the actions.

Skovsmose (2005) hints towards going beyond merely identifying singular actions and the attached intentions. He seems to suggest that it is also important to identify *activities* (or actions grouped together, as defined by Skovsmose) and the intentions behind such activities. Thus, when analyzing an individual’s process of proving, the authors also went beyond identifying individual actions (and the resources involved) and identified groups of actions and resources that seemed to be tied together with a common intention or intentions. Analogous to Skovsmose’s notion of activities, the authors of this report defined these groups of actions and resources as *bundles*. More specifically, bundles are defined as subsections of the proving process that consist of groups of actions and resources that are clustered together by identifiable intentions. These identifiable intentions are nested within the assumed larger goal of proving the statement in question.

The theoretical constructs of bundles, the associated intentions, and the constituent actions and resources were used to describe the use of an individual’s (either an expert or a novice) use of mathematical knowledge in their dynamic proving process. Specifically, the research questions that guided the larger study were:

1. In what ways are expert provers’ bundles similar to or different from novice provers’ bundles during the process of proving mathematical statements?
2. In what ways are the expert provers’ intentions associated with bundles during the process of proving mathematical statements similar to or different from novice provers’ intentions associated with the bundles during the process of proving?

Methods

The research questions described previously do not fall under the category of validating an existing theory of how individuals prove. Instead the study was about investigating the bundles (and the component resources and actions) and their associated intentions of different groups of individuals (expert and novice provers of mathematical statements) in the process of proving a mathematical statement. The focus of the study lent itself to the adoption of certain *grounded theory* methods to guide data collection and data analysis, specifically the data analysis strategies of *open coding, axial coding, selective coding,* and *constant comparative analysis* (Charmaz, 2006; Strauss & Corbin, 2008).

A group of novice provers of mathematics and a relatively more expert group were recruited. The novice group included five undergraduate students who had all successfully completed at least one proof–based course in real analysis. Placing such a requirement allowed the researchers to be confident that the members of the novice group (hereby referred to a Novice Prover or NP) had been exposed to an introduction to various mathematical proof strategies, such as proof by mathematical induction, proof by contradiction, and proof by first principles. The expert group included five doctoral students in mathematics, all of whom had successfully passed their department's doctoral qualifying examinations. By requiring that the members of the expert group (hereby referred to an Expert Provers or EP) to have passed doctoral qualifying examinations ensured their experience with doing mathematics, and more specifically in proving and in the generation of proofs.

All ten participants (NPs and EPs together) were each presented with five real analysis statements (see Figure 1) in an interview setting using a think–aloud protocol. As is shown in the figure, the directions for all five tasks were to "Validate or refute the following statement." Presenting this as the direction, and not presenting the more traditional direction of "Prove that …", ensured that the initial opinion of the participants about the truth of the mathematical statement was not merely due to the format of the directions of the task statement. Also, the statements of Tasks 1 and 4 are invalid as stated in Figure 1. If the provers came up with a valid proof of why the statements were invalid, then the statements were amended to make them valid and the provers were again asked to validate or refute the newly amended statement.

1. Validate or refute the following statement:
   Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $0 < a_n \leq a_{2n} + a_{2n+1}$, $\forall\, n \in \mathbb{Z}$ & $n \geq 1$. Then the series $\sum_{n=1}^{\infty} a_n$ converges.

2. Validate or refute the following statement:
   There exists a function g that satisfies all of the following properties:

   (a) $g(x) = x \cdot g\left(\frac{1}{x}\right)$, for all real numbers $x \neq 0$, and

   (b) $g(x) + g(y) = 1 + g(x + y)$, for all real numbers x and y.

3. Validate or refute the following statement:
   Let $f(x) = \cos(\sqrt{1}x) + \cos(\sqrt{2}x) + \cos(\sqrt{3}x) + \cdots + \cos(\sqrt{n}x)$, where $n \in \mathbb{Z}^+$. Then $f(x)$ is a periodic function for only $n = 1$.

4. Validate or refute the following statement:
   Suppose $p(x)$ is a polynomial with integer coefficients and such that $p(x)=3$ has three distinct integer solutions. Then the equation $p(x)=4$ has at least one integer solution.

5. Validate or refute the following statement:
   Suppose $p(x)$ is a polynomial with integer coefficients such that the sum of all of its coefficients is an odd integer, and its constant coefficient is also odd. Then there is no such polynomial with at least one integer root.

Figure 1. The five tasks selected for the present study.

The interviews were transcribed and then initially coded for instances of actions and resources (open coding phase), and then the actions and resources were organized into bundles and the associated intentions (axial coding phase). The intentions were either explicitly stated by the participants or were inferred from the mathematical work of the participants. In final phase of coding (selective coding), the identified bundles and associated intentions were used to generate characterizations of how an individual (an EP or a NP) and subsequently how the groups (EP or NP) prove a mathematical statement.

Results

Only one of the claims generated by the findings of the larger study is reported here. This claim involves how the expert and novice provers are similar and different in their use of a counterexample and more specifically in their intentions behind searching for counterexamples. The authors found that if the prover deemed the resource of "counterexample" to be relevant to the task context, then he or she (whether an expert or a novice prover) brought the resource of "counterexample" to bear in the proving process. Both the expert and novice provers recognize the implication of finding a counterexample for the validity of the task statement. In other words, the statement in question is invalid if the prover successfully generated a counterexample. For instance, both Cassie[1] (NP) and Yanni (EP) successfully generated counterexamples for the invalid statement of Task 1 (see statement of task in Figure 1). Cassie's counterexample sequence $\{a_n\} = n,\ \ n \in \mathbb{Z}^+,$ and Yanni's counterexample sequence $\{a_n\} = 5,\ \ n \in \mathbb{Z}^+$ both satisfy the required condition ($0 < a_n \leq a_{2n} + a_{2n+1}, \forall\, n \in \mathbb{Z}\ \&\ n \geq 1$), but the respective series $\sum_{n=1}^{\infty} a_n$ clearly diverges. Also, both Yanni and Cassie were convinced about the validity of their counterexamples and as a result were convinced that Task 1 as stated was false.

Even though both the expert provers and the novice provers knew to search for counterexamples to possibly refute a given statement, the expert provers generated and examined successive counterexamples regardless of whether or not they believed a counterexample existed. They did this by varying conditions or constraints they placed on mathematical objects to initiate new searches to generate and test for counterexamples. The experts' intentions behind searching for counterexamples seemed to be more than to succeed in finding a counterexample, and seemed to include gaining insight into why the task statement may or may not be valid. For instance, when asked why he was searching for counterexamples, James (EP) stated, "… the best–case scenario of being able to find a counterexample if [the statement of Task 4] was not true. … if I try a couple times and I find that I can't find the counterexample, this might give me some sort of insight into why it is true let's say." Similarly, Zander (EP) explained his thought process in looking for a counterexample for Task 1 by stating, "if I'm trying to formally show that this sequence is always gonna diverge, I can use something that I found in the [failed] search for a counterexample and that inability is actually um the key to why it always diverges." In contrast, novice provers did not persevere in the search for a counterexample. Instead, they seemed to require a prior rationale for why the statement might be invalid. For example, when Derek (NP) was asked why he did not search for a counterexample in his work for Task 4, he stated, "Before I think of a counterexample I usually try to think of why I'm picking that counterexample. And I couldn't really think of any reason … Usually you have to have a reason why you want the counterexample. … I didn't think of just like randomly searching for a counterexample. That's usually not the most efficient way to do it."

To illustrate the point that expert provers use the unsuccessful searches for counterexamples, consider the work of Zander (EP) on Task 5 (see statement of task in Figure 1). For the sake of brevity, Figure 2 illustrates Zander's process of repeatedly and unsuccessfully searching for a counterexample and what he subsequently learned from searching for counterexamples (CEs) that allowed him to generate an argument that validated

[1] All names used in this report are pseudonyms.

the statement of Task 5. Like Zander, other expert provers also used what they learned from unsuccessful searches for counterexamples to gain some insight about the conditions present in a given task statement that then allowed them to validate the statement.

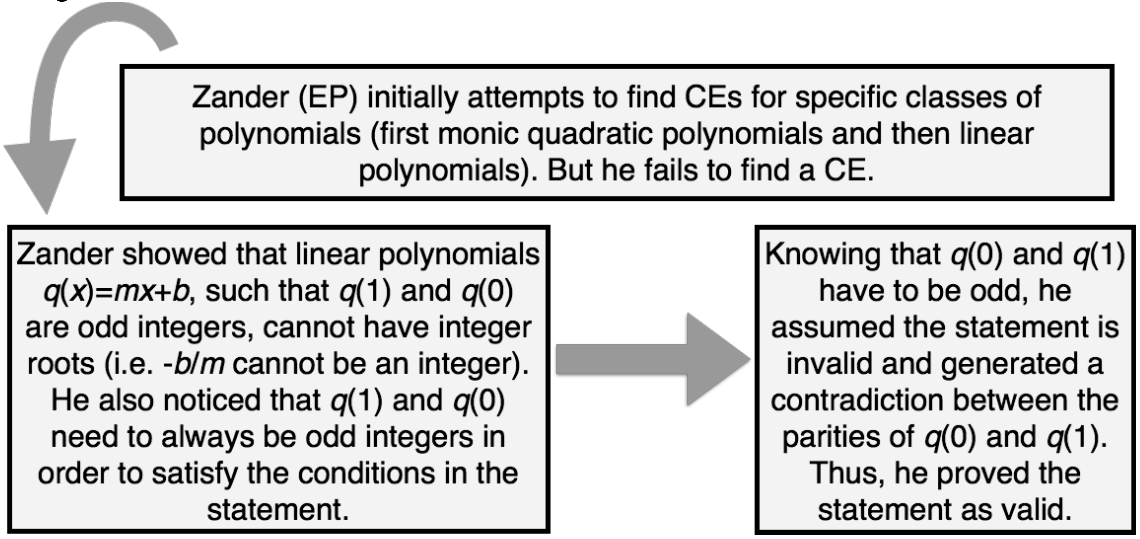


Figure 2. Zander's process of trying to find counterexamples and then subsequently using what he learned from the process to validate the statement of Task 5.

Significance

The characterizations of expert proving behavior related to the use of and searches for counterexamples that were developed based on the results of the reported study can be used to further the ongoing conversation about why expert proving behavior is important to study. Previous research in the field has focused on the static product of proof and to a lesser extent, on the more dynamic process of proving. As such, previous research has produced useful schemes by which to describe and categorize the types of resources used in producing proofs (e.g., Peled & Zaslavsky's (1997) scheme for categorizing counterexamples). However, the current study had a general focus on the proving process (as separate from the product of proof) and more specifically on the provers' intentions behind the bundles involved in the proving process. The focus on bundles, including the intentions associated with the bundles, allowed for a refined look at the nature of the differences between expert and novice proving behavior. Specifically, the result reported here provides evidence for how superficial similarities between expert provers' and novice provers' use of counterexamples may belie deeper differences in their intentions behind the generation and use of these counterexamples.

References

Ayalon, M., & Even, R. (2008). Deductive reasoning: In the eye of the beholder. *Educational Studies in Mathematics, 69,* 235–247.

Blum, W., & Kirsch, A. (1991). Preformal proving: Examples and reflections. *Educational Studies in Mathematics, 22*, 183–203.

Charmaz, K. (2006). *Constructing grounded theory: A practical guide through qualitative research.* Thousand Oaks, CA: SAGE Publications.

Chin, E.–T., & Tall, D. (2002). Proof as a formal procept in advanced mathematical thinking. In F.–L. Lin (Ed.), *Proceedings of the International Conference on Mathematics: Understanding Proving and Proving to Understand* (pp. 212–221). Taipei, Taiwan: National Taiwan Normal University.

Conference Board of the Mathematical Sciences. (2001). *The mathematical education of teachers*. Washington DC: American Mathematical Society.

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics, 44*, 5–23.

Harel, G., & Sowder, L. (2007). Towards a comprehensive perspective on proof. In F. Lester (Ed.), *Second handbook of research on mathematical teaching and learning* (pp. 805–842). Washington, DC: NCTM.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1–28). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Karunakaran, S. S. (2014). *Comparing bundles and associated intentions of expert and novice provers during the process of proving* (Unpublished doctoral dissertation). The Pennsylania State University, University Park, PA.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, *27*, 29–63. doi:10.2307/1163068

National Council of Teachers of Mathematics. (2000). *Principles and standards of school mathematics*. Reston, VA: NCTM.

Peled, J., & Zaslavsky, O. (1997). Counter–examples that (only) prove and counter–examples that (also) explain. *Focus on Learning Problems in Mathematics, 19*(3), 49–61.

Raman, M. J. (2002). *Proof and justification in collegiate calculus* (Unpublished doctoral dissertation). University of California, Berkeley, Berkeley, CA.

Romberg, T. A. (1992). Problematic features of the school mathematics curriculum. In P. W. Jackson (Ed.), *Handbook of research on curriculum: A project of the American Educational Research Association*. New York, NY: Macmillan.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and leaning* (pp. 334–370). New York, NY: Macmillan.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 20–26.

Skovsmose, O. (2005). Meaning in mathematics education. In J. Kilpatrick, C. Hoyles, O. Skovsmose, & P. Valero (Eds.), *Meaning in mathematics education* (83–100). New York, NY: Springer.

Strauss, J., & Corbin, A. (2008). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (3rd ed.) [Kindle edition]. Los Angeles, CA: Sage Publications.

Stylianou, D. A. (2002). On the interaction of visualization and analysis: The negotiation of a visual representation in expert problem solving. *Journal of Mathematical Behavior, 21*, 303–317. doi:10.1016/S0732–3123(02)00131–1

Weber, K. (2001). Student difficulty in constructing proof: The need for strategic knowledge. *Educational Studies in Mathematics, 48*, 101–119.

Weber, K., & Mejia–Ramos, J. P. (2011). Why and how mathematicians read proofs: an exploratory study. *Educational Studies in Mathematics, 76*, 329–344.

Wilkerson–Jerde, M. H., & Wilensky, U. J. (2011). How do mathematicians learn math?: Resources and acts for constructing and understanding mathematics. *Educational Studies in Mathematics, 78*, 21–43. doi:10.1007/s10649–011–9306–5

An Analysis of Sociomathematical Norms of Proof Schemes

Brian Katz
Augustana College

Rebecca Post
Augustana College

Milos Savic
University of Oklahoma

John Paul Cook
University of Science and Arts of Oklahoma

We report on a case study aimed at researching the social interactions of a classroom focusing on the certainty of mathematical claims and justifications. Blending Harel and Sowder's (1998) concept of "proof schemes" with Yackel and Cobb's (1996) "sociomathematical norms," we aim to expand on Fukawa-Connelly's (2012) research on sociomathematical norms of proof presentations. Preliminary analysis of classroom interaction and student interview transcripts from a proof-based, axiomatic geometry course suggests the presence of sociomathematical norms related to argumentation that lie outside of proof validation that facilitate renegotiating proof schemes.

Key words: emergent sociomathematical norms, proof schemes, axiomatic geometry

Introduction

In recent years, researchers have gained significant insight into the teaching and learning of mathematics by looking at patterns of social interactions in mathematics classrooms. By attending to ways that class time and attention are spent, patterns of engagement and responsibility are observed (Fukawa-Connelly, 2012); by attending to the ways that claims are supported and rejected, patterns of meaning-making and proof validation are observed (Yackel, Rasmussen, & King, 2000). This project aims to add to this growing list of observed patterns and our understanding of learning by investigating the ways that the classroom community evaluates the certainty of its collective knowledge and the associated approaches to justification by synthesizing research on beliefs about certainty, called proof schemes (Harel & Sowder, 1998), with research on classroom social patterns, called sociomathematical norms (Yackel & Cobb, 1996).

Background and Theoretical Framework

In their foundational work, Harel and Sowder (1998) introduce the concept of a proof scheme, which "consists of what constitutes ascertaining and persuading for that person" (p. 244). The researchers identified three types of proof schemes, which are separated by the kinds of evidence utilized: (i) external proof schemes rely on authority for evidence, (ii) empirical proof schemes on induction, and (iii) analytic proof schemes on deduction.

The definition of proof schemes is "deliberately psychological and student-centered", yet the researchers are careful to place it "in a given social context" (p. 244). Harel and Sowder (1998) conclude with a call for educational reforms that help students build axiomatic proof schemes, but they do not propose a mechanism for the social classroom to interact with the psychological schemes. For this connection, we turn to the emergent perspective (Cobb & Yackel, 1996) and the concept of sociomathematical norms (Yackel & Cobb, 1996). In the emergent perspective, there is a "reflexive relationship between the social constructs [norms] and their psychological correlates" (Yackel, Rasmussen, & King, 2000). Norms are defined as "normative interactions in the classroom" (Yackel, Rasmussen, & King, 2000) and sociomathematical norms are the subset of norms that are specific to the discipline of mathematics.

Research groups have operationalized the definitions of social and sociomathematical norms in divergent settings including Differential Equations (Yackel, Rasmussen & King, 2000) and second grade mathematics (Yackel & Cobb, 1996), however, very little of the research on sociomathematical norms concerns upper-level proof-based courses. Fukawa-Connelly (2012) studied norms in a student-centered course in Abstract Algebra. The class activity was conceptualized as "making believe" (Ball & Bass, 2000), meaning that the class community was collectively "coming to conviction" (Fukawa-Connelly, 2012), a term that blends proof validation and scheme. Two of the norms described, *using only peer-validated knowledge* and *justifying new inferences based on previous ones*, can be summarized as using an analytic proof scheme. The norm of *convincing oneself* is described as a personal kind of proof validation, and the student interviews explicitly contrasted this activity with memorizing or transcribing without understanding, indicating that the norm goes beyond validating and into ascertaining. The goal of this project is to push this connection further: can we see patterns from the classroom of the students renegotiating what should convince them and to what extent they are convinced? Our research questions are continuations of the questions of Fukawa-Connelly (2012).

Research Questions

· What norms are enacted in an undergraduate advanced mathematics classroom that encourage and facilitate classroom discussions about the certainty of mathematical knowledge?

· What beliefs about their roles as learners of mathematics and the nature of mathematics do the students hold and enact that reflect these norms?

· How do the norms and beliefs support students to engage in appropriate activities for proof-writing or making sense of presented proofs?

Methods

The students participating in the study were enrolled in a junior-level inquiry-based course at a Midwestern small liberal arts college that explored Euclidean and hyperbolic geometries as well as philosophy of mathematics. The course met for 75 minutes, three times per week, for 10 weeks. There were 11 junior mathematics majors enrolled in the course, 8 of whom were mathematics education majors. Most classes consisted of cycles in which the students attempted to prove theorems in small groups of 2-3 at the board, followed by a presentation and group discussion of the arguments.

Data for this preliminary report come from transcripts of video of small group and whole class discussions and audio of individual student interviews. The first phase of analysis followed Rasmussen & Stephan (2008), identifying episodes that discussed epistemological themes, were pivotal for student argumentation, or were cited across the course. Pattern definitions were articulated collaboratively, and the process was repeated until understanding of the episodes stabilized (Cobb et al., 2000).

Results

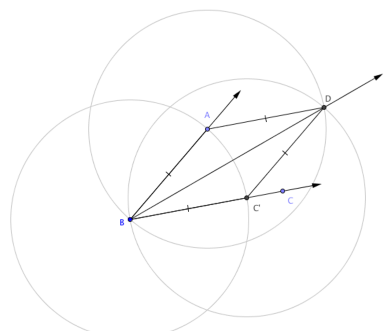
In addition to confirming many of the norms described by other researchers, we observed one new norm that we will call *meta-argumentation*: each member of the community is responsible for evaluating approaches to justifying claims and using that evaluation to guide inquiry and communication. This norm is essentially social in the sense that it requires members of the community to participate in the evaluation; we observed four associated sociomathematical norms that describe appropriate participation in this mathematical context (See Table 1).

Table 1: *Four Associated Sociomathematical Norms*

N1	We describe the extent of our current knowledge and toolkit, including the holes in this collection.
N2	We connect this description of our un/knowns with the demands of proving a claim to select an approach.
N3	We compare and contrast the tools of various disciplines to select an approach.
N4	We use aesthetic judgments to choose simpler and more general approaches.

We will focus on a short episode from the fourth day of class containing enactments of norms N1 and N2. In the episode, the students had just constructed an angle bisector; the student presenters could not justify their claim, so the instructor asks the whole class how the argument *must* proceed (see Figure 1).

Figure 1: *Transcript of the Episode*



I: OK. Yep, they're congruent, but why? What results do we have whose conclusion is "therefore these angles are congruent"?
BP: Congruent triangles.
I: Which we call what? [DV]?
DV: Congruent parts of congruent figures are congruent.
I: That's the ONLY result we have so far that says "therefore two angles are congruent", right? So that HAS to be your second to last step. So? Do you know any angles are congruent already?
All: (head shaking)
I: You don't know anything about angles here, so what is our only other... and line segments, we're not going to get there just with line segments, right? We have three assumptions. We have Axioms 1, 2, and 3. Axiom 1 says segments are congruent if their lengths are, but that's never going to get us to figures being congruent, right? We have side-side-side and side-angle-side. What do you think, [GV]?
GV: Side-side-side.
I: Can you put those two angles inside triangles that are congruent by side-side-side? Can you see those two angles as inside two triangles that are congruent by side-side-side?
EJ: Kind of.
KG: Yeah. Because this line (BD) is shared by both the triangles so it's the same.

This episode makes it clear how N2 relies on N1: in order for the students to match the demands with an approach, they need to call to mind all possible tools available to them. The argument being developed requires two observations, each of which is an instance of N2 that is developed Socratically by the instructor and students. In the first instance, the group realizes that the argument must end with "[corresponding] parts of congruent figures are congruent" because "that's the only result" with the correct conclusion. The same pairing of N1-N2 appears as the group realizes that they should attempt to develop a situation in which they can use the side-side-side axiom to get congruent figures. The instructor prompts the norms in this early episode, but the students do engage and will eventually enact them spontaneously as the course progresses. Moreover, by the end of the term, the students enact N1 by having quick and reliable access to over 150 statements, which they reference simply by number.

Discussion and Questions

As Harel and Sowder (1998) point out, "the proof schemes held by an individual are inseparable from her or his sense of what it means to do mathematics" (p. 235). This is precisely what is being negotiated by the norms we have described above. Moreover, these norms appear to focus the discussion and reflection to support development of axiomatic proof schemes. Norm N1 asks the students to hold a complete list of truths in their minds. In practice, this makes the students, rather than an external authority, responsible for this information; because this information is familiar from prior mathematics courses, the students must also consciously separate their beliefs from the accepted truths. The students demonstrated "concern for the origin of the truths" (p. 247) by tagging these truths in the discourse as assumptions (axioms), choices (definitions), and conclusions (theorems). In fact, in the interviews, one student commented that "we're pushed to not accept that and actually have concrete evidence based on logic that things are, in fact, true or, in fact, un-provable, or false, or whatever".

Norm N2 encodes the "goal-oriented and intended generality" (Harel & Sowder, 1998, p. 258) of arguments in early analytic proof schemes while making sure that "the focus of the study is on the structure itself, not on the axiom system" (p. 273) by asking students to step back and consider *what* they might be able to know and *how*. Moreover, this metacognitive stance provides the reflection needed to interiorize aspects of proof. In the interviews, one student claimed, "I always begin by kind of thinking about where I need to end up and how I'm gonna get there…[and] then try to say, 'Oh! What do I know? What- what'd we learn before this? How can that help me?'" This process is a version of Selden and Selden's (2013) formal-rhetorical part, which seems helpful for the generative habit of novel proof production (Marty, 1991).

Norm N3 is the most obviously related to proof schemes: the most common enactment of this norm involved an explicit discussion of how mathematical certainty required deduction while tools like intuition, perception, measurement, and methods of science could only provide induction or were susceptible to hidden assumptions. Norm N4 often arises "when a [student] understands at least in principle a mathematical justification must have started originally from undefined terms and axioms" (Harel & Sowder, 1998, p. 273) and seeks to participate in these otherwise arbitrary choices.

Questions: What other meta-proving sociomathematical norms are there? Could these norms be seen outside geometry, perhaps in intro-to-proof courses? What pedagogical ways can a sociomathematical norm of proving be inserted into the classroom? How can we leverage our data to understand the development of these norms?

References

Ball, D., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. *Multiple Perspectives on the Teaching and Learning of Mathematics*, 83-104.

Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2001). Participating in classroom mathematical practices. *Journal of the Learning Sciences, 10* (1–2), 113–163.

Fukawa-Connelly, T. (2012). Classroom sociomathematical norms for proof presentation in undergraduate in abstract algebra. *Journal of Mathematical Behavior , 31*, 401-416.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.). *CBMS Issues in Mathematics Education, Vol. 7: Research in Collegiate Mathematics Education III* (pp. 234-283). Providence, RI: American Mathematical Society.

Marty, R. (1991). Getting to eureka! Higher order reasoning in math. *College Teaching , 39* (1), 3-6.

Rasmussen, C., & Stephan, M. (2008). A methodology for documenting collective activity. *Handbook of innovative design research in science, technology, engineering, mathematics (STEM) education* , 195-215.

Selden, A., & Selden, J. (2013). Proof and Problem Solving at University Level. *The Mathematics Enthusiast , 10* (1&2), 303-334.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education* , 458-477.

Yackel, E., Rasmussen, C., & King, K. (2000). Social and sociomathematical norms in an advanced undergraduate mathematics course. *Journal of Mathematical Behavior , 19* (3), 275-287.

Students' generalizations from single to multivariable limits

Sarah Kerrigan, Erin Glover, Eric Weber, and Allison Dorko
Oregon State University

Studies indicate that students struggle with generalizing across mathematical contexts, yet little research on generalization has been conducted in post secondary settings. This poster presentation reports on students' generalization in the context of limits. Understanding limits is an essential part in understanding calculus. We conducted interviews in order to characterize students' generalizations as they made sense of limits in single and multivariable settings.

Keywords: *generalizing, calculus, single variable calculus, multi-variable calculus, limit interview study*

Introduction and Relation to Literature

Generalizing is important to mathematics learning because it can foster conjecturing and justification in the classroom (Lannin, 2005). Much of the existing body of research on generalization is related to algebra and function (e.g., Carraher, Schliemann, Brizuela, & Earnest, 2006; Ellis, 2007; Ellis, 2011). These studies suggest that generalizing within these various mathematical contexts is challenging, yet little research on generalization has been conducted in post secondary settings. Findings from what research that does exist indicates that students have difficulty generalizing between their informal notions of functions in 2-space and functions in 3-space (Dorko & Weber, 2014; Kabael, 2011; Trigueros and Martinez-Planell, 2010; Weber, 2014; Yerushalmy, 1997). This poster presentation reports on students' generalizations in the context of limit because understanding of limit (and the skills associated with limits) is an essential part of what it means to understand calculus (Sofronas, DeFranco, Vinsonhaler, Gorgievski, Schroeder, & Hamelin, 2011). This work originates from the NSF-funded the *Generalizing Across Multiple Mathematical Areas* (GAMMA) project the goal of which is uncovering generalizing in the context of advanced algebra, geometry, calculus, and combinatorics.

Data Collection and Theoretical Perspective

Participants for this study were recruited from multivariable calculus classes and compensated for the hour-long interview. The interview protocol asked questions to probe students' understanding about generalizing their informal notions of limit to the formal definition of limit, and their generalization of limit from single variable to multivariable functions. The interviews were audio recorded, video recorded, and student written work was collected using Pencast's Livescribe technology. All interviews will be transcribed and analyzed. The research team is guided Lobato's (2012) actor-oriented transfer perspective in order characterize generalization based on what students see as similar across mathematical situations rather than conventional ideas about what counts as a correct generalization. The analysis will be conducted using Ellis' (2007) generalizations framework, which describes specific the actions students engage in as they generalize.

Preliminary Results

Early findings indicate that students have a difficult time generalizing informal ideas about limit from single variable to multi variable contexts. Furthermore, when presented with the formal definition of limit students, struggled to make sense of the notation. When prompted to find similarities and extend ideas between their student-generated representations (e.g., graphs, symbolic notation) to representations of a higher dimension, often the students' ideas were often unproductive as they were uncertain as to what they were measuring when using a notion of a limit. The results of this work will support uncovering what generalizations in calculus look like and how students' ideas about single-variable limits might be leveraged in generalizing ideas from single variable to multivariable contexts.

References

Carraher, D. W., Schliemann, A. D., Brizuela, B. M., & Earnest, D. (2006). Arithmetic and Algebra in Early Mathematics Education. *Journal for Research in Mathematics Education*, *37*(2), 87–115. doi:10.2307/30034843

Dorko, A., Weber, E. (2014). Generalising calculus ideas from two dimensions to three: how multivariable calculus students think about domain and range. *Research in Mathematics Education.* doi: 10.1080/14794802.2014.919873

Ellis, A. B. (2007). A Taxonomy for Categorizing Generalizations: Generalizing Actions and Reflection Generalizations. *Journal of the Learning Sciences*, *16*(2), 221–262.

Ellis, A. B. (2011). Algebra in the middle school: Developing functional relationships through quantitative reasoning *Early Algebraization* (pp. 215-238): Springer.

Kabael, T. U. (2011). Generalizing Single Variable Functions to Two-Variable Functions, Function Machine and APOS. *Educational Sciences: Theory and Practice, 11*(1), 484-499.

Lannin, J. K. (2005). Generalization and Justification: The Challenge of Introducing Algebraic Reasoning through Patterning Activities. *Mathematical Thinking and Learning: An International Journal, 7*(3), 231-258.

Lobato, J. (2012). The Actor-Oriented Transfer Perspective and Its Contributions to Educational Research and Practice. *Educational Psychologist, 47*(3), 232-247.

Sofronas, K. S., DeFranco, T. C., Vinsonhaler, C., Gorgievski, N., Schroeder, L., & Hamelin, C. (2011). What does it mean for a student to understand the first-year calculus? Perspectives of 24 experts. *The Journal of Mathematical Behavior*, *30*(2), 131–148.

Trigueros, M., & Martinez-Planell, R. (2010). Geometrical Representations in the Learning of Two-Variable Functions.*Educational Studies In Mathematics*, *73*(1), 3-19.

Weber, E., & Thompson, P. W. (2014). Students' images of two-variable functions and their graphs. *Educational Studies in Mathematics, 87*(1), 67-85.

Yerushalmy, M. (1997). Designing representations: Reasoning about functions of two variables. *Journal for Research in Mathematics Education*, 431-466.

Investigating the effectiveness of an instructional video game for calculus: Mission prime

Authors
Affiliations

Introduction

Effective educational video games have been both created and researched in the K-12 literature (e.g., Riconscente, 2013), yet instructional games created for undergraduate mathematics and the relevant research in their effectiveness is scarce. As a team of researchers and game developers, we created an instructional video game, named Mission Prime, based on the concept of optimization in calculus. The game brings players into a visually complex and engaging world in which they perform tasks involving essential calculus content (in this case, optimization). We aimed to research the effectiveness of the game as compared to both homework and to no treatment and believe that, if given an environment that has visual-spatial manipulatives and no punishment for incorrectness, students could enhance their understanding of the mathematical content.

Background/Framework

Calculus and technology have been intertwined for the last 40 years to assist in students' understanding (Tall, 2010). Technology allows affordances not necessarily easily visualized or written, such as multiple representations (Porzio, 1999), dynamic visual interactions (Tall, 2009), and computational relief (Meel, 1998). Mathematics education research in computer-assisted learning has shown that students can be successful due to the variety of visual representation (Kidron & Zehavi, 2002; Thompson, Byerly, & Hatfield, 2013). Digital game-based learning, coupled with calculus concepts, can incorporate many of the affordances technology achieves.

Digital game-based learning and educational games provide a space were learners can engage in authentic, interactive, adaptive environments that help promote learning in a number of ways. Video games allow players to be situated into particular "roles" in which they can solve complex problems in a low-risk environment that provides ample feedback on both an on-demand and just-in-time basis (Gee, 2003; Squire, Barnett, Grant, & Higginbotham, 2004). In the gaming-ineducation literature, video games have been shown to be an effective means of teaching mathematics (Kebritchi, Hirumi, & Bai, 2010; Chen & Ren, 2013). Interactive engagement has been shown to improve performance in STEM courses over traditional methods (Hake, 1998).

While developing technological tools for mathematics content, the pedagogical aspect needs to be considered. According to Drijvers, Boon, and Van Reeuwijk (2010), there are "three main didactical functionalities for digital technology: (1) the tool function for doing mathematics, which refers to outsourcing work that could also be done by hand, (2) the function of learning environment for practicing skills, and (3) the function of learning environment for fostering the development of conceptual understanding" (as cited in Drijvers, 2012, p. 486). Although there has been a significant amount of research on computer-based algebra systems (CAS) in undergraduate mathematics education (Thomas & Holton, 2003), video games might also be a way to achieve these three major functionalities. In fact, there have been calls for educational video games to be created to assist the development of students' understanding of mathematics (Devlin, 2011); however, to our knowledge, no research has been done with video games designed specifically for undergraduate mathematics education.

Research Questions

Is playing *Mission Prime* more effective in promoting a) conceptual understanding, and b) calculation skills of optimization than in the traditional settings?

Development of *Mission Prime*

A cross-collaboration between the Department of Mathematics, the Digital Game-Based Learning group, and the Department of Communication at a large research university was established with funding from the Provost's office to explore whether instructional games can enhance student learning in introductory STEM courses. The mathematics group consisted of one graduate student, two RUME researchers, and one mathematician. The Digital Game-Based Learning group consisted of a team from the education research and development center on campus including the program director, producer, lead developer, quality assurance lead, art director, system software engineer, and an instructional designer. The communication group consisted of a post-doctoral student and a communication researcher. The team was presented with the opportunity to develop an instructional video game on a calculus topic. The guidelines of the funding required the topic to be one in which students traditionally struggle and to be self-contained enough to be completed in one hour of video game play. Therefore, the team designed a game on the concept of optimization in calculus.

The game, *Mission Prime*, involves a settler setting up various aspects of a space colony using limited resources and working within the parameters of the planetary environment. The game involves four sequential scenarios (entitled Fence, Box, Cone, and River) that can be solved using optimization. The player is given a tutorial that describes the game. The tutorial also guides players to visualization tools that can change the viewpoint from 3D to 2D, rotate the viewpoint, and change solution parameters so that the player can form conjectures about optimal solutions either before or during the problem-solving process. Once the tutorial is complete, the game begins. Throughout the game, both performance and formative feedback messages are given to help students keep track of their progress and adjust their strategies. A screenshot of the video game is located in Figure 1.

Mission Prime is designed to allow players to focus on learning the optimization concepts without having to stop to perform complex calculations. The player is given a menu of formulas and selects the relevant choices to put onto a workspace. Next, the player can insert constraint values into the formulas and can combine the formulas to arrive at a function of a single variable. The player can select from operations to perform, including the key choices "take the derivative" and "find the root". In attempting to maintain engagement in gameplay, the selected computations are done by the computer. When the player believes he/she has arrived at the solution, the player submits the answer. If the answer is correct, they move to the next scenario and if not, they are given feedback asking them to re-evaluate and try again. Feedback messages such as "This function is missing necessary information" or "The expression should contain only one variable" assist the player if their problem-solving strategy needs readjustment.

Figure 1: Screenshot of *Mission Prime*

Method

The study used between-subject experimental design with two treatments and a third control group. This means that different groups of students received the two different treatments while the control group received no treatment. The goal was to compare the effectiveness of the game against a more traditional mathematics learning activity such as doing practice problems. Participants were randomly assigned to one of the three condition groups to reduce the possible influence of extraneous factors. The *game* condition (G) involved playing the educational video game Mission Prime for sixty minutes or until all four scenarios have been completed. Students in the *practice* condition (P) were given a set of four optimization problems similar to those in the game (with diagrams provided as needed) and were allowed sixty minutes to complete the problems. Students in the *control* condition (C) had no treatment activity.

In the testing phase, 132 students from Calculus II courses at a large Midwestern university were recruited using small amounts of extra credit offered by four professors, regardless of the condition assigned to the students. Of these, seven students did not complete their session and were removed from the study. The test was performed during scheduled ninety minute sessions in a testing computer lab. Of the 125 students who completed the study, there were 50 assigned to G, 38 to P, and 37 to C. Through demographic data, 82.9% of the students were freshmen, 65.8% were male, and the average age was 19.39.

At the beginning of their session, participants were all given a preliminary survey about their background and demographic information, and also took an Attitudes Toward Mathematics Inventory (Tapia & Marsh, 2004). They were randomly assigned to one of the three conditions. After the activities, P and G groups took a post-test that included solving two optimization problems (the *computation problems*) and answering two questions about the concept of optimization and how it fits in with calculus (the *conceptual problems*) (See Table 1). C subjects took the post-test survey and then played the game so their experimental session was the same length as the P and G groups.

Table 1: The four problems given to students after activities

Conceptual Problems	Calculation Problems
1. What do you think are the important concepts in optimization?	2. Find the positive real numbers x and y that satisfy the equation 2x+4y = 24 and which have the largest possible product.
4. Explain how and why we use derivatives to solve optimization problems.	3. Find the dimensions of a box with a square base and open top that holds a volume of 36 cubic ft. and which requires the least amount of material to build the box. (An image of a cube with sides labeled was provided)

When solving the two computation problems, students showed their work on paper and entered the final answer into the computer. This enabled the coding team, comprised of the three mathematicians and a graduate student, to assess both correctness of the answer and to assign a code to the quality of the written work. The written work for computations and responses to the conceptual problems were coded using the rubric shown in Appendix 1. In both situations, the coding teams were blind to the condition for each response. An example of one student's written work, which was scored a 2, on problem 3, is located in Figure 2.

$V = (w^2)(h)$

$36 = (w^2)(h)$ $\qquad 36 = (\frac{1}{10})^2(h)$

$SA = 5w^2$ $\qquad 36 = w^8$

$= 10w$ $\qquad 0 = 2w^2$

$\frac{1}{10} = w$ $\qquad 18 = w^2$

Figure 2: An example of student work on problem 3

While the game group played the game, data was collected indicating their behaviors in the game including the frequency of different actions, the number of feedback messages they received, the duration of time on each action and scenario, whether they used the visual manipulation tools, and how long they spent manipulating the visual representations. The participants' perceived immersion, enjoyment, and sense of control were measured with a subset of the cognitive absorption scale survey developed by Agarwal and Karahanna (2000).

Results

Conceptual expression

Students' conceptual understanding was measured using the two concept questions on the post-test, employing the coding scheme described in the Methodology section. Students in the G group scored significantly higher on the two conceptual problems than students in the other two conditions (P, C). To obtain this result, an ANOVA was run with conceptual understanding as the dependent variable and condition as the independent variable.

The result showed that there was a significant difference between the three conditions in general, $F(2, 113)=5.22$, $p=.007$, eta squared = .08. The low value of p indicates that there is only a minute chance that these differences in sample group means are due to random chance rather than an actual difference in underlying group means. Because our groups were large and students were randomly assigned to the groups, we conclude that the differences are due to the treatments and the test is valid. Post-hoc comparison of means using Tukey's test showed that G (M=2.10, SD=1.26) was significantly higher than P (M=1.57, SD=1.04) and C (M=1.32, SD=1.01). P was not significantly different from C (p=.747) which means students in the P group did not do significantly better than those in the control group.

Computational skills

Students' computational abilities were measured using the two computational questions on the post-test. The calculation score was created using the correctness of their answers (which were entered by the students in the computer) and the coded assessment of their written work, as described in the Methods section. The findings indicate that playing an hour of the game may not be more effective in improving the students' calculation skills than doing practice problems or no treatment at all. An ANOVA was conducted in which the experimental condition was entered as the independent variable, and the calculation score was used as the dependent variable.

The result showed that there were no significant difference between the three conditions, $F(2, 113)=.20$, $p=.818$. The larger value of p (greater than .05) suggests that G (M=1.30, SD=1.57) did not perform significantly better than P (M=1.39, SD=1.28) or C (M=1.18, SD=1.17) in terms of calculation skills.

Discussion/Limitations/Future Research

The results suggest that playing the game *Mission Prime* can benefit students' conceptual understanding but not necessarily their computational skills. Since the game was designed to alleviate the need for student computation when posed with optimization scenarios, the results seem to agree with the design philosophy. The encouraging result is the significant gain in conceptual understanding. This study suggests that there is a potential for educational games to be useful at the undergraduate level. To our understanding, this is the first time that an educational game has been created for an undergraduate mathematics course, as well as studied, and we find that there are indicators suggesting its efficacy.

However, there were certain limitations to the study. This game was a short game on a single self-contained topic, so there is no information of sustained conceptual understanding. The students in the study were from Calculus II sections in the latter portion of the semester. Optimization is usually described in Calculus I, so it is possible that students did not recall either conceptual or computational understandings. Due to the randomized assignment of students to groups, however, this influence is expected to be the same across groups.

During the study, we were interested in student usage of the visualization tools of the game. We tracked how often students used the visualization scrolling feature and how much time they spent using that tool. Every student used the visualization tool at least once. The visualization tool was used heavily in the first scenario as students became familiar with the game. After the first scenario, use of the visualization tool increased with increasing difficulty of the scenario. Although visualization was an important aspect in game development, we felt we did not fully have the appropriate information to come to a conclusion about the effectiveness of this manipulative. Hence, visualization is an area of future research.

Due to an extension in funding, the calculus game will expand to cover more topics, including skills from algebra and pre-calculus. We conjecture that an instructional game would be most useful in the classroom when it is utilized often throughout the course. Instructional games allow students to do trial and error, respond to instant feedback, and to better visualize a concept or problem. A calculus course contains many topics (functions, limits, linear approximation, related rates) in which we predict an expanded version of *Mission Prime* could provide a valuable complement to traditional instruction and homework.

Future research may also involve students at other institutions playing the game, and examining the impact of the game on students who are concurrently learning the concept of optimization, in contrast to students who had learned optimization previously. We also plan to investigate whether use of *Mission Prime's* visualization tools increases students' conceptual understanding by performing a more qualitative study in the near future

Acknowledgments

The Mission Prime project was funded from the Provost's Office at our university. This project would not have been possible without the expertise of the game development team housed in the educational research and development center of our university.

Bibliography

Agarwal, R., & Karahanna, E. (2000). Time flies when you're having fun: Cognitive absorption and beliefs about information technology usage. *MIS Quarterly*, 665-694.

Chen, M.-P., & Ren, H.-Y. (2013). Designing an RPG game for learning mathematics conepts. *Second IIAI International conference on Advanced Applied Informatics*, 217-220.

Devlin, K. (2011). *Mathematics education for a new era: Video games as a medium for learning.* Natick, MA, USA: CRC Press.

Drijvers, P., Boon, P., & Van Reeuwijk, M. (2010). Algebra and technology. In P. Drijvers, *Secondary Algebra Education* (pp. 179-202). Rotterdam, The Netherlands: Sense.

Gee, J. P. (2003). *What video games have to teach us about learning and literacy.* New York, NY, USA: Palgraw Macmillan.

Hake, R. (1998). Interactive-engagement versus traditional methods: A six-thousand-student survey of mechanics test data for introductory physics courses. *American Jouranl of Physics*, 64-74.

Kebritchi, M., Hirumi, A., & Bai, H. (2010). The effects of modern mathematics computer games on mathematics achievement and class motivation. *Computers & Education*, 427-443.

Kidron, I., & Zehavi, N. (2002). The Role of Animation in Teaching the Limit Concept. *International Journal of Computer Algebra in Mathematics Education, 9* (3), 205-227.

Meel, D. (1998). Honors students' calculus understandings: Comparing calculus & Mathematica and traditional calculus students. *CBMS Issues in Mathematics Education, 7*, 163-215.

Porzio, D. (1999). Effects of differing emphases in the use of multiple representations and technology on students' understanding of calculus concepts. *Focus on Learning Problems in Mathematics, 21* (3), 1-29.

Riconscente, M. (2013). Results from a controlled study of the iPad fraction game Motion Math. *Games and Culture, 8* (4), 186-214.

Squire, K., Barnett, M., Grant, J. M., & Higginbotham, T. (2004). Electromagnetism supercharged!: Learning physics with digital simulation games. *Proceedings of the 6th international conference on Learning sciences* (pp. 513-520). International Society of the Learning Sciences.

Tall, D. (2010). A sensible approach to the calculus. *Paper presented at the The National and International Meeting on the Teaching of Calculus.* Pueblo, Mexico.

Tall, D. (2009). Dynamic mathematics and the blending of knowledge structures in the calculus. *ZDM, 41* (4), 481-492.

Tapia, M., & Marsh, G. E. (2004). An instrument to measure mathematics attitudes. *Academic Exchange Quarterly, 8* (2), 16-21.

Thomas, M., & Holton, D. (2003). Technology as a tool for teaching undergraduate mathematics. In A. Bishop, M. Clements, C. Keitel, J. Kilpatrick, & F. K. Leung, *Second International Handbook of Mathematics Education* (Vol. 1, pp. 347-390). Dordrecht: Kluwer.

Thompson, P. W., Byerly, C., & Hatfield, N. (2013). A Conceptual Approach to Calculus Made Possible by Technology. *Computers in the Schools, 30*, 124-147.

Appendix 1: Coding rubric for both conceptual and calculation student work

Conceptual Problems	Calculation Problems
Question 1: 0 – No relevance to problem solving or optimization 1 – Problem solving of any type without reference to calculus or max/min 2 – Derivatives 3 – Maximum/minimum, "best/efficient," and critical points 4 – Combining both derivatives and critical points or maximum and minimum	Question 2: 0 – Irrelevant or no answer 1 – Work that indicates minimal understanding (a picture or similar); trial and error; did not appear to use derivatives 2 – Tried to differentiate a multivariable expression; requires P=xy or similar 3 – Incorrect derivative (of a function of one variable), essentially correct solution with algebra errors 4 – Correct and included all the steps necessary: product equation, plug in constraint, differentiate, find critical points.
Question 4: 0 – irrelevant; discussed general techniques of optimization but not related to derivatives 1 – true statements about derivatives but irrelevant; apparent understanding of optimization but little or incorrect mention of derivatives; rates of change mentioned but applied incorrectly or incompletely 2 – "to find minimum and maximum"; "find critical points" but doesn't mention max/min 3 – "to find critical points to find the min/max"; listing steps that were successful in the game without any indication of why; missing what a critical point is or how to find one 4 – "zero derivative implies min/max"; "rate of change is 0 so the function is (maybe) max/min."; requires explicit mention of slope or rate of change	Question 3: 0 - Nothing relevant; only formula with no manipulation 1 – Volume formula with some manipulation (algebra or calculus) but no mention of surface area 2 – Both an area and a volume function present; some attempt at combing the equations; might have incorrectly tried a derivative; a differentiation attempt is not necessary for a two 3 – Derivative of an area function that is correct or nearly correct; missing the part where the derivative is set to 0 4 – correct method and correct answer; correct method but small mistake produced a wrong answer; this might include an incorrect surface area equation but steps are otherwise correct

A mathematics teacher educator's use of technology in a content course focused on covariational reasoning

Kevin LaForest
University of Georgia

The actions of mathematics teacher educators (MTEs) are an underrepresented area of mathematics education research. Likewise, researchers have argued that a pressing area of need in mathematics education is investigating how to support students' covariational reasoning. The purpose of this proposed study is to investigate an MTE's use of technology to engender covariational reasoning in a content course designed for pre-service teachers. Through the lens of Carlson et al.'s (2002) covariation framework and related theories, I will analyze interview and observation data in order to provide insights into the ways in which an MTE conceptualizes the use of technology in his classroom as well as how he plans to implement it to engender covariational reasoning. Additionally, I will focus on the way in which the MTE implements the technology including how students in the classroom reason about quantities that vary through the instructor's use of technology.

Key words: Covariational Reasoning, Technology, Mathematics Teacher Educators

Introduction

A growing body of literature (e.g., Carlson et al., 2002; Smith & Thompson, 2008; Ellis, 2011) identifies covariational reasoning – coordinating how two quantities vary in tandem – as critical for students' success in secondary and undergraduate mathematics. However, covariational reasoning and, more generally, reasoning about relationships between quantities do not receive significant attention in United States curricula (Saldanha & Thompson, 1998). The fact that covariational and quantitative reasoning are not central to student and teacher meanings is not entirely surprising, as researchers (e.g., Oehrtman, Carlson, & Thompson, 2008; Moore, 2010) have repeatedly and emphatically characterized US school mathematics as failing to develop meanings that stem from such reasoning, even in those students who are deemed successful. In light of the importance of covariational and quantitative reasoning for students' success in mathematics in combination with the fact that students are experiencing instructional experiences mostly devoid of such reasoning, an important area of research is identifying how to engender students' covariational and quantitative reasoning.

I argue that one way to engender and support students' covariational reasoning is through the use of technology. Over the past two decades, the proliferation of technology has been expansive and the use of technology in the classroom has been increasing as well (Kaput, 1992; Zbiek et. al, 2007). This societal change raises several questions about how emerging technology can be implemented to support student learning. The study proposed here investigates a mathematics teacher educator's (MTE) use of technology in a content course for pre-service teachers (PSTs). The primary research questions driving this study are: (a) according to an MTE, what role does technology play in the design of a PST content course? (b) How does the MTE implement technology in his classroom? (c) How does the use of technology in the content course support or inhibit PSTs' covariational reasoning abilities?

Technology and Covarying Quantities

In writing the introduction to the "Technology in the Mathematics Curriculum" section of the *Third International Handbook of Mathematics Education*, Leung (2013) stated:

> In many cases, the use of technology to study mathematics has changed the very nature of the mathematics we are studying. So technology in the mathematics curriculum should

> not be characterized by how the evolving technology will have an impact on the learning and teaching of mathematics from the curricula of previous eras. Rather, curriculum and teaching and learning methods will need to be regularly reconceptualized to take advantage of the power of modern technology to improve mathematics education in, possibly, spectacular ways. (p. 523)

In other words, technology provides us a way to transform the content that we teach and the ways of thinking that students construct. Through the use of innovative software, a teacher can bring to life the changing of two covarying quantities dynamically. More generally, using technology enables one to take advantage of media that support dynamic imagery in a user.

Kaput (1992) differentiates extensively between traditional and dynamic computer media and how they can affect cognition. Dynamic media offer a situation in which variation is easy to achieve. An example of this would be creating sketches in Geometer's Sketchpad [GSP] (Jackiw, 2006). Using GSP, a user can drag vertices and sides around nearly instantaneously to see a variety of possibilities. This is not possible in a drawing of the situation, as that medium requires the user to draw each instantiation. Dynamic media make it easier for students to "see" such concepts as variation because instead of being constrained to static media intended to represent variation, they can experience variation in experiential time. Repeated experiences with such activities can help students construct variation cognitively.

Thompson, Byerley, and Hatfield (2013) described that they used the software program Graphing Calculator (Avitzur, 2011) in their classroom and embedded it throughout all of their teaching. One example in which using the software program shaped their instruction involved implementation of a bottle problem: students were asked to imagine a graph of the volume of water in a bottle in relation to the water's height as it fills. The authors used Graphing Calculator to create a setting in which the students could control and experiment with the bottle's shape. Also, they could animate the applet to watch the bottle fill with water. A graphical representation of the relationship between height and volume can also be shown as the bottle fills. The authors' intention was to engender their students' covariational reasoning abilities by providing a dynamic environment where teachers can draw students' attention to how height and volume vary in tandem. The students can, in a sense, enact and perceive the variation themselves with the click of a button. These activities can help the students re-present the situation in the future without needing the technological support.

Theoretical Framework

In explaining the various ways students think about covarying quantities, Carlson et al. (2002) detailed a framework for describing the mental actions involved in applying covariational reasoning. Their framework consists of five distinct but related mental actions (denoted MA1, MA2, MA3, MA4, and MA5) that they consider characteristic of covariational reasoning. These mental actions are described fully in Figure 1. One must be careful in classification, however; just because one exhibits a higher numbered action (e.g., MA5) does not mean he or she has also engaged in a lower numbered action (e.g., MA3).

Researchers (e.g., Castillo-Garsow, Johnson, & Moore, 2013; Thompson, 2011) have recently extended the work on students' covariational reasoning by identifying differences in the way students imagine two quantities covarying. They found that students think about variation in both chunky and smooth ways. In chunky thinking, measuring change is a result of counting the occurrences of equal-sized chunks representing completed change. On the other hand, in order to conceptualize smooth variation one must imagine a change *in progress*; that is, variables continuously take on different values in the flow of time (Castillo-Garsow, Johnson, & Moore, 2013). Castillo-Garsow et al. argued that smooth thinking is a more powerful root for students' change than chunky thinking. One reason for this could be because smooth thinking corresponds with our experiential reality. However, this is often overlooked in school mathematics. For instance, common curricula approaches to function

Mental action	Description of mental action	Behaviors
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other	• Labeling the axes with verbal indications of coordinating the two variables (e.g., y changes with changes in x)
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	• Constructing an increasing straight line • Verbalizing an awareness of the direction of change of the output while considering changes in the input
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	• Plotting points/constructing secant lines • Verbalizing an awareness of the amount of change of the output while considering changes in the input
Mental Action 4 (MA4)	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.	• Constructing contiguous secant lines for the domain • Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function	• Constructing a smooth curve with clear indications of concavity changes • Verbalizing an awareness of the instantaneous changes in the rate of change for the entire domain of the function (direction of concavities and inflection points are correct)

Figure 1. Carlson et al.'s (2002, p. 357) covariation framework.

foreground chunky images of change, if they foreground variation at all. Castillo-Garsow (2012) finds that introduction of technology into classrooms can shift this trend. He writes,

> Computers are capable of displaying animations that occur within a student's experiential time, helping them to imagine continuous change … By performing extremely rapid calculations, computers can create the illusion of constantly changing numbers helping students build an instantaneous process conception of measurement. (p. 68)

These types of animations become objects that can create classroom conversations about the quantities involved, the relationships between quantities, and how quantities covary, while providing students an opportunity to perceive something changing smoothly. In addition to engendering students' reasoning through visual representations, Castillo-Garsow believes that continuous quantitative reasoning will be aided through the touch screen capabilities of many tablets. The ability to build the animation yourself or adjust quantities with your own finger is very powerful. It can give students the feel of "smoothness" rather than of discrete quantities.

Methods

The subjects of this study will be an MTE and two PSTs at a large university in the southeast United States. The MTE has been chosen on a volunteer basis and is the instructor of the first content course in a secondary mathematics education program. The two PSTs will be chosen on a volunteer basis and are students in that same course.

In order to model how the MTE conceptualizes the use of technology in his classroom to engender covariational reasoning, I will conduct semi-structured interviews (Roulston, 2010) with the MTE. To determine the ways in which the MTE implements technology, I will observe his actions in the classroom. Finally, to explore how the use of technology engenders and supports PSTs' covariational reasoning, I will conduct task-based interviews (Goldin, 2000) with the PSTs throughout the semester. All sessions will be videotaped and digitized.

Results

At the time of writing this proposal, data has not been collected or analyzed on the primary foci of this work. However, previous studies have involved data collection and analyses from the class described above to characterize students' quantitative and covariational reasoning (Moore, Paoletti, & Musgrave, 2013; Moore et al., 2013). We have often found that PSTs entering the program have difficulties conceptualizing covarying quantities. When providing PSTs with a drawing of a bottle and asking them to graph the relationship between the height of the water in the bottle and the volume of water in the bottle, several PSTs had difficulty imagining the bottle filling in ways that were productive to the task. No technology was used in the task implementation, but hypothetically the use of technology could have supported the PSTs' covariational reasoning as shown in Thompson, Byerley, and Hatfield (2013). The following is a classification of a hypothetical student going through the problem.

Using a subset of Carlson et al.'s (2002) framework, a hypothetical PST exhibiting thinking in terms of MA1 on this problem would be able to see by using the program that the height changes with volume. The technology helps because rather than just being told the two quantities covary, PSTs can watch the animated bottle fill with liquid and identify these as two changing quantities. A PST exhibiting MA2 would explain that the volume increases as height increases. Also, watching the bottle fill might support the students imagining these two quantities covarying continuously and smoothly. Additionally, the students' control over the change in height allows them to envision the change in volume in a way they would not have been able to do in a static environment. In order for a PST to exhibit MA3 on this problem, he or she would have to be able to distinguish the amount of change in volume there was for a change in the height. The technology affords the students an opportunity to virtually fill up the bottle and consider how the height or volume change, respectively. This, when combined with the smooth imagery supported at the MA2 level, might help students coordinate smooth and chunky images of change in ways compatible with what Thompson frames as continuous covariation (Thompson, 2011). One could argue that someone could bring in physical bottles and do the same thing, but, in general, you cannot easily adjust the shape of a physical bottle.

Implications of the Study and Questions for the Audience

The study, when completed, will have implications on the community of MTEs, a group of people that has been studied minimally compared to other populations in the mathematics education field. MTEs are responsible for preparing PSTs and thus determining how to provide them with experiences that focus on ways of thinking important to the teaching and learning of mathematics is a pressing area of need. The current study will contribute to this area by providing insights into technological supports that engender and grow students' covariational reasoning. The insights will not only produce research knowledge in this area, but also instructional materials relevant to teachers in secondary and elementary mathematics as well as MTEs. Another implication builds off of Castillo-Garsow's (2012) comments about the impact of technology on students' chunky and smooth images of variation. The work will provide insights into which technological tools support each image of variation and also perhaps more importantly, how these images may be coordinated as discussed by Thompson (2011). Similarly, the study will provide evidence for or against the claim that smooth thinking of covariation can grow through an MTE's use of dynamic technology.

Some questions for the audience to answer include: (a) What other data points, if any, do you feel would be interesting to investigate or would add to the credibility of the study? (b) What methods should I use to coordinate the different data collection foci? (c) What other implications stem from an investigation such as the one proposed here?

References

Avitzur, R. (2011). Graphing Calculator (Version 4.0). Berkeley, CA: Pacific Tech.

Carlson, M. (1998). A cross-sectional investigation of the development of the function concept. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *Research in collegiate mathematics education, III. Issues in Mathematics Education,* 7, 115-162.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-378.

Castillo-Garsow, C.C. (2012). Continuous quantitative reasoning. In Mayes, R., Bonillia, R., Hatfield, L. L., and Belbase, S. (Eds.), *Quantitative reasoning and Mathematical modeling: A driver for STEM Integrated Education and Teaching in Context. WISDOMe Monographs, Volume 2* (pp. 55-73). Laramie, WY: University of Wyoming Press.

Castillo-Garsow, C., Johnson, H. L., & Moore, K. C. (2013). Chunky and smooth images of change. *For the Learning of Mathematics*, 33(3), 31–37.

Ellis, A. (2011). Middle school algebra from a functional perspective: A conceptual analysis of quadratic functions. In Wiest, L. R., & Lamberg, T. (Eds.). *Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Reno, NV: University of Nevada, Reno.

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 517–545). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Jackiw, N. (2006). The Geometer's Sketchpad. Berkley, CA: Key Curriculum Press.

Kaput, J. (1992). Technology and mathematics education. In D. A. Grouws (ed.), *Handbook of research on mathematics teaching and learning* (pp. 515-556). New York: Macmillan.

Leung, F. (2013). Introduction to Section C: Technology in the mathematics curriculum. In *Third International Handbook of Mathematics Education* (pp. 517-524). Springer New York.

Moore, K. C. (2010). The role of quantitative and covariational reasoning in developing precalculus students' images of central concepts of trigonometry. Ph.D. Dissertation. Arizona State University: USA.

Moore, K. C., Liss II, D. R., Silverman, J., Paoletti, T., LaForest, K. R., & Musgrave, S. (2013). Pre-service teachers' meanings and non-canonical graphs. In Martinez, M. & Castro Superfine, A. (Eds.), Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 441-448). Chicago, IL: University of Illinois at Chicago.

Moore, K. C., Paoletti, T., & Musgrave, S. (2013). Covariational reasoning and invariance among coordinate systems. *The Journal of Mathematical Behavior*, 32(3), 461-473.

Oehrtman, M., Carlson, M., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. Rasmussen (Eds.), *Making the Connection: Research and Teaching in Undergraduate Mathematics Education* (pp. 27-42). Washington, D.C.: MAA.

Roulston, K. (2010). *Reflective interviewing: A guide to theory and practice*. London: Sage.

Saldanha, L., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensen, K. R. Dawkins, M. Blanton, W. N. Coulombe, J. Kolb, K. Norwood, & L. Stiff (Eds.), *Proceedings of the 20th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 298–303). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Smith III, J., & Thompson, P. W. (2008). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, Carraher D.W. & Blanton M.L. (Eds.), *Algebra in the early grades* (pp. 95-132). New York, NY: Lawrence Erlbaum Associates.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education. WISDOMe Mongraphs* (Vol. 1, pp. 33-57). Laramie, WY: University of Wyoming.

Thompson, P. W., Byerley, C., & Hatfield, N. (2013). A conceptual approach to calculus made possible by technology. *Computers in the Schools, 30*, 124-147.

Zbiek, R. M., Heid, M. K., Blume, G., & Dick, T. P. (2007). Research on technology in mathematics education: The perspective of constructs. In F. K. Lester (ed.) *Second handbook of research on mathematics teaching and learning* (pp. 1169-1207). Charlotte, NC: Information Age Publishing.

The influence of function and variable on students' understanding of calculus optimization problems

Renee LaRue
West Virginia University

Nicole Engelke
West Virginia University

For this study, we aim to answer the following questions: 1) What conceptual knowledge do students need to have to be able to correctly solve optimization problems? 2) What weaknesses do students demonstrate when solving optimization problems? 3) How can we address these weaknesses and improve the teaching of optimization problems in calculus? In this paper, we discuss preliminary findings from this study, focusing on the responses of four first semester calculus students as they solve a basic optimization problem during a semi-structured interview. In particular, we observe how students' understanding of function and variable influences their understanding of optimization problems. We believe we may be able to use APOS theory (Asiala et al, 1996) as a lens for studying how students understand optimization problems and begin to explore that in this paper.

Key Words: Calculus, Optimization, Function, Variable

Introduction

The goal of our study is to better understand how students think about and understand optimization problems in calculus. In particular, what conceptual knowledge do students need to have to be able to correctly solve optimization problems? What weaknesses do students demonstrate when solving optimization problems? Additionally, how can we address these weaknesses and improve the teaching of optimization problems in calculus? This paper will begin to answer these questions.

To solve an optimization problem, a student must use geometry and algebra skills to construct a function that represents the situation described in the problem. Often, there is more than one functional relationship involved and the student must make strategic decisions to find an appropriate single-variable function. Once the student has defined such a function, he or she will use calculus and algebra to find the absolute maximum(s) and/or minimum(s) of the function. Overall, the four students who took part in this study were very good at the algebraic manipulations, but all had trouble constructing the initial function. This is the focus of our paper.

Background and Literature

Functions and variables play a critical role in optimization problems, since the goal is to construct a function that models the given situation and then find the absolute maximum or minimum of that function. There is a large body of literature devoted to researching student understanding of functions (Breidenbach, Dubinsky, Hawks & Nichols, 1992; Monk, 1992; Carlson, 1998; Dubinsky & Harel, 1992; Vinner & Dreyfus, 1989), but the majority of this research is limited to situations in which the students have a function presented to them. Optimization problems provide interesting insight into students' understandings of functions and variables because the students must construct the functions themselves, often using function composition and variables other than the standard choices of x and y.

The APOS theory defined by Asiala et al (1996) is useful for analyzing students' understanding of function, a key aspect of optimization problems. We believe APOS theory may also provide a good lens for studying students' understanding of optimization problems. Asiala et al (1996) define an *action* view of a transformation as one in which a student "can carry out the transformation only by reacting to external cues that give precise details on what steps to take" (1996, p. 9). Asiala et al (1996) define a *process* view as one in which students "can reflect on, describe, or even reverse the steps of the transformation without actually performing those steps" (p. 10). They further note, "a process is perceived by the individual as being internal, and under one's control, rather than as something one does in response to external cues" (1996, p. 10).

Research suggests that understanding the relationships between variables is difficult for many students (Trigeuros & Ursini, 2003; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; White & Mitchelmore, 1996). Symbolizing a functional relationship based on information given in a problem – something that is clearly necessary for optimization problems – is often a source of difficulty for students (Trigueros & Ursini, 2003; White & Mitchelmore, 1996). Understanding the covariational nature of the variables in an equation is challenging for many students (Carlson et al, 2002; Trigueros & Ursini, 2003). Trigueros and Urisini (2003) noted that students tend to have more difficulties when working with more than one variable in the same problem.

Methods

We conducted interviews with seven first and second semester calculus students. The interviews were semi-structured interviews in which students solved optimization problems and answered questions about prerequisite material related to optimization problems. We asked the students to think aloud as much as possible as they solved the problems, and asked follow-up questions if more detail was necessary. We also asked probing questions such as, "Why did you decide to do this?" and "Why are you allowed to do that?" These interviews were video recorded and then transcribed for analysis. This paper will focus on the four first semester calculus students' responses to the optimization problem given in Figure 1. Information that is key to the problem is also stated in Figure 1. All names (Adam, Rob, Jill, Joe) are pseudonyms.

Problem 1: *A rectangular garden of area 200 ft^2 is to be fenced off against rabbits. Find the dimensions that will require the least amount of fencing if a barn already protects one side of the garden.*

Key information:

Total amount of fencing for the garden: $P = 2x + y$

$$A = 200\,ft^2 \Rightarrow xy = 200 \Rightarrow y = \frac{200}{x}$$

$P(x) = 2x + \dfrac{200}{x}$ is a single-variable function representing the perimeter in terms of x

Figure 1: Optimization problem used in interview along with its key information

Preliminary Results and Analysis

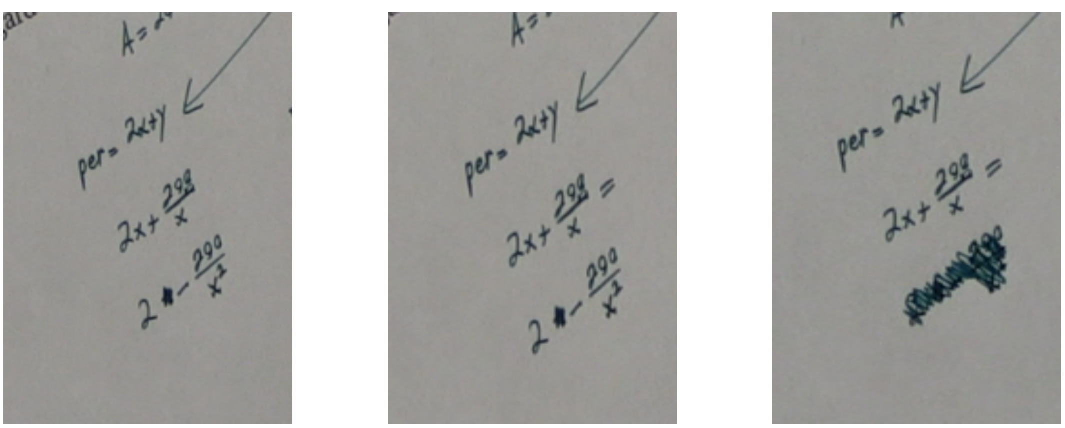
The Equal Sign Dilemma

Three of the four students had difficulty writing complete equations with information on both sides of the equal signs. For two of these students, this presented enough of a stumbling block that they did not know how to proceed.

When attempting to set up his function, Adam wrote, "$2x + y =$ " with nothing written to the right of the equal sign. When asked about it, he said, "I don't know what it's equal to now. I think, uh..." and then he moved on to the next part of the problem. He was still able to complete the problem, but said later that he did not understand why he was doing certain steps. He said, "I know that's how I did it, so it will be that way," and "they've taught us that's how it should be."

Jill wrote "$\text{minimize materials} = 2y + x$" which seemed like a good first step, but then when it came time to find a function representing perimeter, she did not know what to do. She said, "I mean I have another formula involving *y* here, but I don't know what it equals," pointing at the above statement. Even after the interviewer suggested she use the letter *M* to represent "amount of material," she remained hesitant, saying, "Even then I don't think that'd help me. That's just adding another variable." At this point, she did not know what to do next, so the researcher pushed her harder to try using the variable *M*. As soon as she wrote $M = \frac{400}{x} + x$, something changed and she was able to do the rest of the problem.

Joe had similar problems when trying to come up with a function (see Figure 2). He began by writing $\text{per} = 2x + y$ and then wrote $2x + \frac{200}{x}$ without including an equal sign. He knew he needed to differentiate something associated with perimeter, so he wrote $2 - \frac{200}{x^2}$. Reflecting back on what he had written, he said, "that's the equation for perimeter in terms of x" and then seemed to hesitate. When asked about his hesitation, he said $2x + \frac{200}{x}$ was "not an equation like that" and then scratched out the differentiated expression below it. At this point, he did not know how to proceed, and the researcher had to intervene to get him to continue.



No equal sign | Equal Sign | Marks out "derivative"

Figure 2: Joe's initial attempts at constructing a function for perimeter

The Function Notation Dilemma

Three of the four students also had difficulty using appropriate function notation when constructing their functions. Rob wrote $P(ft)$ for his perimeter function. When asked what $P(2) = 204$ meant, he said it meant the length of the rectangle was 204, even though the function was giving the perimeter of the rectangle in terms of its length (meaning the rectangle had a perimeter of 204 when its length was 2).

Joe also had difficulty with the function notation. He was still trying to figure out what to write on the other side of the equation containing the expression for perimeter, and the researcher suggested using the letter P. As a result, he started to write $2x + \frac{200}{x} = f(P)$, but then realized that wouldn't work because "that's not the variables in there." Then he said that maybe it would just be "f of something," but wasn't able to figure out what that something would be.

Adam also had trouble deciding how to express the perimeter as a function. He began to write $2x + y = f(x)$, but then claimed that this would not work because $f(x)$ equals just y, not $2x + y$. When asked to explain, he said, "We know $f(x) = y$. That's a common thing."

Discussion

Once the students overcame the obstacles related to constructing the optimizing function, all four were able to solve the rest of the problem easily. Constructing the function, however, proved to be very difficult for them. After they had completed the problem, it was clear that they still did not know why they had performed certain operations, emphasizing that a correct answer does not imply correct or complete understanding of the process. Both Rob and Adam stated they believed they had done the problem correctly, but did not understand everything they had done.

We believe we can use APOS to analyze students' understanding of optimization problems. A student might have learned specific steps to follow to solve the problem, but may not fully understand how those steps are related. We refer to this as an *action* view. A student with a *process* view is able to solve the problem using an intuitive understanding of the problem solving process, rather than relying on a series of memorized steps. Suppose the next step in the problem is to take the derivative of the function. A student with an action view might think, "after I come up with a function, the next step is to take the derivative," while a student with a process view might think, "I came up with this function from the information in the problem, and if I want to find the absolute maximum or minimum of it, I can differentiate this function and use that to find the $x-$value where we have a horizontal tangent line." The student with the process view has a broader, deeper understanding of the problem.

Future Research and Implications for Teaching

In the next phase of data collection, we plan to conduct another series of interviews with first semester calculus students, and this time we will also observe the class instruction on the days students learn optimization. The above students all had action views of optimization problems. We hypothesize that the examples presented in class are leading them to develop this action view, and we would like to see what, if anything, is being done to move them from an action to a process view. Instruction needs to focus on the "whys" of optimization, not just the "hows." Taking time to have the students wrestle with the function construction process is worthwhile.

Questions

- How could we connect the object and schema part of the APOS theory to optimization problems, or is there another theoretical perspective that might work better?
- When comparing classroom instruction to students' solutions to optimization problems, what should we look for?

References

Asiala, M., Brown, A., Devries, D. J., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A Framework for Research and Curriculum Development in Undergraduate Mathematics Education. *CBMS Issues in Mathematics Education, 6*, 1-30.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the Process Conception of Function. *Educational Studies in Mathematics, 23*(3), 247-285.

Carlson, M. P. (1998). A Cross-Sectional Investigation of the Development of the Function Concept. *CBMS Issues in Mathematics Education, 7*, 114-162.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying Covariational Reasoning While Modeling Dynamic Events: A Framework and a Study. *Journal for Research in Mathematics Education, 33*(5), 352-378.

Dubinsky, E., & Harel, G. (1992). The Nature of the Process Conception of Function. *The Concept of Function, Aspects of Epistemology and Pedagogy, MAA Notes, 25*, 85-106.

Monk, S. (1992). Students' Understanding of a Function Given by a Physical Model. *The Concept of function: Aspects of Epistemology and Pedagogy, MAA Notes, 25*, 175-194.

Trigueros, M., & Ursini, S. (2003). First-year Undergraduates' Difficulties in Working with Different Uses of Variable. In A. Selden, E. Dubinsky, G. Harel & F. Hitt (Eds.), *Research in Collegiate Mathematics Education* (Vol. 12, pp. 1-29). Providence, RI: American Mathematical Society.

Vinner, S., & Dreyfus, T. (1989). Images and Definitions for the Concept of Function. *Journal for Research in Mathematics Education, 20*(4), 356-366.

White, P., & Mitchelmore, M. (1996). Conceptual Knowledge in Introductory Calculus. *Journal for Research in Mathematics Education, 27*(1), 79-95.

THE TEXTBOOK, THE TEACHER, AND THE DERIVATIVE: EXAMINING COLLEGE INSTRUCTORS' USE OF THEIR TEXTBOOK AND DESCRIPTIONS OF DERIVATIVES IN A FIRST SEMESTER CALCULUS CLASS

Initial qualitative work with five community college calculus instructors indicated that these teachers not only modified their textbook when introducing the concept of the derivative, but also formally and informally evaluated their text. In addition, during classroom observations and interviews, these teachers did not distinguish clearly between the idea of the derivative as an object or an operator. This preliminary report details these findings and proposes questions for further research.

Keywords: Calculus Teachers, Textbook Use, Derivative

Of the 3,858,000 college students taking mathematics in 2010, 11% of them were enrolled in a first semester calculus course (Blair, Kirkman, & Maxwell, 2013). It is the culmination of mathematics courses for some students and the beginning of more intensive math courses for others. Calculus is often positioned as a gatekeeper course for students wishing to enter STEM fields (Treisman, 1992; Blair, Kirkman, & Maxwell, 2013). In addition, there is a body of research on the teaching and learning of calculus (e.g. Ferrini-Mundy & Graham, 1994; Hallet, 2006; Siyepu, 2013; Bressoud, Carlson, Mesa & Rasmussen, 2013).

Within calculus, the primary topics taught include limits, derivatives and integrals (Sofronas, et al, 2011). Of these three, derivatives are introduced early and used throughout a typical calculus sequence. Knowledge of derivatives is necessary to understand anti-derivatives and to evaluate integrals using anti-derivatives. They are also used in defining Taylor series, which are in turn used to approximate functions.

This study contributes to knowledge about post-secondary mathematics instructors' use of their textbooks by asking the following research questions:

1) How do the textbook and online supplemental materials support the teaching of derivatives in first semester calculus?
2) How do faculty use these materials as a resource for the planning and teaching of derivatives in first semester calculus?
3) How do teachers describe a derivative for themselves and for their students?

Analytical Frameworks

There is limited information about how post-secondary mathematics teachers use their textbooks during planning and teaching (Stark, 2000; Hora and Ferrare, 2012), and some information about both calculus textbooks (Mesa, 2010) and calculus teaching (Kaput 1997; Hsu, Murphy & Treisman 2008, Natarajan & Bennett, 2014), but very little is known specifically on calculus teachers' uses of textbooks.

Brown (2009) describes the interaction between teachers and the curriculum. He also states that teachers' interactions with curricular materials can be understood in terms of three different degrees of artifact appropriation: Offloading, Adapting and Improvising. See Table 1 for a description and example of these codes.

Brown’s framework for teachers’ interaction with curriculum materials provided a framework that allowed me to examine the role of the textbook in both planning and in execution, around a single topic or for a semester.

I identified two possible frameworks (Zandieh, 2000; Park, 2013) for analyzing how teachers describe a derivative. Both frameworks enabled an examination of teachers’ conceptions and the textbook’s introduction of derivatives. Michelle Zandieh (2000) developed a theoretical framework to analyze how students understood the concept of derivative. As shown in Figure 1, she considers the conception of the derivative in two dimensions: the process-object layer and the contexts for the derivative.

	Contexts				
Process-object Layer	Graphical	Verbal	Paradigmatic Physical	Symbolic	Other
	Slope	Rate	Velocity	Difference Quotient	
Ratio					
Limit					
Function					

Figure 1: Framework for the concept of derivative. From Zandieh (2000) p. 106.

The process-object layer refers to the concept of a derivative, based on the formal definition, $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$, which can be seen as a ratio, a limit, and a function. Zandieh draws on work from Sfard (1992), to describe each of these layers as both an object—something to be acted on, and a process - something to do. For example, a ratio can be considered a fraction (object) or the *process* of division of the denominator into the numerator. The second dimension refers to how the derivative can be described in various contexts (e.g., slope, rate of change, velocity,…).

The second relevant framework for thinking about the conception of a derivative comes from work done by J. Park (2013). She describes four developmental stages of thinking about the derivative as a function: a point-specific value (Stage 1), a collection of values at multiple points (Stage 2), a function (Stage 3), and an operator (Stage 4). This framework differs significantly from Zandieh’s, in that Park focuses on the differences between conceptions of the derivative as an object and a process. The object (stages 1-3) and process (stage 4) are not seen as a pair, but as separate developmental stages in the understanding of the derivative. Park emphasizes the difference in understanding between the derivative at a point and the derivative as a function, something mentioned in other literature (Bezuidenhout, 1998), but not delved into by Zandieh. According to Park, understanding the derivative as a function (and not just an object) is important and nontrivial for students. From these frameworks, I focused on the idea of derivative as limit, slope, rate or function, and then as a process or object.

Methods

Five out of six teachers of first semester calculus at a Midwestern community college participated in this study. I scheduled two observations for every teacher except one, (Charles, who taught on-line) when derivatives were being taught. I collected all handouts and quizzes used in the lesson. Four of the five participants required the same textbook. One teacher, Duncan, stated that he did not require any calculus textbook. I interviewed each teacher immediately following my observations and then formally outside of class.

The same textbook (Larson & Edwards, 2011) was recommended by the college for all participants. All but Duncan required that students have a copy of the textbook, either physically or electronically. I compared the textbook with what teachers said they did, and I analyzed the textbooks introduction to derivatives.

Analysis and Findings

The analysis of the data included coding field notes and transcripts of interviews along two dimensions, first for use of textbook and second for understanding of derivatives. I used a combination of top-down and then bottom-up coding, after I realized that the analytical frameworks from the literature were not capturing all relevant information. An example of coding for textbook use is shown in Table 1.

Code descriptions for textbook use

Code	Description	Example
Offloading	using the materials as written, without change (offload some of the work of planning and teaching onto the textbook)	"Which, I mean, which is one of the wonderful things about a textbook is—like I said before—is that bank of problems. I rely on those a lot." (Charles, 610 - 611)
Adapting	using the materials, but with some modification	"I look at the examples in the textbook, but the main reason I look at them is I don't want to use them… Because I just, I like students to have that as an alternative" (Bruce, 259-263). In this case, Bruce is describing how his examples are deliberately different from those in the textbook. Because his choice of examples depends on the examples in the textbook, it is coded as adapting.
Improvising	does not use the materials	"the textbook gives a lot of practical applications, but there are some practical applications [like depreciation] that I just have to tell them about in the classroom." (Arthur, 619 – 621)

Table 1: Initial Textbook Use Codes

In addition to these three codes, there were multiple references to the textbook that did not fit this framework. Initially I categorized uses that did not fit these three categories as other. Table 2 shows the code frequencies for each teacher.

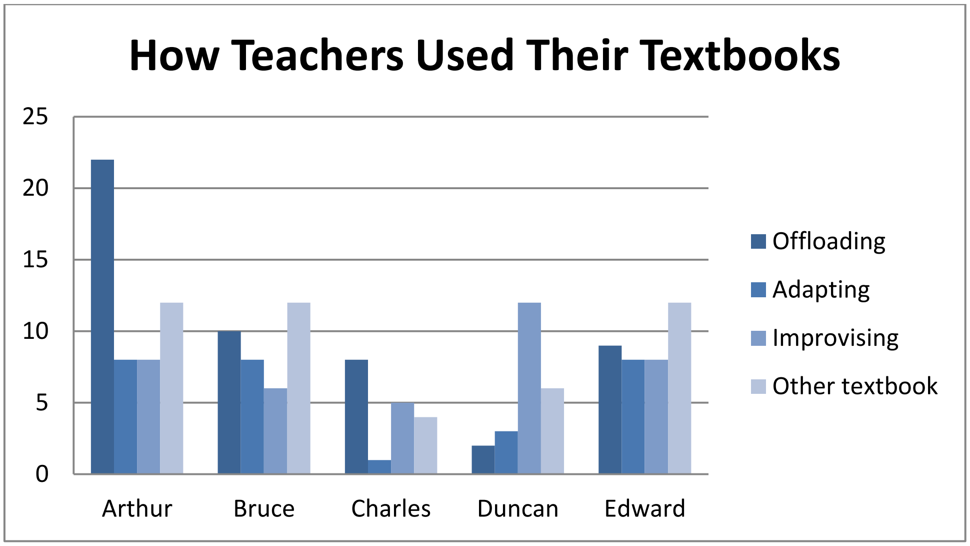


Table 2: Frequency of textbook codes by teacher

The data align well with how teachers talked about their relationship with their textbook. For example, Duncan said he did not use the textbook, and his highest code is "improvising", with very few "offloading" codes. On the other hand, Arthur said that he really liked the textbook. Accordingly, his data has a spike in offloading, where he uses the text as written. Charles teaches an on-line class, so there were no observations to code, and fewer codes overall. There were two main types of comments that were coded as "other textbook". One group of comments referred to how teachers evaluated the textbook. The second group included speculations about how students do or should use their textbook.

Derivative Conceptions

I initially coded transcripts and field notes for mentions of the derivative as a limit, as a rate of change, or as the slope of a tangent line (adapted from Park, 2013 and Zandieh, 2000). I found that these codes were insufficient. I added a code called "other derivative" to capture mentions of the derivative that did not fit this framework. The analysis of the observation field notes on 'other derivative' revealed two conceptions of derivative—the derivative as an object that can be acted on, and the derivative as a process (Zandieh, 2000).

Teachers often see 'derivative' as a process (e.g., differentiation) as well as an object. Arthur, Bruce, Charles, and Duncan say they expect students to give an example of differentiation (a process) when they are asked to define derivative (an object). During Edward's first observation, he described the derivative as "nothing more than a difference of functions and a limit", implying an object. Later in the same lesson he said: "the derivative is not distributive," suggesting that the derivative is a process (field notes, Edward, observation 1). Similarly, during Bruce's first observation, he described $e^y = x$ as "one huge derivative" (object) and then immediately instructs students to "take the derivative of both sides" (process) (field notes, Bruce, observation 1).

Questions:

1) After analyzing and interpreting the data, I propose extending Brown's (2009) framework by adding two categories, evaluating and referencing. The initial three categories of offloading, adapting and improvising remain as defined by Brown. Evaluating would include instances where teachers discuss why or how they choose a textbook. It would also include how teachers

evaluate a textbook both in terms of the process they use as well as the opinions they offer. The final category, referencing, may include such things as deferring to the textbook for mathematical notation and referencing the textbook as a way to indicate upcoming topics. The problem is that referencing is very similar to offloading. What are some arguments for and against having referencing as a category by itself?

2) Given that the concept of derivative is challenging, and the word "derivative" can be used for both a process and an object, what are some implications for textbook writing and teaching? How can we, as teachers of calculus and teachers of future calculus teachers, help our students make this distinction?

3) How is the process/object distinction of derivative similar to - and different from - the distinction between the derivative at a point (usually a numeric value) and the derivative on an interval (a function)?

BIBLIOGRAPHY

Blair, R., Kirkman, E., & Maxwell, J. (2013). *Statistical abstract of undergraduate programs in the mathematical sciences in the United States: Fall 2010 CBMS Survey*. Washington, DC: American Mathematical Society.

Bezuidenhout, J. (1998). First-year university students' understanding of rate of change. *International Journal of Mathematical Education in Science and Technology*, *29*(3), 389-399.

Bressoud, D. M., Carlson, M. P., Mesa, V., & Rasmussen, C. (2013). The calculus student: insights from the Mathematical Association of America national study. *International Journal of Mathematical Education in Science and Technology*, *44*(5), 685-698.

Brown, M. W. (2009). The teacher – tool relationship: Theorizing the design and use of curriculum materials. In J. T. Remillard, B. Herbel-Eisenmann, & G. Lloyd (Eds.), Mathematics Teachers at Work (pp. 17 – 36). New York: Routledge.

Ferrini-Mundy, J., & Graham, K. (1994). Research in calculus learning: Understanding of limits, derivatives, and integrals. *MAA Notes*, 31-46.

Hora, M., & Ferrare, J. (2012). Instructional systems of practice: A multidimensional analysis of math and science undergraduate course planning and classroom teaching. *Journal of Learning Sciences*

Hsu, E., Murphy, T. J., & Treisman, U. (2008). Supporting high achievement in introductory mathematics courses: what we have learned from 30 years of the Emerging Scholars Program. *Making the Connection: Research and Teaching in Undergraduate Mathematics Education, MAA Notes*, *73*, 205-220.

Kaput, J. J. (1997). Rethinking calculus: Learning and thinking. *American Mathematical Monthly*, 731-737.

Larson, R., & Edwards, B. (2011). *Calculus of a Single Variable: Early Transcendental Functions*. Mason, OH. Cengage Learning.

Mesa, V. (2010). Strategies for controlling the work in mathematics textbooks for introductory calculus. *Research in Collegiate Mathematics Education*, *16*, 235-265.

Mesa, V., & Griffiths, B. (2012). Textbook mediation of teaching: an example from tertiary mathematics instructors. *Educational Studies in Mathematics, 1*(79), 85-107.

Natarajan, R., & Bennett, A. (2014). Improving Student Learning of Calculus Topics via Modified Just-in-Time Teaching Methods. *PRIMUS*, *24*(2), 149-159.

Park, J. & Flores, A. (2012). Transition from derivative at a point to derivative as a function. Proceedings of the 2012 PME-NA meeting, Kalamazoo, MI.

Park, J. (2013). Is the derivative a function? If so, how do students talk about it?. *International Journal of Mathematical Education in Science and Technology*, *44*(5), 624-640.

Siyepu, S. W. (2013). An exploration of students' errors in derivatives in a university of technology. *The Journal of Mathematical Behavior*, *32*(3), 577-592.

Sofronas, K. S., DeFranco, T. C., Vinsonhaler, C., Gorgievski, N., Schroeder, L., & Hamelin, C. (2011). What does it mean for a student to understand the first-year calculus? Perspectives of 24 experts. *The Journal of Mathematical Behavior*, 30(2), 131-148.

Stark, J. (2000). Planning introductory college courses: Content, context and form. *Instructional Science,* 28, 413-438

Treisman, U. (1992). Studying students studying calculus: A look at the lives of minority mathematics students in college. *College Mathematics Journal*, 362-372.

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. *CBMS Issues in Mathematics Education*, *8*, 103-122.

Prospective Secondary Mathematics Teachers' (PSMTs') Understanding of Abstract Mathematical Notions

Younhee Lee
The Pennsylvania State University

Kilpatrick (1987) noted, "successful teaching, like successful communication, depends on having a good model of the other" (p. 17). Successful teaching of collegiate mathematics for PSMTs will necessitate understanding of what knowledge PSMTs may bring with them to the learning of collegiate mathematics and through what processes PSMTs construct their knowledge. Thus, in-depth analysis of PSMTs' mathematics is of significant importance to our field for providing PSMTs with meaningful learning experiences in mathematics content courses. In this study, I intend to investigate how PSMTs construct their own knowledge of abstract mathematical notions (e.g., polynomial rings, irreducible polynomial, factorization, minimal polynomials) and, second, the difficulties that they encounter while constructing their knowledge of abstract mathematical notions.

Key words: Teachers' knowledge of collegiate mathematics, horizon knowledge of mathematics, abstract mathematical notions, polynomial rings

Rationale and Perspectives

An effective teacher education program ought to facilitate teachers' ability to transform and connect advanced mathematical knowledge to teaching practices (Conference Board of the Mathematical Sciences, 2012). However, some research studies (Begle, 1979; Monk, 1994; Zazkis & Leikin, 2010) suggest that PSMTs' mathematical knowledge as constructed in their mathematics content courses may not be related to or useful for their future teaching practices. I view this problem is not because college mathematics is inherently disconnected from school mathematics but rather because the ways in which PSMTs construct their advanced mathematical knowledge may not necessarily broaden their *horizon knowledge of mathematics* – "an awareness of the large mathematical landscape in which the present experience and instruction is situated" (Ball & Bass, 2009). However, as Ball and Bass pointed out, "we do not know how horizon knowledge can be helpfully acquired and developed." I believe this study can be an essential step toward addressing this gap in the field by investigating how PSMTs develop their horizon knowledge (specifically, in terms of polynomial rings).

Plan for Methodology

This research will employ 'teaching interview' (Hershkowitz, Schwarz & Dreyfus, 2001) as a way to understand how six individual PSMTs grow in their knowledge of abstract mathematical notions. Hershkowitz et al. define teaching interview as an interview in which the interviewer asks questions with didactic purposes: "(a) to cause [student] to explain what she was doing and why and (b) to induce her to reflect on what she was doing and thus possibly progress beyond the point she would have reached without the interviewer" (p. 204). Design of this study will be governed by a theoretical and methodological framework of 'Abstraction in Context' (Dreyfus, Hershkowitz & Schwarz, 2015), the main premise of which is that the process of abstraction involves three observable epistemic actions 'recognizing, building-with, and constructing.' Guiding the emergence of new constructs based on the context with which PSMTs might be familiar will be a central approach to designing the tasks for the teaching interviews (See Appendix for example).

References

Ball, D. L. & Bass, H. (2009) *With an eye on the mathematical horizon: knowing mathematics for teaching to learners' mathematical futures*. Paper presented at the 43rd Jahrestagung der Gesellschaft für Didaktik der Mathematik, Oldenburg, Germany. Retrieved 16 Oct 2014 from http://www.fachportal-paedagogik.de/fis_bildung/suche/fis_set.html?FId=889839

Begle, E. G. (1979) *Critical variables in mathematics education: Findings from a survey of the empirical literature*. Washington, DC: Mathematical Association of America.

Conference Board of the Mathematical Sciences [CBMS] (2012). *The Mathematical Education of Teachers II*. Providence, RI: American Mathematical Society and Mathematical Association of America.

Dreyfus, T., Hershkowitz, R., & Schwarz, B. B. (2015). The nested epistemic actions model for abstraction in context – Theory as a methodological tool and methodological tool as theory. In A. Bikner-Ahsbahs, C. Knipping, & N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education: Examples of methodology and methods* (Advances in mathematics education series). New York: Springer (in press).

Hershkowitz, R., Schwarz, B. B., & Dreyfus, T. (2001). Abstraction in context: epistemic actions. *Journal for Research in Mathematics Education, 32*, 195–222.

Kilpatrick, J. (1987). What constructivism might be in mathematics education. In *Proceedings of the Eleventh Annual Conference of the International Group for the Psychology of Mathematics Education, Vol. 1*, pp. 3-27. Montreal.

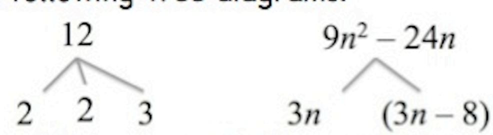
Monk, D. H. (1994) Subject area preparation of secondary mathematics and science teachers and student achievement. *Economics of Education Review, 13*, 125–145.

Zazkis, R., & Leikin, R. (2010). Advanced mathematical knowledge in teaching practice: Perceptions of secondary mathematics teachers. *Mathematical Thinking and Learning, 12*, 263-281.

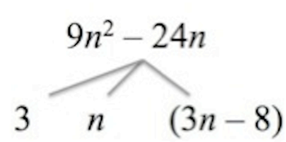
Appendix: Example of Interview Questions

Question 4-4. How do you think Ms. Charlotte in the following situation should address student's question?

During a lesson on quadratic functions and equations (in Algebra 2 class), Ms. Charlotte was using the analogy between prime factorization of integers and factorization of quadratic polynomials as shown in the following tree diagrams.



One student asked, "Can 3n be further divided into 3 and n?" as follows:



Unconventional use of mathematical language in undergraduate students' proof writing

Kristen Lew Juan Pablo Mejia-Ramos
Rutgers University

Introduction

There is a dearth of research on students' use of mathematical language, particularly when writing proofs at the undergraduate level. In this exploratory study, we analyze written student exams (N=149) from an introductory proof course in order to identify different ways in which students' use of written mathematical language differs from mathematicians' writing in formal mathematical settings.

Key words: proof writing, mathematical language, undergraduate mathematics education

Researchers in mathematics education have found that students have difficulties with mathematical proofs at both the high school and undergraduate levels (Senk, 1985; Selden & Selden, 1987; Moore, 1994; Bills & Tall, 1998; Weber, 2001). One of these difficulties is related to their inability to correctly employ mathematical language (Moore, 1994). Indeed, researchers (in both mathematics education and linguistics) have argued that the use of mathematical language poses serious challenges in the learning of mathematics in general (Ervynck, 1992; Kane, 1968; Pimm, 1987). With respect to the language used in university level mathematics, Selden and Selden (2014) discussed how mathematicians write proofs for publication. However, we are not aware of any studies considering the use of language in proof writing at the undergraduate level. As a result, we investigate the following research question in this study: What are some of the unconventional ways in which undergraduate students use mathematical language when writing proofs?

Methods

For this study, 149 written student exams (32 first exams, 63 second exams, and 54 final exams) were collected from four different instructors of an introductory proof course at a public, research university in the US. Exam tasks involving student proof writing were analyzed for use of mathematica language that the researcher believed to be unconventional according to the way that mathematicians communicate in formal writing settings. In this exploratory study, data were analyzed using open coding in the style of Strauss and Corbin (1998).

Findings

Fifteen categories of unconventional uses of mathematical language emerged from the data. A sample of these categories is listed with descriptions and examples in Table 1.

Categories	Description of the Category	Example
Unspecified variables	Using variables without first specifying the set to which they belong.	In a proof regarding rational numbers, the student writes: "Let x=a/b." However, the student never specifies what a and b are.
Overspecified variables	Variables that are unnecessarily specified to be in a specific, rather than generic, set.	In a proof about generic sets A, B, C, where elements $a \in A$, $b \in B$, and $c \in C$, the student writes: "For generic sets A, B, C, let $a, b, c \in \mathbb{N}$." However, the elements need not be in $\mathbb{N}$.
Using formal propositional logic	Using phrases of propositional logic in a proof - for example, using logical symbols.	The student uses logical operators within the text of a proof, for example: "Let $(\exists x \epsilon B) \wedge (\exists y, z \in C)$."

Mixing mathematical notation and text	Using mathematical symbols in prose.	The student uses mathematical symbols within prose, for example: "the product will be ≥ 0" or "since $X \subseteq A$ and $\subseteq B$"
Lay speak	Using informal/non-mathematical words within a proof.	The student uses informal language to describe mathematical phenomena, for example in a proof regarding the Pigeonhole Principle, a student writes: "So the worst case after 21 selections all of the bins will be full, the 22 selections it will go into a bin already filled."
Non-statements	Sentences that lack meaning.	The student writes subordinate clauses or nonsensical statements, for example: "Let $\forall x \in \mathbb{R}$." Or "If $(R \circ S)^{-1}$."
Stating definitions	Providing entire definitions within a proof.	The student includes entire definition statements within the text of a proof, for example including the following in a proof about increasing real functions: "If $f(y) \geq f(x)$ for all $y > x$, where $y, x \in \mathbb{R}$, then the function f is said to be increasing."

Table 1. Different categories of students' unusual use of mathematical language.

In Table 2, we list the number of exams in our sample containing each one of the categories listed in Table 1. Table 2 also indicates the number of final exams exhibiting each one of those unusual ways of using mathematical language. While these findings are purely exploratory and we do not know to what degree these unconventional uses of mathematical language are found in the proof writing of the larger population of undergraduate students, these findings suggest that some of these uses may be common and may persist throughout an introduction to proof course.

Categories of Students Breaking Mathematical Norms	**All Exams (out of 149)**		**Final Exams (out of 54)**	
Unspecified variables	43	28.9%	19	35.2%
Overspecified variables	3	2.0%	3	5.6%
Using formal propositional logic	32	21.5%	6	11.1%
Mixing mathematical notation and text	8	5.4%	3	5.6%
Non-statements	7	4.7%	2	3.7%
Lay speak	42	28.2%	10	18.5%
Stating definitions	33	22.1%	6	11.1%

Table 2. Number of exams exhibiting categories of students' unusual mathematical writing.

Discussion and Future Research

This study serves as a first step towards understanding undergraduate students' use of mathematical language in proof writing, and opens avenues for further research. In particular, this study has led us to design studies to investigate how undergraduate students and mathematicians view these categories of mathematical language usage, the extent to which such usage is considered to be unconventional in expert mathematical practice, and the extent to which it affect how undergraduate students and mathematicians evaluate student constructed proofs.

References

Bills, L., & Tall, D. (1998, July). Operable definitions in advanced mathematics: The case of the least upper bound. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd*

Conference of the International Group for the Psychology of Mathematics Education, (Vol. 2, pp. 104-111). Stellenbosch, South Africa: University of Stellenbosch.
Ervynck, G. (1992). Mathematics as a foreign language. In W. Geeslin & K. Graham (Eds.), *Proceedings of the 16th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 217-233). Durham, USA: University of New Hampshire.
Kane, R. B. (1968). The readability of mathematical English. *Journal of Research in Science Teaching*, *5*(3), 296-298.
Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in mathematics*, *27*(3), 249-266.
Morgan, C. (1998). *Writing mathematically: The discourse of investigation* (Vol. 9). Psychology Press.
Pimm, D. (1987). *Speaking mathematically: Communication in mathematics classrooms*. London: Routledge & Kegan Paul.
Selden, A., & Selden, J. (1987). Errors and misconceptions in college level theorem proving. In *Proceedings of the second international seminar on misconceptions and educational strategies in science and mathematics* (Vol. 3, pp. 457-470).
Selden A. & Selden, J. (2014). The genre of proof. In K. Weber (Ed.), Reflections on justification and proof. In M. Fried & T Dreyfus (Eds), *Mathematics & Mathematics Education: Searching for Common Ground* (pp. 248-251). Netherlands: Springer.
Strauss, A. & Corbin, J. (1998). *Basics of Qualitative Research. Techniques and Procedures for Developing Grounded Theory.* Thousand Oaks, USA: Sage Publications, Inc..
Senk, S. L. (1985). How well do students write geometry proofs?. *The Mathematics Teacher*, *78*(6), 448-456.
Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, *48*(1), 101-119.

Modeling outcomes in combinatorial problem solving: The case of combinations

Elise Lockwood, Oregon State University
Craig A. Swinyard, University of Portland
John S. Caughman, Portland State University

In an effort to understand ways to help students solve counting problems successfully, we conducted a paired teaching experiment in which two students reinvented four counting formulas by generalizing their work from an initial set of basic problems. Subsequent to reinventing these four formulas, they solved all but one counting problem correctly, regularly drawing upon outcomes and displaying a set-oriented perspective. In this paper, we report on the problem that they missed, which involved combinations (the Bits problem: How many 256-bit binary strings contain exactly 75 0's?). We describe a key aspect of their activity that we refer to as combinatorial encoding of outcomes, and we use this language to analyze the student work. We discuss the importance of encoding as an informal way to articulate bijections, and we suggest avenues for future work and pedagogical implications.

Keywords: Combinatorics, Modeling, Encoding, Discrete mathematics, Teaching experiment

Introduction and Motivation

Counting problems are easy to state and they provide rich opportunities to engage with deep mathematical reasoning. Researchers acknowledge the value of combinatorial tasks, noting that they often cannot be easily solved with rote or procedural reasoning (e.g., Kapur, 1970). Their non-algorithmic nature makes difficult to solve correctly, giving them the potential to foster deep mathematical thinking. Annin and Lai (2010) note that, "mathematics teachers are often asked, 'What is the most difficult topic for you to teach?' Our answer is teaching students to count" (p. 403). The overarching goal of our research is to understand how best to support students in solving counting problems. In a recent teaching experiment (reported in Author, Date), two students reinvented four counting formulas by generalizing their work from an initial set of basic problems. Subsequent to reinventing these four formulas, they solved all but one counting problem correctly. In this paper, we report on the problem that they missed involving combinations. We describe a key aspect of their activity that we refer to as *combinatorial encoding of outcomes*. Our work is guided by the following questions: *What role does combinatorial encoding serve for students as they solve counting problems, and what explanatory power does it give researchers to describe students' activity?*

Theoretical Perspective and Relevant Literature

A number of researchers acknowledge student difficulties with counting, both characterizing specific errors and potentially difficult issues (e.g., Batanero, Navarro-Pelayo, & Godino, 1997; Hadar & Hadass, 1981) and reporting low success rates (typically lower than 50%) among students at a variety of levels (e.g., Author, Date; Eizenberg & Zaslavsky, 2004). Some researchers (e.g., English, 1991) have articulated strategies that they have observed among students and ways of thinking that seem productive for students (Author, Date; Halani, 2012). In recent work (Author, Date), we have suggested that sets of outcomes are key aspects of students' counting, and that more explicit attention (by researchers, teachers, and students) should be paid to outcomes in considering combinatorial reasoning.

Counting as a modeling and encoding activity. "Mathematical modeling" is a familiar phrase in mathematics education, although definitions for what it entails vary widely. In English's (2003) definition, a model "is used to describe, make sense of, explain or predict the behavior of some complex system." (p. 229). In such work, students typically engage in real-life tasks that represent complex, real world situations. Some researchers have also studied how students model and interpret word problems (e.g., Wyndham & Saljo, 1997). Greer (1997) discusses the "peculiar nature of school word problems" (p. 294) and highlights the tendency of students to struggle with such problems because they focus on syntactic cues or oversimplify the situation. We find the language of modeling, especially as Greer uses it, to be helpful, because generally we consider counting to be an exercise in modeling. While these researchers have worked with school children solving problems involving a single operation, there are some parallels to be drawn with our work with undergraduates solving combinatorial tasks. Counting problems are typically situated in some context, and modeling requires some abstraction of the relevant details of the problem. Our students similarly must interpret the problem and determine what is being counted in order to enumerate the set of outcomes. We thus use the language of modeling, and particularly work by Greer (1997), in considering aspects of modeling specific word problems.

Enumerating a set of outcomes also requires the student to make a decision about how to *encode* each outcome. Psychology researchers use the term encoding in a specific way – for example, Prather and Alibali, (2011) define it as, "the uptake of information from the environment into working memory" (p. 355). This is not how we take the encoding of outcomes. Rather, we take encoding outcomes to be a particular combinatorial activity of determining the nature of what is being counted. We propose that when a student models a particular outcome while solving a counting problem, there are two levels on which encoding occurs. First, there is the matter of writing down an appropriate representation for how to write and catalog an outcome. Second, there is the matter of appropriately determining what constitutes a desirable outcome. As an example, in solving the problem "How many ways are there to pick a committee of five people from 20 people?", the first kind of encoding might involve representing the people as the numbers 1 through 20. Doing so is appropriate because numbers are distinguishable (as people are), and they can easily be written down and manipulated. An error with this aspect of encoding might involve modeling the people (who are distinguishable) as indistinguishable objects such as identical circles or x's. The second aspect of encoding would involve students then using that representation to articulate what a desirable outcome is – in this case, recognizing that they want to count sets (and not sequences) of numbers. In other words, even once an appropriate representation is determined, it is not a given (and it is not always easy) for students to correctly articulate what they are trying to count. Hadar and Hadass (1981) note that in order to solve a counting problem correctly students must realize that a particular outcome is "not in the set of combinations to be considered" (p. 435). This aligns with our observation that properly encoding the nature of an outcome is a key step in the counting process.

Notice that at the most sophisticated level, the two types of encoding together involve turning the problem at hand into another problem that the counter knows how to solve. In the above problem, the encoding fosters an abstraction to counting subsets of distinguishable objects, which is exactly what a binomial coefficient represents. Thus, modeling a counting problem involves formulating a given problem in terms of an isomorphic problem, creating a bijection between the stated problem and a given problem that one is able to count. Mamona-Downs and Downs (2004) targeted this technique, noting that the creation of such bijections is a key aspect of counting. We believe that successful counters encode outcomes so as to leverage a known solution or technique by the informal establishment of a bijection. We want to build upon the work of Mamona-Downs and Downs by exploring how such bijections arise for students through the informal processes of modeling and encoding outcomes.

A model of students' combinatorial thinking. We also frame our work within a model of combinatorial thinking (Author, Date), which describes key components of students' counting. The model presents the relationships between counting formulas, counting processes, and sets of outcomes. Prior work has extended the model and has emphasized the importance of the relationship between counting processes and sets of outcomes. In our presentation, and we will further extend of this model, particularly elaborating the component of *counting processes*.

Methods

This paper reports on phenomena that emerged during a paired teaching experiment (Steffe & Thompson, 2000), which was designed to have the two students (Thomas and Robin, pseudonyms) reinvent four basic counting formulas through engaging with a variety of counting tasks. The participants in this study were two above-average students who had recently completed an integral calculus course taught by the second author. They were chosen because they had no formal experience with combinatorics but had strong mathematical backgrounds, and because they had displayed a propensity for articulating their mathematical reasoning aloud.

The teaching experiment consisted of ten 90-minute sessions and proceeded in two phases. During Phase 1 (Sessions 1-3) the students reasoned about and solved ten relatively elementary counting problems. During Phase 2 (Sessions 4-10), the students encountered more challenging tasks, both in sophistication and in terms of the size of the set of outcomes. The aim of Phase 2 was for the students to reinvent each of the four basic counting formulas, which we report elsewhere (Author, Date). For this paper, we focus specifically on the only problem in the entire teaching experiment that they never ultimately answered correctly. Our analysis supports us in providing an explanation for why this problem was problematic and in shedding light on important aspects of combinatorial encoding in the process.

The analysis of data occurred at multiple levels. As the teaching experiment proceeded, we conducted an *ongoing analysis* that included reviewing the videotape of each session and constructing a content log. For analysis pertaining to this paper, we paid particular attention to the episode involving the one problem the students solved incorrectly. We transcribed this episode, reviewed and discussed it at length as a research team, and refined our understandings and descriptions of key aspects of the students' reasoning. We considered a number of possible ways to frame and explain this episode, ultimately deciding that the students' choices about *encoding* and *modeling* were central to the phenomenon we observed.

Results

We have previously reported (Author, Date) that Thomas and Robin were very successful in their work. They solved 26 out of 27 (96%) problems correctly, which is remarkable, given typical success rates on counting problems (e.g., Eizenberg & Zaslavsky, 2004). This included solving ten initial problems correctly, eight more problems correctly during the reinvention phase, and then eight out of nine problems correctly when they were given new counting problems after they had reinvented the formulas. We laud their success, and we focus here on the one problem they could not answer because we feel that there are instructive insights to be gained by exploring why this one particular problem proved to be difficult for them.

The students successfully reinvented the formula for combinations (i.e., selecting r objects from n distinct objects), and their development of this formula was closely tied to their reinvention of the formula for permutations. The students recognized that combinations correspond to "groups" of arrangements that are actually identical to each other. For example, a combination containing a, b, and c corresponds to a "group" of six permutations: abc, acb, bac, bca, cab, cba. They saw the need for dividing out by the sizes of such groups, arriving at a derivation of the combination formula $\frac{n!}{(n-r)!r!}$ from the permutation formula $\frac{n!}{(n-r)!}$. As evidence that this was a fundamental aspect of their understanding of the combinations formula, they had the following exchange (Figure 1) when solving the Lollipops problem: *There are eight children, and there are three identical lollipops to give to the children. How many ways could the lollipops be distributed if no child can have more than one lollipop?*

<Insert Figure 1 and Figure 2>

Their awareness that outcome "123" is the same as "321" and that such outcomes are part of the same "group" was an important aspect of their ability to solve combination problems. They looked for this type of "groups" structure in other problems and routinely used their combination formula to correctly solve choosing problems in other contextual situations.

The students' work on the Bits problem. After their successful reinvention of the formulas, we gave the students nine additional problems to assess how they would employ the formulas they had reinvented. They solved a number of problems (including problems involving combinations) correctly using their newly reinvented formulas. However, despite working for over an hour, they were unable to solve the Bits problem: *Consider binary strings that are 256 bits long. How many 256-bit strings contain exactly 75 0's?* A person familiar with combinations may see the connection quickly; each such binary string uniquely corresponds with a subset consisting of 75 positions (chosen out of 256) that will contain 0's, and the remaining 181 positions will contain 1's. Via this encoding, the answer is seen to be $\binom{256}{75}$. Our students did not approach the problem in this way. We briefly outline and then discuss their work to provide evidence for what we think happened.

First, for many minutes the students tried to list some outcomes, but creating a smaller case was not trivial for them. They looked for patterns as they had in the past, and initially it was not clear that they even knew what they were counting. Over time they understood that they were looking for sequences with a particular number of zeros, and so they looked for patterns having exactly one zero or exactly two zeros in several small cases (such as in strings of length 2, 3, and 4). Searching for patterns in the outcomes was consistent with their mathematical activity throughout the teaching experiment, though here it did not lead them down a productive path.

The following episode sheds light on their encoding of outcomes. They had written out a number of small cases and were looking for patterns when they found that there were 1, 3, and 6 strings that had exactly two zeros for strings of length 2, 3, and 4 respectively. By noticing an additive pattern, they guessed that there might be 10 strings of length 5 that had exactly two zeros, but they did not want to write out 32 total outcomes to test whether this would be true. We intervened and asked them if they could just write the ten outcomes that only had two zeros. We hoped that focusing on the positions of the zeros might alert them that the positions of the zeros are an important aspect of the problem. The following exchange occurred (Figure 3):

<Insert Figures 3, 4a and 4b>

It is worth noting that the students did encode the outcomes correctly in some sense. (In Figure 4a they wrote down what they were trying to count for a smaller case and correctly wrote down all ten 5-digit numbers with exactly two zeros. In Figure 4b they wrote out examples of actual outcomes). However, they did not encode the problem in a way that enabled them to use a tool they already possessed. Specifically, they did not see that this problem, too, could involve combinations, simply by choosing which *positions* contain zeros. They did not appear to use a key aspect of their understanding of "groups" problems (leveraging the phrase "123 is the same as 321" as they had in other problems) because they were thinking of sets of identical zeros, not as sets of distinguishable positions. As a result, they never answered the problem correctly.

Discussion

In terms of the model (Author, Date), this episode raises a number of points. For the Bits problem, the students could not see how the encoded outcomes related to a specific tool or technique to which they already had access (binomial coefficients). It is not uncommon to see this type of struggle among novices, and deeper insights could come about with more experience. Nonetheless, this analysis provides a way to talk about what happened for the students. There is a key issue related to their ability to see the problem at hand as being closely related to the combinations formula that they already had at their disposal. This is an encoding issue – not of correctly encoding the nature of the outcome (they were sufficient at that, even on this problem) – but of being able to encode the outcome in such a way as they could use it and apply tools to which they already had access.

The language of encoding provides terminology for the informal process that students employ, and it reflects a more formal activity in which mathematicians commonly engage. Formally naming a bjection is an effective way of counting that characterizes how expert mathematicians typically count (Mamona-Downs & Downs, 2004) – they will formulate a one-to-one, onto function between the set of outcomes in question and some set they know how to count. When encoding, students must interpret a contextualized problem and parse out a way to articulate and work with the outcomes. In an informal way, they are creating a bijection, and this modeling and encoding language describes such student activity.

The Bits problem revealed an instance in which that encoding broke down. In this problem, we see it was not that they lacked a set-oriented perspective, but rather they did not have an encoding technique that would allow them to match up anything they knew how to count with the problem. What we believe is happening in the Bits problem is that the students did not recognize that they could use their formula for combinations to select something that was not explicitly stated in the problem – that they could pick locations as a stage in their counting process. In most other cases, they could directly use their encoding to determine the type of problem they on which they were working. Here, though, the nature of the outcome did not so clearly reveal a problem type, because the objects that were being chosen were not the numbers themselves, but rather the positions that were being chosen. This meant the students could not use what they had relied on previously in determining that it was a combinations problem, because they only saw that 0, 0, 0 is the same as 0, 0, 0, no matter where the zeros were placed. Had they thought to model the problem as picking something not explicitly stated in the problem (choosing locations), they might have recognized that having zeros in locations 1, 2, 3 was the same as having locations in 3, 2, 1, and so combinations would be an appropriate operation here.

It is important to emphasize that typically there is no single correct way to encode or think about a counting problem. However, there are often more or less efficient ways to model and encode counting problems. A key step in this is turning an outcome into something that is usable. The students' encoding of binary numbers was not technically incorrect, but here, unfortunately, by literally encoding ones and zeros (and not encoding positions) they were not able to leverage their understanding of combinations to solve the problem correctly.

Implications and Future Directions

One takeaway from our study is that we have a better sense of what combinatorial content must be taught to students. We may teach students formulas (even through reinvention) and we can try to foster a set-oriented perspective (which helps resolve common counting issues), but there are a number of useful techniques of modeling and encoding that relate to building bijections, even informally. As researchers, we must continue to investigate effective pedagogical means by which to teach such techniques to students. Those who teach counting problems should also be aware that students may struggle with encoding. It may be useful to be explicit about the encoding process, and to have students articulate a bijection between the set in question and a set they are able to count. We also advise teachers to model their own formulation of bijections for students, although we acknowledge that more work must be undertaken to better understand effective ways to help students learn such ideas.

References

Author, Date.

Annin, S. A., & Lai, K. S. (2010). Common errors in counting problems. *Mathematics Teacher, 103*(6), 402-409.

Batanero, C., Navarro-Pelayo, V., & Godino, J. (1997). Effect of the implicit combinatorial model on combinatorial reasoning in secondary school pupils. *Educational Studies in Mathematics, 32*, 181-199.

Eizenberg, M. M., & Zaslavsky, O. (2004). Students' verification strategies for combinatorial problems. *Mathematical Thinking and Learning, 6*(1), 15-36.

English, L. D. (1991). Young children's combinatorics strategies. *Educational Studies in Mathematics, 22,* 451-47.

English, L. D. (2003). Reconciling theory, research, and practice: A models and modeling perspective. *Educational Studies in Mathematics, 54*(2/3), 225-248.

Greer, B. (1997). Modelling reality in mathematics classrooms: The case of word problems. *Learning and Instruction, 7*(4), 293-307.

Hadar, N., & Hadass, R. (1981). The road to solve combinatorial problems is strewn with pitfalls. *Educational Studies in Mathematics, 12,* 435-443.

Halani, A. (2012). Students' ways of thinking about enumerative combinatorics solution sets: The odometer category. In the *Electronic Proceedings for the Fifteenth Special Interest Group of the MAA on Research on Undergraduate Mathematics Education.* Portland, OR: Portland State University.

Kapur, J. N. (1970). Combinatorial analysis and school mathematics. *Educational Studies in Mathematics, 3*(1), 111-127.

Mamona-Downs, J. & Downs, M. (2004). Realization of techniques in problem solving: the construction of bijections for enumeration tasks. *Educational Studies in Mathematics, 56,* 235-253.

Prather, R. & Alibali, M. W. (2011) Children's acquisition of arithmetic principle: The role of experience. *Journal of Cognition and Development, 12*(3), 332-354.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education*. Mahwah, NJ: Lawrence Erlbaum Associates.

Wyndhamn, J. & Saljo, R. (1997). Word problems and mathematical reasoning – the study of children's mastery of reference and meaning in textual realities. *Learning and Instruction, 7*(4), 361-382.

Thomas:	Like the groups maybe?
Robin:	Yeah.
Thomas:	Because they're the same lollipop. If you give child 1, 2, 3 a lollipop, it'd be the same as giving child 3, 2, 1 a lollipop.
Robin:	Mm-hmm.

Figure 1 – The students' exchange on the Lollipop problem

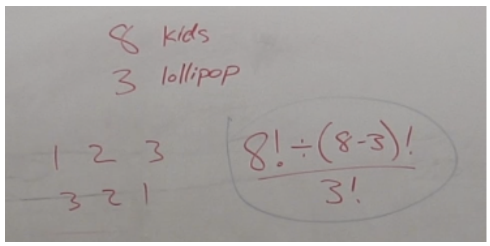
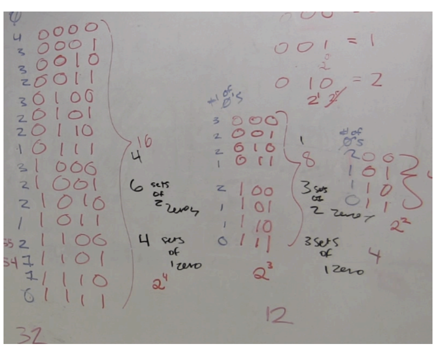


Figure 2 – The Lollipop problem

Int:	Could you also see if you would get ten that would have two zeros? Instead of writing all 32, just write the ten that have two zeros?
Robin:	Sure. Oh shoot, okay, I don't know if I can.
Int.:	I mean you could certainly still write out all 32 if you want.
Robin:	Uh, I'd probably have to.
Thomas:	Yeah I think that way might be easier.

Figure 3 – The students' exchange on the Bits problem



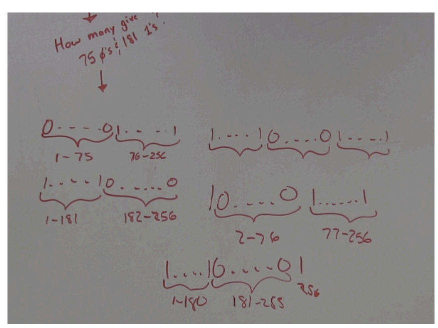


Figure 4a and 4b – Writing outcomes in the Bits problem

Mathematicians' views of mathematical practice

Elise Lockwood
Oregon State University

Eric Weber
Oregon State University

Learning a mathematical practice, such as problem solving, is different from learning mathematical content. Realizing that mathematical practices are a fundamental aspect of engaging in mathematical activity, we seek to better understand the nature of mathematical practices, as well as how they are perceived by those who teach them. In this paper, we explore these issues with university mathematicians. In particular, we focus on explaining how mathematicians think about, learn, and teach mathematical practices. We consider mathematicians' interpretations of various mathematical practices and consider how those interpretations may influence their goals for instruction perspectives on student thinking. Specifically we seek to know how mathematicians understand, think about, and practically address the teaching and learning of mathematical practice.

Key Words: Mathematical practice, Problem solving, Mathematicians

Background and Research Question

What it means to do mathematics, or engage in mathematical practice, is ostensibly at the heart of many endeavors in mathematics education research and mathematics teaching. The desire to teach students to "think mathematically" is a major influence on how we train scientists across disciplines. What does it mean to think mathematically? There are numerous descriptions of it, but we draw on Schoenfeld (1992) who notes that it involves the "mathematical point of view" and "competence with tools of the trade" (1992, p. 337), which support the development of creative and systematic thinking which are valuable to many fields and situations beyond mathematics itself. The desire to engender this point of view and various competencies associated with it are often discussed using the term mathematical practice. For instance, the CCSSM (Common Core State Standards for Mathematics) reflects the importance of mathematical thinking in the form of eight mathematical practices, which "describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years" (National Governors Association Center for Best Practices, 2010). These practices, then, serve as a guide toward helping students develop a mathematical point of view as they progress through school mathematics. In turn, these practices are designed to support the development of students' mathematical thinking.

As these eight standards for mathematical practice illustrate, there are many elements to thinking mathematically. However, in our experience with both schoolteachers and university instructors, these "practices" are often ambiguously defined and, as a result, are left open to interpretation of an individual. Given the subjective interpretations possible for words like modeling, problem solving and quantitative reasoning (all mathematical practices in the CCSSM), this is perhaps not surprising. Because practices are open to interpretation, it seems crucial to understand at a local level how these understandings of mathematical practice may translate into instruction, and subsequently, student learning.

We explored these issues with university mathematicians, and we are interested in learning how mathematicians think about, learn, and teach mathematical practices. In this presentation, we consider mathematicians' interpretations of various mathematical practices and consider how those interpretations may influence their goals for instruction perspectives on student thinking. Specifically our research question is: *How do mathematicians understand, think about, and practically address the teaching and learning of mathematical practice?*

Literature Review: Mathematical Thinking and Practice are Hard to Define

According to the CCSSM (2010), "The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education" (p. 6). We are not aware of a clear definition of what constitutes mathematical practice, and this is perhaps not surprising. Describing what it means to be a member of a community is a complicated task, as Burton (1999a, 1999b, 2001) documented in studies of mathematician's perceptions of knowing mathematics. Burton documented the various ways in which mathematicians identify themselves as part of different mathematical communities, and how that membership affected their perception of what it meant to know and learn mathematics. For instance, consider the case of mathematics and the variety of people who we might identify as mathematicians. Some practice pure mathematics, others focus on applied mathematics, and within these arenas, geometers, algebraists, analysts and others might engage with practices differently. It seems reasonable to conclude that there may be no general definition of mathematical practice. Instead, it is necessary to study certain types of actions or mental acts that are generally agreed upon as a mathematical practice, such as modeling, problem solving, or proving. However, breaking down mathematical practice to these levels still entails a degree of ambiguity.

Indeed, as an example, there has been transformational work on mathematical problem solving and modeling, two of the core practices described by the CCSSM. Yet there is not a clear definition of what constitutes mathematical problem solving or mathematical modeling. For instance, Schoenfeld wrote that, "As the literature summary will make clear, problem solving has been used with multiple meanings that range from working rote exercises to doing mathematics as a professional" (Schoenfeld, 1992, p. 338). Many others have proposed characterizations of problem solving, including Carlson and Bloom (2005) who focus on it as a complicated web of mental actions that express themselves in behavior. Similarly, there are multiple meanings for mathematical modeling that emerge from only a cursory reading of the math education literature, which we do not have the space to discuss here (Doerr & English, 2003; Thompson, 2011).

In light of such ambiguity, we must consider individuals' interpretations of mathematical practice in order to begin to construct a model of mathematical practice at a general level. It is also important to consider in what ways these interpretations influence perceptions of how students develop the ability to think mathematically. We use the remainder of the paper to focus on individuals' interpretations and their implications for teaching and learning mathematics.

Theoretical Lens: Mathematical Practice and Mental Acts

Throughout this paper we describe mathematical practice from the perspectives of various mathematicians. However, in a number of places we describe mathematical practice from our own perspective, and it is important to make clear our position on what mathematical practice entails. We view mathematical practice not as certain actions or behaviors that one carries out but rather as a set of cognitive processes that express themselves in certain types of behavior. Specifically, we think of mathematical practice as a mental act that expresses itself in consistent types of behavior across problem solving scenarios. Our position largely aligns with Harel (2008), who articulated the notion of a mental act and its usefulness for thinking about mathematical practice. Harel noted that “Humans’ reasoning involves numerous *mental acts* such as interpreting, conjecturing, inferring, proving, explaining, structuring, generalizing, applying, predicting, classifying, searching, and problem solving.” (Harel, 2008, p. 3) He described mental acts as basic aspects of cognition, not specific to mathematics, that help us describe and characterize humans’ intellectual activity. We argued earlier that one’s conceptualization of a mathematical practice, like modeling or problem solving, likely influences instructional goals and students’ learning. In framing our study, this description of mathematical practice provides a mechanism to explain that influence. Specifically, a mathematical practice (similar to a mental act), influences the development of content knowledge across situations and within specific problem solving scenarios. Now, if instructors have an image of what is entailed in a specific mental act, their instruction will support the development of certain types of mathematical knowledge more than others. If multiple instructors have different images of what a mental act entails, then different types of mathematical knowledge might be supported.

Method

Seven mathematicians volunteered to participate in a single hour-long clinical interview (Clement, 2000; Goldin, 2000) and were compensated for their time. All of the mathematicians were currently employed by universities or colleges in the Western United States and had taught a variety of mathematics courses to undergraduates and graduate students. The interviews were each roughly an hour in length and were audio-recorded. Due to space, in this proposal we report on four of the mathematicians’ responses.

Our interests in this study were to gain insight into how mathematicians understood, thought about, and practically addressed the teaching and learning of mathematical practice. In thinking about how to accomplish this, we felt that explicitly defining “mathematical practice” might be more difficult and imprecise than discussing specific practices. Instead, we opted to focus on three specific practices, and then at the end we could more broadly talk about practices since they had specific examples of practices. We chose three practices as specific examples: problem solving, justifying, and modeling, and here we report on problem solving and justifying. To be clear, we did not choose these practices because we wanted to learn about these practices in and of themselves. Rather, they served as a means to communicate about practices with the mathematicians and allowed us to paint a broader picture about the teaching and learning of mathematical practice.

We designed the clinical interview questions to target our two primary research questions, listed in Figure 1 below. In many cases, we asked follow up questions to clarify the initial responses to the questions. In other cases, the mathematicians brought up their own questions that they considered over during the course of the interview.

<Insert Figure 1 here>

Throughout each interview, we recorded audio and written work from each of the interviews using a LiveScribe Pen, which provides a recording consisting of synced audio and written work. Each interview was transcribed, and both authors reviewed the transcripts, re-listened to audio excerpts, and summarized and synthesized themes from the data. We discussed the findings for each individual practice, and then looked across results from the practices to uncover common themes that characterized the mathematicians’ views of mathematical practice.

Results and Discussion

Due to space we only provide results for problem solving for four of the mathematicians, although we summarize and draw broader conclusions from the entire data set. These results highlight common themes that emerged from mathematicians' responses about problem solving.

Problem Solving

Problem Solving involves finding an answer to something not previously known. The mathematicians tended to define problem solving as involving finding an answer to something unknown or not seen before (see Figure 2). There were many similarities in the mathematicians' responses to questions about problem solving, revealing some key themes about the value (and challenge) of developing the practice of problem solving.

<Insert Figure 2 here>

Problem solving is teachable, but it is difficult to teach. First, the mathematicians all seemed to acknowledge that problem solving was a teachable and learnable skill and is something that could be developed in students over time. The mathematicians suggested that heuristics (some called them guidelines) are a key aspect of problem solving. For example, M1 said, " I do believe in heuristics, so there's sort of a bunch of steps you can do that give you a concrete leg up to solve the problem, like break it down into smaller steps, like I said try examples, do experiments." M2 agreed, explaining that problem solving techniques might include activities like "starting small, trying to carefully state the question that you're trying to answer, trying to rephrase the question, trying to answer a slightly simpler question, taking note of what you know so far – I think there are lot of heuristics like that." For some of the mathematicians these heuristics are very practical things, like "looking it up on the internet and write programs on your computer" (M1) or leveraging existing literature (M3).

Although they indicated that problem solving is teachable and important, the mathematicians also talked about difficulty in teaching problem solving. For instance, while the mathematicians did see problem solving as involving heuristics, they suggest that students do not always seem to have those tools readily available for their use. M3 said that, "But one interesting thing that I find students don't seem to have are the tools, like actual steps that are involved in problem solving…I feel like a lot of students just come into college not really having an idea of how to do that...So I think part of it is, in my opinion they need to be kind of given guidelines in some ways." M1 also said on several occasions that he was unsure of how to teach problem solving effectively, saying "I don't know that I'm confident enough as an educator to form opinions" and that "I don't really understand the means by which students learn to do this."

When asked if they intentionally think about teaching problem solving as they prep their classes, the mathematicians said that they generally consider aspects of problem solving as they prep their classes, although this is not necessarily manifest as explicitly teaching PS to their students. M2's response in Figure 3 shows how his beliefs about problem solving can shape, even implicitly, how he presents work in his classroom.

<Insert Figure 3 here>

Additionally, mathematicians did not seem sure about actually to incorporate some of these ideas practically in the classroom. M1 in particular was very transparent about not exactly knowing how to teach problem solving. He says, "I would do that if I knew how. I mean I don't know if, the reason I don't say 'we're going to spend today doing problem solving' is because I don't know what that lecture would look like and if I did, I'd be very happy to deliver it." An underlying theme of their responses is that while mathematicians might not be sure exactly how learning problem solving happens for students, they agree they incorporate it somehow in their preparation for teaching, often implicitly. They all indicate that they at least think about problem solving as they prepare to teach, but in practice, while guidelines are important, they do not know how best to go about explicitly teaching those heuristics. That is, they might model problem solving for the students or intentionally give problems that might require problem solving heuristics, but they might not be overtly explicit about teaching heuristics and problem solving guidelines explicitly in their teaching.

A variety of experiences contributed to mathematicians' development as problem solvers. In reflecting on their own experiences with problem solving, we gain some insight from the mathematicians about how problem solving might be developed. Some of them indicated the importance of having some success solving a problem, or having some formative experience of successfully working on and solving a problem. In developing such an experience, M1 emphasized the importance of picking the right problem. "Sure, I mean well first of all you choose problems that you are liable to be able to solve… I would say one of your biggest, as a researcher, part of the goal, part of the most delicate part is knowing which problems to be thinking about" (M1). For others, developing the ability to problem solve was closely tied to experience with solving problems, even if it was through observing experts "However, I feel like I've learned a lot of problem solving just in the course of, just by experience: struggling with things, watching other people solve things, talking to other people, explaining things, and learning" (M2) or working closely with others "I would say a lot of it was on my own and with peers… we're just kind of on our own, and we just banded together and figured out how to do the problems" (M4).

In sum, the mathematicians felt that problem solving was important, they desired to incorporate it into their classrooms, but they did not always know how to do this. The influences that affected their own development as problem solvers were varied, and there did not seem to be one specific path for how the practice of problem solving is developed in students. While this is not surprising, it underscores the difficulty that mathematics teachers and learners (at a variety of levels) face in attempting to teach a practice like problem solving.

Broader reflections on practices

Although we only reported data on the problem solving portion of the interviews, these results are in line with the mathematicians' responses to modeling and justifying. Some specific aspects of the respective practices differed (modeling was less consistently defined, for example, and the mathematicians seemed to feel that justifying was perhaps easier to teach). Overall, though, the mathematicians acknowledged that these are important practices but that it is not at all trivial to learn them and to become adept at them. In particular, the mathematicians emphasized the large amount of experience and practice time required to get a handle on such activities, which could take months, years, or longer. In light of this, a common theme was that the mathematicians were concerned about assessing these practices. They acknowledged that for themselves, determining whether or not someone has learned a practice like problem solving is much harder to gauge then whether or not particular content has been learned. Some of them suggested that alternative ways to assess would be ideal (such as oral examination) but acknowledged that an exam would not always be feasible. In general, the mathematicians valued the mathematical practices as much or more than the development of specific content knowledge, though they often argued the practices were grounded within content. They also seemed to perceive these practices as developing over longer spans of time and across more courses than content knowledge might.

Conclusion and Avenues for Further Research

The mathematicians we interviewed spoke to the inherent importance of (and difficulty in) teaching and learning mathematical practices. They believed that both content and practices are important enough so as to warrant attention by teachers and students, and some of them suggested that they did not feel equipped to know how to teach practices effectively. In subsequent work we seek to understand more clearly how other populations view mathematical practices. We have conducted interviews with other sets of mathematics instructors, and we hope to compare and contrast the mathematicians' responses with pre-service and in-service teachers, community college instructors, and mathematics education researchers. The overall aim is to better understand the nature and perceptions of mathematical practices so as ultimately to help improve their teaching and learning.

References

Burton, L. (1999a). Exploring and reporting upon the content and diversity of mathematicians' views and practices. *For the Learning of Mathematics*, 36-38.

Burton, L. (1999b). The practices of mathematicians: What do they tell us about coming to know mathematics? *Educational Studies in Mathematics, 37*, 121-143.

Burton, L. (2001). Research mathematicians as learners—and what mathematics education can learn from them. *British Educational Research Journal, 27*(5), 589-599.

Carlson, M. P., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational Studies in Mathematics, 58*(1), 45-75.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 547-589). Mahwah, NJ: Lawrence Erlbaum.

Doerr, H. M., & English, L. D. (2003). A modeling perspective on students' mathematical reasoning about data. *Journal for Research in Mathematics Education*, 110-136.

Goldin, G. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education*. Mahwah, NJ: Lawrence Erlbaum.

Harel, G. (2008). What is mathematics? A pedagogical answer to a philosophical question. *Current issues in the philosophy of mathematics from the perspective of mathematicians*, 265-290.

National Governors Association Center for Best Practices. (2010). Common Core Standards for Mathematics.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning* (pp. 334-370). New York: Macmillan Publishing Company.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education* (pp. 33-57). Laramie, WY: University of Wyoming.

1. How do you interpret or characterize problem solving [or other practice]?
2. How do you think a student develops the ability to practice X and do you think that it is possible for students to learn problem solving [or other practice]?
3. Do you think it's possible to teach problem solving [or other practice] to students? Do you think it's worthwhile to teach practice X to students?
4. Let's talk about your own experience. Please reflect on how you do problem solve [or other practice] now, and how you personally (over the course of your schooling and/or career) came to learn how to problem solve [or other practice]?
5. Do you consciously attend to teaching problem solving [or other practice] when you're teaching? Do you give thought to it when you're thinking about creating lesson plans, etc.? Why or why not?
6. What distinction do you think the CCSSM is making by including both content and process standards? How do you interpret that distinction, and do you think such a distinction is important?

Figure 1 – Representative interview questions

M1: So that's when I need the answer to something and I don't know it immediately.
M2: Well, I guess it's kind of trying to find an answer to a question you don't know.
M3: Use some sort of process to try to find out more information about the problem or the system.

Figure 2: The mathematicians' characterizations of problem solving

M2: I probably don't do it [consider problem solving in prepping to teach] as much as I feel like I should. So I, philosophically I think I should do a lot of it, and I think just realistically a lot of times I feel time pressure to just get the content across. That said, my disposition toward modeling problem solving does, not part of my prepping, but a large part of my classes, fully half of the period oftentimes, maybe even embarrassingly more than that, will often be just spent asking about questions, talking to students about what they tried, and then modeling for them how I would solve it, without much preparation, I like to do that off the cuff so that they can hear my own thinking. And uh so that is not something I consciously prep for, but I reserve that time in class to do it, and I think students learn probably more from me doing that than even necessarily when I am lecturing content.

Figure 3 – M2's incorporation of problem solving into his classroom practice

Conceptualizing the notion of a task network

Ami Mamolo University of Ontario Institute of Technology	Robyn Ruttenberg-Rozen York University	Walter Whiteley York University

We develop a theoretical model for conceptualizing the restructuring of computational / numerical tasks, usually considered advanced, with a network of spatial visual representations designed to support geometric reasoning and conceptual development. Through our restructuring of the well-known "popcorn box problem," we illustrate key developmental understandings related to optimization and rate of change, as well as the possible conceptual blends afforded by a networked spatial visual approach.

Key words: Spatial Visual Reasoning; Task Network; Conceptual Development; Task Design

This paper stems from research around how university mathematics departments may support and enhance the mathematical preparation of undergraduates and prospective teachers. Current research has begun to focus on the relevance and benefits of spatial visual reasoning in advancing mathematical understanding at various stages in students' development (e.g., Natsheh & Karshenty, 2014; Uttal et al., 2013). Spatial visual approaches allow for wider accessibility of "advanced" mathematical concepts than typical mathematics investigations usually offer, both vertically across various courses and horizontally across the different strands and concepts within a particular course. As a context for our work, we restructure the familiar popcorn box problem to promote conceptual networks for optimization and rate of change, two topics required for calculus and many other fields. We develop a model for conceptualizing the restructuring of tasks such that relevant spatial visual approaches can enrich algebraic ones while encouraging fluent 'switching' amongst the different representations. Our objectives were motivated by our broader research which has shown how prospective teachers were able to make sense of optimization and rate of change in more connected, conceptual ways when they engaged with representations that elicited a geometric, spatial visual approach. This present theoretical research considers the following question: *How can we think about the design of tasks which foster and apply spatial visual reasoning so as to support conceptual development and understanding?*

We introduce and develop the construct of a *task network*, where we use the conception of "task" developed during the 22nd ICMI study on task design. Specifically the term task is taken to mean a teacher designed purposeful 'thing to do' using tools for students in order to activate an interactive tool-based environment to produce mathematical experiences. In developing this construct we found ourselves networking on multiple counts, including mathematical representations, theoretical perspectives, as well as our disparate experiences and expertise as mathematician, mathematics educator, and practicing school teacher. We offer the construct of a task network as a tool for thinking about and researching task design. At the undergraduate level, our network offers ways of incorporating spatial visual approaches to traditionally computation-heavy courses, such as calculus. For the mathematical preparation of prospective teachers, our construct offers a way to emphasize important connections across school curricula up to and including university mathematics.

Background and Context

The origins of this research began with a perceived need to foster relational understanding of rates of change and optimization. The popcorn box problem (Figure 1) is a long-standing favorite for introducing these, typically via computation and data display. Such numeric-centered approaches were found to be disconnected from the key underlying conceptual structure of optimization problems, even amongst university graduates (Whiteley, 2012). Indeed, students have found rate of change and optimization especially challenging because of a lack of conceptual understanding of these topics (Herbert & Pierce, 2008; Swanagan, 2012). While research has recommended utilizing spatial visual tools to increase relational understanding in calculus (e.g., Berry & Nyman, 2003; Tall, 2007), geometric contexts can pose difficulties for rate problems (Martin 2000). Considering these difficulties, researchers (e.g. Cuoco & Goldenberg, 1997; Tchoshanov et al. 2002) have recommended building the underlying conceptual understandings of rate of change and optimization early in school. This motivates the importance for undergraduate instruction to provide prospective teachers with experiences that may foster relational understanding in such ways as it may be applied both to their current learning, as well as their future teaching.

Given a square sheet of material, cut equal squares from the corners and fold up the sides to make an open-top box. How large should the square cut-outs be to make the box contain maximum volume?

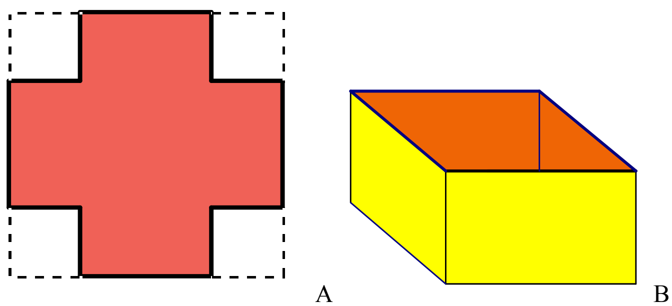


Figure 1: The popcorn box problem

We rely on Presmeg's (2006) definition of spatial visual reasoning, which identifies it as the process of "constructing and transforming both visual mental imagery and all of the inscriptions of a spatial nature that may be implicated in doing mathematics" (p. 206). Reasoning in this way can help learners "see like a mathematician" (Whiteley, 2012). To recast this problem from a numeric activity to a spatial visual one, the concepts of optimization and change in volume may be represented by tactile models in order to invite spatial visual reasoning in the form of geometric transformations, physical movements, symmetry, and comparison of dimensions, which could give direct and convincing evidence as to whether the volume lost is smaller than the volume gained between two similar boxes – leading to locating the potential maximum. In resonance with Sullivan and colleagues' (2013) notion of a purposeful representational task, the recast problem uses a tactile model or representation to demonstrate a mathematical idea, which was then explored, along with a network of associated representations. The key representations developed for this task included pairs of plastic boxes (Figure 2) cut out from identical rigid square pieces of plastic sheets, as per the description in the original task. For a detailed discussion of the materials and their affordances for conceptual development, see Whiteley and Mamolo (2011, 2013).

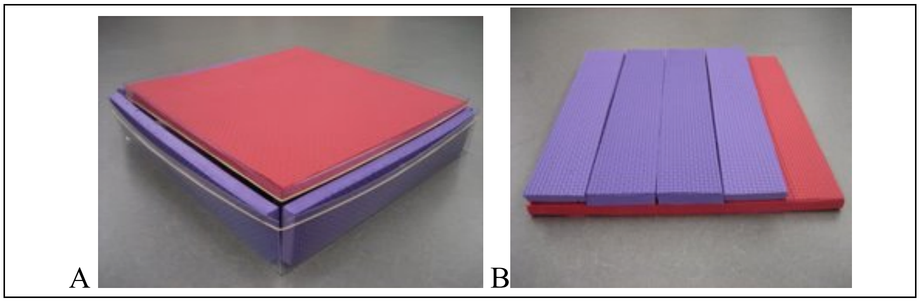


Figure 2 Models

This exploration is constrained by the thickness of the foam and thus supports understanding of average rates of change, but not instantaneous rates of change. One approach is to make the differences of the boxes very small – the thickness of Bristol board, for instance. This provides a physical form of taking limits of changes in cut size by focusing attention on comparing surface areas. A graphical representation using the Geometer's Sketchpad (Figure 3) can highlight important connections between average rates of change and slopes of secant lines, and instantaneous rates of change and slope of tangent lines – notions which are made visible and a source of reasoning through the networking of geometric spatial visual and graphical representations.

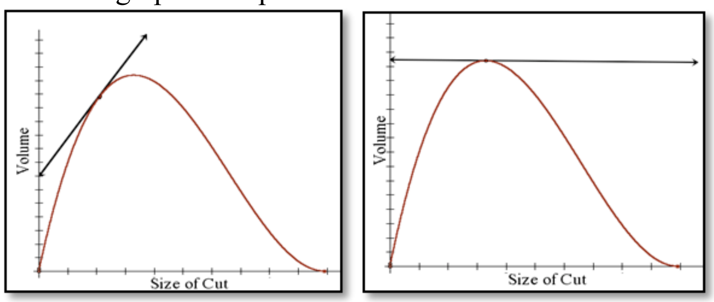


Figure 3: Graphical representations and instantaneous rates of change

In what follows, we develop the construct of a task network by applying fundamental constructs in graph theory to integrate and connect theoretical perspectives that inform task design on three counts: how an individual may reason and 'create' new knowledge, what key concepts may require an ontological shift for understanding, and what supports (physical or otherwise) may foster such shifts and knowledge creation.

Theoretical Underpinnings

In considering mathematical content, we rely on Simon's (2006) construct of key developmental understandings; conceptual blending as developed by Fauconnier and Turner (2002) sheds light on the processes used by learners to conceptualize new (for them) mathematics; and we integrate these to inform the construction of a task network.

Simon (2006) introduces KDUs "to emphasize particular aspects of teaching for understanding and to offer a construct that could be used to frame the identification of conceptual learning goals in mathematics" (p.360). The construct was developed through the coordination of social and cognitive perspectives on learning mathematics with the intent to shed new light on ways of thinking about understanding. KDUs involve a change, or conceptual advance, in students' mathematical reasoning that often cannot be acquired as the result of an explanation or demonstration. It is an important advance in the development of a concept, and "identifies a qualitative shift in students' ability to think about and perceive particular mathematical relationships" (p.363-4). In particular, KDUs afford new ways to think about and perceive mathematical relationships, which otherwise might not be available.

In the context of spatial visual approaches to optimizing rates of change, we note that the ability to think about changes in volume as entities which may be compared, and that the result of the comparison reveals information necessary for obtaining the optimum volume, is an important KDU. In the context of task design, we draw on KDUs to inform the provisions and representations included within the task that may be manipulated or acted upon by the learner so that he or she may develop the intended understanding. In their work with teachers, Sinclair, et al., (2011) identify KDUs for spatial visual reasoning in tasks which incorporate multiple representations. They highlight making connections between 3-D and 2-D representations, and noticing mathematically significant details and ignoring "distractors" such as physical imperfections of 3-D models. Thus, we made deliberate choices in the representations of mathematical relationships in order to support thinking about change in volume as an entity. We see a KDU as a new inference – a new piece of mathematical understanding – that is accessible to individuals through their negotiation of new learning experiences. The theory of conceptual blending informs our understanding of how such an inference may develop, and thus also influences the choices in task design.

Fauconnier and Turner (1998, 2002) offer a theory of conceptual blending to describe how new inferences can arise when two representations and associated ways of reasoning (or 'input spaces') are brought together in a 'blended concept'. The blend can be thought of as a mapping which combines features of the input spaces and projects them onto a third (newly formed) mental space – the output space. In a blending process, some features of the input spaces are mapped, while others are not, thus directing focus of attention and reducing the overall cognitive load for further reasoning. Blends are used to conceptualize actual things such as computer viruses, fictional things such as talking bananas, and impossible things such as time travel. Although sometimes bizarre, "the inferences generated inside [conceptual blends] are often useful and [can] lead to productive changes in the conceptualizer's knowledge base" (Coulson & Oakley, 2005, p.1513). Blending is not a metaphorical or analogical map, rather it is a specific way to combine and infer from and about information from two or more input spaces (Fauconnier & Turner, 2002). The partial representations from an individual's perceptions and concepts that are contained in the prior mental spaces blend by "the establishment and exploitation of mappings, the activation of background knowledge, and frequently involve the use of mental imagery" (Coulson & Oakley, 2005, p.1513).

An emergent blended space arises in three ways: "through composition of projections from the inputs, through completion based on independently recruited frames and scenarios, and through elaboration" (Fauconnier and Turner 2002 p. 48, emphasis as in original). Specifically, composition creates new relations not previously existent in the separate input spaces, while completion allows the composite structure in the blended space to be thought of as part of a larger structure in the blend, and elaboration, or 'running the blend' consists of cognitive work performed within the blend to exploit and elaborate upon the composite structure (Fauconnier, 1997, p.150-1). The blend continues to offer the individual ways to access each of the original representations, in a flexible manner.

The means by which one theoretical perspective informed another is quite subtle and varied. We share our approach in combining these perspectives, elaborating on the details in the context of our task network model in the following section. Briefly, our design process included considerations of what different input spaces are, or could be, available for a learner to draw from, such that the experiences, images, and representations of those input spaces afford the composition of a conceptual blend that may yield a particular KDU.

A Task Network: The Model

An integral feature of our task network is the consideration of the interplay between the intended teaching of the task and the constructed learning of the student (Stein & Lane, 1996). In negotiating and articulating relationships amongst (intended) understanding to be developed, the cognitive processes by which such understanding may develop, and the pedagogical considerations and affordances that may support such processes inform our construct of a *task network*. We borrow terminology from graph theory and illustrate the construct in a generic form in Figure 4. Specifically, a task network includes:

Nodes – these are the fundamental units from which graphs are formed. In graph theory they may be treated as 'featureless' or they may have an internal structure, representing concepts or classes of objects. The nodes in our model have structure and represent "learning centres" which are then networked. This network of nodes is what we describe as our *task network*, and we illustrate it with examples below. Zooming in, each node may also be represented by a network of KDUs, conceptual blending, and mathematical representations. That is, each node was designed around a particular representation and context, which were chosen to illuminate or support an intended KDU, and as such form external or physical input spaces which may be projected by the learner in the creation of a new conceptual blend. Zooming back out again, the intended KDUs for each node were chosen with the task network in mind – every new inference accessible to learners through their negotiation of a particular node was intended to support further inferences that could eventually lead into a conceptual understanding of optimising rates of change in a geometric context.

Edges – these connect nodes and, in a mixed graph, may be either directed or undirected. A directed edge is one with orientation, it can be thought of as an edge that proceeds from one node to another. An undirected edge has no orientation and links nodes without distinguishing one as a predecessor of the other. In our model, we use edges to represent links between KDUs. When the edge is directed it indicates that to acquire a particular KDU for our task, some previous key understanding must have been developed first. The flow of these edges begins in each task network with the tail (in our case a paper folding activity) and ends at the head (the popcorn box investigation). This flow is scaffolded by the inclusion of 'intermediate' nodes which are linked to and from the head and tail, as well as amongst themselves. For the most part, these intermediate nodes are linked amongst themselves with undirected edges – i.e., the KDU developed in one node need not precede the KDU developed in another. In our ongoing research, we explore implications for learning if these edges are treated as undirected versus directed.

Arcs – this is another term for directed edge, yet we distinguish it for our purposes by applying the terminology to the blending process, rather than the progression of KDUs. These arcs may flow in the same direction as the directed (KDU) edges in our model, they may flow in the opposite direction, and they may loop around, revisiting some prior node to thicken understanding at the head. In our model, when the arc aligns with the directed (KDU) edge, we do not distinguish (visually) between the two. Each node may serve as an input space for conceptual blending as it relates to the KDUs of any other node (in a sense forming 'loops' or 'multi-edges'). Thus, within a task network there are multiple possible trajectories a learner may follow to acquire the intended KDUs, and correspondingly, multiple possible blends afforded.

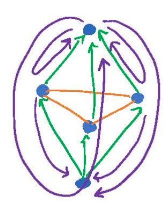


Figure 4: A generic task network – nodes are depicted in blue, undirected edges in orange, directed edges in green, and arcs in purple

Task networks for spatial visual reasoning

Table 1 presents some examples of task networks that have restructured the original popcorn box problem to promote conceptual understanding via spatial visual approaches. We highlight some of the possible conceptual blends afforded by the task networks as they apply to undergraduate mathematics students and prospective teachers. In each example, the head node is the popcorn box activity (PB) described above, and the tail node is a paper folding activity (PF) which involves predicting the shape of the optimal box, creating that box, and then refining the prediction via creating a second box. The intermediate nodes include derivative computations (C), graphical representations such as those from Figure 3 (G), measurement explorations (M) and calculations (C□), and volume comparisons via filling (F). To reduce the complexity of presenting the models, we do not identify all of the arcs representing possible blending (depicted in purple), but one may imagine them there.

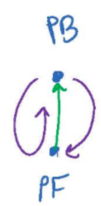
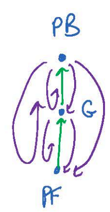
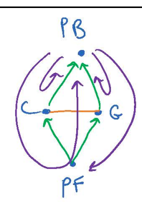
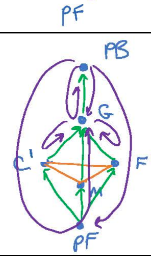
Task Network	Intended KDUs and Possible Blends
	KDU: changes in volume may be compared directly (as encapsulated entities of loss and gain), and the result of this comparison reveals the direction of change of cut of a given box necessary for moving towards the optimum volume Blends: compose - to explore change physically by starting with a pair of representative examples (e.g. boxes) and focus primarily on the change in volumes (ΔV for the secant in symbols); complete – to to focus on the sign of the change, with a simple physical comparison via foam inserts, to determine which changes will make the volume larger.
	KDU: negotiating physical constraints of 3-D models; switching between 2-D and 3-D representations (Sinclair et al., 2011) Blends: compose – to explore average rates of change (constrained by the thickness of the foam); complete – to consider sources of error – and minimize them via "thinner" foam or Bristol board (an informal invitation to a limit process, which is natural in the physical models, and illustrated by the graph)
	Blends: elaborate - to consider what boxes cannot be the optimum and why (rate of change is not zero in both the models and the symbols); elaborate and combine – to eliminate the boxes with non-zero rates of changes and determine the single box shape which remains as the optimum (in both the model and the symbols)
	KDUs (for mathematics-teaching): different representations afford different ways to understand content; spatial visual approaches can meaningfully scaffold advanced content for young learners Blends: compose – to explore connections amongst different representations across nodes; complete – to connect "advanced" representations with scaffolded ones relevant to school curriculum

Table 1

Concluding Remarks

A blend is both an internal cognitive process and a cultural artefact. Once achieved by someone and shared, a new blend becomes a possible cognitive approach requiring less cognitive load for others who have the appropriate parts to develop their own internal blend. Our task network provides a construct for conceptualising the design of learning experiences that could support for such transmission, as the external representations focus attention on key ideas listed above. The use of spatial visual representations further support shared conversations about the blended concept, the process and the reasoning. For undergraduate students, our task networks afford blends where key processes are experienced multiple times, in various representations, supporting the development of both procedural steps and conceptual reasoning. Students can create an 'immediate' balanced blend which supports flexible approaches to solving problems (composition and completion). For prospective teachers, the spatial visual paradigm available through our task networks can be blended with the prior formula-based procedural calculus knowledge, to re-infuse it with sense and visual estimation that grounds the algebraic solutions to geometric optimization (elaboration and completion). Blends and KDUs specific to the content knowledge relevant to teaching mathematics are also fostered through the explicit use and unpacking of a task network – shedding new light on how prospective teachers may view knowledge construction and understanding in mathematics. The specific task networks discussed also provide opportunities to un-pack the concepts of calculus, and they invite reflection to develop a more flexible and 'thicker' conceptual basis for the study of change and optimization (completing a blend). Individuals are creating (composing) a new blend between a sometimes fragile symbolic sense of the processes of calculus, and a novel spatial sense of optimizing and checking optima in geometric problems, which will strengthen both (elaboration).

We suggest that our construct of a task network offers a novel lens through which to design learning experiences that are faithful to the mathematical thinking performed by mathematicians on two counts. First, our emphasis is on the networking of ideas, content, and representations, rather than on sequences or trajectories as proposed by other models. This shifts attention away from discussions of what content or experiences "should come first, second, or third" towards consideration of how integrating content and experiences can support student learning. Second, and relatedly, our task network supports and fosters fluent "switching" between and amongst representations. This switching is described as essential for both problem solving and problem posing, and provides the individual with deeper insight into the underlying structural aspects of numerical/symbolic computations. Our broader research attends to this notion of switching as it relates to mathematical thinking, learning, and communication. We further suggest that via graphing, our spatial visual approach to networking theories of conceptual blending and key developmental understandings provided us with a new insight into how to interpret and foster conceptual development. The choice of looking at task design through multiple lenses was influenced by Simon (2009) who highlighted the advantages of considering a situation through different lenses, where "each lens affords a different view of the same situation" (p. 484). In our case, the lens of KDUs and the lens of conceptual development are deeply related and through our own "switching" of representations (lenses) we were afforded a more in-depth appreciation of how (and which) different representations could foster understanding and connections for a variety of learners. We acknowledge that other lenses may provide alternative interpretations and insights by taking into account different constructs.

References

Berry, J. S., & Nyman, M. A. (2003). Promoting students' graphical understanding of the calculus. *The Journal of Mathematical Behavior*, *22*(4), 479-495

Coulson, S., & Oakley, T. (2005). Blending and coded meaning: Literal and figurative meaning in cognitive semantics. *Journal of Pragmatics*, *37*(10), 1510-1536.

Cuoco, A.A., & Goldenberg, E.P. (1997). Dynamic geometry as a bridge from Euclidean geometry to analysis. In King, J., & Schattschneider, D. (Eds.). *Geometry Turned on: dynamic software in learning, teaching, and research* (No. 41). (pp. 33-44). The

Mathematical Association of America (MAA).
Fauconnier, G. (1997). *Mappings in thought and language*. Cambridge: Cambridge University Press.
Fauconnier, G., & Turner, M. (1998). Conceptual integration networks. *Cognitive science*, *22*(2), 133-187.
Fauconnier, G., & Turner, M. (2002). *The way we think*. New York, NY: Basic Books.
Herbert, S., & Pierce, R. (2008). An 'emergent model' for rate of change. *International Journal of Computers for Mathematical Learning*, *13*(3), 231-249.
Martin, T. (2000). Calculus students' ability to solve geometric related-rates problems. *Mathematics Education Research Journal*, *12*(2), 74-91.
Natsheh, I., & Karsenty, R. (2014). Exploring the potential role of visual reasoning tasks among inexperienced solvers. *ZDM-The International Journal on Mathematics Education*, 46(1), 109-122
Presmeg, N. C. (2006). Research on visualization in learning and teaching mathematics. In A.Gutiérrez & P. Boero (Eds.), Handbook of research on the psychology of mathematics education (pp. 205-235). Rotterdam: Sense Publishers
Simon, M. A. (2006). Key developmental understandings in mathematics: A direction for investigating and establishing learning goals. *Mathematical Thinking and Learning*, *8*(4), 359-371.
Simon, M. (2009). Amidst multiple theories of learning in mathematics education. *Journal for Research in Mathematics Education, 40*, 477-490.
Sinclair, M., Mamolo, A., & Whiteley, W.(2011). Designing spatial visual tasks for research: The case of the filling task. *Educational Studies in Mathematics*, *78*(2), 135-163.
Stein, M. K., & Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project. *Educational Research and Evaluation*, *2*(1), 50-80.
Sullivan, P., Clarke, D., & Clarke, B. (2013). *Teaching with tasks for effective mathematics learning*. New York, NY: Springer.
Swanagan, B. S. (2012). *The Impact of Students' Understanding of Derivatives on Their Performance While Solving Optimization Problems* (Unpublished doctoral dissertation). University of Georgia, Athens, GA.
Tall, D. O. (2007). Developing a theory of mathematical growth. ZDM- The International Journal on Mathematics Education, 39(1-2), 145-154.
Tchoshanov, M., Blake, S., & Duval, A. (2002). Preparing teachers for a new challenge: Teaching Calculus concepts in middle grades. In Proceedings of the Second International Conference on the Teaching of Mathematics (at the undergraduate level), Hersonissos, Crete, Greece.
Uttal, D. H., Meadow, N. G., Tipton, E., Hand, L. L., Alden, A. R., Warren, C., & Newcombe, N.S. (2013). The malleability of spatial skills: a meta-analysis of training studies. *Psychological bulletin*, *139*(2), 352-402.
Whiteley, W. (2012). Mathematical Modeling as Conceptual Blending: Exploring an Example within Mathematics Education. In Bockaravo, M., Danesi, M., & Núñez, R. *Cognitive Science and Interdisciplinary Approaches to Mathematical Cognition.*
Whiteley, W., & Mamolo, A. (2011). The Popcorn Box Activity and Reasoning about Optimization. *Mathematics Teacher*, *105*(6), 420-426.
Whiteley, W. & Mamolo, A. (2013). Optimizing through geometric reasoning supported by 3-D models: Visual representations of change. In C. Margolinas (Ed.). *Task Design in Mathematics Education: Proceedings of the ICMI 22 Conference,* pp.129-140, Oxford, UK

Instructional practice and student persistence after Calculus I

Lisa Mantini
Oklahoma State University

Kitty DeBock
Oklahoma State University

Barbara Trigalet
Texas Academy of Biomedical Sciences

Classroom teaching in multiple sections of Calculus I at a large comprehensive research university was observed and coded using the Teaching Dimensions Observation Protocol (TDOP). Within lecture-based methods, multiple teaching styles were identified ranging from low to moderate to high engagement, sometimes including desk work or group work. A sample population of students from all three engagement groups was followed for one year in order to analyze persistence rates into Calculus II and retention rates in a STEM major. No significant differences were found in the retention rates either at the University or in STEM majors across groups, with an average of 43% of STEM majors having switched out of STEM or having dropped out of the University after one year. However, the group experiencing higher engagement instruction in Calculus I was found to have significantly higher grades in Calculus I and also in Calculus II.

Key words: [Calculus, classroom observations, student performance, student persistence, STEM student retention, teaching dimensions observation protocol]

Introduction and Literature Review

For mathematically intensive college majors, Calculus I is a difficult but fundamental course typically taken by freshmen (Speer, Smith, & Horvath, 2010). There is a large need for graduates in the United States in all of the STEM disciplines (Bressoud, 2011) and it is essential to successful completion of these degrees that students have a good understanding of the concepts taught in Calculus I (Bressoud, Carlson, Mesa, & Rasmussen, 2013). The 2012 PCAST report projected a need for one million more STEM graduates in the US over the next several years. However, attrition rates are high, with only 38% of students who entered STEM majors in 1993 earning a STEM Bachelor's degree within six years ("Degrees of Success", 2010). By comparison, 73.5% of students who began in a non-STEM field in 2004 completed a Bachelor's degree. The goals of the PCAST report require a 34% increase in the rate of production of STEM graduates, but this could be accomplished by increasing persistence during the first year of college.

Student attrition from STEM majors most often occurs in the first or second year of college (Seymour & Hewitt, 1997). Students most often leave STEM majors because of poor instruction in their mathematics and science courses, with calculus instruction and curriculum often cited as a primary reason for students' discontinued STEM enrollment (Larsen et al., 2013; Lichtenstein et al., 2007). Active-learning teaching strategies appear to have a positive impact on students' ability to understand mathematics concepts (Rasmussen & Kwon, 2007; Schoenfeld, 2002). A study on the teaching of introductory physics courses found that high engagement teaching methods increased student understanding, confidence, and persistence (Crouch & Mazur, 2001). A recent paper showed that a single introductory course that used high engagement teaching methods could have a long-term impact and help retain students in a physics major (Watkins & Mazur, 2013).

This study seeks to contribute to a growing body of research on actual classroom practice, as well as to determine the interaction between actual classroom practices and student achievement in Calculus I and Calculus II and retention in the STEM disciplines.

Research Questions

The research questions addressed by this study are:

1. Do the instructional practices observed in Calculus I affect student persistence in continuing on to Calculus II and student success in Calculus II?
2. Do the instructional practices observed in Calculus I affect student persistence in a STEM major?

Theoretical Perspective

Constructivism refers to the idea that each person constructs meaning as he or she learns (Hein, 1991), that knowledge is of our own making, and that active experience is essential to learning (Kivinen & Ristela, 2003; Moll, 2004). In recent decades numerous criticisms of undergraduate educational practices have claimed that many faculty members fail to provide instruction which is adequately engaging and rigorous (Arum & Roksa, 2011; Bok, 2005; Boyer, 1990). Many efforts at reforming collegiate instruction attempt to engage students more effectively in learning through such techniques as peer learning (Mazur, 1997) and inquiry-based learning (Rasmussen, Kwon, Allen Marrongelle, & Burtch, 2006). Yet faculty members are slow to adopt these research-based teaching methods for many reasons (Hora & Ferrare, 2014). The Teaching Dimensions Observation Protocol (TDOP) is based on the *instructional systems-of-practice* framework which situates classroom behavior within networks of artifacts called systems of practice and within distinct disciplinary and organizational cultures (Hora & Ferrare, 2014). This instrument views teaching as a multidimensional practice affected by the discipline, the institution, and the practitioners themselves, and allows a finer grain of analysis of many of the facets of instructional practice. Several recent sources suggest that much more research is needed on college-level teaching practice in mathematics (Speer, Smith, & Horvath, 2010; Bressoud, 2012). While the instructional practices we observed all fit broadly into the category of lecture-based instruction, when viewed through the lens of the TDOP we are able to distinguish nuances of practice which appear to impact student outcomes significantly.

Research Methodology

In this section we describe the setting of the study, the data we collected, and our method of data analysis.

Setting and Participants: This study was conducted at a large, comprehensive research university in the Midwest. At this institution, the Calculus I course covers basic one-variable differential calculus and the introduction of the integral. Calculus II then covers techniques of integration, applications of integration, and sequences and series. At the time of this study, Calculus I was a four-credit course taught in sections of 35-45 students each. During the semester in which study data was collected there were eleven instructors in Calculus I, with experience levels ranging from many decades of experience teaching Calculus to relatively new Ph.D.'s with a few years of experience to graduate teaching assistants. Two instructors were teaching their own section of Calculus I for the first time; all others had prior experience as an independent instructor in Calculus I. Most of the instructors volunteered as study participants and were observed for this study using the Teaching Dimensions Observation Protocol (TDOP).

The TDOP coding instrument records classroom behaviors in five categories during each two-minute interval of a class period. These categories (Hora & Ferrare, 2010) include Instructional Methods such as lecture, discussion, and group work; Pedagogical Moves such as organizational comments, assessments, and emphasis; Instructor-Student Interactions; student Cognitive Engagement observed such as problem solving, verbal articulation of ideas, or real-world connections; and use of Instructional Technology including a chalk or white board or digital tablet such as a document camera. In this study many TDOP codes varied little among instructors, who all used lecture methods with hand-written visuals on a chalkboard or whiteboard extensively. But several codes were significantly correlated with student performance on either the uniform final exam or with normalized gain on the Calculus Concepts Inventory (CCI), including Student Group Work (SGW), Desk Work (DW), verbal Articulation of ideas by students (ART), or Movement of the instructor around the room (MOV), mostly during times when students were performing desk work or group work. Interestingly, these codes are all indicating time intervals during which students were highly engaged, in agreement with numerous studies citing links of higher engagement instructional practices with improved student learning (Code, Kohler, Piccolo, & MacLean, 2012; Crouch & Mazur, 2001; Epstein, 2007; Kivinen & Ristela, 2003; Moll, 2004).

We decided to use these significant TDOP codes in combination in order to more effectively differentiate the type and level of student engagement we were observing among Calculus I instructors. The first new combination code introduced, SVB, indicates a time interval in which students were observed articulating ideas or responding to or asking questions; the TDOP codes included were articulation (ART), student novel question (SNQ), student comprehension question (SCQ), and student response (SR). The second combination code is SWK, indicating that students were observed either working individually or in small groups on problems at their desks; the TDOP codes included were desk work (DW) or student group work (SGW). The final new code SENG indicates that one or the other of SVB or SWK was observed. The proportion of two-minute intervals in which the combined SENG code was observed ranged from 20% to 90% of the time. Clusters of sections with similar proportions of the SENG code being observed were sorted into three groups for purposes of further analysis, the Low Engagement group (SENG from 20% to 40% of time intervals), Moderate Engagement group (SENG from 50%-60%), or High Engagement (SENG from 80%-90%). No sections reported percentages outside of these ranges.

For this study two representative sections from each of the engagement groups were selected for further analysis of the persistence data. The total number of students in this study is 245, with from 78 to 86 students in each engagement group. Analysis of persistence data from the remaining sections is ongoing.

Engagement Profile	**SENG* Values**	**Number of Sections Observed**	**Sections Selected for Further Study**	**Number of Student Participants**
Low	20% - 40%	4	2	81
Moderate	50% - 60 %	3	2	86
High	80% - 90%	4	2	78

Data Collection and Analysis: Additional data was gathered for the students in the selected sections including Calculus II grade (if any), entry test scores such as math ACT subscore and ALEKS placement test score, major at the time of enrollment in Calculus I, and major one year later. The ALEKS placement test is a commercial pre-calculus skill inventory taken by all entering freshmen designed to place students into either Calculus I or the appropriate pre-calculus course. The data were tabulated to show distribution of grades A-F and W for each section studied and grouped totals and averages for low, medium, and high engagement groups. The average grade point and DWF rates were calculated for each group. An ANOVA statistical test was run on SPSS software to test for significant differences in the three groups of engagement for the Calculus II grade point average. ANOVA was run again to test for significant difference in the ACT and ALEKS means for the three groups.

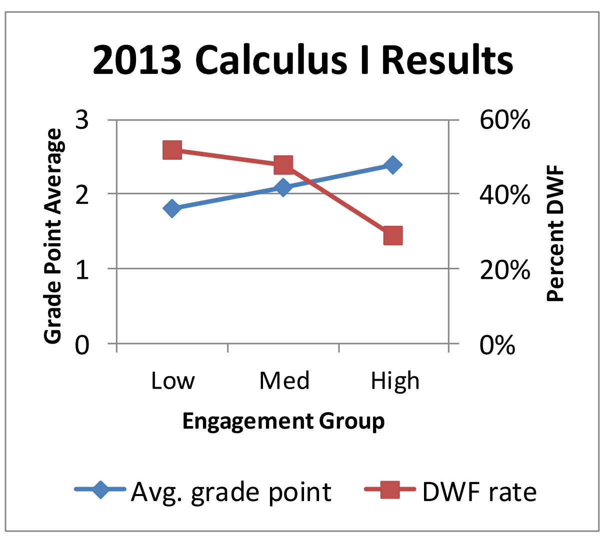
Research Results

A prior analysis of all sections from the full data set (Mantini, Trigalet, & Davis, 2014) reported on the significant correlations found between code SENG, the overall section engagement level, and student performance on the uniform final exam in Calculus I, and also on the significant correlation between code SWK, students seen working during a class period, and the section's Normalized Gain score on the Calculus Concepts Inventory. This study will report on an analysis for the reduced data set of the correlation of the overall student engagement level with course grades in Calculus I and student enrollment rates and grades earned in Calculus II. Analysis of the persistence data for the full data set is ongoing.

First we compared the six sections in the study to see if there were significant differences across groups in the mathematical readiness of the students. We ran a one-way ANOVA test on the distribution of ACT Math subscores and ALEKS placement test scores by students in each group for whom scores were available, using SPSS software to determine if there were any significant differences between groups. The results indicated that there were no significant differences for the ACT Math subscores [$F(2,207) = .413$, n.s.] or for the ALEKS scores between groups [$F(2, 168) = .790$, n.s.] and so no post-hoc tests were needed. We conclude that our students in the three groups had similar mathematical preparation and readiness for Calculus I.

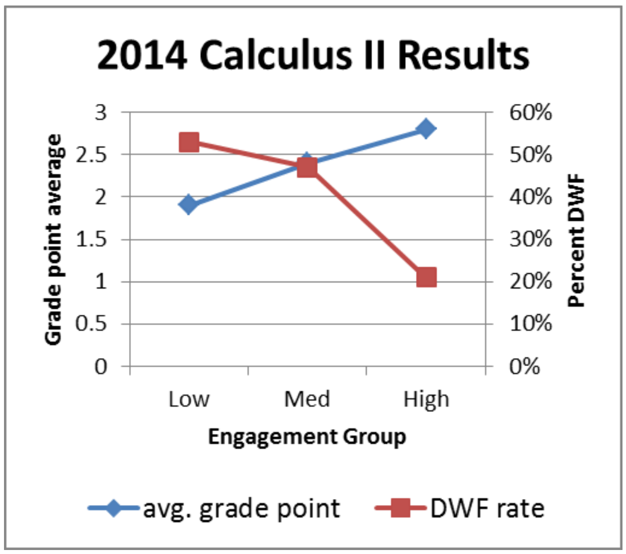
	Engagement Group		
	Low	Moderate	High
Number reporting ACT score	69	78	68
Average Math ACT subscore	25.8	24.2	25.4
Number reporting ALEKS score	62	62	47
Average ALEKS score	62.9	62.4	64.5

Calculus I Grades: The grades students may earn range from A (highest achievement), B, C, or D, to F (failure) or W (withdrawal from the class). We computed grade point averages in each group by assigning 4 points to grades of A, 3 points to B, and so on, with 0 points earned by failing grades of F. We do not consider grades of W when computing grade point averages. We also computed DWF rates in each group, given by the proportion of students in the group earning D, F, or W. We found that the grade point average increased in Calculus I from low to moderate to high engagement groups while DWF rates decreased. Scores are summarized in the graph below.



Calculus II Results: There were 40, 30, and 33 students from the Low, Moderate, and High engagement groups, respectively, who attempted Calculus II within the next two semesters. Of these, the grade point averages were 1.9, 2.4, and 2.8 for students coming from low, moderate, and high engagement instruction in Calculus I, respectively, showing an increase in Calculus II grades earned as the engagement level of their Calculus I instruction increased. Correspondingly, the DWF rates in the low, moderate, and high engagement groups were 53%, 47%, and 21%, respectively, decreasing as the engagement level of the student's Calculus I instruction increased.

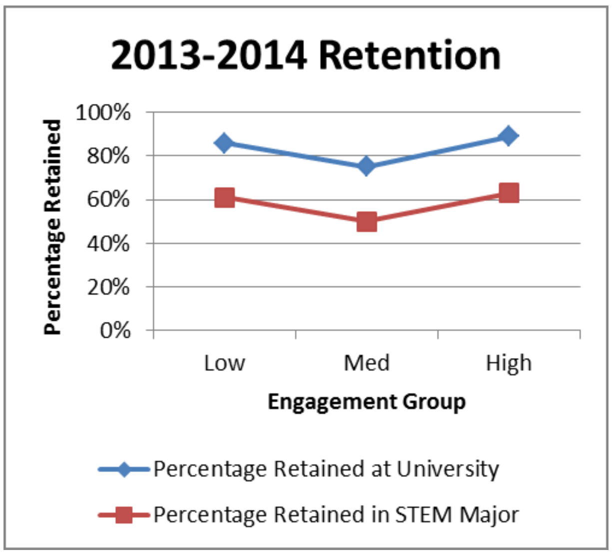
A one-way ANOVA indicated that the grade differences across the three groups were statistically significant, with $F(2, 76) = 3.674$, $p = .03$. Tukey post-hoc comparisons of the groups showed that the high engagement group (M= 2.7, 95% CI[1.36,2.31]) had a significantly higher grade point average than the low engagement group (M=1.84, 95% CI[2.31, 3.10]) with $p = .025$. Comparisons between the moderate engagement group (M = 2.38, 95% CI [1.75, 3.02]) and either the high or low engagement groups indicated no statistically significant difference in grades at $p < .05$.



Persistence in STEM Majors: We define the STEM majors to be majors which require at least Calculus I and II for their students. These include Mathematics, Physics, Chemistry, Geology, and Engineering. Other science majors which require Calculus I only, such as Biochemistry or Architecture, are grouped with non-STEM majors for purposes of this study. We found that initially there were 56, 50, and 43 STEM majors enrolled in the Low, Moderate, and High engagement groups, respectively. One year later we found 34, 25, and 27 of the students from the Low, Moderate, and High engagement groups still enrolled in a STEM major, giving persistence rates of 61%, 50%, and 63%, respectively. On average, 43% of the students in the study had switched out of a STEM major. In fact, 38 of the initial 245 students in the study, or 15.5% of the initial population, had dropped out of the university entirely. These numbers do not indicate a significant difference across the groups, but the numbers of switchers and the number of dropouts are very significant in human terms. The following table summarizes these results, with *%P* indicating the percentage of students who were *Persisters*, students who were still enrolled in a STEM major one year after Calculus I.

	Low			Moderate			High		
	2013	2014	*% P*	2013	2014	*% P*	2013	2014	*% P*
STEM	56	34	**61%**	50	25	**50%**	43	27	**63%**
Non-STEM	16	28		30	35		27	35	
Drop-out		10			20			8	
Total	72	**72**		80	**80**		70	**70**	

The following graph indicates students still enrolled in the University and still enrolled in a STEM major one year after Calculus I.



Discussion

With regard to our first research question, this study found no significant difference across engagement levels among observed sections of Calculus I in the rate at which students enrolled into Calculus II. However, we found a significant difference in the average grades earned in Calculus II, with students who experienced high engagement instruction in Calculus I earned significantly better grades in Calculus II. This is an interesting result given that the six observed sections in this study started with very similar students based on their mathematical readiness in terms of placement test and ACT scores, making it more likely that the observed grade differences are due to the instructional methods experienced. Furthermore, the differences of instructional practice might be considered modest by some in terms of current pedagogical literature since all instructors used lecture-based methods. However, our use of the TDOP instrument enabled us to observe finer nuances of teaching practices within lecture-based instruction, which enabled us to observe these statistically significant differences in student outcomes.

With regard to our second research question, we were not able to observe significant differences in persistence either in students remaining enrolled at the University or in continuing to pursue STEM majors. More study, and a larger sample size, are clearly needed.

References

Arum, R. & Roksa, J. (2011). *Academically adrift: limited learning on college campuses*. Chicago: University of Chicago Press.

Bok, D. (2005). *Our underachieving colleges: A candid look at how much students learn and why they should be learning more*. Princeton, NJ: Princeton University Press.

Boyer, E. (1990). *Scholarship reconsidered: Priorities of the professoriate*. Princeton, NJ: Carnegie Foundation for the Advancement of Teaching.

Bressoud, D. (2011, February 1) Status of the math-intensive majors. [Web log post]. Retrieved from http://www.macalester.edu/~bressoud/pub/launchings/launchings_02_11/launchings_02_11.html#anchor1

Bressoud, D. (2012, July 1) Learning from the physicists. [Web log post]. Retrieved from http://launchings.blogspot.com/2012/07/learning-from-physicists.html

Bressoud, D., Carlson, M., Mesa, V., & Rasmussen, C. (2013). The calculus student: Insights from the Mathematical Association of America national study. *International Journal of Mathematical Education in Science and Technology*, *44*, 1-15.

Code, W., Kohler, D., Piccolo, C., & MacLean, M. (2012). Teaching methods comparison in a large introductory calculus class. In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman, (Eds.) *Proceedings of the 15th Annual Conference on Research in Undergraduate Mathematics Education,* (pp. 375-379).

Crouch, C. H., & Mazur, E. (2001). Peer Instruction: Ten years of experience and results. *American Journal of Physics, 69*(9), 970.Degrees of Success: Bachelor's Degree Completion Rates among Initial STEM Majors. (2010). http://www.heri.ucla.edu/nih/downloads/2010%20-%20Hurtado,%20Eagan,%20Chang%20-%20Degrees%20of%20Success.pdf

Epstein, J. (2007). Development and validation of the calculus concept inventory. In *Proceedings of the Ninth International Conference on Mathematics Education in a Global Community,* (pp. 165-170).

Epstein, J. (2012). *The calculus concept inventory – Measurement of the effect of teaching methodology in mathematics*. Unpublished manuscript, Department of Mathematics, Polytechnic Institute at New York University, Brooklyn, NY.

Ferrini-Mundy, J., & Güçler, B. (2009). Discipline-based efforts to enhance undergraduate STEM education. *New Directions For Teaching And Learning*, *117*, 55-67.

Ganter, S. L. (1999). An evaluation of calculus reform: A preliminary report of a national study. In B. Gold, S. Keith, and W. Marion (Eds.), *Assessment Practices in Undergraduate Mathematics*, (pp. 233-236). Washington, DC: Mathematical Association of America Notes.

Halloun, I. & Hestenes, D. (1985). The initial knowledge state of college physics students. *American Journal of Physics, 53*(11), 1043-1048.

Hein, G. E. (1991, October 15-22). *Constructivist learning theory*. Paper presented at the The Museum and the Needs of People CECA (International Committee of Museum Educators) Conference, Jerusalem Israel.

Hora, M., & Ferrare, J. (2010). The Teaching Dimensions Observation Protocol (TDOP). Madison, WI: University of Wisconsin-Madison, Wisconsin Center for Education Research.

Hora, M. & Ferrare, J. (2013). Instructional systems of practice: A multidimensional analysis of undergraduate math and science course planning and classroom teaching. *Journal of the Learning Sciences*, *22*(2), 212-257.

Hora, M. & Ferrare, J. (2014). Remeasuring postsecondary teaching: how singular categories of instruction obscure the multiple dimensions of classroom practice. Journal of College Science Teaching, 43, 36-41.
Kivinen, O., & Ristela, P. (2003). From constructivism to a pragmatist conception of learning. *Oxford Review of Education, 29*(3), 363-375.
Larsen, S., Johnson, E., & Strand, S. (2013). *Characteristics of Successful Programs in College Calculus: Pilot Case Study*. Paper presented at the 16th Annual Conference on Research in Undergraduate Mathematics Education, Denver, Colorado.
Lichtenstein, G., Loshbaugh, H., Claar, B., Bailey, T., & Sheppard, S. (2007). *Should I stay or should I go? Undergraduates' prior exposure to engineering and their intentions to major*. Paper presented at the American Society for Engineering Education Conference.
Mantini, L, Trigalet, B. and Davis, R.E. (2014). Instructional Practices and Student Performance in Calculus. In T. Fukawa-Connolly, G. Karakok, K. Keene, and M. Zandieh, (Eds.), *Proceedings of the 17th Annual Conference on Research in Undergraduate Mathematics Education,* (pp. 199-210).
Mazur, E. (1997). *Peer instruction: A user's manual*. Upper Saddle River, NJ: Prentice Hall.
Moll, I. (2004). Towards a constructivist Montessori education. *Perspectives in Education, 22*(2), 37-49.
Murray, H.G. (1983). Low-inference classroom teaching behaviors and student ratings of college teaching effectiveness. *Journal of Educational Psychology*, *75*, 138-149.
Porter, A. C. (2002). Measuring the content of instruction: Uses in research and practice. *Educational Researcher*, *31*(7), 3-14.
President's Council of Advisors on Science and Technology (PCAST). *Report to the President Engage to Excel: Producing one million additional college graduates with degrees in Science, Technology, Engineering, and Mathematics*. (2012). Retrieved from http://www.whitehouse.gov/administration/eop/ostp/pcast/docsreports
Rasmussen, C., & Kwon, O. N. (2007). An inquiry-oriented approach to undergraduate mathematics. *The Journal of Mathematical Behavior, 26*(3), 189-194.
Rasmussen, C., Kwon, O. N., Allen, K., Marrongelle, K. & Burtch, M. (2006). Capitalizing on advances in mathematics and K-12 mathematics education in undergraduate mathematics: An inquiry-oriented approach to differential equations. *Asia Pacific Education Review*, *7*(1), 85-93.
Saroyan, A. & Snell, L.S. (1997). Variations in lecturing styles. *Higher Education*, *33*, 85-104.
Sawada, D., Piburn, M. D., Judson, E., Turley, J., Falconer, K., Benford, R. and Bloom, I. (2002). Measuring reform practices in science and mathematics classrooms: The reformed teaching observation protocol. *School Science and Mathematics*, *102*, 245–253.
Schoenfeld, A. H. (2002). Making mathematics work for all children: Issues of standards, testing, and equity. *Educational Researcher, 31* (Jaunuary 2002), 13-25.
Seymour, E. & Hewitt, N. (1997). Talking about leaving: why undergraduates leave the sciences. Westview.
Speer, N. M., Smith III, J. P. & Horvath, A. (2010). Collegiate mathematics teaching: An unexamined practice. *The Journal of Mathematical Behavior*, *29*, 99-114.
Watkins, J., & Mazur, E. (2013). Retaining Students in Science, Technology, Engineering, and Mathematics (STEM) Majors. *Journal of College Science Teaching, 42*(No. 5).

Gains from the incorporation of an approximation framework into calculus instruction

Jason Martin
University of Central Arkansas

Michael Oehrtman
Oklahoma State University

We report on a research-based effort to make calculus conceptually accessible to more students while simultaneously increasing the coherence, rigor, and applicability of the content learned in the courses. Recent studies have indicated that an approximation and error analysis approach to curriculum and instructional design can support a productive and coherent conceptual foundation for students' reasoning about concepts defined in terms of limit. In this study we explore the affects of such an approach to curriculum design systematically implemented in the form of 30 labs spread throughout the first two semesters of calculus. Data taken from pre-tests at the beginning of Calculus 1 and posttests near the end of Calculus 2 indicate conceptual gains above the gains previously observed from students taught without approximation curriculum.

Key words: Calculus, Cooperative Learning, Approximation, Limit

Recently, the President of the United States issued a call for colleges and universities to increase graduation rates for students earning STEM degrees by approximately 30% while simultaneously producing teachers with stronger skills for teaching STEM content (PCAST, 2012; The White House, 2012). Unfortunately, recent studies have documented students dropping out of fundamental courses required by STEM fields or leaving STEM fields altogether (Bressoud, 2012; NCES, 2009; Thompson et al., 2007). For example, in one large institutional study, Thompson et al. (2007) found that 33% of students who earned a C or better in Calculus 1 who's major required Calculus 2 did not persist in Calculus 2. Similarly, 31% of students earning a C or better in Calculus 2 whose major required Calculus 3 did not persist in Calculus 3. Thompson found that many of these strong students were voluntarily leaving the STEM pipeline because they were unsatisfied with the classroom culture in these programs. Bressoud's (2012) analysis of a survey of over 14,000 calculus students at all types of colleges and universities across the United States corroborated these findings, and he noted that "the single greatest factor counteracting this trend that is under the control of the instructor is the quality of teaching as viewed by the students." Bressoud went on to state, "Most students are not engaged by [lecture] format." For engineering, the PCAST report added that "students in traditional lecture courses were twice as likely to leave engineering and three times as likely to drop out of college entirely compared with students taught using techniques that engaged them actively in class" (p. 6). The National Research Council concurred identifying "designing in-class activities to actively engage students" and "[organizing] students in small groups" as the two strongest promising practices for improving STEM instruction (NRC, 2011, pp. 22-23).

In response to these challenges, a comprehensive collection of labs to drive more coherent and engaging instruction in introductory calculus sequences has been created based on a unifying conceptual approximation and error analyses structure (Oehrtman, 2008, 2009). We ask the following:

1) When instructors implement these labs throughout the first two semesters of calculus, are student conceptual gains observed? Are these gains greater than gains from more traditional classes?

2) Is there a shift toward students using approximation models, and if so, does this shift correlate to conceptual gains/or lack thereof?

Criteria for Lab Design

Limit concepts are at the core of mathematics curriculum for STEM majors, but unfortunately decades of research have revealed numerous misconceptions and barriers to students' understanding. Building off of work by Williams (1991, 2001), Oehrtman (2009) identified several cognitive models employed by students that met criteria for emphasis across limit concepts and for sufficient depth to influence students' reasoning. Williams noted that frequently students attempt to reason about limits using intuitive ideas associated with boundaries, motion, and approximation. Oehrtman found that, unlike most other cognitive models employed by students, the structure of students' spontaneous reasoning about approximations shares significant parallels with the logic of formal limit definitions while being simultaneously conceptually accessible and supporting students' productive exploration of concepts in calculus defined in terms of limits. With this in mind, we contend that a false dichotomy exists between a formally sound, structurally robust treatment of calculus on the one hand and a conceptually accessible and applicable approach on the other. By adopting an instructional framework utilizing approximation and error analyses, we designed labs based on criteria listed in Figure 1.

Design Criteria 1. Language, notation, and constructs used in the labs should be conceptually accessible to introductory calculus students.
Design Criteria 2. The structure of students' activity should reflect rigorous limit definitions and arguments without the language and symbolism of formal □-□ and □-*N* notation that is a barrier to most calculus students' understanding.
Design Criteria 3. The labs should present a coherent approach across all concepts defined in terms of limits and effectively support students' exploration into these concepts.
Design Criteria 4. The central quantities and relationships developed in all labs should be coherent across representational systems (especially contextual, graphical, algebraic, and numerical representations)
Design Criteria 5. All labs should foster quantitative reasoning and modeling skills required for STEM fields.
Design Criteria 6. The sequence of labs should establish a strong conceptual foundation for subsequent rigorous development of real analysis.
Design Criteria 7. All labs should be implemented following instructional techniques based on a constructivist theory of concept development.

Figure 1. Design criteria for the labs.

When left unguided, students' applications of intuitive ideas about approximations are highly idiosyncratic (Martin & Oehrtman, 2010a, 2010b; Oehrtman, 2009). Components of a well-developed approximation cognitive model include an unknown actual quantity and known approximations. In addition, for each approximation, there is an associated error,

$$\text{error} = |\text{ actual value} - \text{approximation }|.$$

A bound on the error allows one to use an approximation to restrict the range of possibilities for the actual value. Approximations are judged to be accurate if the error is small, and a good approximation method allows one to improve the accuracy of the approximation until the error is as small as desired. To further systematize students' reasoning concerning approximation ideas and support this accessible yet rigorous approach to calculus instruction, throughout the labs students are engaged in contextualized versions of the questions in Figure 2. These questions can help bring coherence between structural components, reveal operations performed on these components, and highlight relationships among these operations, all of which is foundational for the generation of new understandings (e.g., Piaget, 1970; Glasersfeld, 1995; Ernest, 1998).

Question 1.	Explain why the unknown quantity cannot be computed directly.
Question 2.	Approximate the unknown quantity and determine, if possible, whether your approximation is an underestimate or overestimate
Question 3.	Represent the error in your approximation and determine if there is a way to make the error smaller.
Question 4.	Given an approximation, find a useful bound on the error.
Question 5.	Given an error bound, find a sufficiently accurate approximation.
Question 6.	Explain how to find an approximation within any predetermined bound.

Figure 2. Approximation questions consistent across most labs.

Method & Analysis

Nine instructors piloted up to 30 labs in 14 different first and second semester calculus classrooms at seven different institutions. Instructors participated in a 3-day summer workshop and in weekly online meetings to maintain the fidelity of lab implementation consistent with design criteria. From these classes, 361 students participated in previously developed pre and posttest instruments. These instruments were the Calculus Concepts Assessment (CCA) to measure shifts in students' understanding of the central concepts of calculus and the Limit Models Assessment (LMA) to measure shifts in the cognitive models employed by students to reason with these calculus concepts. The CCA is a 32-item assessment developed using the same techniques used in the development of the Precalculus Concept Assessment (Engelke, Oehrtman, & Carlson, 2005; Carlson, Oehrtman, & Engelke, 2010) to ensure answers and distractors were chosen based on specific identifiable reasoning patterns. The 8-item LMA was developed following Model Analysis Theory from physics education research where modeling the structure of student thinking is viewed as a mixture of previously identified cognitive models with varying probabilities of being activated (Bao & Reddish, 2001; Bao, Hogg, & Zollman, 2001). We computed normalized average gains on the CCA (the ratio of the actual average gain to the maximum possible average gain) for comparison with our database normalized average gains of 0.2 on a prior version of the CCA. Our database consisted of 252 students from 15 Calculus 1 classes feeding into 12 Calculus 2 classes. Our version of the CCA was slightly modified from the version used for the database to include more difficult items (which was further evidenced by a lower percentage of students getting items correct on our CCA pre-test). We also correlated students' normalized average gains with the predominance of approximation models as assessed by the post-course LMA to determine if the development of approximation models was related to conceptual gains/or lack thereof. Additional analyses at the instructor and item levels are ongoing.

Preliminary Results

Tables 1 and 2 summarize preliminary aggregate assessment results for students that took both pre and posttests of the CCA and the LMA.

Table 1 shows conceptual gains on the CCA of 3.5 points for Calculus 1 (14% normalized gain) and 2.0 points for Calculus 2 (10% normalized gain). Currently our gains from pre-Calculus 1 to post Calculus 2 appear to be around 30%. Our previous database average for gains through Calculus 1 and 2 not using the lab materials is 20%, so the students in these pilot classes are still outperforming this baseline.

Table 1.
Calculus Students Taking Both the Pretest and Posttest Calculus Concepts Assessment in the 2013-2014 Academic Year

Course	Pretest CCA	Posttest CCA
Calculus 1 (n = 170)	7	10.5
Calculus 2 (n = 106)	11.9	13.9

Table 2.
Calculus Students Taking Both the Pretest and Posttest Limit Models Assessment in the 2013-2014 Academic Year

Course	Pretest LMA	Models	Posttest LMA
Calculus 1 (n = 195)	$\lambda_1 = 0.77 \quad v_1^2 = \begin{bmatrix} 0.29 \\ 0.12 \\ 0.17 \\ 0.21 \\ 0.21 \end{bmatrix}$	approximation approx distractor collapse ∞ as a number physcial limitation	$\lambda_1 = 0.77 \quad v_1^2 = \begin{bmatrix} 0.36 \\ 0.10 \\ 0.17 \\ 0.21 \\ 0.18 \end{bmatrix}$
Calculus 2 (n = 105)	$\lambda_1 = 0.78 \quad v_1^2 = \begin{bmatrix} 0.37 \\ 0.10 \\ 0.17 \\ 0.22 \\ 0.14 \end{bmatrix}$	approximation approx distractor collapse ∞ as a number physcial limitation	$\lambda_1 = 0.78 \quad v_1^2 = \begin{bmatrix} 0.42 \\ 0.07 \\ 0.18 \\ 0.20 \\ 0.12 \end{bmatrix}$

Table 2 shows a notable shift toward using approximation models when reasoning about limits, 29% to 36% for the class model during Calculus 1 and 37% to 42% for the class model during Calculus 2. All eigenvalues were 0.77 to 0.78 indicating the class models are a good representation of the mixed models employed by most students. Simultaneously students became less likely to select items that used the same terminology about approximations and limits without the appropriate conceptual structure. The only model that appeared consistently detrimental to students' understanding in Oehrtman (2009) was physical limitation, which decreased in frequency. Collapse and infinity as number metaphors were often productive for students, even if not entirely mathematically correct, and their prevalence remained roughly flat according to Table 2.

Discussion & Questions

Our original target for the end of the project was a 50% gain. Although we chose to give a harder version of the test than we originally planned, we still hope to improve the normalized gain to near 50% over the next two years of this study. To help meet this goal, we are currently in the process of incorporating new resources for instructors and students, such as online applets to support students in conceiving of and relating relevant quantities from physical situations to the calculus modeling such situations. How might we best study and measure the effects of implementing such resources?

The lowest CCA gains and smallest LMA shifts were in classes with the least experienced teachers. What are possible links between experience and effective implementation of research-based curricula?

Acknowledgment

This material is based upon work supported by the National Science Foundation under Awards DUE #1245021 and DUE #1245178.

References

Bao, L. & Reddish, E. (2001). Model analysis: Assessing the dynamics of student learning. Submitted to *Cognition and Instruction.* Internet: http://www.physics.ohiostate.edu/~lbao/archive/papers/MAPre_01-0805.pdf. Accessed on 5/3/05.

Bao, L., Hogg, K., & Zollman, D. (2001). Model analysis of fine structures of student models: An example with Newton's third law. *American Journal of Physics, 69,* 45-53.

Bressoud, D. (2012). Factors influencing STEM preparedness: From Algebra to Calculus. National Council of Teachers of Mathematics Research Presession, Philadelphia, PA, April 2012. Internet: http://www.macalester.edu/~bressoud/talks.

Carlson, M., Oehrtman, M., Engelke, N. (2010). The precalculus concept assessment (PCA) instrument: A tool for assessing reasoning patterns, understandings, and knowledge of precalculus level students. *Cognition and Instruction*, *28*(2), 113-145.

Engelke, N., Carlson, M., & Oehrtman, M. (2005). Composition of functions: Precalculus students' understandings, In G. M. Lloyd, M. R. Wilson, J. L. M. Wilkins, & S. L. Behm (Eds.), *Proceedings of the 27th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* [CD-ROM]. Eugene, OR: All Academic.

Ernest, P. (1998). *Social constructivism as a philosophy of mathematics*. State University of New York Press, Albany, NY.

Glasersfeld, E. von (1995). *Radical constructivism: A way of learning and knowing.* Falmer Press: London.

Martin, J., & Oehrtman, M. (2010a). Part / whole metaphor for the concept of convergence of Taylor series. In P. Brosnan, E. Diana, & L. Flevares (Eds.) *Proceedings of the Thirty Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 97-104). Columbus, Ohio: The Ohio State University.

Martin, J., & Oehrtman, M. (2010b). Strong metaphors for the concept of convergence of Taylor series. In *Proceedings of the Thirteenth Conference on Research in Undergraduate Mathematics Education*, Raleigh, NC: North Carolina State University.

NCES (2009). *Students who study science, technology, engineering, and mathematics (STEM) in postsecondary education*. National Center for Education Statistics. Washington, DC.

NRC (2011). *Promising practices in undergraduate science, technology, engineering, and mathematics education: Summary of two workshops*. Natalie Nielsen, Rapporteur. Planning

committee on evidence on selected innovations in undergraduate STEM education, National Research Council. Washington, DC: The National Academies Press.
Oehrtman, M. (2008). Layers of abstraction: Theory and design for the instruction of limit concepts. In M. P. Carlson & C. Rasmussen (Eds.), *Making the Connection: Research and Teaching in Undergraduate Mathematics Education* (MAA Notes, Vol. 73, pp. 65-80). Washington, DC: Mathematical Association of America.
Oehrtman, M. (2009). Collapsing dimensions, physical limitations, and other student metaphors for limit concepts. *Journal for Research in Mathematics Education*, 40(4), 396-426.
Piaget, J. (1970). *Structuralism*. (C. Maschler, Trans.). New York: Basic Books, Inc.
PCAST (2012). *Report to the President: Engage to excel: Producing one million additional college graduates with degrees in science, technology, engineering, and mathematics*. President's Council of Advisors on Science and Technology. Washington, DC.
Thompson, P. W., Castillo-Chavez, C., Culbertson, R. J., Flores, A., Greely, R., Haag, S., et al. (2007). Failing the future: Problems of persistence and retention in science, technology, engineering, and mathematics majors at Arizona State University. Tempe, AZ: Office of the Provost.
The White House, Office of the Press Secretary. (2012). President Obama to Host White House Science Fair [Press Release]. Retreived from http://www.whitehouse.gov/the-press-office/2012/02/07/president-obama-host-white-house-science-fair.
Williams, S. (1991). Models of limit held by college calculus students. *Journal for Research in Mathematics Education, 22*(3), 219-236.
Williams, S. (2001). Predications of the limit concept: An application of repertory grids. *Journal for Research in Mathematics Education, 32,* 341-367.

Formal logic and the production and validation of proof by university level students

Sarah Mathieu-Soucy
Université du Québec à Montréal

The practical role and the contribution of formal logic in mathematics is not clear. Some, for example Poincaré (1905), consider that logic is essential to mathematics and others, for example Dieudonné (1987), consider that logic is not useful to mathematics. Mathematicians Thurston (1994) and Thom (1967) claim that their basic (intuitive and theoretical) knowledge of logic is sufficient for their work and that they use different techniques instead which comes, at least in part, from their experience. When it comes to university mathematics students, who don't have as much experience, where do they get the knowledge necessary to do mathematics without making any logical error? Selden & Selden (1999) noted that concepts studied in most beginner courses in formal logic, like Venn diagrams or truth tables, aren't that useful in the everyday mathematics students have to perform. Also, complex logical statements can often be written in multiple simple statements so that the person manipulating them doesn't need to control all the more complex aspects of formal logic. However, among students, gaps in knowledge of formal logic are one of the causes of difficulties in validating and producing proofs (Selden & Selden, 1995). In sum, assessing the usefulness and the necessity of logic in the production and validation of proofs is quite difficult. Hence, it appears worthwhile to address this question: how does knowledge of formal logic changes the way undergraduate mathematics students produce and validate proofs?

To approach this question, we examine different aspects of mathematics that could help us characterize mathematical work, proofs in our case. First, we usually agree that in order to do mathematics, we need to combine intuition and rigor (which includes logic). But what is intuition? In our work, intuition is a feeling that imposes itself to an individual without being able to explain why. This knowledge arises subjectively to an individual as being true (Fischbein, 1982, 1987). Also, it comes from the experiences of each individual and it can be mathematically incorrect. As far as how an individual constructs a proof, we take into consideration 2 different types of productions: a semantic or a syntactic production (Weber & Alcock, 2004). The former relates to a production where the individual uses his intuitive understanding of concepts and meaningful representations of objects. The latter relates to a production where the individual uses only definitions, relations and properties. Finally, regarding the use of logic in mathematical work, we recognize that logical considerations are absent or nearly so from the discourse of educators and textbooks at the beginning of university and consequently from the work of students (Durand-Guerrier & Arsac, 2003). Such considerations are replaced by contextualized reasoning rules, specific to a certain field of mathematics. Their use seems to be directed by the mathematical knowledge of the individual or his mathematical expertise.

With these possible characterizations in mind, we developed a methodology in 2 phases involving university students from Quebec in the second half of a 3-year mathematics program (20-21 years old). First, we evaluated their level of knowledge in formal logic with a written test. Then, considering those results and their academic background in logic, we formed 4 different teams of 2 students to move to the second phase: "task based interviews" (Goldin, 1997). Our analysis of their work suggests that a course in logic changes the way students produce and validate proofs more significantly than a higher level of knowledge in formal logic. Indeed, academic background in logic seems to increase the alertness to logical characteristics (unconscious noticing of logical specifications) and also to slow their progress in unfamiliar context problem. The poster will present more thoroughly the context, the tasked involved in the methodology and the main findings.

References

Dieudonné J. (1987) *Pour l'honneur de l'esprit humain,* Paris : Hachette

Durand-Guerrier, V. & Arsac, G. (2003). Méthodes de raisonnement et leurs modélisations logiques. Spécificité de l'analyse. Quelles implications didactiques ? *Recherches En Didactique Des Mathématiques*, *23*(3), 295–342.

Fischbein, E. (1982). Intuition and Proof. *For the Learning of Mathematics*, *3*(2), 9–19.

Fischbein, E. (1987). *Intuition in Science and Mathematics. An educational Approach* (p. 225). Dordrecht : D. Reidel Publishing Cie.

Goldin, G. (1997). Observing Mathematical Problem Solving through Task-Based Interviews. *Journal for Research in Mathematics Education. Monograph*, *9*, 45–62.

Poincaré, H. (1905). *La valeur de la science*. (Flammarion, Ed.). Paris.

Selden, A. & Selden, J. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*, *29*(2), 123–151.

Selden, A. & Selden, J. (1999). The role of logic in the validation of mathematical proofs.

Thom, R. (1974). Mathématiques modernes et mathématiques de toujours, suivi de Les mathématiques « modernes », une erreur pédagogique et philosophique ? In R. Jaulin (Ed.), *Pourquoi la mathématique ?* (pp. 39–88). Paris: Éditions 10-18.

Thurston, W. P. (1994). On proof and progress in mathematics. *Bulletin of the American Mathematical Society*, *30*(2), 161–177.

Weber, K. & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics*, *56*(2-3), 209–234.

Mathematics majors' example and diagram usage when writing calculus proofs

Authors
Affiliation

Introduction

We report on a study in which we observed 73 mathematics majors completing seven proving tasks in calculus. We use these data to empirically address several hypotheses from the undergraduate proving literature. The key findings from this study include: (a) Nearly all participants introduced diagrams and examples on multiple tasks, (b) few students relied predominantly on either semantic reasoning or syntactic reasoning, and (c) there was little correlation between one's propensity to use examples or diagrams and one's mathematical achievement, either on the proof-writing tasks or on GPA in advanced mathematics courses.
Each finding is inconsistent with claims from the mathematics education literature. These inconsistencies are discussed at the end of the paper.

Key words: Examples, Diagrams, Visual Reasoning, Proof, Calculus.

While proof is expected to play a central role in elementary and secondary classrooms (e.g., NCTM, 2000, Schoenfeld, 1994), proof assumes even greater importance in advanced mathematics courses. In these courses, a primary goal of instruction is to increase students' ability to write proofs about the course content, and assessments of students' understanding of this content are largely composed of proving tasks.

There is a large number of studies that suggest that mathematics majors have difficulty writing proofs, even after completing courses in advanced mathematics (e.g., Hart, 1994; Iannone & Inglis, 2010; Ko & Knuth, 2009; Moore, 1994; Author). In each study, students were asked to complete a set of proving tasks and their frequency of success was under 50%. In many studies, students' performance was alarmingly poor. For instance, Ko and Knuth (2009) found that none of the 36 mathematics majors in their study could successfully complete any of the three assigned proving tasks in the study.

To account for these difficulties, many researchers have investigated how the use of examples or diagrams may facilitate students' proof-writing abilities (e.g., Gibson, 1998; Lockwood, Ellis, Dogan, Williams, & Knuth, 2013; Sandefur, Mason, Stylianides, & Watson, 2013; Author) and suggested that some students' proving difficulties may be due to their reluctance to employ diagrams and examples (e.g., Moore, 1994; Raman, 2003; Author). However, as we will argue, claims about students' propensity to employ this type of reasoning and their success at doing so are necessarily tentative due to the small sample sizes typically employed in studies of mathematics majors' proof productions. In this paper, we seek to address these questions using a larger sample of students (73 mathematics majors) and a larger number of tasks (seven) than are used in most other studies on proving processes.

Theoretical perspective

In recent years, a number of researchers have remarked that a student might approach a proof construction task in two qualitatively different ways. Weber and Alcock (2004) distinguished between a student who writes a proof with a focus on formal deduction, logical manipulation, and calculation, and a student whose proof is based on inferences drawn from informal representations of mathematical concepts. They call the former a syntactic proof production and the latter a semantic proof production. Several authors have made similar theoretical distinctions (e.g., Burton, 2004; Garuti, Boero, & Lemut, 1998; Raman, 2003; Vinner, 1991).

A commonality in each of these distinctions is an awareness that there is a difference between the process used to write a proof (which may be formal or informal) and the product obtained (where the proof must satisfy certain formal constraints, cf. Boero, 1999). Although one should not draw inferences from diagrams or generalize from specific examples in the formal proof that one submits, such reasoning can be vital in the process of constructing this proof. Another shared feature amongst these constructs is that they distinguish, either explicitly or implicitly, between the types of representations that one uses to construct proofs, with the informal proof productions employing diagrams and considering specific examples and the formal productions relying on formal definitions and calculation. In this paper, we follow Weber and Alcock (2009), who defined *syntactic reasoning* (or formal reasoning) to be reasoning based on representations of concepts that would be permissible in a proof (definitions, formulas, mathematical assertions written with appropriate clarification in a verbal-symbolic mode) and *semantic reasoning* (or informal reasoning) to be based on representations of concepts that would not be permissible in a proof (including graphs, diagrams, and examples).

Related literature

The research literature on students' proving processes, which typically consists of studies with small samples, has produced a number of theoretical contributions, including the generation of valuable constructs, explanations for why students have difficulty with proof, and suggestions for how students' proof construction might be improved. However, because of the lack of studies with larger samples, it has been problematic to make claims about the proving behaviors of the larger population of mathematics majors. Instead, such contributions are typically framed as hypotheses that warrant further testing in studies with larger samples. The goal of the current paper is to investigate some of the hypotheses that pertain to semantic and syntactic reasoning with a larger study. In this section, we list the evidence for three such hypotheses and then pose research question that we aim to address.

1. Mathematics majors are reluctant to engage in semantic reasoning when writing proofs. In several small-scale studies, Moore (1994), Author, and Weber and Alcock (2004) observed that most mathematics majors generally did not consider examples or diagrams when given a proof construction task and, indeed, attributed their lack of example usage as one reason why they struggled to write proofs. In another small-scale study, Raman (2003) reached a similar conclusion and postulated that students' reluctance to engage in writing proofs based on key ideas was due to students' epistemology. Raman (2003) conjectured that students believed that there was no link between conceptual/visual reasoning and the process of construction a proof. Consistent with this viewpoint, some researchers have suggested that students have overgeneralized the maxims "you cannot prove by pictures" (e.g., Author) and "you cannot prove by example" (e.g., Harel, 2001) to infer that diagrams and examples are not useful in the process of writing a proof.

2. *Most mathematics majors either show a strong propensity or a strong reluctance to engage in semantic reasoning.* Some researchers have claimed that many mathematics majors have a "proving style". That is, these students will either engage with semantic reasoning frequently on proving tasks, or alternatively, they will rarely do so and rely on syntactic reasoning instead. For instance, Alcock and Simpson (2004, 2005) and Pinto and Tall (1999) each conducted semester-long studies on students' reasoning in a real analysis course. In both cases, the research teams observed that throughout the semester, roughly half of their sample continually showed multiple markers of visual reasoning on the tasks they were asked to complete (including proof-writing tasks) and the other half of the sample rarely or never did so. Alcock and Inglis (2008) contended that these findings suggest that most students can be classified as syntactic reasoners or semantic reasoners. For a more comprehensive review of this literature, see Author.

3. There is a relationship between undergraduates' diagram and examples usage and their success on proof-writing tasks. Several researchers have reported qualitative studies illustrating how mathematics majors used diagrams (e.g., Gibson, 1998) and examples (e.g., Sandefur, Mason, Stylianides, & Watson, 2013) to successfully write proofs. These authors' in-depth analysis showed how these students' use of informal representations allowed them to make progress in proof construction tasks. Gibson (1998) makes this point clear by noting, "using diagrams helped students complete sub-tasks that they were not able to complete while working with verbal-symbolic representation systems alone" (p. 284). These findings, coupled with the findings of Author and Moore (1994) who blamed students' failure to write proofs partially on a predominance of syntactic reasoning, suggest mathematics majors perform better on this type of tasks when they use diagrams and examples. Furthermore, these findings are buttressed by studies in which mathematicians have been observed to effectively use examples and diagrams on proving tasks (e.g., Lockwood, Ellis, Dogan, Williams, & Knuth, 2013; Schoenfeld, 1985). On the other hand, this hypothesis is challenged by findings from researchers who reported seeing no link between students' propensity to engage in semantic reasoning and their mathematical achievement (e.g., Alcock & Simpson, 2004, 2005; Author; Pinto & Tall, 1999).

Research questions

We use data from videotapes of 73 students writing proofs to address the following questions.

(1) What is the distribution of students' use of semantic reasoning?
   a. Do many students use semantic reasoning on few of the tasks that they are given, signifying a reluctance to use semantic reasoning?
   b. Do many students use semantic reasoning on all of the proving tasks they were assigned or none of the proving tasks that they were assigned, which would suggest that they have a strong propensity to rely on or not use semantic reasoning?

(2) Is the frequency of students' use of semantic reasoning correlated with their academic achievement (i.e., proof writing success and grades earned in proof-oriented mathematics courses)?

Methods

Participants. Participants were recruited from a large state university in the northeastern United States over the course of four semesters. After each semester, every mathematics major who completed a proof-based course in linear algebra was sent an e-mail inviting them to participate in this study in exchange for financial compensation, of which 73 students agreed to participate. This linear algebra course was typically taken by mathematics majors in their senior year and a transition-to-proof course was a prerequisite for taking this class.

Procedure. In this study, participants were asked to "think aloud" as they completed seven calculus proof construction tasks (given in the Appendix). They were video-recorded as they completed these tasks and they were asked to write up their final proofs as if they were handing them in for the final exam of a course. Participants were given the seven calculus proving tasks, one at a time, in a randomized order. For each task, they had two resources available. First, they were permitted to use graphing calculator software on a computer, which permitted them to perform basic arithmetic operations and graph functions with a high degree of resolution. Second, they were given a packet that described the concepts referred to in the proving tasks. For each concept, the formal definition, an example, and when appropriate, a graphical interpretation of the concept were provided. For instance, for the concept of even function, the packed provided the formal definition (a function $f(x)$ was even if $f(x)=f(-x)$ for all real-values x), $f(x)=x^2$ as an example of an even function, and the graphical interpretation that functions were even if their graphs were reflected across the *y*-

axis. This was done to minimize the role that (a lack of) content knowledge would play in participants' proof productions.

Participants were given ten to fifteen minutes to complete each task. Participants were allowed to stop working on a task at any time they either felt that they had obtained a written solution for the problem, or felt that they could no longer make any productive progress on the task. After ten minutes had elapsed, and if the interviewer felt no productive progress was being made, the interviewer would suggest that the participant move on to the next problem. After fifteen minutes, the work on the problem was terminated. Participants generally stopped making progress before this point was reached. This timing is similar to students' exams, where they are typically asked to write four or five proofs in a 50-minute period.

Analysis. For the sake of brevity, we describe only a simplified subset of our analytical scheme here. The complete scheme, as well as its development and rationale, is provided in Author.

Two individual coders each coded 60% of the dataset. To ensure inter-rater reliability, 20% of the dataset (chosen at random) was coded by both coders. In the simplified coding scheme, the coders noted when participants generated an informal representation of a mathematical concept that could not be used in a chain of deductions in a formal proof (e.g. graphs, diagrams, and examples of the concept). Whenever this occurred we coded the corresponding proof attempt as a semantic proof attempt. In cases where this did not occur, the proof attempt was coded as a syntactic proof attempt. This simplified coding scheme was advocated by Alcock and Inglis (2009). One objection that we (and others) have raised is that this dichotomous scheme is too coarse for some analytical purposes (e.g., Sandefur, Mason, Stylianides, & Watson, 2013; Author). Therefore, we also created and used a more flexible and fine-grained scheme that is too lengthy to describe in this report. Happily, our results were consistent across the semantic-syntactic coding schemes that we used. The coders also scored each proof as being (a) completely correct, (b) mostly correct with minor and inconsequential errors, (c) an attempt with significant errors, but one in which substantial progress was made, and (d) an attempt in which no substantial progress was made. Proofs that were coded as (a) and (b) were labeled as "correct", otherwise they were labeled as "incorrect". Finally, participants in this study had all completed a transition-to-proof course, real analysis, and a second course in linear algebra. Their Math GPA score was their aggregate GPA in these three courses and was used as a measure of course achievement.

Results

A summary of the measures we used is presented in Table 1 and the distribution of students' semantic proof productions is given in Table 2.

Table 1. Summary of measures used in this study

Measure	Min	Range Max	Mean	Standard Deviation	Inter-rater Reliability*
Valid Proof	0	6	1.77	1.49	κ = .90 (96%)
Math GPA	1.33	4.00	3.23	1.75	N/A
Semantic Proofs	1	7	4.25	1.55	κ =.91 (96%)

*- Inter-rater reliability was done on a per-item basis. Cohen's kappa (κ) was used with absolute levels of agreement in parentheses.

Table 2. Distribution of students' productions

Number of semantic proof productions	0	1	2	3	4	5	6	7
Number of students	0	5	4	12	21	13	14	4

Table 2 is inconsistent with two hypotheses made in the literature. The first is that students are reluctant to use semantic reasoning on proof-writing tasks. All students used this on at least one task and 52 of the 73 students (71%) used semantic reasoning on the majority of the tasks. The second is that many students would be extreme in their use of semantic reasoning. The majority of the students-- 46 out of 73 (63%)-- used semantic reasoning on 3, 4, or 5 of the tasks. This grouping in the middle is not what one would predict if students consistently engaged in semantic reasoning or consistently declined to do so.

We found no strong relationship between students' use of semantic reasoning and their mathematical achievement. The correlation between the number of students' semantic proof productions and correct proofs was weak (r=.013) as was the correlation between the students' semantic score and Math GPA (r=.119). We considered the possibility that perhaps that extreme use of semantic reasoning was deleterious by conducting curvilinear regressions using the students' semantic score as the independent value and correct proofs and Math GPA as independent variables but found no significant correlation (R^2=.002 and R^2=.012 respectively). Finally, we separated semantic usage by diagrams and formulas (for specific example objects) but still found no positive relationship between these factors and mathematical achievement (r <.12 in all cases).

As a final note, we conducted similar analyses using several continuous measures for students' semantic-syntactic scores based on the amount of semantic reasoning that students used in their proof attempts and obtained similar results.

Discussion

The results of this study are inconsistent with several hypotheses about students' proof-writing behavior that are posed in the literature. First, students in our study did not appear to be reluctant to use semantic reasoning in their proof writing. We note that two of the studies in which this reluctance was observed were situated in the domains of group theory (Author) and elementary logic (Moore, 1994). It might be the case that what these researchers observed was not reluctance *per se*, but rather students lacking access to useful informal representations of the concepts they were studying, an account we advanced to explain one students' behavior in Author.

Second, the fact that students did not exhibit "proving styles", or show a strong consistency with their semantic reasoning usage (or lack thereof), contradicts some claims in the mathematics education literature. However, this would not be surprising to some educational psychologists. In summaries of the literature, Coffield et al. (2004) and Kirschner and Marrienboer (2013) noted that both teachers and researchers often classified students as having a particular cognitive style on insufficient evidence and these narratives can skew the researcher or teacher's perceptions of these students. In particular, judgments about students' cognitive styles are frequently made based on students' self-reports, or on their behavior on a small number of tasks (sometimes a single task), both of which are considered to be unreliable sources of evidence (Coffield et al, 2004). In Author, we reported that many studies claiming to document a consistency in one's propensity to use semantic reasoning suffered from the limitations that Coffield et al. described.

The most significant result from this study is the limited relationship between students' use of semantic reasoning and their mathematical achievement. This result challenges the calls of many to give more emphasis to example-based or diagrammatic reasoning when proof-writing (Garuti, Boero, & Lemut, 1999; Gibson, 1998; Sandefur, Mason, Stylianides, & Watson, 2013; Raman, 2003; Zaslavsky, 2014).

Of course, the current study is not without its limitations-- most notably, the tasks in this study were in only one domain (calculus) and it is possible that different results might have been obtained in other domains such as number theory or topology. To avoid misinterpretation, we do not wish to claim that we have conclusively answered the question on the relationship between semantic reasoning and mathematical achievement. Nonetheless, we can offer a conjecture for why our results appear to contradict much of the literature. The papers cited above each consisted of showing qualitative studies on how examples and diagrams *can* be effective. We think the theoretical claims of these studies are correct-- both examples and diagrams offer affordances for proof-writing that the formal-symbolic notation of mathematics does not, and students can sometimes take advantage of these affordances. However, there are also studies showing how the use of examples and diagrams can cause students *difficulties* and can *inhibit* proof productions (e.g, Author; Pedemonte, 2007). When we ask the question of whether using examples or diagrams correlates with mathematical achievement, we are no longer just asking if these results *can* help students write proofs (they can), but whether the benefits of using examples and diagrams offset the challenges that are associated with using them. Our data suggest that they do not. Of course this does not mean that the suggestion for mathematics majors to use examples and diagrams in their proof writing is wrong; it only implies the suggestion needs to be more nuanced. Students must learn how to use these *effectively* which we believe involves first obtaining a more fine-grained understanding of how mathematicians use examples and diagrams, and the difficulties that students face when they try to use them.

References

Alcock, L. & Inglis, M. (2008). Doctoral students' use of examples in evaluating and proving conjectures. *Educational Studies in Mathematics 69*, 111-129.

Alcock, L. & Inglis, M. (2009). Representation systems and undergraduate proof productions. *Journal of Mathematical Behavior*, 28, 209-211.

Alcock, L. & Simpson, A. P. (2004). Convergence of sequences and series: Interactions between visual reasoning and the learner's beliefs about their own role. *Educational Studies in Mathematics, 57*(1),1-32.

Alcock, L. & Simpson, A. (2005). Convergence of sequences and series 2: Interactions between non-visual reasoning and the learner's beliefs about their own role. *Educational Studies in Mathematics, 58*, 77-110.

Boero, P. (1999, July/August). Argumentation and mathematical proof: A complex, productive, unavoidable relationship in mathematics and mathematics education. *International Newsletter on the Teaching and Learning of Mathematical Proof.* Retrieved August 2, 2014, from http://www.lettredelapreuve.it/OldPreuve/Newsletter/990708Theme/990708ThemeUK.html

Burton, L. (2004). *Mathematicians as Enquirers: Learning about Learning Mathematics.* Berlin: Springer.

Coffield, F., Moseley, D., Hall, E., & Ecclestone, K. (2004). *Learning styles and pedagogy in post-16 learning: A systematic and critical review.* London: Learning & Skills Research Centre.

Garuti, R., Boero, P., & Lemut, E. (1998). Cognitive unity of theorems and difficulty of proof. *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 345-352). Stellenbosch, South Africa.

Gibson, D. (1998). Students' use of diagrams to develop proofs in an introductory real analysis. *Research in Collegiate Mathematics Education 2*, 284-307.

Harel, G. (2001). The development of mathematical induction as a proof scheme: A model for DNR-based instruction. In S. Campbell & R. Zaskis (Eds.), *Learning and teaching number theory: Research in cognition and instruction* (pp. 185-212). New Jersey: Ablex Publishing Corporation.

Hart, E. (1994). A conceptual analysis of the proof writing performance of expert and novice students in elementary group theory. In J. Kaput, and Dubinsky, E. (Ed.), *Research issues in mathematics learning: Preliminary analyses and results* (pp. 49-62). Washington: Mathematical Association of America.

Kirschner, P. & Marrienboer, J. (2013). Do learners really know best? Urban legends in education. *Educational Psychologist*, 48, 169-183.

Ko, Y. & Knuth, E. (2009). Undergraduate mathematics majors' writing performance producing proofs and counterexamples about continuous functions. *Journal of Mathematical Behavior*, 28, 68-77.

Lockwood, E., Ellis, A. B., Dogan, M. F., Williams, C., & Knuth, E. (2012). A framework for mathematicians' example-related activity when exploring and proving mathematical conjectures. In *Proceedings of the 34th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 151-158).

Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics, 27*(3), 249-266.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics*, 66, 23-41.

Pinto, M., & Tall, D. (1999). Student constructions of formal theory: Giving and extracting meaning. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 65–73). Haifa, Israel: PME.

Raman, M. (2003). Key ideas: What are they and how can they help us understand how people view proof? *Educational Studies in Mathematics, 52,* 319-325.

Sandefur, J., Mason, J., Stylianides, G. J., & Watson, A. (2013). Generating and using examples in the proving process. *Educational Studies in Mathematics*, *83*(3), 323-340.

Schoenfeld, A. H. (1985). *Mathematical Problem Solving*. Orlando, FL: Academic Press.

Schoenfeld, A. H. (1994). What do we know about mathematics curricula? *Journal of Mathematical Behavior, 13*, 55-80.

Vinner, S. (1991). The role of definitions in teaching and learning. In. D. Tall (Ed.) *Advanced mathematical thinking. (pp. 65-81).* Dordrecht: Kluwer.

Weber, K. & Alcock, L. J. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics, 56* (3), 209-234.

Weber, K. & Alcock, L. J. (2009). Proof in advanced mathematics classes: Semantic and syntactic reasoning in the representation system of proof. In D.A. Stylianou, M.L. Blanton & E. Knuth (Eds.), *Teaching and Learning Proof Across the Grades: A K-16 Perspective* (pp. 323-338). New York, NY: Routledge.

Zazlavsky, O. (2014). Thinking with and through examples. In *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education*. Vancouver, CA.: University of British Columbia.

Appendix

C1: Suppose $f(0) = f'(0) = 1$. Suppose $f''(x) > 0$ for all positive x. Prove that $f(2) > 2$.

C2: Prove that the only real solution to the equation $x^3 + 5x = 3x^2 + \sin x$ is $x=0$.

C3: Suppose $f(x)$ is a differentiable even function. Prove that $f'(x)$ is an odd function.

C4: Prove that $a^2 + ab + b^2 \geq 0$ for all real numbers *a* and *b*.

C5: Suppose $f''(x) > 0$ for all real numbers *x*. Suppose *a* and *b* are real numbers with $a < b$. Define $g(x)$ as the line through the points $(a, f(a))$ and $(b, f(b))$. Prove that for all $x \in [a,b]$, $f(x) \leq g(x)$.

C6: Prove that $\int_{-a}^{a} \sin^3(x)\,dx = 0$ for any real number *a*.

C7: Let f be differentiable on $[0,1]$, and suppose that $f(0) = 0$ and f' is increasing on $[0,1]$. Prove that $g(x) = \frac{f(x)}{x}$ is increasing on $(0,1)$.

Determining What To Assess: A Methodology For Concept Domain Analysis As Applied To Group Theory

Kathleen Melhuish
Portland State University

With the rise of concept inventory style student assessment, focus has been placed on research-based refinement of assessment questions. Of equal importance is how one arrives at the tasks meant to reflect a given construct. This report will consider how this might be done in a research-based manner. The creation of a Group Concept Inventory (with particular attention to isomorphism) will be utilized to illustrate the approach. The methodology attempts to add rigor to vague suggestions of using textbooks, consulting experts, and referencing literature when developing assessment tasks. Various aspects of understanding concepts and attention to which concepts are essential to group (at an introductory level) will be discussed. The methodology and accompanying example will detail a Delphi process for expert consensus, a narrative and exercise textbook analysis and a thorough literature exploration.

Key words: Assessment, Abstract Algebra, Concepts, Textbook Analysis

Accurately assessing student understanding remains of primary concern in education fields. Often in mathematics, assessing procedures becomes the focus to the detriment of assessing conceptual understanding. This is unsurprising for two reasons: 1.) An emphasis on conceptual understanding is relatively new (especially at the tertiary level). 2.) Traditional summative assessment methods are rarely targeting these understandings. To illustrate this point, consider the introductory calculus course. There has been a huge push for reform and conceptual assessment, and yet a recent analysis of exams indicated nearly all tasks were procedural in nature (Tallman & Carlson, 2012).

In advanced mathematics courses (such as abstract algebra and analysis), there has not that degree of a push towards conceptual understanding. These courses tend to focus on the formal proof. While assessing students based on formal proofs is certainly vital for their growth as mathematicians, we do a disservice if it is to the neglect of concept understanding. In fact, the abstract algebra literature shows that students often lack complete and accurate understandings of many group theory topics (Dubinsky, 1997; Hazzan, 1999; Leron, Hazzan, & Zazkis, 1995). Having robust understandings of these topics should be an essential goal in an introductory course. Further, research has shown the vital role understanding of concepts can play within the formal proving activity (see Moore, 1994; Weber & Alcock, 2004).

One successful model for shifting the focus to concepts comes in the form of the Force Concept Inventory (Hestenes, Wells, & Swackhamer, 1992) from physics education. The inventory is a multiple-choice assessment whose goal was to determine if students' had a "coherent conceptual system" (p. 14) underlying their understanding of Newtonian physics. The Force Concept Inventory is credited with helping to instigate the reform movement (Savinainen & Scott, 2002) due to its conceptual focus, ease of use and widespread adoption. In mathematics, several concept assessments have been modeled after the inventory, notably calculus (Epstein, 2007) and precalculus (Carlson, Oehrtman, & Engelke, 2010).

Such an instrument in advanced mathematics courses does not exist. Recently, Larsen, Bartlo and Johnson (2014) called for the "creation of validated assessment instruments" (p. 709) in group theory for evaluating instructional innovations. A concept-driven multiple choice instrument could allow for the scaling up of explorations of student understandings of concepts, provide a tool for instructors to quickly have information about their students' understanding, provide a way to assess programs, and optimistically, bring awareness to this conversation we are often not having in mathematics departments.

With the neglect of concepts in traditional assessment, developing this type of instrument requires a method for determining what is relevant to the concept at hand (that of groups at an introductory level) and what is representative. It is impossible to make an instrument that tests every facet and so important decisions are made at the beginning of this process. Methodologies for these assessments often begin in earnest when discussing the creation of questions leaving the earlier decisions of what types of content and tasks to include vague. This report will provide a more rigorous methodology for domain analysis (prior to task creation) utilizing expert opinion, textbook analysis and research literature to provide a well-rounded view of concepts essential to *group* at the introductory level.

Assessments and Domain Analysis

Lindell, Peark, and Foster's (2007) meta-analysis of concept inventories provided a list of typical steps during concept inventory creation (see Table 1). While these steps existed, many were not explained in rigorous fashion. The designs focus largely on the latter steps where student interviews are used to provide arguments that student thinking is being reflected in the tasks and answer selections (see Carlson, Oehrtman & Engelke (2010) for a detailed methodology for steps 4-9). Lindell, Peark and Foster identified *content validity* defined as, "the degree to which an inventory measures the content covered in the content domain" (p. 15) as often incomplete or vaguely addressed.

Of particular importance and neglect is step 2. Lindell, Peark and Foster found many researchers relying on their own expertise to determine the concept domain. Typically, if addressed at all, concept inventories referenced utilizing textbooks, experts or literature as ways to determine the concept domain. In Messick's (1995) discussion of validity, he referred to this process as *domain analysis.* The analysis involves identifying foundational concepts, essential task, and established areas of difficulty. Mislevy, Steinberg, and Almond (2004) describe domain analysis as:

> …marshaling substantive information about the domain—bringing together knowledge from any number of sources and the beginning to organize beliefs, theories, research, subject-matter expertise, instructional materials, exemplars from other assessments and so on (p. 7).

Because student understanding in abstract algebra is not a well-explored domain, determining the concept domain requires a triangulation of sources to establish validity. This work aims to describe an explicit methodology for step 2.

Table 1

Steps for Concept Inventory Development

1. Identify a purpose
2. Determine the concept domain
3. Prepare test specifications
4. Construct initial pool of items
5. Have items reviewed- revise as necessary
6. Hold preliminary field testing of items – revise as necessary
7. Field test on large sample representation of the examinee population
8. Determine statistical properties of item scores- eliminate inappropriate items
9. Design and conduct reliability and validity studies (Lindell et al., 2007, p. 15)

Conceptual Understanding in Advanced Mathematics

As noted by Lindell et al. (2007), the precursor to domain analysis, is identifying a purpose. For the case presented here, the goal is to capture a measure of conceptual understanding in group theory. I will be adopting Star's (2005) approach where conceptual knowledge includes both Hiebert and Lefvre's (1986) knowledge rich in relationships and "knowledge of concepts" (p. 408). Concepts in advanced mathematics have often been discussed in terms of *objects.* Understanding of objects can arise in various ways including compressing of processes, abstracting structure from experience, and working from a formal definition (see Tall's (2004) three worlds of mathematics for a thorough discussion.) Knowledge needs to be compressed into *thinkable concepts* in order to "build a coherent connected mental structure to put significant ideas together" (Tall, 2007, p. 153).

Tall and Vinner's (1981) seminal work on concept image and definition unveiled that student understanding of concepts extends far beyond their knowledge of definition and includes "all mental pictures and associate properties and processes" (p. 152). Further, concept images do not require coherence and often only portions are evoked at a given time. Concept inventories aim to capture coherence. Savinainen and Viiri (2008) introduced a framework to reflect conceptual coherence as found in the Force Concept Inventory which included three dimensions: relating different concepts, being able to apply knowledge in appropriate contexts and being able to move between representations. This framework is similar to Biehler's (2005) meanings of mathematical concepts where he identified the *domain of application, relations* to other concepts and *representations* available for working with the concept. Attending to understanding of concepts is a multi-faceted exploration.

The Methodology for Domain Analysis

This methodology will include three phases for domain analysis: expert consensus, analyzing textbooks, and scouring literature.

The Delphi Technique

The Delphi Technique is a process for reaching a conclusion for ill-defined problem via expert consensus. In subjective areas, experts can play an essential role when building consensus. The Delphi technique is unique in its ability to allow for experts to reflect on each other's opinions while mitigating for perceived hierarchy in opinion which can occur in round-table discussions (Streveler, Olds, Miller, & Nelson, 2003). The technique typically consists of four rounds were opinions are shared and rankings compiled. My rounds were as follows:

Pass 1: Experts were asked to compile a list of concepts they think are essential in introductory group theory.

Pass 2: A list was compiled of all concepts mentioned by at least two experts. The experts then ranked each topic on a scale from 1-10 for difficulty and importance.

Pass 3: The experts were provided with the 25, 50, and 75 percentile scores from both categories and asked to rank, again. During this pass, the experts provided justifications for any ranking outside of the 25-75 percent range.

Pass 4: Experts were provided with the same numerical information as well as the new justifications and asked to provide a final ranking.

It is essential to choose a heterogeneous panel to best capture a multitude of experience. Of the 13 panelists who participated all had taught this type of course (ranging from 2.5 to 25 times). The panel had 4 algebra textbook authors; 5 mathematics education researchers who have published related to abstract algebra; 8 mathematician instructors (with research focuses ranging from math history and analysis to group theory specialists.) A measure of center is used to make the final decision on inclusion. For this panel, any topic that had a mean of at least a 9 in importance was identified to be part of further analysis.

Textbook Analysis

The textbook analysis consisted of first establishing which textbooks are currently in use via a random sample. Of the 1,244 schools with a mathematics major, a random sample of 294 institutions were surveyed regarding textbook usage (95% CI with interval of +/- 5%) In schools where the textbook is not uniform, the textbook most recently utilized was included. Any textbook used by at least 20 schools was included for analysis. (This number was eventually lowered to include the 4th most popular textbook.)

Table 2

Representation Codes

Group - Verbal Description	Map – Symbolic Rule
Group - Symbolic Name	Map – Defined Element-Wise
Group - Table	Map – Function Diagram
Group – Elements and operation	Map - Defined on Generating Set
Group – Set Builder Notation	Map – Verbal Description
Group – Cayley Digraph	Map – Visual Other
Group – Geometric Representation	
Group – Defined by generating set	

Each of the representative texts was analyzed based on the list of concepts identified by the expert panel. The analysis was driven by the identified components of conceptual understanding discussed above. The narrative was analyzed with attention to examples and informal/formal definitions. Each example was coded for representation, purpose, and topic. The representation codes were adapted from Mesa's (2004) functions including tables, symbolic and verbal, but were elaborated and refined through the coding process to more accurately reflect representations in group theory (see Table 2). Both groups and mappings were coded by representation type. The examples' purposes were coded initially based on Fukawa-Connelly and Newton's (2014) group example analysis and similarly evolved based on the textbooks (see Table 3).

Table 3

Example Purposes

Example motivating a definition (EMD)
Example of concept following definition (EFD)
Example illustrating a specific property a concept does or does not have (EP)
Example illustrating how to calculate or determine something (EC)
Example illustrating a proving technique (ET)
Example motivating a theorem (EMT)
Example illustrating a theorem (EIT)
Example using a theorem (EUT)
Example illustrating a notation (EIN)

The relevant exercises were then coded using the same representation codes with the addition of expected student activity. These codes evolved to particularly reflect the purpose of finding activities associated with concepts. The codes included: *using a definition to prove a direct consequence*, *showing an example is an instance (or non-instance) of a concept*, *evaluating if an example is an instance of a concept*, and *switching representations*.

Each example and exercise were coded by the author. To establish reliability, a second coder (a mathematics education PhD student) coded a subset of sections. She coded one complete section for each book with varying topics to best capture a variety of codes. Each example and exercise was given a percentage of agreement. (Calculated via number of agreed upon codes divided by total number of codes from either coder.) The total agreement amongst exercise codes was 83.4% and examples was 81.4%.

Literature Review

Each identified concept was then searched for in mathematics education literature to identify known sources of difficulty. This was done with three parts in order to maximize literature found:

1. A full text search for each concept (ex: isomorphism) in each of the 35 journals identified by SIGMAA on RUME as research-based and relevant to undergraduate mathematics (RUMEonline!, 2011).

2. Title search within conference proceedings available from major mathematics education conferences including PME, RUME, DELTA, CERME and ICME.

3. Search for "Group Theory" and "Abstract Algebra" in ERIC database.

From these results, a list of documented student conceptions for each topic was compiled along with corresponding tasks.

Each of these components was developed with consideration to the purpose of the instrument (the many facets of conceptual understanding of group) and domain analysis goals of identifying what tasks are valued, typical and where known areas of difficulty may exist.

Results

Topics for inclusion as identified by the Delphi process are: *binary operation, associativity, identity, inverse, group, modular groups, abelian groups, cyclic groups, order of a group, order of an element, subgroup, isomorphism, homomorphism, cosets, normal subgroups, quotient groups, Lagrange's Theorem,* and *the Fundamental Homomorphism Theorem*. For brevity's sake, the results will include discussion of one full example: isomorphism as well as short general notes.

The textbook surveying indicated that 32% of schools use Gallian (2009), 15% use Fraleigh (2003), 8% use Gilbert and Gilbert (2008), and 6% use Hungerford (2012). No other textbooks were above 3%. (Although, 5% of schools did not use a textbook for their course.)

Formal and Informal Definitions - Isomorphism: Isomorphism formalizes an idea essential to an advanced understanding of groups: What makes two groups the same? As discussed in Leron, Hazzan, & Zazkis (1995), isomorphism has both a naive and formal definition. Students could conceive as an isomorphism as a relabeling of the same group or/and attend to isomorphism being a bijective homomorphism. This duality was reflected in the narratives of all four textbooks (see Table 4).

Table 4

Informal Characterization of Isomorphism

Textbook	Excerpt
Gilbert & Gilbert (2008)	"They are algebraically the same, although details such as the appearance of elements or the rule defining the operation may vary."
Hungerford (2012)	"At first glance, these groups don't seem the same. But we claim that they are 'essentially the same', except for labels on the elements."
Gallian (2009)	"...the same group is described with different terminology."
Fraleigh (2003)	"These four tables differ only in the names (or symbols) for their elements and in the order that those elements are listed as heads in the tables."

General Trends: Isomorphism was unique in having both formal and informal descriptions. No other concept identified had an informal description found across textbooks. Typically, informal descriptions were lacking or alternately, the informal description was merely what we called a *translation* of the formal language.

Examples – Isomorphisms. The examples across all textbooks predominately served the purpose to provide an instantiation after a definition. Each book also contained at least one motivating example. In three of texts, this motivating example used the idea of language providing an isomorphism (saying the numbers in French or using Roman Numerals does not alter the structure of the number system.) The fourth text began by utilizing group examples from earlier in the book. Fraleigh (2003), and Hungerford (2012) included non-examples which utilized structural properties to show groups were not isomorphic. Gallian (2009) included a non-example of a particular map that did not meet the onto requirement. Gilbert and Gilbert (2008) included no non-examples.

General Trends: The isomorphism section more frequently had motivating examples than other sections. However, the purpose of most examples to just instantiate a definition was consistent across sections and books. Occasionally, non-examples were included (especially in the case of a group), but were far scarcer.

Isomorphism Representations: Three of the four textbooks used tables to highlight the "relabeling" aspect of isomorphism in the narrative. The map representation was more typically a symbolic assignment rule or an explicit list of where each element was mapped. The symbolic rule dominated isomorphism sections (92% of representations). Alternate representations (such as a function diagram) were not found in the isomorphism section.

General Trends: Mapping representations were predominately symbolic throughout the narrative and exercises of the texts. Groups were predominately represented by a symbol or name (with a small minority of verbal descriptions.) While alternate representations (such as tables or function diagrams) were occasionally introduced in narrative, they were rarely used within exercises.

Isomorphism Activity: The expected activities associated with isomorphism reflect some of the dualities from the literature. Some texts emphasized showing (or determining if) a map is an isomorphism whereas others emphasized showing two groups were isomorphic (see table 5). All texts had some number of problems showing two groups are not isomorphic with three of the four texts having a significant amount of exercises of this nature.

Table 5

Expected Student Activity

Activity	Fraleigh (2003)	Gallian (2009)	Gilbert & Gilbert (2008)	Hungerford (2012)
Show a given map is an isomorphism	5%	25%	10%	16%
Show two groups are isomorphic	8%	19.5%	20%	29%
Show two groups are not isomorphic	2.5%	19.5%	20%	29%
Evaluate if a map is an isomorphism	38.5%	0%	0%	0%

Evaluate if two groups are isomorphic	0%	0%	10%	3%
Show isomorphism preserves a property	13%	5.5%	13%	3%
Find an isomorphism between two groups	20%	3%	30%	0%
Other	13%	28%	17%	19%

General Trends: The most frequent activity was showing that some example fulfilled the requirements to be a concept. Next was showing an instance was a non-example and evaluating if an instance was an example of a given concept. Secondary was the use of the definition or theorem (in the case of Lagrange's Theorem and the Fundamental Homomorphism) to arrive at an immediate consequence.

In the Literature: Several papers have addressed isomorphism in the literature. Studies have found students often will determine if groups are isomorphic by considering if they are equi-numerous (Dubinsky, Dautermann, Leron, & Zazkis, 1994; Leron, Hazzan, & Zazkis, 1995; Weber, 2001) Leron, Hazzan and Zazkis further identified that students will use order of elements to determine that two groups are isomorphic and want a unique map.

At this point, the research body is primarily case studies ranging from single students to single classes. These studies have only begun to unravel student conceptions. For example, homomorphism, a concept identified as essential, has largely been unstudied. Its exploration exists guised in its relationships to isomorphism and the Fundamental Homomorphism Theorem. Weber (2001) found undergraduates were not aware of the utility of the theorem in proofs. While Nardi (2001) showed a case of students struggling to untangle the various maps in the theorems. Beyond these connections (and the relationship to isomorphism above), homomorphism remains relatively unknown.

Discussion and Direction of Future Research

The preceding domain analysis provides some tools for mapping the concept domain associated with groups. Representations, activities, examples, formal and informal definitions and documented student conceptions all play an essential role when deciding what tasks to include in an assessment. Domain analysis is an often ignored, but important step towards determining what tasks are representative, reflective and valued.

At this point, tasks can be created. In the isomorphism example, such tasks might include determining if groups are isomorphic aligning with textbook activities. Further, groups that are equi-numerous provide fruitful ground for probing understanding base on the literature search. Tasks can then be evaluated for importance and potential insight into student understanding by returning to the expert panel. Only at this point would a subset of tasks be tested as open-ended survey questions with students.

The above analysis also provides some discussion points for how we currently teach the course as reflected in textbooks. For example, while representations are considered essential to meaning and heavily emphasized in some content areas, texts rarely utilized some of the more visual options available such as tables and diagrams. Further, informal definitions are minimal to non-existent for many of the essential topics identified in group theory. These aspects may reflect the larger issue that conceptual understanding is often not prioritized in these courses. A concept-based assessment could challenge the predominating assumption that formal proofs are sufficient evidence of student understanding.

References

Biehler, R. (2005). Reconstruction of meaning as a didactical task: the concept of function as an example. In *Meaning in Mathematics education* (pp. 61–81). Springer. Retrieved from http://link.springer.com.proxy.lib.pdx.edu/content/pdf/10.1007/0-387-24040-3_5.pdf

Carlson, M., Oehrtman, M., & Engelke, N. (2010). The precalculus concept assessment: A tool for assessing students' reasoning abilities and understandings. *Cognition and Instruction*, *28*(2), 113–145.

Delmas, R., Garfield, J., Ooms, A., & Chance, B. (2007). Assessing students' conceptual understanding after a first course in statistics. *Statistics Education Research Journal*, *6*(2), 28–58.

Dubinsky, E. (1997). Special Issue: An investigation of students' understanding of abstract algebra (binary operations, groups and subgroups) and the use of abstract algebra to build other structures (through cosets, normality, quotient groups). *The Journal of Mathematical Behavior*, *16*(3), 181–309.

Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On learning fundamental concepts of group theory. *Educational Studies in Mathematics*, *27*(3), 267–305.

Epstein, J. (2007). Development and validation of the Calculus Concept Inventory. In *Proceedings of the Ninth International Conference on Mathematics Education in a Global Community* (Vol. 9, pp. 165–170). Charlotte, NC. Retrieved from http://math.unipa.it/~grim/21_project/21_charlotte_EpsteinPaperEdit.pdf

Fraleigh, J. B. (2003). *A first course in abstract algebra*. Pearson Education India.

Fukawa-Connelly, T. P., & Newton, C. (2014). Analyzing the teaching of advanced mathematics courses via the enacted example space. *Educational Studies in Mathematics*, 1–27.

Gallian, J. (2009). *Contemporary abstract algebra*. Boston, MA: Cengage Learning.

Gilbert, L. & Gilbert, J. (2008). *Elements of Modern Algebra* (7 edition.). Belmont, CA: Cengage Learning.

Hazzan, O. (1999). Reducing abstraction level when learning abstract algebra concepts. *Educational Studies in Mathematics*, *40*(1), 71–90.

Hestenes, D., Wells, M., & Swackhamer, G. (1992). Force concept inventory. *The Physics Teacher*, *30*(3), 141–158.

Hungerford, T. (2012). *Abstract algebra: an introduction*. Boston, MA: Cengage Learning.

Larsen, S., Johnson, E., & Bartlo, J. (2013). Designing and scaling up an innovation in abstract algebra. *The Journal of Mathematical Behavior*, *32*(4), 693–711.

Leron, U., Hazzan, O., & Zazkis, R. (1995). Learning group isomorphism: A crossroads of many concepts. *Educational Studies in Mathematics*, *29*(2), 153–174.

Lindell, R. S., Peak, E., & Foster, T. M. (2007). Are they all created equal? A comparison of different concept inventory development methodologies. In *2006 Physics Education Research Conference* (Vol. 883, pp. 14–17).

Mesa, V. (2004). Characterizing practices associated with functions in middle school textbooks: An empirical approach. *Educational Studies in Mathematics*, *56*(2-3), 255–286.

Messick, S. (1995). Validity of psychological assessment: validation of inferences from persons' responses and performances as scientific inquiry into score meaning. *American Psychologist*, *50*(9), 741.

Mislevy, R. J., Steinberg, L. S., & Almond, R. G. (2003). Focus article: On the structure of educational assessments. *Measurement: Interdisciplinary Research and Perspectives*, *1*(1), 3–62.

Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics*, *27*(3), 249–266.

Nardi, E. (2000). Mathematics Undergraduates' Responses to Semantic Abbreviations,"Geometric"Images and Multi-Level Abstractions in Group Theory. *Educational Studies in Mathematics*, *43*(2), 169–189.

RUMEonline! - Periodic Publications in Mathematics Education. (2011). Retrieved August 28, 2014, from http://sigmaa.maa.org/rume/journals.html

Savinainen, A., & Scott, P. (2002). The Force Concept Inventory: a tool for monitoring student learning. *Physics Education*, *37*(1), 45.

Savinainen, A., & Viiri, J. (2008). The Force Concept Inventory as a Measure of Students Conceptual Coherence. *International Journal of Science and Mathematics Education*, *6*(4), 719–740. doi:10.1007/s10763-007-9103-x

Sinclair, N., Watson, A., Zazkis, R., & Mason, J. (2011). The structuring of personal example spaces. *The Journal of Mathematical Behavior*, *30*(4), 291–303.

Streveler, R. A., Olds, B. M., Miller, R. L., & Nelson, M. A. (2003). Using a Delphi study to identify the most difficult concepts for students to master in thermal and transport science. In *Proceedings of the Annual Conference of the American Society for Engineering Education*. Retrieved from http://www.thermalinventory.com/papers/2003DelphiStudy.pdf

Tall, D. (2004). Building theories: The three worlds of mathematics. *For the Learning of Mathematics*, 29–32.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151–169.

Tallman, M., & Carlson, M. P. (2012). A characterization of calculus I final exams in US colleges and universities. In *proceedings of the 15th annual conference on research in undergraduate mathematics education* (pp. 217–226).

Thompson, D. R., Senk, S. L., & Johnson, G. J. (2012). Opportunities to Learn Reasoning and Proof in High School Mathematics Textbooks. *Journal for Research in Mathematics Education*, *43*(3), 253–295.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, *48*(1), 101–119.

Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics*, *56*(2-3), 209–234.

Beyond Good Teaching: The Benefits And Challenges Of Implementing Ambitious Teaching

Kathleen Melhuish
Portland State University

Erin Glover
Oregon State University

Sean Larsen
Portland State University

Annie Bergman
Portland State University

Lampert et al. (2010) define ambitious teaching as teaching designed to achieve the ambitious goals of conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive dispositions. They argue that this kind of teaching necessarily involves actively engaging students. Yet, lecture continues to dominate introductory college calculus courses throughout the country. In this poster, we draw on data from an national project (Characteristics of Successful Calculus Programs) to explore both national trends regarding ambitious teaching practices in calculus and present two case studies that give some important insights into how department-wide ambitious teaching can be instituted and sustained.

Key words: Calculus, Ambitious teaching, Active Learning

Lampert et al. (2010) define ambitious teaching as teaching designed to achieve the ambitious goals of conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive dispositions. They argue that this kind of teaching necessarily involves active learning in which students interact with their instructor and classmates. In fact, Freeman et al. (2014) argued that active learning should be the preferred teaching practice based on a meta-analysis of 225 studies. Yet, lecture continues to dominate introductory college calculus courses throughout the country. In this poster, we will present data both on national trends regarding ambitious teaching practices in calculus, as well as present two case study institutions that have embraced ambitious pedagogy in calculus with a continued push towards innovation.

Background and National Trends

This exploration into ambitious teaching is part of a larger project aimed to identify features of college calculus programs that make them successful (Bressoud, Carlson, Mesa, & Rasmussen, 2013). Calculus instructors and students across the country were surveyed with the aim of identifying successful schools based on student outcome variables including persistence, grades, interest, confidence and enjoyment. Eighteen institutions of varying types were then selected as case study schools. Site visits to these institutions featured classroom observations as well as interviews with instructors, students, administrators, and others who were involved with the calculus programs. In total, over 5,500 students completed a survey at the end of their Calculus 1 course. This survey included questions about the types of pedagogical activities in the class. As seen in Figure 1, lecturing was a frequent activity both nationally and within the selected schools. However, Figure 2 illustrates a distinct difference: the selected school more frequently had students working together.

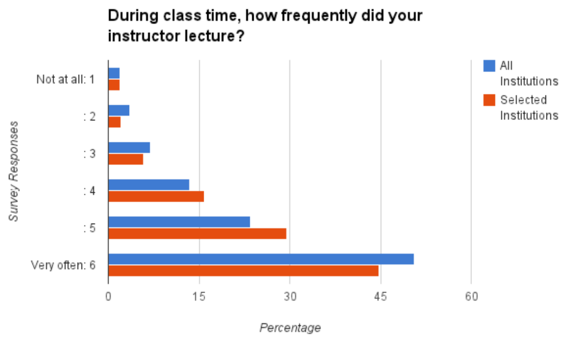


Figure 1. Frequency of lectures from national sample vs. selected institutions.

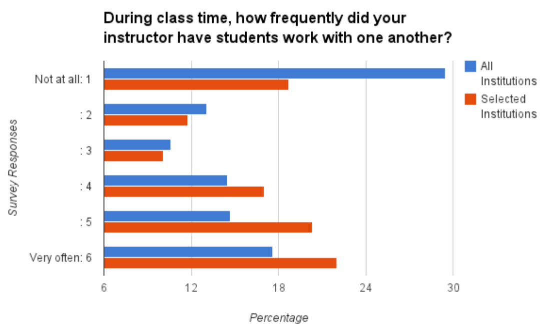


Figure 2. Frequency of students working together from national sample vs. selected institutions.

Case Study Results

We will present two cases of substantial ambitious teaching we observed on our site visits. The first is a case of *sustained ambitious teaching* (PhD 3) while the second is a case of *technology-supported ambitious teaching* (BA 1.) At PhD 3, a large research university, administrators, instructors, and students all described group work, an emphasis on conceptual understanding, and a consistent requirement for students to explain mathematics and their own thinking. This ambitious pedagogy was supported in several ways including a graduate student instructor training program that clearly articulated the conceptual and pedagogical goals of the course. This institution also used the Calculus Concept Inventory (Epstein, 2007) to monitor student conceptual understanding resulting in evidence that the instructional approach was successful. This resulted in administrative pressure to move to a more economically efficient approach.

At BA 1, a private university, instructors and administrators have a history of embracing innovation, particularly technological innovation. Currently, the institution is flipping their calculus classes with the primary goal of increasing student engagement. Technology usage goes beyond flipping and includes the usage of clickers and ipads. Instructors and students commented on the usage of technology to provide deeper exploration of problems and applications, and to share student strategies and facilitate whole-class discussion. Both the instructors and administration have actively pursued changes that would increase access to mathematics for all student learners.

The two cases provide evidence that it is possible to institutionalize ambitious pedagogy in introductory calculus. The flipped classes at BA 1, and the small-group learning model at PhD 3 provide two models for systematically incorporating ambitious teaching.

References

Bressoud, D. M., Carlson, M. P., Mesa, V., & Rasmussen, C. (2013). The calculus student: insights from the Mathematical Association of America national study. *International Journal of Mathematical Education in Science and Technology*, 1–15.

Epstein, J. (2007). Development and validation of the Calculus Concept Inventory. *In Proceedings of the Ninth International Conference on Mathematics Education in a Global Community* (Vol. 9, pp. 165-170). Charlotte, NC.

Freeman, S., Eddy, S., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences.*

Lampert, M., Beasley, H., Ghousseini, H., Kazemi, E., & Franke, M. (2010). Using designed instructional activities to enable novices to manage ambitious mathematics teaching. In *Instructional explanations in the disciplines* (pp. 129-141). Springer US

The purpose of calculus I labs: Instructor, TA, and student beliefs and practices

Yuliya Melnikova
Texas State University

Currently, political and economic demand for students graduating with Science Technology Engineering and Mathematics (STEM) degrees is high, but unfortunately, a large percentage of students switch to non-STEM majors in the first year of study. Roadblock courses, such as Calculus I, can contribute to poor retention rates due to classroom environment and instructor practices. Current research suggests recitation sessions (or labs) led by teaching assistants (TAs) can positively impact student retention rates.

This study investigates the role of labs in Calculus I instruction. Through classroom observation the researcher investigated the practices of TAs and through interviews the researcher explored how beliefs about the purpose of Calculus I labs by the lead instructor, TA, and students compared to one another as well as to the practices observed. Preliminary findings on the alignment of participant views and classroom practices will be presented, and implications for increasing student retention rates will be discussed.

Key words: [Calculus, Recitation Sessions, Teaching Assistant (TA), STEM retention]

Introduction/Literature Review

Engage to Excel, published by the President's Council of Advisors on Science and Technology (PCAST, 2012) called for one million more college graduates with degrees in science, technology, engineering, and mathematics (STEM) in the next decade. Similarly, businesses are advocating for more STEM graduates in order to have a skilled workforce (BHEF, 2010; TAP, 2008). While there is a growing demand for STEM majors, the number of STEM degrees awarded is not increasing proportionately to the overall number of degrees in the United States (BHEF, 2010). A little over half of students entering a STEM major complete their degree (NCES, 2009) with a large percentage switching to non-STEM majors in the first year or two (Hilton & Lee, 1988; Seymour & Hewitt, 1997).

Calculus, as part of undergraduate mathematics, has been a topic of conversation and reform since the 1980s and was viewed as a filter to the STEM pipeline, which blocks access to STEM careers (Steen, 1987; Tucker, 2013). Based on a MAA study of Calculus I in 2010, 72 percent of students enrolled in Calculus I have STEM career goals, and 50 percent earned an 'A' or 'B', which would indicate a likelihood of success in further mathematics courses (Bressoud, Carlson, Mesa, & Rasmussen, 2013). As seen with this data, only half of students are performing well in Calculus I courses, so calculus remains as a roadblock or a filter to the STEM pipeline for the other half.

Studies have addressed why students leave STEM majors and have found that students who stay in STEM as well as switch out of STEM majors report the "cold" classroom environment, dull lectures, and indifferent instructors to be main issues of concern (Seymour & Hewitt, 1997; Strenta, Elliot, Adiar, Matier, & Scott, 1994; Tobias, 1990). Furthermore, the students who switch and students who persist in STEM study do not differ academically or behaviorally, which challenges the notion that introductory STEM courses "weed out" unprepared or academically challenged students (Seymour & Hewitt, 1997). Research by Tresiman (1992) supports similar conclusions.

To increase STEM student retention rates, the structure of classrooms needs to be changed by trying to actively engage students (Daempfle, 2002). If faculty will not or are not able change the classroom structure, Seymour (2002) suggests a middle path where faculty "use recitation sessions run by teaching assistants as a way to insert more active learning, and formative assessment, into an otherwise unchanged lecture and lab pedagogy" (p. 87).

Approximately, a quarter of Calculus I courses have recitation sessions (labs) led by a teaching assistant (TA) (Bressoud, 2011). Those run by TAs have been shown to incorporate more active learning and to increase student satisfaction and retention (Muzaka, 2009; O'Neal, Wright, Cook, Perorazio & Purkiss, 2007). If students perceive the TA as approachable and able to provide a comfortable learning environment, then a lab component might improve the "chilly classroom" issue described in Seymour and Hewitt (1997). Thus, a lab run by a TA might provide a way to remediate the issues cited by students who switched out of STEM majors. Further research is needed to investigate how the lab environment can be best suited to the needs of the students.

Using a case study of Calculus I at a large, four-year university in central Texas, the following study investigated the current state of lab instruction and classroom environment of a single mathematics department. By selecting one section of Calculus I, an embedded case study examined how the lab portion (recitation session or workshop) is viewed by its participants (which include the instructor, the TA, and students) and how those views align between the participants and the practices in the lab. The study aimed to answer the following research questions

1. What is the state of Calculus I labs at a single university?
2. What purpose does the lab serve according to instructor, TA, and student?
    - How are the views of the purpose aligned or misaligned?
    - How are the views aligned or misaligned with the classroom practices?

Theoretical Framework

Taylor and Newton (2013) found that when implementing institutional change at a university, alignment of goals in the stakeholders (administrators, instructors, students) was key for integrating blended learning. Senior leadership served as a principal facilitator, while lack of a concise definition of the goal was a barrier. This research considers the classroom as the unit of change and therefore the key stakeholders are presumed to be the faculty in charge, the TA, and the students. For that reason, these groups were interviewed to better understand their goals as well how those goals align. The goals were then compared with observations of classroom instruction and environment to better understand whether current practices aligned with factors supported by research from the literature which may lead to student satisfaction and possible retention in STEM.

Methodology

To answer the research questions, a qualitative case study design was utilized, and data was collected using observations and interviews. A large, public four-year university in central Texas was selected to serve as a case study since it represented a bounded system (Merriam, 2009). The structure of a Calculus I course at this university has a lecture component lead by a faculty member (lecturer or professor) and a lab component directed by a TA (graduate or undergraduate student) with a maximum enrollment of 48 students. Six out of seven sections of Calculus I offered in the spring semester were observed in order to create an observation protocol to assist with data collection and to describe the state of labs. Using a single Calculus I section as an embedded case study, the instructor, the TA, and three students were interviewed to investigate what is the perceived purpose of the lab component and to see how their views align. Observations were then conducted using the protocol on the embedded case to see how the participants' views aligned with the practices in the lab.

Existing observation protocols (RTOP, ITC, QMI) were considered, but did not suit the needs of the present study. In order to construct an observation protocol, each lab section was observed by the researcher. The classrooms were observed one month into the semester to allow time for the TA and students to establish classroom norms in order to observe a "typical" lab. The researcher took open field notes in order to document as many different aspects of the calculus lab as possible. The field notes were then analyzed using open and axial coding (Merriam, 2009) from which three themes emerged: content, interaction, and participation. Content aims to observe which topics were covered and who selects the material to be presented. Interaction describes how the students and the TA interact with each other, and participation refers to the level of engagement of the students in the classroom. From these themes, the observation protocol was created to facilitate future observations and allow the researcher to count instances of events that occurred. The observation protocol was then piloted by observing each TA's lab again near the end of the semester. Through five rounds of use, the protocol was able to serve as a useful tool to aide in data collection.

One section was selected to serve as an embedded case study. Semi-structured interviews were conducted with the instructor, the TA, and three students to elicit their views on the purpose of the lab component. The interviews were then transcribed and coded to look for themes. Qualitative analysis of the interviews found four themes; assessment, recitation, comfortable learning environment, and communication. The lab serves as a time for assessment through quizzes and a way for the students to prepare for exams. Recitation is also a significant component of the lab with more time and more exposure to different calculus problems. Since the lab is led by a TA, who is also a (graduate) student, the students are able to communicate more comfortably with the TA. This creates a comfortable learning environment in which the students are able to ask questions and work together. The interview themes were then compared with the observations of the classroom.

Results

The researcher found the labs to be TA-centered, with the TA selecting and working out the problems on the board and reviewing lecture material. If given an opportunity to work on a problem on their own, the students worked independently with little evidence of group work or collaboration. Within the multiple sections of Calculus I, there was a large variation in the amount of instances of students asking questions during a single class (1-7 instances) and interacting one-on-one with the TAs (0-16 instances). The level of student engagement also varied across the lab sections. In some labs the students took notes and appeared to be actively listening, while in others, the students socialized with classmates or used cell phones.

From the interviews, the views held by the instructor, the TA, and the students were aligned in two of the four themes; recitation and communication. All of the participants viewed working problems (recitation) as part of the purpose or structure of lab, and the TA was viewed as someone with who the students could easily communicate. The other two themes, assessment and comfortable learning environment, had some misalignment. The instructor and the TA perceived the daily quizzes as formative assessment and as a way for students to self-assess. However, the students viewed the quizzes as part of the daily routine and a way to enforce attendance. While all of the participants viewed the lab as a comfortable learning environment where students can ask questions, the instructor viewed the lab as a time for group work, but this view was not explicitly shared by the TA or the students.

The observations of the embedded case study (one instructor, one TA, and three students) compared with the interview results revealed differences between the stated purpose of lab and the observed routine of the lab. Even though all of the interviewed students mentioned the comfortable atmosphere and the ability to ask questions, almost no students actually utilized the opportunity. During the observed lab, only one student asked a question. The TA frequently gave the students the opportunity to work on a problem before working it out on the board. When prompted to work on the problem, the students worked silently and independently, which is contrary to the instructor's view that the lab should utilize group work, and nor was it evident in observations.

Conclusion

The results of the study show a variation in the practices of the lab, a misaligned of views shared by the instructor, TA, and students, and differences in the stated views and observed practices. While some views may be aligned with research, there is much room for improvement especially within the practices of the classroom. Since the classroom environment and interaction with TAs have shown to aid with STEM retention, enhancing the lab component of Calculus I would be beneficial to all students.

Audience Questions

1. Are there any suggestions on the causes of misalignment between instructors, TAs, and students?
2. Are there any suggestions for future research?

References

Bressoud, D. M. (2011). The calculus I instructor. *Launchings*. Retrieved from http://www.maa.org/columns/launchings/launchings_06_11.html

Bressoud, D., Carlson, M., Mesa, V., & Rasmussen, C. (2013). The calculus student: Insights from the MAA national study. *International Journal of Mathematical Education in Science and Technology*, *44* (5), 685 – 698.

Business Higher Education Forum (2010). Increasing the number of STEM graduates: Insights from the U.S. STEM education & modeling project. Retrieve from http://www.bhef.com/sites/g/files/g829556/f/report_2010_increasing_the_number_of_stem_grads.pdf

Daempfle, P. A. (2002). An analysis of the high attrition rates among first year college science, math and engineering majors. U.S. Dept. of Education, Office of Educational Research and Improvement, Educational Resources Information Center (ERIC).

Fausett, L.V. & Knoll, C. (1991). Effective use of teaching assistants in first year calculus. *PRIMUS*, *1*(4), 407-414.

Feder, M. (2012, December 18). One decade, one million more STEM graduates [Blog post]. Retrieved from http://www.whitehouse.gov/blog/2012/12/18/one-decade-one-million-more-stem-graduates

Hilton, T. L., & Lee, V. E. (1988). Student interest and persistence in science: Changes in the educational pipeline in the last decade. *Journal of Higher Education*, *59*(5), 510–526.

Merriam, S. B. (2009). *Qualitative research: A guide to design and implementation*. San Fransico, CA: Jossey-Bass.

Muzaka, V. (2009). The niche of graduate teaching assistant (GTAs): Perceptions and challenges. *Teaching in Higher Education*, *41*(1), 1-12.

National Center for Education Statistics. (2009). NCES 2009-161. Students who study Science, Technology, Engineering, and Mathematics (STEM) in postsecondary education. Washington, DC: U.S Department of Education, Institute for Education Sciences. Retrieved from http://nces.ed.gov/pubs2009/2009161.pdf

O'Neal, C., Wright, M., Cook, C., Perorazio, T., & Purkiss, J. (2007). The impact of teaching assistant on student retention in the sciences. *Journal of College Science Teaching*, *36*(5), 24-29.

President's Council of Advisors on Science and Technology. (2012). Engage to excel: Producing one million additional college graduates with degrees in science, technology, engineering, and mathematics [Report to President]. Retrieved from http://www.whitehouse.gov/sites/default/files/microsites/ostp/pcast-engage-to-excel-final_feb.pdf

Seymour, E. (2002). Tracking the processes of change in US undergraduate education in science, mathematics, engineering, and technology. *Science Education*, *86*(1), 79-105.

Seymour, E. & Hewitt, N. (1997). Talking about leaving: Why undergraduates leave the sciences. Boulder, CO: Westview Press.

Steen, L. (1987). Calculus for a new century: A pump not a filter. Washington, D.C.: Mathematical Association of America.

Strenta, C., Elliot, R., Russell, A., Matier, M., & Scott, J. (1994). Choosing and leaving science in highly selective institutions. *Research in Higher Education*, *35*(5), 513-537.

Tapping Americas Potential. (2008, July 15). Business leaders call for progress in advancing U.S. innovation by strengthening Science, Technology, Engineering and Math [Press release]. Retrieved from http://tapcoalition.org/news/pdf/tap_progress_press_release.pdf

Taylor, J.A. & Newton, D. (2013). Beyond blended learning: A case study of institutional change at an Australian regional university. *Internet and Higher Education*, *18*, 54-60.

Tobias, S. (1990). They're not dumb: They're different: A new "tier of talent" for science. *Change*, *22*(4), 10-30.

Treisman, U. (1992). Studying students studying calculus: A look at the lives of minority mathematics students in college. *The College Mathematics Journal*, *23*(5), 362-372.

Tucker, A. (2013). The history of the undergraduate program in mathematics in the United States. *American Mathematical Monthly*, *120*(8). Retrieved from: http://www.maa.org/sites/default/files/pdf/CUPM/pdf/MAAUndergradHistory.pdf

Implementing inquiry-oriented instructional materials: A comparison of two classrooms

Hayley Milbourne
San Diego State University

Prior research in linear algebra education has focused on documenting and understanding the difficulties students have with specific topics. In more recent years, the research has started to shift towards developing instructional methods to address these issues. In this study, I explore the ways in which two instructors implement inquiry-oriented materials focused on span and linear (in)dependence. One of the instructors had prior experience with these materials and the other did not. Through an analysis of video recordings of these classes, I use the Inquiry-Oriented Discourse Moves framework to analyze how each instructor conducts whole-class discussion and the affordances these discussions provide their students.

Key words: Linear Algebra, Opportunities to Learn, Inquiry-Oriented Instruction

Over the past decade, educational research in linear algebra has been focused on student understanding in linear algebra (e.g. Britton & Henderson, 2009; Lapp, Melvin, & Berry, 2010; Stewart & Thomas, 2009; Wawro, Sweeney, & Rabin, 2011). Several different theoretical perspectives have been used for analyzing student understanding on a variety of topics, the most common one being APOS (Action Process Object Schema; Dubinsky & McDonald, 2001). However, the overarching theme within these studies has been that students struggle with linear algebra mostly because it is the first time they are expected to understand new definitions of abstract concepts and use them to construct proofs (Britton & Henderson, 2009). To address this, several groups of researchers have begun studying different curricula aimed at helping smooth student difficulties with the subject (e.g. Gueudet-Chartier, 2006; Love, Hodge, Grandgenett, & Swift, 2014; Trigueros, Possani, Lozano, & Sandoval, 2009).

The work presented in this paper is from a larger project, which is developing a sequence of tasks for central topics in linear algebra, including span and linear (in)dependence, transformations, systems of equations, and eigentheory. Currently, the task sequence for span and linear (in)dependence has been used at several different institutions and has shown promise for promoting student understanding (Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012). A great deal of the success of these units, however, relies on fidelity of implementation. To provide support in the implementation of the units, this project is creating instructional materials based on the model suggested by Lockwood, Johnson, and Larsen (2013). The instructor support materials provide, among other things, suggestions on leading whole class discussion to allow for students to engage in argumentation. The phrase "discursive moves" is use to capture the ways in which an instructor's utterances prompt or curtail student argumentation.

In this study I examine how different instructors use the instructional materials in the implementation of the unit on linear (in)dependence. Specifically, I am interested in understanding the different opportunities afforded to the students to engage in the materials by instructors with different levels of experience with the implementation of the unit. The research questions that are guiding this work are:

1. What different discursive moves do these two teachers use and what differing opportunities do these discursive moves afford learners?

2. How can the differences between the two classrooms inform ways in which to support instructors utilizing an inquiry-oriented instructional approach for the first time?

Literature

Inquiry-oriented instruction has been shown to increase student conceptual understanding and produce equivalent performance on computational tasks, as well provide students with a stronger foundation for subsequent coursework (Freeman et al., 2014; Kogan & Laursen, 2014; Rasmussen, Kwon, Allen, Marrongelle, & Burtch, 2006). The level of cognitive demand is critical to such success. For example, Stein and Lane (1996) found greater learning gains when students are given cognitively demanding tasks that have multiple solution methods. However, simply providing instructors with the materials for an inquiry-oriented approach does not ensure these aforementioned gains. Research has shown that teachers with little experience with the inquiry-oriented style struggle with its implementation (Wagner, Speer, & Rossa, 2007).

A central premise of this study is that the classroom learning environment both enables and constrains student learning (Cobb & Yackel, 1996). A classroom that provides students with more opportunities to participate in classroom activities provide students with more opportunities to learn (Bagley, 2014). Gresalfi, Barnes, and Cross (2012) conceptualize opportunities for learning as affordances (Gibson, 1979). Gibson discussed the variety of actions made possible by an object as the affordances of that object, such as a chair affords sitting but a door does not. In a similar way, the tasks used in a classroom and the teaching strategies implemented offer different affordances for student engagement with the material. However, simply having an affordance does not necessitate that the student will take it up. To understand the discursive affordances offered by the teacher and whether or not they are taken up by the student, I use a coding scheme known as the Inquiry-Oriented Discursive Moves (IODM) (Rasmussen, Kwon, & Marrongelle, 2009).

This particular coding scheme focuses on the utterances of the instructor in whole class discussions. Each code is based on the types of discursive moves made by the teacher and therefore offers insight into what types of opportunities the teacher is providing the students to participate in argumentation. Moreover, the way each utterance by the instructor is coded is influenced by how the students respond. The way the students interpret the utterance is the way it influences the classroom discourse so in this way, whether or not the students take up the affordance offered by the instructor through their discursive move is considered. Table 1 provides an explanation of a few of the different codes used in the IODM coding scheme. Specifically, these codes are from the discursive move of Questioning. The complete framework consists of four main categories of discursive moves, each with four subcategories.

Inquiry-oriented Instructional Materials

As stated previously, this study is part of a larger project focused on designing and disseminating an inquiry-oriented curriculum for linear algebra, including instructional support materials for the implementation of the units. The creation of these units was inspired by the instructional design theory of Realistic Mathematics Education (RME). A main tenet of RME is that mathematics is first and foremost a human activity (Freudenthal, 1971). Furthermore, the use of experientially-real problems provides students with anchoring points with which to participate in the reinvention of the mathematics (Gravemeijer & Doorman, 1999). A guiding RME design heuristic is "emergent models." According to Zandieh and Rasmussen (2010), "the intention of the emergent model heuristic is to create a sequence of tasks in which students first develop *models-of* their mathematical activity, which later become *models-for* more sophisticated mathematical reasoning" (p. 58).

In an inquiry-oriented classroom, the instructor leads classroom discussions in ways that help the students engage in argumentation and mathematization. To this end, the instructional materials were written to include the rational for each unit, ideas for implementing the materials, and insight into possible student responses.

Table 1

IODM Coding Scheme: Questioning

Code	Description
Q1: Evaluating	The intention is to check for understanding against what the teacher sees as an expected response.
Q2a: Clarifying – speaker	Purpose of the request is to seek clarification of detail (either for the teacher or for others) what a students is saying. Request for clarification is directed to the speaker.
Q2b: Clarifying – other	Purpose of the request is to seek clarification of detail (either for the teacher or for others) what a students is saying. Request for clarification is directed to someone other than the speaker.
Q3a: Explaining – own	Intention is for student(s) to share ideas (however tentative). Could be in question or request form. Requests to explain your thinking or the thinking of your group.
Q3b: Explaining - other	Intention is for student(s) to share ideas (however tentative). Could be in question or request form. Requests to explain or comment on another student's or group's thinking.

Participants & Data Sources

This study involved two instructors (HM and NS) at two different institutions. Each instructor was paired with a member of the research team that was involved in the development of the materials to provide support in their implementation. Interviews were conducted before, during, and after the implementation of each of the units and each class was videotaped. The data used in the project described here is from the classroom videos in Fall 2013. NS used the materials in previous years and taught Linear Algebra three times prior. HM had not used the materials before but was familiar with the first unit from discussions with researchers involved in the creation of the materials. At the time of the data collection, she was teaching Linear Algebra for the second time. These two instructors were chosen based on their varying levels of experience with the materials and similar amount of experience teaching the course. The videos of their classrooms during the implementation of the first unit, the Magic Carpet Ride (MCR), were transcribed for analysis. Consistent with the research questions that focus on the class as a whole, only the whole class discussions were analyzed.

Methods of Analysis and Initial Results

Each classroom transcript was coded using the IODM coding scheme by two researchers and checked for consistency. This process enabled me to identify what discursive moves are used by each instructor and in turn provides information on what types of opportunities are afforded the students in each classroom. Differences noted between the two will help inform future revisions of the instructor materials for the unit.

Coding is currently in progress and thus only initial findings are discussed. The two classrooms under analysis were quite different from one another in format. In the classroom taught by NS, the instructor more experienced with the materials, the lecture portion of the class usually lasted about 3 to 5 minutes at a time and occurred two to three times during the class. During whole class discussions, NS typically had the students describe their results and compare their methods with those around them. In contrast, the class taught by HM followed a more IRE (initiation, response, examination) pattern throughout the lecture portion. Students also worked in groups but usually for a shorter amount of time compared to NS.

As an example of the discursive differences between the two classrooms, consider the types of questions asked by each instructor during the second day of implementing the materials. Table 2 shows the codes used for describing the types of questions asked by each instructor as well as their frequency in the second day of the course. A description of the codes is given in Table 1.

Table 2

IODM Coding Scheme: Frequency of Questioning

Code	NS	HM
Q1: Evaluating	5	12
Q2a: Clarifying – speaker	6	8
Q2b: Clarifying – other	1	5
Q3a: Explaining – own	13	6
Q3b: Explaining – other	0	0

Both teachers asked many questions during whole class discussions but the type of questions asked was quite different. Each of these different questions provided students with different opportunities to learn and participate in argumentation. For example, consider the following discursive move by NS: “OK, so what does that entail? What did you do, what was your first step?” Within the classroom context, the students were presenting their solutions to the first task in the MCR sequence. NS requested the group to explain their solution method to the class. For this reason, this utterance was coded as a Q3a, which covers questions or requests to explain their own thinking. The students in the group requested to explain are provided an opportunity to participate in argumentation and share their ideas. Moreover, the class is now given the opportunity to see a different solution method.

As a contrasting example, consider the following discursive move by HM: “What do you think? So we have negative time when we’re not starting at the origin. Right? Or we have negative direction. Yeah?” Within the classroom context, the students were trying to decide what a negative scalar meant when it multiplied a direction vector in the MCR sequence. This utterance was coded as a Q1 as the instructor was searching for a specific answer from the students. After this utterance, students began to give explanations for negative time but the instructor continued to ask about negative direction. While this does provide the students with the opportunity to argue their point of view, since the instructor continued to prompt for negative direction, the students were not able to necessarily make the distinction for themselves.

Questions for Audience Discussion

- One of the main goals is to find ways in which the instructional materials can support instructors who are implementing these units, and possible utilizing the inquiry-oriented instructional approach for the first time. What other factors that could affect how these materials are implemented in the classroom have not yet been considered here?

- What supports can we provide to instructors implementing these inquiry-oriented materials that encourage the use of discursive moves affording opportunities to learn?

References

Bagley, S. (2014). *Improving student success in calculus: A comparison of four college calculus classes* (Unpublished doctoral dissertation). San Diego State University, San Diego, CA.

Britton, S., & Henderson, J. (2009). Linear algebra revisited: an attempt to understand students' conceptual difficulties. *International Journal of Mathematical Education in Science and Technology*, *40*(7), 963–974.

E. Dubinsky and M. McDonald. APOS: A constructivist theory of learning in undergraduate mathematics education. In D. Holton (Ed.), The teaching and learning of mathematics at university level: An ICMI study, pages 275–282. Kluwer Academic Publisher, 2001.

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences*, *111*(23), 8410-8415.

Freudenthal, H. (1971). Geometry between the devil and the deep sea. *Educational Studies in Mathematics*, *3*, 413–435.

Gibson, J. J. (1979). *The ecological approach to visual perception*. Boston: Houghton Mifflin.

Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics*, *39*(1-3), 111–129.

Gresalfi, M. S., Barnes, J., & Cross, D. (2012). When does an opportunity become an opportunity? Unpacking classroom practice through the lens of ecological psychology. *Educational Studies in Mathematics*, *80*, 249–267.

Gueudet-Chartier, G. (2006). Using geometry to teach and learn linear algebra. *Research in Collegiate Mathematics Education*, *13*, 171–195.

Kogan, M., & Laursen, S. L. (2014). Assessing long-term effects of inquiry-based learning: A case study from college mathematics. *Innovative Higher Education*, *39*(3), 183-199. DOI:10.1007/s10755-013-9269-9

Lapp, D., Melvin, N., & Berry, J. (2010). Student connections of linear algebra concepts: An analysis of concept maps. *International Journal of Mathematical Education in Science and Technology*, *41*(1), 1–18.

Lockwood, E., Johnson, E., & Larsen, S. (2013). Developing instructor support materials for an inquiry-oriented curriculum. *The Journal of Mathematical Behavior*, *32*, 776–790.

Love, B., Hodge, A., Grandgenett, N., & Swift, A. (2014). Student learning and perceptions in a flipped linear algebra course. *International Journal of Mathematical Education in Science and Technology*, *45*(3), 317–324.

Rasmussen, C., Kwon, O. N., Allen, K., Marrongelle, K., & Burtch, M. (2006). Capitalizing on advances in mathematics and K-12 mathematics education in undergraduate mathematics: An inquiry-oriented approach to differential equations. *Asia Pacific Education Review*, *7*(1), 85–93.

Rasmussen, C., Kwon, O. N., & Marrongelle, K. (2009). A framework for interpreting inquiry-oriented teaching: Opporunities for student and teacher learning. Presented at

the Annual Meeting of the American Educational Research Association, San Diego, CA.
Stein, M. K., & Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project. *Educational Research and Evaluation*, *2*(1), 50–80.
Stewart, S., & Thomas, M. O. J. (2009). A framework for mathematical thinking: the case of linear algebra. *International Journal of Mathematical Education in Science and Technology*, *40*(7), 951–961.
Trigueros, M., Possani, E., Lozano, M.-D., & Sandoval, I. (2009). Learning systems of linear equations through modeling. In *In Search for Theories in Mathematics Education* (pp. 225–232). Thessaloniki, Greece.
Wagner, J. F., Speer, N. M., & Rossa, B. (2007). Beyond mathematical content knowledge: A mathematician's knowledge needed for teaching an inquiry-oriented differential equations course. *Journal of Mathematical Behavior*, *26*, 247–266.
Wawro, M., Rasmussen, C., Zandieh, M., Sweeney, G. F., & Larson, C. (2012). An inquiry-oriented approach to span and linear independence: The case of the magic carpet ride sequence. *PRIMUS: Problems, Resourses, and Issues in Mathematics Undergraduate Studies*, *22*(8), 577–599.
Wawro, M., Sweeney, G. F., & Rabin, J. (2011). Subspace in linear algebra: Investigating students' concept images and interations with the formal definition. *Educational Studies in Mathematics*, *78*, 1–19.
Zandieh, M., & Rasmussen, C. (2010). Defining as a mathematical activity: A framework for characterizing progress from informal to more formal ways of reasoning. *The Journal of Mathematical Behavior*, *29*(2), 57–75.

Students' understanding of composition of functions using model analysis

David Miller — West Virginia University
Nicole Engelke-Infante — West Virginia University
Solomon Adu — West Virginia University

Abstract: *Model analysis is a quantitative research method used in physics education research to analyze and interpret the meaning of students' incorrect responses on a well-designed research-based multiple-choice test. We have adapted this method to study students' understanding of function composition when functions are represented graphical. Model analysis accounts for the fact that students may hold more than one idea or conception at a time, and may use different ideas and concepts in response to different situations. It is uniquely suited to study students' understanding of function composition, as students often hold multiple, sometimes conflicting misconceptions on function composition, which they may use at different times. Model analysis can capture information on self-consistency of a student's responses. We collected data from a calculus class before and after the class reviewed composition of functions. We find that model analysis offers insights not offered by traditional statistics.*

Key words: Model Analysis, Function Composition, Pre-calculus, Traditional Statistics, and Consistency

One of the most fundamental courses for science and engineering is calculus (Bressoud, Carlson, Mesa, & Rasmussen, 2013). To be adequately prepared for calculus, one needs to have a strong understanding of algebra and trigonometry (i.e. pre-calculus). It has been noted that students bring knowledge from their life experiences and instruction in prior classes (Bao & Redish, 2006). These prior experiences and knowledge have an effect on how they interpret what they are taught in mathematics. Bao and Redish points out in physics, which can be transferred over to mathematics, that "student knowledge may be locally coherent" where "different contexts can activate different bits of knowledge." These bits of knowledge are usually limited in earlier learning stages for students and they only have a few (usually two or three) alternative conceptions for any particular mathematics topic. In this article, we will use the definition of mental model from Boa and Redish (2006) that states it is "a robust and coherent knowledge element or strongly associated set of knowledge elements" (p.3).

Students' prior knowledge does not always line up to what is being taught in a mathematics class. Students have misconceptions that are developed by a partial or naïve understanding of a particular mathematics topic. These misconceptions may cause students to answer questions that are similar, from an expert viewpoint, in an inconsistent manner. Instructors might not be able to reveal, when examining performance, whether a student is consistent or inconsistent in answering questions that are similar. Furthermore, instructors would usually not glean from traditional statistics whether the majority of the students answer these similar questions in the same way, if there are groups of students that answer it in the same way, or if there are groups of students that answer in the same way but in some circumstances they answer one way and in other circumstances they answer another way. Traditional statistics would not allow for such detail information to be revealed and this study will use model analysis to reveal the consistencies, inconsistencies and the prominent model states for the pre-calculus topic of function composition.

Literature Review

Functions are one of the central mathematical concepts that play an important role in higher-level mathematics. The concept of function has been the focus of numerous studies (Carlson, 1998; Breidenbach et al., 1992; Dreyfus & Eisenberg, 1982, Leinhardt et al., 1990, Selden & Selden, 1992; Sierpinska, 1992; Vinner & Dreyfus, 1989; Monk, 1992; Monk, 1989; Dubinsky & Harel, 1992, Cooney & Wilson, 1993; Ferrini-Mundy & Graham, 1991). There have been few studies that focused on composition of functions. These studies have focused on students difficulty with function composition with explicit formulas (Sfard ,1992), functions represented by words (Bowling, 2009), and their relation to students' understanding of the chain rule (Cotrill, 1999). Also Carlson et al. (2010) stressed that a process view of a function (see next section) is important in understanding function composition and Vidakovic (1996) claimed that subjects with schemas for function composition and inverse functions are able coordinate them to obtain a new process.

Engelke et al. (2005) investigated students' understanding of function composition based on an analysis of a subset of questions from the PCA (Carlson et al., 2010). Their study focused on functions that were represented by formulas, tables, graphs, and words. Table 1 shows the percentages of students that answered the function composition questions correctly when presented different function representations on two different versions of the PCA (Carlson et al., 2010a). The high percentage of students that answered the function composition question when given a formula showed that students had a solid action view of functions (see next section). However, the declining performance on the remaining questions showed that students did not have the process view of functions that was needed to be successful with function composition (Engelke, 2005).

Precalculus Data		**Composition (% Correct)**				
Version	N	Algebraic Formula	Graph	Table	Square Context	Circle Context
G	379	91%	45%	50%	20%	9%
H	652	94%	43%	41%	25%	17%

Table 1: Data from PCA (Engelke et al., 2005)

Theoretical Framework

A mental model is a robust and coherent knowledge element or strongly associated set of knowledge elements. Mental models may be correct or incorrect and are sometimes simple while other times complex. When given a sequence of questions over a particular mathematical topic, an expert will use a single, coherent mental model while a student (novice) may use one or more incorrect mental models. The incorrect mental models are activated for a variety of reasons. There are a variety of model situations that can happen when students are asked a question. Sometimes one single model is activated which could be correct or incorrect, while other times multiple models are activated and a student has to make a decision on which model will be used to determine their answer. It is in this circumstance where they have to make a decision and is where the student can become confused and use different mental models at different times.

Model analysis is a quantitative research method that can be used to measure student's understanding and interpretation of concepts they learn in the classroom. When using model analysis, researchers take into account the previously obtained knowledge of the students and how this may affect the way in which students answer questions. We will apply the theory developed by Bao & Redish (2006) to the models in each of the categories explained in the methodology section. For simplicity, we will discuss the model analysis with respect to the models of category B – function composition for functions given by a graph. The five steps of model analysis (Bao & Redish, 2006) are:

I. **Identify common student models.** This occurs through in-depth interviews.
II. **Design a multiple-choice instrument.** We will use a modified version of the Pre-Calculus Assessment (PCA) (Carlson, Oehrtman, & Engelke, 2010).

III. **Characterize a student's responses with a vector in a linear "model space."** Since there are five models (MS0, MS1, MS2, MS3, MS4) (see data section for the models in Category B) we will state each student response as a vector with five components.

IV. **Create a "density matrix."** We create a matrix for each student response by taking the outer product of the vector in III and its' transpose. The matrices for all questions for a particular student are combined to get a density matrix for each student. All density matrices are combined to get a class model density matrix.

V. **Analyze eigenvalues and eigenvectors.** The eigenvalues and eigenvectors will reveal the prominent model states in the class. When one eigenvalue is larger than 0.65, then there is a significant prominent model state. When there are two relative large eigenvalues compared to the other eigenvalues and the corresponding eigenvectors are orthogonal, then there are two prominent model states.

For more details on these three steps, refer to (Bao & Redish, 2006). This paper will concentrate on analyzing the results of IV and V.

APOS Theory (Asiala, Brown, DeVries, Dubinsky, and Mathews, 1997) provides another part of the framework for this study and refers to the concepts of Action, Process, Objects, and Schema. An Action is a transformation of objects using a series of steps (or cues) that an individual has to externally refer to (or be aware of). A Process is an action that is not directed by external stimuli (it is internalized). Asiala et al. (1997) state "an individual who has a process conception of a transformation can reflect on, describe, or even reverse the steps of the transformation without actually performing those steps" (p. 7). One can think that when a process is perceived as an entity upon which actions and processes can be made, it becomes an Object. Finally, a collection of interconnected processes and objects organized in a structured manner form a Schema.

Research Questions

1) What additional information does model analysis give us, if any, over the traditional statistical information about composition of function stated in Engelke et al. (2005)?
2) What does this additional information tell us about students' understanding of composition of functions?

Methodology

Participants and Setting

In the spring 2014 semester, students in first semester calculus that incorporates pre-calculus enrolled at a large public northeast research university, were asked complete a modified version of the PCA (Carlson et. al., 2010a) that focused on composition of functions. This version of the PCA consisted of 18 questions on function composition broken up into five categories: Category A - formulas, Category B - graphs, Category C - tables, Category E – words, and Category D. Category D consisted of information about the composite function and either outer or inner function and asked to find the inner or outer function, respectively, at a point. There were 4 questions in Category A, B, and C, and 3 questions in Category D and E. Our instrument was given to 93 students prior to reviewing function composition in class (pre-test) and 83 students after they had reviewed function composition in class (post-test). This resulted in 50 students who took both pre-test and post-test that we will use as data points for the model analysis.

Since the PCA is a valid and reliable instrument, we modified (in collaboration with the second author) the instrument to include more questions across each of the function representation forms. In order for us to identify the different kinds of mental models students used in answering one additional composition of function question, we video recorded interviews with10 students on 3 new function composition questions. After analysis, we were able to determine five common models students used in answering this question and we used these models as multiple choice questions for Category D.

Data

We present the data for one of the four categories investigated by model analysis – functions given by graphs. The other three categories will be presented in an expanded paper. We present Category B since it is similar to Category C (functions given by tables) and the results show there is some inconsistency, along with two prominent model states.

Figure 1 below shows the student model states for Category B. The question for category B is to use the graphical representation of two functions $f(x)$ and $g(x)$ to evaluate either $f(g(x))$ or $g(f(x))$. The incorrect models are the **reverse, ignoring input function, and ignoring outside function models**. Students that use the **reverse model**, calculate $g(f(x))$ instead of $f(g(x))$ or vice versa. Students that use the **ignoring input function model** treat $f(g(x))$ as $f(x)$ by ignoring the input function. Students that use **ignoring outside function model** evaluate $f(g(x))$ as $g(x)$ by ignoring the outside function f.

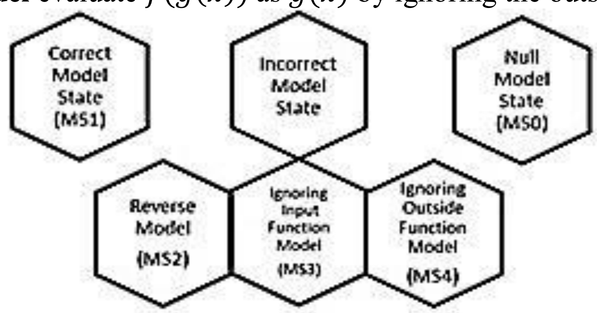


Figure 1 Student Models for Function of Composition Given by Graphs

Table 2 below, presents the class density matrix, eigenvalues and eigenvectors derived from model analysis for the Pre-Test. From the diagonal entries in the density matrix table, **8%** of the students answered the questions using ***Null Model (MS0)***, **42%** used the ***Correct Model (MS1)***, **7%** used the ***Reverse Model (MS2)***, **22%** used the ***Ignoring Input Function Model (MS3)*** and **21%** used the ***Ignoring Outside Function Model (MS4)*** (Diagonal entries of the table give the percentages of students that chose each model). Consistency can be examined by looking at the off-diagonal entries relative to the diagonal entries. Boa and Redish (2006) state that "large off-diagonal elements indicate low consistency (large mixing) for individual students in their model use" (p.9). The model statistic is $p = \frac{OD_{MSX,MSY}}{\sqrt{D_{MS\,X}D_{MS\,Y}}}$ and when it is larger than 0.5 there is a significant mixing between model state X and model state Y, where OD and, D_{MSX} and D_{MSY} stands for off-diagonal entry in the a_{XY} entry of the matrix and the diagonal entries corresponding to row X and Y, respectively. For the pre-test, the only significant mixing occurs between MS3 and MS4. The eigenvalues of **0.29 & 0.49** are both smaller than **0.65**, are much larger than all other eigenvalues, and hence tell us the corresponding orthogonal eigenvectors, where its entries are squared, are the prominent model states for the class (Bao & Redish, 2006). The two prominent class model states are shown in figure 2.

$$\overrightarrow{v_{pre\,B_1}} = \begin{pmatrix} 0.0064 \\ 0.3249 \\ 0.0121 \\ 0.3721 \\ 0.2809 \end{pmatrix} \quad \& \quad \overrightarrow{v_{pre\,B_2}} = \begin{pmatrix} 0.0144 \\ 0.6724 \\ 0.0225 \\ 0.1296 \\ 0.16 \end{pmatrix}$$

Figure 2: Prominent Model States for the Pre-test

The second of these vectors, associated with **0.49** eigenvalue, shows that a group of the students in the class are in the model state where they have a probability of **0.67** that they will choose the correct answer and a probability of **0.32** that they choose the other alternative (incorrect) models. The first of these vectors, associated with **0.29** eigenvalue, indicates that another group of students have a probability of **0.32** that they choose the correct answer and a probability of **0.68** that they choose the other alternative (incorrect) models. There are other groups of students that have other model states in which they do not choose the correct answer with a high probability, however, these groups are relative small compared to the other two groups. That is, the first two eigenvalues are much bigger than all other eigenvalues and therefore there corresponding orthogonal eigenvectors are two prominent groups.

Density Matrix	MS 0	MS 1	MS 2	MS 3	MS 4
MS 0 (null model)	0.08	0.03	0.01	0.02	0.04
MS 1 (correct model)	0.03	0.42	0.04	0.05	0.07
MS 2 (reverse model)	0.01	0.04	0.07	0.04	0.04
MS 3 (ignore input function model)	0.02	0.05	0.04	0.22	0.12
MS 4 (ignore output function model)	0.04	0.07	0.04	0.12	0.21
Eigenvalue	0.06	0.07	0.10	0.29	0.49
Eigenvector	0.02	-0.89	-0.44	0.08	0.12
	0.07	0.02	0.07	-0.57	0.82
	-0.98	0.00	0.01	0.11	0.15
	0.13	-0.21	0.66	0.61	0.36
	0.12	0.41	-0.61	0.53	0.40

Table 2 Common Pre-Test Category B

Table 3 below, present the class density matrix, eigenvalues and eigenvectors derived from model analysis for Post-Test. The diagonal entries in the density matrix table shows that, **8%** of the students answered the questions using ***Null Model (MS0)***, **48%** used the ***Correct Model (MS1)***, **6%** used the ***Reverse Model (MS2)***, **21%** used the ***Multiplication Model (MS3)*** and **18%** used the ***Square or Reverse Square Model (MS4)***. Examining the off-diagonals entries to the diagonal entries, the only significant mixing again occurs between MS3 and MS4. The eigenvalues of **0.30 & 0.53** are both smaller than **0.65** and hence tells us the corresponding orthogonal eigenvectors, where its entries are squared, are the prominent model states for the class. The prominent class model states are shown in figure 3.

$$\overrightarrow{v_{post\ B_1}} = \begin{pmatrix} 0.0016 \\ 0.1764 \\ 0.0121 \\ 0.3969 \\ 0.4096 \end{pmatrix} \ and\ \overrightarrow{v_{post\ B_2}} = \begin{pmatrix} 0.0121 \\ 0.81 \\ 0.01 \\ 0.1225 \\ 0.0529 \end{pmatrix}.$$

Figure 3: Prominent Model States for the Post-Test

The second of these vectors, associated with the **0.53** eigenvalue, shows that a group of the students in the class are in this model state where they have the probability of **0.81** to choose the correct answer and a probability of **0.19** that they choose the other alternative (incorrect) models. The first of these vectors, associated with the **0.30** eigenvalue, indicates that another group of students have the probability of **0.17** that they choose the correct answer and a probability of **0.83** that they choose the other alternative (incorrect) models.

Density Matrix	MS 0	MS 1	MS 2	MS 3	MS 4
MS 0 (null model)	0.08	0.04	0.01	0.02	0.02
MS 1 (correct model)	0.04	0.48	0.03	0.08	0.03
MS 2 (reverse model)	0.01	0.03	0.06	0.03	0.03
MS 3 (ignore input function model)	0.02	0.08	0.03	0.21	0.14
MS 4 (ignore output function model)	0.02	0.03	0.03	0.14	0.18

Eigenvalue	0.05	0.05	0.08	0.30	0.53
Eigenvector	0.01	0.28	0.95	0.04	0.11
	-0.09	-0.04	-0.07	-0.42	0.90
	0.64	-0.73	0.19	0.11	0.10
	0.45	0.48	-0.21	0.63	0.35
	-0.61	-0.40	0.07	0.64	0.23

Table 3 Common Post-Test Category B

Results and Discussion

Engelke et al. (2005) reported that $\mathbf{45\%}$ of the **379** students that took version G of the PCA and $\mathbf{43\%}$ of the **652** students that took version H of the PCA could correctly answer the composition of functions given a graph question. Our data shows that if we look at all the students that took both the pre and post-test modified PCA, $\mathbf{42\%}$ of the students on the pre-test could correctly answer the composition of functions graph problem. Engelke (2005) states the percentages for the incorrect models on version H and our data is very similar for the correct, reverse, and ignore input models. There was some difference in the ignore output function (10% for PCA while 21% for our data) and null model (13% for PCA while 7% for our data), but they are still somewhat similar. Although our data supports Engelke et al. (2005) results, it adds a little more information on the consistency that students choose the same model within all questions on Category B, most but not all the time. In terms of pre-test, the only inconsistency between two models occurred between MS 3 and MS 4 where the p-value ($\boldsymbol{p = 0.5583 > 0.5}$) was significant (Boa & Redish, 2006). This says that some students that chose the answers corresponding with the incorrect model MS3 and while other times chose the answer that corresponding with the incorrect model MS4. The data in Engelke (2005) did not reveal this inconsistency.

For the post-test for category B, we have very similar results. For our data, $\mathbf{48\%}$ of the students answered the composition of functions given a graph correct. This is slightly higher ($\mathbf{6\%}$) than for the pre-test and slightly higher (**4.26%**) than the weighted average (**43.74%**) of version G and H for Engelke et al.'s (2005) study. There is consistency in student's model states except when considering model state 3 and 4 where the p-value was **p=0.72 > 0.5**. When we examine the eigenvalue and corresponding eigenvectors, we gain even more information. Figure 4 shows the prominent model state for the pre and post-test.

$$\overrightarrow{v_{pre\ B_1}} = \begin{pmatrix} 0.0064 \\ 0.3249 \\ 0.0121 \\ 0.3721 \\ 0.2809 \end{pmatrix}, \qquad \overrightarrow{v_{pre\ B_2}} = \begin{pmatrix} 0.0144 \\ 0.6724 \\ 0.0225 \\ 0.1296 \\ 0.16 \end{pmatrix}, \qquad and$$

$$\overrightarrow{v_{post\ B_1}} = \begin{pmatrix} 0.0016 \\ 0.1764 \\ 0.0121 \\ 0.3969 \\ 0.4096 \end{pmatrix}, \qquad \overrightarrow{v_{post\ B_2}} = \begin{pmatrix} 0.0121 \\ 0.81 \\ 0.01 \\ 0.1225 \\ 0.0529 \end{pmatrix}.$$

Figure 4: Prominent Model States for the Pre and Post-Test

When comparing the pre and post for the second of these two groups, we have a larger group of students (not necessarily the same students or the same size) that had a probability of choosing the correct model state (**0.67-pre** vs. **0.81-post**) after they have reviewed composition of functions given by graph in class, while fewer students had the probability of choosing model states 2, 3 or 4 (**0.31** vs. **0.19**, respectively). For the first of these two groups, we have a smaller group of students that had a probability of choosing the correct model state (**0.32** vs. **0.18**) and a probability of choosing model state 3 or 4 (**0.65** vs. **0.81**, respectively).

The probability of choosing MS2 was about the same. Therefore we have two groups whose model state vectors are quite different. We also see that there is very little inconsistency other than a mixed model state for MS3 and MS4.

Conclusion

This study adds additional data to the study by Engelke et al. (2005). It shows that model analysis yields much information beyond that of traditional statistics. We gain information about students' consistencies/inconsistencies and the prominent model state vector for the class which give us a view of what students' understand with regard to function composition.

First of all for function composition given by a graph, students' model state vectors are in the mixed region for both the pre and post-test. That is, as a class, they sometimes answer correctly while other times answer incorrectly. However, model analysis tells us there is a mixing between incorrect model states 3 and 4 where students sometimes answer one way while other time answer another way when answer function composition questions involving graphs. In addition, there were two prominent model states that developed from the data. On the pre-test, one of these groups (the larger one) had a probability of around two-thirds that they would answer correctly, but had misconceptions where they ignored the input function or ignored the outside function with a probability of 0.30. On the post-test the larger group choose the correct answer with a probability of 0.81, the misconception on ignoring the outside function was reduced (from a probability of 0.16 to 0.053), and the misconception of ignoring the input function remained at the same probability level. It was a different story for the other of the two prominent groups. On the pre-test, this group had a probability of about one-third that they would answer correctly, but had misconceptions where they ignored the input function or ignored the outside function with a probability of 0.65. On the post-test this group choose the correct answer with probability 0.18 and they had a stronger misconception of ignored the input function or ignored the outside function with a probability of almost 0.80.

In summary, we see that model analysis can reveal information about the consistency and inconsistency of how students answer questions and can illuminate the model states for prominent groups of students in the class. This information can be used in teaching, not only to know what the misconceptions are, but information on to what extent. This could lead to teaching interventions (in class or outside of class) to dispel the misconceptions in order to solidify the students' knowledge of a fundamental concept in mathematics.

References

Asiala, M., Brown, A., DeVries, D. J., Dubinsky, E., Mathews, D., & Thomas, K. (1997). A framework for research and curriculum development in undergraduate mathematics education. *MAA NOTES*, 37-54.

Bao, L., & Redish, E. F. (2006). Model analysis: Representing and assessing the dynamics of student learning. *Physical Review Special Topics-Physics Education Research*, *2*(1), 010103.

Bowling, S. (2009). Precalculus Student Understanding of Function Composition. In *Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

Breidenbach, D., Dubinsky, E., Hawks, J., and Nicholas, D., (1992). Development of the Process Conception of Function, Educational Studies in Mathematics (1992), 247-285.

Bressoud, D.M., Carlson, M., Mesa, V., & Rasmussen, C. (2013). The calculus student: insights from the Mathematical Association of America national study. *International*

Journal of Mathematical Education in Science and Technology, 44(5), 685-698. doi: 10.1080/0020739X.2013.798874.
Carlson, M. (1998). A cross-sectional investigation of the development of the function concept. *Research in Collegiate Mathematics Education III, Conference Board of the Mathematical Sciences, Issues in Mathematics Education*, *7*(2), 114–162.
Carlson, M., Oehrtman, M., & Engelke, N. (2010a). The Precalculus Concept Assessment: A Tool for Assessing Reasoning Abilities and Understandings of Precalculus Level Students. *Cognition and Instruction, 28*(2), 113-145. doi: 10.1080/07370001003676587
Carlson, M., Madison, B., & West, R. (2010b). The Calculus Concept Readiness (CCR) Instrument: Assessing Student Readiness for Calculus. *arXiv preprint arXiv:1010.2719*.
Cooney, T. and Wilson, M., Teachers' Thinking about Functions: Historical and Research Perspectives, Integrating Research on the Graphical Representation of Function, Lawrence Erilbeum Associates, Hillsdale, NJ, 1993, pp. 131-138.
Cottrill, J. (1999). Students' understanding of the concept of chain rule in first year calculus and the relation to their understanding of composition of functions. Unpublished Doctoral Dissertation, Purdue University.
Dreyfus, T. and Eisenberg, T. (1984). Intuitions on Functions, Journal of Experimental Education, 52, 77-85.
Dubinsky, E., & Harel, G. (1992). The nature of the process conception of function. In E. Dubinsky & G. Harel (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 85-106). Washington, DC: Mathematical Association of America.
Engelke, N., Oehrtman, M., & Carlson, M. (2005, October 20-23). Composition of Functions: Precalculus Students' Understandings. Paper presented at the 27th Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education, Roanoke, VA.
Ferrini-Mundy, J., & Graham, K. G. (1991). An overview of the calculus curriculum reform effort: Issues for learning, teaching, and curriculum development. American Mathematical Monthly, 98, 627–635.
Leinhardt, G, Zaslavsky, O., and Stein, M.K. (1990). Functions, Graphs, and Graphing: Tasks, Learning and Teaching, Review of Educational Research, 60(1), 11-64.
Monk, S. (1989). A Framework for Describing Student Understanding of Functions, Paper presented at the American Educational Research Association, San Francisco.
Monk, S. (1992). Students' Understanding of a Function Given by a Physical Model, The Concept of Function, Aspects of Epistemology and Pedagogy, MAA Notes 25, 175 – 194.
Selden, A. and Selden, J. (1992). Research Perspectives on Conceptions of Functions: Summary and Overview, The Concept of Function, Aspects of Epistemology and Pedagogy, MAA Notes 25, 1-16.
Sfard, A . (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin, Educational Studies in Mathematics, v.22, p. 1-36.
Sierpinska, A . (1992). On understanding the notion of function, In E. Dubinsky and G. Harel (eds.), The Concept of Function. Aspects of Epistemology and Pedagogy, (Dutch) MAA Notes, vol. 25, pp. 25-58.
Vidakovic, D. (1996). Learning the concept of inverse function. *Journal of Computers in Mathematics and Science Technology, 15*(3), 295-318.
Vinner, S. and Dreyfus, T. (1989). Images and Definitions for the Concept of Function, Journal for Research in Mathematics Education, 20(4), 356-366.

The Effects of Using Spreadsheets in Business Calculus on Student Attitudes

Melissa Mills
Oklahoma State University

This study investigates the effects of using spreadsheets in Business Calculus. Both the computer and non-computer sections were taught using reformed curricula that focused on business applications, the use of realistic data sets, and conceptual understanding. This study compares student attitudes towards mathematics in both spreadsheet and non-spreadsheet sections as measured by pre- and post- surveys, student interviews, and examination data.

Key words: business calculus, student attitudes, spreadsheets

Literature Review

A significant body of research has examined Calculus for STEM students (e.g. Asiala, Cottrill, Dubinsky & Schwingendorf, 1997; Bressoud, Carlson, Mesa & Rasmussen, 2013; Carlson, Jacobs, Coe, Larsen & Hsu, 2002; Gonzalez-Martin & Camacho, 2004; Gravemeijer & Doorman, 1999; Oehrtman, 2009; Thompson, 1994; Zandieh, 2000). However, the research on Applied Calculus and Business Calculus in particular is far less extensive (e.g. Garner & Garner, 2001; Liang & Martin, 2008). Before calculus reform, applied calculus was typically a watered-down version of theoretical calculus, consisting mostly of problems devoid of any application context (Garner & Garner, 2001).

Early efforts to incorporate technology in Applied Calculus provided evidence that students can understand calculus concepts before mastering procedural skills (e.g. Heid, 1988; Judson, 1990). The use of technology shifts the instructional emphasis from computation to analysis, which is significant because students in Applied Calculus often have weak algebraic skills. The emphasis on conceptual skills afforded by the use of technology was shown to give students more confidence and flexibility when speaking about the concepts of Calculus, though it did not show any measurable gains on the written examinations (Heid, 1988).

These findings led to a call for re-sequencing the material so that the concepts are presented via application problems before the computational techniques are presented. The Calculus Consortium produced the Hughes-Hallett text (Hughes-Hallett, 1996), which increases the use of applications and modeling with realistic data, and presents conceptual ideas before computational techniques. The MAA's CRAFTY report for Business and Management (Lamoreux, 2004) recommends an emphasis on modeling and application problems in Business Calculus, along with the use of standard business technology (spreadsheets). The Hughes-Hallett text meets many of these recommendations and is considered a "reformed" text (Garner & Garner, 2001), though it does not incorporate the use of spreadsheets.

Although there have been a few efforts to incorporate spreadsheets into the Business Calculus curriculum (Felkel & Richardson, 2009; Lamoreux & Thompson, 2003; Liang & Martin, 2008), the trend is not widespread, and there has been little research into the effectiveness of incorporating spreadsheets in particular.

Studies investigating the effectiveness of calculus courses often focus on student attitudes towards mathematics, students' conceptual understanding, and students' ability to perform on examinations (Depaolo & Mclaren, 2006; Bressoud, et. al., 2013; Heid, 1988; Garner & Garner, 2001). Student performance in a reformed Business Calculus course (using the Hughes-Hallett, 1996 text) showed an increase in their ability to correctly answer conceptual questions and showed more confidence when explaining calculus concepts (Garner & Garner, 2001). It has been shown that attitudes towards mathematics got worse after taking Calculus I. The study showed a decrease in confidence, decrease in enjoyment, and a decrease in desire to continue in mathematics (Bressoud et. al., 2013). However, it has been shown that student attitudes towards business calculus can improve after doing a unit on Calculus in a quantitative analysis course taught by Business faculty (Depaolo & Mclaren, 2006).

This study investigates the impact of using spreadsheets in a reformed Business Calculus course. In particular, the study compares two sections of a reformed Business Calculus course taught using spreadsheets with two sections of a reformed Business Calculus course using graphing calculators. The analysis focuses on student attitudes and their performance on comparable exam items.

Research Questions

1. Does using spreadsheets in Business Calculus have an effect on student attitudes towards mathematics?

2. How does using spreadsheets contribute to students' opinions of the usefulness of a Business Calculus course?

Methods

The data were collected at a large comprehensive research university in the Midwest during the summer 2014 semester. Four sections of Business Calculus were offered during the summer session. Two of these sections used a reform curriculum (Hughes-Hallett, 2010) and the other two used a different reform curriculum that is structured around the use of spreadsheets (May & Bart, 2012). All four sections covered differential calculus and its applications to business (optimization, marginal analysis, elasticity, and linear approximation). The instructors of the four sections agreed to teach the chapter on multivariable functions (including partial derivatives and optimization) in lieu of the chapter on integration.

One of the computer sections was taught by the author and the other was taught by a graduate student. Because of the use of spreadsheets, it is possible to routinely construct mathematical models from data. The text attempts to use examples that students should view as relevant to business and notation that is consistent with their other business classes. Because many of the students are not familiar with using spreadsheets to do computations, the text also covers the necessary spreadsheet skills to do the problems in each section. In class, the instructor would lecture, going back and forth between working examples in spreadsheets and presenting concepts on the chalkboard or projector. The students took notes and followed along with the spreadsheet examples on their own laptops. Because of the small size of the sections (no more than 28 students each), the instructors were able to help troubleshoot students' technical problems and occasionally hold whole-class discussions. Students turned in both paper homework assignments and spreadsheet assignments. They were allowed to use spreadsheets on most quizzes, and their exams were divided into a "computer portion" and a "non-computer portion."

Both non-computer sections were taught by graduate students that had previous experience teaching the course using the Hughes-Hallett (2010) text. The Hughes-Hallett text is a reform text that focuses on real applications and conceptual understanding of the calculus content. The text has a balance between skills and concepts, and uses the "rule of four," which encourages students to think about functions in different ways (graphically, numerically, symbolically, and verbally). The text is technologically agnostic, although in these particular sections the instructors used graphing calculators occasionally. The instructors presented the material via lecture, providing prepared notes for students to print off and fill in during class. The instructors used graphing calculators in class, and permitted students to use graphing calculators on all quizzes and examinations.

The data collected for this study include pre and post surveys of student attitudes, written exams, and post-course student interviews. All students enrolled in Business Calculus were asked to participate in the study. The participants included 46 students from the computer sections and 38 students from non-computer sections.

Three students from the computer sections and three from the non-computer sections volunteered to participate in 15-minute interviews. The interview addressed the students' perceptions of the usefulness of the course as a whole, asked that they comment about different aspects of the course, and addressed their understanding of the concept of derivative. The interviews took place in the three days prior to the final examination.

To measure student attitudes towards mathematics, I used the attitudes portion of the CSPCC instrument (Bressoud, et. al., 2013). The specific questions used in this analysis can be found in the Appendix.

Results

Preliminary data analysis concentrated on the interview data and four items on the questionnaire. The four questionnaire items addressed: perceived ability to succeed, confidence, perceived understanding, and enjoyment were all measured on a six-point Likert scale.

Business Calculus students are often described as "math phobic" (Lamoreux, 2004), while students who take Calculus 1 have chosen more math-intensive majors. The data show that Business Calculus students, in general, have less enjoyment in mathematics than do Calculus 1 students (see Bressoud, et. al, 2014). The participants have a high level of confidence in their knowledge and abilities. The passing rate (grade C or better) for all sections in the summer semester was 83.7%. Note that many of the students who failed or withdrew were not available to take the post-survey.

Table 1. *Student attitudes from pre- and post-survey. Percentage agreeing with each of the statements.*

	Non-computer Pre-survey (n=36)	Non-computer Post-Survey (n=32)	Computer Pre-survey (n=44)	Computer Post-Survey (n=43)
1. I believe that I have the knowledge and abilities to succeed in this course.	92%	100%	89%	95%
2. I am confident in my mathematical abilities.	72%	84%	91%	88%
3. I understand the mathematics that I have studied.	81%	94%	98%	88%
4. I enjoy doing mathematics.	58%	71%	61%	67%

The percentage change was computed for students for whom pre-and post-surveys can be matched. The items were on a six point Likert scale, so the change was computed by taking their post-response minus their pre-response and dividing by six as presented in Table 2. There is positive change for both sections on all of the items except for item 3 in the computer section. Interview data from six participants suggests that the students perceive that they understand the mathematics more when they write it on paper. This could be a possible explanation for why students in the computer section did not see a gain in their perceived understanding.

Table 2. *Average percentage change for students for whom pre- and post- surveys can be matched*

	Non-Computer	Computer
1. I believe that I have the knowledge and abilities to succeed in this course.	2.69%	5.69%
2. I am confident in my mathematical abilities.	1.08%	0.40%
3. I understand the mathematics that I have studied.	3.76%	-1.98%
4. I enjoy doing mathematics.	3.33%	6.35%

The interviews focused on the students' perception of the course as a whole and the usefulness of the course in their business classes and future careers. The small number of interviews suggest that students view learning the mathematics with spreadsheets as directly applicable to their future careers, while students who did not use spreadsheets were not as certain that they would use the material in the future.

The three students from the computer sections all agreed that they would use the spreadsheet knowledge that they gained in the course. One student said, that "the course was really beneficial, as far as getting something out of it. Especially learning spreadsheets. I mean, that is, I can't stress how important that is when you get out of here. That's what everything is. Everything is done with spreadsheets." Another student said, "in the business world, you do a lot with Excel. I feel like learning it in this class has upped my confidence of how to do spreadsheets"

The three students in the non-computer course were not certain that they would use the concepts taught in the course. When asked if taking this course would help him in his future job, one student said "I'm not sure it will." Another student said, "Um, to an extent. I'm not 100% sure… [but] it was really fun to learn all of those things, even if I'm not going to use them." The third student said, "I have a genuine interest in math, so I think I can kind of tie it in to what I might be using [in the future]."

Continuing analysis focuses on the remainder of the questionnaire data and a comparison of the examination data from both computer and non-computer sections.

Questions

1. This study had a relatively small number of interviews. What are some suggestions for interview questions that can be asked when I do this study on a larger scale?
2. Examination data was also collected from students in both sections. What differences should I attend to when comparing student work from both sections?

References

Asiala, M., Cottrill, J., Dubinsky, E., & Schwingendorf, K. (1997). The development of students' graphical understanding of the derivative. *Journal of Mathematical Behavior, 16*(4), 399-431.

Bressoud, D., Carlson, M., Mesa, V., & Rasmussen, C. (2013). The calculus student: insights from the Mathematical Association of America national study. *International Journal of Mathematical Education in Science and Technology, 44*(5), 685-698.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-378.

Depaolo, C. & Mclaren, C. (2006). The relationship between attitudes and performance in business calculus. *INFORMS Transactions on Education, 6*(2), 8-22.
Felkel, B. & Richardson, R. (2009). *Networked Business Math*, Kendall Hunt.
Garner, B. & Garner, L. (2001). Retention of concepts and skills in traditional and reformed applied calculus. *Mathematics Education Research Journal, 13*(3), 165-184.
Gonzalez-Martin, A. & Camacho, M. (2004). What is first-year Mathematics students' actual knowledge about improper integrals? *International Journal of Mathematical Education in Science and Technology, 35*(1), 73-89.
Gravemeijer, K. & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics, 39*(1), 111-129.
Heid, M. K. (1988) Resequencing skills and concepts in applied calculus using the computer as a tool. *Journal for Research in Mathematics Education, 19*(1), 3-25.
Hughes-Hallett, D., Gleason, A.M., Lock, P.F., Flath, D., et al. (2010). *Applied Calculus, Fourth Edition.* Hoboken, NJ: John Wiley & Sons.
Hughes-Hallett, D., Gleason, A.M., Lock, P.F., Flath, D., Gordon, S.P., Lomen, D.O., et al. (1996). *Applied Calculus, preliminary edition.* New York, NY: John Wiley & Sons.
Judson, P. (1990). Elementary business calculus with computer algebra. *Journal of Mathematical Behavior, 9*, 153-157.
Lamoreux, C. & Thompson, R. (2003). *Mathematics for Business Decisions,* MAA.
Lamoreux, C. (2004). Report for business and management. In CRAFTY *The Curriculum Foundations Project: Voices of the Partner Disciplines.* Washington, DC: Mathematical Association of America, available online at http://www.maa.org/cupm/crafty/cf_project.html.
Liang, J. & Martin, L. (2008). An Excel-aided method for teaching calculus-based business mathematics. *College Teaching Methods & Styles Journal, 4*(11), 11-23.
May, M. & Bart, A. (2012). *Business Calculus with Excel,* (preprint).
Oehrtman, M. (2009). Collapsing dimensions, physical limitation, and other student metaphors for limit concepts. *Journal for Research in Mathematics Education, 40*(4), 396-426.
Thompson, P. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics, 26*(2), 229-274.
Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. *Research in Collegiate Mathematics Education. IV. CBMS Issues in Mathematics Education*, 103-127.

Appendix

Please select the most appropriate answer.

	Strongly Disagree	Disagree	Slightly Disagree	Slightly Agree	Agree	Strongly Agree
I believe I have the knowledge and abilities to succeed in this course.	O	O	O	O	O	O
I am confident in my mathematical abilities.	O	O	O	O	O	O
I understand the mathematics that I have studied.	O	O	O	O	O	O
I enjoy doing mathematics.	O	O	O	O	O	O

Students' Reasoning about Marginal Change in an Economic Context

Thembinkosi Mkhatshwa and Helen Doerr

Syracuse University

This study reports on how ten undergraduate students enrolled in business calculus reasoned about marginal change (marginal cost, marginal revenue, and marginal profit) while engaged in a task-based interview followed by a semi-structured interview. The study had two major findings: (1) students had difficulty distinguishing between marginal cost and approximate marginal cost and (2) students conflated marginal cost and marginal revenue with total cost and total revenue respectively. For future research, we might consider investigating how students' understanding of marginal change impacts their ability to solve optimization problems situated in the context of cost, revenue, and profit.

Key words: Marginal Change, Business Calculus, Rates of Change

Literature Review

Students' understanding of rates of change (average and instantaneous rate of change) in a motion context is well documented in the research literature (Bery & Nyman, 2003; Beichner, 1994; Carslon, Jacobs, Coe, Larsen, & Hsu, 2002; Cetin, 2009; Monk, 1992). Research also exists on students' understanding of rates of change in non-motion contexts such as fluid flow and temperature change (Carslon et al., 2002), light intensity and voltage drop in a discharging capacitor (Doerr, Ärlebäck, & O'Neil, 2013), and banking (Wilhelm & Confrey, 2003). However, research on students' understanding of rates of change, and in particular marginal change, in an economic context is scarce, a gap that this study attempted to address through the following research questions:

1. What do students' responses to marginal economic analysis (cost, revenue, and profit) problems reveal about their understanding of marginal change and rates of change?
2. What does students' reasoning about marginal change reveal about their understanding of the derivative?

Theoretical Framework

This study used a multiple representation's framework (Davis, 2007) that served as a lens through which we were able to analyze students' reasoning about marginal change in the real-world context of cost, revenue, and profit in three representations: (1) graph, (2) table, and (3) text.

Research Methodology

The study followed a qualitative research design. Data were collected using task-based interviews (Goldin, 2000) and semi-structured interviews. Open coding and Davis's (2007) framework were used to code the data.

Results of the Research

There were three major results in the study: (1) nearly all the students conflated marginal cost with total cost and marginal revenue with total revenue (2) students had difficulty explaining approximate marginal cost let alone distinguishing it from marginal cost, a result that can be attributed to a limited understanding of the concept of the derivative, and (3) nearly all the students reasoned correctly about marginal change information rooted in a task about a major airline within the immediate context of the task while about half of them reasoned correctly beyond the immediate context of the task.

Discussion and Conclusions

Most of the students demonstrated an understanding of marginal change as an amount of change and not as a rate of change which is problematic since one of the major goals of a business calculus course is to help students understand marginal change as a rate of change. It would be important to investigate the effectiveness of integrating a modeling perspective (Lesh & Zawojewski, 2007) in the teaching of business calculus in helping students develop an understanding of marginal change as a rate of change. It would be important for future research to investigate how students' understanding of marginal change impact (enhance/limits) their ability to solve optimization problems that are situated in an economic context.

References

Beichner, R. (1994). Testing student interpretation of kinematics graphs. *American Journal of Physics, 62*(8), 750–755.

Berry, J. S., & Nyman, M. A. (2003). Promoting students' graphical understanding of the calculus. *The Journal of Mathematical Behavior*, *22*(4), 479-495.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 352-378.

Çetin, N. (2009). The ability of students to comprehend the function-derivative relationship with regard to problems from their real life. *Primus*, *19*(3), 232-244.

Davis, J. D. (2007). Real-world contexts, multiple representations, student-invented terminology, and y-intercept. *Mathematical Thinking and Learning*, *9*(4), 387-418.

Doerr, H. M., Ärlebäck, J. B., & O'Neil, A. H. (2013). Intepreting and Communicating about Phenomena with Negative Rates of Change. In *120th ASEE Annual Conference & Exposition.*

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517-545). Mahwah, NJ: Laurence Erlbaum Associates.

Lesh, R., & Zawojewski, J. (2007). Problem solving and modeling. *Second handbook of research on mathematics teaching and learning*, *2*, 763-804.

Monk, S. (1992). Students' understanding of a function given by a physical model. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy, MAA Notes, Vol.25* (pp. 175-193). Washington, DC: Mathematical Association of America.

Wilhelm, J. A., & Confrey, J. (2003). Projecting rate of change in the context of motion onto the context of money. *International Journal of Mathematical Education in Science and Technology*, *34*(6), 887-904.

Bidirectionality and covariational reasoning

Kevin C. Moore
University of Georgia

Teo Paoletti
University of Georgia

Students' thinking about quantities that vary in tandem remains an important area of mathematics education research due to its implications for student success in mathematics. In this paper, we expand on a way of thinking about covarying quantities, called bidirectional reasoning, in ways not detailed in prior research. A student thinking bidirectionally understands two quantities varying so that the conceived relationship does not have an inherent dependency; the student understands and anticipates that quantities exist and covary simultaneously. After describing bidirectional reasoning and connecting to informing theories, we draw from our work with undergraduate students to illustrate a student reasoning bidirectionally. Because this paper serves as a (re)introduction to bidirectional reasoning and relationships, we close with potential implications of students' bidirectional reasoning, and we hypothesize productive lines of future research.

Key words: Covariational reasoning; Quantitative reasoning; Function; Cognitive Research

Students' covariational reasoning–thinking about how quantities change in tandem–is critical to their developing sophisticated ways of thinking about numerous important mathematical ideas including rate of change, function, and accumulation (Castillo-Garsow, 2012; Ellis, 2007; Johnson, 2012; Moore, 2014a; P. W. Thompson, 1994). Despite the importance of covariational reasoning for students' learning, even top-performing students encounter difficulties thinking covariationally (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). Moreover, numerous researchers have argued there is much to learn about students' covariational reasoning including identifying productive ways of thinking covariationally (Castillo-Garsow, Johnson, & Moore, 2013; P. W. Thompson, 2011). We respond to this call by expanding on a way of thinking about quantities and their relationships termed bidirectional reasoning (A. G. Thompson & Thompson, 1996). We first describe theories that inform our detailing of bidirectional reasoning. After explaining bidirectional reasoning in terms of its entailed mental actions, and drawing from our work to better understand students' and teachers' covariational reasoning, we provide an example of student activity that suggests her reasoning bidirectionally. Because our main purpose is to reintroduce bidirectional reasoning into the conversation on students' covariational reasoning, we close with ideas for future research and the possible implications of students' bidirectional reasoning.

Theoretical Background

Over the past two decades, an increasing number of researchers have made contributions to the literature base on students' covariational reasoning (Carlson et al., 2002; Castillo- Garsow, 2012; Confrey & Smith, 1995; Ellis, 2007; Johnson, 2012; Oehrtman, Carlson, & Thompson, 2008), both with respect to students' understandings of various topical areas (e.g., functions, rate of change, and the fundamental theorem of calculus) and their enactment of important mental processes (e.g., generalization, modeling, and problem solving). Despite these researchers' common intention of understanding students' covariational reasoning, their treatments of covariation are varied. For instance, Confrey and Smith (1994, 1995) approached covariation in terms of students' reasoning about discrete numerical values, finding patterns in these values, and interpolating patterns between them. In contrast, Thompson and Saldanha (Saldanha & Thompson, 1998; P. W. Thompson, 2011) described students' covariation in ways that entail continuous images of change that are magnitude based, and thus not constrained to the availability of numerical data (e.g., specified values).

Carlson et al. (2002) extended the body of literature on covariational reasoning by providing a framework of mental actions (relevant to numerical patterns and magnitudes) that describe specific ways that students reason about and coordinate quantities that vary in tandem.

We interpret our work to align with Thompson, Saldanha, and Carlson et al., as well as other researchers who have built on these researchers' explication of covariation (see: Castillo-Garsow (2012); Johnson (2012)). We focus on covariation in terms of the ways students reason with (continuous) magnitudes because we have found it valuable when characterizing students' covariational reasoning, as well as in articulating productive ways of thinking covariationally (Moore, Paoletti, & Musgrave, 2013; Moore & Thompson, submitted). Despite our focus on covariation in terms of imagining varying magnitudes, we emphasize that the different covariation foci complement each other in numerous ways. For instance, researchers have argued that students' capacity to imagine quantities as continuously changing magnitudes has significant implications for the sophistication of their thinking discretely and numerically. A detailed description of the complementary nature of these frameworks is found elsewhere (Castillo-Garsow, 2010; Castillo-Garsow et al., 2013).

To illustrate an image of covariation based in magnitudes, consider an individual boarding a Ferris wheel ride at the bottom and taking a ride. One can envision that *as* the amount the rider rotates increases, that individual's distance from the ground *simultaneously* increases (MA2: Carlson et al. (2002)). As the ride begins, one can also reason that for successive equal changes in the amount the individual has rotated on the ride, the individual's distance from the ground will increase by less for each equal change in rotation (*Figure 1*) (MA3: Carlson et al. (2002)). As this example illustrates, an individual need not reason about numerical values to conceptualize how quantities covary; in fact, *Figure 1* does not include labeled numerical values. Instead, an individual envisions covarying magnitudes while understanding the magnitudes as measureable at all instantiations of the covariation. Because such reasoning is not constrained to numerical values, students who think in this way develop productive conceptions of function (Moore, 2014a). Namely, students who think about magnitudes are able to move past *action views* of function (Dubinsky & Harel, 1992) because they are not constrained to calculating specific input-output values or determining an analytic rule that relates numerical values.

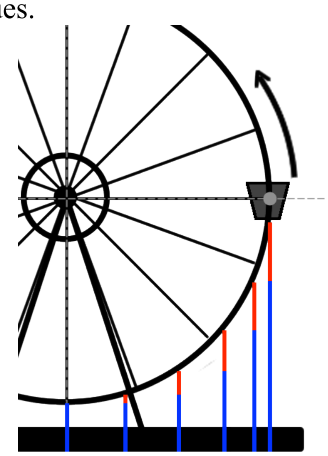


Figure 1. A quantity increasing by less for each equal change in rotation.

Bidirectional Reasoning

Saldanha and Thompson (1998) defined simultaneous (continuous) covariation as someone maintaining a sustained image of two quantities changing in tandem so, "one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value" (p. 298). Describing a teaching experiment on speed with a student named Ann, A. G. Thompson and Thompson (1996) explained:

Pat aimed to have Ann see increases in Rabbit's amount of distance and amount of time as happening simultaneously. His intention is depicted in [*Figure 2*]. The bi-directional relationship which constitutes covariation entails thinking of both quantities varying simultaneously *without* a necessary dependency. Rather, if one focuses first on distance, one can determine time; if one focuses first on time, one can determine distance. (p. 8)

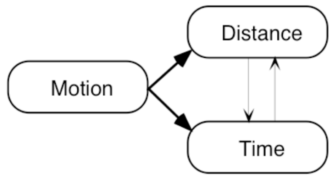


Figure 2. Motion as entailing a bidirectional relationship between distance and time (A. G. Thompson & Thompson, 1996, p. 8).

We interpret Saldanha and Thompson's (1998) phrasing of "*either* [emphasis added] quantity's value" to be inline with A. G. Thompson and Thompson's (1996) description of bidirectional reasoning as an image of "quantities varying simultaneously *without* a necessary dependency." In our work, and like Thompson and Thompson inferred when working with Ann, we have found that students tend to conceive relationships in ways that entail an inherent dependency. Students also think about changes in one quantity *asynchronously producing or causing* changes in the other quantity (*Figure 3a*). Due to the implications of thinking about relationships in this way, which we allude to below, we highlight that a student who conceives a bidirectional relationship does not imagine change asynchronously or entailing inherent input-output quantities. Rather, students' bidirectional reasoning entails constructing and re-constructing images of covarying quantities so that they anticipate quantities existing and covarying simultaneously (Saldanha & Thompson, 1998) (*Figure 3b*).

Due to understanding covariation as occurring simultaneously, a student who has a bidirectional relationship in mind understands that the same covariational relationship exists regardless of the order in which one considers the quantities. Returning to the Ferris wheel example, a student understands that the relationship between the rider's angle of rotation and distance from the ground is not influenced by the order in which they consider the covarying quantities. More generally, when imagining quantity Q_A changing by some magnitude $\Delta||Q\_A||_{2,1}$, they anticipate quantity Q_B simultaneously changing by some magnitude $\Delta||Q\_B||_{2,1}$ (sequence 1 in *Figure 3b*). At the same time, the student understands that they could have first imagined Q_B changing by the magnitude $\Delta||Q\_B||_{2,1}$ while anticipating and understanding Q_A changing by that same magnitude $\Delta||Q\_A||_{2,1}$.

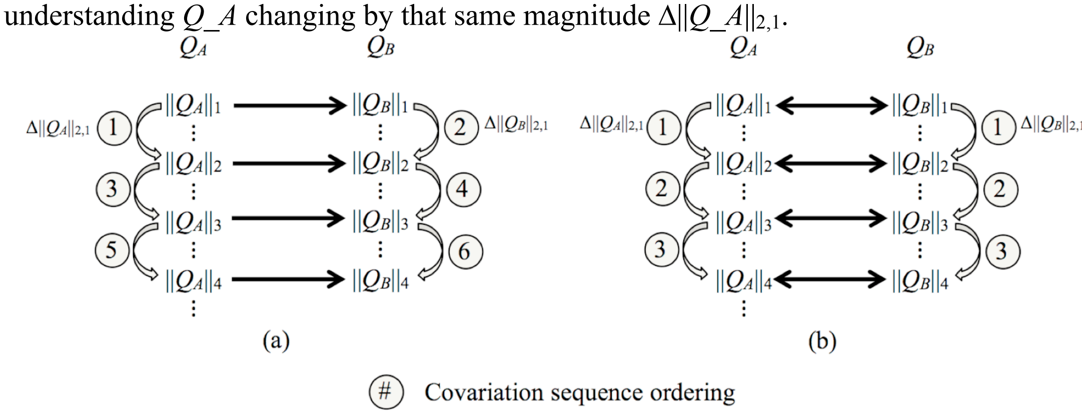


Figure 3. A (a) unidirectional and (b) bidirectional relationship.[1]

Bidirectional Reasoning and Cartesian Graphs

The Cartesian coordinate system (CCS) is based on the projection of two quantities (directed lengths). Hence, students' bidirectional reasoning is relevant to their conceptualization of graphs in the CCS. A student thinking bidirectionally understands a graph in the CCS as constituted by points representing projected quantities that covary simultaneously. The student is not constrained to thinking about the graph in a way that entails an inherent input-output dependency or orientation.

In our work with students, particularly at the undergraduate level, we have found their reasoning about relationships in the CCS to be unidirectional; inherent to their ways of thinking is understanding the quantity represented along the vertical axis as dependent on the quantity represented along the horizontal axis (Moore, Silverman, Paoletti, & LaForest, 2014). Denoting the horizontal axis as the independent/input quantity is a convention common to the teaching of mathematics, but it is often not a convention with respect to students' understanding of the CCS. Instead, students understand it as a necessary part of graphs. Although we have predominantly observed undergraduate students constrained to this way of thinking, by engaging them in repeated and frequent opportunities to reason about graphs without emphasizing a dependency between quantities, we have been able to support them reasoning about the CCS bidirectionally. That is, they come to understand the convention of input on the horizontal axis as exactly that: a convention or a choice, with the opposite choice just as mathematically viable. Choosing an axis to be the input quantity is not necessary to graph a relationship between simultaneously covarying quantities. The same relationship exists regardless of the chosen input-output quantities or axes orientation.

Bidirectional Reasoning and Events

When investigating students' covariational reasoning in the context of problem solving, modeling, and graphing, we have found their bidirectional reasoning to be germane to characterizing how they conceive an event (e.g., a Ferris wheel rotating or two runners racing) in terms of quantities and their relationships. Moreover, we have found undergraduate students are more prone to construct images of events (versus graphs in the CCS) that entail bidirectional relationships. Corroborating Saldanha and Thompson's (1998) description of covariation as developmental, we have found it is through students repeated engagement coordinating two quantities in strategic (but arbitrary with respect to dependency) ways that they come to understand an event as entailing quantities that vary bidirectionally.

As an example, a student modeling the motion of a Ferris wheel rider might first take note of various 'landmark' distances of the rider from the ground (e.g., maximum or minimum distances), how the distance increases or decreases between these landmarks, and then determine corresponding changes in the angle of rotation. Subsequently, the student might seek to understand the rate at which the rider's distance from the ground changes by considering equal changes in the rider's angle of rotation. With equal changes in the rider's angle of rotation constructed, the student compares how the rider's distance changes over a particular interval of the trip (*Figure 1*). Over the course of her or his activity, the student alternates the order in which she considers each quantity's variation, all the while holding in mind that a change in one quantity's magnitude necessitates an associated, simultaneous change in the other quantity's magnitude.

[1] We note that changes in quantities' magnitudes (the sequences in *Figure 3*) might be imagined continuously (smoothly) or discretely (chunky) (Castillo-Garsow et al., 2013).

An Example of Student Work

Our work developing second-order models of students' and teachers' covariational reasoning (Moore, 2014a, 2014b; Moore et al., 2013; Moore et al., 2014) informs our descriptions above. Our focus on students' bidirectional reasoning is partially motivated by this work, and specifically our observations that students who have a greater propensity and capacity to reason bidirectionally with respect to *both* a coordinate system and an event seem to form more productive ways of thinking about important mathematical concepts including rate of change, function, and graphing. Supporting the claims by Carlson, Larsen, and Lesh (2003), these students also have more success modeling dynamic events due to their tendency to maintain a sustained focus on reasoning with quantities in flexible ways.

To illustrate, consider Arya, an undergraduate mathematics education student, explaining how she created a graph by coordinating the volume and height of water in a bottle as the bottle fills. She initially marked the picture of the bottle where the shape of the bottle began to change (*Figure 4*) and described how height and volume covary between these marks. For instance, Arya stated, "From here to here [*pointing to second tick mark from the bottom to the third*] again for equal changes in height our volume is increasing. But increasing this time at an increasing rate cause the bottle is getting bigger [*using her hands to indicate the bottle is getting wider*] so it can hold more water for equal changes in height" (MA3: Carlson et al. (2002)). To represent this relationship on her graph, Arya drew dashed vertical segments representing volume and solid vertical segments indicating amounts of change (*Figure 4*).

After describing each section of the graph in relation to the bottle as above, Arya spontaneously described how the height of the bottle would change for equal changes in volume. For example, using blue and red markers, she shaded areas indicating equal amounts of volume (*Figure 5*), stating, "It was like pushed in [*using her hands to indicate the bottle gets thinner*], for equal, if I put in the same amount it's taking up more height of the glass… So it got bigger. That's bigger [*shading next area in green*]… That's equal the volume but it's taking up more height. Equal changes of volume, our height is increasing at an increasing rate." Then denoting this relationship on the graph, Arya marked equal changes along the vertical axis and signified the increasing changes of height with blue segments (*Figure 5*) (MA3: Carlson et al. (2002)).

Over the course of her activity, and both in the context of the graph and a diagram of the situation, Arya alternated the order in which she imagined the quantities covarying. Throughout her descriptions, she described both quantities varying; to Arya, discussing one quantity varying necessitated discussing the other quantity varying because the quantities covaried simultaneously. Thus, we take Arya's activity to suggest her reasoning bidirectionally with respect to both the event of filling a bottle with water and in creating a CCS graph.

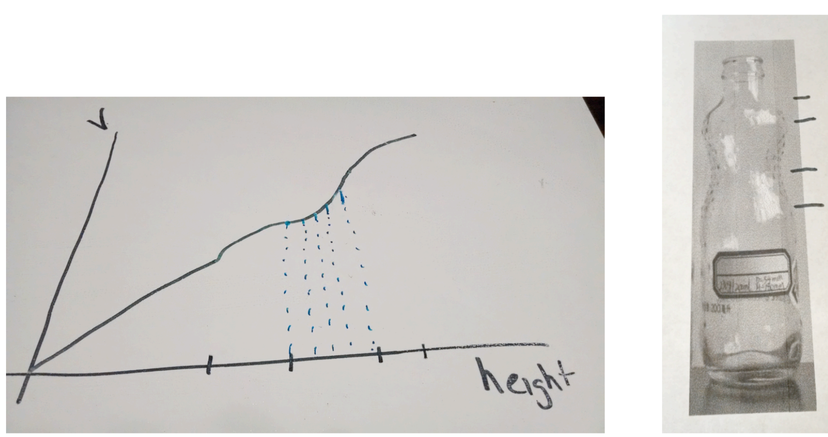


Figure 4. Arya considers how the volume changes for equal changes of height.

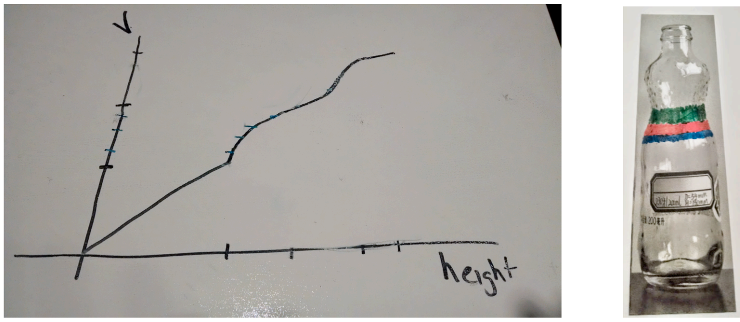


Figure 5. Arya considers how the height changes for equal changes of volume.

Implications and Moving Forward

We envision that future investigations into students' bidirectional reasoning will have wide-ranging implications, the most obvious of which is related to students' ways of thinking about function. Function remains a content strand in which students' difficulties impact their success in mathematics and other STEM fields (Oehrtman et al., 2008). Complicating the matter, students often construct ways of thinking about function that do not entail generalized or invertible processes (Oehrtman et al., 2008). We hypothesize that students who are supported in thinking bidirectionally will construct more productive ways of thinking about function than those who are not. A student who has the opportunity to repeatedly coordinate covarying quantities so that he or she comes to understand quantities as varying simultaneously as Saldanha and Thompson (1998) described essentially establishes a relationship that entails both a function and its inverse. Additionally, if these opportunities include reasoning bidirectionally about a coordinate system, the student can conceptualize a graph as representing a function and its inverse relationship (or inverse function if the relationship is strictly monotonic) simultaneously (Moore et al., 2014). For instance, Arya constituted both the event of filling up a bottle and the corresponding graph in terms of covariation that entailed understanding the height as dependent on volume, as well as volume as dependent on height.

Although we have focused exclusively on graphical representations thus far, when working with students like Arya we have also noted that their reasoning bidirectionally appears to support them in conceiving analytic forms (e.g., equations) in productive ways. With a bidirectional relationship in mind, the student understands associated analytic forms as writing a statement about simultaneous covariation; to write an equation is to write a relationship between quantities' values that remains true as the quantities covary simultaneously (Izsák, 2003). Furthermore, if the analytic form implies dependency (e.g., y = *some expression in* x), a student thinking bidirectionally understands that the relationship can be thought of with respect to the other dependency and anticipate rewriting the analytic form to imply this dependency (e.g., x = *some expression in* y). With respect to both graphical and analytic forms, 'inverse function' is not an action that must be carried out (e.g., reflecting a graph or switching/solving for another variable), but instead is entailed in the conceptualization of a relationship with multiple choices of dependency.

In addition to investigating relationships between students' bidirectional reasoning and their ways of thinking about function, we envision productive future lines of inquiry to be:

How does students' reasoning bidirectionally influence their ability to model dynamic events? Arya's activity suggests that a students' bidirectional reasoning can support their alternating between a dynamic event and a graph modeling the event while holding in mind an invariant relationship. We imagine a different outcome if a student was constrained to reasoning about a relationship or coordinate system with a particular dependency. As Carlson et al. (2003) illustrated, students conceive situations in idiosyncratic ways and their ability to model situations is likely to improve if they can think about relationships in flexible ways.

What connections exist between students' bidirectional reasoning and related research on students' operational or reversible thinking? In investigating students' multiplicative reasoning, Hackenberg (2010) drew on Piagetian notions of operations and mental actions to characterize the students' reasoning with reversible relationships. Hackenberg noted that some students were able to anticipate their activity, while other students were constrained to "carry out some activity, review its results, and then carry out more activity" (pg. 383). We find these differences to be related to the above discussion on inverse function (e.g., inverse as anticipating a change of dependency versus inverse as carrying out an action), and thus hypothesize that connections exist between students' reasoning about reversible multiplicative relationships and students' bidirectional reasoning.

What situations or opportunities support students' bidirectional reasoning? We have found traditional function approaches to foreground unidirectional relationships and instill particular conventions in ways that inhibit students' bidirectional reasoning. Hence, determining situations that afford and promote bidirectional reasoning is a pressing need. One potential approach is using events in which quantities 'naturally' covary simultaneously (e.g., a Ferris wheel ride or a filling a bottle) while repeatedly and frequently prompting students to consider variation in both directions with respect to the event and mathematical representations, as opposed to emphasizing unidirectional functional relationships.

How might students' bidirectional reasoning influence their longitudinal mathematical development? Students' bidirectional reasoning likely has implications for students' ways of thinking for implicit differentiation, implicit functions, and differential equations, as each require reasoning about quantities taking on multiple or ill-defined dependency roles. Also, we conjecture that students' bidirectional reasoning is related to their longitudinal development of coordinate systems. For instance, we hypothesize that coming to understand a three-dimensional graph as an emergent record of varying quantities (see Weber and Thompson (2014)) entails bidirectional reasoning as it requires coordinating variables in different orders while holding in mind that the three quantities vary simultaneously (e.g., fixing z and conceiving how x and y vary simultaneously while anticipating that for any x-y pair, one could fix x or y and consider how z varies simultaneously with y or x, respectively).

References

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-378.

Carlson, M. P., Larsen, S., & Lesh, R. A. (2003). Integrating a models and modeling perspective with existing research and practice. In R. Lesh & H. M. Doerr (Eds.), *Beyond Constructivism: Models and Modeling Perspectives on Mathematics Problem Solving, Learning, and Teaching.* (pp. 465-478). Mahwah, NJ: Lawrence Erlbaum Associates.

Castillo-Garsow, C. (2010). *Teaching the Verhulst model: A teaching experiment in covariational reasoning and exponential growth*. Ph.D. Dissertation. Arizona State University: USA.

Castillo-Garsow, C. (2012). Continuous quantitative reasoning. In R. Mayes & L. L. Hatfield (Eds.), *Quantitative Reasoning and Mathematical Modeling: A Driver for STEM Integrated Education and Teaching in Context* (pp. 55-73). Laramie, WY: University of Wyoming.

Castillo-Garsow, C., Johnson, H. L., & Moore, K. C. (2013). Chunky and smooth images of change. *For the Learning of Mathematics, 33*(3), 31-37.

Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics, 26*, 135-164.
Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(66-86).
Dubinsky, E., & Harel, G. (1992). The nature of the process conception of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 85-106). Washington, D.C.: Mathematical Association of America.
Ellis, A. B. (2007). The influence of reasoning with emergent quantities on students' generalizations. *Cognition and Instruction, 25*(4), 439-478.
Hackenberg, A. J. (2010). Students' reasoning with reversible multiplicative relationships. *Cognition and Instruction, 28*(4), 383-432.
Izsák, A. (2003). "We want a statement that Is always true": Criteria for good algebraic representations and the development of modeling knowledge. *Journal for Research in Mathematics Education, 34*(3), 191-227.
Johnson, H. L. (2012). Reasoning about variation in the intensity of change in covarying quantities involved in rate of change. *The Journal of Mathematical Behavior, 31*(3), 313-330.
Moore, K. C. (2014a). Quantitative reasoning and the sine function: The case of Zac. *Journal for Research in Mathematics Education, 45*(1), 102-138.
Moore, K. C. (2014b). Signals, symbols, and representational activity. In L. P. Steffe, K. C. Moore, L. L. Hatfield, & S. Belbase (Eds.), *Epistemic algebraic students: Emerging models of students' algebraic knowing* (pp. 211-235). Laramie, WY: University of Wyoming.
Moore, K. C., Paoletti, T., & Musgrave, S. (2013). Covariational reasoning and invariance among coordinate systems. *The Journal of Mathematical Behavior, 32*(3), 461-473.
Moore, K. C., Silverman, J., Paoletti, T., & LaForest, K. R. (2014). Breaking conventions to support quantitative reasoning. *Mathematics Teacher Educator, 2*(2), 141-157.
Moore, K. C., & Thompson, P. W. (submitted). Static and emergent shape thinking.
Oehrtman, M., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. L. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 27-42). Washington, D.C.: Mathematical Association of America.
Saldanha, L. A., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensen, K. R. Dawkings, M. Blanton, W. N. Coulombe, J. Kolb, K. Norwood, & L. Stiff (Eds.), *Proceedings of the 20th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 298-303). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.
Thompson, A. G., & Thompson, P. W. (1996). Talking about rates conceptually, Part II: Mathematical knowledge for teaching. *Journal for Research in Mathematics Education, 27*(1), 2-24.
Thompson, P. W. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics, 26*(2-3), 229-274.
Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. Chamberlin, L. L. Hatfield, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOM^e* (pp. 33-57). Laramie, WY: University of Wyoming.
Weber, E. D., & Thompson, P. W. (2014). Students' images of two-variable functions and their graphs. *Educational Studies in Mathematics, 87*(1), 67-85.

Shape thinking and students' graphing activity

Kevin C. Moore
University of Georgia

Patrick W. Thompson
Arizona State University

We describe a construct called shape thinking that characterizes individuals' ways of thinking about graphs. We introduce shape thinking in two forms–static and emergent–that have materialized in our work with students and teachers over the past two decades. Static shape thinking entails thinking of a graph as an object in and of itself, and as having properties that the student associates with learned facts. Emergent shape thinking entails envisioning a graph in terms of what is made (a trace) and how it is made (covarying quantities). We provide illustrations of the shape thinking forms using examples from data that we have gathered with secondary students, teachers, and undergraduate students. We close with future research and teaching directions with respect to students' shape thinking.

Key words: Graphing; Function; Covariational reasoning; Quantitative reasoning

Students' and teachers' graphing activity remains a critical focal area in mathematics education, as their difficulties with graphs have short- and long-term consequences for their success in mathematics and other STEM fields (Oehrtman, Carlson, & Thompson, 2008). Despite their difficulties, students do construct stable and organized ways of thinking over the course of their schooling. Numerous researchers (including ourselves) have claimed that students develop ways of thinking about functions and their graphs that often lack a basis in reasoning about generalized relationships or processes between quantities' values (Dubinsky & Wilson, 2013; Lobato & Siebert, 2002; Oehrtman et al., 2008; Thompson, 1994b, 1994c). If graphs are intended to be representations of related quantities under a coordinate system (with a coordinate system itself being an organization of quantities), then we must ask:

1. *If students do not see a graph representing a relationship between quantities, then what do they think it represents?*
2. *What do we intend students to understand that a graph represents?*
3. *What ways of thinking are involved in understanding a graph as representing a relationship between quantities' values?*

We elaborate on a construct, called *shape thinking,* that we and others (Weber, 2012) have found useful in addressing each of these questions, both clarifying different ways of thinking students hold for graphs and characterizing a productive way of thinking about graphs as emergent relationships between quantities. We discuss shape thinking in two forms–*static* and *emergent*–that clarify important differences among students' understandings of graphs. In detailing the two forms of shape thinking, we draw illustrations of shape thinking from prior studies over the past two decades. For this reason, our purpose is not to report a single study, nor to report on the development or progress of a particular set of individuals. Instead, our purpose is to address the important questions above by describing distinguishable ways of thinking that students and teachers have for graphs.

Two Vignettes

We introduce the forms of shape thinking with two vignettes from clinical interviews (Goldin, 2000) of undergraduate students. Each vignette is a response to the prompt: *A middle-school student graphed the relation defined by* $y = 3x$ *as shown in Figure 1. How might he/she have been thinking when producing the graph?*

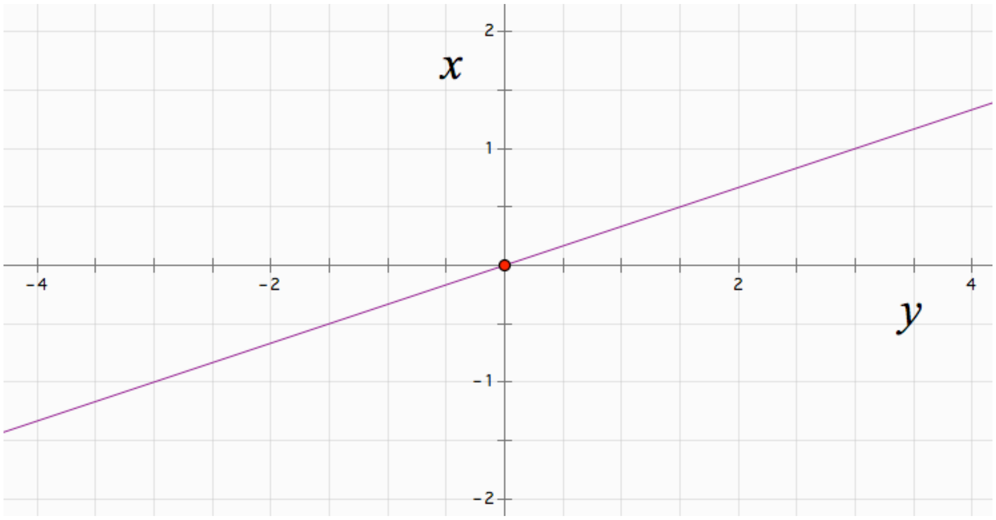


Figure 1. The middle-school student's graph.

Vignette 1

Student 1: He's thinking like this [*turning paper 90 degrees counter clockwise*]. But that's still not right because this is now a negative slope [*tracing the line*].

Int.: What if the student said, "Here's [*rotating back to Figure 1*] how I'm thinking"?

Student 1: The only way I can think of it is like this [*turning paper 90 degrees counter clockwise*] and it's still wrong because after I turn it, this [*laying the marker on the line sloping downward left to right*] is now a negative slope. When I was in middle school we learned a trick to remember positive, negative, no slope, and zero [*making hand motions to indicate directions*]. It's stuck with me so it's important to know which direction the slopes are going, where the slopes are.

Vignette 2

Student 2: He's thinking the rate of change is three, so the change in y is three times the change in x. He put the horizontal axis as y, so whatever he increased by in x, he increased by three times that in y [*tracing her pen to the right three and up one from the origin, then right three and up one from that point*]. He's right.

Int: So what if I do this [*turning paper 90 degrees counter clockwise*]?

Student 2: Well, the relationship is the same because the graph rotated with the axes, so it's still y equals three x. Change in x, change in y [*indicating an arbitrary change in x and a corresponding change in y*]. Change in y is three times change in x.

Student 1 and Student 2 each understood the graph, but their understandings were quite different–they assimilated the graph to different schemes. Student 1 drew on ways of thinking for slope (or rate of change) that were based in perceptual cues, such as thinking a line falling downward left-to-right unquestionably means negative slope. Because Student 1 associated properties of slope with a line's direction and location (e.g., "where the slopes were"), he concluded that rotating the graph changed the slope and, in his understanding, changed the represented relationship. Student 2 examined the graph in terms of how quantities varied in tandem within the axes orientation *as given*. By examining the graph in terms of covarying quantities, she understood that changing the axes' orientation or rotating the sheet of paper does not change the represented relationship. We consider Student 1's actions, which focused on perceptual cues and global properties of shape, to be indicative of *static* shape thinking. We consider Student 2's actions, which focused on the graph as an emergent object constituted by images of covarying quantities, to be indicative of *emergent* shape thinking.

Theoretical Framing

Our interest is to characterize persons' meanings and ways of thinking. To do so, we draw on Thompson and Harel's (Thompson, Carlson, Byerley, & Hatfield, 2014) description of understanding, meaning, and ways of thinking, which has roots in Piaget's (2001) notions of action, operation, scheme, and image. *Understanding* is an in-the-moment state of equilibrium, which may occur from assimilation to a scheme or from a functional accommodation specific to that moment in time. A *Meaning* is the space of implications that the moment of understanding brings forth—actions that the current understanding implies. *Ways of thinking* are "when a person has developed a pattern for utilizing specific meanings…in reasoning about particular ideas" (Thompson et al., 2014, p. 12). Returning to Student 1 and 2, each student's activity suggests that they had constructed specific meanings and ways of thinking about graphs in terms of slope or rate of change.

Data Sources and Motivation

The shape thinking construct originated during the second author's work with middle grade to post-secondary students and teachers. This work included clinical interviews (Goldin, 2000), teaching experiments (Steffe & Thompson, 2000), and professional development, during which he noted students and teachers holding particular dispositions toward understanding graphs. As our research moved forward, differences in students' and teachers' dispositions toward graphing became more apparent during studies in which we explored supporting students' and teachers' covariational and quantitative reasoning (Moore, 2014; Moore, Paoletti, & Musgrave, 2013; Thompson, 1994b, 2013). We often found students' and teachers' ways of thinking about graphs to have little connection to images of covariation, leading us to elaborate on the shape thinking construct. For instance, Weber (2012) introduced notions of *expert* and *novice shape thinking* when characterizing students' reasoning about rate of change in the context of multi-variable functions. Most relevant to the work here, the term static shape thinking emerged as a way to reference a student interpreting a graph statically and giving it meaning by way of association with learned facts, and emergent shape thinking emerged as a way to reference a student interpreting a graph through schemes and images of quantitative and covariational reasoning.[1]

Our motivation for introducing shape thinking and its forms also stems from our asking what a graph represents *to a student*. As mathematics teachers, our instruction on functions and relationships became more productive for student learning once we developed an ear for whether students were thinking about a graph as a static image and object in and of itself or as a trace of quantities having covaried. In what follows, we describe and illustrate both forms of shape thinking. Our illustrations originate from ongoing and retrospective conceptual analyses (Thompson, 2008; von Glasersfeld, 1995) of these clinical interviews, teaching experiments, and professional development.

Static Shape Thinking

Static shape thinking involves operating on a graph as an object in and of itself, essentially treating a graph as a piece of wire (graph-as-wire). Static shape thinking entails assimilations and actions based on perceptual cues and the perceptual shape of a graph. Because static shape thinking entails thinking of the graph as an object, meanings associated with static shape thinking treat mathematical attributes as properties of the graph-as-wire. For instance, in his moment of understanding, Student 1 (Vignette 1) assimilated slope (or rate of

[1] See (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Smith III & Thompson, 2008; Thompson, 1994a, 2011) for more extensive treatments of quantitative and covariational reasoning.

change) more as a property of the graph-as-wire as he perceived it (e.g., the wire rising or falling left to right) than as a measure of how one quantity changes with respect to another.

Another example of static shape thinking is when students treat equations, names, or analytic rules as facts of shape regardless of its coordinate system or orientation. For instance, Excerpts 1 and 2 show responses by two mathematics education undergraduate students to the prompt that a secondary student named Ralph thought that the graph in *Figure 2* displays the inverse sine function because, "*…we are graphing the inverse of the sine function, we just think about x as the output and y as the input*". We designed the task so that Ralph's claim captured the understanding that $y = \sin(x)$ implies $x = \arcsin(y)$ with the appropriate restriction on x. More generally, Ralph's claim rests on the fact that a graphical representation of a single-valued function can be thought of as simultaneously representing the function and its inverse relation (or its inverse function if the original function is strictly monotonic).

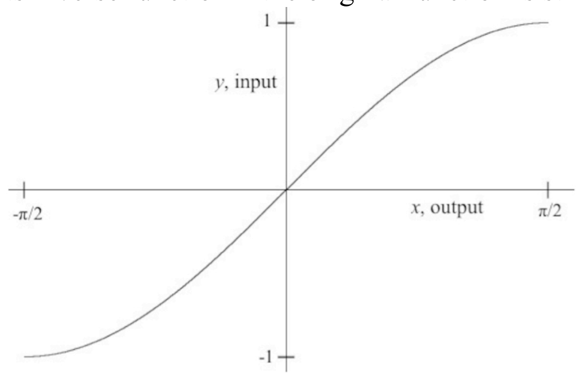


Figure 2. Graph and prompt posed to the PSTs.

Excerpt 1: *Sansa's response to Ralph's statement.*

Sansa: You can't just label it like that…I feel like he's missing the whole concept of a graph…I know you can call whatever axis you know if you are doing time and weight or volume or whatever. You can flip-flop those and be OK. But not necessarily with the sine graph. A sine graph's…a graph everyone knows about.

Excerpt 2: *Brienne's response to Ralph's statement.*

Brienne: I'm thinking this just kind of looks like…the plain sine graph (*laughs*). Which is going to be different. So, no… I guess what I'm like thinking, like struggling with thinking is that like, like I don't know if, or like an inverse function, like the graph of an inverse function, like, can't be the same as the original graph.

To Sansa, the given graph was "a sine graph…everyone knows about." We understood her to mean that the graph was a shape all mathematics students should know exclusively as "the sine graph". Brienne anticipated that a graph of the inverse sine function should appear, in shape, different than that of the given shape because a different function is being graphed. Both students' ways of thinking involved associating a shape with a named function such that no other function could name that shape; the students understood a function's name as a fact of shape (e.g., graph-as-wire).

Emergent Shape Thinking

Emergent shape thinking involves understanding a graph *simultaneously* as what is made (a trace) and how it is made (covariation). As opposed to assimilating a graph as a static object, emergent shape thinking entails assimilating a graph as a trace in progress (or envisioning an already produced graph in terms of replaying its emergence), with the trace being a record of the relationship between covarying quantities. Because we are limited to a static medium in this paper and cannot convey a trace in progress, we convey this way of thinking through snapshots of an emergent trace (*Figure 3*). Emergent shape thinking is more complex than depicted because it entails imagining what happened between snapshots.

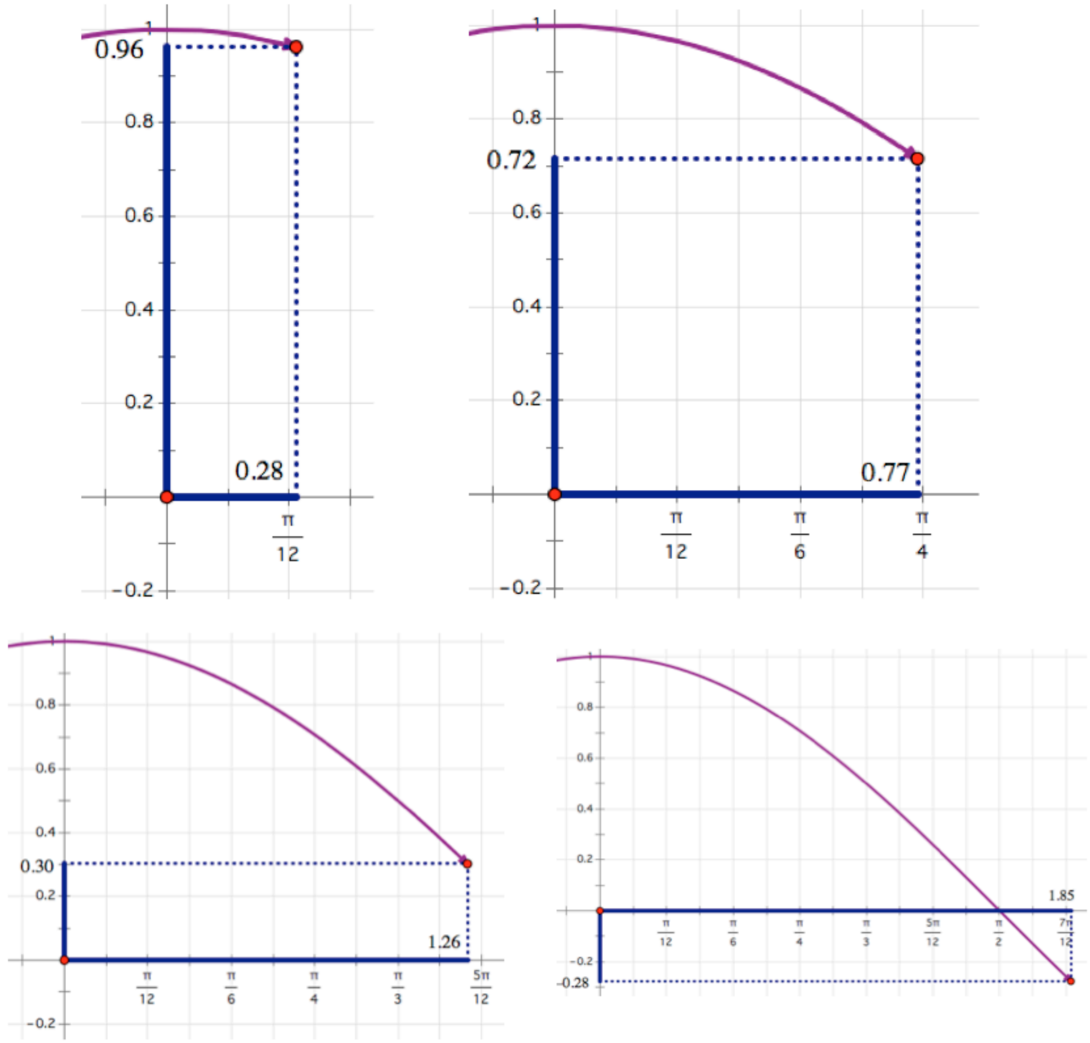


Figure 3. Instantiations of emergent shape thinking.

Because emergent shape thinking entails understanding a graph as an emergent trace of covarying quantities, meanings associated with emergent shape thinking treat mathematical attributes as properties of covariation. To Student 2 (Vignette 2), rate of change was a property of how x and y change together regardless of the graph's orientation. As another example, Shae, a mathematics education undergraduate, said that the graph in Figure 2 represents both $y = \sin(x)$ and $\sin^{-1}(x) = y$. The interviewer then asked Shae how the given graph related to a conventional graph of the arcsine function (e.g., input on the horizontal axis). He asked this question to determine whether presenting Shae with a different shape but the same stated function could perturb her enough that she evoke static shape thinking. Shae went on to describe that both graphs represent "the same thing" (Excerpt 3).

Excerpt 3: *Shae compares nonstandard and standard graphs of arcsine.*

Shae: You could just like disregard the y and x for a minute, and just look at, like, angle measures. So it's like here [*referring to graph of* $\sin^{-1}(x) = y$], with equal changes of angle measures [*denoting equal changes along the vertical axis*] my vertical distance is increasing at a decreasing rate [*tracing graph*]. And then show them here [*referring to graph of* $\sin^{-1}(y) = x$] it's doing the exact same thing. With equal changes of angle measures [*denoting equal changes along the horizontal axis*] my vertical distance is increasing at a decreasing rate [*tracing graph*]. So even though the curves, like, this one looks like it's concave up [*referring to graph of* $\sin^{-1}(x) =$

y from $0 < x < 1$] and this one concave down [*referring to graph of* $\sin^{-1}(y) = x$ from $0 < x < \pi/2$], it's still showing the same thing. [*Shae denotes equivalent changes on each graph as shown in Figure 4*]

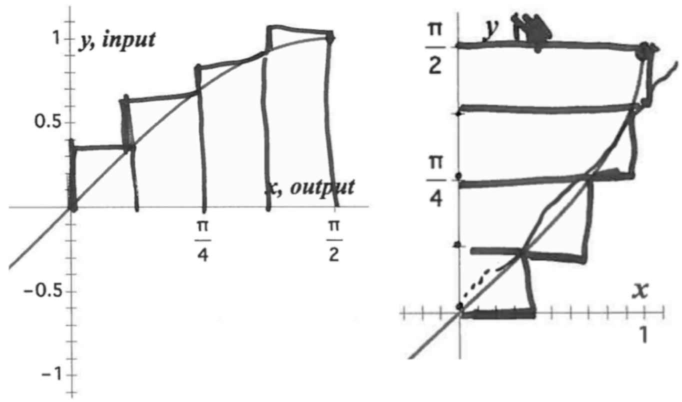


Figure 4. Two graphs that represent one relationship.

Shae reasoned that both graphs convey that some quantity increases at a decreasing rate as another quantity increases in successive equal amounts, while at the same time representing the respective quantities on her graphs (*Figure 4*). By conceiving each graph as an equivalent emergent relationship, Shae understood that a "concave up" trace conveys the same information as a "concave down" trace if the axes are switched. In contrast to Sansa and Brienne's understandings, Shae understood the traces she perceived as representing both the (restricted) sine and inverse sine functions. The function names were not names of a shape; they named a covariational relationship.

Reasoning about or with Quantities

A notable feature of students' static shape thinking is that images of covariation and points as projected quantities represented along the axes are absent from their in-the-moment thinking. In fact, coordinate axes and their representation of quantities are not critical to static shape thinking except in their labeling and orientation–static shape thinking's basis in observables and shape necessitates the labeling and orientations in which the students' abstracted their ways of thinking. In making the claim that images of covariation are absent from their in-the-moment thinking, we do not mean that students engaging in static shape thinking cannot or will not think about quantities and their relationships. Static shape thinking can entail drawing inferences about quantities and their relationships, where these inferences are drawn from a graph's appearance or shape. Hence, static shape thinking does not exclude reasoning about quantities, but the type of reasoning is indexical: a particular shape or property of shape implies something about quantities. Said another way, information about quantities and their relationships are implications of assimilation. To illustrate, students exhibiting actions like Student 1 (Vignette 1) can describe slope in terms of covarying quantities, but this is done after drawing inferences about the slope of the line from perceptual cues (e.g., using that a line falls left to right line to infer a negative slope and conclude y decreases in some manner as x increases).

Students who are limited to thinking about graphs in terms of static shapes can learn to associate information about quantities and their relationships with particular shapes. Students thinking statically are limited to making empirical abstractions from their activity. Students who are capable of thinking about graphs emergently gain insight into relationships between quantities that are more organic to the quantities and relationships. Also, students thinking about graphs emergently are positioned to reflect on their reasoning to form abstractions and generalizations from their reasoning. Hence, unlike associations abstracted through static shape thinking, relationships abstracted through emergent shape thinking are not constrained to a particular labeling and orientation. When changing coordinate conventions and systems, the shape associated with a particular relationship changes in a visual sense. But, the mental operations involved in emergent shape thinking enable constituting a trace in any coordinate system as representative of the same covariational relationship given that the student understands the coordinate system's quantitative structure.

Future Directions for Shape Thinking

Students' shape thinking raises a number of new directions for mathematics education research and teaching. First, researching students' capabilities and constraints when limited to static shape thinking or capable of emergent shape thinking will form a productive line of inquiry. We consider this line of inquiry to offer a new perspective on multiple representations by enabling researchers to be clearer about what a graph represents *to a student*, and thus what students understand multiple representations to be representations of. We find that emergent shape thinking enables students to move among representations while maintaining a subjective sense of invariance in the form of covarying quantities (see Excerpt 3 and Moore et al. (2013)), thus supporting them in conceiving the 'something' that multiple representations are to represent. Second, because static shape thinking can entail inferences about quantities and their relationships, meanings associated with static shape thinking might stem from abstractions partly involving emergent shape thinking. A student who is capable and prone to think about graphs emergently might come to know a family of graphs so well that that thinking statically about a graph implies what they know about its emergence. Understanding these relationships between students' static and emergent thinking will entail exploring the ways that students' static and emergent thinking do or do not complement each other. Such explorations will provide insights into how students' ways of thinking about graphs develop and will contribute additional clarity to the notions that Weber (2012) identified as expert and novice shape thinking. Lastly, we see our work providing a lens to organize and frame curricular approaches more carefully with respect to intended student ways of thinking for graphs and their functions. For instance, we hypothesize that supporting students in thinking emergently will better position them to envision functions as mappings once they can envision completed covariation as having produced all possible pairs of values associated with a mapping. On the other hand, we hypothesize that an approach to function that introduces a parent shape and treats other functions as translations of the parent shape is more likely to promote students thinking of graphs as objects in and of themselves. We find it difficult to envision students making sense of translations as more than moving a graph-as-wire to different locations in the plane according to learned rules without a robust understanding of graphs as having emerged through covariation.

References

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-378.

Dubinsky, E., & Wilson, R. T. (2013). High school students' understanding of the function concept. *The Journal of Mathematical Behavior, 32*(1), 83-101.

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517-545). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Lobato, J., & Siebert, D. (2002). Quantitative reasoning in a reconceived view of transfer. *The Journal of Mathematical Behavior, 21*, 87-116.

Moore, K. C. (2014). Quantitative reasoning and the sine function: The case of Zac. *Journal for Research in Mathematics Education, 45*(1), 102-138.
Moore, K. C., Paoletti, T., & Musgrave, S. (2013). Covariational reasoning and invariance among coordinate systems. *The Journal of Mathematical Behavior, 32*(3), 461-473.
Oehrtman, M., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. L. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 27-42). Washington, D.C.: Mathematical Association of America.
Piaget, J. (2001). *Studies in reflecting abstraction*. Hove, UK: Psychology Press Ltd.
Smith III, J., & Thompson, P. W. (2008). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the Early Grades* (pp. 95-132). New York, NY: Lawrence Erlbaum Associates.
Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 267-307). Hillside, NJ: Erlbaum.
Thompson, P. W. (1994a). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 179–234). Albany, NY: SUNY Press.
Thompson, P. W. (1994b). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics, 26*(2-3), 229-274.
Thompson, P. W. (1994c). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Schoenfeld & J. J. Kaput (Eds.), *Research in Collegiate Mathematics Education: Issues in Mathematics Education* (Vol. 4, pp. 21-44). Providence, RI: American Mathematical Society.
Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sépulveda (Eds.), *Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 31-49). Morélia, Mexico: PME.
Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. Chamberlin, L. L. Hatfield & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOM^e* (pp. 33-57). Laramie, WY: University of Wyoming.
Thompson, P. W. (2013). In the absence of meaning. In K. Leatham (Ed.), *Vital directions for research in mathematics education* (pp. 57-93). New York, NY: Springer.
Thompson, P. W., Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In L. P. Steffe, K. C. Moore, L. L. Hatfield & S. Belbase (Eds.), *Epistemic algebraic students: Emerging models of students' algebraic knowing* (pp. 1-24). Laramie, WY: University of Wyoming.
von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Washington, D.C.: Falmer Press.
Weber, E. D. (2012). *Students' ways of thinking about two-variable functions and rate of change in space*. Ph.D. Dissertation. Arizona State University: USA.

When Mathematicians Grade Students' Proofs, Why Don't the Scores Agree?

Robert C. Moore

Andrews University

Abstract: This poster reports on a study of practices that mathematics professors use to grade proofs. In an initial study, four mathematicians evaluated and scored six proofs of elementary theorems written by students in a discrete mathematics or geometry course. The results indicated that, while the professors generally agreed in their overall evaluations of the proofs, the scores varied substantially. A follow-up study delved more deeply into the reasons for the spread in the scores. This poster presents four reasons why the professors did not always agree in their scoring of the proofs: (a) performance errors, (b) disposition toward grading, (c) judgments about the student's level of understanding and the seriousness of errors, and (d) contextual factors.

Key words: Proof Evaluation, Proof Grading, Teaching Proof

To teach undergraduate mathematics majors how to write proofs that meet an acceptable level of rigor and clarity, mathematics professors require students to write proofs for homework and tests. The professors then grade the proofs by writing marks and comments on them, assigning a score to each proof, and returning the papers to the students. This grading process involves judgments about validity, judgments about the seriousness of the errors, and evaluation of surface features such as the proper use of mathematical language and notation. Thus, grading students' proofs is a complex and important teaching practice. Despite its pervasive role in undergraduate mathematics teaching, proof grading has received little research attention (Speer, N. M., Smith III, J. P., & Horvath, A., 2010).

Related Research

An essential part of grading a proof is checking that it is logically correct, or valid, and another part is evaluating it for clarity and readability. Yet recent studies have shown that mathematicians differ on what constitutes a valid proof (Inglis, Mejía-Ramos, Weber, and Alcock, 2013) and that they often agree, but occasionally disagree to a remarkable extent, on whether specific revisions of a proof improve its clarity for pedagogical purposes (Lai, Weber, and Mejía-Ramos, 2012). These studies lead us to expect that mathematicians will differ in their grading of students' proofs.

Brown & Michel (2010) developed a rubric based on readability, validity, and fluency (RVF) for grading proofs and other mathematical writing. We may ask if other mathematicians agree with this rubric and, if they use it to grade students' proofs, do they arrive at consistent scores?

Methodology and Results

An initial study addressed the following question: *Do mathematics professors agree in their evaluation and scoring of students' proofs?* In individual interviews that lasted about an hour, four mathematicians evaluated and scored five or six proofs written by students. An open coding system (grounded theory) was used to analyze the interview data. The results showed much agreement on the characteristics of a well-written proof, but the scores they assigned to the proofs varied considerably (Table 1).

A follow-up study was designed to delve more deeply into the reasons for the variation in the scores and answer the following questions: (a) *Why did the scores vary?* and (b) *Were there performance errors,* i.e., errors due to overlooking flaws in the proofs? The four professors regraded three of the original six proofs after seeing the other professors' comments and corrections, but not their scores, and also graded a new proof, Proof 7 (Figure 1), which had a logical flaw, namely, it proved the converse of the statement. The overall scores for Proof 7 were quite consistent, but the validity scores varied greatly (Table 2). The analysis of the data identified four main reasons the scores varied: (a) performance errors, (b) disposition toward grading, (c) judgments about the student's level of understanding and the seriousness of errors, and (d) contextual factors. The poster will further explain these four reasons.

Implications

This study calls attention to proof grading as an important teaching practice, and it raises again the question posed by Inglis et al. (2013) as to whether students are receiving a consistent message about the nature of validity as well as other aspects of well-written proofs. A question for further research is how differences in professors' grading practices affect students' progress in mathematics?

References

Brown, D. E. & Michel, S. (2010, February). *Assessing proofs with rubrics: The RVF method*. Paper presented at the 13th Annual Conference on Research in Undergraduate Mathematics Education, Raleigh, NC.

Inglis, M., Mejía-Ramos, J. P., Weber, K., & Alcock, L. (2013). On mathematicians' different standards when evaluating elementary proofs. *Topics in Cognitive Science, 5*, 270-282.

Lai, Y., Weber, K., & Mejía-Ramos, J. P. (2012). Mathematicians' perspectives on features of a good pedagogical proof. *Cognition & Instruction, 30*(2), 146-169.

Speer, N. M., Smith III, J. P., & Horvath, A. (2010). Collegiate mathematics teaching: An unexamined practice. *The Journal of Mathematical Behavior, 29*(2), 99-114.

Table 2

Scores Assigned to Proof 7 by the Professors

Professor	Validity 0-5	Clarity 0-5	Fluency 0-5	Understanding 0-5	Overall score 0-10
A	3	5	5	5	7
B	1	4	4	3	6
C	2	4	5	3	7
D	4	3	3	3	6

Task 7: Inequality

If x is a positive real number, then the sum of x and its reciprocal is greater than or equal to 2, that is, $x + \frac{1}{x} \geq 2$.

Proof 7:

Multiply both sides by x.

Then $x^2 + 1 \geq 2x$, so $x^2 - 2x + 1 \geq 0$.

This means that $(x-1)^2 \geq 0$,

which is clearly true, so the

theorem has been proved. ∎

Figure 1. Task 7 and Proof 7

Table 1

Scores Assigned to the Proofs by the Professors

Professor	Proof 1	Proof 2	Proof 3	Proof 4	Proof 5	Proof 6
A	10	9.8	9.5	10	9.5	9.5
B	10	5	6	7	7	5
C	8	8	9	8	8	—
D	9	9	8	7	7	8
Mean	9.25	7.95	8.13	8.00	7.88	7.50
Range	2.0	4.8	3.5	3.0	2.5	4.0

Cluster analysis of STEM gender differences.

Ian M. Mouzon, Iowa State University
Xuan Hien Nguyen, Iowa State University
Ulrike Genschel, Iowa State University
Andrea J. Kaplan, Iowa State University
Alicia Carriquiry, Iowa State University
Charlotte Mann, Carleton College

Abstract: In this project, we form, describe, and study groups (or clusters) of students based on their academic history prior to their first semester in college. These clusters allow us to examine the effect of gender on a student's academic career decisions, such as course and major selection, while controlling for the level of preparation. We begin with an overview of standard hierarchical clustering and discuss the pitfalls of a straightforward application with our data. We then describe how to adjust the technique in order to form stable clusters. Using these clusters, we find that course selection and STEM retention are related to a student's gender, with female students more likely to leave STEM early than male students with the same level of preparation and college grades.

Keywords: Clustering, STEM Retention, Gender, Academic Background

Introduction

The national concern to produce more qualified professionals in the STEM fields is ongoing (Seymour & Hewitt, 1997; Daempfle, 2003; Olson & Riordan, 2012). The push for more graduates in Science, Engineering, and Math has resulted in more students choosing to major in STEM as they enter college (Huang, Taddese, & Walter, 2000; Seymour, 2002). However, 40% of these students change majors or drop out of college completely (Astin & Astin, 1992).

Some of this attrition is due to students being unprepared and earning low grades in their preparatory math courses (Strenta, Elliott, Adair, Matier, & Scott, 1994); but even students who perform well leave because of the perceived poor instruction (Seymour & Hewitt, 1997; Brainard & Carlin, 1997). The introductory mathematics and statistics courses therefore play an important role in the retention of students through their first and second years.

Female students in STEM majors

In addition, disproportionally few female students choose to go into the STEM fields. The reasons most often cited for this disparity are the lack of preparation of female students, lack of confidence and cultural bias. Women take fewer math courses in high school than men. However, once the measures of developed abilities were taken into account, gender added little to the prediction of choice (to go into STEM) (Strenta et al., 1994). Zafar (2013) and Heilbronner (2009) found that the gender gap is due to gender differences in preferences and taste.

Strenta et al. (1994) noted that the most significant factor predicting losses of STEM majors are low grades earned in science courses during their first two years of study. With grades held equal, gender was not a significant predictor of persistence in engineering and biology but was such in a category that included the physical sciences and mathematics. These results can hide a possible gender bias. Compared to male students with the same entering math ability, female students perform substantially less well in their intro math and science courses (Carrell, Page, & West, 2009). Female in STEM suffer a loss of self-confidence and lower career aspiration (Rogers, 1993) and classroom experience can be much different for women (Henes, 1994). For example, the study of Carrell et al. (2009) suggests that while the gender of the professor has only limited impact on male students, it has a powerful effect on the performance of female students in math and science classes. Female students also leave earlier than other populations (Min, Zhang, Long, Anderson, & Ohland, 2011).

Research questions

1. How can we efficiently describe a student's academic background?
2. How does academic history influence a student's academic decisions?
3. How do students with the same academic history differ in terms of academic decisions and outcomes by gender?

Creating clusters based on academic history

In order to include a student's academic background in a meaningful way, we would like to produce clusters that group students with similar academic backgrounds together instead of using the variables directly. This has to do with the features of the variables traditionally available to researchers.

Student academic background variables

The data provided to us includes several details that give a summary description of the students academic background. Primarily, we will be using the total number of science credits taken by a student in high school (credits in biology, chemistry, physics), the student's scores on standardized tests (at the moment limited to ACT math and English subscores and the ACT composite score), and the student's high school rank (a percentile score with higher scores representing better position in the class).

Complications in these variables

Although these variables have the potential to sketch an academic history for a student, individually they may be less useful. For instance, a student's percentile rank is a result not just of their own academic ability but of the academic ability of their classmates and the academic rigor of their school (and considering that this could misrepresent poor students at poor schools as well as gifted students at advanced schools, rank has a very complicated interpretation). Less complicated is the student's score on the ACT sections. ACT scores are ordinal variables and high scores indicate higher performance. Still, the numeric score may be problematic when used in a model. Although comparisons between individual students may be unambiguous, more complicated comparisons are less clear. For example, is the difference in ability between a student with a 25 and a student with a 29 as meaningful as the difference in ability between a student with a 29 and a student with a 33? Surely not - and hoping that it would be is an abuse of what is plainly the arbitrary nature of the values assigned to the determined levels of performance.

Complications among these variables

Adding to the complication, we gain valuable information from the variety of ways these vague values work together. For instance, we know that some students excel in math and science while performing poorly in subject areas such as history or English. Similarly, some students have a difficult time with the routine of school (for a variety of reasons, sympathetic or not) and have poor class ranks but are able to imbibe enough information to perform well on standardized exams.

However, not all combinations of these variables may identify distinct groups. By clustering, we hope to work through these combinations to find justifiable groups.

Traditional clustering

The first steps in many clustering techniques, as described by, for instance, Hastie et al. (2009) consist of selecting the variables that will be used to define the clusters, choosing a clustering algorithm (hierarchical cluster with complete linkage, for example), and applying statistical tests (or ad hoc methods) in order to determine the number of clusters.

This simple approach does not work in our specific situation (and many others in which the variables chosen for clustering have ordinal response values) because of a few complications.

Ties and their effect on cluster creation

One important feature of our dataset that prevents a straight-forward analysis is the structure of many of our variables. One of the complications of using ordinal data is that several times during the clustering process, ties occur when attempting to determine which clusters should be merged together.

How a tie is decided changes the membership of clusters and can change the future mergers. Most statistical software determines ties based on the order of the data. This can lead to dramatic changes in cluster membership from one permutation of the rows of the dataset to another, which can be seen as evidence in favor of a different clustering mechanism (either a different number of clusters to be selected or a different method of creating clusters).

This is illustrated in Figure 1, which shows the changes in the cluster to which incoming students are assigned for 10 permutations. All the students depicted were initially grouped into the same cluster, but reordering the data rows caused the structure of the clusters to change going from one permutation to the next. In most permutations, the students that initially were grouped together are seen to form three clusters.

While some indicators suggest that a five-cluster arrangement could be valid, these indicators are deceptive. There are strong differences resulting from nothing more than ties making any five cluster model suspect. The resulting clusters lack stability and inclusion in them is based more on chance than on consistency with the other members of the clusters.

For this reason, the resulting clusters are not valid and would inevitably lead to poor results.

Finding stable and valid clusters

The reality is, as can be seen above, that there are more than five clusters. There are multiple students that are consistently being clustered together meaning that although a five clustering structure is unstable, there could be a stable clustering structure that is not strongly affected by ties, one that consistently identifies the clusters that seem visible in the graph above.

We can determine the stable clustering structure by examining the behavior of a few statistics which measure the order (or equivalently disorder) within the clusters created by each permutation of the data observations with various cluster numbers. Figure 2 shows the cubic clustering criterion (CCC), pseudo F statistics, and the pseudo t^2 statistics for structure with one to twelve clusters calculated over 10 possible permutations of the data.

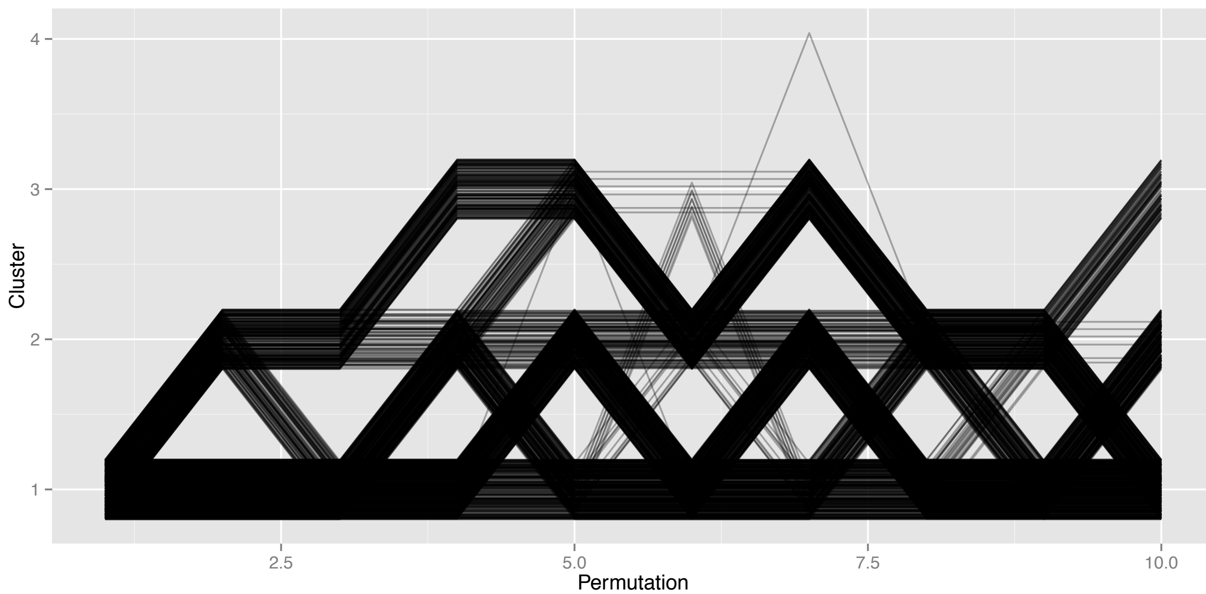


Figure 1: Permuting data and its influence on clustering. Cluster membership can be determined by tie-breakers, which are settled using the order of the rows of the dataset. Shown are cluster assignment (the y-axis) of students (represented by the lines) as the permutation of the rows of the dataset is changed for students who were initially assigned to the same cluster. Going left to right, we see that as the permutation changes, what was once a single cluster splits to form what might more readily be identified as three clusters.

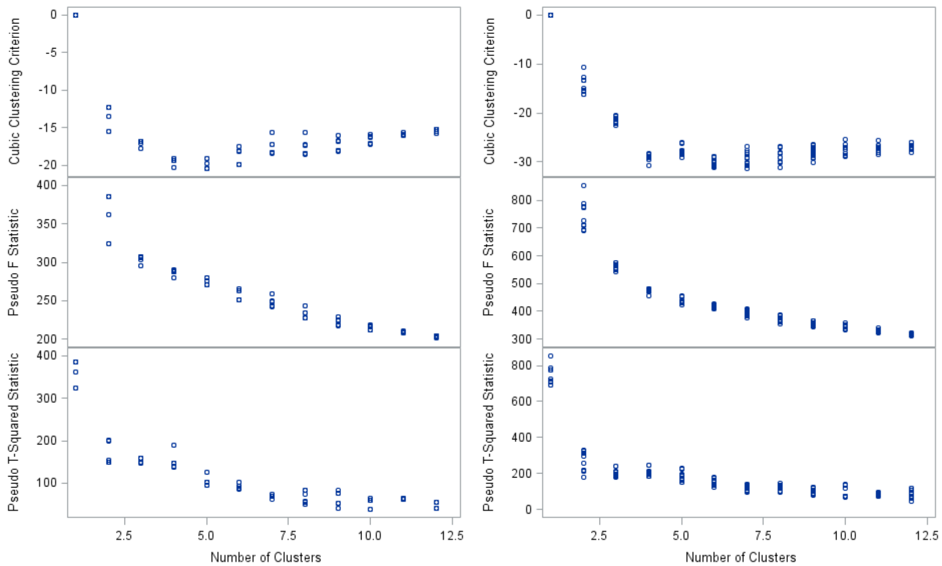


Figure 2: Illustration of the method used to find the number of clusters that is resistant to issues relating to ties. The students have been divided into two groups, those who have had calculus (on the left) and those who have not had calculus (on the right). For each permutation of the data the identified statistics are calculated for one to twelve cluster arrangements of the data. There is evidence in favor of five clusters within the calculus student population (the pseudo t^2) and another five clusters within the non-Calculus students (as seen in the local maximum for the CCC graph). This indicates a stable 10 cluster split structure.

Once the split 10 cluster model was determined to be stable, 1000 permutations were created to give 1000 cluster assignments for each observation. One of the permutations was randomly selected and used as the final clustering assignment.

Describing the clusters

The clusters can be described in terms of the variables used to create them:

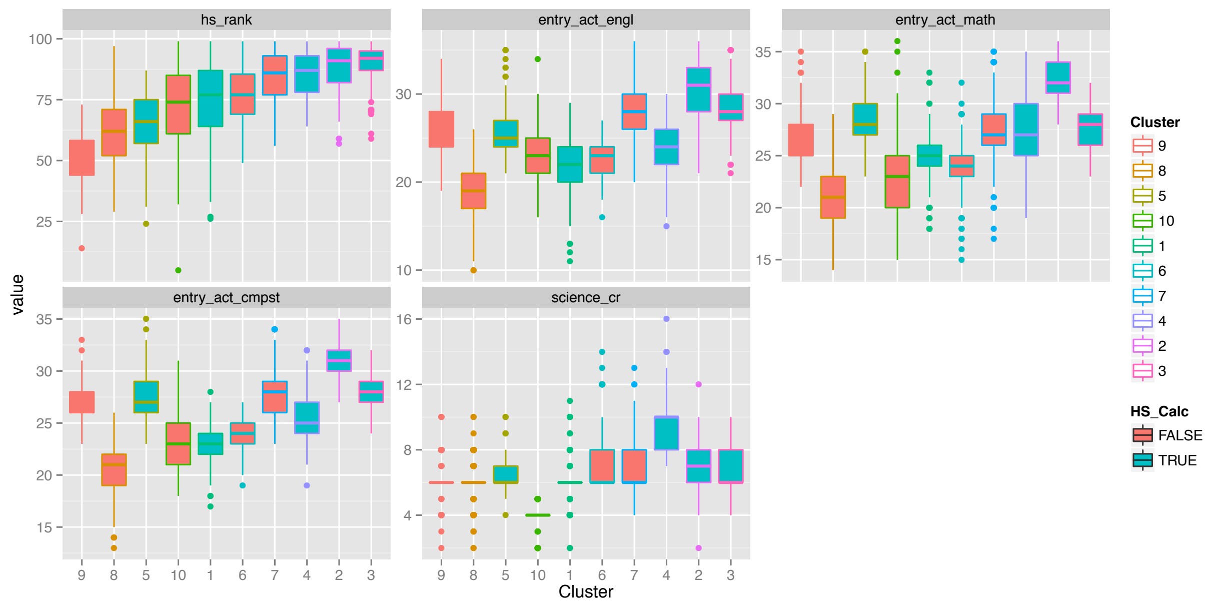


Figure 3: Clustering by variable. The organization of students into clusters based on ACT scores, class rank and science credits. In this case, boxplots are filled by whether or not the students in that cluster took calculus in high school (blue) or did not (red). Clusters are identified by both color and the ID on the x-axis.

The clusters can be described (ordered by increasing high school rank) as follows:

Cluster	Description
9	Poor students, no calculus, excellent test takers, overall less science
8	Bad students, no calculus, worst test takers, overall less science
5	Bad students, calculus, better test takers, overall less science credits
10	Good students, no calculus, fair test takers, fewest science credits
1	Good students, calculus, fair test takers, overall less science
6	Good students, no calculus, fair test takers, overall more science
7	Great students, no calculus, good at math and english, overall more science
4	Great students, calculus, good at math but bad at english, most science credits
2	Top students, calculus, good test takers, overall more science
3	Top students, calculus, poor test takers, overall more science

With a stable clustering structure we can begin to examine the effect of gender on outcome while accounting for students with the same academic background.

Models using the split 10 clustering structure and conclusions

This clustering structure can be used to model STEM retention rates.

Taking all students entering Iowa State University for the first time in Fall 2012 declaring a STEM major and enrolling in Calculus I, we can model their persistence in STEM into their second, third, and fourth semester can be modeled using a multinomial logistic model (where the timing of dropping out is the response and the baseline group are male students in cluster 2).

The simplest model considers only two possible outcomes: the student is still a STEM major at the start of the second semester or they are not. The parameter estimates of this model are found in Table 1 (non significant interactions and effects have been ignored).

In this case, we see that female students are more likely than male students to leave STEM fields during their first semester (about 34% more likely), even when accounting for preparation (cluster) and performance (PFX, where passing the course is the baseline, F is for failing the course, and X for dropping the course).

Analysis of Maximum Likelihood Estimates
(Comparing the second semester to the baseline)

Parameter		Estimate	Std. Error	Chi-Sq	Pr $<$ ChiSq
Intercept		-1.4244	0.1514	88.54	$<$ 0.0001
Gender	F	0.2930	0.1368	4.59	0.0322
Cluster	7	-0.6743	0.3089	4.77	0.0290
PFX	F	0.4284	0.1690	6.43	0.0112
	X	0.6250	0.2031	9.47	0.0021

This implies that early female STEM retention problems are not the result of preparedness. Extending this model to include multiple possible outcomes we see that female students are still more likely than male students to leave STEM early (about 27% more likely than similarly prepared and equally successful male students), but much less likely to leave STEM during the second semester, implying that female students have a tendency to stay in STEM if they can survive the early semesters (about 29% more likely to stay).

Next steps

At this point, we have a model that uses cluster, gender, and grade. The clusters and gender are known prior to the first semester, but we only know the grade at the end of the semester. For that reason, we would like to build a model that uses predicted grades instead of the actual earned grades. We have a proposed model for that, including how to code the responses (Pass, Fail, Drop as distinct categories) and the inputs (cluster, number of semester credit hours, and factor scores from the a self-efficacy survey). This would use the clustering variables twice, once to predict grade and once to predict STEM retention.

We also know that the clusters change slightly from permutation to permutation. We have suggested running the model for several clustering schemes related to the analysis. One issue we are having is that we do not have enough data. We are just using students who started in Fall 2012 and took calculus. This means that we can not use a `clogits` type model in our analysis, even though dropping out in the first semester is clearly worse than dropping out in the second semester, i.e., the response we are modeling is ordered. We are planning to avoid this by fitting one model for students dropping out in the first semester and one model for students dropping out in the first year. Simplifying the model response should give us the ability to fit a simple (instead of multinomial) logistic model and possibly include interactions.

References

Astin, A. W., & Astin, H. S. (1992). Undergraduate science education: The impact of different college environments on the educational pipeline in the sciences. final report.

Brainard, S. G., & Carlin, L. (1997). A longitudinal study of undergraduate women in engineering and science. In *Frontiers in education conference, 1997. 27th annual conference. teaching and learning in an era of change. proceedings.* (Vol. 1, pp. 134–143).

Carrell, S. E., Page, M. E., & West, J. E. (2009). *Sex and science: How professor gender perpetuates the gender gap* (Tech. Rep.). National Bureau of Economic Research.

Daempfle, P. A., Ph. D. (2003). An analysis of the high attrition rates among first year college science, math, and engineering majors. *Journal of College Student Retention: Research, Theory and Practice*, *5*(1), 37–52.

Heilbronner, N. N. (2009). *Pathways in stem: Factors affecting the retention and attrition of talented men and women from the stem pipeline.* ERIC.

Henes, R. (1994). *Creating gender equity in your teaching*. College of Engineering, University of California, Davis.

Huang, G., Taddese, N., & Walter, E. (2000). Entry and persistence of women and minorities in college science and engineering education. *Education Statistics Quarterly*, *2*(3), 59–60.

Min, Y., Zhang, G., Long, R. A., Anderson, T. J., & Ohland, M. W. (2011). Nonparametric survival analysis of the loss rate of undergraduate engineering students. *Journal of Engineering Education*, *100*(2), 349–373.

Olson, S., & Riordan, D. G. (2012). Engage to excel: Producing one million additional college graduates with degrees in science, technology, engineering, and mathematics. report to the president. *Executive Office of the President*.

Rogers, J. M. (1993). *A program of deliberate psychological education for undergraduate females in engineering through role-taking*. Unpublished doctoral dissertation, North Carolina State University.

Seymour, E. (2002). Tracking the processes of change in us undergraduate education in science, mathematics, engineering, and technology. *Science Education*, *86*(1), 79–105.

Seymour, E., & Hewitt, N. M. (1997). *Talking about leaving: Why undergraduates leave the sciences* (Vol. 12). Westview Press Boulder, CO.

Strenta, A. C., Elliott, R., Adair, R., Matier, M., & Scott, J. (1994). Choosing and leaving science in highly selective institutions. *Research in higher education*, *35*(5), 513–547.

Zafar, B. (2013). College major choice and the gender gap. *Journal of Human Resources*, *48*(3), 545–595.

Hastie, T., Tibshirani, R., Friedman, J., Hastie, T., Friedman, J., & Tibshirani, R. (2009). *The elements of statistical learning* (Vol. 2) (No. 1). Springer.

Teachers' meanings for the substitution principle

Stacy Musgrave, Arizona State University
Neil Hatfield, Arizona State University
Patrick Thompson, Arizona State University

Structure sense is foundational to mathematical thinking. This report explores high school math teachers' meanings for the substitution principle, a sub-category of structure sense that research previously identified as sources of difficulty for students. A focus on meanings reflects our belief that teachers' meanings directly impact the mathematical meanings students develop. We suggest ways of thinking that could lead to various response types as a resource for teacher educators to design professional development targeting improved structure sense for teachers.

Keywords: Structure Sense, Representational Equivalence, High School Math Teacher, Substitution Principle, Mathematical Meanings

Structure is a foundational component of mathematics; one could (over) simplify the work of a mathematician as the study of the structure of objects and relationships between those objects. As such, developing structure sense is fundamental to the experience of math students. One powerful component of structure is representational equivalence, which splits into two categories: transformational equivalence and substitution equivalence. Transformational equivalence refers to the equivalence-preserving transformations one may perform on a mathematical object. For instance, while solving an equation, one ought to perform actions on the equation that do not alter the original relationship (e.g. multiply *both* sides of an equation by the same non-zero value). Substitution equivalence, called the substitution principle by other authors, points to the underlying structural "sameness" that holds when substituting a compound term for a variable or a variable for a compound term (Novotná & Hoch, 2008). For example, one might substitute u for $5x-1$ in the equation $(5x-1)^2-3(5x-1)=-2$ to highlight its quadratic nature. In this report, we discuss potential difficulties in applying the substitution principle in an abstract setting to manipulate an expression, as well as implications for this in teaching and learning mathematics.

While the focus of this report centers on the substitution principle, it hints at a broader issue of structure sense. The term "structure sense", coined by Linchevski and Livneh (1999) to signify the use of arithmetic structures in the transition to algebra, and broadened by Hoch (2003), references the "ability to recognize algebraic structure and to use the appropriate features of that structure in the given context as a guide for choosing which operations to perform " (p. 2). Note that the ability to recognize and utilize structure refers to actions that apply across all contexts of school mathematics. Hoch and Dreyfus (2006) demonstrated what this ability might look like in specific contexts, grounding this general definition in a way that could be useful for guiding student learning and curriculum design. The blanket term, however, captures the fact that the notion of structure *is* broad and spans every level of mathematics. Students conceive relationships between concepts, objects and techniques as they transition from course to course. Their development of these relationships supports students' awareness of structure (Mason, Stephens, & Watson, 2009).

Extant literature reveals that many students do not develop structure sense (Hoch, 2003; Hoch & Dreyfus, 2006; Linchevski & Livneh, 1999; Novotná & Hoch, 2008; Novotná, Stehlikova, & Hoch, 2006; Tall & Thomas, 1991). Student performance, documented at various stages of mathematical experience in the aforementioned studies, points to a lack of attention to and ability to employ structural qualities of mathematical objects. We suggest one explanation for this missing piece of students' mathematical development: teachers might not provide experiences that allow students to develop structure sense. We further suspect that this is not a conscious decision on the part of teachers, but rather a result of the fact they themselves do not possess robust structure sense.

This conjecture is problematic in light of the Common Core State Standards' call for students *to identify* structure, meaning a student must be aware that structure is something to look for in representations of mathematical objects, and for students *to act* in accordance with that structure. These are subtle, yet key, distinctions that teachers must be able to make. More often than not, student behavior leans towards acting rather than reflecting on actions. For instance, order of operations is frequently taught in the context of *calculating* values of expressions rather than identifying implicit structure. This compounds the difficulty Thompson and Thompson (1987) identified with regard to students' work with algebraic expressions. Expressions can be "structured explicitly by the use of parentheses, [or] implicitly by assuming conventions for the order in which we perform arithmetic operations" (p. 248). The standard practice of relying on order of operations to imply the structure of an expression means that students must first be aware that structure is something to which they should attend. Only then can they use their internalized conventions to determine the expression's structure. Tall and Thomas (1991) describe another student obstacle in determining structure as the *process-product obstacle*. The obstacle is that students must simultaneously view an algebraic expression as representing the process of a computation and the product of that process. Many students' difficulties stem from focusing on expressions as representing the process of computing rather than the reflecting on the expression as representing the result of computing.

In this report, we provide evidence that many in-service teachers have difficulty with structure sense. We focus specifically on the substitution principle (i.e. taking a complex expression as one object), a subcategory of structure sense that research points to as a common area of struggle for students (Hoch & Dreyfus, 2006; Novotná & Hoch, 2008). We believe that teachers cannot support students in developing richer meanings than the ones the teachers possess, making it imperative to understand the nature of teachers' mathematical meanings. With this understanding, teacher educators can devise ways to help teachers improve both their structure sense and their awareness of its importance for students' mathematical learning. By describing teachers' struggles with the substitution principle and possible sources of difficulty, we hope to identify not only task-specific difficulties, but also identify ways of thinking that lead to teachers' difficulties. Investigating teachers' meanings regarding structure will also allow us to identify potential sources of students' difficulties, thus giving a more comprehensive perspective on the issue of students' development of structure sense.

Theoretical Framework

We view an individual's meanings as her means to organize her experiences and, once formed, as organizers of her experience. Through repeated reasoning and reconstruction, an individual constructs schemes to organize experiences in an internally consistent way (Piaget & Garcia, 1991; Thompson, 2013; Thompson, Carlson, Byerley, & Hatfield, 2013). For example, part of an individual's meaning for a mathematical expression is how she sees its structure. One person might see "$x/2y$" as $(x/2)y$ while another might see it as $x/(2y)$. These two people hold different meanings for the given expression, and the consequences for such differences can be profound.

We take as given, subject to future investigation, that a teacher's meanings can be more or less productive in classroom instruction, with productive meanings supporting students' development of coherent mathematical meanings and ways of thinking. Investigations of teachers' mathematical meanings can inform professional development efforts to help teachers promote productive meanings and coherence in mathematics instruction (Musgrave & Thompson, 2014; Simon & Blume, 1994; Thompson, 2013).

Methodology

Our team of mathematics educators and mathematicians created a diagnostic tool called the Mathematical Meanings for Teaching Secondary Mathematics (MMTsm) in order to address this issue of investigating teachers' meanings. The MMTsm consists of tasks that provide teachers the opportunity to interpret the given scenario and respond according to their meanings. Our team designed, tested, interviewed, and refined items for approximately two years prior to giving these tasks in the summer of 2013. In this report, we concentrate on one item for which the substitution principle is foundational to reasoning about the problem.

We scored teachers' responses to each item in accordance to a scoring rubric. After collecting data in summer 2013 we developed an open coding scheme based on roughly 140 teachers' responses to categorize ways of thinking. We supplemented the coding process with teacher interviews during the yearlong process of developing scoring rubrics. We drew upon both the data and prior research related to how students and teachers understand the various ideas items were designed to tap. Once identified, we organized themes and ways of thinking into levels according to productivity for student learning to form an initial rubric. A group of 10 people from two institutions scored 10 responses to each item and discussed possible improvements to the rubric; we iterated the scoring-refining process until reaching a consensus. At this point, we tested the inter-rater reliability of each rubric with an external group and made adjustments to each rubric until we reached 100% agreement. The team then held a two-day scorer-training workshop on using the rubrics. Upon submission of scores, a team member verified each score for compliance to the appropriate rubric for each item. Any adjustments made followed the specifications of the rubrics. A more detailed description of the method for creating tasks and rubrics attentive to mathematical meanings can be found in Thompson (in press).

We administered the MMTsm to high school mathematics teachers involved in professional development programs from two states in the United States. Eighty-four (84) teachers took a version of the MMTsm containing the item discussed below. The teachers had varying backgrounds with regard to a number of demographic variables. Table 1 shows the distribution of teachers' highest degree obtained along with their major of study. Teachers under "STE" majored in science, technology or engineering. The "Other" major category includes all other majors, such as Business Administration and elementary education. Approximately two-thirds of the teachers majored in mathematics or mathematics education; with another 11% being other STEM related majors.

Table 1. Teachers' Highest Degree Obtained vs. Major.

	Math	MathEd	STE	Other	Total
Bachelor's	9	11	4	8	32
Master's	16	21	5	10	52
Total	25	32	9	18	84

The Task

Thompson and Thompson (1987) describe a common student difficulty in viewing a sub-expression as a single object. In their study, students used a computer program to manipulate expressions into equivalent forms (e.g. transform $(z - q)*u$ into $z*u - q*u$ by selecting and applying appropriate identities and transformations). Several tasks requiring students to view a sub-expression as a unit proved challenging for students, likely because that particular mental activity requires students to focus on the structure of an expression by mentally grouping parts of it as one object. Our research team adapted an item from Thompson and Thompson's 1987 study (Figure 1) for use on the MMTsm.

Δ is an operation with the following property For all real numbers, a, b, and c, $(a \,\Delta\, b) \,\Delta\, c = a \,\Delta\, (b \,\Delta\, c)$. Let u, v, w, and z be real numbers. Can this property of Δ be applied to the expression below? If yes, demonstrate. If no, explain. $(u \,\Delta\, v) \,\Delta\, (w \,\Delta\, z)$

Figure 1. Associative Property Task. © 2014 Arizona Board of Regents. Used with permission.

Teachers could use tasks similar to the one in Figure 1 to guide classroom conversations about expressions, particularly focusing students' attention on structure while discussing how to view the expression in multiple equivalent ways. We thus administered the task in Figure 1 to teachers to gain insight on how they might respond in a situation necessitating the grouping of compound terms as one object. The task defines a property for an operation Δ in terms of three variables and asks if the property can be applied to an expression with four variables. In order to reason that the stated property of Δ applies to the expression, one must reason that $w\Delta z$ can be viewed as one object while viewing $u\Delta v$ as an operation on two objects, or similarly, view $u\Delta v$ as one object while considering $w\Delta z$ as an operation on two objects. We categorized teachers' responses according to a scoring rubric, which we will describe further in the next section.

Results and Discussion

Table 2 shows our classification of teachers' responses to Figure 1. Responses categorized in the top two levels include those that attend to structure in a way that shows that the property applies. The distinguishing trait between the top two levels is the quality of demonstration. Namely, responses at the highest level (Level 3) explicitly show how a sub-expression is treated as one object. We put at Level 2 responses that correctly applied the property without explicitly showing groupings (Type 1) because we believe that simply providing the answer without showing how it is achieved would be less supportive of students who are still developing the skill of identifying compound terms as one object. We also put at Level 2 any response that would have been put at Level 3, but which then showed further work that introduced ambiguity (Type 2). Level 1 captures two types of responses. The first is that the teacher claimed the property does not apply (13 of the 25 teachers at Level 1). The second contains responses that suggested that the teacher thought that the property meant they could move parentheses any way one wishes. Level 0 also captures a variety of responses. Specifically, if a teacher changed the order of the variables in the expression, stated the need to know the definition of Δ, claimed that Δ stood for addition or multiplication, substituted numbers for any of the variables, or wrote a final expression not containing exactly 3 "Δ" symbols, his or her response was scored at Level 0.

The Associative Property Task was atypical among items in our assessment with regard to results varying based on undergraduate major. Table 3 shows the distribution of teachers' responses by level and by undergraduate major. The distribution of responses by teachers with degrees in math, math education, and STE are similar distributions across levels. However, the distribution of responses by teachers with "Other" majors is noticeably different, with a disproportionately large number of teachers holding degrees in the "Other" category providing low-level responses. We suspect that the ways of thinking required to provide a high-level response are not regularly practiced outside of STEM fields. While teachers with “Other” majors have likely practiced grouping objects mentally, they may not have repeated experiences doing so in an abstract setting using symbolic manipulation.

Table 2. Associative Property Task Sample Responses by Level

Response Level	Sample Responses	Number of Responses
Level 3	$(u \Delta v) \Delta (\underbrace{w \Delta z}_{c})$ $(u \Delta v) \Delta c = u \Delta (v \Delta c)$ $= u \Delta (v \Delta (w \Delta z))$	14
Level 2	Type 1: Correctly applied property without demonstrating how the property applies. Sure: $u \Delta (v \Delta (w \Delta z))$ Type 2: Contains elements of Level 3 response, but final answer does not serve the purpose of demonstrating how the property applies. $(u \Delta v) \Delta (w \Delta z)$ let $u = a$ $v = b$ $w \Delta z = c$ $(a \Delta b) \circ (c)$ $a \Delta (b \Delta c)$ $a \Delta (b \Delta w \Delta z)$ $\boxed{u \Delta (v \Delta w \Delta z)}$	15
Level 1	Type 1: Teacher said property does not apply $U \Delta V$ is one expression and $W \Delta Z$ is one expression. We can't demonstrate the associative property with only 2 expressions. Type 2: Placement of parentheses is inconsistent with the given property $u \Delta (v \Delta w) \Delta z$	25
Level 0	Substituting numbers or replacing Δ with a known arithmetic operation: associative prop. $(a+b)+c = a+(b+c)$ $a=1$ $b=2$ $c=3$ $(u+v)+(w+z)$ $(1+2)+3 = 1+(2+3)$ $(u \Delta v) \Delta (w \Delta z) = u \Delta (v \Delta w) \Delta z$ yes Changing order of variables and omitting Δ: $(u \Delta w)(v \Delta z)$	28

Table 3. Responses to Associative Property Task by Major

	Math	Math Ed	STE	Other	Total
Level 3	5	6	2	1	14
Level 2	5	6	2	2	15
Level 1	7	13	1	4	25
Level 0	6	7	4	11	28
No Response	2	0	0	0	2
Total	25	32	9	18	84

Our data suggests two common sources of struggle on this item. The first is the abstract nature of the problem itself; Δ is an unknown operator and variables are used instead of numbers. Indeed, 21 of the 28 Level 0 responses suggested the need to use numbers, or expressed a need to know what operation Δ represents, like multiplication or addition (see Table 2, Level 0, first response). The second source identified in the data is the difficulty in viewing similar structures $u\Delta v$ *and* $w\Delta z$ in two different ways *simultaneously*—one as two objects and the other as one object. In fact, some teachers responded in a way that showed they only saw two objects or that they saw four objects (Figure 2).

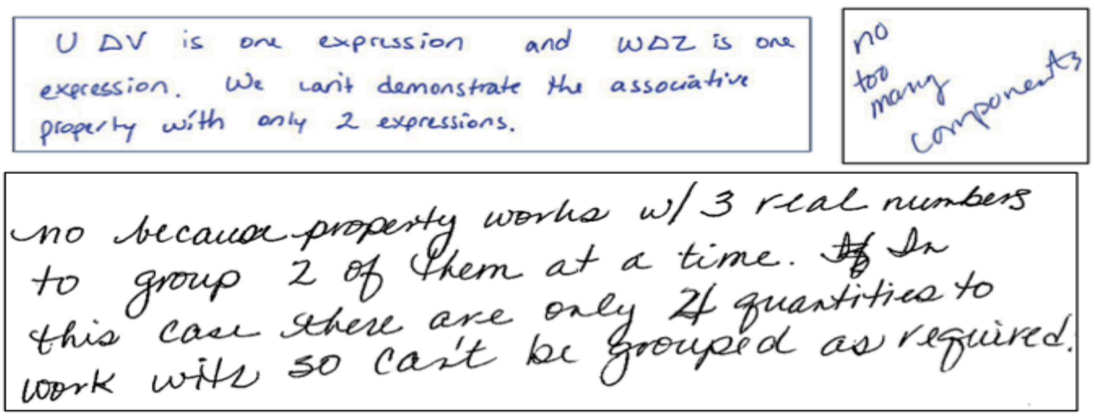


Figure 2. Sample Teacher Responses Demonstrating Conceiving Only of an Even Number of Terms. All teachers have a degree in Math Education.

Conclusions

In this report, we categorized teachers' responses on a task designed to reveal meanings for the substitution principle in the context of structuring an expression. We take teachers' responses as samples of their in-the-moment meanings in the given context with the hope of revealing ways of thinking that might be addressed profitably in professional development. In particular, by identifying teachers' ways of thinking and possible areas of difficulty, we are able to identify how teachers are positioned to support students develop meanings for structure sense. We warn the reader, however, that even if a teacher provided a high-level response, this does not mean the teacher will consciously support the development of structure sense in his or her students. As Novotná and Hoch (2008) and Mason et al. (2009) stated, it is not enough for teachers to just possess well-developed structure sense. They must also be *reflectively aware* of their structure sense, and make it a goal to foster its development in their students. Only then can teachers begin to make decisions about classroom activities and conversation that could support the development of structure sense in students.

The data shared here, along with data on five additional structure tasks from the MMTsm, suggest that low-level structure sense among teachers is commonplace. Such a statement is relatively unsurprising and extremely unhelpful, however. What is useful is that our data suggests that most teachers' responses appear to be driven by context rather than structural reasoning. Compartmentalized meanings could be part of the triggers that Hoch (2003) described as playing a role in how an individual classifies objects and properties into structures. Likewise, our findings support Novotná and Hoch's (2008) warning that "this lack of awareness [of structure sense among students] may return with [them as] teachers back to schools" (p. 102). Our research suggests that teachers operate on tasks based predominantly on contextual cues rather than structural awareness. We suggest future research explore *how* teachers rely on context instead of structure sense to approach problems. With such information, researchers and professional development professionals could begin creating tasks that would support teachers in developing structural awareness. In particular, future research needs to investigate how to generate attention to structure as something that is important and useful within the teaching community. Teachers with this belief will be better poised to support students in using the various facets of structure sense to guide decision-making processes in the act of solving problems.

Admittedly, our study is limited by the fact that our assessment was designed to explore teachers' meanings for a variety of content areas, so we only have six structure items to draw upon for analysis—and only one of which we shared here. Future research should extend the exploration of teachers' structure sense in a more focused fashion. For instance, one might use tasks designed specifically to distinguish between context specific reasoning and structural awareness, and conduct follow-up interviews of teachers aimed at eliciting their thinking. With insight into ways of thinking from such studies, researchers could then develop pedagogical items to be used in professional development and explore the effect of drawing teachers' attention to structure on those teachers' activity in their classrooms.

Our assessment and the corresponding analyses of responses aim to support professional developers in gauging the mathematical meanings by which teachers operate. We propose that classifying common ways of thinking, both productive and less productive in the normative sense, gives necessary information that professional developers need to support teachers in developing richer meanings and ways of thinking. We hope this approach of working with teachers' meanings for the substitution principle will increase the field's awareness of structure sense, thereby positioning mathematics educators to help teachers to better support students' development of structure sense.

References

Hoch, M. (2003). *Structure sense.* Paper presented at the Third Conference for European Research in Mathematics Education, Bellaria, Italy.

Hoch, M., & Dreyfus, T. (2006). *Structure sense vesus manipulation skills: an unexpected result.* Paper presented at the 30th Conference of the International Group for the Psychology of Mathematics Education, Prague, Czech Republic.

Linchevski, Liora, & Livneh, D. (1999). Structure sense: the relationship between algebraic and numerical contexts. *Educational Studies in Mathematics, 40*(2), 173-196.

Mason, J., Stephens, M., & Watson, A. (2009). Appreciating mathematical structure for all. *Mathematics Education Research Journal, 21*(2), 10-32.

Musgrave, Stacy, & Thompson, Patrick W. (2014). *Function notation as an idiom*. Paper presented at the 38th Conference of the International Group for the Psychology of Mathematics Education, Vancouver, Canada.

Novotná, J., & Hoch, M. (2008). How structure sense for algebraic expressions or equations is related to structure sense for abstract algebra. *Mathematics Education Research Journal, 20*(2), 93-104.

Novotná, J., Stehlikova, N., & Hoch, M. (2006). *Structure sense for university algebra.* Paper presented at the 30th Conference of the International Group for the Psychology of Mathematics Education, Prague, Czech Republic.

Piaget, Jean, & Garcia, R. (1991). *Toward a Logic of Meanings*. Hillsdale, NJ: Lawrence Erlbaum.

Practices, National Governors Association Center for Best, & Officers, Council of Chief State School. (2010). *Common core state standards for mathematics*. Washington, D. C.: National Governors Association Center for Best Practices, Council of Chief State School Officers.

Simon, M. A., & Blume, G. W. (1994). Building and understanding multiplicative relationships: A study of prospective elementary teachers. *Journal for Research in Mathematics Education, 25*(5), 472-494.

Tall, David, & Thomas, M. (1991). Encouraging versatile thinking in algebra using the computer. *Educational Studies in Mathematics, 22*, 125-147.

Thompson, Patrick W. (2013). In the absence of meaning. In K. Leatham (Ed.), *Vital Directions for Research in Mathematics Education* (pp. 57-93). New York: Springer.

Thompson, Patrick W. (in press). Researching mathematical meanings for teaching. In L. D. English & D. Kirshner (Eds.), *Third Handbook of International Research in Mathematics Education*. London: Taylor and Francis.

Thompson, Patrick W., Carlson, Marilyn P., Byerley, Cameron, & Hatfield, Neil J. (2013). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In K. C. Moore, L. P. Steffe & L. L. Hatfield (Eds.), *WISDOMe Mongraphs* (Vol. 4). Laramie, WY: University of Wyoming.

Thompson, Patrick W., & Thompson, A. G. (1987). *Computer presentations of structure in algebra.* Paper presented at the Eleventh Annual Meeting of the International Group for the Psychology of Mathematics Education, Montréal, Canada.

Calculus students’ meanings for difference

Stacy Musgrave
Arizona State University

Neil Hatfield
Arizona State University

Patrick Thompson
Arizona State University

Students learn the mathematical operation of subtraction beginning in elementary school, along with key vocabulary to talk about that operation. However, the meanings that students develop for the word “difference” continue to play a role well into students’ study of undergraduate mathematics. In particular, a meaning for “difference” as representing a change in a quantity is essential to understanding and communicating about foundational ideas in Calculus. In this preliminary report, we consider meanings about the word “difference” held by calculus students as revealed on a pre-test in an on-going study designed to explore Calculus students’ structure sense. We further propose potential consequences for those meanings and describe methods to be used in data collection for the remainder of the Fall 2014 semester.

Keywords: Calculus Students, Mathematical Meanings, Difference

Students learn about the mathematical operation of subtraction beginning in elementary school. Along with learning to perform subtraction to calculate values of expressions, students learn vocabulary words, such as “difference”, “minuend” and “subtrahend” to refer to the structure of expressions involving subtraction, the quantity from which another is to be subtracted, and the quantity to be subtracted, respectively. Such words and attention to structure, however, often fall to the wayside in classroom conversation. Students also learn the other arithmetic operations of addition, multiplication and division, then proceed to learn the Order of Operations. Treatment of all these topics traditionally falls in the context of computing values of expressions, rather than identifying the structure of such expressions. For instance, when a student is given an expression like 3 – 5 + 6, more often than not, he is asked to simplify. If he follows the Order of Operations, he will first compute 3 – 5 to get -2 and write a new expression -2 + 6. Notice that as soon as he replaces 3 – 5 with -2, he loses the structural information of where the -2 came from. Likewise, when the student adds -2 and 6 to get 4, all information about where the value 4 came from is lost. Note that as this student engages in this type of computational activity repeatedly, he is likely to develop a meaning for the Order of Operations and the operations themselves that they are a call to *do* something. Exclusive engagement in computational activities will hinder the students’ development of meanings for Order of Operations as a means to describe the structure of the expression (e.g. the above expression is a sum in which the first addend is the difference 3 – 5 and the second addend is 6).

The act of describing the structure of an expression can be thought of in terms of a larger area of study: students’ structure sense. Extant literature shows that students have weak structure sense, both before and after completing coursework at the university level (Hoch & Dreyfus, 2006; Novotná & Hoch, 2008). We suspect that weak structure sense, particularly the awareness of and ability to identify structure of expressions, is a major contributor to the common struggle of Calculus students in applying appropriate techniques of differentiation and integration. After all, it is common to hear a student say, “I have memorized all the rules of differentiation, but when you give me those crazy functions, I don’t know which one to use.”

The broader purpose of our on-going study is to explore Calculus students’ structure sense, particularly with regard to whether or not they attend to the structure of functions. Namely, when given a function, do students recognize the structure of the function rule as a sum, difference, product, or quotient? How do students' meanings for Order of Operations play a role in identifying structure? What activities can a teacher engage in to draw students' attention to structure? Does attention to structure alleviate typical student struggles with applying rules of differentiation and integration techniques?

In this report, we focus specifically on students' meanings for "difference" as revealed on a pre-test. We describe consequences for these meanings in the teaching and learning of Calculus ideas.

Theoretical Framework

We consider the meanings an individual develops as his means to organize his experiences, and once developed, those meanings serve as organizers of new experiences. Creating meanings entails constructing a scheme through repeated reasoning and reconstruction to organize experiences in a way that is internally consistent (Piaget & Garcia, 1991; Thompson, 2013; Thompson, Carlson, Byerley, & Hatfield, 2013). For instance, an individual's meaning for Order of Operations might be entirely situated in the context of computing the value of an expression and entail recollecting the acronym PEMDAS (Parentheses, Exponents, Multiplication, Division, Addition, Subtraction). Such a meaning might inhibit that individual's ability to make sense of the structure of an expression containing both numbers and variables.

Methods

At the time of writing this report, we are in the initial stages of data collection for a study that will conclude at the end of the Fall 2014 semester. We have collected data from 201 Calculus I students at a large research university in the southwestern United States. The design of the Calculus course is distinct from a conventional introductory Calculus course in that the curriculum is research-based and designed with the explicit intent of supporting students in developing rich meanings for the foundational ideas of calculus.

Participants. There are two sections of the specially designed Calculus I course. One section has 52 students and is taught by the lead author. The other section has 149 students and is taught by a senior instructor who has 3 semesters of experience teaching this particular course, and played an integral role in creating the student materials for the course. The students were unaware of the unique design and goals of the course when they registered for the class.

Data Collection. The students completed an 11-item pre-test that investigates their meanings for Order of Operations. Selected tasks from this pre-test and some preliminary results are discussed in the Preliminary Results section. During the remainder of the Fall 2014 semester, data will primarily be gathered from the 52-student section. The instructor of this section will record her lectures (capturing audio and the screen projected for students to see), which will be tailored to explicitly draw students' attention to structural qualities of expressions and functions. She will select 6 students from the pool of volunteers to conduct interviews to probe their thinking further and pilot tasks to be used during whole-class instruction. Student work on assignments related to supporting structure sense will be scanned as a reference to gauge students' tendency to employ structural awareness in their work. At the end of the semester, students from both sections will complete a post-test aimed to reassess students' meanings for order of operations and to see if there is a connection between those meanings and performance on differentiation and integration tasks. Data from interviews and the post-test will be presented in the event that this proposal is accepted.

Preliminary Results

In the discussion that follows, we describe 2 tasks on the pre-test that explicitly investigate students' meanings for difference and discuss preliminary results from 83 students' responses.

Task 1: What does it mean to say an expression is a difference?

We first asked students to answer the question, "What does it mean to say an expression is a difference?" Of the 83 responses coded so far, only 18 did not include the words "subtract" or "subtraction". For 6 students, "subtraction" or "to subtract" was the entire response. Formally, however, three pieces comprise a difference: a minuend, a minus sign and a subtrahend. In the context of Calculus, the minuend and subtrahend have significance; they represent values for measures of given quantities. For instance, one might be given a function f that describes the number of feet traveled by a rocket relative to the number of seconds elapsed since being launched. With the given information, one might symbolically represent the distance traveled by the rocket during the first 10 seconds of the launch by writing the difference $f(10) - f(0)$. It is necessary to imagine two quantities to produce this expression. We did not expect students to use language like "quantity" or "minuend", in their responses, but even after relaxing our criteria to determine how many objects (e.g. "numbers", "expressions", "something") mentioned in their response, 26% of responses contained a reference to fewer than 2 objects (Table 1).

Table 1. Number of objects described in a difference

Number of objects mentioned in describing a difference	Number of Responses (out of 83)
None	20 (24%)
Exactly 1	2 (2%)
"One or more" or "more than one" (as signaled by the use of pluralized words like "number*s*", "expression*s*")	19 (23%)
Exactly 2	31 (37%)
"Two or more"	10 (12%)
Unclear	1 (1%)

Task 2: Identifying differences in a mathematical sentence

In order to see how students operationalize the meaning they have for difference in the context of identifying structure, we asked students to identify differences in a mathematical sentence (Figure 1). We anticipated that students would at least rely on the number of subtraction

List each difference that you see in the mathematical sentence given below.

$$d(x) = \frac{f(x+h) - f(x)}{(x+h) - x} + e^{7-x} - 3\cos(2+x)$$

Figure 1. Identifying Differences Task

symbols (4) to determine the number of differences they should list. However, 33% of students (27 of 83) only listed 3 differences. Most of these students listed (with some variation on which parts of the differences they identified, as discussed below) $f(x+h) - f(x)$, $(x+h) - x$ and $e^{7-x} - 3\cos(2+x)$. Eight people did not respond or otherwise gave responses we could not interpret, four people listed more than 4 differences and 41 students listed exactly 4 differences.

Only 5 students listed the four differences we identified, namely: $f(x+h)-f(x)$, $(x+h)-x$, $7-x$, and $\left(\frac{f(x+h)-f(x)}{(x+h)-x}+e^{7-x}\right)-3\cos(2+x)$.

Table 2. Types of responses to Identifying Differences Task

Characteristic of Response	Number of Responses (out of 83)
Listed 4 differences	41 (49%)
Listed 3 differences	27 (33%)
Listed more than 4 differences	4 (5%)
Wrote e^{7-x} instead of $7-x$	14 (17%)
Wrote an expression containing only the minus sign and the subtrahend (e.g. $-x$ instead of $7-x$)	12 (14%)
Pointed to or circled the minus signs	11 (13%)
No response/Researchers could not interpret response	8 (10%)

Note: students' responses may be listed in multiple categories; the counts (percentages) will not add to 83 (100%).

Table 2 summarizes a few interesting points regarding students' responses. Eleven of 83 students *only* pointed to, or circled, the subtraction symbols. For these 11 students, the subtraction symbol is the difference, rather than the whole expression comprised of the minuend, minus sign and subtrahend. Another 12 students wrote the minus symbol and the subtrahend without the minuend when listing the differences they identified. We suspect students in these two groups may struggle to make meaning of discussions held in class regarding changes in quantities. In particular, for a student who only circled the minus signs in Figure 1, thinking about a difference does not entail imagining a quantity, two values of that quantity and a comparison of those values. Yet holding all these things in mind is essential to reasoning about changes in quantities in Calculus, one of the foundational components to the idea of rate of change and, hence, the Fundamental Theorem of Calculus.

Cross-task Comparisons

An emphasis on the operation subtraction while discussing "differences" may explain the 11 students' responses that reference *only* the minus sign. Further, a meaning for subtraction as "take away" could explain the other 12 students' writing only the minus sign and the subtrahend. Table 3 on the next page shows several of these students' responses. We plan to conduct follow-up interviews to further probe these students' thinking.

Questions for the Audience

In Table 3, it appears that Emily used a dash as a bullet point to list differences. Other students did this as well. Ought we consider Brett's and Cindy's responses to also reflect using a dash as a bullet instead of a minus sign? Does this change the consequences for meanings?

While we only focused on the meaning for difference in this report, we also have data related to students' meanings for the other arithmetic operations and the Order of Operations. Many students tended to rearrange symbols to "show" structure via spatial arrangement. For instance, when presented with a single-line expression $x + 3 / 7 * y$, the student would rewrite the expression as a stacked fraction. Does this constrain students' meanings? Or is a reliance on visual cues an acceptable activity for students, since writing expressions on one line is typically reserved for typing mathematics (an activity most students never do)?

Table 3. Student responses to both tasks

Student	Response to Task 1	Response to Task 2
Ally	Subtraction	$d(x)=\frac{f(x+h)-f(x)}{(x+h)-x}+e^{7-x}-3\cos(2+x)$
Brett	subtraction of given numbers or letters	$-f(x)$ $-3(\cos 2*x)$ $-x$
Cindy	total subtracted amount	$-f(x)$ e^{7-x} $-x$ $-3\cos$
Danny	This is a number that was previously subtracted from.	$d(x)=\frac{f(x+h)-f(x)}{(x+h)-x}+e^{7-x}-3\cos(2+x)$ all ~~arrows~~ arrows point to a difference
Emily	ı to say an expression is a difference? The expression is a subtraction equation	$-F(x+h)-F(x)$ $-(x+h)-x$ $-e^{7-x}$ $-e^{7-x}-3\cos(2+x)$

Note: All names are pseudonyms.

References

Hoch, M., & Dreyfus, T. (2006). *Structure sense vesus manipulation skills: an unexpected result.* Paper presented at the 30th Conference of the International Group for the Psychology of Mathematics Education, Prague, Czech Republic.

Novotná, J., Stehlikova, N., & Hoch, M. (2006). *Structure sense for university algebra.* Paper presented at the 30th Conference of the International Group for the Psychology of Mathematics Education, Prague, Czech Republic.

Piaget, Jean, & Garcia, R. (1991). *Toward a Logic of Meanings*. Hillsdale, NJ: Lawrence Erlbaum.

Thompson, Patrick W. (2013). In the absence of meaning. In K. Leatham (Ed.), *Vital Directions for Research in Mathematics Education* (pp. 57-93). New York: Springer.

Thompson, Patrick W., Carlson, Marilyn P., Byerley, Cameron, & Hatfield, Neil J. (2013). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In K. C. Moore, L. P. Steffe & L. L. Hatfield (Eds.), *WISDOMe Mongraphs* (Vol. 4). Laramie, WY: University of Wyoming.

An investigation of beginning mathematics graduate teaching assistants' teaching philosophies

Kedar Nepal
Mercer University

This qualitative study is an investigation of the teaching philosophies of beginning mathematics graduate teaching assistants (MGTAs). The study considered the cases of two domestic and two international MGTAs. Three teaching philosophy statements from each of the participants were collected at three different stages of a semester-long teaching assistant preparation program (pre-service phase). Three one-on-one interviews were conducted with each participant in the following four semesters (in-service phase) after the conclusion of the pre-service preparation program course. These audio-recorded interviews were transcribed and analyzed using the constant comparative method. Beginning teaching philosophies of these participants and how their philosophies changed over time, during both the pre-service and the in-service phases, will be briefly discussed. The factors that influenced their teaching philosophies in both phases will also be discussed.

Key words: [Mathematics Graduate Teaching Assistants, TA Preparation, Teaching Philosophy, Undergraduate Mathematics Instruction]

Introduction

Mathematics Graduate Teaching Assistants (MGTAs) in many US universities play significant roles in undergraduate mathematics instruction (Belnap, 2005; Speer, Gutmann & Murphy, 2005). The MGTA population consists of both domestic and international students. International MGTAs now make up a sizable portion of graduate students in mathematics. Regardless of where these MGTAs come from, they have tremendous influence on undergraduate students' experiences with mathematics (Speer, Gutmann, & Murphy, 2005).

Research shows that MGTAs' beliefs and philosophy about teaching and learning have a significant impact on their classroom practices and decisions (Kim, 2011; Speer, 2008; Thompson, 1992). Since MGTAs come from a diverse background, they bring their own experiences, perspectives, beliefs, and philosophies about teaching mathematics and student learning with them. Unlike other instructors, they are mathematics instructors who are also fulltime graduate students. Since teachers' philosophies of teaching change over time, a careful examination of beginning and evolving teaching philosophies may provide insight into the support structures necessary to facilitate effective classroom instruction (Simmons et al., 1999). This study attempts to describe the beginning and changing philosophies of teaching of a few purposefully selected MGTAs at a large public university in the midwestern US. The study was guided by the context-based adult learning theory, an extension of Vygotsky's sociocultural learning theory. Participants' learning about teaching and learning mathematics was considered within the cognitive apprenticeship model.

The goals of the study were to answer the following research questions: What are the teaching philosophies of beginning MGTAs? How do their philosophies evolve during the pre-service phase? How do their evolving philosophies of teaching change during the transitional in-service phase? What are the major contributing factors that affect MGTAs and their teaching philosophies during the pre-service and in-service phases?

Methods

The researcher observed all sessions of a semester-long MGTA preparation course during the fall 2012 semester. Four MGTA participants, David, Andrew, Rebecca and Jennifer (all pseudonyms), were selected using a purposeful sampling method. See Appendices A and B for a short description of the course and the participants. Two participants were domestic and two were international, and each category included one male and a female participant.

Three teaching philosophy statements (TPSs) were collected from each participant at three different stages: at the beginning, middle, and end of the semester. These TPSs were course assignments, required by the instructor, who also discussed the writing prompts for these TPSs (see Appendix C) with the researcher. They were directed to revise their earlier TPSs based on their learning from the course and also from their reflection of practicum experiences. Following the conclusion of this course, they taught lower division mathematics courses, either as instructors with full responsibility or as recitation leaders. Three one-one-one audio-taped interviews were conducted with each participant in the following three semesters. The pre-service TPSs and the transcribed interviews were analyzed using the constant-comparative method (Strauss & Corbin, 1990) using open coding techniques. The data from each interview were analyzed before conducting the subsequent interviews.

Results and Discussion

The participants expressed varying opinions in both the pre-service and the in-service phases. In addition, their teaching philosophies evolved differently over time during both phases. This is a case study, and because of the subjective nature of the participants' opinions, it would not be appropriate to generalize and draw conclusions about what novice MGTAs believe. However, some of the themes found in the participants' TPSs were shared by all of them. Below is a summary of the results obtained from data analysis. Because of space constraints, it was not possible to describe all the identified themes in detail.

Pre-service phase. As found in studies involving pre-service and beginning teachers (Stuart & Thurlow, 2000), these MGTAs also had simplistic views of teaching and did not seem to realize what it takes to be an effective teacher. For example, summarizing his TPS I, David wrote: "One can be an effective or a successful teacher if he prepares well on the subject matter before going to the class, develops positive attitudes, has high expectations (for students), and employs fairness in his teaching." The beginning TPSs of the participants had a few elements in common, such as giving importance to conceptual understanding and creating a good learning environment. They reiterated these beliefs even during the in-service phase. However, they wrote very little that was specific to mathematics teaching.

All participants wrote in TPS I that creating an effective learning environment is essential, although they had varying opinions about what constitutes such an environment. Andrew, Rebecca, and Jennifer wrote that they could motivate students to learn mathematics by relating concepts to other fields. David also expressed this belief later, during the in-service phase. The domestic students, Andrew and Rebecca, wrote that instructors should engage students in the learning and problem solving processes. An interesting element found from the analysis was that both international students wrote that treating students equally would help to create the desired learning environment. Rebecca was the only MGTA who stated that employing student-centered instruction would enhance the learning environment.

David, Andrew, and Rebecca wrote that teachers should have high expectations for students. Based on the teaching they experienced as students, they expressed that they had been pushed to work harder and succeed because their professors used challenging problems for assignments and exams. Only two participants, David and Andrew, wrote about the importance of being prepared before going into the classroom. But David only emphasized content preparation, while Andrew underscored the importance of developing lesson plans and creating challenging homework and exams, although he realized that his other obligations as a graduate student might make that impossible. Jennifer also wrote that the instructors should have enough knowledge of mathematical content but neither of the domestic students wrote anything about this topic. All the participants except David wrote that providing out-of-class support was an essential supplement to classroom teaching. Later during the in-service phase, David also stated that providing such support is important.

MGTAs mostly described the teaching behaviors of their past teachers (role model or "ineffective") instead of writing their own opinions, beliefs, and plans in their initial teaching statements. Although some aspects of these beginning MGTAs' teaching philosophies mirrored existing research about the beliefs of pre-service teachers, some of the themes were not at all common in the existing literature. The elements that were not common were their belief in having high expectations for students, the international MGTAs' belief in equal treatment of students, and Andrew's belief in the importance of "coping with" institutional culture.

Data analysis showed that the philosophies of these participants changed very little during this period, from TPS I to TPS III. At the point of TPS II, they generally made only minor revisions and added a few more beliefs to what they had described in TPS I. Jennifer, for example, added only that instructors should smile at students to make them feel comfortable. Rebecca acknowledged the importance of being prepared, a change that she attributed to the observation of her mentor's classes. David wrote that instructors should also prepare teaching materials and employ "effective teaching techniques," in addition to preparing mathematical content they are going to teach. Both of them added that instructors should be well-organized, both in and out of the classroom. David and Andrew added that active classroom interaction contributes to a good learning environment, and that instructors should have a positive attitude. David added that instructors should be caring, but still maintain the role of an authority figure. He also wrote that teachers should be able to explain mathematical concepts clearly to the students. All of these changes were relatively minor, with all participants' major beliefs remaining unchanged. All four participants' belief in creating a comfortable learning environment had actually been strengthened during this period.

At the point of TPS III, near the end of the pre-service phase, all the participants except Andrew developed or reinforced their belief that they should be well prepared before teaching. Jennifer attributed this change to her teaching presentation experience. Both male participants, David and Andrew, said that they would grow as effective teachers as they gain more experience. David realized that people should have a passion for teaching if they want to succeed in the profession. Andrew added that he would reflect on his own teaching and try to improve. Andrew and Rebecca added that instructors should share their passion for mathematics if they want to motivate students and create an effective learning environment. They wrote that they would encourage their students to seek out-of-class support during office hours, but Andrew's earlier belief in employing a tough-love attitude did not change.

In summary, small changes were detected in the TPSs of these participants during the pre-service preparation program. For example, Jennifer only added in TPS II and TPS III that she needed to make eye contact and smile at students to make them feel comfortable. Her situation was unique in that she was a pre-service college instructor in the US, but she already had five years of college teaching experience in her home country. However, she attributed her philosophical changes to her observation of undergraduate classes taught by her faculty mentor during the pre-service preparation program. There are few interesting observations worth noting here. First, if she had five years of college teaching experience in her home country, why was this the first time that she realized the need to make eye contact with her students and smile at them? Second, if it was, was that all she learned from a semester long seminar and practicum experiences in the course? She noted in TPS III that she needed to be a little more prepared, which she attributed to her teaching presentation experience in the course. But she could have already realized the need to be prepared from her five years of teaching experience.

In-service phase. Many common themes were detected in the teaching philosophies of all four participants as they entered into the in-service phase and started to teach. They all considered creating an effective learning environment, being prepared, and providing out-of-class support as fundamental components of their teaching philosophies.

All four participants felt strongly that they would develop as better teachers as they gained more experience. They said that the best way to learn about teaching is to experience teaching as a real classroom instructor. When asked to express their opinions on the importance of professional development activities, they said that these might have little impact in their future teaching. Even though they said (during the interviews) that they learned many things from the pre-service preparation course, they did not attribute their philosophical changes much to that course. They said that the course mostly contributed to alleviating their anxiety and increasing their confidence as beginning teachers. This finding is consistent with findings from earlier research with MGTAs (Belnap, 2005; Harris, Froman, & Surles, 2009). They said that the math department did a good job in helping them during the first semester, but interestingly, they all felt that they would need little further support. This is not consistent with previous research that found TAs believed that their preparation program did not adequately prepare them for classroom teaching (Moore, 1991). It is, however, consistent with previous findings that MGTAs do not see the importance of pre-service preparation and any other professional development programs, because they believe that the only way to learn about teaching is by experience (Chae, Lim, & Fisher, 2009; Harris, Froman, & Surles, 2009). It is also possible that many MGTAs might have not been consciously aware of how their pre-service preparation programs affected them (Belnap, 2005).

The participants said that they changed some of their perspectives about teaching because of their teaching experience during the in-service phase, especially because of the students they taught. Jennifer said that the students are the best source from which to learn, and that teaching evaluations would help her to improve her future teaching. They all expressed that teachers' efforts alone are insufficient, and that students should also put effort into their learning. For example, Andrew said during interview III that students should "bust their butt" in order to succeed. Without being asked, Andrew said that we are trying to replace hard work by "pampering students". He said that some concepts are difficult to learn because they are difficult, and the students should work hard. All of them said that they want to learn from their own experience and mistakes. Even though they would like to have help when needed, they said that they want to teach their way and become the kind of instructor they aspire to be. They did not seem to like teaching recitation sections, or coordinated courses with similar pacing across all sections, and following the instructions of the course coordinators.

One significant change detected in the teaching philosophies of David, Andrew, and Rebecca was that they would like to become more authoritative, to prevent students from treating them as peers because of their small age difference. As all participants (except Andrew) became more experienced teachers during the in-service phase, they gradually reinforced their earlier belief that they need to be prepared for class. Andrew said he would only focus on reviewing the content before going to teach. He said that he does not do lesson plans or any other preparation, and needs only to know the relevant mathematical content, and be able to provide examples "on the fly".

Andrew and Rebecca said that they realized the importance of using technology but rarely had enough time to use it in the classroom, a finding that matched previous studies with beginning K-12 teachers (Schuck, Aubusson, Buchanan, & Russel, 2012). None of the four participants had discussed the use of technology in any of their pre-service TPSs.

There was some variation in participants' expressed beliefs about creating an effective learning environment. All the participants said, in some or all the interviews, that students can be motivated by showing applications of mathematical concepts to other fields. They said that use of humor could contribute to a fun learning environment, and mentioned that instructors should try to change students' mindset that 'math is hard'. Andrew, David, and Rebecca said that sharing their passion for mathematics would positively impact how the students respond to them. Being caring, inspiring, and encouraging were other ways they all thought they could create a good learning environment.

The participants expressed that interaction, student engagement, and collaborative learning all contribute to a good learning environment. Jennifer, Rebecca, and David said that their course coordinators encouraged them to employ collaborative learning, and supported their attempts to do so, but Andrew said that he learned from his own teaching experience. Other researchers had also found that the participants from a multi-day workshop identified a change in their teaching philosophies from teacher-centered to student-centered instruction (Schussler et al., 2011; White et al., 2012). But the participants in the existing research attributed such change to their learning of educational theory in addition to the feedback from experienced faculty (Schussler et al., 2011). David, Jennifer and Rebecca confirmed their learning from their course coordinators but none of the participants attributed their newfound appreciation for collaborative learning to any of several reading assignments from the preparation course.

International MGTAs, David and Jennifer, carried over a philosophy of treating students equally from the pre-service phase. They had developed this belief from the teaching they had experienced as students in their respective home countries. They perceived that they would have been much better instructors if they were more proficient in English, and had a stronger understanding of American culture. They stated that American students seemed unwilling to interact with them, because of cultural differences and their (self-perceived) inadequate proficiency in English. These challenges of international MGTAs due to cultural differences were also identified by Chae, Lim, and Fisher (2009). Research on international teaching assistants has also suggested that differences in communication styles and behaviors may contribute to the negative interactions and misunderstandings with their students, and hence their perceived decreased effectiveness (Liu, Sellnow, & Venette, 2006; McCroskey, 2003). However, the cultural differences and their inadequate English language proficiency may not be the only cause of the development of the belief about the importance of interacting with the students. Even the two domestic MGTAs in this study realized the importance of frequent student-teacher interaction because they said that their students rarely visited them during their office hours.

All participants but Jennifer expressed at all three stages of the in-service phase that having high expectations for students would force them to work harder, a belief they carried over from the pre-service phase. Andrew even said that his philosophy was to employ a tough-love attitude to force students to realize their responsibility and to put effort into their education. Rebecca, however, started to feel less strongly about this belief as she gained more teaching experience. For example, she said that she does not like to intimidate her students by giving very challenging problems at first, but would start by assigning easier problems, then slowly build up students' confidence to solve more difficult problems. Andrew also seemed to have softened somewhat, as he underscored the need to be patient and show compassion for students' difficulties. However, he also said that he would become tougher with students who do not work but keep complaining about the level of effort needed to successfully learn mathematics. The participants said that they developed the belief of having high expectation for students from the impressions of their role model teachers from the past.

Factors. The foundation of MGTAs' initial teaching philosophies stemmed primarily from their past experience as students, especially by their role model teachers. However, their in-service teaching philosophies were influenced mostly by their teaching experience. Other influencing factors included observation of their faculty mentors' classes, their own undergraduate students, MGTA peers, and course coordinators. MGTAs said that the preparation program had little impact in their teaching philosophies.

The data used in this study is only an approximation of what these participants believe or think. What they have in mind might not have been written in the TPSs or expressed during the interviews. Therefore, it is not appropriate to conclude what they really think or believe based on this data set alone. Also, the changes detected might not have been caused solely by their learning during the pre-service or in-service phases. Their TPSs could have been different if they had been given different writing prompts, or instructed to draft new statements instead of revising their old ones. Similarly, what they reported during interviews might have changed with different interview protocols or a different interviewer. Also, beliefs that were first detected in the in-service interviews might also have been held during the pre-service phase but not reported, as it would have been impossible to report all one's beliefs about teaching in the 1-2 page TPSs. Moreover, there is no guarantee that the novice instructors' classroom practices are always consistent with the expressed beliefs. However, understanding their perspectives could inform restructuring of some of the professional development activities in the existing pre-service MGTA preparation programs.

References

Belnap, J. (2005). *Putting TAs into context: Understanding the graduate mathematics teaching assistant.* Retrieved from *ProQuest Dissertations and Theses.* (305024286)

Bonk, C. J., & Kim, K. A. (1998). Extending sociocultural theory to adult learning. In M. C. Smith & T. Pourchot (Eds.), *Adult Learning and Development: Perspectives From Educational Psychology* (pp. 67-88). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Brandt, B. L., Farmer, J. A., & Buckmaster, A. (1993). Cognitive apprenticeship approach to helping adults learn. *New Directions for Adult and Continuing Education, 1993*(59), 69-78.

Chae, J., Lim, J. H., & Fisher, M. H. (2009). Teaching mathematics at the college level: International TAs' transitional experiences. *Primus, 19*(3), 245-259.

Chism, N. V. N. (1998). Developing a philosophy of teaching statement. *Essays on Teaching Excellence: Toward the Best in the Academy*, *9*, 1–3.Teaching mathematics at the college level: International TAs' transitional experiences. *Primus, 19*(3), 245-259.

Brandt, B. L, Farmer, J. A., & Buckmaster, A. (1993). Cognitive apprenticeship approach to helping adults learn. *New Directions for Adult and Continuing Education,* 59, 69-78.

Hansman, C. A. (2001). Context-based adult learning. *New Directions for Adult and Continuing Education,* 89, 43-51.

Harris, G., Froman, J., F., & Surles, J. (2009). The professional development of graduate mathematics teaching assistants. *International Journal of Mathematical Education in Science & Technology, 40*(1), 157-172. doi: 10.1080/00207390802514493

Jenkins, C. (2011). Authenticity through reflexivity: Connecting teaching philosophy and practice. *Australian Journal of Adult Learning, 51*, 72-89.

Jones, J. L. (1993). TA training: From the TA's point of view. *Innovative Higher Education, 18, 147-161.*

Kim, M. (2011). *Differences in beliefs and teaching practices between international and U.S. domestic mathematics teaching assistants.* Retrieved from ProQuest Dissertations and Theses. (885228899)

Kung, D., & Speer, N. (2007). Mathematics teaching assistants learning to teach: Recasting early teaching experiences as rich learning opportunities. *Electronic Proceedings for the Tenth Special Interest Group of the Mathematical Association of America on Research on Undergraduate Mathematics Education.*

Liu, M., Sellnow, D. D., & Venette, S. (2006). Integrating nonnatives as teachers: Patterns and perceptions of compliance-gaining strategies. *Communication Education, 55*(2), 208-217.

McCroskey, L. (2003). Relationships of instructional communication styles of domestic and foreign instructors with instructional outcomes. *Journal of Intercultural Communication Research, 32*(2), 75-96.

McCroskey, L. (2002). Domestic and international college instructors: An examination of perceived differences and their correlates. *Journal of Intercultural Communication Research, 31*(2), 63-83.

Schussler, E. E., Rowland, F. E., Distel, C. A., Bauman, J. M., Keppler, M. L., Kawarasaki, Y.,…, & Salem, H. (2011). Promoting the development of graduate students' teaching philosophy statements. *Journal of College Science Teaching, 40*(3), 32-35.

Schuck, S., Aubusson, P., Buchanan, J., & Russell, T. (2012). Lessons Learnt from Stories of Beginning Teachers. In *Beginning Teaching* (pp. 133-143). Springer Netherlands.

Simmons, P. E., Emory, A., Carter, T., Coker, T., Finnegan, B., Crockett, D., Labuda, K. (1999). Beginning teachers: Beliefs and classroom actions. *Journal of Research in Science Teaching, 36*(8), 930-54.

Speer, N. M. (2008). Connecting Beliefs and Practices: A Fine-Grained Analysis of a College Mathematics Teacher's Collections of Beliefs and Their Relationship to His Instructional Practices. *Cognition and Instruction,* 26(2), 218-267.

Speer, N., Gutmann, T., Murphy,T. J. (2005). Mathematics teaching assistant preparation and development. *College Teaching, 53*(2), 75-80.

Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127–146). New York: Macmillan.

Vygotsky, L.S. (1978). *Mind in the Society: The Development of Higher Psychological Processes.* Cambridge, MA: Harvard University Press.

Appendix A. Seminar and Practicum in Teaching College Mathematics

Seminar and Practicum in Teaching College Mathematics (SPTCM) was a mandatory semester-long preparation program for beginning MGTAs during the first semester of their graduate program. They did not have any teaching or grading assignments during the semester but they were required to participate in this professional development program. MGTAs learned pedagogical knowledge and other classroom management skills through a combination of weekly seminar discussions and classroom practicum experiences.

MGTAs were expected to complete all out of class assignments such as written assignments (syllabi, lessons, exams, papers, etc.) and retain them as part of their course portfolios. They were placed with experienced instructors who served as their mentors. They were required to observe their mentors' classes and visit their offices for completing assigned activities and asking any questions. Mentors submitted an evaluation of their MGTAs' performance at the end of the semester. MGTAs prepared and delivered actual classroom presentations under the direct supervision of their mentors. They were required to write reflections of their own presentations, discuss these reflections with peer MGTAs (who also wrote reflective comments related to their observations of the presenter), and revisit their presented lessons.

Almost every weekly seminar began with a repertoire of student issues related to decision-making and classroom management initiated by the instructor or MGTAs. They were encouraged to express their opinion about how they would respond or act to such student issues. MGTAs were required to learn routine activities such as preparing syllabi, writing exams, using technology, maintaining grade book, and posting student grades from their mentors. During the discussions, they were encouraged to share their learning from the mentors, and share their observations, questions, and reflections they had noted from class observations. MGTAs were involved in activities such as grading actual student homework and exams. In the following week, they are required to present and justify their grading algorithm. This activity provided them an opportunity to learn how others see things differently and also to reflect on their own decisions.

Besides several other reading and writing assignments, MGTAs were assigned a particularly introspective assignment related to 'Developing Your Philosophy of Teaching'.

Appendix B. Description of the Participants

David. David was a 23-year-old first year Master's degree student during the fall 2012 semester. He was an international student who came from a south Asian country. He completed high school in his home country before pursuing a bachelor's degree in a medium-sized university in the mid-western US. He planned to pursue a Ph.D. degree in applied mathematics and his career goal was to become a mathematics professor. He was fluent in spoken English, but with a noticeable accent.

Andrew. Andrew was a 28-year-old domestic graduate student who was working on a Ph.D. in pure mathematics. He grew up in the southwestern US, where he was home schooled during the last three years of high school. He then went to a nearby junior college, before transferring to a university, where he finished his undergraduate degree. He said that he had a lengthy undergraduate experience because he switched his major a couple of times.

Rebecca. Rebecca was a 23-year-old masters' student in applied mathematics and did not have a plan to purse her Ph.D. She completed her undergraduate degree in mathematics and information technology at a small Catholic University in the mid-western US. She also graduated with a minor in accounting. Her career goal was to work in the private sector, but said that she would return to teaching if she did not enjoy the private sector.

Jennifer. Jennifer was a 30-year-old international student who completed her high school, undergraduate, and masters' degrees in a northeast Asian country. She was a Ph.D. student in pure mathematics, and her career choice was to become a mathematics professor. She was not fluent in spoken English and therefore had difficulty expressing her opinions. She said that she taught mostly upper division college mathematics courses in her home country for five years but no other participants had any classroom teaching experience.

Appendix C. Prompt for Teaching Philosophy Statement I

Write a short paper of about 2-3 pages discussing what *you* have learned about effective and ineffective teaching from being a student. Describe the teaching of someone who was, in your experience, a particularly effective teacher, and analyze why you think this person succeeded as a teacher. This is just the beginning on your journey to develop your own philosophy of teaching, a philosophy that will probably change several times during your teaching career. The conclusion of your paper should be a thoughtful initial statement of *your* emerging philosophy of teaching. Be sure to include your thoughts on what you believe now.

The effectiveness of clickers in large-enrollment calculus

Xuan Hien Nguyen, Iowa State University
Heather Bolles, Iowa State University
Adrian Jenkins, Iowa State University
Elgin Johnston, Iowa State University

Abstract: *We report the results of a two-year study of the effect of clickers in the large-lecture format through a variety of metrics. These metrics include specific quiz scores (including both conceptual and applicational questions), as well as pass rates (both for the general student body as well as for males and females specifically). We include statistical basis for our findings.*

Keywords: Calculus, Clickers, Peer Instruction, Classroom Research, Large Classes

Introduction

Following the well-publicized report of the successful use of audience response technology (clickers) on the improved learning of students in a large-enrollment physics class (Deslauriers, Schelew, & Wieman, 2011), this study sought to determine if similar effects on students' learning occurred in large-enrollment first semester engineering calculus. Many studies have reported the benefits of using clickers in the classroom including boosted attendance (Mollborn & Hoekstra, 2010; Preszler, Dawe, Shuster, & Shuster, 2007) increased student participation (Lucas, 2009), and students' enjoyment in using the technology in the classroom (Bode, Drane, Kolikant, & Schuller, 2009). Continued, however, is the need for controlled studies which remove the instructor influence and indicate the impact of clicker use on student learning. For example, results of clicker studies show mixed results when considering gender differences. King and Joshi (2008) found that males who actively participated in the clicker questions scored 10 points higher on their final grades than non-active males. The female counterparts, however, only scored about 5 points higher than the non-active females. Hoekstra (2008) indicated that women tend to cooperate together, whereas if someone worked individually during the clicker-prompted interaction phase of class, more often the individual was male.

The following study builds on the generative learning theory which emphasizes the priming of cognitive processes during instruction through questioning. In general, "when students answer questions during learning they are encouraged to select relevant information, mentally organize the material, and integrate it with their prior knowledge" (Mayer et al., 2009, p.53).

Research questions

Despite the many advantages of clickers, it is still unclear if clickers improve student learning. Although a higher enjoyment is a marked advantage in itself, we sought to determine whether students are more successful in classes with clickers. The research questions that guided the design of this study were:

1. What is the effect of clickers on students' understanding of limits and integration? Does the use of clickers with peer discussion help students retain the information presented in the lectures?
2. Are underrepresented (female or minority) students more likely to benefit from the use of clickers in the classroom?
3. Do students in clicker lectures earn better grades than their counterpart in nonclicker lectures, with the level of preparedness held constant? In particular, does the use of clickers influence the passing rate of students in Calculus 1?

Methods

To account for the instructor effect, three instructors each taught a large-enrollment class of Calculus I in Fall 2012 and again in Fall 2013 at Iowa State University. Two of the instructors used clickers during the Fall 2012 session but the third one did not. The clicker usage was reversed in Fall 2013. Instructors taught at the same times of the same days of the week in Fall 2013 as in Fall 2012.

Clicker vs. nonclicker sections. When teaching the clicker sections, each instructor followed the same protocol as informed by the literature (Miller, Santana-Vega, & Terrell, 2006; Mazur, 1997; Crouch & Mazur, 2001; Cline, 2006). Students were asked on average three questions during each 50-minute class session. The questions emphasized calculus concepts, skills, or applications focused on key ideas of calculus with a goal of promoting interaction amongst students. For each clicker question, students were given time to solve the problem individually before submitting their answer via the clicker. Without viewing response results, students then consulted with their classmates to discuss their reasoning and findings. After submitting their answer again, students were shown the results of both polls. Finally, the instructor led discussion of how to solve the problem, highlighting calculus concepts, techniques, and common pitfalls when solving related problems. The total number of points earned on clicker questions accounted for 5% of the overall course grade.

In the nonclicker sections, the same clicker questions were used as examples incorporated during lectures. Students were encouraged to work on the problems, compare their results with classmates, and answer the questions, though no polls were conducted. As in the clicker sections, the instructor highlighted how to solve the problem and emphasized key calculus concepts, techniques, and common pitfalls when solving related problems. No points were awarded for answering any individual question.

Data. Student performance on three quizzes was compared to test for existence of an effect of clickers on student learning. The quizzes addressed concepts related to limits, applications of the derivative, and the First Fundamental Theorem of Calculus and were identical in both fall semesters. The three instructors discussed a grading rubric for each quiz, and each instructor graded the quizzes for his/her own students. In addition to all the grades on quizzes, exams, and overall letter grade for the course given by the instructors, the researchers obtained demographic information, academic grade and ability measure (ACT composite, English, and math, number of high school calculus credits) for each student from the Registrar's office (see Table 1).

The population for the study included all students enrolled in one of the six lectures taught by the three instructors ($N = 1142$). The gender distribution is 22.48% female and 77.52% male.

Results

The preliminary results from the quiz scores are mixed. For one instructor, students in the clicker section outperformed students in the nonclicker section on each of the three quizzes, though not all of the results are statistically significant. For another instructor, students in the nonclicker section performed better on the quizzes than those in the clicker section, though not all were statistically significant. For the third instructor, students in the clicker section outperformed students in the nonclicker section on two of the three quizzes (see Table 2).

For the third instructor, C, the higher mean on the second quiz may be due to the fact that students in the nonclicker lectures have more high school preparation. When we divide instructor

		nonclicker			clicker			t-test
Instructor	Variable	N	Mean	SD	N	Mean	SD	p-value
A	Total number of students	194			192			
	ACT composite score	172	26.35	3.23	182	26.98	3.06	0.063
	ACT english score	172	25.15	4.24	181	26.04	4.40	0.054
	ACT math score	172	27.69	3.56	181	27.51	3.26	0.625
	High School Calc Credit	186	1.27	1.15	188	1.40	1.05	0.254
B	Total number of students	200			201			
	ACT composite score	167	26.25	3.63	178	26.22	3.70	0.947
	ACT english score	166	25.19	4.51	178	25.00	4.68	0.698
	ACT math score	166	26.87	3.70	178	27.07	3.74	0.618
	High School Calc Credit	178	1.44	1.16	186	1.22	1.18	0.070
C	Total number of students	195			203			
	ACT composite score	164	26.95	3.52	169	26.03	3.40	0.016
	ACT english score	164	25.71	4.33	168	25.02	4.73	0.167
	ACT math score	164	27.66	3.52	168	27.09	3.59	0.146
	High School Calc Credit	174	1.54	1.07	186	1.08	1.11	<0.001

Table 1: ACT scores and HS credits by lecture

C's students into two groups, we see that the nonclicker students with 2 or more high school calculus credits outperform the clicker students on quiz 2. There is no significant difference for students with less than 2 high school calculus credits.

As other studies have shown across various subject areas (Lantz, 2010; Yourstone, Kraye, & Albaum, 2008), preliminary evidence in this study shows that the use of clickers may have little overall effect in advancing student learning. One noticeable impact, however, was that student attendance to the class sessions was higher in the clicker sections than in the nonclicker sections. Late into the semester, attendance continued to reach 80-85% of the enrollment for the class. Each instructor experienced this. While no method of taking attendance was implemented in the nonclicker sections, each instructor agrees that noticeably fewer students regularly attended class in the nonclicker sections. The increased engagement of the clicker students is also reflected in the

Instructor		q1	q2	q3
A	nonclicker	9.91*	12.64	9.30
A	clicker	6.89	13.30	8.50
A	t-test p-value	<0.001	0.305	0.135
B	nonclicker	6.76	12.99	8.41
B	clicker	7.58	15.48*	8.87
B	t-test p-value	0.110	<0.001	0.343
C	nonclicker	6.38	13.72*	9.77
C	clicker	7.31*	11.70	10.14
C	t-test p-value	0.050	<0.001	0.383

Table 2: Averages for the quiz scores. The t-test compares the clicker and nonclicker lectures.

student evaluations, with higher numbers for the clicker lectures.

Students in clicker sections pass at a higher rate than in nonclicker sections (see Table 3). Here, we define passing as a grade of C- or higher, which is the prerequisite grade for going on to Calculus 2. A students does not pass if he/she gets a D, an F, or drops the course. The effect is more pronounced for female students, although it is not statistically significant.

		female				male			
	all	all	pass	nonpass	pass rate	all	pass	nonpass	pass rate
nonclicker	568	110	67	43	60.91%	458	293	165	63.97%
clicker	574	144	99	45	68.75%	430	278	152	64.65%

Table 3: Frequency table of the numbers of students who passed and students who did not pass by gender.

The median ACT composite score for our group of 1032 students is 27 (we do not have ACT scores for all the students). When we split the students into two groups, one with high ACT (≥ 27) and the other one with low ACT score (< 27), the passing rate for male students in the high ACT group are comparable (see Table 4). For female students, a Pearson's chi-squared test comparing

		female				male			
	all	all	pass	npass	pass rate	all	pass	npass	pass rate
nonclicker	264	53	40	13	75.47%	211	165	46	78.20%
clicker	261	61	54	7	88.52%	200	155	45	77.50%

Table 4: Frequency table for students with ACT ≥ 27.

the 2x2 table composed of the pass and nonpass columns shows that the higher passing rate for the clicker lectures is significant with 90% confidence ($\chi^2 = 3.34$ with one degree of freedom; p-value = 0.06761).

We now look at similar data for the low ACT group. Table 5 shows that the passing rate is 5% higher for the clicker lectures (and 10% for female students). The results are not statistically significant. One of the reasons is the low number of female students. A Pearson's chi-squared test

	all				female		male	
	all	pass	fail	drop	all	pass rate	all	pass rate
nonclicker	239	117	80	42	45	44.44%	194	50.00%
clicker	268	145	62	61	73	54.79%	195	53.85%

Table 5: Frequency table for students with ACT < 27.

on the fail and drop columns give us a $\chi^2 = 5.783$ and a p-value = 0.01619 which shows that, with the use of clickers, the students who did not pass the course are more likely to drop the course than fail.

Future work and questions

One of the problems we encountered during the study was the switch of the calculus textbook between Fall 2012 and Fall 2013. In order to obtain more consistent data, the three instructors are teaching Calculus 1 again in the Fall 2014, reverting to the same setting as in the Fall 2012.

Question 1. Is there another direction or result to consider when analyzing the data? We have ethnicity data, ACT scores, the admission term of the students, as well as the semester they took calculus 2 (if they took it at all). With the higher number of participants, we hope to have significant results as to the effect of clickers on female students as well as be able to look at different ethnic groups.

Question 2. Originally, we had hoped to judge students' ability in calculus 1 by their performance on the same three quizzes given each year. Even though a lot of effort was made to grade the quizzes consistently, the wide range of scores lent too much variability. It was not possible to identify a model for the quiz scores. This is the reason why we started looking at passing and not passing numbers. What are some suggestions on statistical modeling? What should we take as a measure of success?

References

Bode, M., Drane, D., Kolikant, Y. B.-D., & Schuller, M. (2009). A clicker approach to teaching calculus. *Notices of the AMS*, *56*(2), 253–256.

Cline, K. S. (2006). Sharing teaching ideas: Classroom voting in mathematics. *Mathematics Teacher*, *100*(2), 100–104.

Crouch, C. H., & Mazur, E. (2001). Peer instruction: Ten years of experience and results. *American Journal of Physics*, *69*(9), 970–977.

Deslauriers, L., Schelew, E., & Wieman, C. (2011). Improved learning in a large-enrollment physics class. *science*, *332*(6031), 862–864.

Hoekstra, A. (2008). Vibrant student voices: Exploring effects of the use of clickers in large college courses. *Learning, Media and Technology*, *33*(4), 329–341.

King, D. B., & Joshi, S. (2008). Gender differences in the use and effectiveness of personal response devices. *Journal of Science Education and Technology*, *17*(6), 544–552.

Lantz, M. E. (2010). The use of clickers in the classroom: Teaching innovation or merely an amusing novelty? *Computers in Human Behavior*, *26*(4), 556–561.

Lucas, A. (2009). Using peer instruction and i-clickers to enhance student participation in calculus. *Primus*, *19*(3), 219–231.

Mayer, R. E., Stull, A., DeLeeuw, K., Almeroth, K., Bimber, B., Chun, D., et al. (2009). Clickers in college classrooms: Fostering learning with questioning methods in large lecture classes. *Contemporary Educational Psychology*, *34*(1), 51–57.

Mazur, E. (1997). *Peer instruction*. Upper Saddle River, NJ: Prentice Hall.

Miller, R. L., Santana-Vega, E., & Terrell, M. S. (2006). Can good questions and peer discussion improve calculus instruction? *Problems, Resources, and Issues in Mathematics Undergraduate Studies*, *16*(3), 193–203.

Mollborn, S., & Hoekstra, A. (2010). a meeting of minds using clickers for critical thinking and discussion in large sociology classes. *Teaching Sociology*, *38*(1), 18–27.

Preszler, R. W., Dawe, A., Shuster, C. B., & Shuster, M. (2007). Assessment of the effects of student response systems on student learning and attitudes over a broad range of biology courses. *CBE-Life Sciences Education*, *6*(1), 29–41.

Yourstone, S. A., Kraye, H. S., & Albaum, G. (2008). Classroom questioning with immediate electronic response: Do clickers improve learning? *Decision Sciences Journal of Innovative Education*, *6*(1), 75–88.

Differentiating instances of knowledge of content and students (KCT): Responding to student conjecture

Authors
Affiliation

This study investigates the nature of preservice elementary teachers' knowledge of content and teaching (KCT) through responses to hypothetical student scenarios. Participants demonstrated two types of KCT: (1) specific KCT, which drew from the specific mathematics of the student scenario and most resembles the construct described by Ball, Thames, & Phelps (2008); and (2) general KCT, which presented itself as "canned" mathematical pedagogy. Examples and influences upon each type of KCT are explored to promote discussion on the differentiation of KCT and for further refinement of the concept of general KCT.

Key words: Pedagogical Content Knowledge (PCK), Knowledge of Content and Teaching (KCT), Preservice Elementary Teachers

The research suggests that pedagogical content knowledge (PCK) is important for teaching (e.g., Ball, Thames, & Phelps, 2008; Campbell, Nishio, & Smith et al., 2014; Shulman, 1986). To better inform the research community on mathematics PCK, Ball and colleagues have developed a model for teachers' Mathematical Knowledge for Teaching (MKT). However, much is still unknown about the constructs of MKT and the various educational influences on teachers' development of MKT. Part of a larger case study, this report explores the nature of preservice elementary teachers' knowledge of content and teaching (KCT), a construct of MKT, as well as any influences on their KCT.

Background

The research suggests that a teacher's mathematical PCK impacts teacher effectiveness (e.g., Campbell, Nishio, & Smith et al., 2014; Hill, Rowan, & Ball, 2005). Drawing from Shulman's (1986) original definition of PCK, Ball, Thames, and Phelps (2008) and Hill, Ball, and Schilling (2008) have conceptualized three components of mathematical PCK in their model of MKT: knowledge of content and students (KCS), knowledge of content and teachers (KCT), and knowledge of curriculum. According to Ball, Thames, & Phelps, KCT "combines knowing about teaching and knowing about mathematics, … [such as] sequenc[ing] particular content for instruction… evaluat[ing] the instructional advantages and disadvantages of representations used to teach a specific idea and identify[ing] what different methods and procedures afford instructionally. Each of these tasks requires an interaction between specific mathematical understanding and an understanding of pedagogical issues that affect student learning… Each of these decisions requires coordination between the mathematics at stake and the instructional purposes at play" (p. 401).

The emergent perspective (Cobb & Yackel, 1996) served as the lens for collecting and analyzing data. I primarily used the psychological lens since the bulk of the data represent individual conceptions. On the other hand, via the social lens I explored the classroom norms, expectations, and experiences that framed participants' perspectives on mathematics teaching and learning. I also drew from Ball and colleagues' conceptualization of mathematical PCK (e.g., Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008) in designing my interview tasks to elicit PCK and again to analyze responses.

Methodology

This interpretive case study (Merriam, 1998) centered on preservice elementary teachers who were seeking a mathematics concentration and enrolled in a number theory course. The majority of class time in this course was spent working on problem sets collaboratively (which were occasionally geared towards elementary school applications of number theory), and the instructor encouraged basic explanations or picture proofs. Otherwise, there was no overt connection made to elementary teaching. Three out of six of the interview participants had previously taken a number and operations course where collaborative, inquiry-based learning was the norm. The other three interview participants had taken a mathematics education course designed for elementary education majors with a mathematics concentration; it also focused on number and operations content using collaborative, inquiry-based learning methods. As part of the course, these three participants served as peer tutors in the number and operations course.

Data for this study came from multiple sources: classroom observational notes, student coursework, as well as responses from two sets of one-on-one task-based interviews, which served as the focus of the data analysis. Many of the tasks posed hypothetical student scenarios, designed to elicit PCK in number theory. To elicit KCT specifically, I would ask participants how they would respond to the students in the scenarios. I would also ask participants to reflect on why they responded in that way. While some might argue that teachers may only demonstrate true KCT in the classroom, others suggest that demonstrations of KCT in a clinical interview may be a sort of pre-knowledge or a subset of the knowledge they could demonstrate in the classroom (Hauk, Jackson, & Noblet, 2010). Even Hill (2010), a contributor of MKT, developed and implemented PCK test items that proposed to elicit KCT. Constant-comparative coding (Corbin & Strauss, 2008) was used as part of the coding process. Among my efforts to ensure trustworthiness, I used member checking during the interviews and data triangulation afterwards.

Results: Responding to Students

Many of the interview tasks posing hypothetical student scenarios were four-fold. Participants were asked to: (1) validate the student's reasoning; (2) determine what the student did and did not understand about the concept/task at hand, as well as reasons why this might be the case; (3) respond to the student in a way that would further her/his understand; (4) explain their reasoning for their responses to the student. While the first and second facets of the task elicit specialized content knowledge (SCK) and KCS, respectively, the third and fourth facets were meant to elicit KCT and insight into that KCT. While I anticipated to see both stronger and weaker instances of KCT, I was more intrigued by the emergent categories of responses to students. I present evidence for these partitions here to promote discussion about one category in particular during the RUME presentation.

General Pedagogical Knowledge

Most of the student scenario tasks required participants to address the mathematical underpinnings of the student claim or conjecture. I presumed that participants' responses to the student in each scenario would draw heavily from that. However, participants occasionally demonstrated general pedagogical knowledge in their responses. In one such task, hypothetical students Talisa and Tom each factored 540 in different ways, each proclaiming that their way was correct. When asked how they would respond to Talisa and Tom, *all* of the participants in part suggested that they would have the students resolve the conflict by explaining their reasoning to each other.

General pedagogical knowledge consists of "strategies of classroom management and organization that appear to transcend subject matter" (Shulman, 1987, p. 8). According to Morine-Dershimer & Kent (1999), there are three areas contributing to general PK: classroom organization and management, instructional models and strategies, and classroom communication and discourse. Participants' responses to these two tasks certainly qualify as classroom discourse that transcends subject matter, as having students explain their reasoning to their classmates can be an important pedagogical tool in many subject areas.

When asked why they responded to Talisa and Tom in this way, most of the participants explained that they learn better that way. For example, Cara said “by having them explain to each other, they'll have a better understanding of what they're doing, and they also get to see another method.” Isla said “instead of me being the boss of the classroom … have them work together. And it always seems more having it come from another classmate.”

Specific KCT

After identifying the general pedagogical responses to students, the rest of the responses were instances of KCT. However, some of these responses specifically pertained to the content at hand, while others appeared to be more canned, i.e., applicable to many student claims or conjectures across many math content areas. What I classified as “specific KCT” most closely resembled examples posed by Ball and colleagues. For example, when hypothetical student Shayna claims that 1 is a prime number, because its only factors are 1 and itself, Gwen’s response was: “I'd just try to tell her that prime factors have 2 different numbers, they don't have just the 1.” When asked why she responded in this way, Gwen replied that this is what Shayna didn’t understand, and that it was important for her to understand that prime numbers have exactly two factor to “help [her] conceptualize a little bit better.” Instances of specific KCT drew on the participants’ specialized content knowledge and KCS.

General KCT

In response to quite a few of the hypothetical student conjectures, participants usually responded with “I would give her/him a counterexample” or “I would guide her/him to recognize a counterexample”. For example, hypothetical student Mark claimed that the product of any two numbers was also their least common multiple, and all six participants suggested that they would respond to Mark with a counterexample strategy. When asked how she knew to respond to Mark in this way, Lucy said “it seems like students… if you just say ‘no, that's wrong’ they're obviously going to question why. And you want to be able to show them.” This appeared to be a canned response for what to do if a student has an incorrect conjecture. It is so general that it transcends mathematics content areas, much like pedagogical knowledge transcends subject matter areas. Thus, I categorized this type of response as “general KCT”, or general mathematics pedagogy.

Another less frequent example of general KCT was suggesting that students model the concept using manipulatives. For example, in addition to her suggestion that Lucy would use a counterexample with Mark, she suggested that Mark could explore the idea with manipulatives because “it’s often times best for them to see it for themselves… manipulative help with a variety of learners.” While this may be more specific to general mathematics pedagogy at the elementary or middle school level, it is still a rather canned response that does not address the specific mathematics of Mark’s conjecture.

Discussion

The structure of the student scenario tasks used to elicit KCT certainly limits the types of KCT responses that participants provided. However, there appears to be a clear partition between the instances of specific KCT versus general KCT. In some ways, general KCT had more in common with general pedagogical knowledge than with specific KCT. Participants’ beliefs on how students best learn was most influential on their instances of pedagogical knowledge and participants’ beliefs on how students best learn *math* were most influential on their instances of general KCT. In other words, participants’ epistemological perspectives greatly affected their pedagogical knowledge and general KCT. In contrast, participants' specialized content knowledge and KCS were the strongest influences on specific KCT. The existence of such a partition in KCT suggests that teacher educators may need different strategies in aiding the development these types of knowledge. At RUME, I plan to present on my observations about the different types of KCT, including evidence and influences, as well as pose the following questions to the audience:

Is it reasonable to suggest this partition in the construct of KCT, and how might I gather additional evidence in support for or against it?

How can I improve my conceptualization of General KCT to make it more useful, and useful for what?

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Campbell, P. F., Nishio, M., & Smith, T. M. et al. (2014). The relationship between teachers' mathematical content and pedagogical knowledge, teachers' perceptions, and student achievement. *Journal for Research in Mathematics Educaiton, 45*(4), 419-459.

Corbin, J., & Strauss, A. (2008). *Basics of qualitative research: Techniques and procedures for developing grounded theory*. Thousand Oaks, CA: Sage.

Hauk, S., Jackson, B & Noblet, K. (2010). No teacher left behind: Pedagogical content knowledge and mathematics teacher professional development. *Proceedings of the 13th Annual Conference on Research in Undergraduate Mathematics Education. Raleigh, NC.*

Hill (2010). The nature and predictors of elementary teachers' mathematical knowledge for teaching. *Journal for Research in Mathematics Education, 41*(5), 513-545.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.

Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *Elementary School Journal, 105*, 11-30.

Merriam, S. B. (1998). *Qualitative Research and Case Study Applications in Education*, Jossey-Bass, San Francisco, CA.

Morine-Dershimer, G. & Kent, T. (1999). The complex nature and sources of teachers' pedagogical knowledge. In *Examining Pedagogical Content Knowledge*. Science & Technology Education Library, 6, 21-50.

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15(2)*, 4-14.

Cobb, P. & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31*(3/4), 175-190.

Neural correlates for action-object theories

Anderson Norton
Virginia Tech

Research from an action-object perspective is well positioned to benefit from the emerging field of mathematics educational neuroscience. In this theoretical paper, we review some relevant findings from neuroscience studies and interpret them from an action-object perspective. This interpretation demonstrates a strong alignment of action-object theories and neuroscience findings, thus affirming many aspects of action-object perspectives on mathematical development. Neuroscience also serves to further elaborate and generalize reflective abstraction—the basis for action-object theories—as a mechanism for mathematical development. Specifically, we can begin to understand the changes in neural functioning associated with the objectification of action. This understanding helps explain some of the limitations teachers experience when attempting to provoke and support students' constructions of actions and objects.

Key Words: APOS Theory, Neuroscience, Reflective Abstraction, Reification

"Mathematics is the science of actions without objects, and for that, of objects we can define through action." Paul Valéry (1973, p. 811).

Action-object theories have played a prominent role in research on undergraduate mathematics education (e.g., Dubinsky & McDonald, 2001; Leron, Hazzan, & Zazkis, 1995; Sfard, 1992). We can trace the origins of APOS and reification, in particular, to a pursuit of a neo-Piagetian theory that would accommodate models of thinking and learning in advanced mathematics (Dubinski, 1991; Sfard, 1991). In general, action-object theories have sought to elaborate on reflective abstraction (Piaget, 1970) by explaining how mathematical objects arise through students' activity, at all stages of development (Tall, Thomas, Davis, Gray, & Simpson, 2000). At the same time, the fields of mathematics education and cognitive neuroscience have been brought closer through advancements in neuroimaging technology and the efforts of interdisciplinary researchers (Campbell, 2006; Fischer, 2009).

Mathematics educational neuroscience is a quickly emerging interdisciplinary field that promises mathematics educators new methods for testing and refining theories of learning. Although most studies focus on basic computation, the field includes far-reaching findings that address numerous domains of mathematics. For example, a recent study using functional magnetic resonance imaging (fMRI) with college mathematics students indicates that determining the equivalency of two algebraic equations recruits the same neural and cognitive resources as translating between algebraic equations and their graphs (Thomas, Wilson,Corballis, Lim, & Yoon, 2010).

The purpose of this theoretical paper is to examine neural correlates of mathematical activity from an action-object perspective in order to better understand the neural and cognitive mechanisms that might undergird the objectification of action. At that same time, we can consider how well action-object theories explain neural phenomena. Related questions include the following:

• How might mathematical actions and objects be associated with neural functioning?

- What can neuroscience tell us about the process of *interiorization*, *condensation*, and *reification* of action?
- Does neuroscience provide clues for how mathematics educators might support students' construction of new mathematical objects?

Action-Object Theories

Piaget's (1970) epistemology distinguishes logic and mathematics from all other forms of knowledge. The basis for this distinction is that the objects of mathematics are comprised of actions that, as objects, can be acted upon. Figure 1 illustrates the idea by indicating (1) a process by which actions become objectified (top arrow) and (2) further action on previously constructed objects (bottom arrow). Piaget referred to the process of objectification as *reflective abstraction,* as elaborated by action-object theories such as APOS theory and reification.

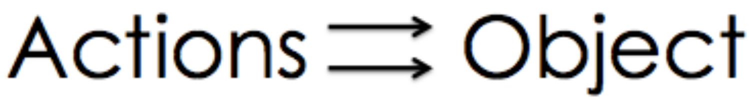


Figure 1. Actions and object.

APOS Theory

Dubinsky and colleagues (e.g., Dubinsky & Lewin, 1986) developed APOS theory as a means of applying Piaget's (1970) constructivist epistemology to research on undergraduate mathematics education. In particular, they demonstrate how mathematical actions may become reflectively abstracted as advanced mathematical objects. Their central tenet is that "mathematical knowledge consists in an individual's tendency to deal with perceived mathematical problem situations by constructing mental *actions*, *processes*, and *objects* and organizing them into *schemas* to make sense of the situations and solve the problems" (Dubinsky & McDonald, 2001, p. 2). In this framework, *actions* are defined as transformations of tangible objects (including diagrams and written symbols). Reflecting on such actions allows the individual to internalize them as mental *processes* that the individual can imagine performing, without the need for tangible objects. The process becomes an *object* for an individual when he or she can symbolize it and purposefully act upon it. "Finally, a *schema* for a particular mathematical concept is an individual's collection of actions, processes, objects, and other schemas which are linked by some general principles to form a framework in the individual's mind" (p. 3).

Reification

Sfard (1991; 1992) further elaborated on Piaget's (1970) notion of reflective abstraction by prescribing three stages through which students progress from engaging in mathematical processes to producing mathematical objects. To illustrate, Sfard provided an extended example from the historical development of number: from natural numbers, to positive rational numbers, to positive real numbers, to real numbers, and finally to complex numbers. She argued that each step-wise development has depended upon stages of *interiorization*, *condensation*, and *reification* (Sfard, 1992). In the production of rational numbers, processes involving the division of natural numbers become *interiorized* so that they "can be carried out in mental representation" (p. 18, from Piaget, 1970). Then they are *condensed* so that they can be combined with other processes, such as measurement. Finally, they are *reified*, or objectified, as a static structure on which to perform further processes, as in the development of positive real numbers.

Mathematics Educational Neuroscience

Neural activity associated with mathematics is generally localized to the frontal and parietal lobes within the neocortex—the outer region of the human brain. As illustrated in Figure 2, the frontal lobe (orange and red) covers the front half of the brain and meets the parietal lobe (blue and green) at the sensory-motor cortex (red and blue). The motor cortex (red) initiates all voluntary body movement; the somatosensory cortex (blue) receives nervous signals back from the rest of the body, regarding tactile or kinesthetic experience. Within the parietal lobe, there is an inferior lobule and a superior lobule, separated by the intraparietal sulcus (purple segment). Neuroscience findings regarding each of these areas have implications for mathematical learning and development. Here, we briefly review three of these findings: neural activity associated with observing actions on physical objects; the role of the intraparietal sulcus in mathematical activity; and frontal-parietal coherence associated with cognitively demanding tasks.

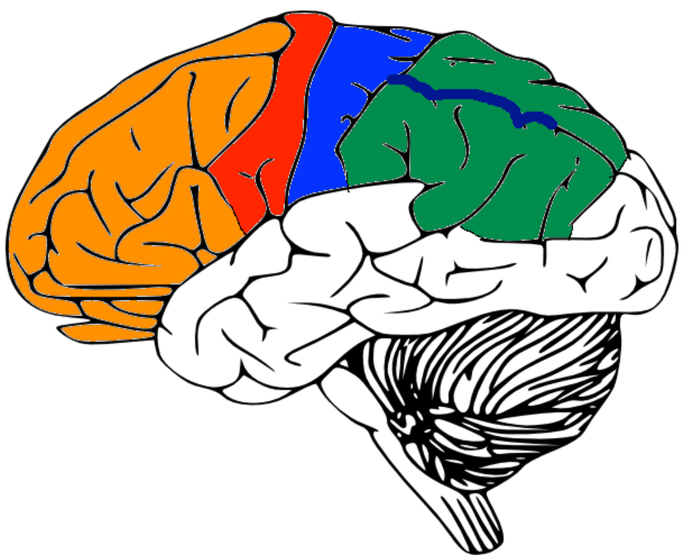


Figure 2. Frontal and parietal lobes.

Actions and Objects

One of the most relevant studies for action-object theory did not involve mathematical tasks at all (Buccino et al, 2001). Rather, the researchers asked 12 young adults to observe another adult performing physical actions with his foot, hand, or mouth. In some of the situations, the observed subject simply moved those body parts, and in others he acted on an object (e.g., kicking a ball). Meanwhile, the researchers conducted fMRI scans of the observer. They found that observation activated the specific area of the premotor cortex (the area of the parietal lobe directly in front of the motor cortex) corresponding to moving the respective body part, as if the observer were planning to move that same part of the body. Moreover, if the observed subject was acting on a physical object, areas of the observer's parietal lobe were activated as well—just behind the area of the somatosensory cortex corresponding to the moving body part. Figure 3 indicates the activated areas of the fontal and parietal lobes for the foot (blue), hand (orange), and mouth (red).

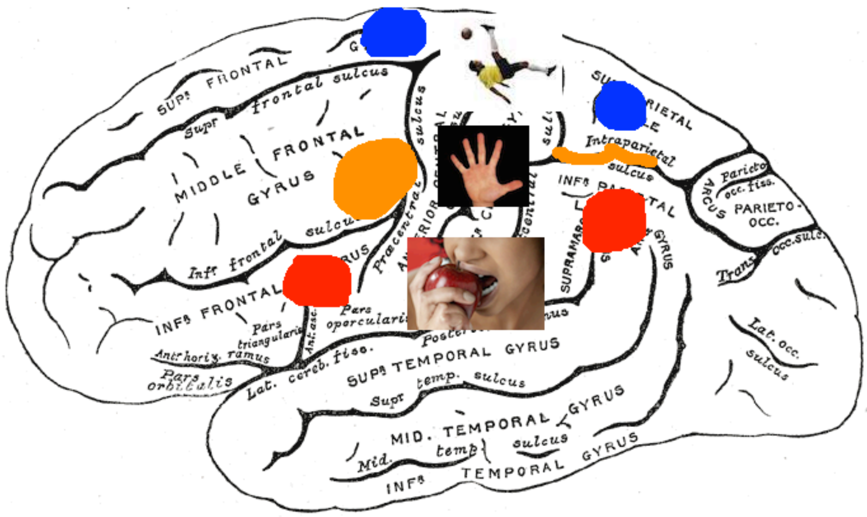


Figure 3. Actions and objects in the brain.

Digits

Note that acting on an object with the mouth (e.g., biting an apple) is associated with neural activity in the inferior parietal lobule, and acting on an object with the foot is associated with neural activity in the superior parietal lobule. The hand is associated with the valley that separates those two lobules—the intraparietal sulcus. This area of the brain has been implicated repeatedly in studies of counting, arithmetic computations, and size comparisons (e.g., Dehaene, 1997; Rosenberg-Lee, Lovett, & Anderson, 2009). The common association between acting on an object with one's hands and mathematical activity should come as no surprise: children learn to count with their fingers; historically, cultures have adopted a base-10 system because of the number of fingers humans typically have; and providing opportunities for students to manipulate objects with their hands is the pedagogical basis for many learning tools used in mathematics education (Raje, Krach, & Kaplan, 2013).

Frontal-Parietal Coherence

Neuroscience studies of participants performing mathematical activities consistently demonstrate the recruitment of additional cognitive resources when solving more challenging tasks (Ischebeck, Schocke, & Delazer, 2009; Thomas, Wilson, Corballis, Lim, & Yoon, 2010). For example, Ansari and Dhital (2006) found age-related differences between children and adults when asked to determine the cardinality of a collection of dots. Neural activity in the intraparietal sulcus was associated with cognitive activity among both groups, but the children's cognitive activity was associated with more neural activity in the frontal lobe, especially the anterior cingulate gyrus (associated with resolving conflict) and the prefrontal lobe (associated with working memory). These findings align with studies of frontal-parietal coherence, which indicate that resources in the frontal and parietal lobes work in concert when during tasks with higher cognitive demand, as those tasks require greater use of working memory and executive function (Sauseng, Klimesch, Schabus, & Doppelmayr, 2005).

Discussion

Action-object theories describe a theoretical process through which students construct mathematics. Such theories have been used to model mathematical development within multiple domains of undergraduate mathematics, including function (Sfard, 1992) and abstract algebra

(Asiala, Dubinsky, Mathews, Morics, & Oktac, 1997). Theoretical models need not correspond with any ontological reality—their only criterion being usefulness in explaining and predicting behavior (von Glasersfeld & Steffe, 1991). However, correspondence between action-object theories and models of neurological activity could provide clues about the process of reflective abstraction on which action-object theories seek to elaborate.

We have reviewed three relevant findings from neuroscience studies related to mathematics. The first of these findings implicates the role of the parietal lobe in imagined activity on physical objects (Buccino et al., 2001). "It is generally accepted that the fundamental role of the parietal lobe is to describe objects for action" (p. 404). With regard to action-object theories, Buccino and colleagues contribute further by suggesting that observed or imagined actions recruit the same areas of the parietal lobe as they would in physically performing the action themselves. This corresponds to the idea that the first stage of objectification—the interiorization of the action (Sfard, 1992) or the formation of a process from the action (Dubinski & McDonald, 2001)—relies upon students' experience in performing the action themselves, and not simply observing it. Otherwise, there will be no neurological referent for making sense of the observed action. Attempts to teach students a new action through demonstration or visualization will likely fail unless the students have already learned to perform the action themselves.

The second and third findings take the first finding further by suggesting that later stages of objectification (or reification) may release actions from associated imagined activity. Specifically, neuroscience studies have consistently demonstrated an association between mathematical activity (especially number and magnitude) and neural activity in the intraparietal sulcus. The intraparietal sulcus aligns with the area of the somatosensory cortex associated with the hand (see Figure 3); however, the area of the *premotor* cortex associated with the hand is *not* activated during experts' mathematical activity (Ansari & Dhital, 2006). Together, these findings indicate a frontal to parietal shift in neurological activity associated with objectifying objects.

Children learn to count using their fingers—the nearly ubiquitous manipulative on which most number systems were developed (Burton, 2007). At later stages of development, children no longer need to count and can use the results of counting for further activity (Steffe, 911). For example, they can consider the quantity formed by subtracting 8 from 13 using a variety of strategies that do not involve motor or premotor activity, but still involve activity in the intraparietal sulcus (Chochon, Cohen, Van De Moortele, & Dehaene, 1999). In other words, as students approach the stage at which they can treat numbers like 8 and 13 as objects to act upon, premotor (frontal) activity associated with the hand dissipates but the activity in the associated parietal area (the intraparietal sulcus) remains. Additional frontal resources are not needed until new cognitive demands arise.

If processes or interiorized actions correspond to activity in the premotor cortex and objects correspond to activity in the parietal lobe, reification (or objectification of action in general) corresponds with a functional or structural reorganization in the parietal lobe induced by coherent activity across the frontal and parietal lobes. Indeed, when children engage in cognitively demanding tasks in which they must act on existing objects (e.g., numbers) in new ways, we find increased frontal lobe activity (Ischebeck, Schocke, Delazer, 2009). This neural activity should dissipate as the new imagined activity is *condensed* (Sfard, 1992), until the object of activity is transformed or a new object is constructed. With this correspondence in mind, we now turn to its pedagogical implications.

Implications for Undergraduate Mathematics Education

Researchers approaching undergraduate mathematics education from an action-object perspective have sought to identify actions and objects within particular domains. Within abstract algebra, researchers have identified set and function as prerequisite objects for further activity, such as forming groups, isomorphisms, cosets, and quotient groups (Dubinsky, Dautermann, Leron, & Zazkis, 1994; Leron, Hazzan, & Zazkis, 1995). In a study with high school teachers, Dubinsky, Dautermann, Leron, and Zazkis (1994) noted that the teachers tended to begin by treating groups as sets on which to act and only later considered the role of a binary operator (function) in defining groups as objects. They further noted the need for a concept of isomorphism in order to construct "group as an equivalence class of isomorphic pairs [of sets and functions]" (Dubinsky et al., 1994, p. 290). The teachers were generally not successful in constructing quotient groups, which the researchers attribute to difficulty in objectifying the process of forming cosets—a prerequisite construction for treating cosets as elements of a group.

In light of neuroscience findings, we should expect an extended period of development following the construction of one mathematical object before an action applied to that object can itself become objectified. On the other hand, students can begin acting on an object as soon as it is objectified. These claims are included as findings in the study cited above, but neuroscience provides a basis for generalizing them to all of mathematics. Furthermore, if the new mathematical activity is not already available to the student, they need to physically engage in it until they can perform it in imagination. Only at that point can visual representations call the action to mind. This claim, too, aligns with previous research in the domain of abstract algebra (Asiala et al., 1997).

References

Ansari, D., & Dhital, B. (2006). Age-related changes in the activation of the intraparietal sulcus during nonsymbolic magnitude processing: An event-related functional magnetic resonance imaging study. *Journal of Cognitive Neuroscience, 18*(11), 1820-1828.

Asiala, M., Dubinsky, E., Matthews, D. M., Morics, S. & Okatc, A. (1997). Development of students' understanding of cosets, normality, and quotient groups. *The Journal of Mathematical Behavior, 16*(3), 241-309.

Buccino, G., Binkofski, F., Fink, G. R., Fadiga, L., Fogassi, L., Gallese, V., ... & Freund, H. J. (2001). Action observation activates premotor and parietal areas in a somatotopic manner: an fMRI study. *European journal of neuroscience*, *13*(2), 400-404.

Burton, D. M. (2007). *The history of mathematics: An introduction* (6th ed.). New York, NY: McGraw Hill.

Campbell, S. R. (2006). Defining mathematics educational neuroscience. In *Proceedings of the 28th Annual Meeting of the North American Chapter of the International Group for Psychology in Mathematics Education (PMENA)* (Vol. 2, pp. 442-449).

Chochon, F., Cohen, L., Van De Moortele, P. F., & Dehaene, S. (1999). Differential contributions of the left and right inferior parietal lobules to number processing. *Journal of Cognitive Neuroscience*, *11*(6), 617-630.

Dehaene, S. (1997). *The number sense: How the mind creates mathematics*. Oxford: Oxford University Press.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 95-123). Netherlands: Kluwer Academic Publishers.

Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On learning fundamental concepts of group theory. *Educational Studies in Mathematics*, *27*(3), 267-305.

Dubinsky, E., & Lewin, P. (1986). Reflective abstraction and mathematics education: The genetic decomposition of induction and compactness. *Journal of Mathematical Behavior, 5*, 55-92.

Dubinsky, E., & McDonald, M. A. (2001). APOS: Constructivist theory of learning in undergraduate mathematics education research. *The Teaching and Learning of Mathematics at University Level, 7*(3), 275-282.

Fischer, K. (2009). Mind, brain, and education: Building a scientific groundwork for learning and teaching. *International Mind, Brain, and Education Society*, *3*(1), 3-16.

Ischebeck, A., Schocke, M., & Delazer, M. (2009). The processing and representation of fractions within the brain: An fMRI investigation. *Neuroimage, 47*, 403-412.

Leron, U., Hazzan, O., & Zazkis, R. (1995*).* Learning group isomorphism: A crossroads of many concepts. *Educational Studies in Mathematics, 29, 153-174.*

Piaget, J. (1970). *Genetic epistemology* (E. Duckworth, Trans.). New York: Norton.

Raje, S., Krach, M., & Kaplan, G. (2013) Connection spatial reasoning ideas in mathematics and chemistry. *Mathematics Teacher, 107,* 220-224.

Rosenberg-Lee, M., Lovett, M. C., & Anderson, J. R. (2009). Neural correlates of arithmetic calculation strategies. *Cognitive, Affective, & Behavioral Neuroscience*, *9*(3), 270-285.

Sauseng, P., Klimesch, W., Schabus, M., & Doppelmayr, M. (2005). Fronto-parietal EEG coherence in theta and upper alpha reflect central executive functions of working memory. *International Journal of Psychophysiology, 57,* 97-103.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on process and objects on different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1-36.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification – the case of function. In E. Dubinsky & G. Harel (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy* (pp. 59-84). Washington, Mathematical Association of America.

Steffe, L. P. (1991). Operations that generate quantity. *Learning and Individual Differences*, *3*(1), 61-82.

Tall, D., Thomas, M., Davis, G., Gray, E., & Simpson, A. (2000). What is the object of an encapsulation of a process? *Journal of Mathematical Behavior, 18*(2), 223-241.

Thomas, M. O., Wilson, A. J., Corballis, M. C., Lim, V. K., & Yoon, C. (2010). Evidence from cognitive neuroscience for the role of graphical and algebraic representations in understanding function. *ZDM*, *42*(6), 607-619.

Valéry, P. (1974). *Cahiers, Vol. II*. Paris: Gallimard, 1974.

von Glasersfeld, E., & Steffe, L. P. (1991). Conceptual models in educational research and practice. *Journal of Educational Thought, 25*(2), 91-103.

Partial unpacking and the use of truth tables in an inquiry-based transition-to-proofs course

Jeffrey D. Pair
Middle Tennessee State University

Sarah K. Bleiler
Middle Tennessee State University

During our research into an inquiry-based-transition to proofs course we observed that several students used truth tables in unique ways. Typically the students translated mathematical statements into propositional logic formalism and used the truth tables as tools in their mathematical activity. Student activity varied from using truth tables to show that a statement was a tautology, to obtain conviction in the truth of the statement, to demonstrate equivalence, or to formulate a conjecture. We hypothesize that the unique use of truth tables emerged because it was the classroom community's responsibility to socially negotiate what counted as a proof. We intend to present some of our preliminary findings, and inquire as to what further research into these cases might be worthy of pursuit.

Key words: Transition to Proof; Truth Tables; Inquiry-Based Learning; Logic

Instruction in formal logic is frequently included in the curriculum of undergraduate transition-to-proof courses. However, in the field, there is no consensus as to whether such instruction is central to students' learning of proof (Epp, 2003). Some authors have begun to explore the role of formal logic in the teaching and learning of proof (Durand-Guerrier, Boero, Douek, Epp, Tanguay, 2012), in particular with respect to truth tables. For example, Epp (2003) discussed some limited roles of truth tables for the teaching of proof, but primarily emphasized that she required students to accompany the use of truth tables with explanations to support conceptual understanding and guard against the mechanistic use of formal logic. Hawthorne (2014) found that some of the undergraduate students he interviewed exhibited a compartmentalized focus on one or two lines of a truth table. He noted that those students who held a holistic understanding of truth tables could productively use a truth table as an organizational tool. Overall, few studies have examined how truth tables may or may not support student learning of proof. Further insight into how students use formal logic and truth tables to understand mathematical proof could aid the field in moving toward a comprehensive framework for proving (Durand-Guerrier et al., 2012).

Conceptual Framework

Building on the work of Selden and Selden (1995) and Harel and Sowder (2007), Brown (2013) introduced the construct of partial unpacking to describe a unique way students utilized propositional logic to make sense of indirect proof. She observed students employing a symbolic proof scheme (Harel and Sowder, 2007) as they translated mathematical statements into propositional logic statements and compared their truth values. The process of translating the statements into propositional logic was similar to the process of unpacking described by Selden and Selden (1995) in which an informal statement is translated into an equivalent statement using predicate logic. Brown (2013) described a partial unpacking as the process by which a student translates a mathematical statement into propositional logic symbolism (devoid of quantifiers) in order to understand the logical structure of the statement. Brown argued that the students' use of partial unpacking to determine statement equivalence aided in understanding indirect proof and was a possible benefit of the symbolic proof scheme. In our research, we have identified what we believe to be students' use of partial unpacking with the use of truth tables during proof construction.

Methodology

As part of an internal grant project at a large Southeastern University, we collected data from a transition-to-proofs course with the intent of answering the following question: What are the opportunities for learning about proof that occur in an inquiry-based learning (IBL) proofs course? Thirteen students from the course participated in the study. Nine were mathematics majors and four were mathematics minors. Data collected included video of each class session, student work, exit tickets, and an end of the semester assignment related to roles of proof (de Villiers, 1990). Students typically worked individually outside of class to complete their problem sets (modified from Taylor, 2007) and then worked collaboratively during class either critiquing the arguments of their classmates, or sharing arguments and creating a group proof. As the instructor rarely lectured, and the course problem sets rarely contained a model of a completed proof, students were required to socially negotiate what counted as proof. Many of the course activities supported students in this negotiation. Early in the course the students engaged in an activity which resulted in their creation of a class rubric which was used throughout the semester as a tool for both students and instructor to assess proofs (Bleiler, Ko, Boyle, & Yee, in press). Also the collaborative nature of the classroom allowed the students to justify their thinking and refine their arguments as they critiqued each other's work.

As part of their final exam grade, students in the course were expected to read de Villiers's (1990) paper describing five roles of proof (verification, explanation, systematization, discovery, communication) and write a reflection describing a time during the semester when they recalled engaging in each of these five roles. As we analyzed students' written reflections for the larger research project, we noted that five of the thirteen students wrote about experiences involving truth tables. Additionally, video and student-work data reveal that at least two additional students used truth tables to aid in their construction of proofs. This repeated focus on truth tables in student reflections was somewhat surprising because most of the reflections referred to instances when students used truth tables *outside* of the weeks of instruction dedicated to formal logic. In this regard, students seemed to transfer what they had learned about truth tables, and use truth tables as a tool within other mathematical contexts. In this preliminary report, we investigate the cases where students referred to a use of truth tables in this course. How did the students use the truth tables in the construction of proofs? Did the students consider that truth tables were proofs?

Preliminary Findings

Our data reveal that students in the IBL course considered truth tables an important tool for interpreting mathematical statements and in constructing mathematical proofs. It appears that all eight of these students engaged in what Brown (2013) referred to as a partial unpacking. We have identified three cases of the use of partial unpacking and truth tables that we would like to highlight and share with participants in this session. Our goal is twofold: (1) to foster a discussion about the potential ways that students may use formal logic and truth tables as a tool for meaningful learning of proof, and (2) to seek participant feedback on productive means of moving forward in this line of research.

The first case is a classroom episode in which students used a partial unpacking and truth table in their attempt to prove the transitivity of the subset relation. Although David and his partner Krissy had developed a narrative argument using the definition of subset, they were not satisfied with their argument and attempted to strengthen it by creating a truth table. The video captures David's excitement when he finds that the last column of his truth table consists entirely of "True" entries (demonstrating that his compound statement was a tautology). This use of the truth table was also meaningful for other students in the class as evidenced by Solomon's comments on his end of course assignment:

> The transitivity of implications was something I had a strong conviction on. Suppose A implies B, and B implies C. A must imply C. It almost seems impossible to be wrong. But I never remember having formally proved that statement to myself. I didn't even know that such a thing could be proven. But it was extremely interesting to see that the statement could simply be proved using a truth table. When [David] and [Krissy] showed that table on the [document camera] screen, I thought that was something I had never thought of and immediately applauded.

In their end of the semester assignments, two students recalled a case in which a truth table was used to formulate a conjecture. Susan used partial unpacking and a truth table in her original proof of the following statement: If x and y are odd integers, then so is xy. At the bottom of her solution, she wrote, "data also suggest biconditional equivalence". The video data reveal that after considering her classmate's arguments, Susan did not believe her original approach was correct or valuable. However, the teacher asked her group members to further consider her argument. The subsequent whole-class discussion resulted in the formulation of "Susan's Conjecture": xy is odd if and only if x and y are odd. The students were required to prove the conjecture or find a counterexample for homework. Solutions were discussed in the next class period.

Another student, Millie, frequently used truth tables to support her arguments that she turned in for homework. Initial examination reveals that the method of partial unpacking and construction of a truth table was used to aid Millie in her construction of proofs of theorems from set theory and number theory. These truth tables typically appeared at the end of her written argument, after she had produced a more standard conversational proof. In a reflection paragraph that accompanied Millie's revisions to a homework assignment, she mentioned the use of truth tables as a key strategy that she used throughout the semester when she needed help approaching a proof. She wrote that,

> Every so often, I would get stuck on a problem, and I would not know how to continue or finish the proof, but if I just broke everything down into their definitions, did an example, or created a truth table, I could figure out where I wanted the proof to go and how to get it there. ... It was nice to know that even with set equality, power sets, and functions, the same methods and structures for writing proofs that we learned in the beginning of the semester were still applicable.

Audience Engagement

We will share videos and student work with session participants to provide insight into the ways students used truth tables during the course. We will then ask the audience to discuss our preliminary findings, and provide suggestions for further work. We will pose the following questions for discussion:

1) How do these cases relate to your experiences with students' use of truth tables in undergraduate mathematics courses?
2) From a mathematician's perspective, is there a situation in which it would be appropriate to use truth tables in a mathematical proof?
3) When we teach this class again, we'd like to further investigate how the use of formal logic and truth tables influence students' learning of proof. What suggestions do you have for investigating this topic in greater depth?

4) Is there additional literature or frameworks that we should consider as we progress in this research?

Conclusion

In our preliminary analysis we have identified cases in which students used partial unpacking and the creation of truth tables during proof construction. Our data seem to suggest that formal logic provided a means of conviction in classroom situations where the mathematical authority had been shifted from the teacher to the students. An in-depth analysis of these cases may contribute to our understanding of the proving process and have important implications for proof instruction.

References

Bleiler, S. K., Ko, Y-Y., Boyle, J. D., & Yee, S. P., (in press). Communal development and evolution of a course rubric for proof writing. In C. Suurtamm (Ed.), *Annual perspectives in mathematics education: Assessment to enhance learning and teaching*. Reston, VA: National Council of Teachers of Mathematics.

Brown, S. (2013). Partial unpacking and indirect proofs: A study of students' productive use of the symbolic proof scheme. In S. Brown, G. Karakok, K. H. Roh, & M. Oehrtman (Eds.), *Proceedings of the 16th Annual Conference on Research in Undergraduate Mathematics Education*. Denver, CO.

de Villiers, M. D. (1990). The role and function of proof in mathematics. *Pythagoras, 24,* 17–24.

Durand-Guerrier, V., Boero, P., Douek, N., Epp, S. S., & Tanguay, D. (2012). Examining the role of logic in teaching proof. In G. Hanna & M. de Villiers (Eds.), *Proof and Proving in Mathematics Education* (pp. 369-389). Springer.

Epp, S. S. (2003). The role of logic in teaching proof. *The American Mathematical Monthly, 110,* 886-899.

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning: A Project of the National Council of Teachers of Mathematics* (pp. 805-842). Charlotte, NC: Information Age Publishing.

Hawthorne, C. (2014). Understanding students' conceptualizations of logical tools. In T. Fukawa-Connolly, G. Karakok, K. Keene, and M. Zandieh (Eds.), *Proceedings of the 17th Annual Conference on Research in Undergraduate Mathematics Education*. Denver, CO.

Selden, J., & Selden A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics, 29,* 123-151.

Taylor, R. (2007). Introduction to proof. *Journal of Inquiry-Based Learning in Mathematics, 4*, 1-37.

Students' reasoning when constructing quantitatively rich situations

Teo Paoletti
University of Georgia

Researchers continue to emphasize the importance of examining students' reasoning when constructing situations involving numerous quantities and relationships (e.g., quantitatively rich situations). To explore student reasoning in such situations, I conducted a semester-long teaching experiment with two mathematics education undergraduate students. The teaching experiment sessions were focused on providing the students repeated opportunities to conceptualize quantitatively rich situations. In this proposal, I explore a few themes that emerged through analyses of their activity, characterizing their thinking during their construction of such situations. For instance, the order in which a student coordinated two quantities (e.g., coordinating a change in quantity A then a corresponding change in quantity B versus coordinating a change in quantity B then a corresponding change in quantity A) emerged as critical to their images and their representations of the situation. This and other findings provide important insights into ways students' reason quantitatively and covariationally.

Key words: Quantitative reasoning; Covariational Reasoning; Cognitive Research; Teaching Experiment

From rate of change and linearity (Ellis, 2007; Johnson, 2012) to major calculus concepts (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002), researchers' examinations of students' *quantitative* (Thompson, 2011) and *covariational reasoning* (Carlson et al., 2002) have provided important insights into students' learning of various mathematical concepts. Additionally, researchers have indicated that students' quantitative and covariational reasoning play a major role when the students engage in problem solving and generalizing (Ellis, 2007; Moore & Carlson, 2012). Researchers have also suggested that students are capable of constructing meanings for various mathematical ideas via quantitative and covariational reasoning that provide a foundation for their developing more formal or abstract mathematical understandings (Ellis, Ozgur, Kulow, Williams, & Amidon, 2012; Johnson, 2012). These results have led to numerous researchers calling for increased attention to students' quantitative and covariational reasoning, including exploring students' activity as they engage in situations where they conceive of multiple quantities covarying (Castillo-Garsow, Johnson, & Moore, 2013; Thompson, 2011). Responding to this call, I conducted a semester-long teaching experiment (Steffe & Thompson, 2000) with two undergraduate students in which I engaged them in tasks designed to afford their engaging in *quantitatively rich situations* (Moore, Silverman, Paoletti, & LaForest, 2014; Thompson, 1993). I define quantitatively rich situations to be situations that afford students a natural opportunity to construct quantitative and covariational relationships between numerous quantities (Thompson, 1993). In this report, I describe a few themes that arose as the pair worked to model relationships between covarying quantities. The students' activity and understandings inform the research on quantitative and covariational reasoning by providing information about the complexity of students' reasoning when constructing (and continually re-constructing) quantitatively rich situations.

Theoretical Perspective

A *quantity* is a conceptual entity which an individual constructs as an attribute of an object or phenomena that permits a measurement process (Thompson, 2011). Thompson

(2011) asserted that *quantification* is the process of deciding what it means to measure an attribute of an object or phenomena (e.g. the quantity) and what this measurement means. It is often through an individual's continued attempts to determine how to measure a quantity that quantification occurs (Moore, 2013; Thompson, 2011). When an individual creates relationships between quantities and a network of these quantitative relationships, she creates a *quantitative structure*. An individual engages in *quantitative reasoning* when she analyzes the quantitative structures she constructed (Thompson, 1994). *Covariational reasoning* is defined as an individual's cognitive activities involved in imagining and coordinating two quantities as they *change or vary* in relation to each other (Carlson et al., 2002; Saldanha & Thompson, 1998). Researchers have argued that an individual's quantitative structures and her images of covariation are emergent; an individual initially constructs images of a situation that lack a well-developed quantitative structure. With respect to covariational reasoning, by imaging one quantity changing, then the other, then back to the first, and so on, an individual can create a refined quantitative structure which may include images that enable her to envision the two quantities changing in tandem (Saldanha & Thompson, 1998). Through this iterative process, an individual constructs more sophisticated quantitative and covariational relationships with respect to the quantities that these relationships entail (Carlson et al., 2002; Thompson, 1994).

For the purposes of this work, it is important to note quantitative and covariational reasoning can involve both numerical and non-numerical reasoning (Johnson, 2012). While an individual may use specific numbers when engaging in quantitative and covariational reasoning, these numbers are often unnecessary or understood as arbitrary (e.g., measures in any unit could be used). The essence of quantitative reasoning is non-numerical, having more to do with the comprehension of the quantities in a situation and how they relate to each other in ways that are not inherently tied to specified measures (Smith III & Thompson, 2008).

Methodology

To explore students' quantitative and covariational reasoning when constructing quantitatively rich situations, I conducted a semester-long teaching experiment (Steffe & Thompson, 2000) with two undergraduate students, Arya and Katlyn[1]. The students were enrolled in a secondary education mathematics program at a large state institution in the southern United States. Both were juniors (in credit hours taken) who had successfully completed a calculus sequence and at least two additional courses beyond calculus.

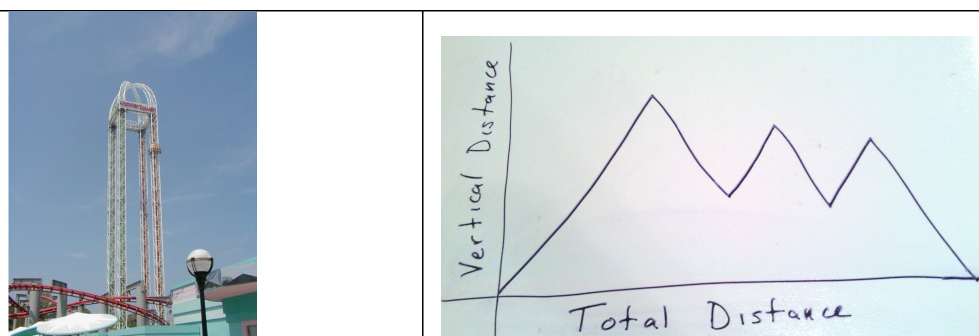
Providing a way to apply the radical constructivist epistemology in research (von Glasersfeld, 1995), I conducted a teaching experiment in order to explore students' mathematical activity and build models of the students' mathematics (Steffe & Thompson, 2000). Teaching experiments serve as an exploratory tool, giving a researcher 'firsthand' experiences with students' mathematics, allowing him to explore the mathematical progress students make over an extended period of time. The researcher's experiences gives him insights into the inherently unknowable bodies of understanding the students have, which is referred to as the *students' mathematics* (Steffe & Thompson, 2000). The models of the students' mathematics the researcher constructs using his interpretations and insights are then referred to as the *mathematics of students*. In order to analyze the data, I conducted a conceptual analysis to develop and refine models of the students' mathematics. As Thompson (2008) described, one purpose of a conceptual analysis is "building models of what students actually know at some specific time and what they comprehend in specific situations" (p. 60). With this goal in mind, I analyzed the recordings from the teaching experiment using an open (generative) and axial (convergent) coding approach (Clement, 2000; Strauss & Corbin,

[1] Gender preserving pseudonyms.

1998). Initially, I analyzed the videos identifying episodes of the students' behaviors and actions that provided insights into each student's understandings. Such instances aided in generating tentative models of the students' mathematics that I could test by searching for confirming or contradicting instances in their activity. When evidence contradicted my constructed models, new hypotheses were made to explain the novel way of operating as well as all prior observations. Using this technique, the data analyses consisted of an iterative process of continually creating, refining, and adjusting hypotheses.

Results

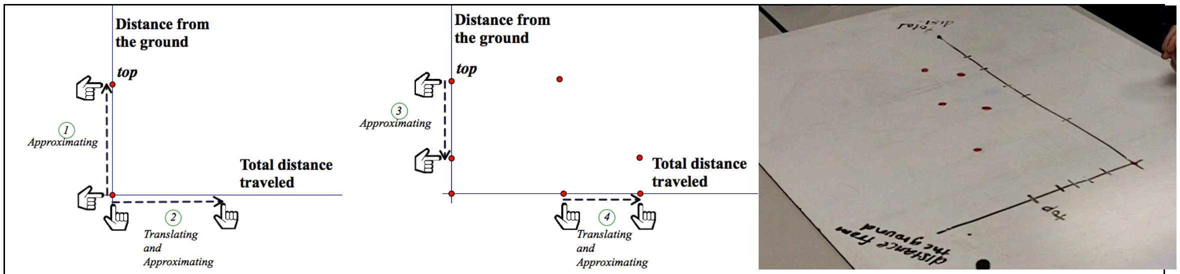
For brevity's sake, I provide data from one teaching session that exemplifies a few themes that emerged from my analyses. The session began with Arya and Katlyn watching a video of an amusement park ride, the Power Tower (Figure 1a), shooting riders up a vertical tower, allowing the riders to drop back towards the ground before shooting them back up, repeating this up-and-down process twice more (for more on this activity see Moore et al. (2014)). While the video played on a loop, we tasked the pair with graphing a rider's total distance traveled (heretofore referred to as distance) and the rider's distance from the ground (referred to as height), with height represented on the vertical axis (for a correct sketch see Figure 1b). The pair first asked if they were given a value for the maximum height of the tower, but decided it sufficed to label a tick mark on the vertical axis as 'top' to represent the maximum height of the rider. After labeling their axes, the pair worked by identifying landmark moments where the ride changed directions with Arya initially stating, "They go all the way to the top *(pointing to the tick mark labeled 'top' on the vertical axis)* and then that's where *(pointing to the location on the horizontal axis equally far from the origin as the point labeled 'top')*, how far they traveled first *(moving her finger horizontally from the origin to location on the horizontal axis she previously pointed to)*." Katlyn then used her fingers to measure the length from the origin to the tick mark labeled 'top' along the vertical axis and attempted to measure this same length from the origin along the horizontal axis (Figure 2a).



(a) (b)

Figure 1: (a) The Power Tower (b) a correct representation of a rider's total distance and vertical distance

Continuing, the pair marked a point on the vertical axis to represent the rider's height where the first drop ended. Katlyn measured the length along the vertical axis from 'top' to this new point and translated this length to the horizontal axis starting at the previous point they had marked to signify the end of the first length they measured (Figure 2b). They continued this process of estimating minimum and maximum heights of the rider by placing points on the vertical axis and measuring the length between consecutive points on this axis. The pair would then coordinate this change in height to an equivalent change in distance by translating the length measured as the change in height to the horizontal axis starting at the last point on this axis. The pair understood that a change in height corresponded to an equal change in distance (in magnitude). Their resultant graph can be seen in in Figure 2c.



(a) (b) (c)

Figure 2: Arya and Katlyn's (a)-(b) activity coordinating changes in distance from the ground to changes in total distance travelled and (c) the resultant graph.

Having points labeled on their graph, with these points stemming from their coordinating the rider's height then distance, the pair sought to determine how to connect the points. As they did so, they re-watched the video and focused on the physical motion of the rider, using words like "fast", "slower", and "shoots up." They then drew a concave down curve from the origin to the first point corresponding to the rider's maximum height. Arya stated, "Okay, so if we think of this as our total distance *(pointing to the label 'total distance' on the horizontal axis)*…We're covering distance pretty quickly here *(pointing to the beginning of the curve)* or we're covering height here, so this is at a pretty fast rate *(motioning her finger from the origin over the beginning of their curve)*. And then as we get close to the top, right here near the top *(motioning over the part of the curve near the first peak)* we start to slow."

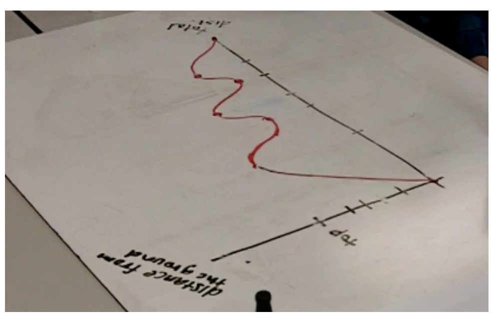


Figure 3: The pairs completed graph

I infer from this statement, and from their activity moving forward, that the pair used the rider's speed to create a graph composed of smooth curves (Figure 3). In fact, although Arya initially referred to the quantity on the horizontal axis as "total distance," neither student explicitly referred to the quantity represented on the horizontal axis during the next three minutes of creating and justifying their graph. Instead, they (knowingly or unknowingly) used time as the quantity on the horizontal axis, and thus (tacitly) coordinated changes in time with the rider's height. This hypothesis is supported by the fact that their graph (Figure 3) accurately represents the relationship between time and the rider's height.

After the pair constructed the smooth graph, a teacher-researcher asked them to re-explain how they plotted their initial points in order to determine if the pair realized the inconsistency he perceived in their two explanations. When posed with this, the pair returned to reasoning that focused on coordinating height and distance over completed intervals of the journey (e.g., a trip up, a trip down, etc.), mimicking their activity represented in Figure 2a-b. Again both Arya and Katlyn first coordinated changes in height then coordinated how that change corresponded to an equal change in distance, representing distance on the horizontal axis. While from the researchers' perspective the pair continued to conflate distance and time, the pair did not notice this as they returned to working with completed intervals of height and distance (e.g., when coordinating completed intervals of height and distance they were not concerned with how height and distance were covarying *within* the interval).

In the above interaction, the pair again accurately described the points by coordinating changes in height first, but did not see any problem with the curvature of their graph. Because of this, the teacher-researcher decided to refocus the students in a way that might lead them to notice their conflation of quantities with respect to their graph. He asked Arya and Katlyn, "So talk to me about this first trip, you said things are speeding up slowing down, can you break it up in terms of amounts of change?" Arya stated "For equal changes in total distance *(Katlyn taps along the horizontal axis indicating equal changes)*. The change in distance from the ground *(Arya points to the 'distance from the ground' label on the vertical axis)* is decreasing *(Katlyn taps locations along the curve)*." Katlyn provided a similar explanation, with both of their explanations accurately describing their *created graph*. However, from the researcher's perspective they provided an incorrect statement about the physical relationship between distances as portrayed in the video.

Shortly after this, the teacher-researcher asked the pair to explain how they were interpreting speed. Arya identified speed as "rate of change" to which the interviewer asked, "Like what's the quantities between?" Katlyn responded that speed is a coordination of "distance over time." Immediately after this, Arya began to question their graph.

Arya: Do we have to do distance over time? Like this graph *(pointing to their graph)* doesn't have to include time necessarily so.
Katlyn: It's like one [complete ride]. Like it's not, it doesn't really, does it really matter?
Arya: But it kind of does because the way you changed the rate that the distance changes is over, distance changes by time.
Katlyn: But I mean even if you said this was like t equals zero *(pointing to origin)* and t equals *(pointing to where the graph intersects the horizontal axis again)*, like if it takes it thirty seconds for them to reach, I don't know. Does it really matter when we're talking about total distance?
Arya: Does time matter?
Katlyn: Yeah.
Arya: Yea but that's how we figured this out *(pointing to the graph)*. What I was thinking is like how else could you do *(motioning along the first curve of the graph)*, is there another way?
Katlyn: To get less steep is that what you mean?
Arya: Yeah, like or *(pause)* could you, is there a different relationship?

From the above interaction I infer that, until this point, both Katlyn and Arya did not distinguish between time and distance when coordinating changes of the quantity represented along the horizontal axis (e.g., they used the horizontal axis to represent time when drawing the smooth curve) (Lines 6-9, 12). During this interaction, Arya realized that their graph did not (directly) represent time (Lines 1-2) and began to question whether they accurately represented the relationship between height and distance (Line 16). Shortly after this interaction when coordinating changes along the horizontal axis the pair explicitly worked with distance, coordinating that, in intervals during which the ride did not change direction, for any change in distance there was an equal change in height (in magnitude). This maintained the relationship they had constructed when first coordinating changes in height.

Conclusions

This study adds to this body of literature on quantitative and covariational reasoning by providing an example of the ways in which students construct quantitatively rich situations. The pair exhibited productive quantitative and covariational reasoning throughout the episode; they constructed numerous quantities and coordinated these quantities as they covaried. Further, the pair's reasoning was non-numerical, instead focusing on measurable magnitudes, which is the essence of quantitative reasoning (Smith III & Thompson, 2008).

Corroborating previous research that has emphasized an increased focus on students' images of problem situations (Moore & Carlson, 2012), the students' images of the situation and their coordination of the quantities were reflexive throughout the episode. Initially, the students found points using their images of the situation and later they described the situation based on their graphical representation. Specifically, when initially plotting points the pair considered height and changes in height. They likened the vertical axis to the tower and used this image to find locations on this axis to represent heights where the rider changed direction. They used these points to indicate changes of directional variation of height as well as to consider specified changes in height and the corresponding changes in distance. Using this method, they found points on the graph representing the rider's height and distance at each maximum and minimum height of the rider through one complete ride.

When Arya and Kaleigh moved to connecting these points a complexity arose as they reasoned about the graph and its relationship to the situation. In this instance they coordinated height with respect to (tacit) time by imagining the situation in terms of the rider's speed. This led them to draw smooth curves between their points. Further, when describing the relationship between height and distance using their created graph, they made statements that accurately described the relationship between these quantities as defined by their created graph. That is, when using the graph to describe 'distance' and 'height', their reasoning focused on the graph in ways that did not attend to an image of the physical situation.

A related explanation for their activity rests with the students' images of covariation, specifically their imagining chunky and smooth images of change (Castillo-Garsow et al., 2013; Thompson, 2011). Their initial reasoning was chunky with their activity focused on coordinating completed intervals of change (e.g., changes between maximum and minimum heights). It was when they attempted to unpack these chunks that they changed from coordinating two distances to coordinating a distance with respect to (tacit) time. This suggests that initially neither their graph, nor their images of the situation, entailed smooth images of change with respect to the posed quantities of distance and height.

These finding have important implications for the use of quantitatively rich situations in the teaching and learning of mathematics. First, it is important that we take students' conceptions of both relationships and graphs seriously, meaning that we consider their graphs as viable from *their perspective* regardless of our intentions. In this study the pair consistently described and created quantitatively correct relationships and graphs, though they were not always the relationships the researcher intended. Further, the pair fluctuated between different conceptions of the situation (and relationships) while working with the same coordinate axes and graph. While other researchers have emphasized the importance of students' mental images of a situation when problem solving (Moore & Carlson, 2012; Thompson, 2011), they have not reported on such conflations that can arise when students use multiple images that rely on different quantities when representing a situation.

Future Research

While providing novel insights into the quantitative and covariational reasoning literature, these findings raise questions as well. Quantitatively rich situations give students the opportunity to grapple with multiple quantities and relationships. In order to construct a robust understanding of these situations, students must construct and consider all of the quantities involved. Therefore, it is not surprising when students conflate these quantities. In this case the pair's incorporation of time resulted in them drawing smooth curves. One possible explanation of this conflation is related to the pairs' prior school experiences. Other researchers have noted that common graphing conventions can influence students' meanings for graphs (Moore et al., 2013). If Arya and Katlyn consistently experienced graphing situations in which time was represented along the horizontal axis, it is possible their meanings for graphs tacitly include time on the horizontal axis (e.g., when creating or interpreting a graph, assume the horizontal axis represents time). This study suggests continuing to explore how students' reasoning differs in situations that do and do not include time could be a productive line of inquiry.

References

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. . (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 5*(33), 26.

Castillo-Garsow, C. W., Johnson, Heather L., & Moore, K. (2013). Chunky and Smooth Images of Change. *For the Learning of Mathematics*.

Clement, J. . (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 547-590). Hillsdale, N.J. : Lawrence Erlbaum.

Ellis, A. B. (2007). The Influence of Reasoning with Emergent Quantities on Students' Generalizations. *Cognition & Instruction, 25*(4), 439-478.

Ellis, A. B., Ozgur, Z., Kulow, T., Williams, C., & Amidon, J. (2012). Quantifying exponential growth: The case of the jactus. In R. Mayes & L. L. Hatfield (Eds.), *Quantitative Reasoning and Mathematical Modeling: A Driver for STEM Integrated Education and Teaching in Context* (pp. 93-112). Laramie: University of Wyoming.

Johnson, Heather L. (2012). Reasoning about variation in the intensity of change in covarying quantities involved in rate of change. *Journal of Mathematical Behavior, 31*(3), 313-330.

Moore, K. (2013). Making sense by measuring arcs: a teaching experiment in angle measure. *Educational Studies in Mathematics, 83*(2), 225-245.

Moore, K., & Carlson, M. (2012). Students' images of problem contexts when solving applied problems. *Journal of Mathematical Behavior, 31*(1), 11.

Moore, K., Liss, D., Silverman, J., Paoletti, T., LaForest, K., & Musgrave, S. (2013). The primacy of mathematical conventions in student meanings. In M. V. Martinez & A. C. Superfine (Eds.), *Proceedings of the 35th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 837-840). Chicago, IL: University of Illinois at Chicago.

Moore, K., Silverman, J., Paoletti, T., & LaForest, K. (2014). Breaking Conventions to Support Quantitative Reasoning. *Mathematics Teacher Educator, 2*(2), 141-162.

Saldanha, L., & Thompson, P. W. (1998). *Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation.* Paper presented at the Proceedings of the Annual Meeting of the Psychology of Mathematics Education - North America, Raleigh, NC.

Smith III, J., & Thompson, P. W. (2008). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the Early Grades* (pp. 95-132). New York, NY: Lawrence Erlbaum Associates.

Steffe, L., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267-307). Hillsday, NJ: Erlbaum.

Strauss, Anselm, & Corbin, Juliet. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory (2nd ed.)*. Thousand Oaks, CA US: Sage Publications, Inc.

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics, 25*(3), 43.

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 179-234). Albany, NYY: SUNY Press.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. Chamberlin, L. L. Hatfield & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOM^e*. Laramie, WY: University of Wyoming.

von Glasersfeld, Ernst. (1995). *Radical Constructivism: A Way of Knowing and Learning. Studies in Mathematics Education Series: 6*.

Pre-Service Teachers' Inverse Function Meanings

Teo Paoletti
University of Georgia

Irma E. Stevens
University of Georgia

Natalie L. F. Hobson
University of Georgia

Kevin C. Moore
University of Georgia

Kevin R. LaForest
University of Georgia

Researchers have indicated that pre-service teachers (along with in-service teachers and college students) typically do not develop productive meanings for function and function inverse. In order to explore pre-service teachers' inverse function meanings further, we conducted clinical interviews with 25 pre-service teachers. In this paper, we include a summary of previous research concerning individuals' inverse function meanings as well as a description of the methodology and theoretical framework we used when making sense of the pre-service teachers' activities. We present and interpret data highlighting the techniques the pre-service teachers used during the tasks. We present our interpretation of these results in relation to common curricular approaches to inverse function. We conclude with implications from our findings and areas for future research.

Keywords: Function; Inverse Function; Pre-service Secondary Teachers; Meanings

Inverse function is important in secondary and post-secondary mathematics, particularly in the study of Calculus, Analysis, and Differential Equations. Inverse function also falls under the function content strand in the Common Core State Standards of Mathematics (National Governors Association Center for Best Practices, 2010), thus setting the expectation that secondary mathematics teachers support their students in constructing productive function inverse meanings. It follows from such an expectation that pre-service teachers should construct and operationalize inverse function meanings that are productive for their future teaching experiences. By supporting their students in constructing productive understanding of inverse function, teachers can prepare their students to develop many ideas in post-secondary mathematics including families of functions (e.g., logarithmic and inverse trigonometric functions) and calculus topics (e.g., implicit differentiation).

Although several researchers have investigated students' function meanings (e.g., Breidenback, Dubinsky, Hawks, and Nichols (1992), Leinhardt, Zaslavsky, and Stein (1990), Oehrtman, Carlson, and Thompson (2008), Thompson (1994)), fewer researchers have explicitly focused on students' inverse function meanings. Researchers who have investigated students' inverse function meanings have argued that college students, pre-service teachers, and in-service teachers have difficulty constructing productive inverse function meanings (Brown & Reynolds, 2007; Carlson & Oehrtman, 2005; Engelke, Oehrtman, & Carlson, 2005; Kimani & Masingila, 2006; Lucus, 2005; Vidakovic, 1996). Our goal is to build on the current body of research by gaining insights into pre-service teachers' (heretofore referred to as students) meanings for inverse function based on their activities during clinical interviews. First, we provide relevant background research and our theoretical perspective. Then, we describe our methods, including an explanation of our task design and its relation to our research goals. We present selected results describing students' activities and techniques when finding, using, or describing inverse functions. We describe the implications that stem from our findings including implications related to common curricular approaches to inverse function. We conclude with limitations of the study and areas for future research.

Background

Vidakovic (1996) presented a genetic decomposition of inverse function (i.e., a description of how students might learn the concept, including possible construction methods for their schemas). Vidakovic's (1996) genetic decomposition closely resembles the formal mathematical definition of inverse function as a property of a set under the operation of function composition; she described a hierarchy that involved students developing schemas in the following order: function, composition of functions, and then inverse function. Students exhibiting the highest level of her hierarchy would be able to coordinate the three aforementioned schemas and to work with inverse functions through this coordination.

Whether implicitly or explicitly, many researchers (Brown & Reynolds, 2007; Kimani & Masingila, 2006; Lucus, 2005; Vidakovic, 1997) who have investigated students' inverse function meanings have maintained Vidakovic's (1996, 1997) focus on function composition as critical to students' development of productive inverse function meanings. However, in practice, it remains to be seen if students develop meanings for inverse function that entail reasoning about the composition of functions. For example, in an attempt to use her proposed genetic decomposition in an instructional sequence, Vidakovic (1997) noted some students "were able to define (in terms of switching x and y) and find (by switching x and y, or by trial and error) the inverse function without understanding the concept of the composition of two functions" (p. 191). We take these results and the results of others (Brown & Reynolds, 2007; Kimani & Masingila, 2006) to indicate that although students were able to determine an analytic representation of the inverse function, they did not use the idea of composition of functions to do this, nor did they relate their products to the notion of function composition. As part of our examination of students' inverse function meanings, we were interested to discover if the students drew on meanings related to the composition of functions. Further, given these previous research findings, we also were interested in characterizing the students' meanings in the event that they did not draw on meanings related to function composition.

Most researchers examining students' inverse function meanings have investigated these meanings without reference to contextualized applications of inverse functions. As such, there has been little research examining how students make sense of the inverse of a contextualized function. Further, those who have used contextualized functions have reported on the added complexities of such situations (Wilson, Adamson, Cox, & O'Bryan, 2011). For instance, when examining an undergraduate mathematic major's conception of function, Philips (2015) noted that when the undergraduate, Britney, was determining the inverse of a given function that related the number of people who enter a theme park (input) to the income made if each person pays \$7 (output), she claimed the inverse function did not exist. She stated, "Common sense-wise it's not invertible. Logically it's not invertible." We take Britney's activity to indicate she understood that a function that inputted income and outputted the number of people who entered the park was not viable as this was not how the situation worked (i.e., income was based on the number of people and not vice versa). Britney's case illustrates one potential added complexity that can arise in student thinking when interpreting the inverse of a contextualized function.

More generally, researchers have shown that students and teachers often hold compartmentalized meanings for inverse function dependent on the function class and/or representation (Brown & Reynolds, 2007; Carlson & Oehrtman, 2005; Engelke et al., 2005; Kimani & Masingila, 2006; Lucus, 2005; Vidakovic, 1996). These same researchers have argued that students' meanings are restricted to carrying out actions particular to analytic or graphical situations. For example, Engelke et al. (2005) reported 652 pre-calculus students' responses to questions on a research based assessment (Carlson, Oehrtman, & Engelke, 2010) that the authors perceived as relating to inverse function. They reported that when given a graph of a function f, 35% of the students were able to find x such that $f(x) = 3$. On the three questions that included inverse function notation, no more than 20% of students in total responded correctly to any given problem and only 1% of the students answered all three of these tasks correctly. We infer from these results that few students in their study maintained inverse function meanings that supported them in correctly addressing all of the tasks.

Theoretical Perspective

Incorporating the tenets of radical constructivism, we approach knowledge as actively built up by an individual in ways idiosyncratic to that individual (von Glasersfeld, 1995). Thus, students' knowledge is fundamentally unknowable to us as researchers; we can only make inferences about their meanings based on our interpretations of their words and actions. When making such inferences, which Steffe and Thompson (2000) referred to as the *mathematics of students*, we rely on definitions that Thompson, Carlson, Byerley, and Hatfield (2014) attributed to Thompson and Harel (in preparation). Specifically, Thompson and Harel defined *understanding* to be a *cognitive state* of equilibrium that results from assimilation. Whereas understanding refers to a cognitive state*, meaning* refers to the scheme(s) associated with an understanding including the scheme's space of implications (Thompson et al., 2014). The students involved in our study likely had multiple instructional experiences prior to the study in which they established meanings for function and inverse function. Our interest was to gain insights into these established meanings and the space of implications associated with their meanings (e.g., did they draw on various schemes that were a connected and coherent part of what the students conceived of as 'inverse function' or draw on meanings that appeared disconnected from our (and possibly their) perspective).

In this paper, we use the term *technique* to describe a student's activity as he or she addressed a single task. We did not make inferences about a student's inverse function meanings based solely on his or her technique for finding an inverse function in one task or representational system. Instead, and as we describe below, we analyzed students' activities both within certain representations or types of tasks as well as across the different representations and tasks types. By designing tasks (see *Task Design*) requesting students to determine, use, and interpret inverse functions in a variety of representations and settings, we conjectured we would be able to build models of the students' inverse function meanings that viably explained the techniques they used when engaging in inverse function tasks. For instance, and consistent with Thompson and Harel's (in preparation) description of knowing, we hoped to characterize the extent that students used techniques in various tasks, representational systems, or settings while holding in mind some meaning that enabled them to see these techniques as connected and invariant. We also hoped to characterize when students used different techniques in isolated and disconnected ways too see if students' meanings for inverse function consisted of techniques isolated to the students' associations with representational systems or contexts.

Methods

In this section, we first describe the subjects and settings of this study. Then, we present our task design, including an explanation of how our task design reflected our goals in relation to our theoretical perspective. Then, we describe our data analysis efforts.

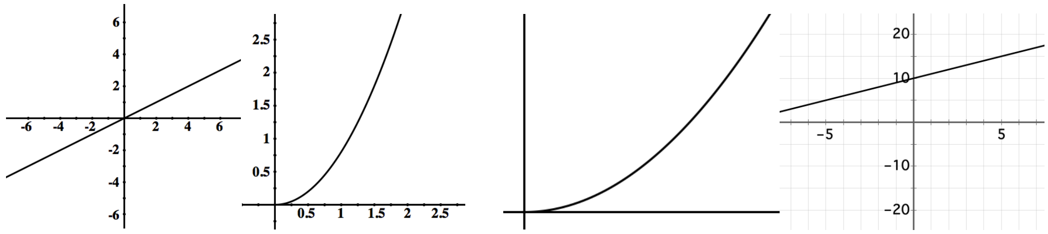
Subjects and Setting

In order to explore undergraduate students' inverse function meanings, we conducted semi-structured clinical interviews (Clement, 2000) with 25 students (18 female, 7 male) over the course of three semesters. The students were enrolled in their first content course as part of the secondary mathematics teacher education program at a university in the southeast United States. Each student had completed a full calculus sequence and two additional mathematics courses (e.g., linear algebra, differential equations, etc.) with a minimum grade of a C in each course prior to enrolling in the aforementioned content course. We conducted the interviews, each lasting between sixty and ninety minutes, before the students started their first pair of courses (content and pedagogy) in their teacher education preparation program.

The interviews were semi-structured task-based clinical interviews (Clement, 2000; Goldin, 2000; Hunting, 1997) consisting of a series of questions that examined their function and inverse function meanings. These interviews consisted of an individual subject working on preplanned tasks while interacting with a researcher (Goldin, 2000). Clinical interviews give a researcher insight into students' meanings without intending to create shifts in their meanings. We note that although we did not intend to promote shifts in students' thinking, unintended shifts may have occurred as tasks perturbed the students' meanings.

Task Design

The interview tasks included decontextualized and contextualized tasks in analytic (equation rule) and graphical representations. These variations enabled us to explore the techniques and meanings students drew on in each of these situations as well as to analyze the students' activities across representations. As an example, we asked students to graph the inverse functions of (decontextualized) graphed functions. These tasks, (recreated in Figure 1 for readability) involved both linear (a and d) and non-linear (b and c) curves as well as graphs with square (a and b), rectangular (d), and unlabeled (c) axes. By using linear and non-linear curves, we intended to examine if students would use different techniques depending on the curve. By providing square, rectangular, and unlabeled axes, we intended to delineate between students who engaged in activity regardless of the axes labeling versus those who engaged in activity attentive to the quantity labels and axes scaling.



(a) (b) (c) (d)

Figure 1: Four Decontextualized Graphical Tasks

Other tasks consisted of decontextualized functions represented analytically. These tasks included identifying if two functions represented analytically are inverses of each other, evaluating $f^{-1}(6)$ after using a given rule for f to determine $f(2) = 6$, and determining x when given $f(x) = 1$ and an analytic rule for $f^{-1}(x)$. By asking the students to determine if two functions were inverses of each other as well as to work with functions and their inverses for specific values, we intended to explore if (and if so the extent to which) the students' inverse function meanings were tied to carrying out particular techniques. For instance, consider the task, "Suppose that $f(x)$ is a one-to-one function whose inverse is $f^{-1}(x) = (x + 1)^3 - 5x^2 + 2$. Find a value of x so that $f(x) = 1$." We designed this task so that determining a rule for $f(x)$ by switching the variables and solving would be unmanageable. As such, we conjectured this task would help us delineate between students with different inverse function meanings. A student who understands inverse functions in relation to function composition might understand that $f^{-1}(f(x)) = x$ and leverage this to find x by evaluating $f^{-1}(1)$. In contrast, a student whose inverse function meanings are restricted to switching the variables and solving might be unable to find the rule for f, leaving them unable to successfully obtain a value for x.

In addition to exploring students' activities in decontextualized tasks, we posed tasks that asked students to determine the inverse of a given graph and analytic rule that defined the relationship between degrees Fahrenheit and degrees Celsius. In this case, we were interested in examining if the students used a technique similar to one used to determine decontextualized inverse functions. Further, we were interested in exploring if (and, if so, how) the students made sense of a contextual meaning for their constructed inverse function.

Finally, we asked students to graph the inverse sine function given a graph of the sine function. The reason we chose the sine function is that there is not an explicit, finite sequence of calculations that relate the input and output values of trigonometric functions. In the absence of an explicit rule, a strategy like switching and solving would not help students graph the function (i.e., $y = \sin(x)$ becomes $x = \sin(y)$ or $y = \sin^{-1}(x)$) without a calculator to provide specific output values for given input values. In all, Table 1 provides a count of how many contextualized, decontextualized, and trigonometric function tasks in which we asked the students to work with inverse functions in both graphical and analytical representations.

Table 1: Inverse Function Task Types

	Graphical	Analytical
Decontextualized	4	3
Contextualized	1	1
Trigonometric	1	0

Lastly, we strategically did not ask the students to define "inverse function" at the onset of the interview because we were concerned that they might rely on their constructed definition throughout all of their activity. Instead, we intended to observe the techniques students used in the moment of making sense of tasks involving (from our perspective) inverse functions. We also intended to gain insights into the extent that the students' techniques were suggestive of a connected system of meanings for inverse functions. For instance, we were interested to determine if the students spontaneously identified some meaning for inverse function that connected her or his activity both within and across representations, contexts, and function types.

Analysis

Each interview was video-taped and transcribed with all of the students' written work digitized. To analyze the data, we used open and axial techniques (Clement, 2000; Strauss & Corbin, 1998) as well as conceptual analyses (Thompson, 2008). Each member of the team analyzed a set of students individually by noting each student's techniques for working with inverse functions in various representations. Afterwards, we met to discuss our observations, looking for common techniques among the students on specific tasks or types of tasks. As patterns developed, we created codes to identify the techniques we observed. Codes were revised or created in order to capture the similar or different techniques we observed both within similar tasks and across varying tasks. When a researcher was unsure how to code a particular instance, we watched this instance collectively and reached an agreement regarding the coding of that student's technique, possibly leading to the refinement of a code or creation of a new code. Through this iterative process, we developed a final set of codes to represent students' techniques, which we describe in this paper.

In addition to creating codes to describe the students' techniques on individual tasks, we looked across tasks to explore if the students exhibited, from our perspective, consistent techniques across representations, contexts, and/or function classes. For instance, if a student defined an inverse function by using the reciprocal in all analytically defined functions, then we classified that student as having a consistent technique when working with inverse functions of analytically defined functions. However, if that same student reflected over the x-axis when asked to graph the inverse of a linear function represented graphically, but then reflected over the line $y = x$ when producing the inverse graph of a non-linear function represented graphically, we classified the student as not having a consistent technique for constructing graphical inverse functions. By describing the students' activities in these various settings and comparing and contrasting their activity in this way, we hoped to gain insights into students' meanings for inverse functions.

Results

In this section, we describe the results from analyzing the students' activities in decontextualized tasks, contextualized tasks, and the trigonometric task. We begin each section with examples of individual student work that exemplify specific techniques. We conclude each section with a summary across all students with respect to the task type.

Results from the Decontextualized Analytical and Graphical Tasks

Caroline is an example of a student who exhibited a consistent technique across the decontextualized graphical tasks (see Figure 2). When addressing these tasks, Caroline relied on switching coordinate values (i.e., the point (a, b) in the original curve became the point (b, a)). When she could not identify coordinate values (bottom-left graph), Caroline reflected the curve over the line $y = x$, and expressed that this reflection produced the same outcome as switching coordinate values. Caroline was one of 11 students who consistently exhibited the technique of switching coordinate values (or reflecting over the line they perceived as $y = x$) for graphing the inverse function of a given graphed function. In total, 19 of the 25 students (see Table 2) exhibited consistent techniques when asked to graph the inverse function of a decontextualized function represented graphically.

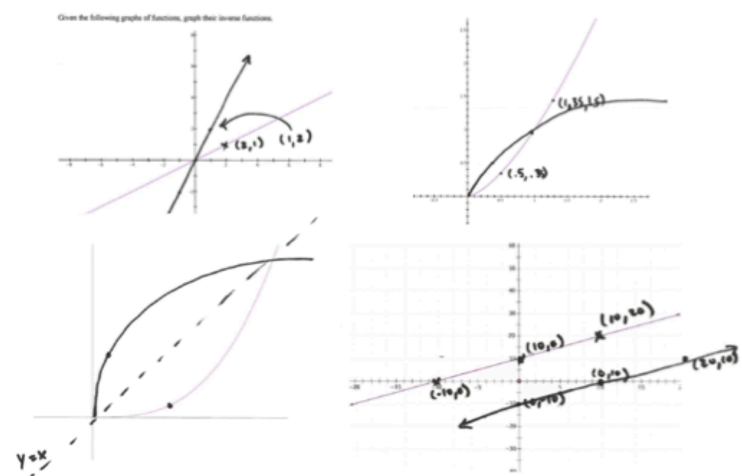
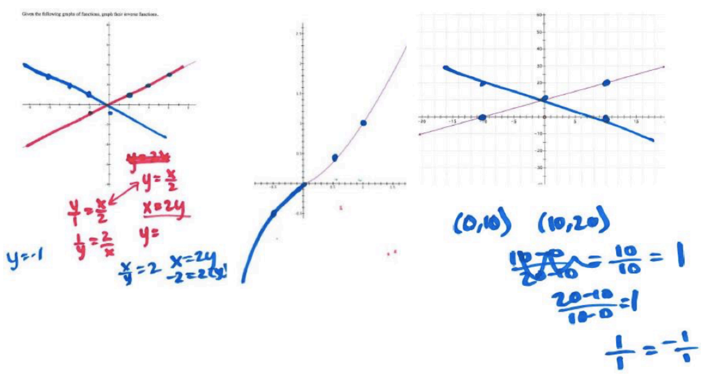


Figure 2: Caroline's solutions to the four decontextualized graphical tasks

In contrast to Caroline, Alyssa is an example of a student who did not use what we perceived to be a consistent technique across the decontextualized graphical tasks. Alyssa first (Figure 3a) attempted to find the slope of the line representing the original function and calculate "one over" this slope to determine what she anticipated to be the slope of the line representing the inverse function. After having difficulty obtaining a graph by executing this technique, she negated the x-values of the initial curve (i.e., the point (a, b) became the point (-a, b) in Figure 3a). Then, in Figure 3b (and for the curve on the graph with unlabeled axes), Alyssa negated both values of a point on the curve (i.e., the point (a, b) in the original became the point ($-a$, $-b$)). Yet, for Figure 3c, she calculated the slope of the line representing the original function, and then she negated the slope to determine the slope of the inverse function while maintaining the same y-intercept. Due to the variety of her techniques, we characterized Alyssa as a student who did not exhibit consistent techniques across decontextualized graphical tasks.[1] In total, six students used inconsistent techniques across decontextualized graphical tasks (see Table 2).



(a) (b) (c)

Figure 3: Alyssa's solutions to the decontextualized graphing tasks

All but one student (24 of 25) exhibited consistent activity across the decontextualized analytical inverse task types. This one student used a function composition to check if two functions represented analytically were inverses of each other. However, the student did not have a technique for determining the analytic representation of an inverse function, thus leaving her unable to solve the tasks that requested she determine an inverse function or interpret inverse function notation in some way.

Table 2. Categorizations of Students' Meanings in Decontextualized Tasks (where # denotes the number of students out of 25 who consistently exhibited the corresponding activity)

Graphical Technique	#	Analytical Technique	#
Switched x and y values or reflected over the line $y = x$	11	Switched x and y (with or without solving for the switched y variable)	15*
Reflected a graph over an axis or a line other than $y = x$	3	Used the reciprocal (e.g., $f^{-1}(x) = 1/f(x)$)	8
Determined an analytic function then solved for x or switched x and y and solved y	3	Solved for x in the original function	1
Created the inverse graph by transforming each point (x, y) to $(x, 1/y)$	1		
Negated slope, and/or reflected concavity	1		

[1] It could be argued that her techniques were consistent in that each involved 'switching' something or performing a transformation of some sort, but Alyssa showed no awareness during her activity that each technique might have some form of underlying invariance.

* Three of the 15 students consistently used a composition of functions method to verify if one function was the inverse of another function and consistently used a 'switch-and-solve' method in the other decontextualized analytical tasks (i.e., determining the analytic representation of an inverse function or computing values).

Table 2 provides a summary of the techniques used by the students who exhibited a consistent technique across the decontextualized graphical or analytical tasks. We highlight that a majority of the students used a technique for finding inverse functions that involved "switching" x and y, a common technique reported in previous research (Kimani & Masingila, 2006; Vidakovic, 1997). In total, eight of the 25 students consistently used a "switching" technique in both the decontextualized graphical and the analytical tasks. An additional 10 students used a "switching" technique in one representation (7 students in analytical, 3 students in graphical) but not in the other representation.

We found the students' responses to the decontextualized analytical task "Suppose that $f(x)$ is a one-to-one function whose inverse is $f^{-1}(x) = (x + 1)^3 - 5x^2 + 2$. Find a value of x so that $f(x) = 1$" especially interesting. Only six students successfully determined the value of x to be 5. Two of these six students, both of whom used function composition to check if two functions were inverses of each other, immediately recognized they could input one for x in the inverse rule to determine x in $f(x)$. The other four students changed notation (i.e., wrote y for $f^{-1}(x)$), switched the variables x and y obtaining $x = (y + 1)^3 - 5y^2 + 2$), and eventually realized they could substitute one for y in their new rule to determine x.

Results from the Contextualized Inverse Function Tasks

To explore the students' activities in contextualized situations, we provided the students with two tasks, one graphical (a graph with degrees Fahrenheit on the vertical axis and degrees Celsius on the horizontal axis) and one analytical (the rule $C(F) = (9/5)F + 32)$). We asked the students to determine the inverse of each function. Recall that our goals were to determine whether a student used the same technique in the contextualized and decontextualized tasks, and to determine if the student made sense of a contextualized meaning for the inverse function.

We present Kate's activity as one example of a student's activity on these tasks (see Figure 4 for Kate's work). Kate was one of the eight students who used a switching technique in both graphical and analytical *decontextualized* task types. She continued to use these techniques on both Celsius-Fahrenheit tasks. However, Kate was perturbed as to how to interpret her constructed inverse functions. In the graphical task, after describing that the point (10, 50) on the given line meant that 10 degrees Celsius corresponded to 50 degrees Fahrenheit, the interviewer asked Kate to explain what the point (50, 10) on her line representing the inverse function represented in context. Kate responded, "That's a good question. If I start at, if I have 50 degrees Celsius, then I have 10 degrees Fahrenheit. That doesn't make sense." After a long pause, Kate pointed to the line representing her inverse function (in red in Figure 4) and said, "I can't wrap my mind around what that means."

Similarly, Kate was unsure how to interpret the analytically represented inverse function she determined ($(5/9)F + 32 = C^{-1}(F)$ in red in Figure 4). As she attempted to interpret the rule, she conjectured if the given function had an input of degrees Fahrenheit, then the inverse function would have an input of degrees Celsius. However, she quickly noted, "I'm thinking you can [find degrees Fahrenheit from degrees Celsius] if you don't take the inverse," indicating she understood that she did not need to switch the variables or construct a different analytic rule to determine Fahrenheit values if given Celsius values. This left Kate unsure of the contextual meaning of her determined inverse function. Hence, although Kate exhibited a consistent technique for determining inverses in both graphical and analytical tasks, she remained perturbed when trying to make sense of the contextualized graph and formula of the inverse functions she obtained using this technique.

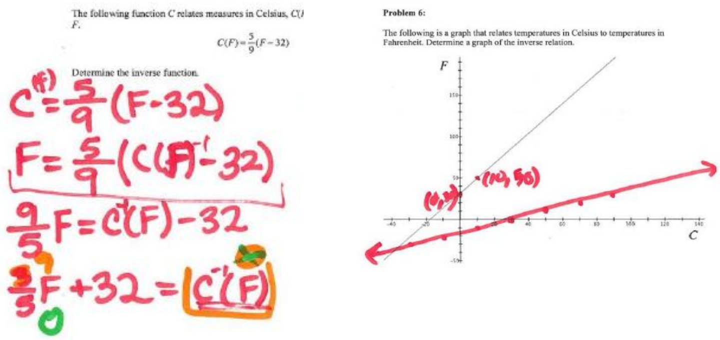


Figure 4: Kate’s solutions to the Celsius/Fahrenheit tasks

Figure 5 represents the number of students who (1) maintained a consistent technique across decontextualized (graphical or analytical) tasks, (2) maintained this same technique in the contextualized task, and (3) interpreted their inverse function as continuing to represent the *given* relationship between degrees Celsius and degrees Fahrenheit. Including Kate, 17 of the 19 students who exhibited a consistent technique in decontextualized graphical tasks used this same technique when given a contextualized graph. In order to make sense of their created graph so that it represented what they considered to be a correct relationship, 10 of these 17 students concluded that the quantity represented on each axes switched (i.e., the Fahrenheit axis became the Celsius axis). Six (including Kate) of the remaining seven students maintained the quantity represented by each axis and had difficulty interpreting their created graph.[2] For example, one of these six students claimed, “I don’t know what the point of having the inverse function of a graph like this… Because you’re not gonna, it’s not gonna give you any useful information, I don’t feel like.” Collectively, these six students were unable to relate their constructed inverse function to the context.

We interpret the results presented in Figure 5 to imply that students experienced more difficulty when attempting to make sense of their constructed analytically defined inverse function. In this case, 21 of the 24 students used the same technique they had used on the decontextualized analytic tasks. Only five of these students made sense of their inverse function in terms of the context, and the other 16 students remained unsure what their inverse function was meant to represent with respect to the context or concluded that their inverse function was not meant to represent the same relationship as the original function. This is not surprising, however, considering 15 of these 21 students continued to use a switching technique that, once applied, no longer maintained the relationship between the temperature measures *unless* the student simultaneously coordinated different variable referents depending on the analytic representation under consideration (i.e., in the original function F = degrees Fahrenheit and C = degrees Celsius and in the inverse function F = degrees Celsius and C = degrees Fahrenheit).

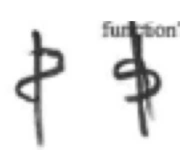
[2] The other student turned her focus to determining and working with analytic functions and did not address context in this task.

Figure 5: Students' graphical (*left*) and analytical (*right*) consistency across decontextualized and contextualized tasks

Results from the Trigonometric Inverse Function Task

We provide one example of a student's activity that is germane to many students' activities when constructing a graph of the inverse sine function given a graph of the sine function; Caroline recalled memorized shapes. Prior to this task, Caroline consistently switched x and y values or reflected over the line $y = x$ in decontextualized graphical tasks (see Figure 2). When addressing this task, she attempted to recall a graph of the inverse sine function that included 'U-shapes', claiming, "[I] never could memorize whether, when it was up and when it was down," and drawing Figure 6a (without identified asymptotes). In order to determine the location of one 'U-shape' in the plane, she recalled and plotted two points she associated with sine inverse ((0,0) and (π/2,1)) and created a 'U- shape'. Caroline continued to refine her graph by adding asymptotes at locations where she identified that $\sin^{-1}(x)$ "does not exist". Caroline then suddenly recalled and drew two other learned shapes (shown in Figure 6b), although she was still unsure which shape was in the correct orientation. After drawing the curves shown in Figure 6(a)-(b), Caroline said, "I don't even remember sine inverses… I have like these two graphs [*pointing to graphs shown in Figure 6(a)-(b)*] that I remember seeing and associating with them inverses and sine. And honestly, I think I used to graph it in my calculator and then justify it to myself, and then I'd be good with it for like however long I needed it." Hence, when asked to find the inverse of a trigonometric function, Caroline abandoned her previously consistent technique in decontextualized graphical situations and instead shifted her focus to recalling memorized shapes. Caroline could not recall which shape corresponded to the inverse sine function and was left perturbed as to how to graph the inverse sine function.

(a)



(b)

Figure 6: Caroline's work on the inverse sine task

Of the 19 students who we classified as exhibiting consistent techniques in decontextualized graphical tasks, only six used the same technique when graphing the inverse sine function. Eight of the other 13 students (including Caroline) attempted to recall a learned graph (i.e., shape) and were unsuccessful (from the researchers' perspective) in creating a graph that accurately represented the inverse sine function. The other five of the 13 students either made no attempt to graph the inverse sine function (one student), graphed $1/\sin(x)$ (one student), or reflected over a single axis (three students).

Conclusions and Discussion

In this section we summarize our results and discuss how these results relate to other researchers' findings. Then, we describe how we interpret this data in relation to the students' inverse function meanings more generally and relate our results to our experiences teaching inverse function at the secondary and postsecondary levels.

Students' techniques when working with inverses functions

With respect to the students' activities when responding to the decontextualized tasks, our findings are consistent with those of previous researchers (Brown & Reynolds, 2007; Engelke et al., 2005; Lucus, 2005; Vidakovic, 1997); many of our students used techniques to determine or work with inverse functions that were reliant on a particular representation or function class. Further, compatible with Vidakovic's (1997) observations, a majority of the students (21 of the 25) did not use composition of functions when determining if two analytically defined functions were inverses of each other. Although the formal mathematical definition of inverse function depends on the composition of functions, this definition was of little use (to the students) when *constructing* inverse functions from given functions.

In this study, we contribute findings related to students' activities and reasoning about the contextual meaning of an inverse function for a given contextualized function (i.e., the function that relates degrees Celsius and degrees Fahrenheit). Only five of the 25 students interpreted their constructed inverse function in both the analytical and graphical representations as maintaining the relationship between degrees Fahrenheit and degrees Celsius. An additional five students interpreted the *graphically* represented inverse function as maintaining this relationship by switching the quantity represented on each axis, but these same five students did not conceive of their analytically defined inverse function, obtained by switching-and-solving, as representing the given relationship between degrees Fahrenheit and degrees Celsius. These students' techniques for constructing an analytic inverse *required* switching variables, but the students did not conceive such switching to include changing the quantitative referents of the variables, leaving them unsure how to interpret their analytically defined inverse function. These results provide evidence supporting Wilson et al. (2011), who argued that the switching-and-solving procedure commonly taught in school mathematics can convolute the meaning of inverse function in contextualized situations. Although students in our study had techniques for determining an inverse function in a contextualized situation, a majority of the students struggled when interpreting their inverse functions in context.

This study also adds to the body of literature on students' understandings of trigonometric functions (Akkoc, 2008; Hertel & Cullen, 2011; Moore & LaForest, 2014; Weber, 2005). Our results indicate that although a majority of students exhibited consistent techniques when graphing decontextualized inverse functions, most of these students did not draw on these techniques when working with decontextualized trigonometric functions. Instead, many students reverted to recalling learned shapes for these functions. One explanation for this is that trigonometric functions require more sophisticated ways of thinking about function than a majority of the function families students encounter in K-12 mathematics (Moore, 2014; Weber, 2005). For instance, the analytic representation of trigonometric functions does not entail an explicit sequence of calculations or operations. Instead, understanding trigonometric functions is contingent on conceiving them as quantitative relationships. However, researchers (e.g., Akkoc (2008); Hertel and Cullen

(2011)) have shown that students' understandings do not support them in thinking of trigonometric functions as a relationship between quantities. Another, possibly related, explanation is that students' meanings for inverse functions consist of a collection of techniques to execute, and thus, the students understood trigonometric functions as having their own set of rules (e.g., memorizing shape-analytic rule pairs).

Students' Inverse Function Meanings

Our results indicate a majority of the students in our study left their K-14 school experiences having constructed stable inverse function meanings, or what Harel and Thompson might call ways of thinking (Thompson et al., 2014). We infer that the students' meanings, or ways of thinking (and the implications of which), were predominantly constrained to carrying out techniques tied to the various representations, contexts, or function classes. In essence, carrying out the techniques in order to obtain a new equation or graph was nearly the entirety of many students' meanings for inverse function in that representation or context. If the students encountered difficulty carrying out their technique and obtaining a new graph or equation, the students were often unable to determine an alternative technique or relevant way of thinking about inverse function. One way to frame this outcome is that a majority of the students' inverse function meanings were the result of pseudo-empirical abstractions (von Glasersfeld, 1995), or generalizations tied to the product of activity and the particulars of that activity.

This idea of students maintaining meanings based in pseudo-empirical abstractions explains why many students exhibited techniques that were inconsistent from the researchers' perspective (and often from the students' perspective). For example, only nine students exhibited techniques when working with decontextualized graphical and analytical tasks that would result in compatible inverse functions. As another example, five students interpreted their constructed inverse graph of the relationship between degrees Celsius and degrees Fahrenheit as maintaining the relationship between temperatures, but did not do so when interpreting their constructed inverse function of the analytical representation. During the study, these students exhibited two inverse function meanings; one meaning involved switching-and-solving to determine the inverse rule of an analytically defined function and the other meaning involved an inverse function maintaining the relationship between quantities represented by the original function. It was not until the students addressed the Celsius/Fahrenheit problems that some students became aware of these two meanings and perceived them as being inconsistent with one another, often leaving the students in a state of perturbation as to what their inverse function represented. We note only one student addressed all of the inverse function tasks in a way that we took to indicate she maintained inverse function meanings that supported her in addressing all of the tasks (e.g., she was consistent across all tasks and also made sense of the contextualized tasks).

Considering results in relationship to curricular approaches to inverse function

As we reflected on our experiences teaching inverse function at the secondary and postsecondary level using curricular materials which cover inverse function, we began to consider these results as far from surprising. Pre-calculus textbooks (e.g., Dugopolski (2009); Larson, Hostetler, and Edwards (2001); Stewart, Redlin, and Watson (2012); Swokowski and Cole (2012)) often have one section dedicated to inverse function. In the texts listed in the previous sentence, each contains an inverse function section that includes a definition for one-to-one functions and introduces the horizontal line test. Each text states the property that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$ either as a definition, theorem, or property of inverse functions. Each text includes multiple examples with techniques students can use to find inverse functions, including special boxes describing the switch-and-solve method for determining an analytic inverse rule and a graph showing a function and its inverse as a reflection over the line $y = x$. But, these texts do not always provide explicit connections between these different techniques for finding inverses. Hence, it is not surprising students develop techniques for finding inverse functions that lack connections across representations.

Furthermore, only one of these texts (Dugopolski (2009)) references a contextualized example in their inverse function narrative, yet each expects students to interpret the meaning of inverse functions in contextualized situations in the problems they pose at the end of the section. Also, we note that it is common to omit inverse trigonometric functions in the initial treatment of inverse functions, instead including these functions in a unit specifically addressing trigonometric functions. Additionally, the treatment of inverse trigonometric functions does not include the same techniques found earlier in the text. It should not surprise us if a student exhibits activity to indicate he or she understands inverse trigonometric functions as being distinct from other inverse functions in the even that a student experiences such a curriculum.

As we consider these curricular approaches to teaching inverse function in relation to the students' activities in this study, we are not surprised that the students often performed techniques without having an underlying meaning that connected these techniques. Similar to the complexities students encounter as they attempt to make sense of the function concept described by Carlson (1998), students are often expected to develop powerful meanings without being given sufficient time or experiences to construct such meanings. In the case of inverse functions, we as educators expect students to learn how to determine if a function has an inverse. We expect students to make sense of inverse function notation (which is strikingly similar to the notation used for exponents). We also expect students to be able to use the definition of inverse function to determine if two functions are inverses of each other via function composition, and to construct inverse functions in a variety of representations. We expect students' to understand that these techniques for constructing inverse functions differ in contextualized and decontextualized analytically defined functions. Further, we expect students to recall all of these ideas when learning about inverse trigonometric (and logarithmic) functions well after the initial treatment of inverse function. We contend that if we expect students to develop powerful inverse function meanings in which they conceive of connections between these techniques and activities in the various representations, we must give them ample opportunities and experiences in which these connections and understanding are meaningful.

Limitations and Future Research

Although our study builds on and supports previous research on students' inverse function meanings, it also raises many questions. A reliance on a procedure creates a predicament for some students attempting to make sense of inverse functions in context. Hence, and following up on the suggestions of Wilson et al. (2011), future researchers may be interested to explore ways to support students in constructing meanings for function and inverse function that are productive in both contextualized and decontextualized situations. Additionally, a productive line of inquiry might be to pursue the use of contextualized situations to support students in constructing productive function and inverse function meanings.

Our data only came from one interview. Thus, our inferences about the students' meanings were limited to that one interaction. Although we were surprised by the students' activities on the inverse sine task, this was the only task in which we asked the students to work an inverse trigonometric function. Because of this, it is difficult for us to conclude why most of the students did not use the same technique used in previous in graphical situations. From our perspective, it would seem that if the students maintained meanings that were consistent in graphing and/or analytic situations, then these meanings would carry into their activity in trigonometric functions. Future researchers may be interested to explore how students' inverse function meanings differ between function classes.

References

Akkoc, H. (2008). Pre-service mathematics teachers' concept image of radian. *International Journal of Mathematical Education in Science and Technology, 39*(7), 857-878.

Breidenback, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the Process Conception of Function. *Educational Studies in Mathematics, 23*(3), 147-185.

Brown, C. A., & Reynolds, B. (2007). Delineating Four Conceptions of Function: A Case of Composition and Inverse. *Conference Papers -- Psychology of Mathematics & Education of North America*, 1-193.

Carlson, M. P. (1998). A cross-sectional investigation of the development of the function concept. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Research in Collegiate Mathematics Education* (Vol. 1, pp. 115-162). Providence, RI: American Mathematical Society.

Carlson, M. P., & Oehrtman, M. C. (2005). Key Aspects of Knowing and Learning the Concept of Function. from Mathematics Assn. of America

Carlson, M. P., Oehrtman, M. C., & Engelke, N. (2010). The precalculus concept assessment (PCA) Instrument: A tool for assessing reasoning patterns, understandings, and knowledge of precalculus level students. *Cognition and Instruction, 28*(2), 113-145.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 547-590). Hillsdale, N.J.: Lawrence Erlbaum.

Dugopolski, M. (2009). *Fundamentals of Precalculus*. Boston, MA: Pearson Education, Inc.

Engelke, N., Oehrtman, M. C., & Carlson, M. P. (2005). Composition of Functions: Precalculus Students' Understandings. *Conference Papers -- Psychology of Mathematics & Education of North America*, 1-8.

Goldin, G. (2000). A Scientific Perspective on Structured, Task-Based Interviews in Mathematics Education Research. In A. E. Kelly & R. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 517-546). Hillsdale, N.J.: Lawrence Erlbaum.

Hertel, J. T., & Cullen, C. (2011). Teaching trigonometry: A directed length approach. In L. R. Wiest & T. Lamberg (Eds.), *Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1400-1407). Reno, NV: University of Nevada, Reno.

Hunting, R. P. (1997). Clinical Interview Methods in Mathematics Education Research and Practice. *Journal of Mathematical Behavior, 16*(2), 20.

Kimani, P. M., & Masingila, J. O. (2006). Calculus Students' Perceptions of the Relationship among the Concepts of Function Transformation, Function Composition, and Function Inverse. *Conference Papers -- Psychology of Mathematics & Education of North America*, 1.

Larson, R., Hostetler, R. P., & Edwards, B. H. (2001). *Precalculus with Limits: A Graphing Approach* (3 ed.). Boston, MA: Houghton Mifflin Company.

Leinhardt, G., Zaslavsky, O., & Stein, M. (1990). Functions, Graphs, and Graphing: Tasks, Learning, and Teaching. *Review of Educational Research, 60*(1), 1-64.

Lucus, C. A. (2005). *Composition of Functions and Inverse Function of a Function: Main ideas as perceived by teachers and preservice teachers.* (Doctor of Philosophy Dissertation), Simon Fraser University, BC, Canada.

Moore, K. C. (2014). Quantitative Reasoning and the Sine Function: The Case of Zac. *Journal for Research in Mathematics Education, 45*(1), 37.

Moore, K. C., & LaForest, K. L. (2014). The circle approach to trigonometry. *Mathematics Teacher, 107*(8), 616-623.

National Governors Association Center for Best Practices, C. o. C. S. S. O. (2010). Common Core State Standards Mathematics (Vol. 2012): National Governors Association Center for Best Practices, Council of Chief State School Officers, Washington D.C.

Oehrtman, M. C., Carlson, M. P., & Thompson, P. W. (2008). Foundational Reasoning abilities that promote coherence in students' function understandings. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics* (pp. 27-42). Washington, D.C.: Mathematical Association of America.

Philips, N. (2015). *Domain, Co-domain and causation: A study of Britney's conception of function.* Paper presented at the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education, Pittsburgh, PA.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267-307). Hillsday, NJ: Erlbaum.

Stewart, J., Redlin, L., & Watson, S. (2012). *Precalculus: Mathematics for Calculus, Sixth Edition* (6th ed.). Belmont, CA: Brooks/Cole Cegage Learning.

Strauss, A. L., & Corbin, J. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory (2nd ed.)*. Thousand Oaks, CA US: Sage Publications, Inc.

Swokowski, E. W., & Cole, J. A. (2012). *Precalculus: Functions and Graphs* (12 ed.): Cengage Learning.

Thompson, P. W. (1994). Students, functions, and the undergraduate mathematics curriculum. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Research in Collegiate Mathematics Education, I* (Vol. 4, pp. 21-44). Providence, RI: American Mathematical Society.

Thompson, P. W. (2008). *Conceptual analysis of mathematical ideas: Some spadework at the foundation of mathematics education.* Paper presented at the Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education, Morelia, Mexico.

Thompson, P. W., Carlson, M. C., Byerley, C., & Hatfield, N. (2014). *Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra.* Paper presented at the Epistemic algebraic students: Emerging models of students' algebraic knowing, Athens, GA.

Thompson, P. W., & Harel, G. (in preparation). Standards of understanding.

Vidakovic, D. (1996). Learning the Concept of Inverse Function. *Journal of Computers in Mathematics and Science Teaching, 15*(3), 295-318.

Vidakovic, D. (1997). Learning the concept of inverse function in a group versus individual environment. In E. Dubinsky, D. Mathews, & B. E. Reynolds (Eds.), *Readings in cooperative learning for undergraduate mathematics.* (Vol. 44, pp. 175-196). Washington, DC US: The Mathematical Association of America.

von Glasersfeld, E. (1995). *Radical Constructivism: A Way of Knowing and Learning. Studies in Mathematics Education Series: 6*.

Weber, K. (2005). Students' understanding of trigonometric functions. *Mathematics Education Research Journal, 17*(3), 21.

Wilson, F. C., Adamson, S., Cox, T., & O'Bryan, A. (2011). Inverse Functions: What our Teachers Didn't Tell Us. *Mathematics Teacher, 104*(7), 7.

Unifying Concepts in the Introductory Linear Algebra Course

Spencer Payton

Washington State University

The introductory linear algebra course provides unique challenges to many undergraduate students. The abstract nature of the course and immense amount of vocabulary offers a stark contrast to the procedural mathematics encountered in calculus and high school mathematics courses (Carlson, 1993; Dorier & Sierpinska, 2001). Many concepts in linear algebra are inherently connected. These connections can be made explicit by the use of a *unifying and generalizing concept*, defined by Dorier (1995). Examples of a unifying concept include the Invertible Matrix Theorem as presented by Lay (1994). In my study, I considered two potential unifying concepts, pivots and solution sets of matrix equations and asked how these concepts influenced student understanding of linear algebra.

My research and teaching throughout this process were informed by the constructivist perspective. As a teacher, I designed my lectures and assignments in an attempt to perturb my students' cognitive structures. To aid in the description of these cognitive structures, I refer to Tall and Vinner's *concept image* and *concept definition*. Further, I adapted a classification system for linear algebraic concept images of span and linear independence from one developed by Wawro and Plaxco (2013). Based on my presentation of the material in my class, two of the concept images described by Wawro and Plaxco were of particular interest to me. The *vector algebraic* concept image consists of interpretations of span or linear independence in the context of vector addition, scalar multiplication, and linear combinations. The *matrix algebraic* concept image consists of interpretations of these terms in the context of properties of matrices and procedures on matrices. Through my research, I describe an additional concept image, one I shall refer to as the *linear systematic* concept image. This concept image involves interpretations of span and linear independence in the context of a linear system. Key words associated with this concept image are system, variables, basic variables, free variables, equation, solution, and consistency.

After a series of observations and data collection in other instructors' classrooms, I conducted a study in my own class. I offered all of my students the chance to participate in the study; of my 105 students, 28 agreed to participate. Throughout the course, I frequently illustrated connections between concepts by referring to pivots and solution sets of matrix equations. I collected data from student responses on worksheets on span and linear independence, a review worksheet, the midterm, and from interviews with two of my participants. I categorized different student responses as evoking a vector algebraic, matrix algebraic, or linear systematic concept image.

Many students displayed vector algebraic and matrix algebraic concept images in their responses. However, several students who referred to pivots as a unifying concept consistently displayed only matrix algebraic concept images. That is, they were unable to think of span or linear independence without referring to pivots. Other students who were not as reliant on pivots were more able to describe span and linear independence multiple ways and showed cognitive flexibility with regard to their evoked concept images. Further, illustrating connections between concepts from the context of solution sets of matrix equations yielded more responses indicative of linear systematic thinking than illustrating the same connections from the context of pivots. Thus, while pivots can be an incredibly useful tool, an emphasis on them as a unifying concept may restrict students to developing concept images that involve little beyond the matrix algebraic. Considering solution sets of matrix equations as a unifying concept seems to allow more cognitive flexibility and serves as a useful reminder of the connection between the matrix and a linear system.

References

Carlson, D. (1993). Teaching linear algebra: Must the fog always roll in? *College Mathematics Journal*, *24*(1), 29–40.

Dorier, J.-L. (1995). Meta level in the teaching of unifying and generalizing concepts in mathematics. *Educational Studies in Mathematics*, *29*(2), 175–97.

Dorier, J.-L., & Sierpinska, A. (2001). Research into the teaching and learning of linear algebra. In D. Holton (Ed.), *The Teaching and Learning of Mathematics at University Level: An ICMI Study*. Netherlands: Kluwer Academic Publishers.

Lay, D. C. (2011). *Linear algebra and its applications* (Fourth ed.). Reading, Mass.: Addison-Wesley.

Wawro, M., & Plaxco, D. (2013). Utilizing Types of Mathematical Activities to Facilitate Characterizing Student Understanding of Span and Linear Independence. In *Proceedings of the 16th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 12 – 26). Denver, CO.

Silence: A Case Study

Matthew Petersen
Portland State University

Silence has been important to many disparate traditions, notably, Zen Buddhism, and Taoism. But it has received relatively little treatment in the mathematics education literature. This paper attempts to begin a conversation on silence, its good and bad uses, and raises the question of whether silence may be an important aspect of mathematics activity. In order begin an answer to that question, it analyzes the contribution that a particular group member's silence produced for the group, leading both to a correct solution, and to a valuation of her group-mate's arguments.

"We must cultivate the courage to stay silent for a while among the people with whom we live, so that when we do speak our voice will have become theirs." –Eugen Rosenstock-Huessy (1966, p. 180).

Silence is given a central location in many disparate spiritual traditions. The *Rule of St. Benedict* claims "The spirit of silence is so important, permission to speak should rarely be granted, even to perfect disciples." Sri Guru Granth Sahib, the Sihk sacred text, exhorts us to "Make silence your ear-rings." Siddhartha, who was to become the Buddha, achieved enlightenment while meditating under the Bodhi tree, in silence. The Dao de Jing says "Those who know do not speak; those who speak do not know." Zen Buddhist monk Tich Nhat Hanh exhorts us that "being is non-action", and "don't just do something, sit there" (2014 p. 20-1).

Modern psychology and sociology, especially ecological psychology ethnomethodology, have also begun to appreciate that stillness and silence, is not simply a lack of motion, the default state of a system without stimulus, but is in fact, a particular form of motion. Thus from an ecological perspective, both Marratto (2012) and Reed (1996) argue that perception is not inherently passive, but is an active taking up the world. A position which would seem to imply that silence is not a "default" position, but requires a particular form of discipline and, in a sense, activity.

More directly, Conversation Analysis (Liddicoat, 2011; Heritage 1984) has long been attentive to the silences inherent in the rhythms of conversation, and their necessity for communication. For instance, conversations show what Conversation Analysts call a "preference structure", in which responses that maintain solidarity—for instance, accepting an invitation—usually occur without delay, and are made explicit at the beginning of the turn to speak, whereas responses that break solidarity, "dispreferred actions"—e.g. declining an invitation—usually begin with pauses, and the dispreferred response is often implicit, or delayed till the end of the turn. Indeed, a delayed response to an invitation is able to be heard as a rejection of the invitation, whereas a quick prompt "no" may be taken as rude (Heritage, 1984).

In contemporary discourse, silence often has a negative connotation, particularly in calls for justice for silenced peoples—and rightly so. As Bakhtin (1986), drawing on Thomas Mann notes, one of the greatest terrors of the Concentration Camps was "an absolute *lack of being heard*", since "there is nothing more terrible than a *lack of response*" (p. 146-7, emphasis in the original): That is, than pure and absolute silence. When we silence various groups, refusing to hear them, and especially, their calls for justice, we do them one of the greatest wrongs imaginable—perhaps, as Martin Luther King Jr. noted, in his "Letter from Birmingham Jail", worse even than those who explicitly commit the injustice. But yet, as Tich Nhat Hanh (2014) notes, our actions in defense of the wronged must be preceded by an inner stillness and peace, cultivated through silent meditation. And, again as he noted, mindful non-action is not nothing, but something (p. 20)—and can even be something subversive (p. 44-5).

In the mathematics education literature, setting aside the social justice literature for the reasons enumerated in the above paragraph, silence, *per se*, has not received much discussion. Some people writing from an embodied perspective have attended to silence(s), (Radford, Bardini & Sabena, 2007; Bautista & Roth, 2012). A few papers have implicitly argued that something like silence can be important for doing mathematics, thus, for instance, Savic (2012) showed that mathematicians often resolve an impasse in their work by going for a walk, or out to lunch, or about department business, and *not thinking* about the mathematics. From a different perspective Gal, Lin & Ying (2007) argued that hidden in the high performance of many East Asian mathematics classrooms is a negative silence of the underachieving.[1] And in a Conversation Analysis journal, Ingram & Elliott, (2014) have shown the rhythms and pauses—and, explicitly, silences—of everyday speech, are an important part of the mathematics classroom. Nevertheless, that silence is given relatively short shrift is shown, if nowhere else, by the usual transcription conventions which pass over silence *in silence*.

This paper attempts to begin to ask whether silence can be a positive good in the classroom, and whether it is perhaps an important aspect of mathematical activity that as researchers we should attend to.

Regarding silence as potentially a good to be cultivated in the classroom: Lang (1985) claims that one of the chief reasons mathematicians do mathematics aesthetic: Mathematics is beautiful, proofs particularly so. There is small body of literature which draw attention to the beauty of mathematics (Sinclair, 2006; Mamolo & Zazkis, 2012), especially Winston (2010). Perhaps reflection on the beauty of mathematics could be aided by times of positive silence.

Regarding silence as an aspect of mathematical activity: Daoists and Zen Buddhists (and perhaps Sikhs) have argued that silence—or what is referred to as no-self (Heisig, 2013)—is an important prerequisite for true action (Hanh, 2014). This leads to the research question in this paper: Is (or "how is") silence a positive aspect of undergraduate mathematics activity?

Methodology

This episode was part of a study of student group work in calculus. As part of the study, groups of students were interviewed, and the group dynamics observed. In order to facilitate video recording, the students were asked to work on a white board. In this interview, three students, two male and one female, at a community college in the Pacific Northwest, were asked to graphically identify three unlabeled functions as the position, velocity, and acceleration functions of a car (figure 1). The students were mostly allowed to work on their own.

[1] This silence is, I believe, somewhat akin to the negative silence the social justice literature seeks to address. And while one of the theses of this paper is that we should attend to silence as potentially important; I do not at all intend to minimize the evil of negative silence.

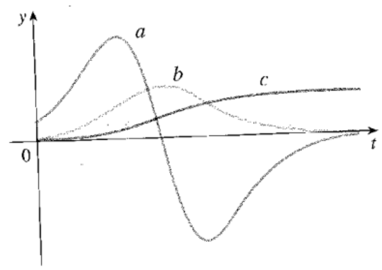


Figure 1. **The figure above shows the graphs of three functions. One is the position function of a car, one is the velocity of the car, and one is its acceleration. Identify each curve (from Stewart, 2012).**

The interview was then transcribed and analyzed using Conversation Analysis (Heritage, 1984, Liddicoat, 2011). Conversation Analysis (CA) is a branch of ethnomethodology which attempts to describe the intersubjective nature of communication: How conversations are co-constructed by the participants precisely through the public enactment of an understanding of the social engagement, an enactment which both shapes the character of preceding dialogue, and which projects future responses.[2] Because of this focus on the structure of the conversation itself, CA does not seek to use a interview to discover hidden variables like internal conceptions, rather, it is the conversation itself that is interesting, and, as in all ethnomethodology (Heritage, 1984) the public means that social actors use to co-construct their interaction.

Furthermore, it is precisely this public character of conversation that allows social researchers to "eavesdrop", and, through careful attention to the prosodic character of utterances, their pitch contour, their rhythms, etc. and the structure of the turns at talk, to discover the internal structure of a conversation.

CA has been used in mathematics education by Roth & Radford (2011) as a tool for describing a classroom episode in detail, with particular sensitivity (in their case) to what they call the cultural-historical nature of mathematics activity.

Results

At the beginning of this sequence, Jason and Katherine have been debating interpretations of the graph. Katherine has been extremely proactive in making her point, several times cutting Andy off to restate her argument, and immediately, and negatively responding to Jason's argument. Her insistence on her argument is seen clearly in the following passage:

```
Jas: When there's an apex (.) when there's a max here and a min
     here (.) that should be zeros on the derivative of that=
Kat: =No zeros would be (.) the zeros would be ah um (.2)
     inflection point?
```

Note that since this is a negative response, and so dispreferred, it is particularly strong: She begins her turn immediately after Jason had finished his, with no pause, and leads with a disagreement, "no".

In spite of this determination that her argument be heard, the final solution was first offered by Katherine, and notably, not in her own voice, but in Jason's. The passage is worth quoting at

[2] As an example of how a response shapes a preceding action, consider the exchange (from Heritage, 1985, p. 255): A: "Why don't you come see me sometimes." B: "Sorry (.) ok; I've been busy recently." B's response treats the original question as an accusation. And if A intended it as an invitation, A will have to treat it as "heard as an accusation", and address that hearing.

length. At this point the group had noticed that graph **a** was the derivative of graph **b,** and had interpreted that to mean that graph **b** was the position function and graph **a,** the velocity (and by default, graph **c**, acceleration). Katherine was arguing that this reading is plausible, since graph **c** seems to be second derivative of **b**, whereas Jason was arguing that something doesn't fit, since graph **c** cannot be the derivative of graph **a**.[3]

1. Jas: You're right that **b** is all concave down. Um=
2. Kat: So there's no points of inflection on **b** so therefore **c**
3. should never cross (.) the x-axis;
4. Jas: But **c**'s the second derivative of ( )
5. Kat: No **c**'s the second derivative (.)
6. Jas: of **b** but **c** is still °just the derivative of **a** right°
7. (2.0)
8. Kat: right
9. → ((1 minute 45 seconds during which Katherine is silent, omitted))
10. And: What do you think of that. ((to Kat))
11. Kat: Well (2.0) so (2.0) does anybody disagree that we've labeled
12. these as the position and the velocity and the acceleration;
13. ((as she says this, she points to the labels "position" and
14. "velocity" they had written earlier))
15. And: [( )]
16. Jas: [( )] Here's what I'll agree with (.) **a** is
17. And: I believe
18. Jas: definitely >the derivative of **b**< (.) **a** is definitely the
19. derivative of **b** (.) And tch, (cocks head to left)
20.→ Kat: So do you believe **c** is the position function; this ((**b**))
21. is velocity and this ((**a**)) is (.2) acceleration;
22. Jas: I think I think >↑**b** ↑is ↑the ↑derivative ↑of ↑**c**<.

At the beginning of this passage, Jason and Katherine acknowledge the other's point (l. 1 and 8), and their argument pauses (l. 7). However, both remain physically tense, with their attention oriented to the mathematics on the board, not to each other or Andy.[4] Though Jason is initially silent like Katherine, Andy draws him into a conversation about the physical significance of the graphs, e.g. negative velocity.

Throughout their discussion, however, Katherine is physically withdrawn from the group and does not, even with her gestures, respond to the Andy and Jason's work. But her body remains tense, oriented to the mathematics. Her response to Andy in line 11 may seem odd—the labels were written on the board—but Jason (l. 17, 18-19) responds to her question by returning the discussion to where it had been in l. 5: They have agreed that **a** is the derivative of **b**, but, as Jason's unfinished statement, exhale, and cocking his head to the left indicate, they had had not been able to make sense of graph **c**, plausibly arguing that it was, and was not the acceleration function. Katherine responds to Jason by suggesting the correct solution, notably, for the first time in the interview.

Jason agrees with her suggestion (l. 22), increasing both the pitch and the speed of his speech—which seems to suggest that he had not thought of that option before, and was *excited* at

[3] Complete transcription conventions are included in an appendix.
[4] Shifting their gaze to each other or to Andy would be an invitation for further speech (Liddicoat, 2011, p. 160).

hearing it. Jason will immediately explain this solution, and the group agrees the solution is correct and conclusive—treating it as conclusive by beginning to go on to the next problem.

Katherine's reentry into the conversation almost immediately led to a solution, and it seems clear that it was the period of silence that was fruitful. At the beginning, she and Jason were both confused about the correct solution, and when she speaks again, she almost immediately suggests it. But the *social* character of her solution is perhaps as important as the fact of the solution.

Katherine does not merely suggest that **c** is the position function, but like in the opening quote, she does this in Jason's voice, leaving aside her, plausible, but erroneous, claim that **c** could be the second derivative of **b**, and *attributing the solution to Jason* ("you" in l. 20), as if this is what his arguments had implied. And Jason agrees not only with the suggested solution, but, by using the first person, that this solution was what he was implicitly suggesting and grasping for, but, as his excitement indicates, couldn't quite reach.

This ability to speak as, it were, with Jason's voice is in marked contrast to her activity prior to her silence, where she was (admirably) insistent on making sure her argument is heard, cutting off Andy to restate her argument, and emphatically responding to Jason's arguments. Thus in the silence, she was able not only to arrive at a correct solution, but to transcend the agonism, and give voice to her interlocutor, helped the group reach the solution in union.

Conclusion

In this episode, Katherine's nearly two minutes of silence seems to have played an important role in the group solution. In silence, she reached a solution to the problem; whereas the noisy business of Jason and Andy only reached a dead-end. Perhaps more importantly, though, after her silence, she had overcome the agonistic conflict, and offered a solution that gave voice to, and affirmed the contribution of her group mate.

This leads to the following three questions:

Are there other sources on silence I missed, or did not consider?

How can we distinguish between good and bad silences?

Does it make sense to research silence, and treat it as a positive aspect of mathematical activity?

Appendix. Transcription Key

[ ]	Overlapping talk (transcript is in courier so overlap can be accurately transcribed)
(0.5)	Length of silence measured to the tenths of a second
(.)	Micro-pause
.	Falling intonation
?	Strong rising intonation
;	Slight, terminal rising intonation
,	Continuing intonation, slightly rising
=	Contiguous utterances, no pause or gap
_	(Underline) emphasis
(( ))	Transcriber's description
()	Speech which is unclear or in doubt in the transcript
:	Prolongation of immediately prior sound
º	Noticeably quieter talk
> <	Fast talk
↑	Raised pitch

References

Bakhtin, M. (1986). *Speech Generas and Other Late Essays*. [Nook Version.]

Bautista, Alfredo, & Roth, Wolff-Michael (2012). Conceptualizing sound as a form of incarnate mathematical consciousness. *Educational Studies in Mathematics* 79: 41-59.

Gal, Hagar, Lin, Fou-Lai, & Ying, Jia-Ming (2008). Listen to the Silence: The Left-Behind Phenomonon as Seen through Classroom Videos and Teachers' Reflections. *International Journal of Science and Mathematics Education*. 7: 405-429.

Khalsa, Sing Sahib Sant Singh, Trans. (N.D.). *English Translation of Siri Guru Granth* Sahib. Tucson, AZ: Hand Made Books, Retrieved from: http://bit.ly/1rBCCcS

Heisig, James (2013). *Nothingness and Desire: an East-West Philosophical Antiphony*. Honolulu, HI: University of Hawai'i Press.

Heritage, John (1984). *Garfinlel & Ethnomethodology*. Polity Press, Cambridge, UK.

Hanh, Thich Nhat (2014). *How to Sit*. Berkley, CA: Parallax Press.

Ingram, Jenni, & Elliott Victoria (2014). Turn taking and 'wait time' in classroom interactions. *Journal of Pragmatics*. 62: 1-12.

King, Martin Luther Jr. (1963). "Letter from a Birmingham Jail". Retreived from: http://www.uscrossier.org/pullias/wp-content/uploads/2012/06/king.pdf

Lang, Serge (1985). *The Beauty of Doing Mathematics: Three Public Dialogues*. New York, NY: Springer.

Laozu (2010). *The Dao de jing*. (R. Eno, Trans.). Indiana University. Retreived from: http://www.indiana.edu/~p374/Daodejing.pdf

Liddicoat, A. J. (2011). *An Introduction to Conversation Analysis*. New York, NY: Continuum.

Mamolo, Ami, & Zazkis, Rina (2012). Stuck on convention: a story of derivative relationships. *Educational Studies in Mathematics*. 18: 161-177.

Marratto, Scott (2012). *The Intercorporeal Self: Merleau-Ponty on Subjectivity*. Albany, NY: State University of New York Press.

Radford, Louis, Bardini, Caroline, & Sabena, Cristina (2007). Perceiving the General: The Multisemiotic Dimension of Students' Algebraic Activity. *Journal for Research in Mathematics Education*. 38, 5: 507-530.

Reed, Edward (1996). *The Necessity of Experience*. New Haven, CT: Yale University Press.

Rosenstock-Huessy (1966). *The Christian Future*. New York, NY: Harper Torchbooks.

Roth, Wolff-Michael & Radford, Louis (2011). *A Cultural-Historical Perspective on Mathematics Teaching and Learning*. Boston, MA: Sense Publishers.

Savic, Milos (2012). What do mathematicians do when they reach a proving impasse?. In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman (Eds.), Proceedings of the Fifteenth Annual Conference on Research in Undergraduate Mathematics Education (p. 531-535). Portland, OR: Portland State University.

Sinclair, Nathalie (2006). *Mathematics and Beauty: Aesthetic Approaches to Teaching Children*. New York, NY: Teachers College Press.

Stewart, J. (2012). *Calculus, Seventh Edition*. Belmont, CA: Brooks/Cole, Cengage Learning.

St. Benedict, Leonard Doyle trans. (2001). *The Rule of St Benedict*. Retrieved from: http://www.osb.org/rb/text/toc.html#toc

Winston, Joe (2010). *Beauty and Education*. New York, NY: Routledge.

Domain, co-domain and causation: A study of Britney's conception of function

Authors
Affiliation

Abstract: Function has been shown to be an important, but difficult concept for students to master (Carlson, Oehrtman & Engelke, 2010; Dubinsky & Harel, 1992). Through a clinical interview with a preservice mathematics teacher, I characterize the ways in which her function definition is able to account for novel relationships between quantities. Utilizing APOS theory, I find that though she is able to exhibit a process view of function, the student struggles to reconcile her definition of function with her intuitions about domain/co-domain and causation. The research is part of a larger study examining the ways in which preservice teachers define function affect their ability to accommodate novel contexts and representations.

Key words: [function, preservice teacher, domain, inverse]

A student's understanding of the concept of function is an integral part of undergraduate mathematics curriculum (Carlson, 1998; Carlson, Oehrtman & Engelke, 2010; Jones, 2006). As important as function is, it is also seems to be that students largely struggle with developing a robust understanding of function (Breidenbach, Dubinsky, Hall, & Nichols, 1992; Dubinsky & Harel, 1992; Dubinsky & Wilson, 2013; Even, 1988; Sfard, 1987; Trigueros, Ursini, & Reyes, 1996). In an APOS perspective of student understanding of function, a student with a process view is able to interiorize their mental actions in such a way that the totality of their actions can take place entirely in the mind or be imagined as taking place (Dubinsky, 1991). In the analysis I will focus on how a preservice teacher's process level identification of function and inverse was dependent on the domain/co-domain and context of the problem situation.

This study focuses on a single senior undergraduate mathematics major, Britney, who was a part of a preservice teacher program at a large state school in southwestern Virginia. This study is part of a larger study on three preservice teachers' understanding of function. The students engaged in a group card sorting activity introduced by Hillen & Malik, focusing on students' categorization of functions (2013). The research team then conducted clinical interviews, emphasizing inquiry into students understanding of function, inverse, domain and co-domain.

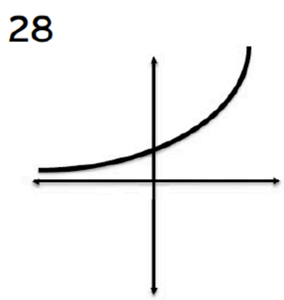


Figure 1

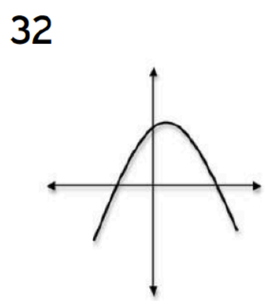


Figure 2

Britney's initial task was to evaluate the relationships in Figures 1 & 2, to determine if the inverse graph was also a function. For both Figures 1 & 2, Britney uses a definition of function as a process where "you plug in one input and you get a output, a uni, a unique output." Her statement indicates a point-wise comparison of the domain to the co-domain; generalizing the function action. This comparison of "input" to "output" is also used when Britney is determining whether an inverse is a function in a tabular representation.

A theme park has an admission cost of \$7 per person. How could you determine the income from admission?

Figure 3

When evaluating Figure 3, Britney initially had trouble determining that the domain and co-domain of the task were discrete, {1,2,3, …} and {7,14,21, …} respectively. Once aware of this, she no longer believes that a mapping between them is a function, saying "[i]t's not a function in the first place because it's just a set of dots." Britney's definition of function competes with her intuitions about discontinuous mappings. For her a function is a relation of a single input to a single output. As such, she has trouble arguing against the inclusion of the mapping as a function: "If you plug in something from the limited domain of natural numbers you're going to get some output no matter what…. so uh, I guess it would be considered a function" She remains uncertain of whether the mapping is a function, but concedes that is. We later find that this is a limited, and possibly temporary, accommodation of her function concept, as she claims that the mapping of the domain to the co-domain in Figure 4 is not a function.

26

The cost of admission to the park is \$1 per person and \$3 for parking the vehicle.

Figure 4

For both Figure 3 and Figure 4, Britney claims that the inverse mapping does not exist. "Common sense-wise, it's not invertible. Logically it's not invertible". She claims that there is no reverse causal relationship between *income* and *number of persons* or *number of vehicles*, implying no mathematical relationship. Britney's use of "input" and "output" in her function definition seems to imply for her a close connection between causation and function/mapping.

In the study, Britney had interiorized the concept of function as a transformation of the elements of a set mapping to the elements of another set. This activity did not need to be made explicit, characterizing this student as having at least a process view of the function concept. Despite this, there were still two significant barriers to Britney's ability to determine whether a mapping is a function and whether it has an inverse: discrete domains/co-domains and causation. Her inability to fully reconcile her definition of function with her strong intuitions against discrete domains/co-domains and for causation seemed to be evident in the other students of the larger study. The coordination of the students' function definition and these intuitions is a topic for further research.

References

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the Process Conception of Function. Educational Studies in Mathematics , 247-285.

Carlson, M. (1998). A Cross-Sectional Investigation of the Development of the Function Concept. CBMS Issues in Mathematics Education , 114-144.

Carlson, M., Oehrtman, M., & Engelke, N. (2010). The Precalculus Concept Assessment: A tool for assessing students' reasoning abilities and understandings. Cognition & Instruction , 113-145.

Dubinsky, E., & Harel, G. (1992). The nature of the process conception of function. In The Concept of Function: Aspects of Epistemology and Pedagogy (pp. 85-106). Mathematical Association of America.

Dubinsky, E., & Wilson, R. (2013). High school students' understanding of the function concept. Journal of Mathematical Behavior , 83-101.

Even, R. (1988). Pre-service teacher's conceptions of the relationship between functions and equations. Proceedings of the Twelfth International Conference for the Psychology of Mathematics Education, (pp. 304-311). Veszprem.

Jones, M. (2006). Demystifying Functions: The Historical and Pedagogical Difficulties of the Concept of Function.

Trigueros, M., Ursini, S., & Reyes, A. (1996). College students' conception of variable. In L. Puig, & A. Gutierrez (Ed.).

Sfard, A. (1987). Two conceptions of mathematical notions: Operational and structural. Proceedings of the Eleventh International Conference for the Psychology of Mathematics Education, (pp. 162-169). Montreal.

Exploring practices and beliefs that shape the teaching of mathematical ways of thinking and doing at university

Alon Pinto
Weizmann Institute of Science, Israel

This study examined two lessons taught in parallel in a real analysis course by two different teaching assistants. The lessons were based on a single lesson-plan but decisions the instructors made prior to and during class took the lessons in substantially different directions. In this paper we focus on one instructor and describe some of the ingenious ways by which he adapted the written curriculum and introduced mathematical practices and ways of thinking in his lesson. We apply Schoenfeld's resources, orientations and goals framework to highlight relationships between the instructors' beliefs and practices, and to propose explanations as to why this instructor implemented the written curriculum and addressed the mathematics the way he did. On the basis of this analysis we suggest a new perspective on the nature of the process by which university instructors make sense of the curriculum, derive and prioritize goals and form their lesson-images.

Key words: Teacher Practice, Teacher Beliefs, Real Analysis, Curriculum Enactment

Introduction

The lessons students learn about mathematics extend far beyond the scope of definitions, procedures, theorems and proofs that are taught and discussed explicitly in classrooms. There is extensive literature (e.g. Ball, 1988; Schoenfeld, 1988, 1992; Yackel & Cobb, 1996) describing how students at K-12 pickup mathematical habits, practices, beliefs and norms through instruction regardless of what is explicitly taught, and how these mathematical ways of thinking and doing are often undesirable. This problem is even more apparent at university, where students are required to unlearn school mathematics and adopt mathematics as practiced by mathematicians. Studies of the secondary-tertiary transition have documented various ways by which students fail to make sense and acquire the practices, norms and language of the professional mathematics community (Dreyfus, 1999; Gueudet, 2008; Jones, 2000; Lim & Selden, 2009). Nevertheless, these tacit ways of thinking and doing are only rarely discussed explicitly in classrooms, as teachers and instructors often consider them beyond the scope of their responsibility, if at all teachable (Selden & Selden, 2005).

The study described in this paper is part of an ongoing research that explores the teaching and learning of mathematical ways of thinking and doing at the collegiate level. This study compared lessons of different instructors teaching the same explicit content, and explored how specific beliefs and practices shaped the mathematics addressed, explicitly or implicitly, in the lessons. In this paper we examine the teaching of one real-analysis instructor, a young graduate student, who adapted the written curriculum in his lessons in ingenious ways that often introduced new mathematical ways of thinking and doing. We examine the instructor's preparation for the first lesson of the semester, describe his practices as he made sense of the mathematics in the curriculum and formed a lesson image, and discuss how his beliefs shaped and guided this process and their consequent impact on the mathematics in the lesson.

Literature Review and Theoretical Framework

Studies that explored the connections between curriculum and student achievement at K-12, have highlighted the central and crucial role of the teacher as the enactor of the curriculum (Remillard & Bryans, 2004), and the deep and profound transition that the curriculum often undergoes from his original written form to its intended form, the teacher's image of the lesson, and to the enacted form, the curriculum that is actually taught at the classroom (Stein, Remillard, & Smith, 2007). While this work has not been extended yet to the university level, a preliminary study, partly reported in (Pinto, 2013), that compared lessons of different instructors enacting the same written curriculum, indicated that the picture at university might be quite different. The lectures that were examined in this study often represented quite faithfully the intended curriculum as inferred from discussions with the instructors and from their lecture notes. This finding highlighted the particularly important role that *lesson images* (Schoenfeld, 2000, p. 250) play at the university. This study examined and compared the practices and beliefs of different university mathematics instructors as they prepare for lectures and form their lesson images.

After an extensive review of research literature, Speer, Smith and Horvath (2010) concluded that empirical research on the actual teaching practices at the university level is virtually nonexistent, and that very little is known about what university math instructors think and do on a daily basis as they perform their teaching work. In their review, Speer et al. called in particular for studies that compare the teaching practices of instructors teaching the same curriculum. Even after this call, empirical studies that compare the practices of different instructors are rare (e.g Nardi, Jaworski, & Hegedus, 2005; Speer, 2005; Viirman, 2014) and most of these studies were not conducted under controlled conditions. One notable exception is a study of different enactments of an innovative inquiry-orientated curriculum by two mathematicians (Wagner & Keene, 2014). This study distinguishes itself by comparing the teaching practices of different instructors enacting in parallel the same curriculum in the same course. Furthermore, whereas the studies reviewed above made some references to the mathematics addressed in the classrooms, their focus was on the teaching approaches and methods. This study investigates practices not only in light of *how* mathematics is taught, but also *what* mathematics is taught, and *why*.

The analysis in this study is based on Schoenfeld's (2011) resources-orientations-goals (ROG) model of goal-oriented decision making. The ROG is an evolving theory that was developed as a tool for explaining how and why teachers make the decisions they make as they teach in terms of their knowledge and other resources, their orientations (e.g. beliefs, views and attitudes) and their goals - the conscious or unconscious aims the teachers are trying to achieve. According to the ROG model, orientations invoke goals and shape their prioritization, and activate and shape the prioritization of relevant resources in the service of these goals. Although relatively new, substantial empirical data has been subsumed under the ROG umbrella (e.g. Hannah et al., 2011; Paterson et al., 2011; Pinto, 2013; Thomas & Yoon, 2013; Törner et al., 2010) and it has proven its usefulness in uncovering connections between particular beliefs and specific instructional practices, inside and outside of the classroom.

Settings and Methods

Data for this study were collected from the lessons of two teaching assistants (TAs) in a real analysis course at a major university in Israel. The students in the course were mostly first year mathematics and computer science undergraduates. This paper focuses on the first lesson of the semester of Amit, one of these TAs. Amit was at the time of the study a math graduate student with 4 years of experience as a TA in various courses, but only once in this specific course, 3 years earlier. The lesson-plan for the lessons was prepared by another TA. The TAs were generally expected to follow this plan, and the students were told they could attend any one of the TA-lessons since they were all "roughly the same". However, the TAs did not have to report on what they actually did in their lessons, and many considered the lesson-plan no more than a recommendation. Thus, it was not unusual for a TA to take liberties and make considerable adaptations to the lesson-plan.

Data collection and analysis in this study were performed in an iterative manner. The author attended the TA-lessons throughout the semester as a non-participant observer, audio taped them and took notes. The first lessons of each TA were transcribed, compared and contrasted with the lesson-plan to highlight ways in which each TA adapted the lesson-plan. These adaptations served as the focal points of discussions during clinical interviews with the TAs that aimed at eliciting the knowledge, beliefs and goals underlying the instructors' practices. The interviews were transcribed and analyzed according to Schoenfeld's ROG framework, and recurring themes were identified. Data from the lessons were analyzed again in search of connections between the TAs' beliefs and goals and the particular mathematical aspects of the lesson-plan addressed, both explicitly and implicitly, in the lessons. Due to limitations of space, we restrict the analysis in this paper to Amit's lesson.

Analysis

We start the analysis with three examples illustrating ways in which Amit introduced in his lessons mathematical ways of thinking and doing that were not specified in the lesson-plan. We then proceed to discuss how and why Amit adapted the lesson and addressed particular mathematical aspects the way he did.

Case 1 – Making sense of a new concept

The lesson-plan suggested several examples of applications of the definition of the derivative, the first of which was example (A):

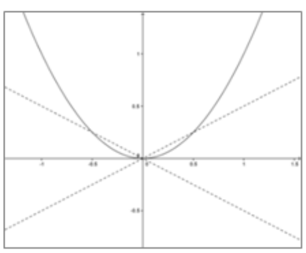
Find the derivative of $f(x)=\sqrt{4x-3}$ at $x_0=3$ (see figure 1 in the next page for details)

Amit decided to skip example (A) altogether. At the beginning of his lesson, after writing the definition of the derivative on the board, Amit said to the students:

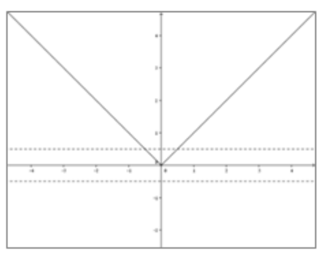
Before getting into examples, I would like us to gain some intuition … we are going to unpack this definition and make a drawing of it, to see what it really means.

Amit then initiated a teaching sequence that he designed, where he unpacked the definition in terms of epsilon and delta, and after a series of algebraic manipulations concluded that if *f '(p)=c* then for every $\varepsilon>0$ there exists some neighborhood of *p* where $f(p)+(c-\varepsilon)(x-p)<f(x)<f(p+(c+\varepsilon)(x-p)$ for every *x*. He then drew **Drawing1** on the board and explained to the students:

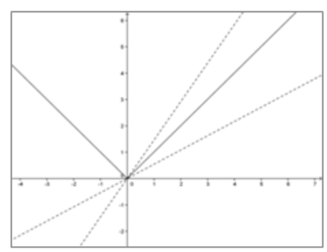
What we see here is that if f is differentiable at p then for any given epsilon f is bounded, in some neighborhood of p, from above and from below by two lines with slopes $c+\varepsilon$ *and* $c-\varepsilon$ *crossing each other at the point (p,f(p)).*



Drawing1



Drawing2



Drawing3

Case 2 – Comparing and contrasting mathematical concepts

After introducing the geometric interpretation of the derivative, Amit turned to discuss differentiability and continuity, inferring from **Drawing1**:

We see that differentiability implies continuity. *When I get closer to 0 (Amit's finger slides on the parabola towards the origin) the graph of the function gets closer and closer to f(0) … it must! I'm not giving here a full formal proof but I think that the fact that differentiability implies continuity is very clear from this drawing.*

At this point Amit made an in-the-moment decision to explore deeper the relationship between differentiability and continuity. He suggested to the students that the geometric interpretation they just learned can be used to gain intuition as to why differentiability is stronger the continuity. He drew **Drawing2** and **Drawing3** and argued orally in geometrical terms, using both drawings as visual aids, that absolute value function is continues but not differentiable at 0. He concluded:

Of course, this is not formal mathematics, but these drawings can help us understand how differentiability is stronger than continuity.

Case 3 – Recognizing mathematical structures and patterns

After the discussion on the absolute value function, Amit continued to example (B) of the lesson-plan (see figure 1): Define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=x\cdot|x|$. Show that *f* is differentiable at every point and find all its derivatives.

After writing the task on the board, Amit paused and then said to the students:

When we saw that the absolute value is continuous but not differentiable, we actually saw that in some sense we can paste together two functions, each one very simple, even linear, and in particular differentiable, such that the new function is continuous but not differentiable […] I claim that if I am careful enough I can paste differentiable functions so that the new function is not only continuous but even differentiable. Let's see an example.

Amit continued to explaining the notion of pasting functions together, which was new to the students, and not part of the course's syllabus. He then suggested that the solution he is about to present can be generalized to a whole class of pasted functions.

Example A: Let $f(x)=\sqrt{4x-3}$ then we find $f'(3)$.

$$\lim_{x\to3}\frac{\sqrt{4x-3}-3}{x-3}=\frac{\sqrt{4x-3}-3}{x-3}\cdot\frac{\sqrt{4x-3}+3}{\sqrt{4x-3}+3}=\frac{4x-12}{(x-3)(\sqrt{4x-3}+3)}=\frac{4}{\sqrt{4x-3}+3}$$

The function $4x-3$ is continuous at every point, and the function $\sqrt{y}$ is continuous at every $y\geq 0$ and therefore $\sqrt{4x-3}$ is continuous at every $x\geq 3/4$ and in particular at 3. It follows (arithmetic of finite limits) that $f'(3)=\lim_{x\to3}\frac{4}{\sqrt{4x-3}+3}=\frac{4}{6}$.

Example B: define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=x\cdot|x|$. We will show that f is differentiable at every point and find it derivatives. First, if $x_0>0$ then $f(x)$ agrees with the function $g(x)=x^2$ in a neighborhood of x_0, and therefore $f'(x_0)=g'(x_0)=2x_0$ (because for every x in the neighborhood of x_0 we have $\frac{f(x)-f(x_0)}{x-x_0}=\frac{g(x)-g(x_0)}{x-x_0}$ and thus the limits are equal). Similarly, if $x_0<0$, then then $f(x)$ agrees with the function $-x^2$ in a neighborhood of x_0, and therefore $f'(x_0)=-2x_0$. Assume then that $x_0=0$, then we need to find the limit $\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{x\to0}\frac{f(x)}{x}$. Since $\lim_{x\to0^+}\frac{f(x)}{x}=\lim_{x\to0^+}\frac{x^2}{x}=\lim_{x\to0^+}x=0$, and $\lim_{x\to0^-}\frac{f(x)}{x}=\lim_{x\to0^+}\frac{-x^2}{x}=\lim_{x\to0^+}-x=0$ it follows that $f'(0)=\lim_{x\to0}\frac{f(x)}{x}=0$ (since both one-sided limits exists and are equal). We conclude that $f'(x)=2|x|$

Figure 1- Examples (A) and (B) as they appeared in the lesson-plan

Explaining Amit's decisions

The three cases above demonstrate various mathematical ways of thinking and doing. In case 1, Amit made a conscious and deliberate decision prior to the lesson to model for his students how a mathematician makes sense of a new definition by playing with it, unpacking it and drawings pictures. In case 2, Amit made an in-the-moment decision to interrupt his plan for the lesson and explore the relationship between two concepts by comparing and contrasting their geometric interpretations in a special case. In case 3, Amit recognized and highlighted a hidden pattern between two different topics in the lesson by introducing a notion from the mathematical horizon of the students, suggesting that a straight-forward solution can in fact serve as a first step in an exploration of an interesting phenomena towards a general theorem.

In what follows we discuss the nature of the process through which Amit interpreted, evaluated and adapted topics in the lesson-plan, and suggest a model that explains his decisions in the three examples above as well as in other cases not discussed in this paper.

Recurrent themes in the discussions with Amit on his decisions and practices indicated that Amit approached topics in the lesson-plan already having in mind a well-formed image of what his lesson would look like, a generic TA-lesson shaped by orientations that were not related to the specific lesson, content or students, but rather to math teaching and learning in more broad contexts, at the university, in TA-lessons and in calculus courses. Thus, Amit adaptations can be explained in terms of attempting to fit topics to the particular features of this generic TA-lesson image.

For example, Amit repeatedly expressed an orientation that TA-lessons should provide students with an experience that is different from the rigorous and coherent experience provided in the lectures; a direct interaction with the mathematical content that cannot be mediated through a notebook. Thus, when going over a topic in the lesson-plan, Amit sought mathematical ideas that call for intuitive reasoning that can serve as the basis for informal discussions about the content. This image of the TA-lessons was supported by Amit's understanding of his role and responsibilities as an instructor who, compared to his students, is an experienced mathematician. Amit repeatedly pointed out that he believes that as someone who has a broad perspective and deep understanding of the content, he should help students see beyond what is taught in lectures, to get a glimpse of mathematical horizons or depths, even if they cannot be fully explained to the students at this stage. Thus, Amit was drawn towards mathematical aspects of the content that the students are not likely to get a sense of just by reading the lecture notes and textbooks. Amit also repeatedly explained that as an experienced mathematician he feels obliged to provide his students with opportunities to see how he himself approaches new mathematical ideas, how he thinks and acts to make sense of them and gain intuition. This goal was also evident in his reflection on his practices of preparing for lessons:

I try to pass on to the students the way I think. The way I do it is to take the task from the lesson-plan and try solving it on my own and then reflecting on what I did. I sometimes polish the answer a bit, but I try to maintain its authenticity. Students often see mathematics as a polished diamond and I try giving them something else, working knowledge.

What emerges from Amit's reflection is that he relies on his own introspection as a main resource for while evaluating topics in the lesson-plan and forming and prioritizing goals:

While preparing for the lesson I usually encounter something which makes me pause. […] I often end up taking this something to class, thinking that if I found it interesting or if it got me confused then it would probably do the same for the students.

To summarize, Amit had in mind an image of a lesson in which the students undergo a profound mathematical experience, the teacher displays his practices and ways of thinking, and the mathematics is informal, intuitive and inspiring. While preparing for the lesson, Amit was reflecting on the way he himself approached the content, and looked for an added value that he could provide as an experienced mathematician - interesting subtle points he can highlight, or mathematical practices and ways of thinking he can model for his students.

We can now return to case (1) and explain Amit's decision to teach the geometric interpretation of the derivative instead of example (A) in terms of measuring both options against Amit's generic TA-lesson image. In his interview, Amit reflected on this decision:

I don't think students benefit much from seeing me just performing algebraic manipulations in front of them; it seems like a waste of time. […] If a student encounters some difficulty while doing the algebra he can just turn to the handed-out solutions and work it out on his own [...] Seeing me working out this example in class may provide some benefit for the students, but only little.

The reason I opened the lesson with the geometric interpretation is that it is not standard. It is a good example of the things I'm drawn to. It is not just textual and it requires a great amount of explanation. It is not something a student would get just by reading the lecture notes.

As it turns out, from Amit's inner perspective, solving example (A) entailed nothing more than algebraic manipulations, which he found not particularly interesting or challenging. He did not sense mathematical practices or ways of thinking worth modeling, no mathematical depths or horizons to highlight, and no intuition to be gained by working out this example. In contrast, the geometric interpretation of the derivative fitted perfectly in Amit's image of a TA-lesson. It involved drawings and required a great amount of explanation; it was a perspective of the derivative that the students were not likely to find on their own; and it represented an authentic mathematical practice that he could model for the students, since he himself reached this interpretation in the course of make sense of the definition.

We conclude this analysis by returning to case (3) and illustrating how generic TA-lesson images can highlight mathematical ways of thinking and doing that remained implicit in the lessons. The context in which Amit presents example (B) has little to do with the solution he ended up presenting to the students, which followed the solution outlined in the lesson-plan. Thus, the reasons for introducing this notion remained quite implicit. The question what led Amit to introduce this task in this particular way, highlighting content which is not part of the course's syllabus, becomes clearer when we consider this decision in terms of adapting the task to Amit's generic TA-lesson. This adaptation elevated the task from the level of a routine and straight forward application of the definition, to the level of investigating an interesting phenomenon – the differentiability of pasted functions. Instead of "just" doing algebraic manipulations on the board, Amit raised a conjecture, refuted it and refined it towards a general theorem, modelling the true work of a mathematician. This introduction also opened the door for an informal and intuitive discussion of a notion from the mathematical horizon of the students – pasting functions – which they were not likely to sense on their own. Pasting function is a very useful tool for generating counter examples, which is a practice which Amit wanted his students to learn. All this, even though it remained implicit in the lesson, made the topic compatible with Amit's generic TA-lesson image and thus worthwhile from his perspective.

Summary

We examined three lesson episodes illustrating different ways by which Amit adapted the lesson-plan and introduced new mathematical practices and ways of thinking. We discussed how Amit's decisions to highlight particular aspects of the content were based on orientations regarding mathematics learning and teaching in very broad contexts, beyond the scope of the specific lesson, the mathematical content or his actual students. We suggested modelling Amit's decisions as adapting the curriculum to a generic lesson image, making it easier to explain how and why certain orientations came to play and how they functioned together in the context of specific teaching practices and decisions. We also discussed how this model can help uncovering and highlighting mathematics that was left implicit in the lessons.

There is extensive research in K-12 levels showing that teacher beliefs have significant influence on teacher practice, curriculum enactment**,** and opportunities for learning. However, the impact of teacher beliefs on teaching and learning at the university level, particularly in the context of mathematical ways of thinking and doing, is not as well understood or documented, and can only be surmised. The study reported here is part of a long-term research project who seeks to explain how math teaching at the university shapes the mathematical ways of thinking and doing that students develop. Exploring the complex relationships between teacher beliefs and the mathematical ways of thinking and doing addressed explicitly and implicitly in the lessons is a first small step in that direction.

References

Ball, D. L. (1988). *Knowledge and reasoning in mathematical pedagogy: Examining what prospective teachers bring to teacher education*. Michigan State University.

Dreyfus, T. (1999). Why Johhny can't prove (with apologies to Morris Kline). *Educational Studies in Mathematics*, *38*(1-3), 85–109.

Gueudet, G. (2008). Investigating the secondary–tertiary transition. *Educational Studies in Mathematics*, *67*(3), 237–254.

Hannah, J., Stewart, S., & Thomas, M. (2011). Analysing lecturer practice: the role of orientations and goals. *International Journal of Mathematical Education in Science and Technology*, *42*(7), 975–984.

Jones, K. (2000). The student experience of mathematical proof at university level. *International Journal of Mathematical Education in Science and Technology*, *31*(1), 53–60.

Lim, K., & Selden, A. (2009). Mathematical habits of mind. In S. L. Swars, D. W. Stinson, & S. Lemons-Smith (Eds.), *Proceedings of the Thirty-first Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1576–1583). Atlanta: Georgia State University.

Nardi, E., Jaworski, B., & Hegedus, S. (2005). A spectrum of pedagogical awareness for undergraduate mathematics: from "tricks" to "techniques." *Journal for Research in Mathematics Education*, *36*(4), 284–316.

Paterson, J., Thomas, M., & Taylor, S. (2011). Decisions, decisions, decisions: what determines the path taken in lectures? *International Journal of Mathematical Education in Science and Technology*, *42*(7), 985–995.

Pinto, A. (2013). Variability in university mathematics reaching: A tale of two instructors. In *Eighth Congress of European Research in Mathematics Education (CERME 8), Antalya, Turkey* (pp. 2416–2425).

Remillard, J. T., & Bryans, M. B. (2004). Teachers' orientations toward mathematics curriculum materials: Implications for teacher learning. *Journal for Research in Mathematics Education*, *35*(5), 352–388.

Schoenfeld, A. H. (1988). When good teaching leads to bad results: The disasters of "Well Taught" mathematics courses. *Educational Psychologist*, *23*(2), 145–166.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In *Handbook for Research on Mathematics Teaching and Learning* (pp. 334–370).

Schoenfeld, A. H. (2000). Models of the teaching process. *Journal of Mathematical Behavior*, *18*(3), 243–261.

Schoenfeld, A. H. (2011). *How we think: A theory of goal-oriented decision making and its educational applications*. New York: Taylor & Francis US.

Selden, A., & Selden, J. (2005). Perspectives on advanced mathematical thinking. *Mathematical Thinking and Learning*, *7*(1), 1–13.

Speer, N. M. (2005). Issues of methods and theory in the study of mathematics teachers' professed and attributed beliefs. *Educational Studies in Mathematics*, *58*(3), 361–391.

Speer, N. M., Smith, J. P., & Horvath, A. (2010). Collegiate mathematics teaching: An unexamined practice. *Journal of Mathematical Behavior*, *29*(2), 99–114.

Stein, M. K., Remillard, J., & Smith, M. S. (2007). How curriculum influences student learning. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 319–369). Information Age Publishing.

Thomas, M., & Yoon, C. (2013). The impact of conflicting goals on mathematical teaching decisions. *Journal of Mathematics Teacher Education*, *17*(3), 227–243.

Törner, G., Rolka, K., Rösken, B., & Sriraman, B. (2010). Understanding a teacher's actions in the classroom by applying Schoenfeld's theory teaching-in-context: Reflecting on goals and beliefs. In B. Sriraman & L. English (Eds.), *Theories of mathematics education* (pp. 401–420). Berlin, Heidelberg: Springer Berlin Heidelberg.

Viirman, O. (2014). The functions of function discourse – university mathematics teaching from a commognitive standpoint. *International Journal of Mathematical Education in Science and Technology*, *45*(4), 512–527.

Wagner, J. F., & Keene, K. A. (2014). Exploring differences in teaching practice when two mathematics instructors enact the same lesson. In *Proceedings of the 35rd Annual Conference of North American Chapter of the International Group for the Psychology of Mathematics Education, Chicago, IL* (pp. 322–336).

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, *27*(4), 458–477.

John's lemma: How one student's proof activity informed his understanding of inverse

David Plaxco
Virginia Tech

Recent discussions in the field have explored proofs' explanatory power. Such research, however, focuses on how a written proof might convey explanation. I present a conjecture that individual proof activity (the development of proofs) might, itself, have explanatory power. I then discuss one student's (John's) activity related to proving that the centralizer for a fixed element in a group (the set of elements that commute with the given element) is a subgroup and how this activity informed his understanding of inverse. During an individual interview, John developed a lemma claiming that the left- and right- inverses of an element are the same element, his proof of which contradicted his previous ways of thinking about inverse. I analyzed John's proof activity using Aberdein's (2006) extension of Toulmin's (1979) model for argumentation in order to better organize his activity, providing an example of how proof activity might itself be explanatory.

Key words: Proof, Abstract Algebra, Inverse, Toulmin model

Educational researchers have clearly established the importance of exploring and discussing students' engagement in and understanding of mathematical proof (Hanna, 2000; Weber, 2010). Indeed, according to Harel and Sowder (2007), "No one questions the importance of proof in mathematics, and in school mathematics" (p. 806). This presentation contributes to this body of work by investigating successful proof activity adopted by one student. The data used in this article are drawn from a larger study of undergraduate students' proof activity and understanding in an abstract algebra context. This research focuses on modeling students' proof activity and conceptual understanding of inverse and identity, investigating relationships between the two. I chose the case presented here to give insight into how proof activity might inform understanding of the concepts used in the proof. This research outlines how one student's (John's) proof activity fostered meaningful change in his understanding of inverse in a group theory context.

Literature

Bell (1976) contends that mathematical proof should provide a sense of "illumination, in that a good proof is expected to convey an insight into why the proposition is true" (p. 24). Similarly, Almeida (2000) and de Villiers (1999) each claim that proofs should explain. These descriptions suggest that these researchers regard proofs as being inherently explanatory. This notion aligns with several other researchers (Hanna 1990; Mancosu, 2001; Steiner, 1978). Weber (2010) discussed a perspective regarding the notion of an explanatory proof that situates a proof's explanatory power relative to the proof reader. In his discussion, Weber describes "a proof that explains as a proof that enables the reader of the proof … to translate the formal argument that he or she is reading to a less formal argument in a separate semantic representation system" (2010, p. 34). This perspective is most clear in his critique of Steiner's discussion of mathematical proof, when he says, "Steiner treats an explanatory proof as a property inherent in the text of the proof rather than an interaction between the proof and its reader" (p. 34). Weber uses this point to draw distinctions between two representational systems that are used in the proof process: formal and informal. Formal representational systems are the signs, notation, and operations that one carries out in abstract thought, whereas informal representational systems largely rely on specific instantiations of concepts used as exemplars.

Considering this discussion, I see value in Weber's (2010) assertion that a proof conveys explanation only to the degree to which an individual is able to glean explanation from a written proof. The focus in the literature, however, is on individuals' understanding of the mathematical arguments in *others'* written proofs. This discussion neglects the individual's process of proving (developing proofs) which could provide an alternative perspective into proofs' explanatory power. The current investigation seeks to gain insight into the explanatory power of the proofs that one *constructs*, rather than the proofs that one might read.

Abstract Algebra curricula provide rich opportunities to explore students' proof activity due to their emphasis on student-generated proof. Early concepts, such as identity and inverse remain pervasive throughout the curriculum as more advanced concepts are defined using inverse and identity. Researchers have suggested that students' understanding of inverse in Abstract Algebra often build on their notions of multiplicative and additive inverse (Brown, et al., 1997; Hazzan, 1999) Novotna & Hoch, 1998) and in the context of symmetry groups (Almeida, 1999; Larsen, 2009, 2013). Larsen (2009, 2013) builds students' understanding in his Inquiry-Oriented Abstract Algebra curriculum (TAAFU) through students' experiences with symmetries of a triangle. Using both additive and multiplicative notation to relate the symmetries of the triangle, the students develop the group axioms from their exploration of triangular symmetries. Larsen (2009) describes the inverse and identity axioms as the most difficult "rules" for students to generate. By having students engage in the development of the group axioms via their work in S_3, Larsen provides a unique insight into students' generalizations from a specific case. Undoubtedly, students' experiences with inverses in other areas of mathematics inform their conceptions of inverse and identity in novel Abstract Algebra situations. It is interesting, then, that the inverse axiom is one of the two most difficult axioms for the students to develop from the Cayley table in Larsen's research. What aspects, then, of conceptual understanding of inverse would explain this difficulty? This question highlights a need to investigate students' developing notions of inverse and identity, specifically as they relate to the TAAFU curriculum. In this presentation, I focus on John's understanding of inverse.

In order to explore John's activity proving about inverse, I will use Toulmin's (1979) model of argumentation. Several researchers have adopted Toulmin models to document proof (Fukawa-Connelly, 2013; Pedemonte, 2007; Weber, Maher, Powell, Lee, 2008). This analytical tool organizes arguments based on the general structure of *claim*, *warrant*, and *backing*. In this structure, the claim is the general statement about which the individual argues. Data is a general rule or principle that supports the claim and a warrant justifies the use of the data to support the claim. More complicated arguments may use *backing*, which supports the warrant; *rebuttal*, which accounts for exceptions to the claim; and *qualifier*, which states the resulting force of the argument (Aberdein, 2006). This structure is typically organized into a diagram similar to a directed graph, with each part of the argument constituting a node and directed edges emanating from the node to the part of the argument that it supports (Figure 1).

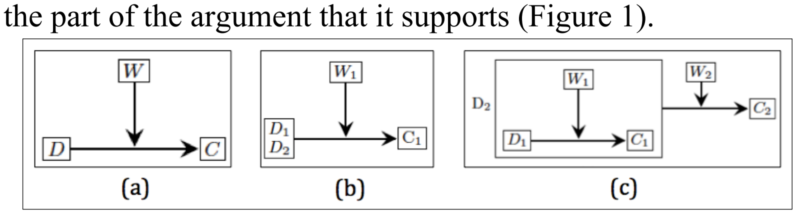


Figure 1: Visual representation of (a) Toulmin model, (b) Linked Toulmin model, and (c) Embedded Toulmin model (Aberdein, 2006, p. 211, 214)

Aberdein (2006) provides a discussion of how Toulmin models might be used to organize proofs, including several examples relating the logical structure of an argument to a Toulmin model. Aberdein includes a set of principles he to coordinate more complicated mathematical arguments in a process he calls *combining layouts*: "(1) treat data and claim as the nodes in a graph or network, (2) allow nodes to contain multiple propositions, (3) any node may function as the data or claim of a new layout, (4) the whole network may be treated as data in a new layout" (p. 213). Figure 1 shows two proposed combined layouts, Linked (Figure 1, b) and Embedded (Figure 1, c), that I are used in the current research. These combined layouts provide for multiple Data (Linked) and Data (Embedded) that are, themselves, claims in another argument.

Theoretical Perspective

As stated, this presentation makes use of data from a larger research project. The broader framing of this work draws on Cobb and Yackel's (1996) interpretive framework for the Emergent Perspective, focusing on connections between sociomathematical norms (for proof) in the classroom community and individuals' mathematical conceptions and activity, connections between classroom mathematical practices and individuals' mathematical conceptions and activity, and relationships between individual mathematical conceptions and activity. This presentation focuses on relationships of the latter type, specifically, exploring proof activity that fosters change in conceptual understanding. Accordingly, the goal of this research is modeling John's proving activity in an Abstract Algebra setting and identifying ways in which this activity informs his understanding of inverse in group theory. It should be noted, however, that John's proving activity took place in an interview setting, which constitutes a community of practice distinct from, although informed by, the practices in the Abstract Algebra classroom of which John was a member. This view is consistent with Cobb and Yackel's (1996) contention that,

> "…it is important to view the students' activity as being socially situated even in settings such as interviews, which are typically associated with psychological paradigms. The psychological analysis would then be conducted against the background of a social analysis that documents the interactively constituted situation in which the student is acting." (p. 185)

The Emergent Perspective thus helps frame the current research as focused on an individual's mathematical understanding and activity situated in an interview setting.

Data Collection and Methods

Data were collected in a Junior-level Inquiry-Oriented Modern Algebra course (Introductory Abstract Algebra). The course met twice a week, for 75 minutes per meeting, over fifteen weeks. The course instructor was an assistant professor in a mathematics department and taught using the Teaching Abstract Algebra for Understanding curriculum (TAAFU; Larsen, 2013). I conducted three (beginning, middle, and end of the semester, respectively), semi-structured individual interviews (forty-five to ninety minutes each) with seven participants. Each interview began by prompting the student to both generally and formally define identity and inverse. The interview protocol then sought to engage each participant in specific mathematical activity aimed to elicit engagement in proof or proof related activity. Throughout the interviews I kept field notes documenting participants' responses to each interview task. I also audio and video recorded each of the interviews.

The data explored here come from John's second (midsemester) interview. Specifically, I focused on John's response to Question 7 of the protocol (Figure 2). This Question asked the participants to prove or disprove whether a defined subset H of a group G was subgroup of G. During the interview, it became clear that John was thinking about inverse in a specific way and that this way of thinking changed for John. I iteratively analyzed John's response to question 7, attending to his argumentation related to the proof as well as his use of, notation of, and statements about inverse. In the first iteration of analysis I selectively transcribed sections of John's proof that developed his argument about whether H was a subgroup of G. These transcriptions were then used to generate a broad Toulmin model of his proof, outlining the major claim, data, and warrants used in his argument. A second iteration focused on each of the data in the larger argument, parsing out selected sub-arguments related to inverse that were used to validate each of the data in the larger argument. Two such sub-arguments emerged as focal points in the proof. The overarching Toulmin model was then modified to include these sub-arguments. Throughout the second iteration, specific aspects of how John thought about and used inverse emerged as important aspects of these sub-arguments. This prompted a subsequent iteration of analysis in which I characterized the various ways he talked about, used, and represented inverse throughout these parts of the proof. Finally, I coordinated one part of the Toulmin model with his ways of thinking about inverses, situating this coordination within the interaction between John and the interviewer.

"Prove or disprove the following: for a group G under operation * and a fixed element $h \in G$, the set $H = \{g \in G : g*h*g^{-1} = h\}$ is a subgroup of G."

Figure 2: Interview 2, Question 7 asks participants to prove about the normalizer of h

Results

In this section, I first briefly detail aspects of John's conceptual understanding that inform an analysis of his proving activity. I then provide a general Toulmin model of John's proof to situate two specific parts of the broader proof: the first to highlight aspects of John's proof activity and understanding of inverse and also to provide an example of how Aberdein's (2006) extension of Toulmin models might be used to organize students' proof activity, the second to demonstrate how John's proof activity informed his understanding of inverse. I support each episode with excerpts of John's interview and a Toulmin model in order to convey a clearer understanding of John's thinking and proof activity throughout the proof. Finally, I will discuss the results and implications of this work for future research.

Throughout the interview, John worked with specific instantiations of groups as well as more abstracted representations of groups and their elements. For instance, John was very comfortable working with real numbers under addition and the group of symmetries of a triangle, evidenced by his frequent use of these groups when describing examples of concepts. John also referred to, proved about, or alluded to multiplication of real numbers, addition of real numbers, addition and multiplication of matrices, integers under addition, and the symmetries of a square. John defined identity and inverse using abstract, more general notation and completed two proofs using abstract notion. John described and used several specific ways of thinking about inverses. These included the cancellation law, self-inverses (elements of order 2 and the identity), the inverse of two elements when operated together ($(ab)^{-1} = b^{-1}a^{-1}$), that an inverse's inverse is the original element ($(a^{-1})^{-1} = a$), and that the identity is its own inverse. Early in the interview, when asked more generally about inverses, John alluded to situations in which a left-inverse and right-inverse might not be the same element. In written instances when an element was concatenated with or operated with its inverse, John either replaced the two elements with a representation of the identity or rewrote the entire statement and omitted the two elements.

John's approach to the proof in Question 7 began with a declaration that he would attempt to prove that H is a subgroup of G (Claim; C). He stated that it is typically easier to try to prove that something is a group and discover that it is not, rather than trying to prove that something is not a group. He had described his general approach for proving that a subset is a subgroup by determining that it satisfied each of the group axioms: H contains an identity (Data 1; D_1), H contains the inverse of each element in H (Data 2; D_2), H satisfies closure under the operation (Data 3; D_3), and H satisfies associativity (Data 4; D_4). From his discussion, the general Linked Toulmin model follows the format in Figure 3. John provided warrant for the broad proof by acknowledging that H satisfied the four group axioms (Warrant; W).

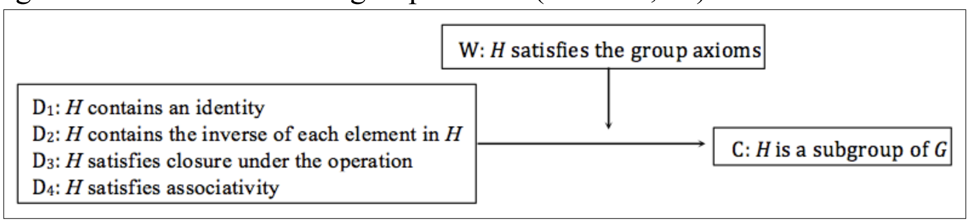


Figure 3: Linked Toulmin model for John's entire proof of Question 7

First, I focus on John's work related to D_1 in order to provide a better sense of how John proves with abstract representations of group elements. In order to verify that the set H contained an identity, John began his 45-second argument by "pretending" that the element g is the identity element. He then quickly said, "Then it really works. I know the identity exists." This is John's first statement of the claim that the identity element (of G) is an element of H ("the identity is in there (pointing at the letter H)". He then wrote the line "Let g, $h \in G$, um, and let g be the identity of G" followed by the equation "$g*h*g^{-1} = h$." On the next line, John rewrote the equation, substituting the letter e for g and g^{-1} and crossing through each e that he had written. As he wrote this, John stated, "and since the inverse of the identity is the identity and this (points to paper) is the identity, you get h equals h. Definition of identity. So we know the identity exists in the set." This sequence of statements is diagrammed in the Toulmin model in Figure 4.

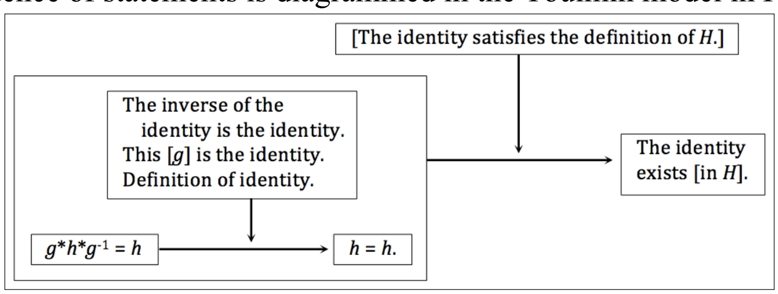


Figure 4: Embedded Toulmin model for D_1, showing an identity exists in H

Notice that John's proof begins with two assumptions, that g and h are elements of G and that g is the identity element of G. Following this, John assumes that g satisfies the relationship necessary for inclusion in H. He then conveys a chain of reasoning that results in the reflexive equation $h = h$. While each step in the sequence follows from the previous step and cites reason for the new statement, John fails to recognize that he has begun his line of reasoning with the statement he intends to prove, rather than beginning with the reflexive relation $h = h$ and deducing that the relation $g*h*g^{-1} = h$ holds. While this can be viewed as a problematic aspect of John's proof, consider the insight into John's ways of thinking about inverse that this excerpt provides. First, notice that John identified the inverse of the identity as the identity. Given the quickness with which he used this fact in the interview, one may infer that John is comfortable with thinking about inverse in this way. Notice also that John quickly rewrote the equation $e*h*e$

$= h$ as $h = h$. This implies that John readily thinks of a composition of an element and the identity as merely just the element, while still citing the definition of the identity in his reasoning.

Next, consider John's development of a lemma used in developing D_2. While beginning his argument for D_2, John manipulated various elements in S_3 and seemed to come to a consensus that inverse elements were contained in *H*, saying, "Oh! I can do something with this." He then stopped and said, "Oh! That was assum- I can't do that, because that's assuming that left and right inverses are the same. I don't know." This prompted a discussion in which John described how he thought about g^{-1} as a left inverse and stated that he didn't know how to use g^{-1} if it's not on the left. The interviewer then asked John what would happen if left inverses were the same as right inverses. John replied, "Is that always true? That could be a thing that's always true." At this point in the interview, John left for a class, and returned later in the day to finish the proof. Upon his return, John announced that he had proved that left and right inverses were indeed the same element. Asked to explain his proof, John wrote out two equations, $a*b = e$ and $b*c = e$, concurrently stating that a is the left inverse of b and c is the right inverse of b. He then concatenated $a*b*c$ and grouped the concatenation as $a*(b*c)$ and $(a*b)*c$. From this, John wrote two lines: $a = a*(b*c)$ and $(a*b)*c = $ c followed by the line $(a*b)*c = a*(b*c)$, stating that this was true because groups are associative. Figure 5 provides a Toulmin model of this proof.

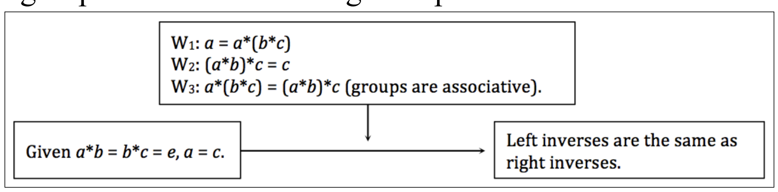


Figure 5: Toulmin model of John's Lemma

Discussion

John's lemma development was roused by his activity with examples in S_3 and his inability to progress in showing that the set *H* contained inverses of its elements. As he pointed out, he had assumed that left- and right- inverses were the same element, which he had previously stated to not always be the case. Prompted by the interviewer, John questioned whether this aspect of his understanding of inverse was valid and, following his proof of the lemma, declared that his new way of thinking about left- and right- inverses was valid. It is important to question whether John would have developed this lemma without prompting from the interviewer. This aspect of the situated interaction cannot be overlooked and must be accounted for. Similarly, the task setting itself prompted John to consider situations that he likely had not considered before (e.g., the definition of the centralizer of an element). Importantly, though, John attributed his changing notions of inverse to his validation of the lemma. This shows how the interview setting and John's willingness to question his own ways of thinking combined to afford an opportunity for him to inform his own understanding of the very concept he was proving about. From this, future research may be carried out exploring and classifying the different types of proof activities and interactions that afford similar opportunities. This work has contributed to the broader research literature by providing an extension of Aberdein's (2006) adaptation of the Toulmin model. To date, only one other research article (Wawro, 2011) has used Aberdein's Toulmin models to analyze students' arguments. The insight that these models lent in parsing out John's proofs warrants additional research in using Aberdein's adaptations.

References

Aberdein, A. (2006). Managing informal mathematical knowledge: Techniques from informal logic. In *Mathematical Knowledge Management* (pp. 208-221). Springer Berlin Heidelberg.
Almeida, D. F. (1999). Visual aspects of understanding group theory. *International journal of mathematical education in science and technology*, *30*(2), 159-166.
Almeida, D. (2000). A survey of mathematics undergraduates' interaction with proof: some implications for mathematics education. *International Journal of Mathematical Education in Science and Technology*, *31*(6), 869-890.
Bell, A. W. (1976). A study of pupils' proof-explanations in mathematical situations. *Educational Studies in Mathematics*, *7*(1), 23-40.
Brown, A., DeVries, D. J., Dubinsky, E., & Thomas, K. (1997). Learning binary operations, groups, and subgroups. *The Journal of Mathematical Behavior*, *16*(3), 187-239.
Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational psychologist*, *31*(3-4), 175-190.
de Villiers, M. (1999). The role and function of proof with Sketchpad. Rethinking proof with the Geometer's Sketchpad, 310.
Fukawa-connelly, T. (2013). Using Toulmin analysis to analyse an instructor's proof presentation in abstract algebra. *International Journal of Mathematical Education in Science and Technology*, (ahead-of-print), 1-14.
Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange*, *21*(1), 6-13.
Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational studies in mathematics*, *44*(1-2), 5-23.
Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. *Second handbook of research on mathematics teaching and learning*, *2*, 805-842.
Hazzan, O. (1999). Reducing abstraction level when learning abstract algebra concepts. *Educational Studies in Mathematics*, *40*(1), 71-90
Larsen, S. (2009). Reinventing the concepts of group and isomorphism: The case of Jessica and Sandra. *The Journal of Mathematical Behavior*, *28*(2), 119-137.
Larsen, S. P. (2013). A local instructional theory for the guided reinvention of the group and isomorphism concepts. *The Journal of Mathematical Behavior*. Pp. 1-14.
Mancosu, P. (2001). Mathematical explanation: Problems and prospects. *Topoi*, *20*(1), 97-117.
Novotná, J., & Hoch, M. (2008). How structure sense for algebraic expressions or equations is related to structure sense for abstract algebra. *Mathematics Education Research Journal*, *20*(2), 93-104.
Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed?. *Educational Studies in Mathematics*, *66*(1), 23-41.
Steiner, M. (1978). Mathematical explanation. *Philosophical Studies*, *34*(2), 135-151.
Toulmin, S. (1979). *The uses of argument*. Cambridge: Cambridge University Press.
Wawro, M. (2011). Individual and collective analyses of the genesis of student reasoning regarding the Invertible Matrix Theorem in linear algebra. *Dissertation Abstracts International*, 72(11). Retrieved May 14, 2012 from Dissertations & Theses: Full Text. (Publication No. AAT 3466728).
Weber, K. (2010). Proofs that develop insight. *For the learning of mathematics*, *30*(1), 32-36.
Weber, K., Maher, C., Powell, A., & Lee, H. S. (2008). Learning opportunities from group discussions: Warrants become the objects of debate. *Educational Studies in Mathematics*, *68*(3), 247-261.

Examining individual and collective level mathematical progress

Chris Rasmussen	Megan Wawro	Michelle Zandieh
San Diego State University	Virginia Tech	Arizona State University

A challenge in mathematics education research is to coordinate different analyses to develop a more comprehensive account of teaching and learning. We contribute to these efforts by expanding the constructs in Cobb and Yackel's (1996) interpretive framework that allow for coordinating social and individual perspectives. This expansion involves four different constructs: disciplinary practices, classroom mathematical practices, individual participation in mathematical activity, and mathematical conceptions that individuals bring to bear in their mathematical work. We illustrate these four constructs for making sense of students' mathematical progress using data from an undergraduate mathematics course in linear algebra.

Key words: Individual, Collective, Emergent Perspective, Linear Algebra

Recent work in mathematics education research has sought to integrate different theoretical perspectives to develop a more comprehensive account of teaching and learning (Bikner-Ahsbahs & Prediger, in press; Cobb, 2007; Hershkowitz, Tabach, Rasmussen, & Dreyfus, 2014; Prediger, Bikner-Ahsbahs, & Arzarello, 2008; Saxe et al., 2009). An early effort at integrating different theoretical perspectives is Cobb and Yackel's (1996) emergent perspective and accompanying interpretive framework. In this paper we expand the interpretive framework for coordinating social and individual perspectives by offering a set of constructs to examine the mathematical progress of both the collective and the individual. We illustrate these constructs by conducting four parallel analyses and make initial steps toward coordinating across the analyses.

The emergent perspective is a version of social constructivism that coordinates the individual cognitive perspective of constructivism (von Glasersfeld, 1995) and the sociocultural perspective based on symbolic interactionism (Blumer, 1969). A primary assumption from this point of view is that mathematical development is a process of active individual construction and a process of mathematical enculturation (Cobb & Yackel, 1996). The interpretive framework, shown in Figure 1, lays out the constructs in the emergent perspective. The significance of accounting for both individual and collective activity is highlighted by Saxe (2002), who points out that, "individual and collective activities are reciprocally related. Individual activities are constitutive of collective practices. At the same time, the joint activity of the collective gives shape and purpose to individuals' goal-directed activities" (p. 276-277).

Our prior work with the interpretative framework (e.g., Rasmussen, Zandieh, & Wawro, 2009; Yackel & Rasmussen, 2002; Yackel, Rasmussen, & King, 2000) has raised our awareness of the opportunity (and need) to go beyond the constructs in the interpretative framework. In particular, we expand the ways we can analyze individual and collective mathematical progress. We use the phrase "mathematical progress" as an umbrella term that admits analyses of collective practices and individual conceptions and activity.

Social Perspective	Individual Perspective
Classroom social norms	Beliefs about own role, others' roles, and the general nature of mathematical activity
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions and activity

Figure 1. The interpretive framework

On the bottom left hand side of the interpretive framework (Figure 1), the construct of classroom mathematical practices is a way to conceptualize the collective mathematical progress of the local classroom community. In particular, such an analysis answers the question: What are the normative ways of reasoning that emerge in a particular classroom? Such normative ways of reasoning are said to be reflexively related to individual students' mathematical conceptions and activity. In prior work that has used the interpretive framework, individual conceptions and activity has been treated as a single construct that frames the ways that individual students participate in classroom mathematical practices (e.g., Bowers, Cobb, & McClain, 1999; Cobb, 1999; Stephan, Cobb, & Gravemeijer, 2003). Such a framing of the individual is, in our view, compatible with what Sfard (1998) refers to as the "participation metaphor" for learning.

In an effort to be more inclusive of a cognitive framing that would posit particular ways that students think about an idea, we split the bottom right hand cell into two constructs, one for individual participation in mathematical activity and one for mathematical conceptions that individual students bring to bear in their mathematical work. With these two constructs for individual progress we now can ask the following two questions: How do individual students contribute to mathematical progress that occurs across small group and whole class settings? And what conceptions do individual students bring to bear in their mathematical work?

Our prior work at the undergraduate level has also highlighted the fact that, in comparison to K-12 students, university mathematics and science majors are more intensely and explicitly participating in the discipline of mathematics. However, the notion of a classroom mathematical practice was never intended to capture the ways in which the emergent, normative ways of reasoning relate to various disciplinary practices (Stephan & Cobb, 2003). In order to more fully account for what often occurs at the undergraduate level, we therefore expand the interpretive framework to explicate how the classroom collective activity reflects and constitutes more general disciplinary practices. Thus we add an additional cell to the bottom left row of the interpretive framework, disciplinary practices. We can now answer two different questions about collective mathematical progress, one related to disciplinary practices (What is the mathematical progress of the classroom community in terms of the disciplinary practices of mathematics?) and one for classroom mathematical practices (What are the normative ways of reasoning that emerge in a particular classroom?).

To summarize, Figure 2 shows our expansion of the bottom row of the interpretive framework, which now entails four different constructs: disciplinary practices, classroom mathematical practices, individual participation in mathematical activity, and mathematical conceptions.

<table>
<tr><th colspan="2">Social Perspective</th><th colspan="2">Individual Perspective</th></tr>
<tr><td colspan="2">Classroom social norms</td><td colspan="2">Beliefs about own role, others' roles, and the general nature of mathematical activity</td></tr>
<tr><td colspan="2">Sociomathematical norms</td><td colspan="2">Mathematical beliefs and values</td></tr>
<tr><td>Disciplinary practices</td><td>Classroom mathematical practices</td><td>Participation in mathematical activity</td><td>Mathematical conceptions</td></tr>
</table>

Figure 2. Expanded interpretive framework

The left hand side of the bottom row comprises two different constructs for examining the mathematical progress of the classroom community, while the right hand side comprises two different constructs for examining the mathematical progress of individual students. The contribution that this expansion makes is in providing researchers with a more comprehensive means of bringing together analyses from social and individual perspectives. In particular, the expanded interpretive framework enables a researcher to answer the questions listed in Figure 3.

Disciplinary practices	Classroom mathematical practices	Participation in mathematical activity	Mathematical conceptions
What is the mathematical progress of the classroom community in terms of the disciplinary practices of mathematics?	What are the normative ways of reasoning that emerge in a particular classroom?	How do individual students contribute to mathematical progress that occurs across small group and whole class settings?	What conceptions do individual students bring to bear in their mathematical work?

Figure 3. Four constructs for analyzing mathematical progress and respective research questions

Setting and Participants

We illustrate the four constructs and address the respective research questions using data from a semester-long classroom teaching experiment (Cobb, 2000) in linear algebra conducted at a large public university. The teaching experiment was part of a larger design research project that explored ways of building on students' current ways of reasoning to develop more formal and conventional ways of reasoning (Wawro, Rasmussen, Zandieh, & Larson, 2013). We selected data from this teaching experiment based on its potential to illustrate all four constructs.

The majority of students in the class had completed at least two semesters of calculus, with some having completed Calculus III or a discrete mathematics course. Most students were in their second or third year of university and had chosen engineering, mathematics, or computer science as their major course of study. We collected data for analysis by videotaping each class session, collecting student written work, and conducting interviews with students throughout the semester. In addition to videorecording whole class discussions, three of the eight small groups were videorecorded; we present analysis here of one of the groups (henceforth referred to as the focus group) and its individual members. We chose to analyze data from this group because the members were particularly open to sharing their thinking and willing to challenge others' ideas, which gave us access to their mathematical thinking without having to rely on interview data.

Theoretical and Methodological Background

Classroom mathematical practices. Classroom mathematical practices refer to the normative ways of reasoning that emerge as learners solve problems, explain their thinking, represent their ideas, etc. By normative we mean that there is empirical evidence that an idea or way of reasoning functions as if it is a mathematical truth in the classroom. This means that particular ideas or ways of reasoning are functioning in classroom discourse *as if* everyone has similar understandings, even though individual differences in understanding may exist. The empirical evidence needed to document normative ways of reasoning is garnered using the approach developed by Rasmussen and Stephan (2008). This approach applies Toulmin's (1958) argumentation scheme to document the mathematical progress using two well-developed criteria.

Disciplinary practices. Disciplinary practices refer to the ways in which mathematicians typically go about their profession. The following disciplinary practices are among those core to the activity of professional mathematicians: defining, algorithmatizing, symbolizing, and theoremizing (Rasmussen, Zandieh, King, & Teppo, 2005). Not all classroom mathematical practices are easily or sensibly characterized in terms of a disciplinary practice. This is because classroom mathematical practices capture the emergent and potentially idiosyncratic collective mathematical progress, whereas a disciplinary practice analysis seeks to analyze collective progress as reflecting and embodying core disciplinary practices. In this report we focus on theoremizing, which encompasses both conjecturing and justifying. Our method for documenting theoremizing builds on the work of Rasmussen, Zandieh, King, and Teppo (2005) and Zandieh and Rasmussen (2010), who analyzed the disciplinary practices of algorithmatizing, symbolizing, and defining.

Mathematical conceptions. As students solve problems, explain their thinking, represent their ideas, and make sense of others' ideas, they necessarily bring forth various conceptions of the ideas being discussed and potentially modify their conceptions. Our analysis of individual student conceptions makes use of analyses from prior work that have characterized different ways that students think about mathematical ideas (e.g., Harel, 1997; Hillel, 2000; Larson & Zandieh, 2013; Selinski, Rasmussen, Wawro, & Zandieh, 2014; Sierpinska, 2000; Stewart & Thomas, 2009; Trigueros & Possani, 2013; Wawro & Plaxco, 2013).

Participation in mathematical activity. This analysis draws on recent work by Krummheuer (2007, 2011), who characterizes individual learning as participation within a mathematics classroom using the constructs of production design and recipient design. In production design, individual speakers take on various roles, which are dependent on the originality of the content and form of the utterance. The title of *author* is given when a speaker is responsible for both the content and formulation of an utterance. The title of *relayer* is assigned when a speaker is not responsible for the originality of either the content nor the formulation of an utterance. A *ghostee* takes part of the content of a previous utterance and attempts to express a new idea, and a *spokesman* is one who attempts to express the content of a previous utterance in his/her own words. Within the recipient design of learning-as-participation, Krummheuer (2011) defines four roles: conversation partner, co-hearer, over-hearer, and eavesdropper. A *conversation partner* is the listener to whom the speaker seems to allocate the subsequent talking turn. Listeners who are also directly addressed but do not seem to be treated as the next speaker are called *co-hearers*. Those who seem tolerated by the speaker but do not participate in the conversation are *over-hearers*, and listeners deliberately excluded by the speaker from conversation are *eavesdroppers.*

Selected Results

The data we use come from three episodes on days 4-6 of the linear algebra class, beginning with the focus group work from day 4 followed by whole class discussions on days 5 and 6. In the full paper we provide more detail on the analyses with each of the four constructs of the expanded interpretive framework. Here we point to a few selected findings, followed by selected analyses that coordinate across constructs. Due to space constraints in this proposal we omit the transcript and simply refer to participant turns with a number.

Selected parallel analyses. In the classroom mathematical practice analysis we identified two normative ways of reasoning that emerged on days 5 and 6. The first was that a set of vectors being linearly dependent means the same thing as being able to return to your original position. We show that this idea was normative because on an earlier class session this idea needed justification (i.e., data and warrants) but in a subsequent discussion the idea was beyond justification (that is, data and warrant dropped off, which is criterion one in our classroom mathematical practice methodology). The second idea that was normative was that having more vectors than dimensions implies the vectors are linearly dependent. Evidence that this was normative comes from criterion two, namely that on day 6 this claim needed to be justified and then on days 9 and 20 we see that this statement idea functions as data for new claims.

For the individual conception analysis, we framed student thinking in terms of ways of thinking about span and linear independence identified in Wawro and Plaxco (2013). These ways of reasoning included travel, geometric, vector algebraic, and matrix algebraic conceptions. By Episode 1 the students had already spent three days of class developing notions of span and linear independence and dependence in a way that cultivated different conceptions. As an example, consider Justin's statements in (1-5). He argued that three vectors in $\mathbf{R}^2$ would be dependent "no matter what … they don't have to be multiples." The reference to multiples indicates a vector algebraic conception in that it refers to the operation of scalar multiplication as a way to compare vectors. He went on to discuss what happens if two vectors are "not on the same line," invoking a geometric conception. He then concluded with language indicating a travel conception, "we can always get to a point where we can get back on the third vector." Other students in the focus group also exhibited travel and vector algebraic conceptions in Episode 1.

Moving to the individual participation analysis, we highlight the small group work detailed in Episode 1. In this episode Justin was the author of both the claim and a justification for any three vectors in $\mathbf{R}^2$ being a linearly dependent set. In line 1 he initially stated his claim. Aziz, who was the one writing the group's ideas on their white board that day, asked Justin for clarification (2). This statement positioned Aziz as a relayer of Justin's idea in (1); furthermore, Justin was Aziz's conversation partner for that statement, whereas the other members of the focus group were co-hearers. For the first portion of (3), in which Justin clarified his claim by saying, "For dependent, as long as you have three vectors," Aziz was his conversation partner. For the remainder of (3) and all of (5), though, when Justin clarified his claim from (1) and added justification for the claim (thus extending his role as an author), all other members of the focus group served as conversation partners.

Lastly, our analysis of the discipline practice of theoremizing reveals the following aspects of students' mathematical work: engaging in a mathematical setting, observing relationships, clarifying and refining stated relationships, arguing for (or against) claims, generalizing, and justifying generalizations. Taken as a whole, these various activities progress from work with particular examples in a particular setting to creating and justifying generalized statements and hence characterizes the mathematical progress of the classroom community in terms of the disciplinary practice of theoremizing. For example, to initiate theoremizing, students engaged in a problem situation in which they constructed examples or struggled to construct examples of sets of vectors with certain properties. As students began to realize under what conditions these examples are or are not possible, they made initial conjectures that eventually led to theorem-like statements and justifications for these statements.

Selected coordinated analysis. To illustrate the coordination of analyses, we consider a portion of the transcript from Episode 3 and discuss how that same data was analyzed with each of the four constructs. In (38) the teacher drew attention to a generalization that had been developed as a conjecture during small group work the previous day. The creation of this generalization is an example of the disciplinary practice of theoremizing because the students were observing mathematical relationships and creating conjectures regarding those relationships. As the teacher asked students to unpack the meaning of this generalization, Justin (39) offered, "If you have more vectors than dimensions, you'll always be able to return to your original position", and Nate (43) agreed. Within the classroom mathematical practice analysis, the first normative way of reasoning detailed was, "A set of vectors being linearly dependent means the same thing as being able to return to your original position." When noticing the two collective-level analyses in conjunction with each other, we see that as students engaged in the mathematical work of justifying a generalization (one aspect of what constitutes theoremizing), a previously established normative way of reasoning (that linear dependence means being able to get back to your original position) was employed in the service of that justifying activity. From the individual mathematical conception construct, Justin's rewording (39) of the generalization as "being able to return to your original position" was consistent with the travel conception of linear dependence because it captured notions of "getting to" or "moving to" locations in the vector space. Within the construct of individual participation, we saw that the teacher's question in (38) positioned Justin to be a spokesman (39), and her request to have Nate comment (40) positioned him as a conversation partner.

Conclusion

In addition to using various combinations of the four constructs to more fully interpret students' mathematical progress, there exist multiple ways in which coordination across the four constructs is possible. For instance, one could choose an individual student within the classroom community and trace his/her utterances for the ways in which they contributed to the emergence of various normative ways of reasoning and/or disciplinary practices. Alternatively, when considering a normative way of reasoning, a researcher could investigate who the various individual students are that are offering the claims, data, warrants, and backing in the Toulmin analysis used to document normative ways of reasoning. How do those contributions coordinate with those students' production design roles within the individual participation construct? For instance, does a student ever utilize an utterance that a different student authored as data for a new claim that he is authoring, and in what ways may that capture or be distinct from other students' individual mathematical conceptions? We also imagine ways to coordinate across the two individual constructs as well as across the two collective constructs. For example, how do patterns over time in how student participation in class sessions relate to growth in their mathematical conceptions? Are different participation patterns correlated with different mathematical growth trajectories? In what ways are particular classroom mathematical practices consistent (or even inconsistent) with various disciplinary practices? Finally, future research could take up more directly the role of the teacher in relation to the four constructs.

We anticipate that future work will more carefully delineate methodological steps needed to carry out the various ways in which analyses using the different combinations of the four constructs can be coordinated. Indeed, we view this report as a first step in developing a more robust theoretical-methodological approach to analyzing individual and collective mathematical progress.

References

Bikner-Ahsbahs, A., & Prediger, S. (in press). *Networking of theories as a research practice in mathematics education.* Switzerland: Springer International Publishing.

Blumer, H. (1969). *Symbolic interactionism: Perspectives and method*. Englewood Cliffs, NJ: Prentice-Hall.

Bowers, J., Cobb, P., & McClain, K. (1999). The evolution of mathematical practices: A case study. *Cognition and Instruction*, *17*(1), 25-66.

Cobb, P. (1999). Individual and collective mathematical development: The case of statistical data analysis. *Mathematical Thinking and Learning*, *1*(1), 5-43.

Cobb, P. (2000). Conducting classroom teaching experiments in collaboration with teachers. In A. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 307–334). Mahwah, NJ: Lawrence Erlbaum Associates.

Cobb, P. (2007). Putting philosophy to work. Coping with multiple theoretical perspectives, In F. K. Lester (Eds.), *Second handbook of research on mathematics teaching and learning* (pp. 3–38). Reston, VA: NCTM.
Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31*, 175-190.
Harel, G. (1997). Linear algebra curriculum study group recommendations: Moving beyond concept definition. In D. Carlson, C. R. Johnson, D. C. Lay, A. D. Porter, A. Watkins, & W. Watkins (Eds.), *Resources for Teaching Linear Algebra* (pp. 106-126). Washington, DC: The Mathematical Association of America.
Hershkowitz, R., Tabach, M., Rasmussen, C., & Dreyfus, T. (2014). Knowledge shifts in a probability classroom - a case study involving coordinating two methodologies. *ZDM - The International Journal on Mathematics Education*, *46*(3), 363-387.
Hillel, J. (2000). Modes of description and the problem of representation in linear algebra. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 191-207). Dordrecht, Netherlands: Kluwer Academic Publishers.
Krummheuer, G. (2007). Argumentation and participation in the primary mathematics classroom: Two episodes and related theoretical abductions. *Journal of Mathematical Behavior, 26*, 60-82.
Krummheuer, G. (2011). Representation of the notion of "learning-as-participation" in everyday situations in mathematics classes. *ZDM - The International Journal on Mathematics Education, 43*, 81-90.
Larson, C., & Zandieh, M. (2013). Three Interpretations of the Matrix Equation *Ax=b. For the Learning of Mathematics, 33*(2), 11-17.
Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM – International Journal for Mathematics Education, 40,* 165–178.
Rasmussen, C., & Stephan, M. (2008). A methodology for documenting collective activity. In A. E. Kelly, R. A. Lesh, & J. Y. Baek (Eds.), *Handbook of innovative design research in science, technology, engineering, mathematics (STEM) education* (pp. 195 - 215). New York, NY: Taylor and Francis.
Rasmussen, C., Zandieh, M., King, K., & Teppo, A. (2005). Advancing mathematical activity: A view of advanced mathematical thinking. *Mathematical Thinking and Learning, 7*, 51-73.
Rasmussen, C., Zandieh, M., & Wawro, M. (2009). How do you know which way the arrows go? The emergence and brokering of a classroom mathematics practice. In W.-M. Roth (Ed.), *Mathematical representations at the interface of the body and culture* (pp. 171-218). Charlotte, NC: Information Age Publishing.
Saxe, G. B. (2002). Children's developing mathematics in collective practices: A framework for analysis. *Journal of the Learning Sciences, 11*, 275-300.
Saxe, G. B., Gearhart, M., Shaughnessy, M., Earnest, D., Cremer, S., Sitabkhan, Y., Platas, L., & Young, A. (2009). A methodological framework and empirical techniques for studying the travel of ideas in classroom communities. In B. B. Schwarz, T. Dreyfus & R. Hershkowitz (Eds.), *Transformation of knowledge through classroom interaction* (pp. 203-222). London, UK: Routledge.
Selinski, N., Rasmussen, C., Wawro, M., & Zandieh, M. (2014). A methodology for using adjacency matrices to analyze the connections students make between concepts: The case of linear algebra. *Journal for Research in Mathematics Education, 45*(5), 550-583.

Sfard, A. (1998). On two metaphors for learning and on the danger of choosing just one. *Educational Researcher, 27*, 4-13.
Stephan, M., & Cobb, P. (2003). The methodological approach to classroom-based research. *Journal for Research in Mathematics Education. Monograph, Vol. 12, 36-50.*
Stephan, M., Cobb, C., & Gravemeijer, K. (2003). Coordinating social and individual analyses: Learning as participation in mathematical practices. *Journal for Research in Mathematics Education. Monograph, Vol. 12,* 67-102.
Stewart, S., & Thomas, M. O. J. (2009). A framework for mathematical thinking: The case of linear algebra. *International Journal of Mathematical Education in Science and Technology, 40*(7), 951-961.
Toulmin, S. (1958). *The uses of argument*. Cambridge, UK: Cambridge University Press.
Trigueros, M., & Possani, E. (2013). Using an economics model for teaching linear algebra. *Linear Algebra and Its Applications, 438*(4), 1779–1792. doi:10.1016/j.laa.2011.04.009
von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Bristol, PA: Falmer Press.
Wawro, M., & Plaxco, P. (2013). Utilizing types of mathematical activities to facilitate characterizing student understanding of span and linear independence. In (Eds.) S. Brown, G. Karakok, K. H. Roh, and M. Oehrtman, *Proceedings of the 16th Annual Conference on Research in Undergraduate Mathematics Education, Volume I* (pp. 1-15), Denver, Colorado.

Analyzing Data from Student Learning

Bernard P. Ricca
St. John Fisher College

Kris H. Green
St. John Fisher College

A full examination of learning or developing systems requires data analysis approaches beyond the commonplace pre-/post-testing. Drawing on graph theory, three particular approaches to the analysis of data – based on adjacency matrices, affiliation networks, and edit distances – can provide additional insight into data. Data analysis methods based on adjacency matrices demonstrate that learning is not unidimensional, and that learning progressions do not always progress monotonically toward desired understandings and also provide insight into the connection between instruction and student learning. The use of affiliation networks provides insight into how students' prior knowledge relates to topics being studied. Careful use of edit distances indicates a likely overestimate of effect sizes in many studies, and also provides evidence that concepts are often created in an ad hoc manner. All of these have implications for curriculum and instruction, and indicate some directions for further inquiry.

Category: Poster Presentation

Key words: Data Analysis Methods, Student Learning, Cognitive Research, Classroom Research

Analyzing Data from Student Learning

Learning often involves learning a process of categorization: students must learn to make choices about the method of integration to use, or what approach to proof will be attempted, and so on. Hence, a fuller understanding of learning requires methods to examine student categorization data. Furthermore, during learning, student classification schemes will likely change over time, and so the data involved in such classification schemes are longitudinal. Although some methods do already exist for studying classifications, such as *card sort* tasks (Diebel, Anderson, & Anderson, 2005), they are not up to the task of working with longitudinal data. Hence, the issue of what additional methods can be developed to look at learning, and what novel insight into situations of development those methods yield is an important one. This poster presents some novel data analysis methods, based in graph theory, and examines their usefulness in some areas of mathematics learning.

Limitations of Existing Approaches

Although pre-/post-test methods can be useful in identifying the existence of changes across an intervention, such methods do not yield much insight into the processes of that change. Longitudinal methods, collecting data at more than two times, can provide some insight (Koopmans, 2014), but many such methods yield information only about the rate of the changes and not always the processes. The use of edit distances (Diebel, Anderson, & Anderson, 2005) can be helpful but neither the relationship of edit distance to learning nor the use of edit distances longitudinally has been previously pursued. These methodological deficiencies are particularly limiting when examining learning, where a detailed understanding of the learning process could provide useful feedback to an instructor.

New Approaches to Data Collection and Analysis

One approach to collecting data on learning is an extension of Vygotsky's (1986) method of *double stimulation*, which Vygotsky used to examine concept development in adolescents. This method probes the development of a learner's classification schemes in response to feedback, and while Vygotsky used qualitative approaches to analyze the data, it is possible to use more quantitative approaches, based in graph theory, to understand the process of learning. In particular, three longitudinal approaches – an examination of changes in edit distances, the use of affiliation networks (which include student-identified category names rather than pre-set categories; see Figure 1), and an examination of correctly and incorrectly grouped items – have been developed to provide new insight into learning (Author & Author, in press).

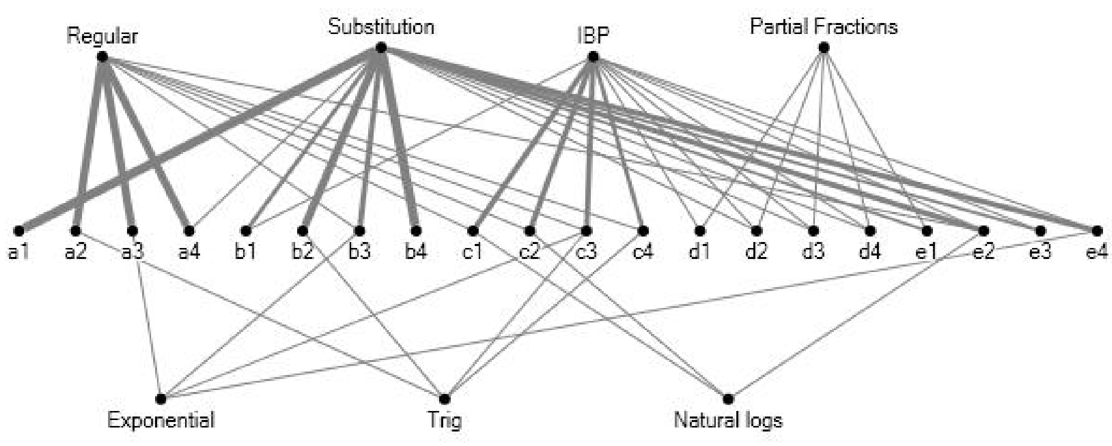


Figure 1. Affiliation network of students' groupings of 20 integration problems

Findings and Implications

In the context of learning techniques of integration in a second semester Calculus course, these new methods provided three insights into learning and raised additional questions about educational research. First, while students overall progressed in their understanding, none of them monotonically improved the number of correctly grouped items while simultaneously reducing the number of incorrectly grouped items, instead seeming to focus on one or the other depending upon the feedback given to the students. (See Figure 2.) Second, the existence of prior concepts that are only a surface feature of the new idea can prove problematic to students, indicating that curriculum sometimes does not attend sufficiently to the limitations of concepts. Third, it appears that students, rather than deriving new understandings directly from existing understandings, instead coordinate components of previous understanding into new ideas on the fly, much as in locomotion hopping is a coordination of the same underlying muscle movements as walking, but is not derived from walking (Thelen & Smith, 1994). In addition to these insights, it appears that commonplace approaches to data collection and analysis may greatly underestimate the variability in the data, and hence overestimate both significance and effect size.

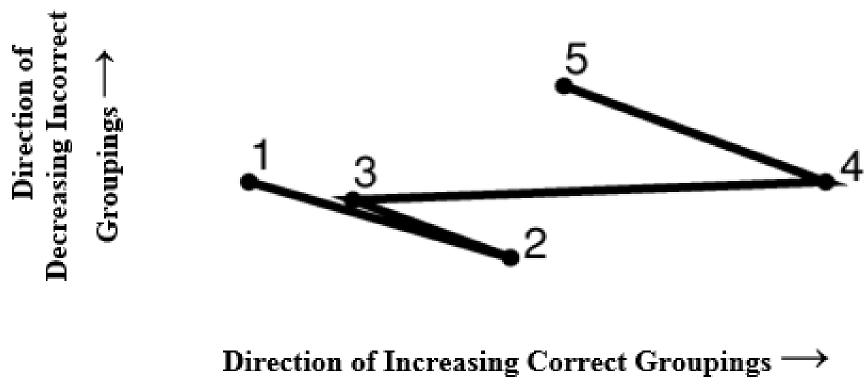


Figure 2. Progression of one student's groupings across five rounds

References

Diebel, K., Anderson, Ruth, & Anderson, Richard (2005). Using edit distance to analyze card sorts. *Expert Systems* 22(3), 129-138.

Koopmans, M. (2014). Nonlinear change and the black box problem in educational research. *Nonlinear Dynamics, Psychology, and Life Sciences*, 18(1), 5-22.

Ricca, B. & Green, K. (in press). Graph theoretic methods for the analysis of data in developing systems. To appear in *Quality and Quantity*. DOI: 10.1007/s11135-014-0089-5

Thelen, E. & Smith, L. (1994). *A dynamic systems approach to the development of cognition and action*. Cambridge, MA: MIT Press.

Vygotsky, L. (1986). *Thought and language (Revised edition)*. Cambridge, MA: MIT Press.

Digging in deep: From instrumental to logical understanding in calculus

Douglas Riley
Birmingham-Southern College

Maria Stadnik
Birmingham-Southern College

Calculus is a foundational sequence in mathematics and many client disciplines, such as physics and engineering. For student success both in mathematics and in these client disciplines, the mathematical background provided must go beyond simple instrumental or procedural skills to a deeper level. For success in higher-level mathematics, students must delve to the level of logical understanding, being able to articulate logical connections between two mathematical concepts. In this study we analyze students' abilities to explain the connection between the limit definitions of derivative and definite integral, and their common geometric interpretations involving slope and area. We also determine whether a group exercise early in the term which reinforces the connection between the derivative and slope enhances students' written responses concerning the connection on the final exam.

Key words: Calculus, Writing, Classroom Research

Skemp (1976; 1979) describes three categories of understanding that students can achieve. The lowest level, "instrumental understanding" or rote memorization, is understanding the rules without comprehending the reasoning behind the rules (Skemp, 1976). Skemp points out that mathematics taught at the level of instrumental understanding is typically easier for a student to comprehend, offers immediate and obvious rewards to the student, and allows the student to obtain answers quickly and reliably. Hiebert and Lefevre (1986) have also used the term "procedural knowledge" for this type of understanding. They describe this as the ability to apply rules, algorithms, or procedures to a formal mathematical language (Hiebert & Lefevre, 1986), typically in a linear manner.

A second, deeper type of understanding Skemp coins "relational understanding" can be viewed as "knowing both what to do and why" (1976). Skemp (1976) points out that relational understanding of a topic has the benefits of making the topic more adaptable to new tasks and easier to remember. Moreover, once students begin to learn at this deeper level, they may find personal satisfaction from obtaining relational knowledge and some may pursue further relational knowledge of other material (Skemp, 1976). Hiebert and Lefevre (1986) have used the term "conceptual knowledge" for this type of understanding, and they point out that students can possess procedural knowledge without truly understanding the underlying concepts, but that they must understand the meaning of the underlying ideas to obtain conceptual knowledge (Porter & Masingila, 2000). This type of understanding is not linear but rather a web or network of knowledge with many kinds of relationships between the ideas.

Skemp (1979) adds a third and still deeper type of understanding, "logical understanding." This can be thought of as understanding a topic well enough to be able to explain it to others (Idris, 2009). Students demonstrate logical understanding when they are able to supply a logical string of inferences as evidence of their understanding (Thomas, 2002). In other words, students are able to communicate a concept by using information given to them along with an appropriate chain of ideas or axioms and arrive at a logical conclusion. Students with logical understanding can prove mathematical statements and influence the understanding of their peers (Idris, 2009).

Several researchers have proposed writing activities as one method to help students to develop deeper levels of understanding in mathematics courses. These writing activities encourage students to develop their own ideas and add to or create meaning from the concepts. The National Council of Teachers of Mathematics (NCTM) has encouraged communication in school-level mathematics since the 1980s (McIntosh & Draper, 2001). In 2000, the NCTM released “Principles and Standards for School Mathematics”, which encouraged school teachers to include writing in mathematics courses in an effort to enhance their mathematical thinking (Stonewater, 2002) as well as their communication skills (McIntosh & Draper, 2001). The Common Core States Standards initiative also includes standards related to constructing “viable arguments” and to justify conclusions and communicate them to others (Common Core, 2010). Barmby, Harries, Higgins & Suggate, (2007) explains, “If we ask students to explain what they are doing in mathematical tasks, then we can try and infer the links that they have made between different mental representations.” Pugalee (1997) adds, “Writing helps build thinking skills for mathematics students as they become accustomed to reflecting and synthesizing as parts of a normal sequence involved in communicating about mathematics.”

This writing theme has also become popular in college calculus courses (Ferrini-Mundy & Graham, 1991) and several studies have been conducted concerning the effects of writing in the college classroom. Qualitative research techniques such as teaching experiments, clinical interviews, and analysis of student errors have been used in areas such as arithmetic, algebra, geometry, and physics, but most research in calculus classrooms is of the large-scale, quantitative variety (Ferrini-Mundy & Graham, 1991). The disadvantage of these large-scale studies is that they do not offer a glimpse into the student learning process. Small-scale classroom research studies in calculus classrooms have been conducted with various interesting results (Stonewater, 2002; Idris, 2009; Porter & Masingila, 2000).

This project continues this theme using writing as a vehicle to help students delve to a deeper level of mathematical understanding, and it also assesses that understanding. We conducted a “classroom research” endeavor (Cross & Steadman, 1996) in entry-level Calculus courses at a small liberal arts college. In particular we ask whether a writing project can enhance students’ logical understanding in calculus. This mirrors a similar query by Porter and Masingila (2000), although they take a different approach to study that question.

Description of the Project

We implemented a classroom research study within the framework of a small, residential, liberal arts college. All students at the institution who took and completed the introductory calculus course in the Spring of 2013 or Fall 2013 are included in the study, a total of 178 students over eight classes (four each term). The same two instructors (the authors) taught all eight sections, and they collaborated on homework, schedule, and tests. The course content from Spring 2013 to Fall 2013 changed little, although the cohort was of course different. The one significant course content change from Spring to Fall was changing the course project from a standard applied optimization project (which was designed to enhance relational understanding) to one involving a comparison of the standard definition of the derivative to a close cousin (which was designed to enhance logical understanding). Our research question was to see whether this change in course content would enhance students’ ability to articulate the connection between the limit definition of the derivative and its geometric interpretation as the slope of the curve on the final exam.

The definition project used in the Fall of 2013 occurred during the first third of the term and was a collaborative group project started in-class and finished outside of class. The product was a group paper which, among other things, required each group of two to three students to explain how the limit definition produced slope for a smooth function. The exercise was designed to be formative, with ample opportunity for peer discussion, small group querying by the instructor, and time for students to investigate verbal explanations of these connections.

The assessment of students' logical understanding of the connection between the definition of the derivative and the slope of the curve occurred on the final exam in an essay format. The specific statement of the essay question was given in advance for both the Spring and Fall terms. Additionally, students were tasked with explaining the connection between the limit definition of the definite integral and its geometric connection to area, again in an essay format provided in advance. The instructors for the course kept the presentation of the material concerning the limit definition of the definite integral and its connection to area as similar as possible from term to term. The assignments from Spring to Fall terms on this topic remained unchanged. The scores on these essays would be used to control for outside differences between cohorts between the Spring and Fall terms.

After the completion of the Fall 2013 term, essays from the final exam from both Spring and Fall terms were copied, randomly numbered, and then mixed by section, instructor and term before being assessed. To each essay we applied a four-point scale rubric along four dimensions: the student's discussion of slope (area), discussion of limits, use of a picture, and demonstration of technical writing. Each author assigned scores along each dimension which essentially followed Skemp's scale of understanding from none demonstrated to instrumental to relational to logical. Each author completed a full assessment and the scores were compared. Of the 712 scores on slope essays (four dimensions times 178 essays), on fewer than twenty did the two authors disagree by two or more levels. These few were reassessed and a consensus score was recorded. There were 220 scores where the authors varied by one level. In this case we essentially "rounded up" to the next level of understanding. So, in particular, if one of the two authors determined an essay demonstrated understanding at the deepest level, then that essay was counted as demonstrating logical understanding.

Results

The authors hoped that the addition of the definition comparison written project in the Fall term would result in the demonstration of a deeper level of logical understanding among these calculus students. Alas the results were not so clean. In fact, one could argue that the addition of the project seems to have reduced the level of performance on this task on the final exam. Along all dimensions the percentage of scores at the logical level of understanding decreased from Spring to Fall term.

Spring 2013 Derivative Essay Data

Level of Understanding	Slope	Limit	Picture	Writing
None demonstrated	19%	37%	1%	10%
Instrumental	14%	20%	37%	40%
Relational	30%	23%	15%	40%
Logical	38%	20%	47%	11%

Fall 2013 Derivative Essay Data

Level of Understanding	Slope	Limit	Picture	Writing
None demonstrated	12%	31%	2%	7%
Instrumental	29%	27%	29%	49%
Relational	23%	31%	41%	34%
Logical	36%	11%	28%	9%

One possible explanation of the decrease in the scores is the differences in the groups of students. In order to try and correct for these differences, a sample of the definite integral essays (approximately one third of the essays) was scored using a similar rubric. The results were mixed, but seem to indicate that the Spring cohort was weaker than the Fall.

Spring 2013 Definite Integral Essay Data (Sample)

Level of Understanding	Area	Limit	Picture	Writing
None demonstrated	9%	41%	3%	9%
Instrumental	41%	28%	44%	63%
Relational	28%	22%	47%	19%
Logical	22%	9%	6%	9%

Fall 2013 Definite Integral Essay Data (Sample)

Level of Understanding	Area	Limit	Picture	Writing
None demonstrated	6%	18%	3%	9%
Instrumental	29%	24%	35%	38%
Relational	44%	35%	35%	35%
Logical	21%	24%	26%	18%

It may be worth noting that the understanding demonstrated along the limit dimension of the definite integral essays was significantly higher in the Fall than the Spring term.

We are forced to conclude that the addition of the writing assignment early in the term did not enhance students' ability to articulate the connection between the limit definition and geometric interpretation in any meaningful way on the final exams. This confirms a similar conclusion drawn by Porter and Masingila (2000).

Discussion

Thomas (2002) and others argue that understanding at the relational level or deeper enhances long-term retention of ideas. Ideally instructors of calculus can provide opportunities for their students to form these long-lasting connections. After the completion of this project, the authors were left with a myriad of questions:

1. Can one expect a group project given early in the term to influence scores on the final exam?
2. Does the effect of giving the essay questions in advance (as was done both terms) and encouraging students to study together swamp any meaningful analysis of the data?
3. Will the skills developed/encouraged by the project translate into higher performance on related tasks? Is the increase in understanding along the limit dimension of the definite integral rubric evidence of this?

Porter and Masingila (2000) suggest that writing tasks did not have a different effect than non-writing activities on students' "procedural ability" and ultimately on their "conceptual understanding." Our results seem to affirm these findings, but more work must be done to verify that the instrument used to measure students' logical understanding is a true measure. Yet the current results do suggest that a group writing project has little long-term effect on learning in the calculus classroom. If validated, this would have implications for the type of effective assignments utilized in our calculus classrooms and potentially at other small liberal arts institutions.

References

Barmby, P., Harries, T., Higgins, S., & Suggate, J. (2007). How can we assess mathematical understanding?. *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematical Education*, *2*, pp. 41-48.

Cross, K.P. & Steadman, M. (1996). *Classroom research: Implementing the scholarship of teaching*. San Francisco: Jossey-Bass.

Ferrini-Mundy, J. & Graham, K.G. (1991). An overview of the calculus reform effort: Issues for learning, teaching, and curriculum development. *The American Mathematical Monthly*, *98(7)*, pp. 627-635.

Hiebert, J. & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: an introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: the case of mathematics* (pp.1-27). Hillsdale, NJ: Lawrence Erlbaum Associates.

Idris, N. (2009, February). Enhancing students' understanding in calculus through writing. *International Electronic Journal of Mathematics Education, 4(1)*, pp. 36-55.

McIntosh, M.E. & Draper, R.J. (2001, October). Using learning logs in mathematics: writing to learn. *Mathematics Teacher, 94(7)*, pp. 554-555.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics*. Washington, DC: Authors.

Porter, M. & Masingila, J. (2000). Examining the effects of writing on conceptual and procedural knowledge in calculus. *Educational Studies in Mathematics, 42(2)*, pp. 175-177.

Pugalee, D.K. (1997, April). Connecting writing to the mathematics curriculum. *Mathematics Teacher, 90(4)*, pp. 308-310.

Skemp, R. R. (1976/2006). Relational understanding and instrumental understanding. *Mathematics Teaching in the Middle School*, 12(2), 88–95. Originally published in *Mathematics Teaching*.

Skemp, R. R. (1979). *Intelligence, learning, and action*. New York:Wiley.

Stonewater, J.K. (2002, November). The mathematics writer's checklist: The development of a preliminary assessment tool for writing in mathematics. *School Science and Mathematics 102(7)*, pp. 324-334.

Thomas, M. O. J. (2002). Versatile thinking in mathematics. In D. O. Tall, & M. O. J. Thomas (Eds.), *Intelligence, learning and understanding in mathematics* (pp. 179–204). Flaxton, Queensland, Australia: Post Pressed.

Undergraduate students' constructions of existence proofs

Kyeong Hah Roh
Arizona State University

Yong Hah Lee
Ewha Womans University

The purpose of this study is to explore students' activities while they construct existence proofs. We focus on three undergraduate students who completed a transition-to-proof course, and analyze their constructions of existence proofs. The results indicate that students' activities for existence proofs were associated with their interpretations of a given statement, strategic knowledge for their proofs, and proving frames recruited. In addition, we discuss how the students' conceptions of proof also play a role in their construction of existence proofs.

Key words: Existence Proof, strategic knowledge for proof, proving frame, conception of proof, algebra of inequalities

Introduction

Existence proof comprises a very important portion of mathematics and it often makes mathematics distinct from other scientific inquiries. For instance, when asked to find a solution to a problem, one might immediately search for a solution without questioning about its existence. However, if the problem does not have a solution, is it meaningful to look for a solution? Indeed, the existence of a solution needs to be considered prior to finding a solution. On the other hand, existence proof is often significant in mathematics even in the case that we cannot formulate explicitly a way to construct a solution to the problem. Proofs of the mean value theorem, the intermediate value theorem, and the minimum-maximum value theorem are such examples in calculus. Although student conceptions of proof and student difficulties with proof in general have been discussed (e.g., Moore, 1994; Harel & Sowder, 1998; Selden & Selden, 2008), there are little empirical studies about students' construction of existence proofs (cf., Barkai et al., 2002; Epp, 2009). In this paper we address the following research question: How do students construct a proof for existence and what are the challenges or difficulties that they encounter to existence proofs? We present our findings collected by observation of three undergraduate students' activities while they were proving the statement (*): There exists a real number a such that for all b with $b>a$, $(3b+1)/(b+5)>2$.

Theoretical Framework and Background Literature

Our theoretical perspective to explore students' existence proofs is aligned with radical constructivism (von Glasersfeld, 1995). From this perspective, we assume that each student would have actively built up his or her own knowledge on existence proof which might differ from others. In particular, our analysis centers on the following three aspects of individual students' knowledge associated with existence proofs: (1) Interpretations of the given statement; (2) strategic knowledge for proof; and (3) proving frames recruited.

A student would prove or disprove a given statement based on his or her *interpretation* of a given statement about existence. However, students' interpretations of a statement are often different from those of mathematicians (e.g., Dawkins & Roh, 2011; Dubinsky & Yiparaki, 2000; Roh & Lee, 2011; Selden & Selden, 1995). In line with the standpoint, we consider a student's interpretation of a given statement as a crucial aspect of the student's construction of existence proofs.

We also employ Weber's (2001) notion of *strategic knowledge* for proof as knowledge chosen by a student in proving a statement. In the case of the statement (*), one might plug in specific values for a and b to check if $b>a$, then $(3b+1)/(b+5)>2$. Plugging in numbers this way is not a legitimate way to prove the statement (*), but can be considered as an initial step in proving it. We thus consider 'plugging in numbers' as the student's strategic knowledge. Another strategic knowledge might be found when 'working backwards' from the inequality

($3b+1)/(b+5) > 2$. Such a process would produce a series of implications (e.g., $(3b+1)/(b+5) > 2 \Rightarrow 3b+1 > 2(b+5) \Rightarrow b > 9$). In fact, by working backwards, the student's proof is not for the conditional 'if $b>a$, then $(3b+1)/(b+5) > 2$' but for its converse, which is problematic. In addition, the first implication in the series is not valid unless $b+5>0$. Nonetheless a student might work backwards to prove the statement (*), in which case we identify 'working backwards' as the student's strategic knowledge. Likewise, our notion of strategic knowledge includes not only mathematically precise definitions and theorems but also a student's informal or heuristic knowledge of mathematical concepts.

By a *proving frame* we refer to an outline or a structure of a proof (Zandieh, Roh, & Knapp, 2014), similar to what Selden and Selden (2009) call the formal-rhetorical part of a proof. There would be at least three different proving frames that a student might recruit for his or her existence proof for a statement in the form of $\exists x(p(x))$: (1) *Constructive proving frame* - construct or create a specific example for x and then verify such a chosen x satisfies $p(x)$; (2) *referential proving frame* - refer to a previous theorem or property to imply the existence of x without constructing a specific x that satisfies $p(x)$; and (3) *proof by contradiction frame* - assume it is not the case that there exists an x that satisfies $p(x)$, and then generate a series of implications to derive a contradiction. Again, we accept students' proving frames even if they may not be legitimate ways of structuring existence proofs.

Research Methodology

This study was conducted at a large public university in 2013. Three undergraduate students, who will be called Dawn, Susan, and Peter in this paper, participated in this study as a voluntary extra-curricular activity. All three students were majoring in mathematics, and had already completed calculus and a transition-to-proof course. During the semester when the students participated in the study, they were taking at least two upper division math courses among geometry, real analysis, and abstract algebra. Data for this study were collected from a half-hour survey and 60-90 minute task-based exploratory interviews. The *survey* consisted of multiple choice and proof writing problems. The multiple choice problems were designed to examine students' understanding of existential quantification. The statement (*) in the introduction section was given to the students as one of the proof writing problems. The main task of the follow-up *interviews* was then to ask the students the following three questions about their proofs for the statement (*): (1) What was your goal in proving the statement (*)? (2) What were the key mathematical ideas that you used in your proof? (3) How did you structure your proof? Students' utterances were then compared with their written proofs and used in our analysis as evidence to support our conjecture on their interpretations of the statement (*), strategic knowledge, and proving frames. All interview sessions were video-taped and the students' written work during the interviews were synchronized with their voice.

Results and Analysis

All students determined the statement (*) to be true, and constructed their proofs of the statement (*). However, their written proofs were to some extent different from each other. In addition, their reasoning behind their written proofs often included different invalid arguments. See Figures 1, 2, and 3 for Dawn's, Peter's, and Susan's written proofs, respectively.

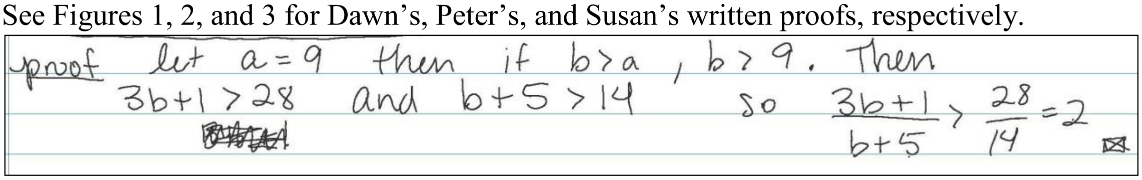
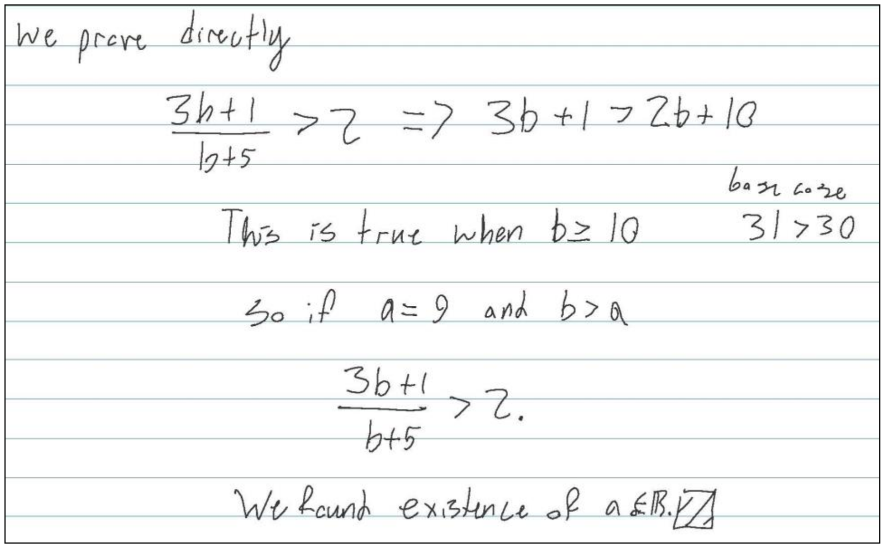


Figure 1 Dawn's proof of the statement (*)



We prove directly

$\frac{3b+1}{b+5} > 2 \Rightarrow 3b+1 > 2b+10$

base case

This is true when $b \geq 10$ $\quad 31 > 30$

So if $a = 9$ and $b > a$

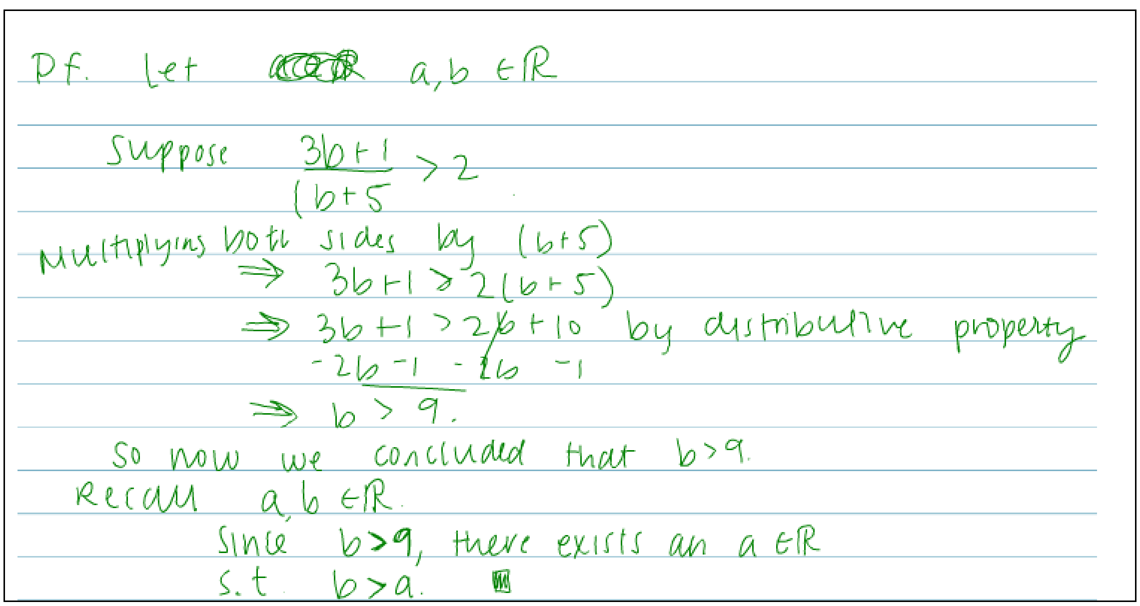
$\frac{3b+1}{b+5} > 2.$

We found existence of $a \in \mathbb{R}$. $\boxed{/}$

Figure 2 Peter's proof of the statement (*)

Interpretations of the given statement

When analyzing the students' interpretations of the statement (*), one of our foci was on their understanding of the relationship between *a* and *b*. Dawn and Peter properly described *a* to be chosen independently from *b*. On the other hand, Susan considered *a* to be dependent on *b*. For instance, she interpreted '$\exists a \in$R such that $\forall b \in$R with $b>a$' in the statement (*) as '$\forall b \in$R, $\exists a \in$R such that $b>a$' (see Figure 3 Susan's proof of the statement (*)). In addition, she often read '$b>a$' not as '*b* is greater than *a*' but as '*a* is less than *b*,' which shows her thinking of *a* in terms of its dependence on *b*. Hence we found Susan's case supports what Dubinsky and Yiparaki (2000) described as students' tendency to interpret EA statements as AE statements.



Pf. Let $a, b \in \mathbb{R}$

Suppose $\frac{3b+1}{(b+5} > 2$

Multiplying both sides by $(b+5)$

$\Rightarrow 3b+1 > 2(b+5)$

$\Rightarrow 3b+1 > 2b+10$ by distributive property

$-2b-1 \quad -2b \quad -1$

$\Rightarrow b > 9.$

So now we concluded that $b>9$.

Recall $a, b \in \mathbb{R}$.

Since $b>9$, there exists an $a \in \mathbb{R}$

s.t $b > a$. $\blacksquare$

Figure 3 Susan's proof of the statement (*)

Strategic knowledge for proof

We examined the students' strategic knowledge for proving the statement (*) by analyzing (1) how they knew of the existence of such an *a*; and (2) how they justified the existence of such an *a*. Regarding the question (1), Dawn *plugged in numbers* 1, 2, 3, 6, 7, 9, 10, and 11 for

b and checked that when b=9, 3b+1 = 2(b+5), and if b=10 or b=11, 3b+1 >2(b+5). Based on the values of b satisfying the inequality, she then concluded that a must exist. On the other hand, Susan and Peter *worked backwards* to determine the existence of a. They started from the inequality (3b+1)/(b+5) >2, implied 3b+1>2(b+5), and concluded that b>9 and hence a must exist. Concerning the question (2), the students recruited mainly their *knowledge of algebra* for justifications. However, we found that their algebraic knowledge for inequalities were often problematic. Dawn drew comparisons between numerators and denominators of (3b+1)/(b+5) and 28/14, and concluded that since 3b+1 >28 and b+5 > 14, (3b+1)/(b+5) > 28/14. In fact, she believed it is always true that if $x_1>y_1$ and $x_2>y_2$, then $x_1/x_2>y_1/y_2$. Susan and Peter justified 'if (3b+1)/(b+5)>2, then b>9' which is in fact the converse of the conditional in the statement (*). In particular, they *supposed* (3b+1)/(b+5)>2 instead of verifying the inequality. Susan explained it because the inequality was given in the statement (*): "*I picked the inequality (3b+1)/(b+5)>2 because that was the one that was given. […] Because it's just given to us, so I'm saying 'suppose.' […] Because they give it to you, so we have no reason to believe that it's not true, and then [I] continue to work on it from there*." Indeed, when saying "given" she referred to the statement (*) that we gave to her whereas the word "given" is often used in proof texts/courses to assume the premise or hypothesis of a conditional statement. Susan seemed to adopt such mathematical convention into her proof by supposing what she thought of as given. However, because of the incompatibility between her meaning of 'given' and its mathematical convention, she came to suppose improperly the inequality (3b+1)/(b+5)>2. In fact, Susan and Peter's intention of working backwards was not to find under which condition the inequality (3b+1)/(b+5)>2 holds; but it was rather to find what (3b+1)/(b+5)>2 would entail. In addition, they multiplied b+5 to the both sides of the inequality (3b+1)/(b+5)>2 without justifying why b+5 is always positive. They rather treated the conditional statement 'if (3b+1)/(b+5)>2, then (3b+1)>2(b+5)' as if it must be true regardless of the value of b.

Proving frames recruited

We identified Dawn and Peter's proving frame recruited for their proofs of the statement (*) as a *constructive proving frame* because they both stated 'a = 9' for their choice of a value for a (Figures 1 & 2). On the other hand, Susan was unsure if choosing an example for a (i.e. constructive proving frame) is a legitimate way to prove the statement (*). Pointing to the phrase 'let a, b' in the first sentence of her proof (Figure 3), Susan explained that she used the word 'let' to express her consideration of a and b as 'random, 'arbitrary', or 'all' cases: "*I'm just trying to show that it's arbitrary. So we are choosing any two random a and b. I'm not saying okay if a=5 and b=7, then this holds true. […] It gives you an example, but it doesn't prove that for all cases, it holds true. […] So 'let' is just my way of saying that it's arbitrarily chosen*." Here, we identified Susan's proving frame recruited as a *generalization proving frame,* by which we mean a proving frame recruited to prove $p(x)$ holds true not for a particular x but for arbitrary x. In addition to a generalization proving frame, Susan also recruited a *referential proving frame*, i.e., she did not construct a specific example of a, but concluded the existence of a by referring to properties of real numbers. However, Susan neither explained exactly how her reference would work nor provided any warrant to her reference.

Concluding Remarks

Table 1 summarizes the main findings of this study in terms of the three aspects discussed in the theoretical framework, with some cells shaded in grey to highlight critical issues on the students' existence proofs. These findings indicate that it is not easy to construct existence proofs properly even for the students who completed a transition-to-proof course.

We also found that the three students had different conceptions of proof and these conceptions were related to the students' activities for proving existence. Susan's rejection of a constructive proving frame for existence proofs was based on her conception that a proof should deal with all cases. Susan might have built up such a conception of proof from her experience with proving statements involving a universal quantifier (e.g., $\forall x\ (p(x))$). On the other hand, Peter believed that it is enough as an existence proof to demonstrate how to find a value for the existence. Since his proof described how he found a value for a in his proof, he did not see any more detailed justifications to be added to his proof. Contrasting to Peter, Dawn insisted that an existence proof does not need to explain how to find such a value for that value to exist. Dawn hence resisted to include her scratch work of plugging in numbers to her proof. An implication of these results is that our current instruction might not have enough emphasis on existence proofs and our students would need more guidance to understand what constitutes a valid proof for existence.

Table 1 Summary of students' proving activities in terms of the theoretical framework

	Interpretation of (*)	**Strategic knowledge**	**Proving frame**
Dawn	EA statement	Plugging in numbers	Constructive proving frame
		Algebra of inequalities	
Susan	AE statement	Working backwards	Generalization proving frame
		Algebra of inequalities	Referential proving frame
Peter	EA statement	Working backwards	Constructive proving frame
		Algebra of inequalities	

References

Barkai, R., Tsamir, P., Tirosh, D., & Dreyfus, T. (2002). Proving or refuting arithmetic claims: The case of elementary school teachers. *Proceedings of the 26th International Group of Psychology of Mathematics Education*, Vol. 2 (pp. 57-64).

Dawkins, P., & Roh, K. (2011). Mechanisms for scientific debate in real analysis classrooms. In L. Wiest & T. Lamberg (Eds.), *Proceedings of the 33rd annual conference of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 820-828), Reno, NV: University of Nevada, Reno.

Dubinsky, E., & Yiparaki, (2000). On student understanding of AE and EA quantification. *Research in Collegiate Mathematics Education,* IV, 239-289.

Epp, S. (2009). Proof issues with existential quantification. In F.-L Lin, F.-J. Hsieh, G. Hanna, & M. de Villiers (Eds.), *Proceedings of ICMT Study 19: Proof and proving in mathematics education,* Vol. 1 (pp. 154-159), Taipei, Taiwan.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in Collegiate Mathematics Education III* (pp. 234-282). Providence, RI: American Mathematical Society.

Moore, R. (1994). Making the transition to formal proof. *Educational Studies in Mathematics, 29,* 123-151.

Roh, K., & Lee, Y. (2011). The Mayan activity: A way of teaching multiple quantifications in logical contexts. *Problems, Resources, and Issues in Mathematics Undergraduate Studies*, *21*(*8*), 1-14.

Selden, A., & Selden, J. (2008). Overcoming students' difficulties in learning to understand and construct proofs. In M. Carlson & C. Rasmussen (Eds.), *MAA notes*. *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 95-110). Washington, DC: MAA.

Selden, A., & Selden, J. (2009). Understanding the proof construction process. In F.-L Lin, F.-J. Hsieh, G. Hanna, & M. de Villiers (Eds.), *Proceedings of ICMT Study 19: Proof and proving in mathematics education,* Vol. 2 (pp. 196-201), Taipei, Taiwan.

Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics, 29,* 123-151.

Von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. London, England: Falmer Press.
Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics, 20,* 356-366.
Zandieh, M., Roh, K., & Knapp, J. (2014). Conceptual blending: Student reasoning when proving "conditional implies conditional" statements. *Journal of Mathematical Behavior, 33*¸ 209-229.

An extended theoretical framework for the concept of derivative

David Roundy	Tevian Dray	Corinne A. Manogue	Joseph F. Wagner	Eric Weber
Oregon State University	Oregon State University	Oregon State University	Xavier University	Oregon State University

This paper extends the theoretical framework for exploring student understanding of the concept of the derivative, which was developed by Zandieh (2000). We expand upon the concept of a physical representation for the derivative by extending Zandieh's map of the territory to provide higher resolution in regions that are of interest to those operating in a physical context. We also introduce the idea of "thick" derivatives, which are ratios of small *but not* infinitesimal *changes, which are practically equivalent to the true derivative.*

Key words: derivative, theoretical framework, physical, experiment

In this theoretical report we extend the theoretical framework for exploring student understanding of the concept of the derivative which was developed by Zandieh (2000). We expand upon the concept of a "physical" representation for the derivative. As with Zandieh's original framework, this work is not meant to explain how or why students learn as they do, nor to propose a learning trajectory. Rather, this work extends Zandieh's "map of the territory," to provide higher resolution in regions that are of interest to those working with derivatives in a physical context. In addition to focusing on the physical context, we discuss challenges that have arisen in applying Zandieh's framework to an understanding of the derivative beyond the level of first-year calculus.

This work is motivated by preliminary results of a project to study understanding of the derivative across STEM fields (Roundy, Weber, Sherer & Manogue 2014b). In the process of interviewing physicists and engineers, we have identified shortcomings that arise when applying Zandieh's framework beyond the level of first-year calculus, and in particular outside the field of mathematics. We have found that the concept image for the derivative of physicists and engineers contains substantial elements that are congruent with the three process-object layers identified by Zandieh, but lead to the introduction of new contexts and representations that could also be productive in the instruction of calculus.

Physicists and engineers live and work in a world full of uncertainty, and are accustomed to use the language of equality where there is actually approximation. This language reflects a somewhat "thicker" concept of the derivative than that held by mathematicians. Where a mathematician would speak of the slope of the secant line as an approximation for the derivative, a physicist or engineer might say that the slope of a line drawn between two carefully chosen measurements of a physical observable *is* the derivative (with some unspecified uncertainty). As we will explain, this "thickness" derives from the impossibility of achieving exact results in physical or numerical contexts. Attempts to estimate a derivative over too small an interval, for example, could result in a highly erroneous estimate of a derivative due to numerical round-off error or limitations in experimental precision.

Theoretical Background

Concept Image

In this work, we extend the theoretical framework of Zandieh (2000), which itself draws on the idea of concept image (Vinner, 1983). Vinner (1983) describes the concept image as the set of properties associated with a concept together with mental pictures of the concept.

Thompson (2013) argues that the development of coherent meanings is at the heart of the mathematics that we want teachers to teach and what we want students to learn. He argued that meanings reside in the minds of the person producing them and the person interpreting them.

Zandieh's framework for the concept of the derivative

Zandieh (2000) introduced a framework for the concept of the derivative, aimed at mapping student concept images at the level of first-year calculus. This framework maps out the *correct* concepts as understood by the mathematical community, and thus does not incorporate incorrect understandings. We reproduce in Fig. 1 below Zandieh's outline of her framework. This table consists of columns corresponding to *representations* or *contexts*, and rows corresponding to *process-object layers*. The *process-object* framework is taken from Sfard (1991), who conceives of mathematics as proceeding through processes acting on objects, with those processes then becoming reified into objects.

Process-object layer	Graphical	Verbal	Physical	Symbolic	Other
	Slope	Rate	Velocity	Difference Quotient	
Ratio					
Limit					
Function					

Figure 1: Zandieh's outline of the framework for the concept of the derivative.

Representations

Each of the representations in Zandieh's table can be used to convey the concepts behind the three process-object layers. She also likens these columns to "contexts" in the sense that each of these provides a context within which we can think about the derivative. In the paragraphs below, we give a brief summary of each position in Fig. 1.

Graphical. The graphical representation of the derivative is slope. At the ratio layer, this is the slope of a secant line between two points on the curve describing a function. When taking the limit, we arrive at the slope of the tangent line at a point. Finally, considering the derivative as a function requires us to recognize that the slope is different for different values of the independent variable.

Verbal. The verbal representation for the derivative discussed by Zandieh is the "rate of change." At the ratio layer, this is expressed as an "average rate of change." When taking the limit, this becomes the "instantaneous rate of change." Understanding this verbal description as a function requires us to visualize the instantaneous rate of change for the inputs over the domain of the function.

Physical. The physical representation, or *paradigmatic physical* representation is velocity: average velocity, instantaneous velocity, and the velocity as a function of time. These physical concepts provide a language that we can use to understand the derivative: a large derivative means "faster" and a varying derivative means there is acceleration going on.

Symbolic. The symbolic representation of the derivative is the formal definition of the derivative in terms of the limit of a difference quotient. In this case, the distinction between the limit layer and the function layer can be subtle. They differ in the recognition that the variable describing the point at which the limit is taken can be treated as the argument of a function. Zandieh expresses this with a notational distinction between x_0 and x.

Other. Finally, we point out that Zandieh explicitly placed in her framework space for additional contexts. In particular, when discussing the physical context, she mentioned that there is a wide set of physical contexts for understanding the derivative. In this paper, we will discuss some of the subtleties we have encountered in investigating understanding of the derivative within the context of a mechanical system (Sherer, Kustusch, Manogue & Roundy 2013, Roundy et al. 2014b).

Extensions to Zandieh's framework

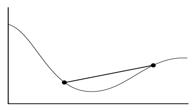
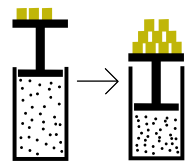
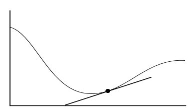
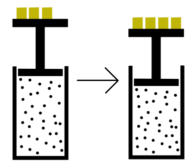
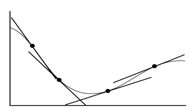
Likwambe and Christiansen (2008) extend Zandieh's framework in three ways. Firstly, they recognize the importance in a concept image that we be able to make connections between different representations, and extend the use of the table to include arrows indicating that a student has made a connection between two representations or ideas. Secondly, they add a "non-layer" row, which indicates a recognition or use of that representation of the derivative without indication of an understanding of any of the three process-objects layers. Finally, Likwambe and Christiansen (2008) added a separate category for what they refer to as *instrumental understanding*, a term taken from Skemp (1978). Instrumental understanding (as opposed to *relational understanding* refers to the knowledge of and ability to follow a procedure. Both Skemp (1978) and Lithner (2003) point out that instrumental understanding is commonly emphasized in both homework assignments and exams. Zandieh explicitly omits instrumental understanding from her framework, but Likwambe and Christiansen (2008) add an additional box for instrumental understanding, in order to include "the only learning exhibited by most of the interviewees."

Extending Zandieh's Framework for the Derivative

In our research on expert understanding of the derivative across disciplines, we have encountered several issues that led us to an extension of Zandieh's framework for the derivative, with a particular focus on physical contexts. We propose a deeper understanding of the "physical" representation, and add an additional "numerical" representation, which fills out the *Rule of Four*: graphical, verbal, symbolic and numerical (Hughes-Hallett et al., 1998). In addition, we follow Likwambe and Christiansen (2008) in adding an *instrumental understanding* category that lives outside the three process-object layers.

Figure 2 shows our framework for the concept of the derivative. This figure is modeled after Fig. 1, the framework of Zandieh, and is best understood in terms of the differences between these two frameworks. We have added one additional column labeled *numerical* (and removed the *Other* column to make space). We have added the *instrumental understanding* of Likwambe and Christiansen (2008) (which is to say, the rules of differentiation) as an entirely separate table, partially to reflect its weak connection to any other aspect of the concept of the derivative.

Finally, we have added into each entry of the table (which Zandieh left blank) an iconic description of the concept meant by that entry. These entries are intended to aide in understanding the table by compactly describing the conception of the derivative indicated by that combination of row and column.

Process-object layer	Graphical	Verbal	Symbolic	Numerical	Physical
	Slope	Rate of Change	Difference Quotient	Ratio of Changes	Measurement
Ratio		"average rate of change"	$\frac{f(x+\Delta x)-f(x)}{\Delta x}$	$\frac{1.00-0.84}{1.5-1.0}$	
Limit		"instantaneous …"	$\lim\limits_{x\to 0} \cdots$	$\frac{0.89-0.84}{1.1-1.0}$	
Function		"… at any point/time"	$f'(x)=\cdots$	x | y | $\frac{dy}{dx}$ 0.0 | 0.00 | 0.96 0.5 | 0.48 | 0.72 1.0 | 0.84 | 0.32 1.5 | 1.00 | -.18	*tedious repetition*

	Symbolic
	Instrumental Understanding
Function	*rules to "take a derivative"*

Figure 2: Our extended framework for the concept of the derivative.

Changes in the framework

In this section, we discuss individually the extensions we have made to Zandieh's framework.

Physical. We begin by noting that the physical examples given by Zandieh (2000) each involve a time derivative: velocity, acceleration, and the time rate of change of temperature. We suggest that although these quantities do reside in a physical context, perhaps at least *some* uses of these phrases properly belong in the realm of verbal representation. We propose here a more "physical" (as opposed to verbal) concept of the physical representation of the derivative.

We define the physical representation for the derivative to be a process to *measure* that derivative (see, for instance Roundy, Kustusch, & Manogue, 2014a; Styer, 1999). Of course, the *concept* does not require us to actually perform a measurement, just to imagine one. However, we note that it is the *process of measurement* itself that is the physical representation. Actually obtaining a numerical measurement would (also) require the use of the numerical representation, and describing the measurement may involve a verbal or graphical representation (Roundy et al., 2014a; Styer, 1999), but the measurement process itself is the physical representation of the concept of the derivative.

As an example, consider the derivative dV/dp of the volume of a piston full of air with respect to the pressure on the piston, as controlled by a set of weights on the piston (illustrated in Fig. 2). At the ratio layer, one can say that you need to measure the volume twice, with two different pressures, and the derivative is the change in volume divided by the change in pressure. The limit layer imposes on this process the idea that the two pressures need to be quite similar in order for this ratio to "be" the derivative in the thick sense used by physicists and engineers. However, it is not desirable to choose *too* small a value for Δp, because this would result in an imprecise measurement, since the change in volume would be too small to be precisely measured, resulting in increased error in the value of the measured derivative. Finally, the function layer requires us to recognize that this ratio will depend on the pressure itself and that to fully explore the derivative, we must perform repeated experiments—or more likely a single experiment in which we gradually add weight to the piston and repeatedly measure its volume.

The physical representation of a derivative can often (but not always) be felt or perceived directly, which leads scientists to give derivatives names such as compressibility, velocity, thermal conductivity, etc. Qualitatively, the derivative dV/dp describes the compressibility of the air: how easy it is to compress. We anticipate that as the piston is compressed at higher pressures, it will require more and more pressure to compress it further. Because the volume cannot be negative, we can conclude on physical grounds that the derivative must eventually approach zero as the pressure increases. .

Numerical. The *numerical* representation is the one member of the Rule of Four (Hughes-Hallett et al., 1998) that was not present in the framework of Zandieh (2000). We recognize a numerical representation of the derivative that is closely allied to but distinct from the physical representation. This representation parallels the formal symbolic concept of the derivative, but differs in ways that are of practical importance in the use of the derivative in the sciences and in numerical analysis.

The numerical concept of the derivative begins with a ratio of change:

$$\frac{y_2 - y_1}{x_2 - x_1},$$

where it is understood that the values in this equation are numerical values. When we take the limit numerically, we do not formally write $\lim_{\Delta x \to 0} \cdots$, and we do not apply a formal procedure. Rather we select a value of Δx that is *small*, where small is understood in terms of the desired precision. As in the case of physical measurements, practically speaking it is possible to make the change Δx *too* small, in this case due to truncation error in a computer or calculator. In this regard, when operating numerically we think of derivatives as having some "thickness," in contrast to the formal definition which requires an infinitesimal limit. Finally, the derivative as function is understood as a sequence of numerical ratios of differences, just as a function can be understood numerically as an array of numbers or set of ordered pairs.

Conclusions

We have extended the framework of Zandieh (2000) in several ways: we have elaborated on the physical representation of the derivative; we have added a numerical representation of the derivative; and we have added space in the framework for the set of rules for finding symbolic derivatives. Each of these changes reflects an expansion of the table to incorporate additional answers to the prompt, "find the derivative." By making use of the numerical representation of the derivative, one can answer the prompt numerically. Similarly, if the derivative is situated in a physical context, one can respond with a measurement process. Both of these responses require a conceptual understanding of the derivative in terms of ratio, limit and function, and involve a certain "thickness" in the derivative. In contrast, as pointed out by Zandieh, the instrumental-understanding approach to "find the derivative" using the rules for symbolic derivatives does *not* require a conceptual understanding of the derivative.

References

Hughes-Hallett, D., Flath, D., Gleason, A., Gordon, S., Lock, P., Lomen, D., . . . others (1998). *Calculus: Single variable* (2nd ed.). John Wiley & Sons Australia, Limited.

Likwambe, B., & Christiansen, I. M. (2008). A case study of the development of in-service teachers' concept images of the derivative. *Pythagoras*(68), 22–31.

Lithner, J. (2003). Students' mathematical reasoning in university textbook exercises. *Educational studies in mathematics*, 52(1), 29–55.

Roundy, D., Kustusch, M. B., & Manogue, C. (2014a). Name the experiment! interpreting thermodynamic derivatives as thought experiments. *American Journal of Physics*, 82(1), 39–46.

Roundy, D., Weber, E., Sherer, G., & Manogue, C. A. (2014b). Experts' Understanding of Partial Derivatives Using the Partial Derivative Machine, *2014 Physics Education Research Conference Proceedings*, in press.

Sherer, G, Kustusch, M. B., Manogue, C. A., & Roundy, D. (2013). The Partial Derivative Machine. *2013 Physics Education Research Conference Proceedings*, 341-344.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational studies in mathematics*, 22(1), 1–36.

Skemp, R. R. (1978). Relational understanding and instrumental understanding. The *Arithmetic Teacher*, 9–15.

Styer, D. F. (1999). A thermodynamic derivative means an experiment. *American Journal of Physics*, 67(12), 1094–1095.

Thompson, P. W. (2013). In the absence of meaning. In K. Leatham (Ed.), *Vital directions for research in mathematics education*. New York, NY: Springer.

Vinner, S. (1983). Concept definition, concept image and the notion of function. International *Journal of Mathematical Education in Science and Technology*, 14, 293-305.

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. *CBMS Issues in Mathematics Education*, 8, 103–122.

The transition from AP to college calculus: Students' perceptions of factors for success

Authors
Affiliation

This study examines similarities and differences in the Advanced Placement and college calculus experience from the student perspective to characterize how taking AP Calculus in high school relates to student success in a college calculus course. Fourteen first-semester college students who had taken the AP Calculus exam were interviewed about their perceptions of, and experiences in the courses. The Academic Performance Determinants Model (Credé & Kuncel, 2008) was used to develop an interview protocol. Qualitative analysis of the interviews revealed four categories of themes about the students' experience: 1) Students' study approaches in the respective classes, 2) Students' self-efficacy and metacognition, 3) The Class format's effect on student success, and 4) Students' beliefs about the cognitive demand of the course. All the themes, their implications for those teaching in AP and college calculus, and the need for further research are presented..

Key words: Calculus, Student success, Advanced Placement

Introduction

It is estimated that over 300,000 students are entering college each year having taken some form of calculus in high school but not entering with credit for Calculus I (Bressoud, 2009). With so many students studying calculus for the second time in college, it is important to determine how the high school experience relates to their performance in the college course. This report will focus on the first part of a larger study that explored the differences in Advanced Placement (AP) and college calculus to better understand the potential impact that taking AP Calculus might have on students who repeat the course in college. This part of the study used semi-structured interviews to examine the relationship between high school and college calculus from the student perspective. The following research questions were addressed: 1. What factors affect student success in calculus courses? 2. How do these factors differ in AP and college calculus?

Review of Literature and Framework

The College Board reported in 2008 that students who repeated Calculus I in college after taking AP Calculus in high school and not receiving credit on the AP exam actually underperformed in Calculus II, in comparison to students who had not taken AP Calculus (Keng & Dodd, 2008). This result, along with the growing number of students in this situation, requires examining how the AP Calculus experience may be affecting this large cohort of students that are retaking calculus I after taking AP Calculus. Bressoud asserts that "we need to know more about the preparation of the students who take calculus in college and what they need in order to succeed once they get to our classes" (2009, p. 23).

Klopfenstein and Thomas (2006) distinguish between college level and college preparatory courses, defining college level courses as those that teach the same material as college courses and college preparatory courses as those that develop skill sets students will need to be successful. These authors suggest that many AP courses are taught as college level courses when they should be college preparatory. The number of contact hours in most AP courses is often 2-3 times that of a typical college course, allowing for more discussion, practice, and exposure to specific types of problems under teachers' supervision. While this gives students an advantage in learning to solve those particular problems, it may also decrease the cognitive demand of the course for students and/or decrease the amount of effort required by the student outside the classroom, in comparison to the college calculus experience (Hong 2009). Hourigan (2007) labels this loss of cognitive demand a "reductionist orientation" and provides evidence that secondary school teachers are much more likely than college instructors to adopt this approach (p. 470).

Students may develop poor study habits or not develop positive ones as a result of this experience. Not surprisingly, study skills, attitudes, habits, and motivation are strongly correlated with success in college, but "exhibit near-zero relationships" to performance in high school (Credé & Kuncel, 2008, p. 442). The results of Anthony's (2000) study clearly indicate a discrepancy in the amount of practice that is expected from instructors and the amount that students believe is necessary. "Insufficient work" was the number one reason instructors gave for failure, while this was ranked 18th by students. 55% of the students surveyed indicated they studied less than 4 hours per week for their mathematics course, which falls far short of the expectation set by most college mathematics departments. This may be due in part to students' high confidence levels. In a study of first-semester college Calculus I students, Bressoud, Carlson, Mesa, and Rasmussen (2010) found that 95% of first-year students in Calculus I enter the course confident they have the ability to succeed. However, the percentage of students that achieve this goal is far lower.

Credé & Kuncel (2008) assert that study skills, habits, attitudes, and motivation are essential factors in student success in college. They propose a model of Academic Performance Determinants that shows that some performance factors affect others, and some are more closely related to success than others. While this model is not specific to mathematics and cannot account for all factors of success in mathematics courses, it provides a lens for examining students' experiences in calculus and interpreting the challenges they face. This model was used to help create the interview protocol discussed below.

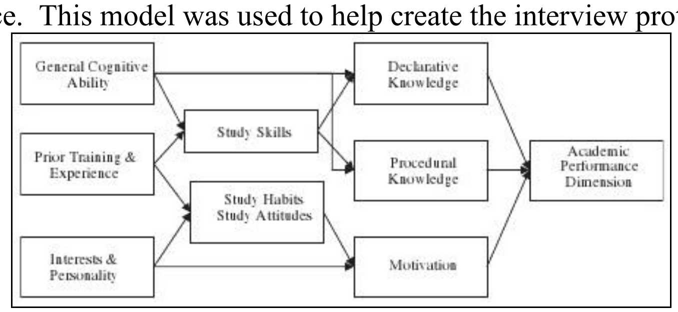


Figure 1. Academic Performance Determinants Model from Credé & Kuncel (2008)

Research Methodology

The fourteen interview participants were college calculus students who were repeating Calculus I after taking AP Calculus in high school. Participants were recruited with the help of instructors at six different universities in the southeastern United States. Two of these were small private universities and four were large public universities. Students were recruited anywhere 8-14 weeks into the 16 week fall 2012 semester. To participate, a student must have taken the AP exam and also must have made a low C or worse on at least one of their recent in-class tests in their college course. Some participants had passed the AP Calculus exam (earned a 3 or 4 out of 5, depending on their institution's standards) and were choosing to repeat the course, while others had not received credit for AP Calculus.

Institution Type	Student Participant	Passed AP test?	Passing Course?
Large Public 1	Jeremy	No	Barely
Large Public 1	Allen	No	Barely
Large Public 1	Frank	No	No
Large Public 1	Michael	No	No
Small Private 1	Katelynn	No	Barely
Small Private 1	Maggie	No	Barely
Small Private 1	Isaac	No	Yes
Large Public 2	Albert	Yes	Yes
Small Private 2	Blake	Yes	Yes
Large Public 3	Jeffrey	Yes	Yes
Large Public 3	Wade	No	Yes
Large Public 3	Erin	Yes	Yes
Large Public 4	Samuel	No	No
Large Public 4	Haley	No	Barely

Figure 2. Characteristics of Student Participants

Interviews were conducted in two parts, with each part lasting approximately 45 minutes. They were conducted in the fall semester of 2012 between mid-October and early December and scheduled as soon as possible after the student agreed to participate. Each interview was conducted face-to-face in an office or group study room on the participant's campus. The results from the first part of the interviews are addressed in this report. The format of the first part of the interviews was semi-structured and conversational (Gall, Gall, & Borg, 2003). Detailed field notes were taken and the interviews were videotaped unless the student requested to be audiotaped instead.

The interview protocol consisted of 17 open-ended questions. The students were asked questions about both their AP and college calculus courses, including "What did your instructor expect from students?" "How do you define success in this course?" and "If you could start the semester over, would you do anything differently?" The questions were influenced by the Academic Performance Determinants Model (Credé & Kuncel, 2008) which suggests that a student's study skills, habits, and attitudes influence their acquisition of knowledge.

The interviews were transcribed and then analyzed. The first step of data analysis involved reading each transcript once. During a second reading, open coding (Strauss & Corbin, 1998) was used to identify any experience potentially common to another participant. These codes were highlighted and noted in the transcripts and upon a third reading they were recorded. Many of these codes were related in some way and the codes were listed in an order that showed connections, but at this point they were still written as distinct codes directly from the transcripts. Axial coding was used to group the codes into themes (Strauss & Corbin, 1998). For example, a theme emerged around the content focus of calculus courses. Some students had discussed a shift from a concrete or procedural focus in high school to a conceptual or abstract focus in college. Others described their experience in terms of "doing' versus "understanding". Still other students highlighted the increased requirement of formal mathematical language and attention to detail on exams. However, all these codes were related to how content was presented or tested in the classroom and were therefore collapsed into a theme called content focus.

Once the codes were sufficiently collapsed into themes, four categories of themes emerged and the themes were grouped into these categories. For example, the theme of content focus, discussed above, was grouped under the category: Course Content and Cognitive Demand. Transcripts were reviewed again to code line-by-line for each theme. If a theme was not found amongst at least 3 participants, it was eliminated.

Results

Seventeen themes emerged that were then grouped into four categories. These categories were 1) Student's Study Approach, 2) Student's Self-efficacy and Metacognition, 3) Class Format, and 4) Cognitive Demand of Course. Note that the first two categories contain student-controlled or student-dependent themes and the last two are specific to the course or instructor. Table --- shows each of the themes and how they were grouped into categories.

CATEGORY	THEMES
COURSE CONTENT AND COGNITIVE DEMAND	1. **Memorization**: Required Memorization Tends to Decrease Motivation. Amount already memorized by students in college varies.
	2. **Pre-Calculus Knowledge**: Trigonometry is more prevalent in college calculus than in high school calculus. It is one of the most challenging parts of college calculus.
	3. **Calculator Use**: Used all the time in some AP courses; these students must learn certain techniques for the first time in college. College courses: students can use calculators on homework and sometimes in class but not on tests.
	4. **Content Focus**: College calculus is more abstract, conceptual, and formal; AP Calculus is more concrete, procedural, and informal.
	5. **Problem Similarity**: AP Courses focus on solving specific types of problems rather than general problem solving strategies. College courses are more likely to test students on items different from those previously seen in class and on assignments.
CLASS FORMAT	1. **Class Size**: Students know to expect larger classes in college but are not prepared for the effects it will have on their participation.
	2. **Classroom Interaction**: Greater interaction is desired by students. High school courses were typically more interactive. Students who had interactive college courses like them as much or more than their high school courses.
	3. **Instructor Relationships**: Relationships with instructors have a larger impact than relationships with other students; relationships with instructors are much more common in high school and motivate students to increase their course goals.
	4. **Structure and Accountability**: Frequent assignments, assessments, and reminders by instructors increase motivation. College courses had less structure and accountability; students are understanding of this, but say it is not helpful.
	5. **Homework Setup**: High school courses assign homework but grade it only for completion. College courses use online only or a combination of online and written assignments; it is counted for a small percent of the course grade.
STUDY APPROACH	1. **Uses of Homework**: Students equate doing homework with studying. Homework in high school is not completed because it is not graded for accuracy. Online assignments in college can be completed by using similar examples available, computational tools, other students that would not be available during a test.
	2. **Preferred Resources**: Students go to their teachers first and classmates second in high school. They use friends and acquaintances first in college and online resources second. Written materials, particularly textbooks, are used as a last resort.
	3. **Lack of Awareness of Study Approach**: College students are sometimes unaware of their ineffective or inappropriate study strategies. Some students who are aware do not take the initiative to change.
SELF-EFFICACY AND METACOGNITION	1. **Calculus Affects Confidence**: For some this happens in AP; for others it happens in college. Some students believe some students are just "math people" or have "mathematical wit." AP Calculus can produce a sense of overconfidence going into college calculus which leads to under-preparedness.
	2. **Online Homework Grades Don't Reflect Understanding**: Students get a false sense of confidence from high scores on online assignments. There are many ways to get high homework scores without knowing how to solve the problem in a testing situation (multiple submissions, worked examples, solutions online, etc.)
	3. **Belief Regarding Performance**: Students give more weight to high school success than poor scores on first college tests.

	4. **Assessing Status in the College Course**: Students are unsure about their status in the course because of not knowing other students and how they are performing in relation to others. They often believe they will be successful, even despite low test scores.

Figure 3. Results from Interview Analysis

In this paper, only one or two themes in each category will be discussed, but the presentation and full paper will address them all.

Content Focus

A prominent difference in the AP course and college course for some participants was the focus on concrete and procedural material in high school versus abstract and conceptual material in college. While not all participants addressed this issue, no one suggested the reverse was true in their experience. A couple of students discussed the amount of time spent in class in college on derivations and proofs. For some, it went beyond a change in lecture emphasis to a change in requirements on exams. Kelley explained how she had been required to memorize that $\lim_{x\to 0}\frac{\sin x}{x} = 1$ in high school but in her college course was required to know how to prove it.

This distinction is similar to how other participants described the difference in their high school and college courses as doing versus understanding, respectively, or "learning how" versus "learning why". Stetson explains than in preparing for tests in his high school class, he'd "know how to do the problem without ever understanding the theory behind it." Other students described their college courses as going more "in depth" than their AP course.

Another related difference that the students identified was an emphasis in the college setting on using proper mathematical language and notation. Emilee described calculus as "nit-picky," and expressed frustration for losing points for not including parentheses that were in fact necessary to make her answer correct. Stetson described having difficulty expressing his answers completely and accurately, explaining that his college instructor often emphasized the importance of being able to "communicate mathematically."

Problem Similarity

Students explained that their AP courses focused on solving specific types of problems, rather than on concepts or general problem solving strategies. They recalled spending a significant amount of class time working old AP exams in preparation for their AP exam. Some instructors regularly included these problems in lecture and on in-class tests. Students were divided in their appreciation for this approach. Some thought it prevented them from understanding the material at a level needed for college, while others explained that had they worked harder to learn how to do these types of problems, they would have learned the material better or at least have gotten credit for the course and not had to repeat it in college.

In contrast to this approach of "teaching-to-the-test" that some students saw in their AP course, the college courses were more likely to test students on items that differed from those seen previously in lecture and assignments. Isaac explained that his study process, which was relatively thorough, did not allow him to prepare for all questions on his college instructor's tests. Erin described these problems that were in some way unique as "curveballs," indicating she did not know to expect these problems.

Classroom Interaction

A majority of the participants discussed the opportunity for interaction and engagement during class as being an important aid to their learning in both high school and college. Several students mentioned more student interaction as the primary change they would make if teaching their own college calculus course.

Perhaps the most interesting aspect of this finding is its consistency amongst participants in various types of college course settings. While the majority of students were part of a traditional lecture-based course, there were a handful of students who described their college courses as very interactive. These students were just as vocal about the importance of this feature as those who only experienced it in high school. Blake, in discussing how he would teach AP Calculus, described aspects of his college course: "interactive, very interactive. Work with the students, a lot of group work. Spend time with each of my students. Less time talking AT them, more time talking with them…[my college instructor] does a good job of talking WITH rather than AT." Keeping students engaged with the instructor and each other is viewed as a positive attribute of instruction both in the high school and college environments. Moreover, the predominance of this type of instruction in the high school environment is seen as a strength of the AP course over college calculus.

Instructor and Student Relationships

A majority of the participants described knowing their AP Calculus teachers well and this being extremely motivating. While knowing their peers tended to mostly affect how the participants behaved in class, having or not having a relationship with their instructor also impacted how they would study for the course outside of class. Frank said that he would have been content with a lower grade than he actually received in his AP course, but because he did not want his teacher to think poorly of him and because he did not want the teacher to think s/he was not an effective teacher, he worked harder than he would have, had this relationship not existed. So students were motivated by these relationships not only to meet their personal goals but to exceed them.

Strong relationships with instructors were much more common in the AP course than in college and the lack of them was viewed by the participants as a negative factor in their college experience. Haley described how she was not motivated as she had been in high school. "I don't know [my teacher] as well so I'm not motivated to do as well…It's terrible to get bad grades, but not because I know him and I'll be embarrassed." The participants were overwhelmingly understanding of this issue; they did not fault their college instructors for not knowing them, but rather seemed to have accepted it as how things have to be. Most students had made little or no effort themselves to get to know their college instructors. One exception was Frank. He talked at length about having very recently gone to office hours and how much it had encouraged him to persist, despite his previous poor test scores. He expressed regret for having not done this earlier in the semester.

Uses of Homework

Participants described using homework assignments in ways not intended by instructors or course designers. In high school it was common for participants to quickly write down something to show their instructor for homework checks when they did not know how to solve the problems in their assignments. Many were able to avoid learning how to do these problems because their instructors gave them review sheets or practice tests that were very similar to the tests and students would only need to learn these problems to be successful on the tests.

In college, some students reported doing their homework but using resources to complete the assignments that removed the challenge and therefore did not allow them to really learn to solve the problems. Sometimes solutions were provided within the homework systems and students would simply copy and paste their numbers into the solution. Other times they would use computational tools to solve problems they were being asked to solve symbolically. The students who reported doing this had, by the point of the interview, decided this was not effective and believed this approach had had a negative effect on their test grades.

Online Homework Grades Do Not Reflect Understanding

Several participants reported being made overconfident about their test preparation in college by their homework assignments. All of these students had regular online assignments that were graded electronically for accuracy. Some of these assignments had solutions provided to very similar problems that could easily be manipulated or copied to obtain full credit. Wade explained, "They have 'show me how to do this problem. When you click this, it changes the numbers. But the 'show me an example,' if you look at the end, you can find the pattern and b.s. the answer." Similarly, Maggie said that she "looked at so many examples in order to figure it out that in the end I didn't really figure it out, I just looked at it and would substitute my own numbers in." She said that if she could start the semester over, "I wouldn't trick myself with my online homework. I'd actually take the time to learn how to do it."

Students struggled with knowing whether they really knew how to do a problem they received credit for in their online assignments. One reason was that they often had multiple submissions for these assignments. Frank said that he understood why this was setup like it was and understood its benefits for learning, but had mixed feelings about its results. He recommended having a companion section of homework where you only get one try because "it puts the pressure on you to really learn it rather than just try a couple things until something works." The other issue with the online homework was how the answer format requirements differed from exams. Samuel's instructor graded the students' work on exams along with their answers but did not do this with homework. Samuel explained, "Even if you write something off to the side [on the test], if it's not exactly mathematically correct, you lose points." This student had completed his early homework assignments mostly "in his head," without writing down full solutions. He believed this set him up to fail. He said, "I theorize that students twenty years ago understood calculus ten times better than anybody in our class does…[online homework] will unfortunately never go away, but I think it's hurting our math students."

Belief Regarding Performance

Some students were convinced their level of understanding of calculus was not reflected in their test grades. For these students, their prior successes in high school outweighed the impact of their college calculus test grades in determining how much they believed they understood the material and how they would ultimately perform in the course, at least for a short period of time at the beginning of the semester. Frank had done poorly on both of his first two tests but was still very confident about his understanding of calculus at the midpoint of the semester. He explained that there was too much memorization and precalculus required in his college course, but he was very comfortable with topics like integration which he had learned in his AP course, so he felt good about his knowledge and ability. Despite having failing test grades, he believed he would ultimately succeed in the course.

Discussion and Conclusion

There is ample opportunity for both AP Calculus and college calculus instructors to learn from one another's best practices, in light of the student perspectives revealed in this study. For example, this study provides evidence that AP Calculus may indeed be administered as a college level rather than a college preparatory course in many high schools. This gives students a sense of overconfidence going into calculus I in college which may reduce their study efforts and diminish their success. However, certain design aspects of AP Calculus such as the opportunity for student interaction in the classroom and the existence of strong relationships between students and teachers can produce very positive effects on student learning and the possibility of increasing these in college classrooms must be studied further.

This study offers results from research that reveals the student perspective on their personal transitions from high school (Advanced Placement) to college calculus. Further research and implementation of what is learned here will allow us to improve teaching at both the high school and college level.

References

Anthony, G. (2000). Factors influencing first year students' success in mathematics. *International Journal of Mathematical Education in Science and Technology, 31*(1), 3-14.

Bressoud, D. (2009). Is the sky still falling? *Notices of the AMS, 56*(1), 20-25.

Bressoud, D. M., Carlson, M. P., Mesa, V., & Rasmussen, C. (2013). The calculus student: insights from the Mathematical Association of America national study. *International Journal of Mathematical Education in Science and Technology*, *44*(5), 685-6

Credé, M. & Kuncel, N. (2008). Study habits, skills, and attitudes: The third pillar supporting collegiate academic performance. *Perspectives on Psychological Science, 3*(6), 425-453.

Gall, M. D., Gall, J. P., & Borg, W. R. (2003). *Educational research: An introduction*. (Seventh ed.). Boston: Allyn and Bacon.

Hong, Y.Y., Kerr, S., Klymchuk, S., McHardy, J., Murphy, P., Spencer, S...Watson, P. (2009). A comparison of teacher and lecturer perspectives on the transition from secondary to tertiary education. *International Journal of Mathematical Education, 40*(7), 877-889.

Hourigan, M., & O'Donoghue, J. (2007). Mathematical under-preparedness: The influence of the pre-tertiary mathematics experience on students' ability to make a successful transition to tertiary level mathematics courses in Ireland. *International Journal of Mathematical Education in Science and Technology, 38*(4), 461-476.

Keng, L, & Dodd, B. (2008). A comparison of college performances of AP and non-AP student groups in 10 subject areas. College Board.

Klopfenstein, K. & Thomas, K. (2006). The advanced placement performance advantage: Fact or fiction? retrieved from http://www.aeaweb.org/annual_mtg_papers/2005/0108_1015_0302.pdf.

Strauss, A., & Corbin, J. (1998). *Basics of qualitative research: Grounded theory procedures and techniques*. (2nd ed.) Thousand Oaks, CA: Sage.

A STUDY OF CONNECTIVISM AS A SUPPORT FOR RESEARCH ON MEANING-MAKING FOR MATHEMATICS

Luciane Mulazani dos Santos
Ivanete Zuchi Siple
Gabriela Cecille Corrêa Lopes
Marnei Luis Mandler

Universidade do Estado de Santa Catarina, Joinville/SC, Brasil

This paper presents ongoing research whose goal is to study theoretical frameworks that support teaching practices carried out in courses that are part of a Bachelor's Degree in mathematics which use information and communication technology resources - such as blogs, social networks and virtual learning environments - for the teaching and learning of mathematics content that is part of the curriculum in colleges and universities. The outline presented here refers to the ongoing study in the Mathematics, Culture, Art and Technology line of research being done by the research group THEM (Spices of History in Mathematics Education) about connectivism as a theoretical approach to teaching and learning.

Key words: Information and communication technology, connectivism, meaning-making

Introducing the Research

Advances in information and communication technology (ICT) have enabled changes in the teaching and learning process in higher education. There are several initiatives, some proposed by institutions and others implemented by teachers, geared towards the use of internet technology to support classroom education. Some examples are virtual learning environments, social networks, blogs, wikis, etc. Finding ourselves in this context of the use of ICT in formal education, we, as teachers of mathematics and mathematics education courses in higher education, feel the need to lay out the reasons for our practices. One of our initiatives is the study we are doing about connectivism as a theoretical approach to teaching and learning. We will present here some of the initial results of our theoretical study with the intention to share experiences and receive contributions for the continuity of our research, which is focused on the processes of knowledge construction and meaning-making for mathematical content. This group's research around the theme of connectivism started in February 2014 led by a senior project in mathematics whose theme was the use of Facebook as a pedagogical architecture in one of the program's courses. One of the theoretical bases used was based on Connectivism.

Connectivism

Two Canadian researchers, George Siemens and Stephen Downes, have presented a set of studies on networked learning, defining the concept of connectivism as a theoretical approach to teaching and learning. This concept was first presented in 2004 in an online text. In 2006, Siemens presented connectivism in the book *Knowing Knowledge*. It was Siemens' interest in the pedagogical possibilities of ICTs that led him, along with Downes, to propose connectivism as a new theory of teaching and learning in the context of knowledge construction within a network, i.e., as an argument for situations in which educational processes connect different information sources and continuous communication. We use the work of Siemens and Downes as theoretical support for the discussion of this research regarding the concept of knowledge construction and networked learning.

Connectivism is a new perspective to discuss teaching and learning, and the integration of principles explored by chaos, network, complexity and self-organization theories. It focuses on education in the digital age and takes into consideration how technology influences the current forms of communication and learning. According to connectivism, learning is a process that occurs within environments where the central elements are constantly changing - not entirely under the control of people. Learning is defined as actionable knowledge that can reside outside of ourselves, for example, within an organization or a database. The focus of learning is on connecting specialized sets of information. (Siemens, 2004).

According to Siemens (2004),

> Behaviorism, cognitivism, and constructivism are the three broad learning theories most often utilized in the creation of instructional environments. These theories, however, were developed in a time when learning was not impacted through technology. Over the last twenty years, technology has reorganized how we live, how we communicate, and how we learn. Learning needs and theories that describe learning principles and processes, should be reflective of underlying social environments. (Siemens, 2004, p.1)

When Siemens says "not impacted through technology," he is talking about the fact that there was no previous existence of the countless resources coming from the internet and ICTs. The fact is that we now have at our disposal a networked world supported by resources such as blogs, websites, social networks, etc. We can be connected with many other people, and those people can also be connected to each other. Access to information is very broad and very fast. If we want, we can access in a few seconds, via the network, information that is on the other side of the world. All this directly affects the processes of teaching and learning, creating the need for new understanding about the theoretical approaches given to these processes.

According to Siemens (2004), the following are some of the new trends in learning that justify the need to rethink the theoretical approaches of behaviorism, cognitivism and constructivism:

> Many learners will move into a variety of different, possibly unrelated fields over the course of their lifetime. Informal learning is a significant aspect of our learning experience. Formal education no longer comprises the majority of our learning. Learning now occurs in a variety of ways – through communities of practice, personal networks, and through completion of work-related tasks. Learning is a continual process, lasting for a lifetime. Learning and work related activities are no longer separate. In many situations, they are the same. Technology is altering (rewiring) our brains. The tools we use define and shape our thinking. The organization and the individual are both learning organisms. Increased attention to knowledge management highlights the need for a theory that attempts to explain the link between individual and organizational learning. Many of the processes previously handled by learning theories (especially in cognitive information processing) can now be off-loaded to, or supported by, technology. Know-how and know-what is being supplemented with know-where (the understanding of where to find knowledge needed). (Siemens, 2004, p.1)

These issues raised by Siemens show some of the aspects that have caused changes in the role of the teacher, the learner and the teaching resources in both formal and informal processes of education. According to Siemens (2004, p. 2), "Behaviorism, cognitivism, and constructivism (built on the epistemological traditions) attempt to address how it is that a person learns" using "the notion that knowledge is an objective (or a state) that is attainable (if not already innate) through either reasoning or experiences.

> A central tenet of most learning theories is that learning occurs inside a person. Even social constructivist views, which hold that learning is a socially enacted process, promotes the principality of the individual (and her/his physical presence – i.e. brain-based) in learning. These theories do not address learning that occurs outside of people (i.e. learning that is stored and manipulated by technology). They also fail to describe how learning happens within organizations. (Siemens, 2004, p. 2).

We see how important it is for the author to highlight the need to study and understand how learning evolves in processes that rely on technology and also those developed within organizations, i.e., in conditions external to the person. According to the author,

> Learning theories are concerned with the actual process of learning, not with the value of what is being learned. In a networked world, the very manner of information that we acquire is worth exploring. The need to evaluate the worthiness of learning something is a meta-skill that is applied before learning itself begins. (Siemens, 2004, p. 2)

This is an important issue raised by the author, which discusses the amount of information to which we have access in the networked world. When there was no such condition (established primarily through the use of technology that we have today), when we had more limited access to information, it was enough to discuss the theories of learning, since the process of discussing the means of access to information for knowledge construction was intrinsic to learning.

Siemens thus justifies the need for new approaches in theoretical approaches to teaching and learning:

> Many important questions are raised when established learning theories are seen through technology. The natural attempt of theorists is to continue to revise and evolve theories as conditions change. At some point, however, the underlying conditions have altered so significantly, that further modification is no longer sensible. An entirely new approach is needed. (Siemens, 2004, p. 2).

Continuing his argument about the need for new theoretical approaches to learning, the author stresses the differences in the context of constructing knowledge through network connections with the support of technology. According to the author, the inclusion of technology and connection-making as learning activities begins to move learning theories into a digital age. We can no longer personally experience and acquire learning. We achieve our skills as a result of the formation of connections. That sums up Siemens' defense for a new theoretical framework to the extent that he argues that when we use technology for constructing networked knowledge, we can no longer speak only in experimentation (behaviorism) and personal construction (cognitivism and constructivism) when discussing learning and teaching theoretically. We must go further, observing and reflecting on theoretical approaches that consider that knowledge construction depends on the formation of connections in the external world, which is greatly impacted by technology. Thus, Siemens presents his definition of connectivism:

> Connectivism is the integration of principles explored by chaos, network, and complexity and self-organization theories. Learning is a process that occurs within nebulous environments of shifting core elements – not entirely under the control of the individual. Learning (defined as actionable knowledge) can reside outside of ourselves (within an organization or a database), is focused on connecting specialized information sets, and the connections that enable us to learn more are more important than our current state of knowing. Connectivism is driven by the understanding that decisions are based on rapidly altering foundations. New information is continually being acquired. The ability to draw distinctions between important and unimportant information is vital. The ability to recognize when new information alters the landscape based on decisions made yesterday is also critical. (Siemens, 2004, p. 3)

Although some authors and studies do not consider connectivism as a learning theory, such a new theoretical approach to learning and teaching leads us to consider the use of new teaching methodologies and learning architectures. Because of this, our research group is looking at connectivism as a theoretical framework to be incorporated into others already in use to investigate knowledge construction and meaning-making for mathematics in higher education.

References

Downes, S. (2007) What Connectivism is. Half an hour. Retrieved March 10, 2014, from http://halfanhour.blogspot.com.br/2007/02/what-connectivism-is.html.

Siemens, G. (2004) Connectivism: A Learning Theory for the Digital Age. *International Journal of Instructional Technology & Distance Learning*. v. 2, n. 1, 2004. Retrieved March 10, 2014, from http://www.itdl.org/Journal/Jan_05/article01.htm.

Mathematicians' views on spatial reasoning in undergraduate and graduate mathematics

Existing research shows there is a strong correlation between spatial reasoning and mathematics achievement in K-12. Is spatial reasoning important then for succeeding in undergraduate and graduate mathematics? I interviewed four (N=4) young mathematicians about their conceptualizations of spatial reasoning and where they potentially saw spatial reasoning in undergraduate and graduate mathematics. Two participants identified themselves as geometers/topologists and the other two identified as algebraists. The two geometers/topologists emphasized understanding 2D representations of 3D shapes and the two algebraists emphasized the notion of dynamics, i.e. mental movement. Among graduate fields of mathematics, geometry/topology was cited most frequently as being spatial and algebra was cited the least. In undergraduate mathematics, all four participants drew connections to multivariable calculus and discussed examples from their teaching of multivariable calculus. This work has some implications for teaching multivariable calculus and discerning the relationship between spatial reasoning and mathematics.

Key words: Graduate Mathematics, Mathematicians' Practice, Spatial Reasoning, Nature of Mathematics

Research over the twentieth century has shown there is a strong correlation between spatial reasoning and mathematics achievement (Uttal et al., 2012). Some studies have looked at the direct link between spatial reasoning and K-12 mathematical content areas, such as geometry (Battista, 1990) or arithmetic (Cheng & Mix, 2012). What is the link between spatial reasoning and undergraduate or even graduate mathematics? Many studies have looked at spatial abilities in children, but the world of mathematicians remains largely unexplored in this regard. My research questions are: 1) How do mathematicians conceptualize the term spatial reasoning? 2) What fields of mathematics do they identify as using spatial reasoning?

I conducted think-alouds/interviews with four (N=4) young mathematicians at a large Midwestern university. I define a young mathematician to be a doctoral student in mathematics. Two participants, Adam and Brian, identified themselves as geometers/topologists. The other two participants, Chris and Danielle, identified themselves as algebraists. In the think-aloud portion, mathematicians were given three paper-and-pencil examples of spatial tasks (mental rotation, cross section of three dimensional objects, and a perspective task), which served as stimuli for talking about spatial reasoning. The subsequent semi-structured interview consisted of questions about their conceptualizations of spatial reasoning and potential connections between spatial reasoning and their research area, other fields of mathematics, and their teaching. Data consisted of transcripts of hour-long audio recordings and field notes.

Transcript data was analyzed using grounded theory techniques to identify key themes in how each participant conceptualized spatial reasoning. The following themes that emerged were visualizing, dynamics, intangibility, mental "jumps," understanding 2D representations of 3D objects, non-rigid transformations, and spatial reasoning as a form of mental evidence. The two geometers/topologists emphasized understanding 2D representations of 3D objects, i.e. inferring 3D properties from the drawn paper-and-pencil figures. The two algebraists emphasized the idea of dynamics, i.e. imagining movement, for a task to use spatial reasoning.

In this poster, I will also present the different fields of mathematics that the mathematicians referenced as using spatial reasoning and examples they gave (see Table 1).

Table 1

Graduate mathematical fields participants identified as using spatial reasoning

Participant	Algebra	Analysis	Graph Theory	Geometry/Topology
Adam	X	X	X	X
Brian		X	X	X
Chris			X	X
Danielle	X	X	X	X

Geometry/Topology was the field most often cited as spatial in nature (with examples) and Algebra was cited the least often. Participants also cited spatial reasoning as helpful in Analysis, for visualizing counterexamples. All four participants also referenced Graph Theory as using spatial reasoning but gave no specific examples. For undergraduate mathematics, all four participants cited multivariable calculus as relying heavily on spatial reasoning and discussed how their students could benefit from improved spatial reasoning in order to understand concepts. This works extends the research on mathematicians' practice and explores deeper how spatial reasoning and mathematics are intertwined.

References

Battista, M. T. (1990). Spatial visualization and gender differences in high school geometry. *Journal for Research in Mathematics Education*, *21*(1), 47–60.

Cheng, YL. and Mix, K. S. (2012). Spatial training improves children's mathematics ability. *Journal of Cognition and Development.*

Uttal, D. H., Meadow, N. G., Tipton, E., Hand, L. L., Alden, A. R., Warren, C., & Newcombe, N. S. (2012). The Malleability of Spatial Skills: A Meta-Analysis of Training Studies. *Psychological Bulletin.*

Developing a creativity-inprogress rubric on proving

Milos Savic
University of Oklahoma
savic@ou.edu

Gulden Karakok
University of Northern Colorado
gulden.karakok@unco.edu

Gail Tang
University of La Verne
gtang@laverne.edu

Houssein El Turkey
University of New Haven
houssein@ou.edu

There is a considerable amount of mathematics education literature on creativity (e.g., Torrance, 1966; Balka, 1974; Silver, 1997), yet there is little discussion of mathematical creativity in undergraduate mathematics education. Specifically, to our knowledge, there is no literature on mathematical creativity in the proving process. We attempt to contribute to the literature by developing a framework in which to discuss mathematical creativity in proving, Creativity-in-Progress Rubric (CPR) on proving. Through previous rubrics on creativity (Leikin, 2009; Rhodes, 2010) and interviews with both mathematicians and undergraduate students, we claim that there are three aspects of creativity in proving: Making Connections, Taking Risks, and Creating Ideas. We demonstrate how to use the rubric with an example of a student's proving process. Finally, we give future research considerations of using the rubric in proof-based classrooms.

Key words: mathematical creativity, proving process, creativity rubric, proof-based courses

Introduction

Many great historical mathematicians often discussed creativity in their work as an illumination that is somewhat unexpected (Poincare, 1946; Hadamard, 1945). Though creativity is one important aspect of professional mathematicians' work, it is a complicated subject to research for mathematics educators, given that there are over 100 different definitions of creativity (Mann, 2005). In fact, a recent publication by the Mathematical Association of America (Borwein, Liljedahl, & Zhai, 2014) demonstrated that many brilliant mathematicians had different ideas about mathematical creativity. Some conceptualizations of creativity focus on emphasizing whether the end product is original and useful (Runco & Jaeger, 2012), while others describe mathematical creativity as a process that involves different modes of thinking, some of an unusual nature (Balka, 1974). Even though the process approach creates difficult hypotheses for testing and evaluating (Torrance, 1966), the product approach may not provide full understanding of the development of mathematical creativity.

Despite the entangled state of mathematics creativity research, Sriraman (2004) stated that, "[i]t is in the best interest of the field of mathematics education that we identify and nurture creative talent in the mathematics classroom" (p. 32). It is also timely to consider investigating this topic as it gains momentum in education agendas (Askew, 2013). However, our examination of the undergraduate mathematics education literature yielded little discussion of how undergraduate students are creative or how creativity can be fostered in the classroom. Furthermore, there is little discussion of how creativity arises in proving processes. In our proposal, we attempt to begin this discussion by proposing a "creativity-in-progress rubric" on proving to gauge undergraduate students' mathematical creativity.

Literature Review

There have been numerous research articles demonstrating the difficulties students have with undergraduate proof-based courses (Moore, 1994; Weber, 2001). Setting up a proof framework (Selden & Selden, 2009) can be one way to alleviate some difficulty. Proof frameworks consist of writing the beginning (usually a re-written hypothesis from the statement of the theorem) and ending (the conclusion of the theorem) of a proof or subproof, coupled with understanding and unpacking the definitions occurring in the conclusion. Hypothetically, this can open up the student to focus more on the problem-solving part of the proof. However, what happens when the student experiences difficulty with the problem-solving part?

A student may encounter the creative aspect of the proving process during this problem-solving portion. Although most students will not create a proof analogous to that of Fermat's Last Theorem, they may experience *relative creativity*. Relative creativity is defined as "the discoveries by a specific person within a specific reference group, to human imagination that creates something new" (Vygotsky, 1982, 1984; as cited by Leikin, 2009, p. 131). That is, students may create something that is new to them or to their peers, but it may not be new to the greater mathematical community. The latter would be considered *absolute creativity* or discoveries at a global level, such as the proof of Fermat's Last Theorem by Andrew Wiles. We claim that explicitly valuing and assessing relative creativity in an undergraduate classroom could help students' learning and develop their own mathematical creativity. The main goal of our research project is to investigate this claim.

Therefore, our research question is:

- *In what ways can instructors make relative creativity explicit to their students and assess it in undergraduate proof-based courses?*

In our first attempt to address this question, we created the "Creativity-in-Progress Rubric" or CPR. Prior to sharing the development process of CPR and its use, in the next section we briefly discuss the theoretical framework that we considered during the development of the CPR.

Theoretical Framework

Simon (2009) hoped that "future researchers…deeply understand many of the [learning] theories available, are aware of the affordances and limitations of each, and use these theories strategically." (p. 488). Kozbelt, Beghetto, and Runco (2010) provided a summary of ten major contemporary theories from a meta-analysis of creativity work: *Developmental*, *Psychometric*, *Economics*, *Stage and Componential Process*, *Cognitive*, *Problem Solving and Expertise-Based*, *Problem Finding*, *Evolutionary*, *Typological*, and *Systems.* The aim was to help researchers to better situate their theoretical assumptions, and possibly discover overlapping areas between theoretical perspectives, which may lead to advancement of our understanding of creativity in general. Both the *Developmental* and *Problem Solving and Expertise* theories, accompanied by Simon's (2009) perspective, guided our project.

The primary assertion of the *Developmental* theory is that creativity develops over time, and the main focus of investigation is a person's process of creativity. It emphasizes the role of environment, in which interaction takes place, to enhance the creativity. The *Problem Solving and Expertise* theory emphasizes the role of an individual's problem solving process and also argues "creative thought ultimately stems from mundane cognitive processes" (Kozbelt et al., 2010, p. 33). The tasks, problem representation, and heuristics are key concepts of this theory. The creation and use of the rubric falls under both theories due to our investigation of classroom environments and exploration of students' heuristics. In both of these theories, *relative creativity* is the main focus, and the *process* of "creating" is mediated by an *interaction* between person and environment.

Development of the CPR

The development of the CPR was initiated on the key pieces (*process* and *interaction*) of the aforementioned theoretical frameworks. It was rigorously constructed through triangulating research-based rubrics, mathematicians' and students' views on mathematical creativity, and students' proving attempts. We considered the Association of American Colleges and Universities (AAC&U) Creative Thinking Value rubric with six categories: *acquiring competencies*, *taking risks*, *solving problems*, *embracing contradictions*, *innovative thinking*, and *connecting, synthesizing, transforming* (Rhodes, 2010). Using this rubric as a starting point, two members of our research group interviewed six mathematicians about their perspectives in teaching and assessing creativity in proof-bases courses. All six mathematicians were given the AAC&U rubric along with three proofs of the same number theory theorem constructed by three students and published in Birky et al., (2011). Their discussions about mathematical creativity and feedback on the adaptability of the AAC&U rubric to the mathematics classroom were influential factors in our first draft of the CPR.

In the mathematical creativity literature, Leikin (2009) created a rubric for mathematical creativity in problem solving that valued three categories: *flexibility*, *fluency*, and *originality*. These categories were influenced by the Torrance (1966; 1988) tests for assessing creativity and Silver's (1997) work with K-12 instructional activities. Finally, one member of our research group interviewed eight students from a transition-to-proof or introduction to proofs course using the same three proofs used in the mathematician interviews. Selected scratch work of students in the course, which was collected via LiveScribe pen technology, were also thoroughly examined to refine the rubric.

Creativity-in-Progress Rubric (CPR) on Proving

Three main categories about creativity in students' proving processes surfaced from the existing rubrics and the mathematicians' and students' data:

- *Making connections* - demonstrating links between multiple representations and/or ideas from the student's current and/or previous course(s).
- *Taking risks* - approaching a proof and demonstrating flexibility in using different or multiple approaches.
- *Creating ideas* - developing original mathematical ideas that are either pertinent to the proof or can be proven.

Note that Leikin's flexibility and originality are inherent in the above descriptions, and fluency can be helpful when making connections between mathematical concepts. Table 1 shows the subcategories that explicate the definitions of each category, and the desired milestones.

We now clarify some terms used in the rubric. A *mathematical representation* in the rubric is used to mean "an external manifestation of mathematical objects" (Pape & Tchoshanov, 2001, p. 119) (e.g., spoken words, written symbols/words, pictures or diagrams, gestures). Specifically, we can represent the symbols $y = x^2$ as a graph with vertex at the origin "pointing" up, or with a gesture of *U* shape "opening" up, or with a table of inputs and outputs. *Proof technique* refers to a method of approaching a proof of a theorem. For example, "proof by contradiction" and "induction" would be two different proof techniques. We mean by *originality* as the ability to create new, novel, unique, and/or unusual ideas, that are relative to the student or the course.

Table 1: Creativity-in-Progress Rubric: On Proving

MAKING CONNECTIONS:	**Beginner**	**Developing**	**Satisfactory**
Between Definitions or Theorems	Introduces few definitions/theorems (some of them may be irrelevant)	Recognizes some relevant definitions/theorems from the course or textbook and attempts to connect them in their proving	Implements definitions/theorems from the course and/or prior knowledge (e.g. a prior course work)
Between Representations	Attempts a connection between two representations	Demonstrates connections between multiple representations either to enhance an idea or help understanding	Utilizes different representations to strengthen the proof
Between Examples	Generates one or two specific examples for the proof	Attempts to make connections between specific and general examples	Able to move freely back and forth between specific and general examples
Between Proof Techniques and Previous Proofs	Does not show a connection between proof techniques of previous proofs	Attempts to utilize a proof technique due to its success in previous proofs	Recognizes an understanding of previous success with proof techniques and groups certain mathematical concepts with certain proof techniques

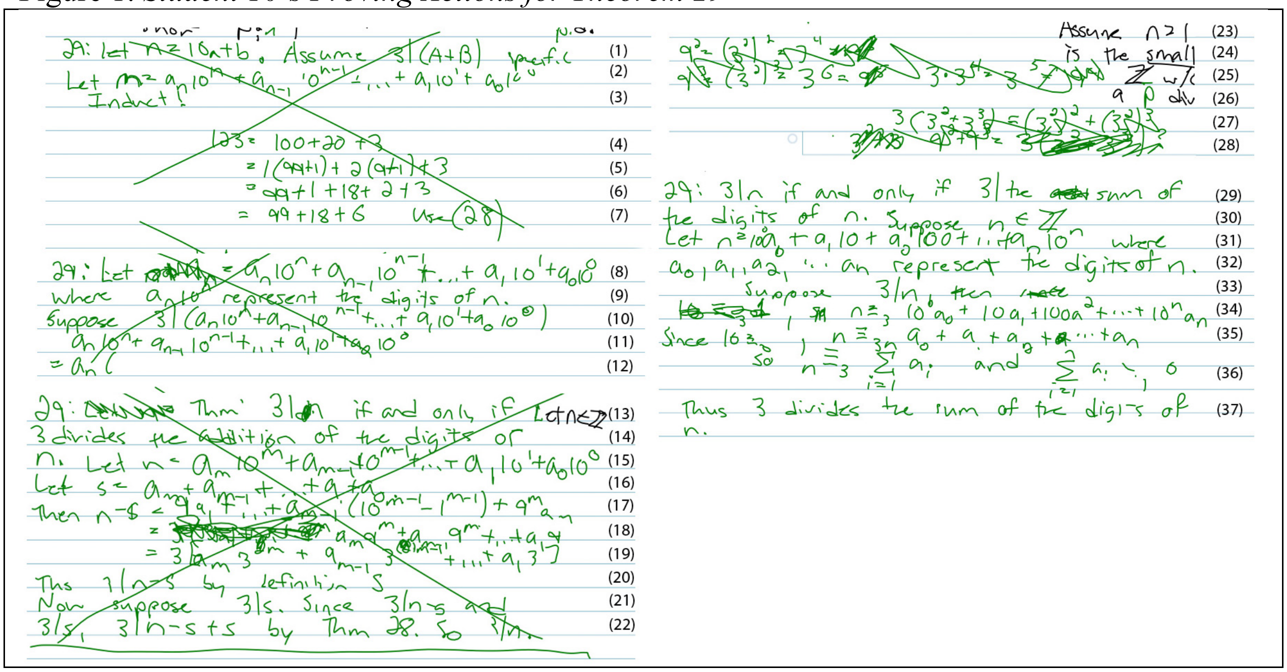
TAKING RISKS:	**Beginner**	**Developing**	**Satisfactory**
Attempting a Proof	Attempts a proof	Attempts a proof with some indication of directed thought	Attempts a proof with some indication of exhaustive thought towards the proof
Proof Technique Flexibility	Attempts one proof technique	Implements a proving technique completely	Scratch work (verbal or written) indicates thinking of different proving approaches.
Completeness	Provides an incomplete proof	Provides a complete argument (either verbal or written) without a rigorous written proof	Provides a complete proof written rigorously
Evaluation of the Attempt	Checks work locally	Recognizes a unsuccessful proving attempt	Recognizes the key idea that makes the proving attempt unsuccessful or successful

CREATING IDEAS:	**Beginner**	**Developing**	**Satisfactory**
Originality	Attempts to create original ideas for the proving attempt	Displays original ideas (for that student) that are somewhat expected but impressive	Creates a whole new idea never expected or unusual for the course
Posing Questions	Poses questions clarifying a statement of a definition or theorem	Poses questions about reasoning within a proof	Poses questions that take account global understanding or modification of hypothesis of the theorem posed
Conjectures	Poses a trivial or incorrect conjecture, or rewords a previous theorem	Extends theorems or definitions in the form of corollaries or poses conjectures from patterns	Poses and attempts to prove a conjecture that leads to or indicates a generalization of prior ideas

Using the CPR: An Example

In an inquiry-based transition-to-proof course at a large Midwestern university, 24 students were given LiveScribe pens, a data collection tool capable of capturing audio and written work in real time. Use of this technology was an intentional attempt to capture the processes of student's proof development. All students were required to do and turn in their homework using the pen and special paper; all homework was downloaded to the professor's computer for both grading and analysis. For example, we analyzed some of the proving actions that Student 10 enacted chronologically for Theorem 29: "If 3 divides the sum of the digits of n, then 3 divides n" for the development of the rubric. This theorem was the third theorem in the number theory section, located after the definition of even and odd numbers, divisibility, (DEF S: $a|b \Leftrightarrow b = na$ for some $n \in \mathbb{Z}$) and theorems (27 and 28, respectively): "If m and n are even numbers, prove that $m + n$ and $m \cdot n$ are even numbers" and "If $a|b$ and $a|c$, then $a|(br + cs)$ for any $r, s \in \mathbb{Z}$." Student 10's first attempt involved thinking about proving by induction and looking at a specific case (123). He finished the attempt by acknowledging that Theorem 28 needed to be used. Prior to Student 10's second attempt, the professor had given the hint: "Let $n = a_m \cdot 10^m + \cdots + a_1 \cdot 10^1 + a_0$." During the third proving attempt, Student 10 tried directly proving the statement by manipulating n algebraically. He then skipped to the next two theorems in the assignment before attempting Theorem 29 again. He attempted the proof four times in the course of two days. Figure 1 shows these attempts.

Figure 1: *Student 10's Proving Actions for Theorem 29*



To use the rubric, you would simply highlight over the gray arrow up to the level in which you believe your student to be. For example, Student 10's Making Connection evaluation is presented in Figure 2.

Figure 2: *Student 10's Making Connections Assessment*

MAKING CONNECTIONS:	Beginner	Developing	Satisfactory
Between Definitions or Theorems	Introduces few definitions/theorems (some of them may be irrelevant)	Recognizes some relevant definitions/theorems from the course or textbook and attempts to connect them in their proving	Implements definitions/theorems from the course and/or prior knowledge (e.g. a prior course work)
Between Representations[1]	Attempts a connection between two representations	Demonstrates connections between multiple representations either to enhance an idea or help understanding	Utilizes different representations to strengthen the proof
Between Examples	Generates one or two specific examples for the proof	Attempts to make connections between specific and general examples	Able to move freely back and forth between specific and general examples
Between Proof Techniques[2] and Previous Proofs	Does not show a connection between proof techniques of previous proofs	Attempts to utilize a proof technique due to its success in previous proofs	Recognizes previous success with proof techniques and groups certain mathematical concepts with certain proof techniques

We assessed Student 10's subcategory "Between Definitions or Theorems" of Making Connections as *satisfactory* since he implemented both Definition S (line 20 in Figure 1) and Theorem 28 (line 22). Student 10 scored *developing* on his proving attempts in the subcategory "Between Representations" since he transitions between writing out a sum (line 35) and representing that sum in sigma notation (line 36), which enhanced his understanding of the proof. He also uses modular arithmetic (line 36), writing that the sums of all the a_is are congruent to 0 (mod 3), another representation of the sum that he believes enhanced the proof. However, we feel as if both representations did not *strengthen* the proof, a requirement for satisfactory. In the subcategory "Between Examples," Student 10 received a developing score, noting that he generated multiple examples to understand his approaches to proving (lines 4-7, 23-28) and attempts to generalize (from lines 4-7 to line 8). Finally, we believe that Student 10 would score *developing* in the subcategory "Between Proof Techniques," due to his use of a similar proving technique (lines 17-22) that was used in the proof of Theorem 28 (not shown above).

Discussion/Future Research

Notice that correctness of a proof is never addressed in the rubric. This decision was based on the interviewed mathematicians' claims that in some instances, more mathematical creativity was generated when they were pursuing conjectures or proofs that were incorrect and that the results from incorrect proofs were somewhat useful (Savic, Karakok, Tang & El Turkey, 2014). Therefore, we claim that one way to teach with the CPR is by valuing and encouraging students to share their incorrect processes in attempting a proof. When students share their proving attempts in the course, the educator can use the CPR to emphasize certain categories that the student is enacting at that moment. However, an educator should be aware of a student's affect or confidence when initially doing this.

Our rubric is intended to assess an individual student's progress in the course, and not to compare students' to each other. We are attempting to value, and ultimately develop, each student's relative mathematical creativity. The CPR can then be used as a formative assessment tool, where a student can monitor his/her own progress and evaluate his/her understanding to ultimately recognize where improvement is needed in his/her creative process.

Our future research is focused on implementing the CPR within the transition-to-proof course in fall 2014, with possibly more in the spring of 2015 at different institutes. We aim to develop and implement different versions of CPR that are applicable to different undergraduate courses. We conjecture that if students focus on developing their own mathematical creativity through formative assessment, then the conceptual and computational understandings will follow. Therefore, we conjecture that students will ultimately self-improve if the CPR is valued and used consistently in undergraduate math courses.

References

Askew, M. Issues in teaching for and assessment of creativity in mathematics and science. In D. Corrigan, R. Gunstone, & A. Jones (Eds.), *Valuing Assessment in Science Education: Pedagogy, Curriculum, Policy*. (pp. 169-182) Dordrecht: Springer-Verlag.

Balka, D. S. (1974). Creative ability in mathematics. *Arithmetic Teacher, 21* (7), 633-638.

Birky, G., Campbell, C. M., Raman, M., Sandefur, J., & Somers, K. (2011). One problem, nine student-produced proofs. *The College Mathematics Journal, 42* (5), 355-360.

Borwein, P., Liljedahl, P., & Zhai, H. (2014). *Mathematicians on Creativity.* Washington D.C., USA: Mathematical Association of America.

Hadamard, J. (1945). *The mathematician's mind.* Princeton: Princeton University Press.

Kozbelt, A., Beghetto, R. A., & & Runco, M. A. (2010). Theories of creativity. In *The Cambridge Handbook of Creativity* (pp. 20-47). New York, NY, USA: Cambridge University Press.

Leikin, R. (2009). Exploring mathematical creativity using multiple solution tasks. In R. Leikin, A. Berman, & B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students* (pp. 129-145). Haifa, Israel: Sense Publishers.

Mann, E. (2005). *Mathematical creativity and school mathematics: Indicators of mathematical creativity in middle school students.* (Doctoral Dissertation). University of Connecticut : Storrs.

Moore, R. (1994). Making the transition to formal proof. *Educational Studies in Mathematics 27*, 249-266.

Pape, S. J., & Tchoshanov, M. A. (2001). The role of representation(s) in developing mathematical understanding. *Theory Into Practice, 40* (2), 118-127.

Poincare, H. (1946). *The foundations of science.* Lancaster, PA: The Science Press.

Rhodes, T. (2010). *Assessing Outcomes and Improving Achievement: Tips and Tools for Using Rubrics.* Washington, DC: Association of American Colleges and Universities.

Runco, M. A., & Jaeger, G. G. (2012). The standard definition of creativity. *Creativity Research Journal, 24* (1), 92-96.

Savic, M., Karakok, G., Tang, G., & El Turkey, H. (2014). How Can We Assess Undergraduate Students' Creativity in Proof and Proving? In *Proceedings of the 8th International Conference on Mathematical Creativity and Giftedness.* Denver, CO.

Selden, J., & Selden, A. (2009). Teaching proof by coordinating aspects of proofs with students' abilities. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and learning proof across the grades: A K-16 perspective* (pp. 339-354). New York, NY: Rutledge.

Silver, E. (1997). Fostering creativity through instruction rich in mathematical problem solving and posing. *ZDM Mathematical Education, 3* , 75-80.

Simon, M. (2009). Amidst multiple theories of learning in mathematics education. *Journal for Research in Mathematics Education, 40* (5), 477-490.

Sriraman, B. (2004). The characteristics of mathematical creativity. *The Mathematics Educator, 14*, 19-34.

Sriraman, B. (2009). The characteristics of mathematical creativity. *ZDM Mathematics Education, 41*, 13-27.

Torrance, E. P. (1988). The nature of creativity as manifest in its testing. In R. J. Sterberg, *The nature of creativity: Contemporary psychological perspectives* (pp. 43-75). New York, NY: Cambridge University Press.

Torrance, E. P. (1966). *The Torrance tests of creative thinking: Technical-norms manual.* Princeton, NJ: Personnel Press.

Vygotsky, L. S. (1984). Imagination and creativity in adolescent. In D. B. Elkonin, *Vol 4: Child Psychology. The Collected Works of L. S. Vygotsky* (pp. 199-219). Moscow, SSSR: Pedagogika.

Vygotsky, L. S. (1982). Imagination and its development in childhood. In V. V. Davydov, *Vol. 2: General Problem of Psychology. The Collected Works of L. S. Vygotsky* (pp. 438-454). Moscow, SSSR: Pedagogika.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics, 48* (1), 101-119.

A theoretical perspective for proof construction

John Selden — New Mexico State University
Annie Selden — New Mexico State University

This theoretical paper suggests a perspective for understanding undergraduate proof construction based on the ideas of conceptual and procedural knowledge, explicit and implicit learning, behavioral schemas, automaticity, working memory, consciousness, and System 1 and System 2 cognition. In particular, we will discuss proving actions, such as the construction of proof frameworks that could be automated, thereby reducing the burden on working memory and enabling university students to devote more resources to the truly hard parts of proofs.

Key words: proof construction, behavioral schemas, automaticity, consciousness, System 1 and System 2 cognition

Introduction

Suppose a mathematician would like to help a student who has had difficulties solving a problem, for example, difficulties involving the chain rule. In this case, it is likely that the mathematician could look at the student's written work, explain how the chain rule works, and provide practice problems. We think students' written work can often be used in this way. However, suppose the student had difficulty, not with content, but with the process of proving. For example, suppose the student had understood the hypotheses of the theorem to be proved, but had failed to focus on the conclusion and unpack its meaning. This can be detrimental to constructing a proof; however, failing to focus on the conclusion typically does not show in a student's written work. Because we are interested in teaching proof construction, we would like to find a finer-grained perspective than just using students' written work. This has led us to the present nascent theoretical perspective based on actions, including mental actions. In the remainder of the introduction, we mention a number of ideas from psychology that we will call on and then foreshadow how they can be used. Then in the paper itself, we consider situations and actions; situation-action links; behavioral schemas; consciousness, implicit learning, and automaticity; decomposing the proving process; seeing similarities, searching and exploring; and implications thereof.

Much has been written in the psychological, neuropsychological, and neuroscience literature about ideas of conceptual and procedural knowledge, explicit and implicit learning, behavioral schemas, automaticity, working memory, consciousness, and System 1 (S1) and System 2 (S2) cognition (e.g., Bargh & Chartrand, 2000; Bargh & Morsella, 2008; Bor, 2012; Cleeremans, 1993; Hassin, Bargh, Engell, & McCulloch, 2009; Stanovich, 2009, Stanovich & West, 2000). In trying to relate these ideas to proof construction, we have discussed procedural knowledge, situation-action links, and behavioral schemas (Selden, McKee, & Selden, 2010; Selden & Selden, 2011) However, more remains to be done in order to weave these ideas into a coherent perspective. In doing this, a key idea is the roles that S1 and S2 cognition can play in proof construction. S1 cognition is fast, unconscious, automatic, effortless, evolutionarily ancient, and places little burden on working memory. In contrast S2 cognition is slow, conscious, effortful, evolutionarily recent, and puts considerable call on working memory (Stanovich & West, 2000). Of the several kinds of consciousness, we are referring to phenomenal consciousness—approximately, reportable experiences.

It is our conjecture that large parts of proof construction can be automated, that is, that one can facilitate mid-level university students in turning parts of S2 cognition into S1 cognition, and that doing so would make more resources, such as working memory, available for the truly hard problems that need to be solved to complete many proofs. In particular, because a proof construction is a sequence of actions, or the results of actions, that arise from situations in the partly completed proof, with practice some of the situations can be "linked" to actions in an automated way. Here is an example of one such possible automated situation-action link. One might be starting to prove a statement having a conclusion of the form *p* or *q*. This would be the situation at the beginning of the proof construction. If one had encountered this situation a few times before, one might readily take an appropriate action, namely, assume *not p* and prove *q* or vice versa. While this action can be warranted by logic, there would no longer be a need to do so.

We are interested both in how various types of knowledge (e.g., implicit, explicit, procedural, conceptual) are used during proof construction, and also in how such knowledge can be constructed. If that were better understood, then it might be possible to facilitate university students' learning through proof construction experiences. Although one can learn some things from lectures, this is almost certainly not the most effective, or efficient, way to learn proof construction. Indeed, inquiry-based transition-to-proof courses seem more effective than lecture-based courses (e.g, Smith, 2006).

The idea that much of the deductive reasoning that occurs during proof construction could become automated may be counterintuitive because many psychologists (Schechter, 2012), and (given the terminology) probably many mathematicians, assume that deductive reasoning is largely S2. Nevertheless, we suggest that, with growing expertise, proof construction can become a combination of S1 and S2.

Situations and Actions

We mean by an (inner) situation in proving, a portion of a partly completed proof construction, perhaps including an interpretation, drawn from long term memory, that can suggest a further action. The interpretation is likely to depend on recognition of the situation which is easier than recall perhaps because fewer brain areas are involved (Cabeza, et al., 1997). An inner situation is unobservable. However, a teacher can often infer an inner situation from the corresponding outer situation, that is, from the, usually written, portion of a partly completed proof.

Here we are using the term, action, broadly, as a response to a situation. We include not only physical actions (e.g., writing a line of a proof), but also mental actions. The latter can include trying to recall something or bringing up a feeling, such as a feeling of caution or of self-efficacy. We also include "meta-actions" meant to alter one's own thinking, such as focusing on another part of a developing proof construction.

Situation-Action Links

If, in several proof constructions in the past, similar situations have corresponded to similar actions, then, just as in classical Pavlovian conditioning, a link may be learned, so that another similar situation yields the corresponding action in future proof constructions without the earlier need for deliberate cognition, that is, the action occurs almost automatically. Use of *situation-action links* strengthens them and after sufficient practice/experience, they can become overlearned, and thus, automatic. Morsella (2009, p. 13) has pointed out, "Regarding skill learning and automaticity, it is known that the neural correlates of novel actions are distinct from those of actions that are overlearned, such as driving or tying one's shoes. Regions [of the brain] primarily responsible for the control of movements during the early stages of skill acquisition are different from the regions that are activated by overlearned actions. In essence, when an action becomes automatized, there is a 'gradual shift from cortical to subcortical involvement …' ". Returning to mental actions, because cognition often involves inner speech, which in turn is connected with the physical control of speech production, the above information on the brain regions involved in skill acquisition is at least a hint that when one has automated a situation-action link that one has not only converted it from S2 to S1 cognition, but also that different parts of the brain are involved in access and retrieval. Something very similar to the above ideas of automaticity in proof construction has been investigated by social psychologists such as Bargh and Chartrand (2000).

Behavioral Schemas

Some actions can be decomposed into sequences of smaller actions. If the action in a situation-action link is minimal with respect to decomposition into smaller actions, then we call it a *behavioral schema*. We see behavioral schemas as partly conceptual knowledge (recognizing the situation) and partly procedural knowledge (the action), and as related to Mason and Spence's (1999) idea of "knowing to act in the moment". We suggest that, in the use of a situation-action link or a behavioral schema, almost always both the situation and the action (or its result) will be at least partly conscious. Also, at least for most people, it appears impossible to "chain together" several behavioral schemas completely outside of consciousness. For example, a person who could do so, should easily be able to start with the equation $3x + 5 = 14$ (which can easily be solved with only three behavioral schemas), and *without bringing anything else to mind*, immediately say $x = 3$. We expect that most people cannot do this, so we wonder what is happening (outside of consciousness) for mathematicians for whom a complex proof is said to come to mind all at once. Such a consideration is beyond the scope of our current perspective, and we think it might be a question for neuroscience or for those studying giftedness.

Implicit Learning of Behavioral Schemas

It appears that the entire process of learning a behavioral schema, as described above, can be implicit. We note a person can acquire a behavioral schema without being aware that this is happening. Indeed, such unintentional, or implicit, learning happens frequently and has been studied by psychologists and neuroscientists (Cleeremans, 1993). In the case of proof construction, it is possible that after the experience of proving a considerable number of theorems in which similar situations occur, an individual might acquire a number of relevant behavioral schema, and as a result, simply not have to think quite so deeply as before about certain portions of the proving process and might, as a consequence, make fewer "wrong turns".

Something similar has been described in the psychology literature regarding the automated actions of everyday life. For example, an experienced driver can reliably stop at a traffic light while carrying on a conversation. But not all automated actions are positive. For example, a person can develop a prejudice without being aware of the acquisition process and can even be unaware of its triggering features (Cleeremans & Jiménez, 2001). This suggests that we should consider the possibility of mathematics students developing similarly unintended negative situation-action links implicitly during mathematics learning, and in particular, during proof construction.

Detrimental Behavioral Schemas

We begin with a simple and perhaps very familiar algebraic error. Many teachers can recall having a student write $\sqrt{(a^2 + b^2)} = a + b$, giving a counterexample to the student, and then having the student making the same error somewhat later. Rather than being a misconception (i.e., believing something that is false), this may well be the result of an implicitly learned behavioral schema. If so, the student would not be thinking very deeply about this calculation when writing it. Furthermore, having previously understood the counterexample would also have little effect in the moment. It seems that to weaken/remove this particular detrimental schema, the triggering situation, $\sqrt{(a^2 + b^2)}$, should occur a number of times when the student can be prevented from automatically writing "$= a + b$" in response. However, this might be difficult to arrange.

For another example of an apparently implicitly learned detrimental behavioral schema, we turn to Sofia, a first-year graduate student in a course meant to help students learn to improve their proving skills. Sofia was a diligent student, but as the course progressed what we came to call an "unreflective guess" schema emerged (Selden, McKee, & Selden, 2010, pp. 211-212). After completing just the formal-rhetorical part of a proof (essentially a proof framework) and realizing there was more to do, Sofia often offered a suggestion that we could not see as being remotely helpful. At first we thought she might be panicking, but on reviewing the videos there was no evidence of that. A first unreflective guess tended to lead to another, and another, and after a while, the proof would not be completed.

In tutoring sessions, instead of trying to understand, and work with, Sofia's unreflective guesses, we tried to prevent them. At what appeared to be the correct time, we offered an alternative suggestion, such as looking up a definition or reviewing the notes. Such positive suggestions eventually stopped the unreflective guesses, and Sofia was observed to have considerably improved in her proving ability by the end of the course.

Decomposing the Proving Process

In order to begin helping students automate certain portions of the proving process by developing positive behavioral schema, we would like to decompose the reasoning part of the proving process and focus on those portions that frequently occur. Some possibilities are: (1) writing the first- and second-level proof frameworks (Selden & Selden, 1995), which themselves can have parts; (2) noting when a conclusion is negatively phrased (e.g., a set is not empty or a number is irrational) and automatically attempting a contrapositive proof or a proof by contradiction; and (3) noting when the conclusion asserts the logical equivalence of two mathematical assertions and knowing there are two implications to prove.

One can also change one's focus by deciding to unpack the conclusion of a theorem, by finding or recalling a relevant definition, or by applying a definition—actions that are part of constructing a second-level proof framework (Selden, Benkhalti, & Selden, 2014; Selden & Selden, 1995). Having done that, one might get a feeling of knowing, of self-efficacy (Selden & Selden, 2014), or even of not knowing what to do next, that is, one might be at an impasse. Upon reaching such an impasse, one might spontaneously do something else for a while. That action may, or may not, include the conscious intent that doing so, and coming back later, might allow one to get a new idea. That is, while one might just give up in frustration (for the moment), one might "know" that, in the past, such alternative actions had been beneficial for getting new ideas.

Proving exercises that we have tried to help students automate, with only modest success to date, are converting formal mathematical definitions into operable definitions (Bills & Tall, 1998). In this regard, we have tried providing "flash cards" for our transition-to-proof course students to practice with. On one side of a typical card was "What can you say if you know $f: X \to Y$ is a function, $A \subseteq Y$, and $x \in f^{-1}(A)$?". On the other side of the card was "$f(x) \in A$". One might think that this sort of translation into an operable form would be automatic (without such practice) given the definition $f^{-1}(A) = \{ x \in X \mid f(x) \in A\}$, but we have found that it is not, even when the definition can be consulted, and does not need to be remembered.

Seeing Similarities, Searching and Exploring

How does one recognize situations as similar? Different people see diverse situations as similar depending both upon their past experiences and upon what they choose to, or happen to, focus on or attend to. While similarities can be extracted implicitly during exposure

(Markman & Gentner, 2005), as teachers and researchers we want to direct students' attention to relevant proving similarities. It would seem that some conceptual knowledge would come to bear when learning to see situations as similar.

For example, we do not, as yet, have suggestions for helping students to "see" that the situations of a set being empty (i.e., having no elements), of a number being irrational (i.e., not rational), and of the primes being infinite (i.e., not finite) are similar. These three situations--empty, irrational, and infinite--do not seem similar until one rephrases them to expose the existence of a negative definition. Unless students rephrase these situations implicitly, or perhaps explicitly, it seems unlikely that they would see this similarity and link these situations, when they are given as conclusions to theorems, to the action of beginning a contrapositive proof or a proof by contradiction.

In addition to automating small portions of the proving process, we would also like to enhance students' searching skills (i.e., their tendency to look for previously proved related results that might be helpful) and to enhance students' tendency and ability to "explore" possibilities when they don't know what to do next. In a previous paper (Selden & Selden, 2014, p. 250), we discussed the kind of exploring that was entailed in proving the rather difficult (for students) *Theorem: If S is a commutative semigroup with no proper ideals, then S is a group*.

Teaching and Research Implications

The above considerations can lead to many possible teaching interventions, not all of which can, or should, be attempted simultaneously if one also wants to obtain data on their effectiveness. This then brings up the question of priorities. What proving actions, of the kinds discussed above are most useful for mid-level university mathematics students to automate, when they are just learning how to construct proofs? Since such students are often asked to prove relatively easy theorems—ones that follow directly from definitions just given—it would seem that noting the kinds of structures that occur most often might be a place to start. Indeed, since every proof can be constructed using a proof framework, we consider constructing proof frameworks as a reasonable place to start.

Also, helping students interpret formal mathematical definitions so that these become operable might be another place to start. This would be helpful because one often needs to unpack a definition into an operable form in order to use it to construct a second-level framework. However, it is becoming clear, that learning to use definitions is a multi-stage process.

Finally, we believe this particular perspective on proving, using situation-action links and behavioral schemas, together with information from psychology and neuroscience, is new to the field.

References

Bills, L., & Tall, D. (1998). Operable definitions in advanced mathematics: The case of the least upper bound. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2 (pp. 104-111). Stellenbosch, South Africa: University of Stellenbosch.

Bargh, J. A., & Chartrand, T. L. (2000). Studying the mind in the middle: A practical guide to priming and automaticity research. In H. T. Reid & C. M. Judd (Eds.), *Handbook of research methods in social psychology* (pp. 253-285). New York: Cambridge University Press.

Bargh, J. A., & Morsella, E. (2008). The unconscious mind. *Perspectives on Psychological Science*, *3*(1), 73-79.

Bor, D. (2012). *The ravenous brain*. New York: Basic Books.

Cabeza, R., Kapur, S., Craik, F. I. M., McIntosh, A. R., Houle, S., and Tulving, E. (1997).

Functional neuroanatomy of recall and recognition: A PET study of episodic memory. *Journal of Cognitive Neuroscience, 9*, 254-265.

Cleeremans, A. (1993). *Mechanisms of implicit learning*: *Connectionist models of sequence processing*. Cambridge, MA: MIT Press.

Cleeremans, A., & Jiménez, L. (2001). Implicit learning and consciousness: A graded, dynamic perspective. Retrieved December 1, 2014, from http://srsc.ulb.ac.be/axcWWW/papers/pdf/01-AXCLJ.pdf.

Hassin, R. R., Bargh, J. A., Engell, A. D., & McCulloch, K. C. (2009). Implicit working memory. *Conscious Cognition*, *18*(3), 665-678.

Markman, A. B., & Gentner, D. (2005). Nonintentional similarity processing. In T. Hassin, J. Bargh, & J. Uleman, *The new unconscious* (pp. 107-137). New York: Oxford University Press.

Mason, J., & Spence, M. (1999). Beyond mere knowledge of mathematics: The importance of knowing-to-act in the moment. *Educational Studies in Mathematics*, *28*(1-3), 135-161.

Morsella, E. (2009). The mechanisms of human action: Introduction and background. In E. Morsella, J. A. Bargh, & P. M. Goldwitzer (Eds.), *Oxford Handbook of Human Action* (pp. 1-34). Oxford: Oxford University Press.

Schechter, J., (2013). Deductive reasoning. In H. Pashler (Ed.), *Encyclopedia of the Mind*. Los Angeles, CA: SAGE Publications.

Selden, A., McKee, K., & Selden, J. (2010). Affect, behavioural schemas, and the proving process. *International Journal of Mathematical Education in Science and Technology*, *41*(2), 199-215.

Selden, A., & Selden, J. (2014). The roles of behavioral schemas, persistence, and self-efficacy in proof construction. In B. Ubuz, C. Hasar, & M. A. Mariotti (Eds.). *Proceedings of the Eighth Congress of the European Society for Research in Mathematics Education* [CERME8] (pp. 246-255). Ankara, Turkey: Middle East Technical University.

Selden, J., Benkhalti, A., & Selden, A. (2014). An analysis of transition-to-proof course students' proof constructions with a view towards course redesign. *Proceedings of the 17th Annual Conference on Research in Undergraduate Mathematics Education*. Available online at http://sigmaa.maa.org/rume/Site/Proceedings.html.

Selden, J., & Selden, A. (2011). The role of procedural knowledge in mathematical reasoning. In B. Ubuz (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4 (pp. 124-152). Ankara, Turkey: Middle East Technical University.

Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*, *29*, 123-151.

Smith, J. C. (2006). A sense-making approach to proof: Strategies of students in traditional and problem-based number theory courses. *Journal of Mathematical Behavior*, *25*, 73-90.

Stanovich, K. E. (2009). Distinguishing the reflective, algorithmic, and autonomous minds: Is it time for a tri-process theory? In J. Evans & K. Frankish (Eds.), *In two minds: Dual processes and beyond* (pp.55-88), Oxford: Oxford University Press.

Stanovich, K. E., & West, R. F. (2000). Individual differences in reasoning: Implications for the rationality debate? *Behavioral and Brain Sciences, 23,* 645-726.

An examination of college students' reasoning about trigonometric functions with multiple representations

Soo Yeon Shin
Minnesota State University

The purpose of this study is to examine how individual college students reason through tasks using trigonometric functions and translate among different types of representations of trigonometric functions across various mathematical tasks.

To examine how individual college students reason through tasks using trigonometric functions among different types of representations, the author used a qualitative embedded multi-case study for this study. An embedded design was used to study various units within an identifiable case. In this study, the tasks served as the cases, with each case/task being purposefully designed to begin in a different one of Duval's (2000, 2004) representation registers (natural language (N), drawings (D), symbolic systems (S), and graphs and mathematical diagrams (G)). Analysis of six participants' work was embedded as sub-units within each of these cases. Data were collected and analyzed under two frameworks, Duval's (2000, 2004) cognitive approach and Lithner's (2004, 2008) mathematical reasoning–imitative reasoning and creative reasoning.

In this study, the multiple-functional registers N and D were used less often by the participants than the mono-functional registers S and G. However, participants used mainly creative reasoning when employing the multiple-functional registers, N and D. Also, it was likely to see registers S and G used together when registers N and D were employed. Registers S and G were often used with imitative reasoning, although the use of register G contributed to several examples of local and global creative reasoning.

Overall, translations among different registers that were based upon creative reasoning were more likely to lead participants to be able to complete given tasks. This study illustrated how college students employed their reasoning during open-ended and unfamiliar tasks, which helped students disclose their reasoning in informal ways without memorizing solutions. The study illustrated some interesting ways in which students were able to creatively use different registers to help them when they became stuck in the register in which they were working. By combining Duval's and Lithners' frameworks together, imitative and creative reasoning were classified in a new approach that could be usefully applied to studies other types of functions in future work.

References

Duval, R. (2000). Basic issues for research in mathematics education. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education* (pp. 155–169). Hiroshima, Japan: Hiroshima University.

Duval, R. (2004). A crucial issue in mathematics education: The ability to change representation register. In M. Niss (Ed.), *Proceedings of the 10th International Conference on Mathematical Education* (pp. 1–17). Roskilde, Denmark: Roskilde University.

Lithner, J. (2004). Mathematical reasoning in calculus textbook exercises. *Journal of Mathematical Behavior*, *23*(4), 405-427. doi:10.1016/j.jmathb.2004.09.003

Lithner, J. (2008). A research framework for creative and imitative reasoning. *Educational Studies in Mathematics*, *67*(3), 255-276. doi:10.1007/s10649-007-9104-2

The generalization of the function schema: The case of parametric functions

Harrison E. Stalvey
Georgia State University

Draga Vidakovic
Georgia State University

This paper reports on an investigation of fifteen second-semester calculus students' understanding of the concept of parametric function, as a special relation from a subset of $\mathbb{R}$ *to a subset of* $\mathbb{R}^2$*. A substantial amount of research has revealed that the concept of function, in general, is very difficult for students to understand. Furthermore, several studies have investigated students' understanding of various types of functions. However, very little is known about how students reason about parametric functions. Employing APOS theory as the guiding theoretical perspective, this paper describes how students reason about parametric functions given in the form* $p(t) = (f(t), g(t))$*. One common misconception that was observed among students is addressed.*

Key words: Parametric function, Calculus students, APOS

Introduction

The concept of function is one of the most fundamental concepts in mathematics. Despite its emphasis in secondary mathematics curriculum, researchers have reported that undergraduate students continually demonstrate an impoverished understanding of the function concept (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Carlson, 1998; Oehrtman, Carlson, & Thompson, 2008; Thompson, 1994; Vinner & Dreyfus, 1989). As a result, there have been calls for "instructional shifts that promote rich conceptions and powerful reasoning abilities" (Oehrtman et al., 2008, p. 27). To serve as a backdrop for developing such rich conceptions, Oehrtman et al. (2008) recommend that students should experience diverse function types with an emphasis on multiple representations, including different coordinate systems (p. 29). Moreover, in the *Principles and Standards for School Mathematics*, the National Council of Teachers of Mathematics has called for curriculum to incorporate parametric equations as representations of functions and relations (NCTM, 2000, p. 296). In order to effectively respond to these instructional calls, it is crucial to consider students' conceptions (and misconceptions) of parametric functions.

Several recent studies have investigated students' understanding of functions more sophisticated than real-valued functions of a single-variable, such as two-variable functions (Kabael, 2011; Martínez-Planell & Trigueros Gaisman, 2012; Trigueros & Martínez-Planell, 2010; Weber & Thompson, 2014), while other studies have explored students' understanding of functions in different coordinate systems (Montiel, Vidakovic, & Kabael, 2008; Montiel, Wilhelmi, Vidakovic, & Elstak, 2009; Moore, Paoletti, & Musgrave, 2013). However, only a few studies have addressed students' understanding of parametric functions (Bishop & John, 2008; Keene, 2007; Trigueros, 2004), and no prior study has focused specifically on calculus students' understanding of parametric functions. The goal of this study was to investigate and document calculus students' conceptual development of the notion of parametric function as a special relation from a subset of $\mathbb{R}$ to a subset of $\mathbb{R}^2$. This report addresses the following research questions:

1. For $p(t) = (f(t), g(t))$, can students perceive p as a function?
2. What are students' misconceptions when reasoning about parametric functions given in the form $p(t) = (f(t), g(t))$?

Theoretical Framework

The theoretical framework guiding this study is Action–Process–Object–Schema (APOS) theory (Asiala et al., 1996). An *action* is a transformation of objects by reacting to external cues that give precise details on what steps to take. When an action is repeated, and the individual reflects on it, the action can be interiorized into a *process*. An individual who has a process conception can reflect on or describe the steps of the transformation without actually performing those steps. Additionally, new processes can be constructed by the means of reversal of a process or the coordination of two or more processes. When an individual becomes aware of the process as a totality and can perform additional actions or processes on it, then the process has been encapsulated into an *object*. Furthermore, objects can be de-encapsulated to obtain the processes from which they came. The individual's collection of actions, processes, and objects organized in a structured manner is his or her *schema*. When an individual learns to apply an existing schema to a wider collection of phenomena, then we can say that the schema has been generalized.

Context and Methodology

This study was conducted in one section of the course *Calculus of One Variable II* at a large public university in the southeastern United States during the fall semester of 2013. Out of forty-four students enrolled, fifteen volunteered to be in the study. These students participated in a semi-structured, video-recorded interview that was approximately 1.5 hours long. Some interviews were conducted in groups of up to three students, while other interviews were individual, depending on students' availability outside of class. The students were first given a questionnaire, which they completed individually on paper prior to the discussion with the interviewer. For each question, the students were asked to explain their solutions verbally on an individual basis, but group discussion was encouraged when it arose. The instructor for the course (one of the authors) was not present during the interviews. Instead, the interviews were conducted by three other instructors, each with a carefully written protocol.

Although the function concept is considered prerequisite knowledge for the calculus sequence, the instructor for the course briefly revisited the definition as a rule that accepts an input and returns a unique output. Then the instructor adapted this definition to define a parametric function as a function that accepts a real number as an input and returns one ordered pair of real numbers as an output, which was supplemental to the way the concept was introduced in the textbook for the course. It should be noted that the interviews were conducted after the students were taught the concept of parametric function, which is a standard topic in the calculus II curriculum where this study took place.

The data reported in this paper is from students' responses to one subquestion of the interview which asked to determine whether $p(t) = (t^2, t^3)$ represents a function. A correct response would affirm p as a function because for one value of t there is a unique ordered pair. It was hypothesized that in order to develop an understanding of parametric functions, an individual should possess schemas for function and $\mathbb{R}^2$. Furthermore, in order to generalize the function schema and view $p(t) = (t^2, t^3)$ as representing a function, it was conjectured that the individual should coordinate his or her schemas for function and $\mathbb{R}^2$.

Results

Out of fifteen students, three affirmed that $p(t) = (t^2, t^3)$ represents a function and gave correct justification, two students gave an affirmative answer with correct justification as a result of prompting from the interviewer, and ten students gave a negative answer or an affirmative answer with incorrect or unclear justification (even with prompting).

Mary is the most descriptive example of a student who without prompting affirmed that p is a function with correct reasoning. She said:

M: Well, if you're thinking of like the point, then you're only going to have one point. So that's why I said it was a function, because for each t like there's only one point that it's going to come out with.

Mary further explained:

M: You get a x output and a y output, but then I thought about it, and then it's like a coordinate on a plane. So it's like for every t that you plug in, you get a point on the plane. So it eventually creates a graph. So that's kinda why I thought it's a function.

In Mary's reasoning, her schema for $\mathbb{R}^2$ is apparent when she described how $p(t) = (t^2, t^3)$ constructs a point in the plane: "You get a x output and a y output ... and then it's like a coordinate on a plane. So it's like for every t that you plug in, you get a point on the plane." When she said, "so it eventually creates a graph," she is treating a point as an object and is imagining the process of the point tracing a curve in the plane. She concluded that p is a function by coordinating her schema for $\mathbb{R}^2$ with her schema for function: "So that's why I said it was a function, because for each t like there's *only one point* that it's going to come *out* with." In this previous statement, Mary is clearly applying the generic definition of function and treating t as the input and the point (x,y) as the output. Because Mary did not need to evaluate $p(t)$ at particular values of t or plot points, she is considered to be coordinating her schemas for function and $\mathbb{R}^2$ at the process level.

Several misconceptions about parametric functions emerged when students were reasoning about the function given by $p(t) = (t^2, t^3)$. These misconceptions fell into four categories: (1) misconceptions about the function value $p(t)$, (2) misconceptions about the domain and range of p, (3) misconceptions about the vertical line test, and (4) misconceptions about the input and output of p. Due to space limitations, this report focuses on the most common misconception, which pertained to the function value $p(t)$.

Misconception: Function value

The misconception about the function value $p(t)$ appeared in a couple of different ways. One way, which was anticipated, pertained to the uniqueness of the function value $p(t)$. Eight students perceived the function value as not unique because evaluating $p(t)$ results in a value for the first component, t^2, and a value for the second component, t^3, which were viewed as two outputs instead of one output in the form of an ordered pair. Other students were not bothered by the possibility of getting different values for the components. In fact, this was their reasoning for affirming p as a function. In particular, two students believed that the values of the components had to be different in order for p to be a function.

In the following excerpt, Lee and Bailey express the misconception about the uniqueness of $p(t)$, while Nicole expresses the misconception about the distinction of the coordinates.

L: I put no for (c) because I remember him (*the instructor*) saying for time you have to have one output for every input. So it looks like for this one you have two outputs for time, which I thought made no sense.

N: I put yes because, like, other than 1 and 0, the other numbers they were different points, and I thought pretty much that they were just points.

I: Okay.
B: I put no simply because one t input and then there are two outputs.

Although Nicole affirmed p as a function, her reasoning was not correct. When she said, "other than 1 and 0, the other numbers were different points," she was referring to the fact that x and y have different values at all values of t except 0 and 1. In particular, she was saying that (t,x) and (t,y) are "different points" except when $t=0$ and $t=1$. This suggests that when determining whether p is a function, Nicole was considering separately the two functions defined by $x(t)$ and $y(t)$, instead of considering (x,y) as a single output of p. This was confirmed later in her written work when she graphed y versus t instead of the plane curve defined by (x,y). Moreover, it seems that Nicole required that x be different from y (at least, most of the time), as this was her reasoning when affirming p as a function. Lee and Bailey, on the other hand, rejected p as a function, claiming that an input value t gives two output values. Furthermore, nowhere in their reasoning did they consider t^2 and t^3 as components of an ordered pair, indicating the lack of coordination of schemas for function and $\mathbb{R}^2$.

As a result of prompting, two of the eight students who initially expressed a misconception about the uniqueness of the function value $p(t)$ were able to resolve their misconception and affirm p as a function with correct reasoning. Lee is an example of a student who resolved this misconception, while Bailey is an example of a student who did not, as illustrated in the next two excerpts.

After Lee, Nicole, and Bailey's initial elaboration about their reasoning when answering this question, the interviewer prompted them to think about the graph of the curve defined by $p(t)$. In the following excerpt, when Nicole describes plugging in values for t to obtain points, Lee and Bailey reconsider their earlier answers that rejected p as a function.

I: How would you go about graphing it?
N: I have no idea [*laughs*].
I: You have no idea? But you're the only one who said it was a function [*laughs*].
N: I mean like ... I just put numbers in ... like if it was, you have $(0,0)$, then (1,1). If you have 2, it would be 4 and 8, something like that ...
L: Oh, it is a function. If you plot the points, it's just saying, like, let's say we choose t to be 1, then the coordinates would $(1,1)$. Then we choose t to be 2. Then then coordinates would be $(4,8)$. And keep going like that ... I didn't even look at it to plot the points. I just saw that there were three ts [*laughs*].
I: Remind us again what your definition of a function was.
B: One input, one output.
I: If you take a single value of t ... Let's take 2 for you. You take 5. And $t=-1$. Do you get more than one input, I mean, more than one output?
B: Yes.
L: You get two numbers, but then you're getting one output as a ordered pair.
I: As an ordered pair. So for each t in ...
L: There's one ordered pair out.
I: [*Speaking to Bailey*] Does that satisfy your definition of a function?
B: Yes.

Lee explained that she rejected p as a function because she "saw that there were three ts." This suggests that she was not applying her schema for $\mathbb{R}^2$ in order to consider the output as an ordered pair. However, when Nicole described plugging in values of t to create ordered pairs, Lee was prompted to coordinate her schemas for function and $\mathbb{R}^2$ and assign a unique ordered pair to each value of t. As a result, Lee was able to perceive p as a function and give the correct justification, "you get two numbers, but then you're getting one output as a ordered pair." Furthermore, after Lee described plugging in values of t, she said, "and keep going like that." This statement indicates that she can imagine evaluating (t^2, t^3) for all values of t, suggestive of a process conception of parametric function.

In the previous excerpt, Bailey also changed his answer to the affirmative, but he did not give any reasoning to suggest that he understood why. In the following excerpt, Bailey changes his answer again, rejecting p as a function. His further reasoning demonstrates a weak schema for $\mathbb{R}^2$. This discussion took place when Bailey was trying to determine if the y-coordinate is a function of the x-coordinate in $p(t) = (t^2, t^3)$.

B: You know what, I don't think (c) is a function now. I'm going back to no.
I: Going back to no?
B: But this can't be [*pause*] this can't be a point.
I: Why not?
B: The variables are the same.
I: So?
B: I don't see how it's an ordered pair.
I: It's in parentheses with a comma.
B: But it's the same variable, t. So how can it be ... it's at two places at the same ...
I: No, it's only at one place. When t is 0, it's at the place $(0,0)$. When t is equal to 1, it's at the ordered pair $(1,1)$. At that one particular spot.
B: Okay. I don't see it, because t is our x-value [*makes hand motions in the air drawing a horizontal line*].
I: Why?
B: Because it's p of t. So if I put in 2, I get 4 and 8.
I: 4 comma 8. You get the ordered pair.
B: But they're both ts. So aren't they both points on the line t?
I: No.
B: No? Okay.

Bailey's response suggests that he is unable to coordinate the processes of $x(t) = t^2$ and $y(t) = t^3$ to construct an ordered pair because he cannot conceive that the components of an ordered pair could be defined by functions. He described plugging in 2 for t and getting the values 4 and 8, which are "both points on the line t." From this statement, it seems that when Bailey evaluated $p(2)$, he constructed two separate ordered pairs, $(2,4)$ and $(2,8)$. Furthermore, referring to t as a line and making a linear hand motion when he stated, "t is our x-value," suggests that he views t as the horizontal axis and t^2 and t^3 as values on the vertical axis. The idea of viewing t as belonging to an axis has been offered in literature to imagine $(x,y,z) = (x(t), y(t), t)$ as a point in space (Keene, 2007; Oehrtman et al., 2008). This idea also agrees with Euler's presentation of non-planar curves in Book II of his *Introductio in Analysin Infinitorum*. However, Bailey would need to first construct a schema for $\mathbb{R}^3$ in order to view t, t^2, and t^3 as values on mutually perpendicular axes.

Conclusion

It was previously hypothesized that an individual should possess schemas for function and $\mathbb{R}^2$ in order to develop an understanding of parametric functions. In particular, it was conjectured that the coordination of these schemas is necessary to generalize the function schema and view $p(t) = (t^2, t^3)$ as representing a function. The results reported in this paper support this hypothesis.

Students who, either with or without prompting, correctly affirmed $p(t) = (t^2, t^2)$ as representing a function appeared to apply both schemas for function and $\mathbb{R}^2$. Meanwhile, students who rejected p as a function appeared to have an underdeveloped schema for function or $\mathbb{R}^2$ (or both). Students with a weak function schema did not appear to use a coherent function definition when discerning whether or not p is a function. On the other hand, students who demonstrated having a well-developed function schema did not always apply (or possibly even possess) a schema for $\mathbb{R}^2$, which led to the misconception that $p(t)$ is two-valued, namely t^2 and t^3. Based on these results, it is recommended that students be provided with opportunities to develop and strengthen their schemas for function and $\mathbb{R}^2$ prior to the study of parametric functions.

References

Asiala, M., Brown, A., DeVries, D. J., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education. In J. Kaput, A. H. Schoenfeld, & E. Dubinsky (Eds.), *Research in collegiate mathematics education II, CBMS Issues in Mathematics Education* (Vol. 6, pp. 1–32). Providence, RI: American Mathematical Society.

Bishop, S., & John, A. (2008). Teaching high school students parametric functions through covariation. Unpublished masters project, Arizona State University, Tempe, AZ.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics*, *23*, 247–285.

Carlson, M. P. (1998). A cross-sectional investigation of the development of the function concept. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *Research in collegiate mathematics education III, CBMS Issues in Mathematics Education* (Vol. 7, pp. 114–162). Providence, RI: American Mathematical Society.

Kabael, T. (2011). Generalizing single variable functions to two-variable functions, function machine and APOS. *Educational Sciences: Theory & Practice*, *11*(1), 484–499.

Keene, K. A. (2007). A characterization of dynamic reasoning: Reasoning with time as parameter. *The Journal of Mathematical Behavior*, *26*, 230–246.

Martínez-Planell, R., & Gaisman, M. T. (2012). Students' understanding of the general notion of a function of two variables. *Educational Studies in Mathematics*, *81*, 365–384.

Montiel, M., Vidakovic, D., & Kabael, T. (2008). Relationship between students' understanding of functions in Cartesian and polar coordinate systems. *Investigations in Mathematics Learning*, *1*(2), 52–70.

Montiel, M., Wilhelmi, M. R., Vidakovic, D., & Elstak, I. (2009). Using the onto-semiotic approach to identify and analyze mathematical meaning when transiting between different coordinate systems in a multivariate context. *Educational Studies in Mathematics*, *72*, 139–160.

Moore, K. C., Paoletti, T., & Musgrave, S. (2013). Covariational reasoning and invariance among coordinate systems. *The Journal of Mathematical Behavior*, *32*, 461–473.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Oehrtman, M., Carlson, M., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education, MAA*

Notes 73 (pp. 27–41). Washington, DC: Mathematical Association of America.

Thompson, P. W. (1994). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *Research in collegiate mathematics education I, CBMS Issues in Mathematics Education* (Vol. 4, pp. 21–44). Providence, RI: American Mathematical Society.

Trigueros, M. (2004). Understanding the meaning and representation of straight line solutions of systems of differential equations. In D. E. McDougall & J. A. Ross (Eds.), *Proceedings of the 26th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 127–134). Toronto: University of Toronto.

Trigueros, M., & Martínez-Planell, R. (2010). Geometrical representations in the learning of two-variable functions. *Educational Studies in Mathematics*, *73*, 3–19.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, *20*(4), 356–366.

Weber, E., & Thompson, P. W. (2014). Students' images of two-variable functions and their graphs. *Educational Studies in Mathematics*.

Using journals to support student learning; The case of an elementary number theory course

Christina Starkey, Hiroko K. Warshauer, and Max L. Warshauer
Texas State University

We present results from a study that examined the use of journal writing by undergraduates in an Honor's Number Theory course and how the journals supported students' learning, attitudinal changes, and proof writing. The 17 undergraduates in this course submitted weekly journals online to their instructor and reflected on their mathematical learning. The instructor provided comments to each of the students' journal submissions that informed him of each student's successes, challenges, issues, and questions. We analyzed the journals and share our preliminary findings on what the journal writing revealed about students' learning and how their mathematical understanding developed over a semester. We include results of the pre-post survey of student attitudes toward mathematics along with interviews of 5 of the students that give additional insight into their experiences in the course. Future work will examine the uses of journals in other courses, and different ways journals support student learning

Keywords: Transitions to proof, Journaling, Number Theory

This poster will report on the results of a mixed methods study of an elementary number theory class in which students kept weekly reflective math journals as they learned how to prove. Students' performance in proof writing has been investigated in recent years (Weber, 2001; Moore, 1994; Raman, 2003). The research indicates that learning to read and write proofs presents a significant challenge to undergraduate students (Weber, 2001; Raman, 2003), and Moore (1994) asserts that the challenge stems from the significant shift in thinking away from a computational view of mathematics necessary for proof competency. To address this, researchers are increasingly calling for shifts away from traditional proof instruction, in which students are presented complete proofs and must reproduce them on exams, to instruction that makes students more active in the proving process (Yoo, 2008; Jones, 2000; Blanton, Stylianou, and David, 2009). This includes providing opportunities for students to reflect on proofs and to write proofs of their own. However, research is still needed to investigate innovative pedagogical approaches to teaching proof and how students' thinking about proof develops. Reflective journaling has been shown to be a unique, valuable tool for supporting students' learning and providing insight into students' thinking in other mathematical domains (Borasi and Rose, 1989; Clark, Waywood, & Stephens, 1993). This study investigated the questions: How do reflective journals support students' learning to prove in an undergraduate elementary number theory course? How do reflective journals demonstrate the development of students' thinking about proof in an undergraduate elementary number theory course?

Methods

The elementary number course in this study allows students at different levels of mathematical maturity to participate and work together. It also provides a context for students to learn how to explore problems deeply and give careful, rigorous mathematical proofs. The 17 undergraduate students in the course wrote weekly journal entries related to the elementary number theory course and submitted them online. The instructor of the course then read and responded to each student's submission. The journal assignments consisted of both structured and unstructured prompts. Our mixed methods study included pre-post attitude surveys, which we analyze quantitatively, using a t-test. A qualitative portion of our data collection included pre-post surveys asking the students' views about mathematical proof and their perceptions about journals; students' open-ended and structured journal responses and task-based interview transcripts with five of the students. The qualitative data were coded according to the framework provided in Borasi and Rose (1989) for the journaling component, and the quantitative attitudes surveys were analyzed using a modified Fennema-Sherman Attitude Scales (Kalder & Lesik, 2011; Tapia & Marsh, 2004). The students' proof attempts from the task-based interviews were coded using Raman's (2003) framework for students' proof ideas.

Results

The ongoing analysis of students' structured and unstructured journal entries, along with the interview data, suggest the students used the unstructured journal assignments primarily as a means to reflect on their feelings about the course material and their learning. However, in the structured journals, the students wrote specifically about the process of proving and their views about mathematics and proving. The entries also reveal students' thinking about their use of definitions, examples, and strategy when attempting to write proofs. All of the interviewees discussed an appreciation for instructor feedback on their journals, and mentioned that they saw changes in instruction based on their journal entries, which supported their learning. Implications of this study suggest that journaling creates an added dimension of communication for students and the instructor to support students learning the course material in a more responsive manner.

References

Blanton, M. L., & Stylianou, D. A., David, M. M. (2009). Understanding instructional scaffolding in classroom discourse on proof. In Despina A. Stylianou, Maria L. Blanton, & Eric J. Knuth (Eds.), *Teaching and learning proof across the grades* (290-307). New York, NY: Routledge.

Borasi, R., & Rose, B. (1989). Journal writing and mathematics instruction. *Educational studies in mathematics, 20*, 347-365.

Clarke, D. J., Waywood, A., & Stephens, M. (1993). Probing the structure of mathematical writing. *Educational studies in mathematics, 25*, 235-250.

Jones, K. (2000). The student experience of mathematical proof at university level. *International journal of mathematical education in science and technology, 31*(1), 53-40.

Kalder, R. S. & Lesik, S. A. (2011). A classification of attitudes and beliefs towards mathematics for secondary mathematics pre-service teachers and elementary pre-service teachers: An exploratory study using latent class analysis. *Issues in the Undergraduate Mathematics Preparation of School Teachers. 5*, 1- 21.

Moore, R. C. (1994). Making the transition to formal proof. *Educational studies in mathematics, 27*(3), 249-266. http://www.jstor.org/stable/3482952

Raman, M. (2003). Key ideas: What are they and how can they help us understand how people view proof? *Educational studies in mathematics, 52*, 319-325.

Tapia, M. & Marsh, G. E. (2004). An instrument to measure mathematics attitudes. *Academic Exchange Quarterly, 8*(2), 16-21.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational studies in mathematics, 48*, 101-119.

Yoo, S. (2008). *Effects of traditional and problem-based instruction on conceptions of proof and pedagogy in undergraduates and prospective mathematics teachers* (Doctoral Dissertation). Retrieved from the University of Texas Libraries Digital Repository. (http://repositories.lib.utexas.edu/handle/2152/17834)

The calculus laboratory: Mathematical thinking in the embodied world

Sepideh Stewart
University of Oklahoma

Keywords: Calculus, Geogebra, Three Worlds Model of Mathematical Thinking

The use of technology in teaching and learning has been explored by many mathematics education researchers. It is believed that, the technology itself is not a magic bullet to attract students to mathematics and more importantly help them to learn.

The research described here is a case study that took place at a large research university, while the author was teaching a section of the Calculus I course. Throughout the semester, the instructor held one lecture per week at the mathematics department's computer laboratory and encouraged students to solve calculus problems using the free software, *Geogebra*. During these sessions, the instructor answered questions and helped students individually or in groups. Almost all students (n=50) attended the laboratory sessions regularly and very seldom missed a class. Students were given approximately three days to complete the laboratory assignments and upload solutions in the dropbox for assessment (10% of total grade). The instructor then graded the assignments using an ipad and posted immediate feedback to students on D2L. The data for this study emerged from an online survey of the Calculus I students enrolled in the course and their reaction to the exposure to the weekly laboratory sessions.

The aim of the study was to allow students to experience calculus first-hand and discover many fascinating aspects of the subject on their own. The underlying theory to guide the research was Tall's (2013) framework of three world of embodied, symbolic and formal mathematical thinking. In his view, the world of conceptual embodiment is based on "our operation as biological creatures, with gestures that convey meaning, perception of objects that recognize properties and patterns...and other forms of figures and diagrams" (2010, p. 22). Embodiment can also be perceived as the construction of complex ideas from sensory experiences, giving body to an abstract idea.

Did visualizing complex and fancy equations such as $y^2 (y^2 - 4)=x^2 (x^2 -5)$ (devil's curve); looking for limits of functions and estimating local maximum and minimum values graphically, help a group of 50 students (non-mathematics majors) in this study to relate to what they were studying and connect them closer to calculus?

As part of a larger visualization project the author is working with a group of three mathematicians, a physicist and a cognitive psychologist to discover more about the embodied world of mathematical thinking and its role and importance in understanding mathematics.

References

Tall, D. O. (2010). Perceptions, operations, and proof in undergraduate mathematics. *Community for Undergraduate Learning in the Mathematical Sciences (CULMS) Newsletter*, 2, 21-28.
Tall, D. O. (2013). *How humans learn to think mathematically: Exploring the three worlds of mathematics*. Cambridge University Press.

Pedagogical challenges of communicating mathematics with students: Living in the formal world of mathematical thinking

Sepideh Stewart Ralf Schmidt
University of Oklahoma
John Paul Cook
University of Sciences and Arts of Oklahoma
Ameya Pitale
University of Oklahoma

In this paper we examine an Abstract Algebra professor and one of his students' thought processes simultaneously as the class was moving toward the proof of the Fundamental Theorem of Galois Theory. We employed Tall's theory of three worlds of mathematical thinking to trace which route (embodied, symbolic, formal) the mathematician was choosing to take his students to the formal world. We will discuss the pedagogical challenges of proving an elegant theory as the events unfolded.

Keywords: Reflections on Teaching, Abstract Algebra, Formal World of Mathematical Thinking

Introduction

Research in pedagogy at the university level is fairly new, and regrettably the amount of communication between mathematicians and mathematics educators on pedagogy is still very limited. According to Byers (2007) "*Many mathematicians usually don't talk about mathematics because talking is not their thing – their thing is the "doing" of mathematics" (p. 7)*. As Dreyfus (1991) suggested, "one place to look for ideas on how to find ways to improve students' understandings is the mind of the working mathematician. Not much has been written on how mathematicians actually work" (p. 29). Two decades later, Speer, Smith, and Horvath (2010) declare that "very little research has focused directly on teaching practice and what teachers do and think daily, in class and out, as they perform their teaching work" (p. 111). In recent years some mathematics professors have been more willing to examine and reflect on their own teaching styles, leading to a growing body of research in this area (Paterson, Thomas, & Taylor, 2011; Hannah, Stewart, & Thomas, 2011; 2013; Kensington-Miller, Yoon, Sneddon, & Stewart, 2013).

In this project, we collaborated with a mathematician to examine his thought processes over the period of two semesters. Since he was an abstract algebraist, naturally our attention shifted on the pedagogical challenges of teaching his subject. In addition to a lack of information about teaching practices in abstract algebra, there is considerable evidence documenting student difficulty with the subject's most basic concepts (Clark, Hemenway, St. John, Tolias, & Vakil, 2007; Dubinsky, Dautermann, Leron, & Zazkis, 1994). This situation has led one group of researchers to starkly conclude that "the teaching of abstract algebra is a disaster" (Leron & Dubinsky, 1995, p. 227). To further investigate what makes this course so challenging, we examined the mathematician's daily mathematical activities through his teaching diaries to understand his way of thinking and possible challenges of teaching abstract algebra that many mathematicians and their students may face. The overarching aim of this study is to investigate how mathematicians live and operate in the formal world of mathematical thinking and, at the same time, communicate their knowledge to their students.

Theoretical Framework

The theoretical framework described in this paper is based on Tall's three world model of embodied, symbolic and formal worlds of mathematical thinking. Tall (2010) defines the worlds as follow: The embodied world is based on "our operation as biological creatures, with gestures that convey meaning, perception of objects that recognise properties and patterns...and other forms of figures and diagrams" (p. 22). Embodiment can also be perceived as giving body to an abstract idea. The symbolic world is the world of practicing sequences of actions which can be achieved effortlessly and accurately. The formal world "builds from lists of axioms expressed formally through sequences of theorems proved deductively with the intention of building a coherent formal knowledge structure" (p. 22). In Tall's view (2013, p. 18), "formal mathematics is more powerful than the mathematics of embodiment and symbolism, which are constrained by the context in which the mathematics is used". He believes that the formal mathematics is "future-proofed in the sense that any system met in the future that satisfies the definitions of a given axiomatic structure will also satisfy all the theorems proved in that structure. The formal mathematics can reveal new embodied and symbolic ways of interpreting mathematics. " (p.18). In his view "research mathematicians will focus attention on the higher demands of research and assert professional standards appropriate at that level" (p. 143), but they must make their subject appealing to all undergraduates, not just to an elite group who are going to become research mathematicians.

We employed this framework to examine which worlds of mathematical thinking the mathematician in this study was accessing the most while teaching certain abstract algebra concepts, and which route he was taking to take his students to the formal world. Furthermore, we will seek to find some of the pedagogical challenges that both the mathematician and the student were facing along the way.

Method

The research described here is a case study of a research mathematician (the second named author, Ralf) that took place at The University of Oklahoma in Fall 2012 and Spring 2013. The research team consisted of two mathematicians and two mathematics educators. Ralf, one of the two mathematicians, was an experienced faculty member who had taught many mathematics courses from college algebra to algebraic geometry. He captured many details in his daily diaries and shared them promptly with the rest of the research team. The journals were brief, often included technical language, and gave an impression to the reader of being present in the class. He wrote journal entries for nearly each class period over the entire two semester course sequence. In these entries, he would note the content discussed in that class period and reflect on any aspects of teaching that came to mind, including preparation, in-class decisions, student engagement, and pedagogical considerations (among others). These were shared immediately with the rest of the research team. Weekly meetings were convened to discuss and reflect upon the journal entries and emerging themes related to these entries and were audio recorded. Additionally, Ralf welcomed unannounced visits to his class by other members of the team. During the course of the two semesters he planned and devised a few teaching experiments in his abstract algebra lectures. His positive attitude toward teaching and education enabled the team to get as close as possible to his way of thinking and interacting in the classroom. In addition, Kim (pseudonym) one of the 15 students in Ralf's class was recruited on a volunteer basis by the first author. To protect her anonymity (and, correspondingly, the authenticity of her journal entries), Kim's identity was not (and have not been) revealed to any other member of the research team. Similarly to her instructor, Kim wrote daily journal entries and supplied them via email to the first author on a weekly basis. These journals were not shown to the rest of the research team until several months after the conclusion of the year-long course sequence. Both journals, Kim's and Ralf's, along with the transcripts from the weekly research meetings were coded using a standard open-coding scheme (Strauss & Corbin, 1998).

The main themes emerging from the data were: (a) pedagogical challenges of *communicating* the "greatness" of a concept (e.g. Galois Theory) to a beginner, (b) difficulties of teaching very abstract concepts (e.g. *Tensor products)* which are hard to explain or break down, (c) having a dynamical class while still being traditional, (d) mediating the disconnect between desire for mathematical elegance and the struggles of a student learning difficult material.

The course

The class we discuss was a two-semester course in abstract algebra for beginning graduate students. The purpose of this class is to provide students with a good foundation in group theory, ring theory and field theory - basic material every mathematician should know. There is a corresponding abstract algebra qualifying exam for the PhD degree, which itself is not part of the class, but for which the class is supposed to prepare. Usually such a course has an enrollment of about 15 students in the first semester, and a slightly lower number in the second semester. In this respect the observed course was rather typical. This was the second time for Ralf to teach this course. The first time, five years earlier, he had presented the course material in the traditional order, i.e., groups - rings - fields. This time, he decided to reverse the order and teach fields - rings - groups. The book "*Algebra: Volume 1, Fields and Galois Theory*" by Falko Lorenz (2006), which takes this approach, was chosen as a textbook for the course. The traditional way seems more logical at first, as the material progresses from objects with minimal structures (groups) to objects with rich structures (fields). However, the reverse approach follows more closely the historical development, and it seems that students have no difficulty, in fact less difficulty, grasping the "rich" structures first. The sequence fields - rings - groups is somewhat simplified, the actual order of things being dictated by the overarching goal to get to the main theorem of Galois Theory. Accordingly, the book chosen does start with the basic theory of algebraic field extensions. In the course taught, the main theorem of Galois Theory was presented about 1/3 into the second semester. In some sense, Galois Theory was the dominating theme of this two-semester course. The theory of algebraic field extensions, the development of the necessary ring-theoretic concepts, the study of separability of polynomials, all leading up a straight path to Galois Theory. The entire concept of "fields first" may be viewed as rooted in a desire to get to the Fundamental Theorem of Galois Theory as fast as possible.

Results and Discussion

In this section, we will analyze Ralf and Kim's journals on the four lectures leading up to the proof of Galois Theory. Ralf had mentioned the word "Galois Theory" several times, and much before getting to the subject itself. He had made comments about a connection between algebraic field extensions and group theory, which probably remained mysterious at the time. Although, these comments were designed to instill a feeling in the students that Galois Theory is something important. It seems that the students knew that they were heading for something "big", without knowing precisely what it was.

Day 1: February 4

In his journals, Ralf wrote: *I started with a review of normal field extensions, a topic from last semester, since this was needed in our first theorem. Then I announced that we are now making the most important definition of the entire course: Galois extensions. Our first theorem was the characterization of Galois extensions as normal and separable, and this was followed by some easy consequences. We haven't proved the main theorem of Galois Theory yet, but already one can see the flavor of the whole theory. Towards the end we were in a position where I could explain to the students the basic principle, the correspondence between groups and field, and how this allows one to study fields using methods of group theory. Students didn't comment on this explicitly, but still I got a feeling this remark made sense to most of them. It was almost a small watershed moment, in the sense that after this class the students should be able to explain to someone else what the basic principle of Galois Theory is, while before this class they couldn't.*

On the same day Kim wrote: *Today, Dr. Schmidt did a quick review over the material we covered in class last time, as well as, integrating a few things from previous classes that we would need for this lecture. (He usually does this every class. It takes only about five minutes, but it really helps me get in my "algebra groove".) He also made sure to point out one of the theorems we learned today as "very important". When a teacher says that, I make sure to highlight, star, and mark it with bold letters. (It has almost been a week, and I have used this theorem multiple times in my homework, the lecture, and even to better understand an algebraic concept.)*

For a research mathematician, the need to make a transition from the formal world of mathematical thinking back to the symbolic and embodied worlds is pedagogically challenging and requires an awareness of students' level of thinking and careful preparation. In one of the research meetings, Ralf said: "*Mathematics drives the class, I don't even think about pedagogy in some sense*." We noted that Kim was aware of the fact that an important theory was mentioned in class but she did not elaborate on it. Ralf suspected that the class was not quite ready for the proof of the main theorem, so he deliberately delayed it to provide the necessary preparations.

Day 2: February 6

On February 6 Ralf wrote: *We proved two more lemmas in our development of Galois Theory. At this stage, the interplay between groups and fields is all-present. The concept of normal subgroup of a group, and the accompanying quotient, was introduced two weeks ago via homework problems. Had we not had this concept, we would have had to define it in this class. This is a nice example of how group theory is naturally motivated through field theory. As was pointed out to me, so far this semester there was a decided lack of examples. The theory remained almost completely abstract. Somehow, it doesn't bother me, but I am not sure how the students think about it. In any case, the abstract phase will soon be over. The next homework sheet will have three instructive examples of Galois extensions over Q. It is easier to find good examples once the main theorem of Galois Theory will be proven.*

On the same day Kim wrote: *Today was pretty much the same as usual - a short recall, where he hits the key points of previous lectures again and a wonderful lecture with diagrams and helpful notes sprinkled throughout. One thing I noted is that he is really good about recalling theorems and propositions from previous lectures to helps us remember why statements are true.*

Ralf had his eyes focused on the formal mathematics and was going to delay giving examples till after the theorem was proved. On the other hand, Kim thought this was just *another* abstract algebra lecture and seemed unaware of the fact that Ralf was preparing the path for the proof of Galois Theory. It is noteworthy that on two occasions (February 4 and 6), the second mathematician involved in this research visited Ralf's class. From the level of the conversation that they had together at the research meeting, we noticed that his take of the classes was completely different from Kim's. He mentioned: "*Yes, I was very much distracted by the math, it was so nice that I was not paying attention to your teaching.*"

Day 3: February 8

On February 8 Ralf wrote: *Today we got right up to the main theorem of Galois Theory for finite extensions. I went slowly and thoroughly, recalling concepts like the normal closure (something from last semester). I asked a lot of questions, and got a reasonable amount of replies. Made me think that the majority of the class is still with me. The amount of interaction with the class made for a good atmosphere today. I spent the last 10 minutes giving some explanations about the new homework sheet, which was handed out today. Since we are not having discussions this semester, I feel this is necessary. Otherwise the* *homeworks and lectures would live side by side in parallel universes, and maybe students would not always see the connections. So I explained how they can use several of the results that were proven today to attack some of the homework problems.*

In this lecture, Ralf used the homework to connect the symbolic and formal worlds of mathematical thinking together and despite going through some hard concepts, interacted well with his students. Unfortunately, we did not receive any journal entries from Kim on that day. At the weekly meeting, Ralf commented that "*Well we are doing Galois Theory now, it's very exciting, very attractive mathematics, it's just going the same style, just old fashioned teaching and I ask some questions and I get some answers, it's going pretty well".*

Day 4: February 11

On February 11 Ralf wrote: *In this class we proved the main theorem of Galois Theory. Looking at my notes right before class, I realized that all the pieces of the proof are in place, and there isn't really much more to do. So I decided to more or less have the students develop the proof. I started the class by not stating the theorem, but writing down some ingredients of the proof, without the students knowing that this is going to be a proof at all. Then I guided them towards the main theorem by asking questions. Within ten minutes the proof was complete, and only then did we state it formally as a theorem. This was followed by an example, in fact our first real example ever. I found this example very instructive, and I hope the students did too. In any case they seemed very alert, maybe having to do with the fact that similar examples are on the current homework sheet. This was a class with a lot of communication back and forth, and I found it to be maybe the best class so far this semester. After weeks of developing theory, it was like bringing in the harvest.*

On the same day Kim wrote: *Today, we are going to prove the main theorem of Galois Theory. That is how Dr. Schmidt started class. With that knowledge, how could one not smile happily at the prospect of learning a foundational theorem to one of her favorite subjects. Yep, I grinned and looked around at my classmates to share in each others' excitement. After that he dived right into the lesson with a theorem which was pretty much intuitive after everything we had learned to this point, but by no means trivial. He then did something really cool. He started by explaining the concept of mapping intermediate extensions of E to subgroups of G(E/K). Then he asked questions to the class that lead us logically to proving this map was a bijection. Then he said, "Thus, we have the main theorem of Galois Theory". It was so "Andrew Wiles" that I wanted to clap. He then gave us the formal theorem. Dr. Schmidt then proceeded to tell about how Galois was thinking in terms of polynomials when he came up with this theorem, not fields. Dr. Schmidt went on to give the definition of the "Galois Group of f over K" or (the way Galois thought of it) "the Galois group of the equation f(x) = 0". It was great!*

Ralf considered this class as the culmination of a great deal of work, and possibly the high point of the entire two-semester course. We noticed that Ralf was grounded firmly into the formal world throughout these four lectures. In the last lecture, Kim finally acknowledged the theory and expressed her excitements regarding its proof.

Concluding Remarks

This study uncovered some of the complexities of communicating abstract algebra to students. Clearly, Ralf believed the need to remain as close as possible to the formal world of mathematical thinking and only on the last day decided to reverse back to the symbolic world by giving some examples. It seemed that Kim felt at home by this approach, however did not have sufficient experience and expertise to follow Ralf's path leading up to the main proof.

Over the past two years, Ralf's daily teaching journals, the weekly research meetings and many hours of discussion while analyzing the data, has revealed a significant amount of insight into his thought processes and has resulted in a fruitful collaboration for everyone involved in this project. Of course, this is but a small portion of his journals and reflections (a full-scale research report is beyond the scope of this proposal).

To understand more about the nature of the formal world of mathematical thinking our future research directions will involve studying Ralf's research activities. This will include analyzing his research journals, attending his seminars and observing some of his research meetings with his collaborators.

References

Byers, W. (2007) *How mathematicians think: Using ambiguity, contradiction, and paradox to create mathematics*, Princeton University press.

Clark, J., Hemenway, C., St. John, D., Tolias, G., & Vakil, R. (2007). Student attitudes toward abstract algebra, *Primus*, *9*(1), 76-96.

Dreyfus, T. (1991). Advanced Mathematical thinking processes. In D. O. Tall (ed.) *Advanced Mathematical Thinking*, (pp. 25-41). Dordrecht: Kluwer.

Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On learning fundamental concepts of group theory, *Educational Studies in Mathematics, 27*(3), 267-305.

Hannah, J., Stewart, S., & Thomas, M. O. J. (2011). Analysing lecturer practice: the role of orientations and goals. *International Journal of Mathematical Education in Science and Technology*, *42*(7), 975-984.

Hannah, J., Stewart, S., & Thomas, M. O. J. (2013). Conflicting goals and decision making: the deliberations of a new lecturer, In Lindmeier, A. M. & Heinze, A. (Eds.). *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 425-432. Kiel, Germany: PME.

Kensington-Miller, B., Yoon, C., Sneddon, J., & Stewart, S. (2013). Changing beliefs about teaching in large undergraduate mathematics classes. *Mathematics Teacher Education and Development, 15*(2), 52-69.

Leron, U., & Dubinsky, E. (1995). An abstract algebra story, American Mathematical Monthly, *102*(3), 227-242.

Lorenz, F. (2006). *Algebra*. *Vol*. *1*. *Fields and Galois Theory*. Translated from the 1987 German edition by Silvio Levy. Universitext. Springer, New York.

Paterson, J., Thomas, M. O. J., & Taylor, S. (2011). Decisions, decisions, decisions: What determines the path taken in lectures*? International Journal of Mathematical Education in Science and Technology*, *42*(7), 985-995.

Speer, N. M., Smith, J. P & Horvath, A. (2010). Collegiate mathematics teaching: An unexamined practice. *Journal of Mathematical Behavior*, *29*, 99–114.

Strauss, A. L., & Corbin, J. (1998). *Basics of qualitative research: Grounded theory procedures and techniques* (2nd ed.). Newbury Park, CA: Sage.

Tall, D.O. (2010). Perceptions, operations, and proof in undergraduate mathematics. *Community for Undergraduate Learning in the Mathematical Sciences (CULMS) Newsletter*, 2, 21-28.

Tall, D. O. (2013). *How humans earn to think mathematically: Exploring the three worlds of mathematics*, Cambridge University Press.

Balancing formal, symbolic and embodied world thinking in first year calculus lectures

Sepideh Stewart Clarissa Thompson Keri Kornelson Lucy Lifschitz Noel Brady
University of Oklahoma Kent State University University of Oklahoma

In this paper we present a mathematics professor's thought processes while teaching Calculus I, as shared through her teaching diaries and later discussed in weekly meetings with a team of two other mathematicians, a mathematics educator, and a cognitive psychologist over the period of a semester. We examine the way she balanced formal and symbolic thinking while encouraging embodied thinking throughout her lectures. Moreover, we will discuss some data obtained from students through interviews, a questionnaire, and end-of-semester course evaluations.

Keywords: Reflections on Teaching, Calculus, Three Worlds Model of Mathematical Thinking, Expert-Novice Distinction, Cognitive Psychology

Introduction

Research in pedagogy at the university level is still in its infancy, and the communication between mathematicians and those outside of the community on pedagogy is often very limited. Byers (2007) states that "many mathematicians usually don't talk about mathematics because talking is not their thing – their thing is the "doing" of mathematics" (p. 7). As Dreyfus (1991) suggested, "one place to look for ideas on how to find ways to improve students' understandings is the mind of the working mathematician. Not much has been written on how mathematicians actually work" (p. 29). Two decades later, a study by Speer, Smith, and Horvath (2010) shows that "very little research has focused directly on teaching practice and what teachers do and think daily, in class and out, as they perform their teaching work" (p. 111). In an attempt to improve this situation, Hodgson (2012), in his plenary lecture at ICME 12, raised the point about the need for a community and forum where mathematicians and mathematics educators can work as closely as possible on teaching and learning mathematics. In recent years, more mathematicians are willing to examine and reflect on their own teaching styles, leading to a growing body of research in this area. For example, a study by Paterson, Thomas, and Taylor (2011) described a supportive and positive association of two groups of mathematicians and mathematics educators that allowed the "cross-fertilization of ideas" (p. 359). Hannah, Stewart, and Thomas (2011, 2013) indicated cases in which professors took careful diaries of their actions and thoughts during linear algebra lectures and reflected on them with the rest of the team. Also, a study by Kensington-Miller, Yoon, Sneddon, and Stewart (2013) showed how a mathematician, with the support of a research team, made changes in his lecturing style by asking well-planned questions while teaching a large undergraduate mathematics course.

In this paper, we will examine a mathematician's daily mathematical activities through her teaching diaries to understand her way of thinking and possible challenges that many mathematicians (and their students) may face. The overarching goal of this project is to investigate how mathematicians live and function in the formal world of mathematical thinking and, at the same time, communicate their knowledge to their students. This research also has implications for instructors in Calculus I classrooms as many students find it difficult to transfer their relevant Calculus knowledge to other upper-division mathematics and science courses (Dray & Manogue, 1999).

Theoretical Framework

Tall introduced a framework based on three worlds of mathematical thinking: the *conceptual embodiment*, *operational symbolism,* and *axiomatic formalism*. The world of conceptual embodiment is based on "our operation as biological creatures, with gestures that convey meaning, perception of objects that recognize properties and patterns...and other forms of figures and diagrams" (Tall, 2010, p. 22). Embodiment can also be perceived as the construction of complex ideas from sensory experiences, giving body to an abstract idea. The world of operational symbolism is the world of practicing sequences of actions that can be achieved effortlessly and accurately. The world of axiomatic formalism "builds from lists of axioms expressed formally through sequences of theorems proved deductively with the intention of building a coherent formal knowledge structure" (p. 22). Tall (2013) suggested that: "Formal mathematics is more powerful than the mathematics of embodiment and symbolism, which are constrained by the context in which the mathematics is used (p. 138). We employed this framework as a means of differentiating and drawing comparisons between the varying levels of mathematical thinking exhibited by the mathematician in her journals (and data from students). We hypothesized that the mathematician would "live" primarily in the formal world, whereas her students would operate mainly in the comfort zone of the symbolic world. How the mathematician promoted the value of the formal world to her students was of particular interest to our research team.

Our research was also guided by the expert-novice distinction in cognitive psychology. The expert mathematician's goal is to get her novice students to think about Calculus in a more expert-like manner (e.g., moving from the symbolic to formal worlds). This is quite an endeavor because, "children, or adults, may be able to reflect on their knowledge when asked (or may not), but it is unusual for an expert to be keenly aware of "how" they are operating on a problem while they are engaged in it" (Bjorklund, 2008). Experts and novices reason about problems and remember critical information quite differently (e.g., Chi, 1978). For example, chess experts were better able to remember the locations of chess pieces that were arranged in meaningful formations than were non-chess experts. However, it is important to note that the chess experts did not have superior memories for non-chess related information and for meaningless chess formations. When asked how to solve various physics problems, experts were more likely to categorize problems on the basis of the abstract formulas used to solve them, whereas novices were likely to categorize problems on the basis of literal, surface-level features (e.g., group all problems together that involved inclined planes vs. springs; Chi, Feltovich, & Glaser, 1981). Content knowledge plays a bigger role in the expert-novice distinction than does IQ (Schneider, Körkel & Weinert, 1989). The expert mathematician's deep level of Calculus content knowledge will allow her to live in the formal world, whereas the students' comparably shallow level of content knowledge will likely constrain the novices to operate in the symbolic world. Our research question is: Given that the mathematician in this study is able to think in all three worlds of mathematical thinking, how does she balance this in her Calculus lectures to facilitate students as they begin to reason beyond the level of symbol manipulation?

Method

The research described here is a case study of a research mathematician (Keri, one of the coauthors) that took place at The University of Oklahoma in Fall 2013. The research team consisted of three mathematicians, a mathematics educator, and a cognitive psychologist forming a community of enquiry to look into a mathematician's daily thought processes while she taught a first-year Calculus course. The data emerged from the research mathematician's daily reflections on her teaching of a Calculus I course, which were made available to the group after each class; the team members' observation of the classes and their comments; weekly discussion meetings of the whole group after reading each of these reflections; an online survey of the Calculus I students and interviews with several Honors students enrolled in the course; and the audio recordings of each meeting which were later transcribed. The mathematics professor in this study was an experienced faculty member who had taught Calculus I approximately 10 times at four different colleges and universities around the U.S. She captured many details in her daily diaries and during weekly meetings, and the rest of the research team, having already read the journals, gave the mathematician an opportunity to discuss her teaching from the past week. This was followed by questions from the research team, which often generated additional discussions. Additionally, the mathematician welcomed unannounced visits to her class by other members of the team. The main themes emerging from the data were: 1) teaching (e.g., preparation, pedagogy, examples, advice, rapport), 2) reflections (e.g., on students, timing, experiences of the instructor), 3) students (e.g., prior knowledge, group work), 4) technology (e.g., Webwork), 5) questions (e.g., asked/answered by instructor and/or students), 6) visualizations (e.g., drawings, Wolfram demonstrations), 7) mathematics (e.g., Intermediate Value Theorem), 8) community of practice (e.g., seeking pedagogical advice outside or inside the research group), and 9) formal assessment (e.g., quizzes, exams). The coded data from the daily journals relating to teaching and reflection comprised 66% of the total data set; all other codes comprised less than 10% of the data each. For the purposes of this paper, we will concentrate on the teaching and reflection codes only. More specifically, we will describe how the instructor consciously thought about the best timing of concrete examples in her Calculus lessons and her reflections (and student reflections) on this teaching technique. We will also concentrate on the instructor's use of visualizations, via chalkboard drawings and online demonstrations, to convey the mathematics in a more tangible manner.

Results and Discussion

Symbolic and Formal World Thinking in Calculus I

As a formal thinker, Keri wanted her students to understand the theories and tried to balance the examples and amount of theory she provided. Was she successful? How did the students react to this approach?

Keri often reflected on students' experiences and level of understanding in her Calculus course. In fact, out of the 87 total reflections coded from Keri's daily class journals, nearly half were reflections where Keri attempted to gauge students' understanding. One of Keri's major reflections was that students greatly valued concrete examples and computations (symbolic world manipulations) and found very little value in the necessary framing discussion of theoretical background (formal thinking). Keri hypothesized that the reason students prefer instructors to work examples is because these are the types of skills often assessed on exams. "*Maybe they have been trained to know that the examples tell them how to do what they are going to be evaluated on, whereas the definitions and the discussion…is the stuff that they can tune out because nobody is ever going to ask them that*." Keri also believed that students' previous high school experience with Calculus was simply at the symbol-manipulation level. A student survey response provided evidence for Keri's hypothesis, "*Teacher focused too much on things that were not related to the tests and left us to learn some things by ourselves and did not provide many advanced examples that would help us either on the homework or test*." However, Keri realized that she often delivered the pertinent background information before presenting the concrete concept instantiations. "*I'm thinking this afternoon about how I often explain all kinds of methods and reasons and background before I do examples. They don't get what I'm talking about until after they've tried some problems. So maybe I should restructure a bit by giving them the routine first,* *letting them try some problems, and then talking about some of the motivations, background, etc*." Because Keri was so attuned to her students' level of understanding and was willing to experiment with some new teaching techniques in her classroom, she decided to flip the order in which she presented theory and examples. "*My reason for trying this was that I felt the students sort of zoned out during my supposedly-helpful introductory remarks, and only woke up when we started doing examples*." Examples have an important place in the learning process for novice undergraduates, expert mathematicians, and those advanced graduate students who fall somewhere between the two anchor points on the novice-expert continuum. Keri noted the value of concrete examples to help advanced mathematics students to better understand very complex concepts. "*Well what's the first thing, you know, when we want students to really dig in and understand a new abstract concept? The first thing we tell them to do is think of an example, a really finite concrete easy to understand, toy, if you want to use the word toy, example. And you run it through that abstract concept. So this is what we're telling our graduate students right. When you get a new crazy abstract concept, right and of course it's the way we learn too, cause how else do you understand anything except by trying to build a little example of it*."

Also, it seems that Keri could relate to her students' preference for concrete examples because she recalled being a student and having a very cursory understanding of the theory as she manipulated the symbols during calculations. It wasn't until later that she established a firm, expert-like grasp on the underlying conceptual nature and interconnections of the mathematics. The vast majority of respondents (71%) in an optional end-of-semester survey agreed or strongly agreed with the statement, "*I learned a lot from doing many in-class examples*." A student response on the end-of-semester mandatory instructor evaluations noted, "*I think Dr. Kornelson did a good job on explaining things thoroughly--sometimes too thoroughly. Sometimes her explanations about what we were doing were so long that I would forget what exactly we were doing in the first place*." Or as one student put it rather simply, "*too much theory based, not enough examples*." Keri knew that students under-valued the importance of theory and possessed an overreliance on examples, so she attempted to intersperse theory and examples more to help students make the connection between the two and begin thinking more like an expert in the discipline.

Embodied Thinking

In an effort to provide the most informative concrete examples, Keri often drew pictures. "*There are certain things where a picture is not so helpful but there are certain kinds of topics… (where) the pictures are incredibly helpful…(because) you'd have a really hard time trying to do it without a picture or a demo on the computer. There are certainly times when students' intuition lead them to exactly the right thing…you can say things that are more precise than that, but their intuition is good in that sense*." Sometimes students copy the images into their notes, but they don't understand the concept at a deep enough level to later make sense of their drawings. Keri mentions that the pictures are only helpful in conjunction with the background information that she's talking about during lecture. *"The students come and show you their notes and it will be exactly what you wrote on the board, and it makes absolutely no sense because whatever it was you were saying they didn't get that down."* Keri is aware of this shortcoming of visualizations and says, *"I wonder if I've written enough to make sure that when they look at the picture a week later they have any idea what it was about."*

We wondered whether Keri, an expert mathematician, attempts to visualize and problem-solve during her own mathematics research. She noted that there aren't many pictures that help to illustrate her own research. *"I don't have the opportunity to be visual very often because most of what I do doesn't have much of a picture that is very illuminating. It's great when there is. I do like to see examples."* Keri noted the pictures may be individualized from one expert to the next, and sometimes there are multiple pictures that might represent the same topic under discussion. *"So I have another picture in my head too but I don't usually give that to the students. But I like that picture too because…it's something…in a setting that looks completely different."*

Further, Keri has insight into what her novice students must experience as they attempt to understand her drawings. When she views the visualizations of another expert devoid of any background or context, the pictures are nearly meaningless. *"I work with a guy…and I'll get this scanned set of notes that he's written…and they'll be these funny diagrams in there. And if I hadn't had a face-to-face conversation with him where I've seen these things develop…these are completely incomprehensible. When we've had a face-to-face meeting I'll know what this weird set of symbols means, but if we haven't had a face-to-face meeting and this is all evolved in his own head outside of a conversation, it's like I've got nothing."* The pictures are a visual representation of what is going on in the expert mathematician's mind. Sometimes Keri urges her students to draw pictures to get them in the ballpark of the right answer, *"The graphs give you an intuition, but they may not tell you the whole story."* At times these visuals are clearly not optimal, *"The picture I see is actually not a great picture…and loses a lot of the insight."* At other times, the pictures are all the proof one needs to solve a problem and no additional proof/theorem is necessary. *"Rolle's Theorem is a fun case…you draw a bunch of pictures and they believe that you have proved it. Even in the middle of class, I was still debating whether I was going to actually write a 'proof proof', but I decided the pictures were essentially the proof."*

Concluding Remarks

Without a doubt, the expert mathematician's need to transition from the formal world of mathematical thinking to the symbolic and embodied worlds while teaching Calculus I is pedagogically challenging. The course demands a keen awareness of students' level of understanding and background, and it is challenging for the instructor to balance the amount of the theory and examples that are presented.

Returning to our research questions, the results of this study provide some insight into the daily thought processes engaged in by a mathematician. Specifically, this paper details the instructor's efforts to help her students access the formal and embodied nature of Calculus. Of course, this is but a small portion of her journals and reflections. After her involvement in this project, Keri mentioned that she would do several things differently the next time she taught the Calculus I course. For example, she would like to implement her idea of starting off with examples followed later by information about the conceptual background of the material sooner in the semester.

Our in-depth analysis of Keri's day-to-day experiences is generalizable to other Calculus I instructors. We would recommend instructors to systematically reflect on what their students know and do not know. This process will help the expert mathematician to build on his novice students' rudimentary understanding of Calculus to make it more expert-like. Although, it is not always clear in which order--embodied, symbolic and formal--the concepts may be introduced in different mathematics courses (Hannah, Stewart, & Thomas, 2014), our study suggests that Calculus instructors might consider making use of diagrams and visualizations to help their students intuitively understand the concept under consideration before the students begin attempting to compute the correct answer. The diagrams can be thought of as a way for students to estimate the correct answer. Estimation is a difficult skill for both children and adults to master (see Siegler, Thompson, & Opfer, 2009), so instructors may need to assist students as they attempt to hone this skill in the context of a Calculus I course. We also believe that if the instructor chooses to include diagrams in teaching, he should ensure that the students have a good explanation in their notes for what was drawn.

This collaboration has been positive because group members are not only focusing on the research mathematician's teaching strategies and thinking processes, but also on how they might improve their own approaches to teaching. Moreover, it has provided a platform allowing mathematicians to freely talk about mathematics and share pedagogical challenges with each other. The presence of a cognitive psychologist has also illuminated some psychological principles that are in effect in the classroom.

It is noteworthy that Keri continued writing daily journals for other classes because she found the process very helpful to her course preparation and reflection. Keri noted that this course gave her the opportunity to "tinker" with some new teaching techniques and gave her the confidence to do a major overhaul of her discrete mathematics course by offering it in a flipped format in Spring 2014 semester. The flipped format allowed Keri to spend more one-on-one time with students as they grappled with particularly difficult course concepts.

Future research could empirically investigate whether providing concrete practice prior to abstract understanding improves course performance in actual classrooms. Results from laboratory studies (Kaminski, Sloutsky, & Heckler, 2008) suggest that abstract understanding of mathematics concepts facilitates transfer of learning more so than does concrete examples. This has real implications for students who value examples over theory and are expected to transfer knowledge learned in Calculus to other relevant mathematics and science courses.

References

Bjorklund, D. F. (2005). *Children's thinking* (4th ed). Belmont, CA: Wadsworth.

Byers, W. (2007) *How mathematicians think: Using ambiguity, contradiction, and paradox to create mathematics*. Princeton University Press.

Chi, M. T. H. (1978). Knowledge structures and memory development. In R. Siegler (Ed.), *Children's thinking: What develops?* (pp. 73-96). Hillsdale, NJ: Erlbaum. Reprinted in: Wozniak, R. H. (1993) *Worlds of Childhood*, (pp. 232-240), Harper Collins College Publishers.

Chi., M. T. H., Feltovich, P. J., & Glaser, R. (1981). Categorization and representation of physics problems by experts and novices. *Cognitive Science, 5*, 121-152.

Dray, T., & Manogue, C. (1999). The Vector Calculus Gap: Mathematics ≠ Physics. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies, 1*, 21-28.

Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. O. Tall (Ed.) *Advanced Mathematical Thinking* (pp. 25-41). Dordrecht: Kluwer.

Hannah, J., Stewart, S., & Thomas, M. O. J. (2011). Analysing lecturer practice: The role of orientations and goals. *International Journal of Mathematical Education in Science and Technology*, *42*, 975-984.

Hannah, J., Stewart, S., & Thomas, M. O. J. (2013). Conflicting goals and decision making: The deliberations of a new lecturer. In Lindmeier, A. M. & Heinze, A. (Eds.) *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 425-432. Kiel, Germany: PME.

Hannah, J., Stewart, S., & Thomas, M. O. J. (2014). Teaching linear algebra in the embodied, symbolic and formal worlds of mathematical thinking: Is there a preferred order? In Oesterle, S., Liljedahl, P., Nicol, C., & Allan, D. (Eds.) *Proceedings of the joint meeting of PME38 and PME-NA 36*, Vol. 3, pp. 241-248. Vancouver, Canada: PME.

Hodgson, B. R. (2012). Whither the mathematics/didactics interconnection? Evolution and challenges of a kaleidoscopic relationship as seen from an ICME perspective. ICME 12 conference, Plenary Presentation, Seoul, South Korea.

Kaminski, J. A., Sloutsky, V. M., & Heckler, A. F. (2008). The advantage of abstract examples in learning math. *Science, 230,* 454-455.

Kensington-Miller, B., Yoon, C., Sneddon, J., & Stewart, S. (In press). Changing beliefs about teaching in large undergraduate mathematics classes. *Mathematics Teacher Education and Development*.

Paterson, J., Thomas, M. O. J., & Taylor, S. (2011). Decisions, decisions, decisions: What determines the path taken in lectures*? International Journal of Mathematical Education in Science and Technology*, *42*, 985-995.

Schneider, W. Körkel, J., & Weinert, F. E. (1989). Domain-specific knowledge and memory performance: A comparison of high- and low-aptitude children. *Journal of Educational Psychology, 81*, 306-312.

Siegler, R. S., Thompson, C. A., & Opfer, J. E. (2009). The logarithmic-to-linear shift: One learning sequence, many tasks, many time scales. *Mind, Brain, & Education, 3*, 143-150.

Speer, N. M., Smith, J. P., & Horvath, A. (2010). Collegiate mathematics teaching: An unexamined practice. *Journal of Mathematical Behavior*, *29*, 99–114.

Tall, D. O. (2010). Perceptions, operations, and proof in undergraduate mathematics. *Community for Undergraduate Learning in the Mathematical Sciences (CULMS) Newsletter*, 2, 21-28.

Tall, D. O. (2013). *How humans learn to think mathematically: Exploring the three worlds of mathematics*. Cambridge University Press.

Components of a formal understanding of limit

Steve Strand
Portland State University

Abstract: This presentation reports on research investigating what mathematical constructs and ways of understanding constitute a formal understanding of limit. This work builds primarily on the genetic decomposition of Swinyard and Larsen (2012), which itself was modified from Cottrill, et al. (1996). Ten undergraduate students were interviewed in a semi-structured, clinical setting. While analysis is in the preliminary stages, the data suggest that the types of tasks given have a significant influence on whether or not formal understanding is demonstrated.

Key words: *Limit, Real Analysis, APOS, RME*

Introduction

The concept of limit has served as the theoretical foundation for the calculus and its applications ever since the work of Cauchy, Bolzano, and others in the early and mid 19th century (Grabiner, 1981). It follows that a formal understanding of the limit concept is essential to any investigation of the theoretical underpinnings of the calculus. As a part of my dissertation project to develop RME-based curriculum for introduction of the formal limit concept in Real Analysis, I am investigating the efficacy of, and hope to expand upon, the genetic decomposition of limit offered by Swinyard and Larsen (2012) (see Appendix A), which was itself modified from Cottril, et al. (1996). My goal with the current phase of the project is to deepen our knowledge of what mathematical constructs and ways of understanding constitute a formal understanding of limit.

I have interviewed ten undergraduate students, with varying degrees of experience with limits, about different aspects of their understanding of the limit concept. The interview tasks were designed to investigate the following questions:

What ways of understanding characterize a formal understanding of limit? In particular:

a) *Is there further evidence that a transition to a range-first perspective is required to have a formal understanding of limit?*
b) *Is there further evidence that the development of the notion of arbitrary closeness is required to have a formal understanding of limit?*

Literature Review

A great deal of research on student understanding of the limit concept has focused on investigating the struggles students face in working with limits and the tools they use to deal with those struggles (Bezuidenhout, 2001; Cornu, 1991; Davis & Vinner, 1986; Moru, 2009; Oehrtman, 2009; Sierpińksa, 1987; Szydlik, 2000; Tall & Schwarzenberger, 1978). The other main area of focus has been investigating the process of students formalizing their understanding of limit (Cottrill, et al., 1996; Oehrtman, Swinyard, & Martin, 2014; Swinyard & Larsen, 2012; Williams, 1991). In line with the aforementioned research, I will consider a student to have a formal understanding of limit if they can use a formal definition of limit to justify limit candidates and to construct formal proofs. That is, the student is able to engage in formal activities with limits using a formal definition. With my research I hope to contribute to our growing understanding of what mathematical constructs and ways of understanding characterize a formal understanding of limits.

One of the first attempts to describe how students progress in their understanding of limits was put forward by Cottrill, et al. (1996). Using the Action, Process, Object, Schema framework (APOS), they described in a step-by-step manner how a student's understanding of limits of functions could deepen and become increasingly formal. In this model, a formal understanding of limit was achieved by abstracting processes at the informal level. This abstraction was exemplified in Step 5 of Cottrill, et al.'s genetic decomposition: "Reconstruct the processes of 3(c) in terms of intervals and inequalities. This is done by introducing numerical estimates of the closeness of approach, in symbols, $0 < | x - a | < \delta$ and $| f(x) - L | < \varepsilon$."

A further development of this model was offered by Swinyard and Larsen (2012). While the first three steps of the model put forward by Cottrill, et al. (1996), remained unchanged, it was Steps 4 – 7 which did not seem to describe what the authors were seeing in their data. Cottrill, et al.'s Step 1 – 3 described *finding* a limit, and rather than an abstraction of these processes, the formal definition described *verifying* a limit candidate. Swinyard and Larsen identified the fundamental shift in purpose from evaluating limits of concrete functions and sequences (less formal) to proving things about limits (more formal) as characterizing the transition to working with limits formally. The recognition of this change in the nature of the activity is what led them to propose a modification to the cognitive model. Specifically, if a student is to be able to work with limits formally, they must be able to reason from a range-first perspective, and they must have developed a reified notion of arbitrary closeness. In the standard definition, this reified notion of arbitrary closeness is represented by choosing a single arbitrary ε to stand in place of all possible measures of closeness.

The primary goal of this research is to verify and expand upon Swinyard and Larsen's (2012) genetic decomposition, as adapted from Cottrill, et al. (1996).

Theoretical Tools

The design of the interview study was guided by a few different resources. For the structure of the interview itself (apart from the content of the tasks), I looked to Zazkis and Hazzan's (1998) framework for designing interview instruments. Based on the authors' descriptions and my research questions, I opted for a semi-structured, clinical setup for the interviews. For the content, Swinyard and Larsen's (2012) genetic decomposition guided the design of the interview tasks. This decomposition uses the APOS framework to describe how a student might develop in their understanding of the limit of a function. Briefly, *actions* are procedures or transformations that are performed on objects, often in a very step-by-step manner. Through interiorization, a sequence of actions can be reflected upon, envisioned, and analyzed without needing to be carried out. When an individual interiorizes a sequence of actions we say that they have constructed a *process*. When an individual is able to reflect on a process as a whole and even apply other actions to that process, we say that the individual has encapsulated that process into an *object*. Actions, processes, and objects can be coordinated into *schema*. For a more thorough introduction to the APOS framework, see Brown, DeVries, Dubinsky, and Thomas (1997).

Together, APOS and Swinyard & Larsen's (2012) genetic decomposition helped me to focus my interview tasks, and will also be useful for analysis. One of my goals with each task was to provide insight into the level of formality (according to the genetic decomposition) at which each student was operating. I anticipate that one way to discern this will be to note in which contexts and to what extent students use the language of actions, processes, or objects (in the APOS sense) to talk about limits.

Data Collection

In order to discern mathematical constructs and ways of understanding that characterized a formal understanding of limit, I interviewed ten students who were currently enrolled in an introductory analysis course, or courses that were upper-division prerequisites to analysis: Linear Algebra, Introduction to Proof, Abstract Algebra. Data consisted of video/audio recordings of the semi-structured, one-on-one clinical interviews, as well as my notes during and immediately following those interviews.

The interview questions were designed to get at different aspects of each student's understanding of limit. The first question asked them to evaluate a limit in algebraic form, while the rest of the questions explicitly asked them to explain or interpret different limit scenarios and statements about limits. The first three questions were somewhat informal, but the remaining six questions asked students to engage in formal activity with limits: explaining how the formal definition captures the idea of a limit, using the formal definition to verify a limit from a graph, using the formal definition to verify that a limit does not exist from a graph, and proving the additivity of limits for functions. From the outset I was interested in two different types of comparisons: 1) each students' language and activity on informal versus formal tasks; and 2) different students' language and activity on informal and formal tasks. That is, comparisons within students across different types of activities, as well as comparisons across students. In this way I hoped to identify mathematical constructs and ways of understanding that characterized a formal understanding of the notion of the limit of a function at a point.

Anticipated Results and Future Research

My objective is to expand our knowledge of what mathematical constructs and ways of understanding constitute a formal understanding of limits of real-valued functions. In particular, I hope that the analysis of my data will allow me to verify, refine, and perhaps expand upon the genetic decomposition of limit offered by Swinyard and Larsen (2012).

For the next steps in my research program, I will use this information to develop a local instructional theory, and an accompanying task sequence, that will explain how students can develop from an informal understanding of limit gained in the calculus sequence to a formal understanding as required for investigations in Real Analysis. This theory has been initially developed using my review of the relevant literature on limit and the design heuristics of Realistic Mathematics Education (RME), and will be further developed and refined in a series of teaching experiments.

Questions:

- What additional theoretical tools might be useful for this data/project?
- What contexts/problems do you use to introduce limits informally/formally?
- What should be some of the big-picture goals of a Real Analysis course?

References

Bezuidenhout, J. (2001). Limits and continuity: Some conceptions of first-year students. *International Journal of Mathematical Education in Science and Technology*, *32*(4), 487–500.

Brown, A., DeVries, D. J., Dubinsky, E., & Thomas, K. (1997). Learning binary operations, groups, and subgroups. *The Journal of Mathematical Behavior*, *16*(3), 187–239. doi:10.1016/S0732-3123(97)90028-6

Cornu, B. (1991). Limits. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 153–166). Dordrecht: Springer Netherlands.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process scheme. *Journal of Mathematical Behavior*, *15*(2), 167–192.

Davis, R. B., & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. *The Journal of Mathematical Behavior*, *5*(3), 281–303.

Grabiner, J. V. (1981). *The Origins of Cauchy's Rigorous Calculus*. Cambridge, Mass.: MIT Press.

Harel, G. (2007). The DNR system as a conceptual framework for curriculum development and instruction. In R. Lesh, E. Hamilton & J. Kaput (Eds.), *Foundations for the Future in Mathematics Education*. Mahwah, NJ: Lawrence Erlbaum Associated

Moru, E. K. (2009). Epistemological obstacles in coming to understand the limit of a function at undergraduate level: A case from the national university of Lesotho. *International Journal of Science and Mathematics Education*, *7*(3), 431–454.

Oehrtman, M. (2009). Collapsing dimensions, physical limitation, and other student metaphors for limit concepts. *Journal for Research in Mathematics Education*, *40*(4), 396–426.

Oehrtman, M., Swinyard, C., & Martin, J. (2014). Problems and solutions in students' reinvention of a definition for sequence convergence. *The Journal of Mathematical Behavior*, *33*, 131–148.

Sierpińska, A. (1987). Humanities students and epistemological obstacles related to limits. *Educational Studies in Mathematics*, *18*(4), 371–397.

Swinyard, C., & Larsen, S. (2012). Coming to understand the formal definition of limit: Insights gained from engaging students in reinvention. *Journal for Research in Mathematics Education*, *43*(4), 465–493. doi:10.5951/jresematheduc.43.4.0465

Szydlik, J. E. (2000). Mathematical beliefs and conceptual understanding of the limit of a function. *Journal for Research in Mathematics Education*, *31*(3), 258–276.

Tall, D., & Schwarzenberger, R. L. E. (1978). Conflicts in the learning of real numbers and limits. *Mathematics Teaching*, *Vol.82*, 44–49.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151–169. doi:10.1007/BF00305619

Appendix A

A genetic decomposition of the limit concept (Swinyard & Larsen, 2012).

1) The action of evaluating f at a single point x that is considered to be close to, or even equal to, a.
2) The action of evaluating the function f at a few points, each successive point closer to a than was the previous point.
3) Construction of a coordinated schema as follows.
   (a) Interiorization of the action of Step 2 to construct a domain process in which x approaches a.
   (b) Construction of a range process in which y approaches L.
   (c) Coordination of (a), (b) via f. That is, the function f is applied to the process of x approaching a to obtain the process of $f(x)$ approaching L.
4) Constructing a mental process in which one tests whether a given candidate is a limit by:
   (a) Choosing a measure of closeness to the limit value L along the y-axis;
   (b) Determining whether there is an interval around the point at which one is taking the limit (i.e., a) for which every function value aside from the one at that point is close enough to L; and,
   (c) Repeating this for smaller and smaller measures of closeness
5) Associating the existence of a limit with the ability to continue (theoretically) this process forever without failing to produce the desired interval about a, or equivalently with the observation that there is no point at which it will be impossible to find such an interval.
6) Encapsulating this process via the concept of arbitrary closeness. This involves realizing that one can establish that the process in 4) will work for every possible measure of closeness by proving that it will work for an arbitrary measure of closeness.

Some preliminary results on the influence of dynamic visualizations on undergraduate calculus learning

Julie M. Skinner Sutton

To produce more STEM graduates in the U.S., improving student success in calculus is crucial; previous research suggests that students with a proclivity to visualize when solving mathematics tasks are not the "star" students in mathematics classrooms. A study of undergraduate curriculum also found that common calculus tasks reinforce procedural understanding. Since incorporating dynamic visualization (DV) provides a possible tool for increasing understanding, we investigate the role of DV in calculus learning at the university level. We examine student understanding of derivative as a rate of change by comparing student experiences when exploring with DV software or engaging with static tasks in individual interviews collected in four episodes over one semester on four students identified as visualizers and five as non-visualizers. Comparisons reveal the emergence of cognitive conflict and its resolution for students encountering the DVs but this resolution is not evident for those only engaged in static work.

Key words: undergraduate calculus, classroom technology, dynamic visualizations

Some preliminary results on the influence of dynamic visualizations on undergraduate calculus learning

Julie M. Skinner Sutton

Increasing student success in calculus addresses the critical need to produce more STEM graduates in the United States. Students failing to obtain a deep understanding of calculus leave the sciences altogether (Carlson, Oehrtman & Thompson, 2008; PCAST 2012). Since incorporating dynamic visualization into the calculus experience provides a possible avenue for increasing student understanding of the concepts in calculus we investigate the role of dynamic visualization in calculus learning. A study of calculus curriculum found that typical calculus tasks stress procedural understanding of calculus with little emphasis on developing and supporting development of conceptual understanding (Lithner, 2004). Also, research suggests that students preferring to visualize when working on mathematical tasks are not the "super stars" in mathematics classrooms (Presmeg, 2006). Student experiences with dynamic visualization software (DVS) may generate the sufficient conflict between unaligned concept images and concept definitions in order to induce the cognitive shift needed to bring these two structures into agreement (Tall & Vinner, 1981; Williams, 1991; Carlson, Oehrtman & Engelke, 2010).

DVS enables students to manipulate a mathematical object dynamically and facilitates visual exploration of mathematical relationships. Thus, our primary research questions center on how the use of DVS influences visualizer and non-visualizer student understanding of derivative as a rate of change of one quantity with respect to another and how experiences with DVS affect students' graphical, analytical, and conceptual understanding of derivatives.

One hundred and ten students enrolled in an introductory calculus course at a large university in the Southwest completed the Mathematical Processing Instrument (Presmeg, 1986). This instrument allowed classification of students into two groups: those who prefer to visualize when doing mathematics (visualizers) and those who prefer not to visualize (non-visualizers). We invited fifteen students to participate in a series of four individual interviews on topics from calculus. Eight were classified as "high visualizers" and the remaining seven were "low visualizers". Twelve students agreed to participate in the study. Half of the students participated in static task interviews and the other half participated in task interviews incorporating DVS. The static interviews centered on typical tasks assigned in calculus. While these tasks may have included a graph or table, they were not posed dynamically nor was DVS offered as a tool for investigation. In contrast, students in the DVS interviews encountered topics analogous to those in the static interviews, but the students explored properties of functions, tangent lines, derivatives, etc., as they manipulated items using DVS.

Preliminary analysis suggests that participants in the DVS interviews hold concept images of derivative that include geometric and algebraic interpretations. Some evidence suggests that these students are more likely to refer to the derivative in terms of an instantaneous rate of change than the students participating in the static interviews. This contrasts with patterns observed in the static interviews where we see little evidence of conceptual understanding of derivative as a rate of change to support students' procedural knowledge. Instead, they appear focused on the "rules" and procedures related to calculating a derivative at a point with little consideration for the "big picture." Evidence suggests that students engaging with DVS experience sufficient conflict between their concept images and concept definitions of calculus constructs to undergo a cognitive change to reconcile the conflict. The data does not show that students in the static interview group experience such a change. Preliminary findings suggest that even those students who prefer not to visualize may benefit from exploring mathematical concepts related to derivative using DVS.

References

Carlson, M., Oehrtman, M., & Engelke, N. (2010). The Precalculus Concept Assessment: A Tool for Assessing Students' reasoning abilities and understanding. *Cognition and Instruction , 28* (2), 113-145.

Carlson, M., Oehrtman, M., & Thompson, P. (2008). Foundational reasoning abilities that promote coherence in students' understandings of function. In M. Carlson, C. Rasmussen, M. Carlson, & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics* (pp. 150-171). Washington, DC: MAA (The Mathematical Assocation of America).

Lithner, J. (2004). Mathematical Reasoning in Calculus Textbook Exercises. *Journal of Mathematical Behavior , 23*, 405-427.

President's Council of Advisors on Science and Technology. (2012). *Engage to Excel: Producing One Million Additonal College Graduates With Degrees In Science, Technology, Engineering and Mathematics.* Washington, DC: United States.

Presmeg, N. C. (1997). Generalization Using Imagery in Mathematics. In L. D. English, *Mathematical reasoning: Analogies, metaphors, and images. Studies in mathematical thinking and learning* (Vol. viii , pp. 299-312). Mahwah, NJ, US: Lawrence Erlbaum Associates Publishers.

Presmeg, N. (2006). Research on Visualization in Learning and Teaching in Mathematics: Emergence from Psychology. In A. Gutierrez, & P. Boero, *Handbook of Research on the Psychology of Mathematics Education: Past Present and Future* (pp. 205-235). Rotterdam, Netherlands: Sense Publishers.

Tall, D., & Vinner, S. (1981). Concept Image and Concept Definition in Mathematics with a particular reference to Limits and Continuity. *Educational Studies in Mathematics , 12*, 151-169.

Williams, S. R. (1991). Models of Limit Held by College Calculus Students. *Journal for Research on Mathematics Education , 22* (3), 219-236.

Examining the pedagogical implications of a secondary teacher's understanding of angle measure

Michael A. Tallman
Arizona State University

This paper reports the results of a series of task-based clinical interviews I conducted to examine how a secondary mathematics teacher's understandings of angle measure afford or constrain his capacity to bring his mathematical knowledge to bear in the context of teaching. The results suggest that the teacher, David, possessed two complementary but conceptually distinct ways of understanding angle measure that he was not consciously aware of having. As a result David was unable to strategically employ his two ways of understanding in novel problem-solving situations and was unable to leverage his understandings in the context of teaching. I also discuss the effectiveness of an instructional intervention designed to support David in becoming aware of his understandings and conclude that engaging teachers in experiences that promote reflected abstraction is one way of supporting them in transforming their mathematical knowledge into a pedagogically efficacious form.

Key words: Mathematical Knowledge for Teaching; Trigonometry; Secondary Mathematics; Clinical Interview.

Introduction

The overwhelming majority of scholarship on teacher knowledge in mathematics education has attended to one, or more, of the following foci: (1) characterizing the nature of mathematical and pedagogical knowledge teachers need to provide students with opportunities to develop a conceptual understanding of mathematics (e.g., Ball, 1990; Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008; Hill & Ball, 2004; Hill, Ball, & Schilling, 2008; Hill, Schilling, & Ball, 2004; Shulman, 1986, 1887); (2) understanding the experiences by which teachers may construct such knowledge (e.g., Harel, 2008; Harel & Lim, 2004; Silverman & Thompson, 2008); and (3) demonstrating the causal link between teacher knowledge and student achievement (e.g., Hill et al., 2008; Hill, Rowan, & Ball, 2005). In other words, research on teacher knowledge in mathematics education has largely focused on what teachers need to know, how they may come to know it, and the effect that this knowledge has on student performance. While this literature makes substantial contributions on several fronts, it appears to be based on the implicit assumption that when teachers develop strong mathematical knowledge, this knowledge is necessarily ready-made to inform their instructional actions.

The present study challenges this assumption by addressing the following research question: How does a secondary mathematics teacher's knowledge of angle measure afford or constrain his capacity to bring this knowledge to bear in the context of teaching?

Methods

The sole participant for the present study was secondary mathematics teacher, David, teaching Honors Algebra II at a large urban public high school in the Southwestern United States. David used the *Pathways Algebra II* curriculum (Carlson, O'Bryan, & Joyner, 2013).

I conducted four task-based clinical interviews (Clement, 2000) to construct a model of David's way of understanding angle measure. Each interview lasted between 60 and 90 minutes and focused on the geometric object of an angle, what it means to measure an angle in radians and degrees, and the condition that a unit of angle measure must satisfy. Many of the tasks from the series of clinical interviews involved the use of *Geometer's Sketchpad*

(Jackiw, 2001). Data collection consisted of obtaining video recordings of the computer screen to capture David's activity using *Geometer's Sketchpad* and video recordings of David's physical actions and written work.

My analysis of the video data began with conducting a pass of open coding (Strauss & Corbin, 1990). I then transcribed all coded instances of each video and performed a line-by-line conceptual analysis (Thompson, 2008) of the transcripts. During this detailed analysis of the transcripts, I attended to explicating hypothetical conceptual operations that explain my interpretation of David's language and actions. I then identified themes in David's reasoning within each interview and compared these themes against relevant data from other interviews. This comparative analysis allowed me to construct a comprehensive and viable model of David's thinking.

Theoretical Perspective

I adopted aspects of Piaget's genetic epistemology—particularly ideas of *abstraction* and *equilibration*—as a theoretical framework in the design and analysis of the series of task-based clinical interviews. Action, according to Piaget, is the catalyst for knowledge development (Piaget, 1967). Piaget explained that higher forms of knowledge derive from abstractions of the subject's actions and the results of his or her actions (Gallagher & Reid, 2002). To characterize and advance David's understanding of angle measure, I designed tasks to engage him in actions that promote his construction of particular understandings and to facilitate abstractions from these actions. My design of such tasks was therefore heavily informed by Piaget's notion of *abstraction* (Piaget, 2001), of which he distinguished five varieties: *empirical*, *pseudo-empirical*, *reflecting*, *reflected*, and *meta-reflection*. The present study was guided primarily *reflecting* and *reflected abstraction*.

Reflecting abstraction involves the reconstruction on a higher cognitive level the coordination of actions from a lower level (Chapman, 1988). Reflecting abstraction is thus an abstraction of actions and occurs in three steps: (1) the *differentiation* of an action from the effect of the action, (2) the *projection* of the action from the level of material action to the level of representation, and (3) the *reorganization* that occurs on the level of representation of the action projected from the level of material action. *Reflected abstraction* involves operating on the actions that result from prior reflecting abstractions at the level of representation, which results in a coherence of actions and operations accompanied by conscious awareness (Piaget, 2001). The subject's ability to assimilate new experiences to the level of representation provides evidence that the subject has engaged in reflected abstraction.

Results

My analysis of the series of task-based clinical interviews suggests that David possessed two complementary but conceptually distinct ways of understanding what it means to measure an angle in radians. David demonstrated a way of understanding angle measure in radians as a fraction of 2π, the total number of radians "in a circle."[1] Consider, for instance, David's remarks in Excerpt 1 from the first clinical interview.

Excerpt 1.

1	Michael	What does it mean to say that an angle has a measure of one radian?
2	David	Then, um (pause). So one radian (pause) means that the complete circumference of a circle is made up of 2π, so 6.28, so one radian is approximately a sixth way around the circle if we're going to do approximates,

[1] Since a radian is a unit of angle measure, the suggestion that there are a certain number of radians "in a circle" is meaningless. My use of this phrase, however, is consistent with my interpretation of David's vague and ill-defined way of understanding what 2π represents.

um, it would be (pause), it would be one over 2π if we wanted to associate a number value …

…

3 Michael So then a similar question: What does it mean to say that an angle has a measure of 2.1 radians?

4 David So to say that it has a measure of 2.1 radians would mean that about, now about a third of the way around a complete circle, if I'm just approximating so that I can kind of understand where it would be in my head, um, without doing any real math, uh, involved.

Based on David's responses in Lines 2 and 4 of Excerpt 1, it does not appear that the word "radian" had any meaning for David beyond "a circle is made up of 2π" of them. Therefore, according to David's way of understanding, to say an angle has a measure of n radians means the angle is $n/(2\pi)$ of the "way around the circle."

Excerpt 2 further supports the claim that David understood angle measure in radians as a fraction of 2π. My question in Line 1 of Excerpt 2 refers to the image displayed in Figure 1.

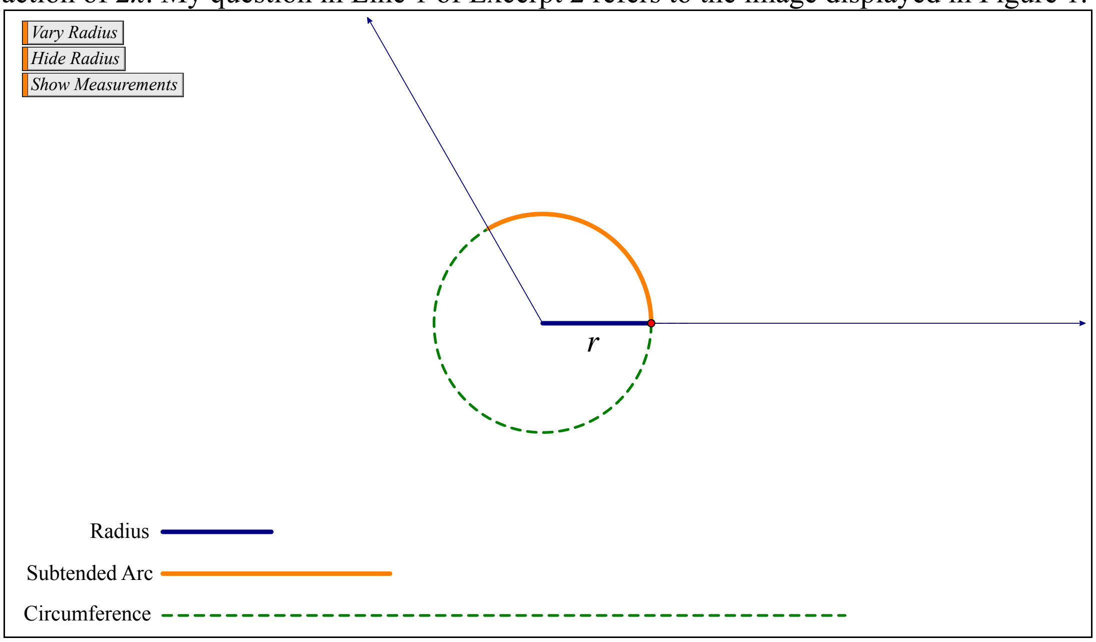


Figure 1. Approximate the measure of the angle.

Excerpt 2.

1 Michael What's approximately the measure of this angle in radians? …

2 David Uh, two, um, two-pi-thirds.

3 Michael Why is that?

4 David Uh, because it's about a third of the, uh, circle. And so, uh, the entire circle in radians would be 2π so it's a third of 2π, which is $\frac{2}{3}\pi$.

Notice that David could have approximated the measure of the angle in at least two different ways given the information provided in Figure 1. For instance, David could have estimated that the subtended arc length is two times as large as the radius, making the measure of the angle approximately two radians. Instead, David estimated that the subtended arc length is a third of the circumference, and thus the measure of the angle must be a third of 2π, the number of radians in a complete circle. David's response further reveals that he understood angle measure in radians as a fraction of 2π.

To determine if David conceptualized angle measure in any way other than as a fraction of 2π, I asked him the same question that I asked in Line 1 of Excerpt 2, but without displaying the length of the circumference at the bottom of the screen. When I asked David to estimate the measure of the angle in radians the second time, he was looking at the image displayed in Figure 2. David responded, "I would say it's about two because there are about two, um, radius lengths in the, um, arc." David's response demonstrates that he was able to reason about angle measure in radians as a multiplicative comparison of the subtended arc length and the radius length. It is worth noting that I simply removed a piece of information that David previously used to estimate the measure of the angle. Without this information David was able to assimilate the task and produce an approximation for the measure of the angle in a completely different way than he had previously.

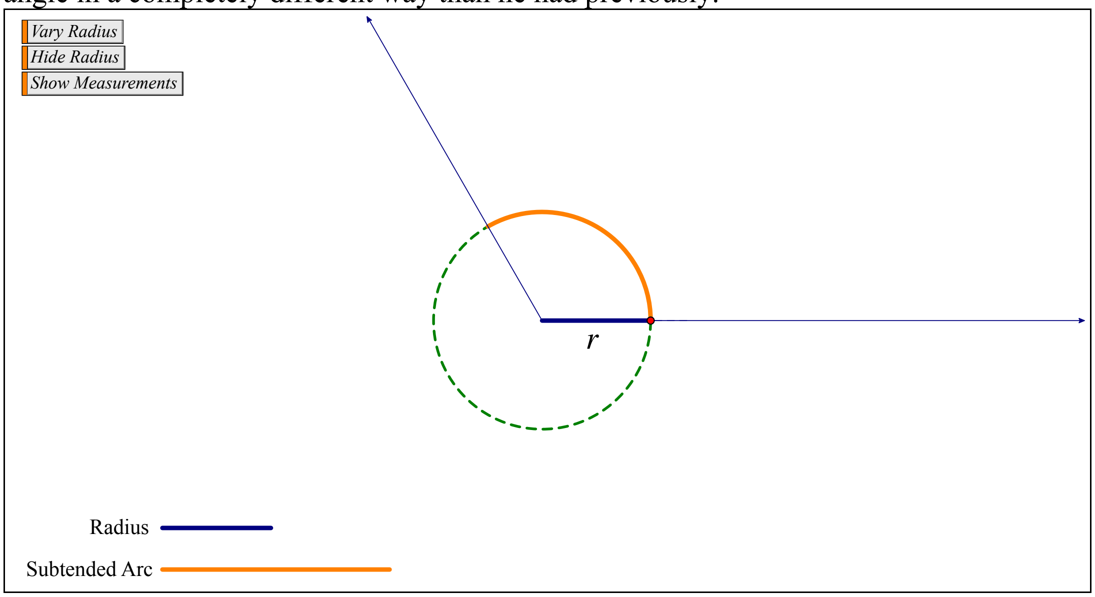


Figure 2. Approximate the measure of the angle (again).

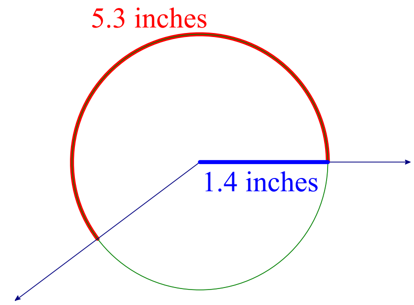
Immediately following David's remark that the measure of the angle in Figure 2 is approximately two radians, I asked him to explain what it means to say that an angle has a measure of 3.92 radians. His response: "I would take the length of the radius, I would multiply the length of the radius by 3.92, and then I would use that to create the length of the arc desired, um, of a circle with the radius that I measured." The way of understanding that David's response suggests differs substantially from that which he demonstrated in Lines 2 and 4 of Excerpt 1 and Lines 2 and 4 of Excerpt 2. David recognized that an angle with a measure of n radians subtends an arc length that is n times as large as the radius of the subtended arc. More generally, David understood angle measure in radians as a measurement of the length of the arc that an angle subtends in units of the radius of the subtended arc.

It is clear that David possessed two complementary but conceptually distinct ways of understanding angle measure in radians, which I refer to as *WoU 1* and *WoU 2*:

(*WoU 1*) To say an angle has a measure of n radians means the angle is $n/(2\pi)$ of the circle centered at the vertex of the angle.

(*WoU 2*) An angle with a measure of n radians subtends an arc length that is n radius lengths, or n times as large as the radius of the subtended arc.

David demonstrated *WoU 2* on several occasions during the first three clinical interviews. One would therefore expect that David would solve the task in Figure 3 by dividing the subtended arc length (5.3 inches) by the radius length.



Suppose an angle has a circle of radius 1.4 inches centered at its vertex and suppose the angle subtends an arc of 5.3 inches. What is the measure of the angle in radians?

Figure 3. What is the measure of the angle in radians?

I asked David the question in Figure 3 towards the end of the third clinical interview and, to my surprise, he did not employ *WoU 2*—an understanding that he utilized on several occasions prior to being asked this question. Instead, David determined the length of the circumference, by computing the product of 2π and 1.4, and divided the subtended arc length (5.3 inches) by the circumference, a ratio that is approximately 0.6. David then claimed that the angle has a measure of 0.6 radians. After a long pause, David was certain that his solution was not correct. He explained that the measure of the angle in radians must be at least π because the angle subtends an arc that is more than half of the circle. After a couple of minutes thinking in silence, David divided 5.3 by 1.4. When I asked David to describe why he divided these values, he explained that since there are 2π radius lengths "around the entire circumference," he would need to multiply the ratio $5.3/(2\pi\cdot 1.4)$ by 2π. David simplified the product $(5.3/(2\pi\cdot 1.4))\cdot 2\pi$ to $5.3/1.4$ but did not appear to see this ratio as representing the number of times the subtended arc length is larger than the radius of the subtended arc. In other words, David did not appear to see the ratio $5.3/1.4$ as the result of measuring the subtended arc in units of the radius. Figure 4 displays David's written work.

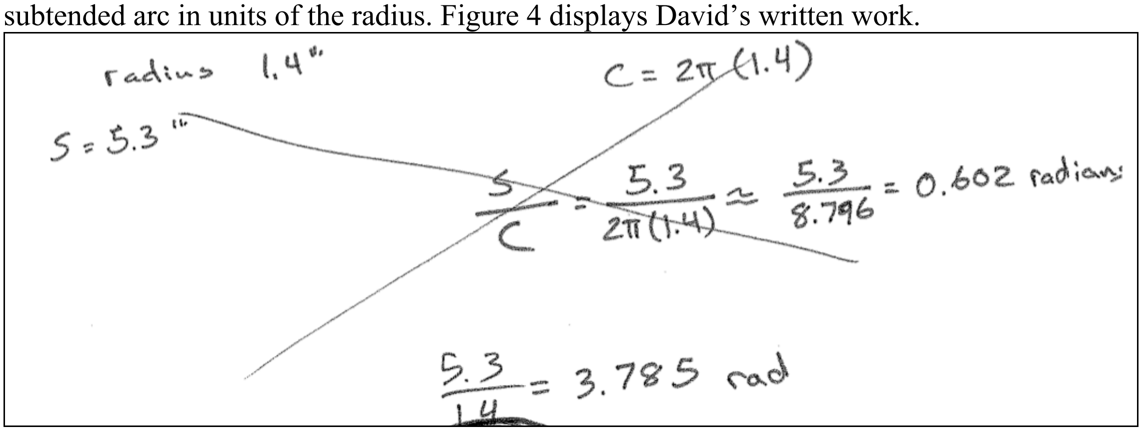


Figure 4. David's written work.

It is noteworthy that David did not utilize *WoU 2* while responding to the task in Figure 3. Moreover, David did not utilize *WoU 1* until he examined the appropriateness of his solution. David's difficulty with this task suggests that his two ways of understanding what it means to measure an angle in radians were not always available to conscious awareness. As a result, David was not equipped to strategically apply the understanding most appropriate for the situation.

I attempted to support David in becoming consciously aware of his two ways of understanding angle measure by providing opportunities for him to perform mental operations on them. In other words, I attempted to engender *reflected abstraction*. I presented David with the following task at the beginning of the fourth clinical interview:

> Courtney claims that measuring an angle in radians means measuring the arc length that the angle subtends in units of the radius of the circle centered at the vertex of the angle. Rebecca says that to measure an angle in radians, you take the length of the arc that the angle subtends divided by the length of the circumference of the circle centered at the vertex of the angle and then multiply this ratio by 2π. Are they both correct?

David was immediately convinced that Courtney's claim (which represents *WoU 2*) is correct but was much more skeptical regarding the accuracy of Rebecca's claim (which represents *WoU 1*). After a few minutes of thinking quietly to himself, he asserted with confidence that both Courtney and Rebecca are correct. David then made the observation in Excerpt 3.

Excerpt 3.

David And actually if we look at what I did on the last question (see task in Figure 3), that is basically what happened. When we do this (points to where he wrote "$\frac{S}{C}=\frac{5.3}{2\pi(1.4)}\approx\frac{5.3}{8.796}=0.602$ radians"), this is the second method that was being described, the ratio of arc length to circumference, and I realized that this is just the percentage of the circle and if I multiplied it by 2π I would be back at the answer. … So this is Rebecca's method (points to the crossed out work in Figure 4) except I didn't multiply it by 2π. … Where this (pointing to where he wrote "5.3/1.4") is taking the arc length and dividing it by the radius measurements to get the answer in, um, radians. … This is the first girl's (Courtney's) method, what we're doing here (points to where he wrote "5.3/1.4"). We're saying we have the arc and we're saying how many of the radiuses go into it. And then here (points to the work that he crossed out in Figure 4) … we figure out the percentage of the circle that we have and then by multiplying it by 2π then we figure out the percentage of radians, or we figure out the number of radians that we have where 2π is basically that there should be 2π radians in a complete circle and this is the percentage of the circle (points to the ratio "5.3/8.796").

Asking David to reflect on the accuracy of Courtney and Rebecca's statements engendered a reflected abstraction by providing an occasion for him to operate on his two ways of understanding what it means to measure an angle in radians. Prior to engaging in this reflected abstraction, David was not aware of having two different ways of understanding what it means to measure an angle in radians. After engaging in this reflected abstraction, David could not only describe the validity of Courtney and Rebecca's claims, but could identify how their meanings for angle measure were represented in his solution to a task that he previously struggled to solve and explain. The ease with which David recognized both *WoU 1* and *WoU 2* in his solution to the previous task, and the fluency with which he strategically utilized these two ways of understanding to solve subsequent tasks, suggests that David had reorganized his two ways of understanding what it means to measure an angle in radians into a coherent scheme and, as a result, had become consciously aware of having these understandings.

Discussion

While David demonstrated two rather useful ways of understanding what it means to measure an angle in radians, he had not previously reflected on these understandings in a way that enabled him to become consciously aware of having them. David was therefore unable to strategically employ his two ways of understanding to solve novel problems.

These results suggest the following pedagogical implication: *Teachers can only leverage understandings in their instruction that they are aware of having*. Teachers' subject matter knowledge is useful only to the extent that it informs the learning opportunities they provide for students. It is of little consequence for teachers to have mathematical knowledge that is incapable of informing their instructional actions. Teachers must have reflected on the mental actions and operations that comprise their mathematical understandings in order to transform implicit subject matter knowledge into a form that maintains pedagogical utility.

Characterizing, at the level of mental activity, the ways of understanding teachers want students to develop allows them to interact with students in productive ways and to employ effective instructional interventions in the moment. Since David was previously unaware of his own mathematical understandings for angle measure, his instruction could not have been informed by an image of the mental actions involved in constructing a desirable way of understanding angle measure. David was therefore ill-prepared to engage students in purposeful learning experiences that seek to promote specific mental activity. This study demonstrates that engaging teachers in experiences that promote reflected abstraction is one way of supporting them in transforming their mathematical knowledge into a form they can leverage to support students' learning.

References

Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *Elementary School Journal, 90*, 449-466.

Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade and how can we decide? *American Educator* 14-22; 43-46.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*, 389-407.

Carlson, M., O'Bryan, A., & Joyner, K. (2013). *Pathways Algebra II: Implementing the Common Core Mathematics Standards* (3rd ed.). Gilbert, AZ: Rational Reasoning, LLC.

Chapman, M. (1988). *Constructive evolution: Origins anddevelopment of Piaget's thought*: Cambridge University Press.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In R. Lesh & A. Kelly (Eds.), *Handbook of Research Methodologies for Science and Mathematics Education* (pp. 547-589). Hillsdale, NJ: Lawerence Erlbaum.

Gallagher, M. J., & Reid, K. (2002). The learning theory of Piaget and Inhelder.

Harel, G. (2008c). What is mathematics? A pedagogical answer to a philosophical question. In R. B. Gold & R. Simons (Eds.), *Current Issues in the Philosophy of Mathematics From the Perspective of Mathematicians*: Mathematical Association of America.

Harel, G., & Lim, K. (2004). *Mathematics teachers' knowledge base: Preliminary results.* Paper presented at the 28th Conference of the International Group for the Psychology of Mathematics Education, Bergen, Norway.

Hill, H. C., & Ball, D. L. (2004). Learning mathematics for teaching: Results from California's mathematics professional development institutes. *Journal for Research in Mathematics Education, 35*(5), 330-351.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372-400.

Hill, H. C., Blunk, M. L., Charalambos, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction, 26*(4), 430-511.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.

Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *The Elementary School Journal, 105*(1), 11-30.

Jackiw, N. (2001). The Geometer's Sketchpad™ (Version 4.0) [Computer software]. Emeryville, CA: KCP Technologies.

Piaget, J. (1967). *Six psychological studies*. New York: Random House.

Piaget, J. (2001). *Studies in reflecting abstraction*. New York, NY: Psychology Press.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 1-22.

Silverman, J., & Thompson, P. W. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education, 11*, 499-511.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*: Sage Publications.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundation of mathematics education. *Proceedings of the annual meeting of the International Group for the Psychology of Mathematics Education, 1*, 45-64.

Exploration of undergraduate students' and mathematicians' perspectives on creativity

Gail Tang
University of La Verne
gtang@laverne.edu

Houssein El Turkey
University of New Haven
houssein@ou.edu

Milos Savic
University of Oklahoma
savic@ou.edu

Gulden Karakok
University of Northern Colorado
gulden.karakok@unco.edu

Mathematical creativity has been an object of discussion in mathematics for some time (Borwein, Liljedahl, & Zhai, 2014). Though there are recent efforts to include creativity in K-12 education agendas (Askew, 2013) and a number of research articles in mathematics education literature about creativity focus K-12 (e.g., Silver, 1997), there is little research in undergraduate mathematics education about creativity. Our study aims to address this issue by examining what mathematicians and undergraduate students think about creativity in proving. We coded six mathematician and eight student interviews using the Creativity in Progress Rubric (CPR) on proving created by the research group. A majority of the students' responses were of the Creating Ideas category of the rubric, while mathematicians were more balanced. We claim that the other two categories (Taking Risks and Making Connections) might need further explicit discussion in classrooms if the transition from student to mathematician is desired.

Key words: creativity, proving, undergraduate students, mathematicians, assessment rubric

Introduction

Exploring mathematical creativity is an ongoing quest of researchers, with the earliest known attempt by two psychologists, Claparède and Flournoy, in 1902 (as cited in Borwein et al., 2014 and Sriraman, 2009). The famous mathematician Hadamard (1945) extended the study of Claparède and Flournoy by surveying (via mail) mathematicians' creative processes. Most recently, Sriraman (2009) and Borwein et al. (2014) reported their findings on mathematicians' perspectives on creativity, indicating its importance for the growth of the field. In particular, Borwein et al. (2014) demonstrated "how people actively engaged in mathematics think about the undertaking as a creative and exciting pursuit" (p. xi). The important role of creativity in mathematicians' work is undeniable. Thus, it is important to understand how creativity can be cultivated while learning mathematics.

Especially while teaching, Mann (2005) stated that avoiding the acknowledgment of creativity could "drive the creatively talented underground, or worse yet, cause them to give up the study of mathematics altogether" (p. 239). In addition, Askew (2013) pointed out the recent efforts in education agendas and stated, "[c]alls for creativity within mathematics and science teaching and learning are not new, but having them enshrined in mandated curricula is relatively recent" (p.169). For such reasons, investigating creativity and understanding its role in teaching mathematics are essential. Although there have been explorations of mathematical creativity of students in K-12 (e.g., Sriraman, 2005, Silver, 1997; Lev-Zamir & Leikin, 2013), there is a need to extend the research efforts to undergraduate-level mathematics teaching and learning.

In our multi-tiered research project, we first investigated the mathematicians' perspectives on creativity, values of creativity in teaching mathematics courses (especially proof-based ones), and possible ways of assessing students' creativity in such courses (Savic, Karakok, Tang, & El Turkey, 2014). Following this first step of the project we developed a Creativity in-Progress Rubric (CPR) on proving (Savic, Karakok, Tang, & El Turkey, 2014). In this paper, we share our results from the second step of our project, which addresses the research question:

What are some similarities and differences between mathematicians and undergraduate students' perspectives on creativity?

Literature Review

Even though the efforts of understanding mathematical creativity have been around for about a century, there is no single definition for it. Mann (2006) reported that there are over 100 different definitions of creativity. Some of these definitions highlight the *end product view* by evaluating the originality and the usefulness (Runco & Jaeger, 2012) of the final solution. On the other hand, some definitions emphasize the *process view*, which aims to understand the mechanism of creativity while a person is engaged in a creative activity (Balka, 1974; Torrance 1966). For example, Pelczer and Rodriguez (2011) pointed out that "it is important that when judging the creativity of a student we pay attention also to the process by which he[/she] arrived to the results and not only to the final problem" (p. 394).

As well as focusing on process, Sriraman (2005) advocated that students' creativity needed to be evaluated according to their prior experiences. This particular point highlights the difference between *absolute* and *relative* creativity; the former one refers to historical inventions or discoveries at a global level and the latter one is defined as, "the discoveries by a specific person within a specific reference group, to human imagination that creates something new" (Vygotsky, 1982, 1984; as cited by Leikin, 2009, p. 131). For example, Sriraman and Liljedahl (2006) use the relativistic perspective to define mathematical creativity at the school level as a process of offering new solutions or insights that are unexpected for the student, with respect to his/her mathematics background or the problems s/he has seen before. This definition acknowledges that students "have moments of creativity that may, or may not, result in the creation of a product they may, or may not, be either useful or novel" (Liljedahl, 2013, p. 256).

Sriraman (2005) argued that these definitions of creativity, which were generated from investigating K-12 students, do not capture the kind employed by mathematicians. Furthermore, we claim that the creativity enacted in an upper-level undergraduate mathematics course is closer to the type of creativity that mathematicians use than that exhibited in K-12 courses. Thus, there are needs to:

1. understand mathematicians' perspectives of creativity in teaching and learning of mathematics in undergraduate level, and
2. investigate the mathematical creativity of undergraduates.

To address the first point, we interviewed mathematicians and developed a rubric, *Creativity in-Progress Rubric (CPR) on proving* (see Authors, 2014 for more detail on development of rubric study). In the next section, we describe the method of our study, in particular, how we implemented this rubric as an analytical framework to address the research question. More precisely, in this study we investigate the similarities and differences between mathematicians' and students' perspectives on creativity in the context of undergraduate mathematics education.

Methods

Participants

Participants of this study were six research mathematicians at two universities and eight undergraduate students who were enrolled in an introduction-to-proofs course. Six research mathematicians were selected for interviews according to their teaching and research experiences. Two professors (both male) that are from a predominantly teaching institute (University X) and four professors (three females, one male) from a large research institute (University Y) participated in the research. Varying from 8 to 30 years of experience, they teach both undergraduate and graduate mathematics courses and conduct research in different areas.

The student participants of this study were enrolled in the introduction-to-proofs course offered Spring 2014 at University Y. This course is a requirement for all engineering and mathematics majors, and can be taken as early as completion of Integral Calculus. From the 24 students enrolled, eight were selected for end-of-semester interviews based on his/her year of study, major, gender, and level in class. There were two freshmen, three sophomores, one junior, and two seniors. Four were mathematics majors, three of which were also double majoring in either Political Science, Management Systems, or Accounting. Two out of the three females in the course participated in the research. Three students received a final grade of A, four received B's and one received a C. All interviews were video- and audio-taped and transcribed.

Data Analysis

The *CPR on proving* is used as an analytical framework to analyze the interview data. The purpose of using *CPR* as an analytic tool had two purposes: (1) to validate the categories created from the mathematician interviews; (2) to tease out similarities and differences between mathematicians' and students' views.

The rubric was constructed through triangulating research-based rubrics, mathematicians' views on mathematical creativity at the undergraduate level, and students' proving processes and attempts in a transition-to-proof course. The Creative Thinking Value rubric, developed by the Association of American Colleges and Universities (AAC&U) (Rhodes, 2010), Leikin's (2009) study implementing another rubric for mathematical creativity in problem solving, and the Torrance tests for assessing creativity (1966; 1988) were cross-analyzed with the interview data from mathematicians. Three main categories about creativity in students' proving processes surfaced from the existing rubrics, mathematician interviews, and students' in-class data:

- *Making connections* - demonstrating links between multiple representations and/or ideas from the student's current and/or previous course(s).
- *Taking risks* - approaching a proof and demonstrate flexibility in using different or multiple approaches.
- *Creating ideas* - developing original mathematical ideas that are either pertinent to the proof or can be proven.

Two researchers on the team separately categorized the student and mathematician interviews using these three categories of the CPR. In particular, they analyzed responses to participants' definitions of creativity and reactions to three student-generated proofs published in Birky et al., (2011). After the separate coding, the two researchers compared their coding and combined their schemes into one in which they were in agreement. Any disagreements were resolved within the entire research team.

When appropriate, student and mathematician quotes were matched to the subcategory criteria in the CPR. Since the CPR is designed to assess students' relative creativity in their proving processes, it was challenging to identify all subcategories of each main category in the CPR. For this reason, quotes were classified generally under one of the three above aspects. For example, Dr. C says, “I'm asking [the students] ... to make connections with things that they haven't made connections before.” This clearly fits into the category Making Connections, but it does not specifically fall under any of the Making Connections subcategories in the CPR.

Results

In this section we provide a sample of quotes from mathematicians and students exemplifying the coding process. Tables 1 and 2 provide an overview of how each group of participants differ in their references to the three different categories of the CPR.

Mathematicians’ Responses

Mathematicians expressed certain ideas on creativity in their interviews. We show some examples of their quotes, themed by the three CPR on proving categories.

Making Connections

- Somehow your mind has to spread out a little bit *to see connections, connections to other theorems you could use* . . . That's creativity also. – Dr. A
- I think when students realize that they can *solve these problems* with things that are not just in this section. It can be *from some other part of the course*. Be somewhat creative. You are encouraged to use everything you know. – Dr. LS

Taking Risks

- [Y]ou're saying, “there is this problem, and I'm going to try this approach. And this approach, I don't even know what the next step should be.” So I think the creativity part of it affects the proof differently. – Dr. B
- I think the creativity comes in thinking about *which technique to use*, what is it that you’re going to do next. You are trying to figure something out. – Dr. D

Creating Ideas

- So I was working on that and there was an implication that the colleague had proved, A implies B. And it occurred to me...*is the converse true? Does B also imply A?* … And then I proved it. And that was a great moment. Because it was some *new discovery*. But the actual creative moment was not the carrying out, … [b]ut having the idea was the spark. … It's that initial moment that is the creative part – Dr. A
- Nobody knew how to do it that whoever did it was creative in the sense that he thought of some kind of *idea or approach that no one ever thought of before*. – Dr. C
- [Proofs 2 and 3 from Birky et al. (2011)] are both *non-routine thinking*. I like both of them. *I'm not sure if I would be able to come up with any of those two proofs* because it requires some *unexpected ideas*. – Dr. DA

Table 1: *Number of Mentions in each Category by Mathematician*

	Creativity Category		
Mathematician	Making Connections	Taking Risks	Creating Ideas
Dr. A	4		3
Dr. B	1	3	6
Dr. C	4		3
Dr. DA	4	1	10

Dr. LS	6	2	3
Dr. D	4	4	2
Total	23	10	27
Percentage	38	17	45

Students' Responses

Students reported their ideas about creativity in their interviews. Some of their quotes are separated into the three categories. S15 denotes Student 15.

Making Connections

- I would have tried to do something similar based on other things we'd proved like this. – S10
- The creativity would be *using the rules that I can hold*, tangible rules that are easily, relatively easily proven to build your way to something new. – S15

Taking Risks

- Creativity in math would mostly be *trial and error…* – S28
- So you have a proof to do – there's sometimes, you know, *tons of different ways to do it* and to be creative is when, you know, you don't just use the simple one necessarily, but uh – uh *think of something else*. – S11
- Most people would think induction, but they're like "okay, I'll try to *think of another way*" – S16

Creating Ideas

- [Proof 3 from Birky et al. (2011)] would be creative just because *it's not necessarily something I would've thought of.* – S5
- Another aspect of being creative is that you have a theorem and then there's different things – different implications of that theorem you can think about…*all the what if's*. When you go down those roads, that's being creative. – S11
- Creative [sic] comes down to *finding certain ideas* to take the uh statement of a uh theorem or a conjecture and just *coming up with a strange way* to take it down to the solution. – S14

Table 2: *Number of Mentions of each Category by Student*

	Creativity Category		
Student	Making Connections	Taking Risks	Creating Ideas
5		3	4
10	1	1	4
11	1	6	6
14			9
15	3	2	8
16		2	2
26			
28	1	3	8
Total	6	17	41
Percentage	9	27	64

Table 2 shows that the mathematicians value Making Connections, Taking Risks, and Creating Ideas 38%, 17%, and 45% of their 60 total responses, respectively. However, students valued Creating Ideas at 64%, followed by Taking Risks at 27%, and finally Making Connections at 9% out of their 64 total responses.

Discussion & Conclusion

The largest difference between the participants is that the Making Connections aspect of creativity was not as valued by the students (9%) as it was by the mathematicians (38%). We claim that students rarely think of connecting mathematical ideas as being creative. In addition to this particular finding, we observed that Creating Ideas was most valued by the two participant sets, which could be a result of the meaning of creativity in general sense. Mathematicians almost equally valued Making Connections (38%) and Creating Ideas (45%). The mathematicians' least-valued category was Taking Risks (17%), perhaps due to the fact that mathematicians commonly take risks in proving in both their research (Burton, 1999) and in proving tasks (Savic, 2012).

This difference in percentages between students and mathematicians could be the result of the varying mathematical experiences between mathematicians and students, or that students may not have been explicitly asked by the instructor to make many connections with the content given in previous courses. The latter reason could be alleviated if some explicit discussion of Making Connections happens in the classroom. The same could be true for increasing the value students see in Taking Risks while being creative. Taking Risks is a category that is slightly difficult to promote, especially in a course that only grades proofs based on correctness. Some students may feel as if they will never create a correct proof so they should not even try. This reason really underscores the importance of focusing on the process view of creativity rather than the product view.

We asked the students if mathematical creativity could or should be assessed in class. All eight participants agreed that creativity should not be graded, due to issues with relative creativity. Student 14 claimed that "I think it would be really hard to pinpoint, um what is creative and what isn't creative, partially because I think it is subjective to each of us, what is creative could be different." We also asked the mathematicians about teaching creativity in proving and they, too, were unanimous in not explicitly grading creativity. We claim that the CPR on proving used to code the participants could also be used as a formative assessment (relative to each student) in proof-based courses, thus alleviating Student 14's worry. The CPR on Proving may both reveal to students aspects of creativity that are valued by the mathematicians and help to foster these values in the classroom. For example, in this study, it is clear that mathematicians highly value Making Connections when discussing creativity, while students' responses with Making Connections was limited. The rubric would communicate this aspect to the students and the instructors could use it ssto help students develop this trait during the course of the class.

In Fall 2014, the CPR will be used to explicitly value creativity in an introduction-to-proofs course. The rubric itself will be available to students so that they may monitor their own progress and evaluate their understandings. Tasks, such as asking for multiple proofs for a theorem or conjecturing and proving new theorems, will be implemented throughout the course to foster mathematical creativity in proving. Though students will not get grades on creativity for those particular tasks, creativity will be worth some portion of the final grade. We conjecture that if the teacher values mathematical creativity, students will value it also.

References

Askew, M. (2013). Issues in teaching for and assessment of creativity in mathematics and science. In D. Corrigan, R. Gunstone, & A. Jones, *Valuing Assessment in Science Education: Pedagogy, Curriculum, Policy* (pp. 168-192). Dordrecht: Springer-Verlag.

Balka, D. S. (1974). Creative ability in mathematics. *Arithmetic Teacher, 21* (7), 633-638.

Birky, G., Campbell, C. M., Raman, M., Sandefur, J., & Somers, K. (2011). One problem, nine student-produced proofs. *The College Mathematics Journal, 42* (5), 355-360.

Borwein, P., Liljedahl, P., & Zhai, H. (2014). *Mathematicians on Creativity.* Washington D.C., USA: Mathematical Association of America.

Burton, L. (1999). The practices of mathematicians: What do they tell us about coming to know mathematics? *Educational Studies in Mathematics, 37*, 121-143.

Byers, W. (2007). *How mathematicians think: Using ambiguity, contradiction, and paradox to create mathematics.* Princeton, NJ: Princeton University Press.

Hadamard, J. (1945). *The mathematician's mind.* Princeton: Princeton University Press.

Leikin, R. (2009). Exploring mathematical creativity using multiple solution tasks. In R. Leikin, A. Berman, & B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students* (pp. 129-145). Haifa, Israel: Sense Publishers.

Lev-Zamir, H., & Leikin, R. (2011). Creative mathematics teaching in the eye of the beholder: Focusing on teachers' conceptions. *Research in Mathematics Education, 13*, 17-32.

Liljedahl, P., & Sriraman, B. (2006). Musings on mathematical creativity. *For the Learning of Mathematics*, 17-19.

Mann, E. (2006). Creativity: The essence of mathematics. *Journal for the Education of the Gifted, 30* (2), 236-260.

Mann, E. (2005). *Mathematical creativity and school mathematics: Indicators of mathematical creativity in middle school students.* (Doctoral Dissertation). University of Connecticut : Storrs.

Pelczer, I., & Rodriguez, F. G. (2011). Creativity assessment in school settings through problem posing tasks. *The Montana Mathematics Enthusiast, 8*, 383-398.

Polya, G. (1957). *How to solve it: A new aspect of mathematical method.* Garden City, NJ: Doubleday.

Rhodes, T. (2010). *Assessing Outcomes and Improving Achievement: Tips and Tools for Using Rubrics.* Washington, DC: Association of American Colleges and Universities.

Runco, M. A., & Jaeger, G. G. (2012). The standard definition of creativity. *Creativity Research Journal, 24* (1), 92-96.

Savic, M. (2012). What do mathematicians do when they reach a proving impasse? In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman, *Proceedings of the 15th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 531-535). Portland, OR: Online at http://sigmaa.maa.org/rume/Site/Proceedings.html.

Savic, M., Karakok, G., Tang, G., & El Turkey, H. (2014). How Can We Assess Undergraduate Students' Creativity in Proof and Proving? In *Proceedings of the 8th International Conference on Mathematical Creativity and Giftedness.* Denver, CO.

Silver, E. (1997). Fostering creativity through instruction rich in mathematical problem solving and posing. *ZDM Mathematical Education, 3*, 75-80.

Sriraman, B. (2005). Are giftedness and creativity synonyms in mathematics? *Prufrock Journal, 17(1)*, 20-36.

Sriraman, B. (2009). The characteristics of mathematical creativity. *ZDM Mathematics Education, 41*, 13-27.

Torrance, E. P. (1988). The nature of creativity as manifest in its testing. In R. J. Sterberg, *The nature of creativity: Contemporary psychological perspectives* (pp. 43-75). New York, NY: Cambridge University Press.

Torrance, E. P. (1966). *The Torrance tests of creative thinking: Technical-norms manual.* Princeton, NJ: Personnel Press.

Vygotsky, L. (1978). *Mind and society: The development of higher mental processes.* Cambridge, MA: Harvard University Press.

Vygotsky, L. S. (1982). Imagination and its development in childhood. In V. V. Davydov, *Vol. 2: General Problem of Psychology. The Collected Works of L. S. Vygotsky* (pp. 438-454). Moscow, SSSR: Pedagogika.

Leveraging historical number system to understand number and operation in base 10.

Eva Thanheiser
Portland State University

Andrew Riffel
Portland State University

Abstract: Historical number systems were leveraged to build preservice teachers (PTs) understanding of base 10. Variation theory states that you cannot know something if all you know is that one thing. Thus to understand numbers and operations in base 10 PTs need to experience number and operations within and beyond base 10. In this study we examined how historical numbers systems can be used to allow PTs to build a conceptual understanding of numbers and operation in base ten by comparing and contrasting various number systems. Thirty-six PTs concepts of number were identified before and after they work through a series of tasks situated in various number systems and compared and contrasted across the systems. Almost all PTs developed a more sophisticated conception of number though these experiences.

Keywords: Elementary Teacher Training, Number and Operation, Variation Theory.

Rationale & Literature Review: Preservice elementary teachers (PTs) often enter our university mathematics classrooms efficient at performing procedures but struggling when asked to explain those procedures conceptually (Ball, 1988; Ma, 1999; Thanheiser, 2009, 2010). Much of this struggle may be related to their limited conceptions of number (Thanheiser, 2009, 2010).

To understand how numbers are composed in the base 10 system and how algorithms take advantage of that composition PTs need to understand the underlying base ten system and how numbers are composed on that system. When we base-ten users read 527, the referent for the entire number is understood to be ones, so 527 is 527 ones. Each digit name also has ones as the referent unit: 500 ones, 20 ones, and 7 ones. But when working with standard algorithms, we usually think of the digits in columns: 5 hundreds, 2 tens, and 7 ones and work with a different referent unit (ones, tens, hundreds, etc.) in each column. The fact that a number can be decomposed into parts that refer to different referent units [e.g., 527 x (1) = 5 x (100) + 2 x (10) + 7 x (1)] can be considered "one of the basic powerful ideas in the invention of the base ten numeration system" (Zazkis & Khoury, 1993, p. 41). Through this decomposition, columns can be treated individually in the context of the algorithms. Because of the underlying structure of the base-ten power sequence on which base-ten numbers are built, each referent unit is 10 times as large as the next lower referent unit; thus, we can group 10 units of a smaller size to make 1 unit of the next larger size and vice versa.

Thus to understand numbers in a way that allow PTs to explain the algorithms built on them they need to understand that a) each digit has two values, a face value indicating the number of groups (i.e. 5 in 527 for the digit 5) and a place value indicating the size of the group (i.e. 100 in 527 for the digit 5), and that the value of the digit is determined by multiplying the face value (number of groups) by the place value (size of group); b) the group sizes are based on powers of ten, thus ten of a smaller unit form one of the next larger and one of a larger unit can be decomposed into ten of the next smaller; and c) numbers can be grouped and regrouped on this system.

Thanheiser (2009) showed that most PTs hold limited conceptions of number restricting them from understanding why the algorithms work. She identified 4 conceptions PTs hold when entering mathematics content courses for teachers (see Table 1). With only 30% of the PTs holding a correct conception (reference units or groups of ones) and only 20% holding the most sophisticated conception (reference units), which builds on the underlying base ten system and is required to explain all aspects of the algorithms. These results have held steady across several studies at the beginning and the end of teacher education programs (Thanheiser, 2010, 2014; Thanheiser, Philipp, Fasteen, Strand, & Mills, 2013) showing that only up to 30% of PTs hold correct conceptions.

Table 1. Definition of conceptions in the context of the standard algorithm for the PTs in Thanheiser's (2009) study

Conception held	Number of PSTs
1. *Reference units.* PSTs with this conception reliably[a] conceive of the reference units for each digit in the number and can relate the reference units to one another; in 389, the 3 can be seen as 3 hundreds or 30 tens or 300 ones, and the 8 can be seen as 8 tens or 80 ones.	3 (20%)
2. *Groups of ones*. PSTs with this conception reliably conceive of all digits in terms of groups of ones; 389 would be 300 ones, 80 ones, and 9 ones. PSTs holding this conception do not conceive of the digits in terms of reference units.	2 (13%)
3. *Concatenated digits plus.* PSTs with this conception conceive of at least one of the digits in the number in terms of an incorrect unit type at least some of the time. They therefore struggle when relating the values of the digits in a number to one another. A PST may correctly conceive of groups of 100 ones for a digit in the hundreds place but incorrectly conceive of ones for the tens place (e.g., 389 would be seen as 300 ones, 8 ones, and 9 ones).	7 (47%)
4. *Concatenated digits only*. PSTs holding this conception conceive of all the digits in terms of only ones (e.g., 548 would be 5 ones, 4 ones, and 8 ones).	3 (20%)

[a] *Reliably* in these definitions means that after the PSTs were first able to draw on a conception in their explanations in a context, they continued to do so in that context.

This aligns with prior research, for example, Ross (2001) showed that PTs' understanding of the digits in a multidigit number is lacking, and Zazkis and Khoury (1993) showed that PTs' understanding of the underlying structure of powers of ten in our base-ten numeration system is deficient.

We also know that children "experience considerable difficulty constructing appropriate number concepts of multidigit numeration and appropriate procedures for multidigit arithmetic'' (Verschaffel, Greer, & De Corte, 2007, p. 565). If we want teachers to be in a position to help children develop appropriate number concepts we need to help PTs develop an understanding of number concepts so they can help children develop a rich mathematical understanding. However, developing this understanding is difficult since once the procedures are in place it is very hard make sense of the underlying mathematics (Pesek & Kirshner, 2000). One way to address this difficulty is to put PTs into a context in which they do not yet have the procedures available and thus have to make sense of the underlying mathematics. This can be accomplished by allowing PTs to explore number and operation in different number systems and then make comparisons between the systems. In this study we examine how historical number systems can be leveraged to help PTs make sense of base 10 by stepping out of it.

Theoretical Framework. Variation theory states that one cannot know something if all one knows is that one thing (Lo, 2012). To really understand something one needs to know what that thing is and what it is not as well as the difference (variation) between those two (Marton, 2009).

"According to Variation Theory, meanings do not originate primarily from sameness, but from difference, with sameness playing a secondary (Marton, 2009) role." (Marton in Lo, 2012 foreword). Thus, to fully understand our number system (built on base 10) one needs to know that system as well as other systems and understand the similarities and differences between the two systems.

In addition to variation it is essential that PTs perceive the tasks that they are given as authentic. One way of making a task more authentic for is by connecting the university classroom to the real world (in the case of PTs the K-12 classroom) (Newman, King, & Carmichael, 2007). Research has shown the importance of authentic tasks; "students who experienced higher levels of authentic instruction and assessment showed higher achievement than students who experienced lower levels of authentic instruction" (Newman et al., 2007, p. vii). PTs in particular are motivated by tasks for which they see a real connection to their future classroom (i.e. they can use that task with children later). Some research has explored the use of alternate bases with PTs to identify conceptions (Zazkis & Khoury, 1993) and the development thereof (Fasteen, Meluish, & Thanheiser, 2015; McClain, 2003; Yackel, Underwood, & Elias, 2007). In this study we use historical number systems as a context to make sense of numbers and operation in base 10 . PTs typically enjoy learning about them and view them as authentic because they are relevant to their future teaching (ancient cultures are part of the K-8 curriculum). We build on some prior work by Thanheiser (2014) and Thanheiser & Rhoads (2009), which examined the use of the base 20 Mayan numeral system as a context to explore shifts in the value of digits when comparing the values of a "one" with one, two, and six zeros attached at the end (Thanheiser, 2014; Thanheiser & Rhoads, 2009). This study showed that the Mayan system allowed the PTs to discuss the value of adjacent digits in a way that is not possible in base 10.

Methods: Data is drawn from two sections of a mathematics content course for preservice teachers with a total of 36 PTs (13 PTs in a summer course and 23 PTs in a regular school year course) who gave consent to have their data used. Both courses were 4 credit courses in a 10-week quarter system. All PTs were interviewed before and after the course to identify their conceptions of number (see Thanheiser et al., 2013 for interview protocol). Thanheiser's 2009 framework (see Table 1) was used to identify the conceptions. The interviews were double coded with an agreement of 88% (the disagreements were resolved through discussion). Both groups experienced the same sequence of tasks described below. All PT work was collected and scanned and read to make sense of how PTs approached each task.

The Tally Activity presented students with the idea that all number systems share one thing in common; they have a symbol for 1 (tally). Students are then presented with a sheet of randomly placed tallies (about five hundred of them) and asked to count those. The goal of the activity is to create an authentic need for grouping (needed to count without loosing track). This activity then leads into number systems developed to record large numbers (grouping systems) in which groups of tallies are represented by new symbols (i.e. the Egyptian System). The variation emphasized between the tally system and other systems is that other systems utilize groups of various sizes while the tally system only has groups of size 1 and that some other systems utilize place value while the Tally System does not. See Table 2 for a comparison between the characteristics of each system described in this paper.

The Egyptian Activity. The Egyptian number system is a grouping system, each digit represents its value independent of placement. A tally represents a one, an arch represents a ten, a coil represents a hundred, etc. The group sizes are successive multiples of ten. Thus 25 can be represented by a tally followed by an arch followed by four tallies followed by another arch, or any other combination of those symbols (including non minimal collections). The activity allowed PTs to explore the idea that while different symbols represent different sized groups; the location of the symbols does not matter in the Egyptian system. PTs were asked to convert numbers between the base 10 and the Egyptian systems. Egyptian numerals were presented in mixed order (not ordered from largest to smallest) to highlight the fact that order doesn't matter. Artifacts of children's mathematical thinking were used to discuss the fact that a symbol for zero is not needed in a grouping system. For example, PTs were first asked to convert 4508 into Egyptian symbols and then viewed a video of children doing the same and discussing how/whether to represent the 0 tens (they decide to leave a space). Then PTs were asked to perform operations (multiplication in particular) in the Egyptian system to highlight the need for a system designed to allow easy computations. The variation emphasized between the Egyptian system and base 10 is that the location of the symbol in base 10 determines the size of the group it represents (ones, tens, hundreds, etc.) while the location is not relevant in the Egyptian system, there is no need for a symbol for zero, and computations are increasingly messy in the Egyptian system.

The Mayan Activity allowed PTs to explore a different base. First students familiarize themselves with the Mayan number system (a base 20 system). Students are presented with the first 30 Mayan numbers and then asked what a one with one zero (20), a one with two zeros (400) and a one with six zeros (64×10^6) represents. After this activity - which is designed to help PTs explicate the underlying base system resulting in a x20 relationship between adjacent digits - PTs are asked to invent addition and subtraction algorithms in the Mayan system. The variation emphasized between the Mayan system and base 10 is the explication of the underlying base (20 vs 10) and the relationship between adjacent unit types as x20 (Mayan) and x10 (base 10).

Compare and contrast the different systems. Once PTs make sense of each of these systems they are asked to describe a grouping system and a place value system and discuss the similarities and differences among them and identify the important aspects of a place value system.

Table 2. Comparison between the characteristics of each system described in this paper.

Number System	Number of Symbols	Can a symbol represent more than one value	Grouping System	Place Value System	Need for zero?	Relationship of adjacent places
Tally	1	No	No	No	No	N/A
Egyptian	Infinite	No	Yes	No	No	N/A
Mayan	2	Yes	No	Yes	Yes	20 to 1
Base 10 system	10	Yes	No	Yes	Yes	10 to 1

Results and Implications for Teaching. Each of the activities helped PTs learn important information about numbers in base 10 by pulling them out of their typical context and discussing the differences (variations). Almost all PTs developed more sophisticate concepts of number in the base ten system throughout the course (see Table 3).

Table 3: PTs conceptions of number before and after the course

Conception	# of PTs before	# of PTs after
Reference units	2	27
Groups of ones	7	4
Concatenated-digits plus	18	4
Concatenated-digits only	9	1

In the Tally activity PTs naturally grouped the tallies by 2s, 5s, 10s, or 20s. If they started with 2 or 5 they often grouped those groups again into larger groups. This activity is a nice motivation activity for grouping and explicates what grouping is used for. It also most often leads to groupings found in historical number systems.

In the Egyptian Activity PTs learned that the place of the symbols doesn't matter, however, for ease of reading and writing numbers they (just like the Egyptians) would order the symbols from largest to smallest. This can then lead to a discussion of how our base ten system has the same underlying grouping structure (ones, tens, hundreds, etc.). Attempting to perform operations such as multiplication on larger Egyptian numbers also highlights the need for the development of a system designed for calculations.

In the Mayan Activity PTs struggled identifying the value of a one with two zeros and a one with six zeros (see Thanheiser 2014 for a more detailed description of those struggles). The most common misconceptions were a one with two zeroes interpreted as 200 (since a one with one zero represented 20 and a zero was appended to that 20) and a one with six zeroes as 20,000,000 (same line of reasoning). These arguments utilized the underlying idea that appending a zero is equivalent to multiplication by 10. While this notion is correct within each base system the PTs mixed systems by interpreting the multiplication by 10 as 10 in base ten (thus making the 20 into 200) rather than interpreting the multiplication by 10 as 10 in Mayan (which would make the 20 into a 400). The power of this task derives from the fact that conceptions, which would not be easily observable in base 10, become visible and can be examined by the PTs (i.e. appending zeros above) when comparing across number systems. This can then prompt a discussion why procedures such as appending zeros work in base 10. Along the same lines regrouping needs to be examined when working on adding and subtracting numbers, and the fact that we ungroup a group of larger size into the next smaller groups is explicated (as it is not hidden behind a procedure). PTs will also often quite naturally invent expanded addition and subtraction algorithms in the context of the Mayan numbers and thus develop algorithms that make sense.

Comparing and contrasting the different systems allows PTs again to compare similarities and differences and thus to build a better understanding of what base 10 is.

References

Ball, D. (1988). *The subject-matter preparation of prospective mathematics teachers: Challenging the myths*. Lansing: Michigan State University.

Fasteen, J., Meluish, K., & Thanheiser, E. (2015). Multiplication by 10five: Making sense of place value structure through an alternate base. *Mathematics Teacher Educator*.

Lo, M. L. (2012). *Variation theory and the improvement of teaching and learning*: University in Gothenburg, Sweden.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in china and the united states*. Mahwah, NJ: Erlbaum.

Marton, F. (2009). Beyond learning as changing participation. *Scandinavian Journal of Educational Research, 53*(2), 211-215.

McClain, K. (2003). Supporting preservice teachers' understanding of place value and multidigit arithmetic. *Mathematical Thinking and Learning, 5*(4), 281-306.

Newman, F., King, M., & Carmichael, D. (2007). *Authentic instruction and assessment: Common standards for rigor and relevance in teaching academic subjects*. Des Moines, IA: Department of Education.

Pesek, D. D., & Kirshner, D. (2000). Interference of instrumental instruction in subsequent relational learning. *Journal for Research in Mathematics Education, 31*, 524-540.

Ross, S. H. (2001). Pre-service elementary teachers and place value: Written assessment using a digit-correspondence task. In R. Speiser, C. A. Maher & C. N. Walter (Eds.), *Proceedings of the twenty-third annual meeting of the north american chapter of the international group for the psychology of mathematics education* (Vol. 2, pp. 897-906). Snowbird, UT: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Thanheiser, E. (2009). Preservice elementary school teachers' conceptions of multidigit whole numbers. *Journal for Research in Mathematics Education, 40*(3), 251 - 281.

Thanheiser, E. (2010). Investigating further preservice teachers' conceptions of multidigit whole numbers: Refining a framework. *Educational Studies in Mathematics, 75*(3), 241-251.

Thanheiser, E. (2014). Developing preservice elementary teachers conceptions with well-designed tasks: Explaining successes and analyzing conceptual difficulties. *Journal for Mathematics Teacher Education*.

Thanheiser, E., Philipp, R., Fasteen, J., Strand, K., & Mills, B. (2013). Preservice-teacher interviews: A tool for motivating mathematics learning. *Mathematics Teacher Educator, 1*(2), 137 - 147.

Thanheiser, E., & Rhoads, K. (2009). Exploring preservice teachers' concpetiosn of numbers via the mayan number system. Brief report. In S. Swars, D. Stinson & S. Lemons-Smith (Eds.), *Proceedings of the thirty-first annual meeting of the north american chapter of the international group for the psychology of mathematics education* (Vol. 5, pp. 1220-1227). Atlanta, GA: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Verschaffel, L., Greer, B., & De Corte, E. (2007). Whole number concepts and operations. In F. K. Lester & M. National Council of Teachers of (Eds.), *Second handbook of research on mathematics teaching and learning : A project of the national council of teachers of mathematics* (pp. 577 - 628). Charlotte, NC: Information Age Pub.

Yackel, E., Underwood, D., & Elias, N. (2007). Mathematical tasks designed to foster a reconceptualized view of early arithmetic. *Journal of Mathematics Teacher Education, 10*(4-6), 351-367.

Zazkis, R., & Khoury, H. A. (1993). Place value and rational number representations: Problem solving in the unfamiliar domain of non-decimals. *Focus on Learning Problems in Mathematics, 15*(1), 38-51.

Psychometric analysis of the Calculus Concept Inventory

Matt Thomas
Ithaca College

Jim Gleason
University of Alabama

Spencer Bagley
University of Northern Colorado

Lisa Rice
Arkansas State University

Nathan Clements
University of Wyoming

Diana White
University of Colorado, Denver

Introduction and motivation:

Concept inventories have become an increasingly popular way to measure conceptual understanding in STEM disciplines. The Force Concept Inventory (FCI; Hestenes, Wells and Swackhamer, 1992) was the first concept inventory to make a significant impact in the undergraduate education community, becoming widely used and significantly affecting the teaching of introductory hi physics. The FCI paved the way for the broad application of analyzing student conceptual understanding of the basic ideas in a STEM subject area (Hake, 1998a, 2007; Hestenes & Wells, 1992; Hestenes et al., 1992); concept inventories have been written for biology, chemistry, and astronomy.

The Calculus Concept Inventory (CCI), developed more recently (Epstein, 2007, 2013) is seeing increasing use. However, the descriptions of the validation and analysis have been less clear than in other concept inventories, and there is a lack of peer-reviewed literature on its development and psychometric analysis. One study showed that the current CCI did not measure a difference between student conceptual knowledge between students in a conceptually-focused class with frequent student group work and those in a traditional lecture-based class, though other measures indicated that a difference existed (Bagley, 2014).

Methods and Results:

Data of over 1500 students at four institutions from the Calculus Concept Inventory was collected using combinations of both pretest and posttest data to avoid bias from floor or ceiling effects. Using an exploratory factor analysis and a scree plot (see Figure 1), we found that the CCI instrument is unidimensional, as opposed to the proposed three dimensions in the development of the instrument (Epstein, 2007), but three of its 22 items had significantly lower loadings than the remaining 19 items (see Table 1). We confirmed this by IRT analysis with these three items not modeled well by either two or three parameter models. The remaining 19 items had acceptable levels of internal reliability to use for group comparisons. However, the three problematic items are items that measure conceptual understanding without computational skills, specialized calculus vocabulary, or specialized calculus notation. This raises questions about the instrument's ability to measure conceptual, rather than simple procedural, knowledge based on increased notational and vocabulary background.

Conclusions:

There is currently no valid and reliable instrument to measure this conceptual understanding of differential calculus, yet such an instrument is essential in the work of determining evidence-based approaches for the teaching and learning of differential calculus. We conclude there is a need to create and validate a criterion-referenced concept inventory on differential calculus. A concept inventory would significantly impact teaching and learning during the first two years of undergraduate STEM students by providing a resource to measure students' conceptual understanding of differential calculus. This can be used for formative and summative assessment during calculus courses. Instructors can make instructional decisions based on the feedback to improve student learning. The resource can also be used by researchers and evaluators to measure growth of student conceptual understanding during a first semester calculus course to compare gains of students in classrooms with differing instructional techniques. Our initial step to developing the concept inventory is create a taxonomy of calculus concepts. This is foundational to constructing items and interview protocols and to start the cyclic process of administering and revising the inventory to establish its validity and reliability.

References

Bagley, S. F. (2014). Improving Student Success in Calculus : A Comparison of Four College Calculus Classes. University of California, San Diego.

Epstein, J. (2007). Development and validation of the Calculus Concept Inventory. In Proceedings of the Ninth International Conference on Mathematics Education in a Global Community (pp. 165–170).

Epstein, J. (2013). The Calculus Concept Inventory - Measurement of the effect of teaching methodology in mathematics. Notices of the AMS, 60(8), 1018–1026.

Hake, R. R. (1998). Interactive-engagement versus traditional methods: A six-thousand-student survey of mechanics test data for introductory physics courses. American Journal of Physics, 66(1), 64–74. doi:10.1119/1.18809

Hake, R. R. (2007). Six lessons from the physics education reform effort. Latin American Journal of Physics Education, 1(1), 24–31.

Hestenes, D., & Wells, M. (1992). A mechanics baseline test. The Physics Teacher, 30, 159–166. doi:10.1119/1.2343498

Hestenes, D., Wells, M., & Swackhamer, G. (1992). Force concept inventory. The Physics Teacher, 30(3), 141–158. doi:10.1119/1.2343497

Figures

Figure 1: Scree Plot for the CCI

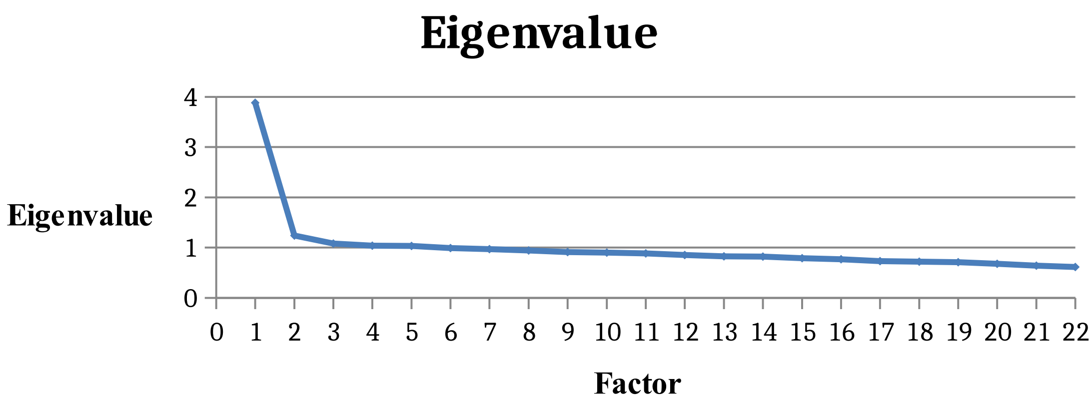


Table 1: Factor Matrix for the CCI for 3-factor and 1-factor models

	Three Factor Model			One Factor Model
	Factor 1	Factor 2	Factor 3	Factor 1
Question 1	.116	.113		.126
Question 2	.525	.197	-.105	.538
Question 3	.485	.174		.500
Question 4	.526		.249	.510
Question 5	.421		-.130	.415
Question 6	.328		.287	.329
Question 7	.362	-.133	.118	.344
Question 8	.465	.129		.475
Question 9	.433			.442
Question 10	.409	.209	-.112	.424
Question 11				
Question 12	.225			.217
Question 13	.380			.386
Question 14	.324			.329
Question 15	.352			.347
Question 16	.290			.299
Question 17	.510			.507
Question 18	.182			.186
Question 19	.314			.322
Question 20	.310	.172		.326
Question 21	.312	.145		.323
Question 22	.456	-.581		.355
Goodness of Fit	Chi-Square: 197.4; df 168; sig. .060			Chi Square: 402.718; df: 209; sig: .000

Extraction Method: Maximum Likelihood

Students' visual attention while answering graphically-based fundamental theorem of calculus questions

Rabindra R. Bajracharya
Oregon State University

John R. Thompson
University of Maine

As part of work on student understanding of the Fundamental Theorem of Calculus (FTC) and definite integrals, we incorporated a technique known as eye tracking to investigate how students attribute their visual attention while answering graphically-based questions. The direction and duration of eye gaze of 17 students was recorded in real time. We analyzed the total proportion of time spent on various question domains (lexicons, equations & symbols, graphs, and question options) as well as on various relevant and irrelevant features of the graphs. We found that the students who responded correctly spent more time on relevant graphical features, whereas those responded incorrectly spent more time on irrelevant graphical features. We also found that student visual attribution depends on types of representations and notations provided in the questions. Most of the eye-tracking results corroborate previously reported written and interview results on student application of the FTC across the mathematics-physics interface.

Key words: Fundamental Theorem of Calculus, Eye tracking, Graphical representations
Physics

Introduction

Studies in undergraduate mathematics education have identified student difficulties with the Fundamental Theorem of Calculus (FTC) and related concepts [1,2,3,9,11,12]. Previous work has shown that students have common types of difficulties with the FTC across mathematics and physics and these difficulties affect students' strategies to solve graphically based FTC problems with physics contexts. Studies in other mathematical and/or physical contexts have shown that representational aspects of a problem play a key role in student problem-solving [7,13,14]. Prior work in cognitive science using physics contexts suggests that some student difficulties stem from student focus on irrelevant question features [5,8]. However, there have been no explicit studies on the impacts of representations, in particular graphical and symbolic, on student strategies for solving these kinds of problems.

We found that students' written and interview responses were cued by various question features. Our initial results led us to explore visual cueing mechanisms and visual attention during problem solving with graphical representations. The main research questions here are:

- To what extent are these difficulties due to difficulties with the graphical representation of the FTC?
- How do students interact visually with the graphical representations?
- Do correct and incorrect responders process graphs differently?

Previous studies have shown that eye tracking is extremely useful in investigating the influence of visual cues on individual's cognitive processing, as an individual's visual attention reflects his/her cognitive processes [4,6,8,10]. We designed and conducted eye-tracking experiments to study individual student's visual behaviors while answering graphically based FTC questions (Fig. 1).

Method

In this study, we investigated the visual attention of 17 individual college students who had at least one semester of calculus. These students were shown a series of graphically based FTC questions, constructed by modifying and expanding previous written and interview questions, on a computer monitor (see Fig. 2) and asked to answer those questions. Total proportions of time spent in different areas of interest (AOI) were computed to design a number of repeated-measure mixed factorial ANOVAs. The single dependent variable was mean percentage of time spent (fixation) on different areas of interest (AOIs), whereas the independent variables were areas of interest, i.e., graph, equations, words, etc. (within-group) and correctness, i.e., correct vs. incorrect (between-groups).

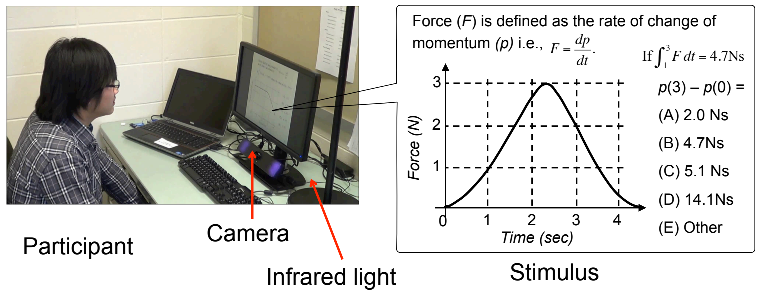


Figure 1. An eye-tracking experiment set-up and a zoomed in FTC question.

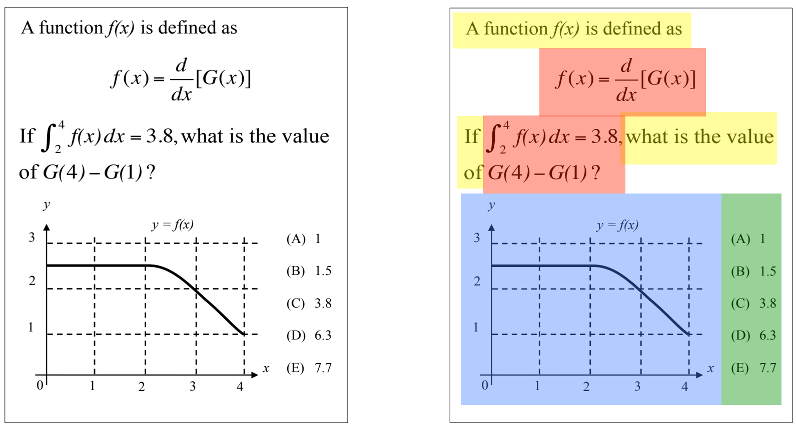


Figure 2. (a) A graphically-based FTC II question without areas of interest shown to students. (b) Same question with areas of interest (AOI) for analysis purpose.

Results

We present following two important findings. First, correct responders spent more time looking at relevant graphical features, whereas incorrect responders spent more time looking at irrelevant graphical features ($p < 0.01$). Second, for particular questions (with rate and integral equations) correct responders spent relatively more time looking at equations than graphs, whereas incorrect responders spent relatively more time looking at graphs than equations ($p = 0.02$). We analyze our findings and interpret them primarily using a model for cognition known as top-down and bottom-up processing theory, which connects cognition to attention [5]. Within this analysis, we identify specific features of the FTC-based questions that affect students' responses. Most of the eye-tracking findings align well with our previously reported written survey and the interview findings and previous research in this area. Eye tracking has promise as a method to investigate issues in RUME that are otherwise difficult to access, such as an individual's cognitive involvement during problem solving.

References

1. Bajracharya, R. R., Wemyss, T. M., & Thompson, J. R. (2012). Student interpretation of the signs of definite integrals using graphical representations. *In 2011 Physics Education Research Conference* (Vol. 1413, pp. 111–114).
2. Beichner, R. J. (1994). Testing student interpretation of kinematics graphs. *American Journal of Physics*, **62**(8), 750–762.

3. Grundmeier, T. A., Hansen, J., & Sousa, E. (2006). An exploration of definition and procedural fluency in integral calculus. *Problems, Resources, and Issues in Mathematics Undergraduate Studies*, **16**(2), 178–191.
4. Lai, M. L., Tsai, M. J., Yang, F. Y., Hsu, C. Y., Liu, T. C., Lee, S. W. Y., . . . Tsai, C. C. (2013). A review of using eye-tracking technology in exploring learning from 2000 to 2012. *Educational Research Review*, **10**, 90–115.
5. Heckler, A. F. (2011). The ubiquitous patterns of incorrect answers to science questions: The role of automatic, bottom-up processes. *Psychology of Learning and Motivation-Advances in Research and Theory*, **55**, 227.
6. Just, M. A., & Carpenter, P. A. (1976). Eye fixations and cognitive processes. *Cognitive Psychology*, **8**(4), 441–480.
7. Kohl, P. B., & Finkelstein, N. D. (2006). Effects of representation on students solving physics problems: A fine-grained characterization. Physical Review Special Topics-Physics Education Research, **2**(1), 010106.
8. Madsen, A., Larson, A. M., Loschky, L. C., & Rebello, N. S. (2012). Differences in visual attention between those who correctly and incorrectly answer physics problems. *Physical Review Special Topics-Physics Education Research*, **8**(1), 010122.
9. McDermott, L. C., Rosenquist, M. L., & Van Zee, E. H. (1987). Student difficulties inconnecting graphs and physics: Examples from kinematics. *American Journal of Physics*, **55**(6), 503–513.
10. Rayner, K. (2009). Eye movements and attention in reading, scene perception, and visual search. The quarterly journal of experimental psychology, **62**(8), 1457–1506.
11. Thompson, P. W. (1994a). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics*, **26**(2-3), 229–274.
12. Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in calculus. Making the connection: Research and teaching in undergraduate mathematics, **73**, 43–52.
13. Vinner, S. (1989). The avoidance of visual considerations in calculus students. *Focus on learning problems in mathematics*, **11**(1), 149–56.
14. Wagner, J. F., Manogue, C. A., & Thompson, J. R. (2012). Representation issues: Using mathematics in upper-division physics. *In 2011 Physics Education Research Conference* (pp. 89–92).

Creating online videos to help students to overcome exam anxiety in statistics class

Anna Titova
Becker College

In this poster I would like to share my ideas on how to help students overcome math anxiety, exam anxiety in particular. Math anxiety is reported to be a major learning obstacle for many students at various levels of learning mathematics; students worry about outcomes of in-class assessments, especially exams. Instructors typically place a lot of weight on exams, so students fear that if they do poorly their grade will go down. This is known as exam anxiety; not necessarily math test anxiety, but putting the two together would certainly multiply students' stress. In this poster I would like to illustrate how videos can be utilized to ease math test anxiety and help improve students' overall performance in math classrooms.

Key words: math anxiety, exam anxiety, videos, statistics, technology.

Many agree that a large number of students reveal a high level of stress when it comes to taking a math course whether or not their major is math-related. However, various majors require students to take an introductory mathematical statistics course. As a result, statistics classrooms are often made up of students with very little motivation. Consequently, this can elevate their anxiety level; students become apprehensive that they will not do well and that it will affect their overall GPA. Mathematics anxiety is a phenomenon that many instructors and researchers have observed. Betz (1978) quotes Richardson & Suinn's definition of math anxiety as "feelings of tension and anxiety that interfere with the manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary life and academic situations."

In statistics courses, students often experience anxiety about understanding the problems because they are word problems. Even those who feel confident in terms of procedural knowledge still have trouble interpreting word problems and understanding the variables they describe. This is especially evident during exams. Mixed with a math anxiety, the exam anxiety – "worrying about the outcome of the test and experiencing negative emotions during the test" (Gharib, Phillips, Mathew, 2012) – contributes to a student's stress level and causes poor performance on exams.

Researchers and instructors have been applying various techniques to help students overcome their anxiety. For instance, Gharib, Phillips, Mathew (2012) are exploring the effect of open-book or cheat-sheet exams on students' performance. Balkam, Nellessen, and Ronney (2013) implemented collaborations during exam preparation in class. Batton (2010) looked at helping students overcome math anxiety using group work.

Rapid evolution of technology in education practices gave me the idea to explore the possibility of using video applications to help students grow more confident in their test preparation. I believe that if a student feels a greater measure of self-assurance, their level of anxiety will be much lower and will result in better grades overall. I came up with the following methodology: every student was given a "mock exam" a few days prior to the in-class exam. Students were told that the number and the types of problems match the actual exam. Students were instructed to take this exam outside of class, and the next day I emailed them a video with the solutions. They were asked to grade their own responses, take notes, and ask questions. This way they had a chance to experience "test-like" conditions and reflect on their own knowledge and responses.

One of my introductory statistics classes was selected for this application and they were given exams on a regular basis. Prior to each exam students received a review sheet with a set of review problems and concepts. The students were expected to study for the exam by working on the problems and discussing any questions with me. This procedure was changed once and students were asked to work on the "mock" exam instead. The average grade for the exam that followed the "mock" was higher than usual. I had a chance to do informal interviews with several students afterward. They were chosen from three groups: those with "weak", "average", and "good" grades (based on previous exam performance). Each student reported that the video increased their confidence, which lowered their stress level when it came time to take the exam, which, in turn, enabled them to get a higher grade.

In future semesters I plan to elaborate more on this method. I am currently working on a questionnaire to measure students' level of both math and test anxiety at the beginning of the course and again after the method described above is implemented. Formal, open-ended interviews will be conducted with students, and samples will be selected from clusters based on students' anxiety level as well as on their performance in class. A more thorough literature review may suggest additional variables or evidence interpretation. Current evidence (average class grade) supports the assumption that this type of exam preparation helps students stress less and perform better on exams, however more data, both qualitative and quantitative, is needed to provide further evidence to fully support this hypothesis.

References:

Betz, N. E. (1977). *Math anxiety: What is it?* Retrieved from http://search.proquest.com/docview/63829975?accountid=35619

Gharib, A., Phillips, W., & Mathew, N. (2012). Cheat sheet or open-book? A comparison of the effects of exam types on performance, retention, and anxiety. *Psychology Research, 2*(8), 469-478. Retrieved from http://search.proquest.com/docview/1312420307?accountid=35619

Balkam, B. E., Nellessen, J. A., & Ronney, H. M. Using collaborative testing to reduce test anxiety in elementary and middle school students. , 164. Retrieved from http://search.proquest.com/docview/1347459740?accountid=35619

Mathematicians' ideas when proving

Melissa Troudt
University of Nortnern Colorado

Gulden Karakok
University of Northern Colorado

Mickael Oehrtman
Oklahoma State University

This study sought to describe the ideas professional mathematicians' find useful in moving their arguments forward while constructing mathematical proof and the context surrounding the development of these ideas. Three research mathematicians completed real analysis tasks while thinking aloud in interview and independent settings recorded through Livescribe technology. Follow-up interviews were also conducted. Data were analyzed for perceived useful ideas and coded based on Dewey's inquiry framework and Toulmin's argumentation model. Toulmin argumentation diagrams were implemented to describe the evolution of the arguments, whereas Dewey's inquiry framework helped to describe the context surrounding the development of the ideas. Preliminary findings show an active inquiry into forming a geometric understanding to develop warrants based on intuition and examples and then working to find backing that can be rendered into a formal argument.

Key words: proof, professional mathematicians, Toulmin argumentation model, inquiry

Mathematicians and graduate students in mathematics have been shown to construct proof using both purely formal reasoning and also constructions that are accompanied by informal reasoning via the mathematicians' instantiations of concepts (Weber & Alcock, 2004; Raman, 2003; Alcock & Inglis, 2008). Inglis, Mejia-Ramos, & Simpson (2007) found mathematics graduate students used warrants based on inductive reasoning (*inductive warrants)*, intuitive observations about or experiments with some kind of mental structure (*structural-intuitive warrants)*, and formal mathematical justifications (*deductive warrants*) when evaluating conjectures about a novel number theory topic. Tall, Yevdokimov, Koichu, Whitely, Kondratieva, and Cheng (2012) described what a proof is for professional mathematicians as "involv[ing] thinking about new situations, focusing on significant aspects, using previous knowledge to put new ideas together in new ways, consider relationships, make conjectures, formulate definitions as necessary and to build a valid argument" (p. 15). Little research has been performed that would describe the context around the formulation of the ideas that the prover finds useful and how these ideas influence the development of the mathematical argument. Looking at the moments where these ideas develop through the perspective of Dewey's theory of inquiry (1938) may grant important information about the context of the generation of these ideas and the purposes that they serve as the argument evolves.

Theoretical Perspective

Since argument has been described as encompassing both informal and formal arguments and also arguments to convince oneself or another, I wish to describe the proving process as an evolving personal argument. The *personal argument* encompasses all thoughts that the individual deems relevant to making progress in proving the statement. It is a subset of the entire concept image of a proof situation. The focus of this study is to describe how mathematicians' personal arguments evolve in that we are looking to see how they incorporate and use new ideas that they view as better enabling their arguments forward.

Toulmin (1958; 2003) developed an approach to analyzing arguments that focuses on the semantic content and structure. Toulmin's scheme classifies statements of an argument into six different categories. The claim (C) is the statement or conclusion to be asserted. The grounds (G) are the foundations on which the argument is based. The warrant (W) is the justification of the link between the grounds and the claim. Backing (B) presents further evidence that the warrant appropriately justifies that the data supports the claim. The modal qualifiers (Q) are statements that indicate the degree of certainty that the arguer believes that the warrant justifies the claims. The rebuttals (R) are statements that present the circumstances under which the claim would not hold. Toulmin's framework for arguments provides a means of describing, structurally, the evolution of the personal argument.

In addition to describing how the personal arguments evolve, this research seeks to describe the context surrounding the development and incorporation of new ideas. Dewey's (1938) theory of inquiry gives us a means of understanding how knowledge is created and how it is perceived useful in problem-solving situations. In periods of inquiry, one is actively engaged in the cyclical process of reflecting on problem situations, applying tools to these situations, and evaluating the effectiveness of the tools (Hickman, 1990). Using the framework I describe the actions performed by the participants noting if the prover perceives a problem and what they perceive the problem to be. If the prover perceives a problem, I attempt to describe process of selecting a tool to apply to the problem, the individual's expected outcome of using the tool, and the individual's perspective of how the action affected the situation. These factors together provide an organization for the context of the situation from the participant's point of view.

Research Questions

Part of a larger project with a greater research agenda, this report focuses on preliminary findings for the research question: What ideas move the argument forward as a professional mathematician's personal argument evolves? Specifically, I address what problematic situation the prover is currently entered into solving when s/he articulates and attains an idea that moves the personal argument forward.

Methods

The participants for this research were three professional mathematicians with faculty appointments at 4-year universities who either specialized in researching or in teaching courses in real analysis. The data collection phase for each participant proceeded as follows: the participant worked on a task or tasks in a task-based interview, continued to work on the task or other tasks on their own, turned in their at home work captured via Livescribe technology (Savic, 2012), participated in a follow-up interview of their work, and repeated this process with new tasks in the next interview (see Table 1). Participants identified potentially challenging tasks from the field of real analysis for themselves and their peers; I provided the "researcher task" if the participants found the peer tasks to be familiar or unproblematic. Note in Table 1 that three of the six total tasks were completed by two participants; namely peer task B, peer task C, and the researcher's task.

Table 1. Sequence of interviews and tasks.

Participant	Interview 1	Interview 2	Interview 3
Participant A	Choose personal task A and peer task A. Work on personal task A.	Stimulated recall of personal task A. Work on peer tasks B and C.	Stimulated recall of peer tasks B and C.
Participant B	Choose personal task B and peer task B. Work on personal task B.	Stimulated recall of personal task B. Work on peer tasks A and C and independent work on Researcher task	Stimulated recall of peer task A and C and researcher task.
Participant C	Choose personal task C	Stimulated recall of	Stimulated recall of

	and peer task C. Work on personal task C and peer task B.	personal task C and peer task B. Work on researcher task.	researcher task.

I analyzed the work that each participant completed during the first interview and between the first two interviews to formulate hypotheses that could be tested in the second interview by asking questions and having participants describe their thinking of the completed tasks. I noted moments where the participant appeared to generate a new idea, to identify a certain tool as useful, or to gain some insight into the problem; and moments where it was unclear what motivated a certain action. The moments where the new ideas occurred acted as markers of transitions in the timeline of the evolution of the argument. I performed initial Toulmin (2003) analyses on the argument between these markers. Reviewing the context surrounding the emergence of the idea, I hypothesized what the participant perceived as problematic.

Primary analysis, informed by the follow-up interviews is still in process. It includes the writing descriptions of each idea that appeared to move the argument forward and the context surrounding the generation of that idea, describing the argument's evolving structure via Toulmin diagrams formulated in the preliminary analysis and informed by the follow-up interviews. Finally, pattern analysis will be conducted across all the ideas of each participant as well as across participants along the common tasks.

I report some findings from Dr. B's work on the researcher's task: *Let f be a continuous function defined on* $I = [a, b]$*. f maps I onto I, f is one-to-one, and f is its own inverse. Show that except for one possibility, f must be monotonically decreasing on I.*

Preliminary Results

It appeared that Dr. B's work involved addressing problems of getting a geometric interpretation of the situation, determining a warrant for his claims, and finding backing for those warrants that could be translated into a deductive argument. He began by, in his words, "chipping away at the geometric restrictions" developing a set of ideas leading to a geometric mental picture and his first physical picture, which he found useful in convincing himself of the claim underlined in the quote below and broken down as Argument 1 in Table 2. The backing of the warrant appeared to be based on the picture and intuition based on structural understandings.

> It would reflect back and forth and in order to be its own inverse and increasing. Its reflection would be the same thing that you started off with. [runs pen over line $y = x$] And you'd have to be right on that line because there's no other way of doing it. So, $f(x) = x$ suffices. That should be our only increasing function. $f(x) = x$ <u>intuitively seems like the only increasing option they're talking about</u>.

He shifted his inquiry to answering the following questions of "*What have I convinced myself geometrically? And how do I prove those things? Why is f of x equals x the only increasing function?"* I interpreted this as his tackling the problem of identifying a warrant backed by mathematical justifications. He identified a warrant based on his intuition. *"So my intuition there, if it wasn't, then if you reflected itself, there would have to be double values of this thing from its reflection."* The idea of potential double values (the potential for f(x) to not be able to pass the vertical line test) posed itself as a possible contradiction to be used in a proof. To find backing for this warrant, he drew a picture and used it as a means of searching for a backing of the idea that an increasing, non-identity function with the given constraints would necessarily violate the vertical line test if reflected across the line y=x. During this inquiry, he was interrupted by a colleague and returned to the problem articulating a new idea that manifested while previously working with the picture. *"So what's going on here? I was* *just about to get on this. I have this nice picture. And on my picture, I can see that if I reflect this type of function, it's not going to be one-to-one."* He then searched for a backing for the one-to-one warrant by returning to and modifying his third picture, but he found this picture did not capture the characteristic of f being its own inverse. He chose to abandon the third picture and restart. When asked about this decision in the follow-up interview, he said, *"I'm trying to think what am I trying to contradict here. I sort of lost track of what I was doing because of when I got interrupted too. I think I'm just realizing I know what I'm doing. Now I just have to start over. I finally realized that's the other fact I need. Because it's its own inverse."* When he restarted, he drew and talked through a fourth picture developing Argument 4 which he was able to translate into algebraic symbols developing Argument 5. Finally, he deemed himself ready to write his arguments in a formal proof.

Table 2. Progression of arguments.

	Data	Claim	Warrant	Backing
Argument 1	f maps I to I is continuous, 1-1, onto, and its own inverse	If f is increasing, then it must be f(x)=x	"there's no other way of doing it";	"its reflection would be the same thing that you started off with"; "you'd have to be right on that line" (first picture)
Argument 2	If f is increasing and has the given restraints	Then it must be f(x)=x	otherwise there would be double values from reflection	Geometric intuition and first picture
Argument 3	If f is increasing and has the given restraints	Then it must be f(x)=x	Otherwise it wouldn't be one-to-one	"on my picture, I can see" (third picture)
Argument 4	If f is increasing and has the given restraints	Then it must be f(x)=x	Otherwise it wouldn't be one-to-one	"it would result in double values if it went through here" (third picture)
Argument 5	f increasing, one-to-one, onto, and f inverse equals f	then f(x)=x	otherwise one would get 2 different values for x=f(f^-1(x))	Fourth picture
Argument 6	f increasing, one-to-one, onto, and f inverse equals f	then f(x)=x	otherwise one would get 2 different values for x=f(f^-1(x))	Algebraic translation and generalization of fourth picture

Discussion

Dr. B appeared to tackle the problems of understanding the situation geometrically and then searching for a deductive warrant. Dr. B's developed geometric understanding contributed to his proposal of possible warrants based on examples and his own intuition based on structural understandings (Inglis, et al., 2007). He attempted to find a logical backing for those warrants by unpacking and testing the ideas without success. His final picture, potentially informed by his earlier work, captured an argument that could be translated into a logical proof.

This initial report simply focuses on the problems addressed by the participant and how they related to the evolution of warrants. It does not yet describe the actions contributing to the generation of new ideas or how the previous arguments may have informed his construction and analysis of the fourth picture. Further investigation will address these and other questions.

Questions for Discussion

Aside from inductive, structural-intuitive, and deductive, what other warrant-types have been discussed in the literature? This research uses two frameworks to understand the proof construction process. Do they appear to complement each other?

References

Alcock, L. & Inglis, M. (2008). Doctoral students' use of examples in evaluating and proving conjectures. *Educational Studies in Mathematics, 69*, 111-129

Crotty, M. (1998). *The foundations of social research*. Thousand Oaks, CA: Sage.

Dewey, J. (1938). *Logic: The Theory of Inquiry*. New York: Henry Holt & Co.

Hickman, L.A. (1990). Knowing as a Technological Artifact. In *John Dewey's Pragmatic Technology*, Bloomington: Indiana University Press, 17-59.

Inglis, M., Mejia-Ramos, J. P., & Simpson, A. (2007). Modeling mathematical argumentation: the importance of qualification. *Educational Studies in Mathematics, 66(1),* 3-21.

Raman, M. (2003). Key ideas: What are they and how can they help us understand people's views of proof? *Educational Studies in Mathematics, 52*(3), 319-325.

Savic, M. (2012). What do mathematicians do when they reach a proving impasse? In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman, *Proceedings of the 15th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 531-535). Portland, OR: Online at http://sigmaa.maa.org/rume/crume2012/RUME_Home/RUME_Conference_Papers_files/RUME_XV_Conference_Papers.pdf.

Tall, D., Yevdokimov, O., Koichu, B., Whiteley, W., Kondratieva, M., & Cheng, Y.-H. (2012). Cognitive Development of Proof. *Proof and Proving in Mathematics Education*, 13–49.Thurston, W.P. (1994). On proof and progress in mathematics. *Bulletin of the American Mathematical Society, 30(2),* 161-177.

Toulmin, S.E., (1958, 2003). *The uses of argument*. Cambridge, UK: University Press.

Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics, 56*, 209-234.

The efficacy of projects and discussions in increasing quantitative literacy outcomes in an online college algebra course

Luke Tunstall
Appalachian State University

This research stems from efforts to infuse quantitative literacy (QL) in an online version of college algebra. College algebra fulfills Appalachian's QL requirement, and is a terminal course for most who take it. In light of the course's traditional content and teaching methods, students often leave with little gained in QL. An online platform provides a unique means of engaging students in quantitative discussions and research, yet little research exists on online courses in the context of QL. The researcher's course includes weekly news discussions as well as "messy" projects requiring data analysis. Students in online and face-to-face sections of the course took the QLRA (developed by the NNN) during the first and final weeks of the fall 2014 semester. There were significant statistical gains in the online students' QLRA performance and mathematical affect but none for the face-to-face students. Implications of this include that project-based learning in an online environment is a promising strategy for fostering QL in terminal math courses.

Key words: Quantitative Literacy, College Algebra, Online Courses, Assessment, General Education

Under leadership from the MAA and National Numeracy Network (among others), quantitative literacy (QL) has garnered significant attention in the twenty-first century (Madison & Steen, 2008). While a number of acceptable frameworks for QL exist, the researcher will use that from the charter of the SIGMAA on QL (2004):

> Quantitative literacy (QL) can be described as the ability to adequately use elementary mathematical tools to interpret and manipulate quantitative data and ideas that arise in an individual's private, civic, and work life. Like reading and writing literacy, quantitative literacy is a habit of mind that is best formed by exposure in many contexts.

Deborah Hallett (2003) suggests that though the foundations of QL are laid in middle-school, it is the *responsibility* of high-school and college faculty to cultivate this knowledge – they are to ensure students receive the "exposure" the above definition calls for. In part, a reason for calling attention to this responsibility is that college algebra – a terminal math course for many – has a reputation for its complicity in failing to foster QL (Steen, 2006). Seeking improvement, Small (2006) believes the course should have little lecture and instead a considerable number of small-group activities. It should focus on real-world, ill-defined modeling rather than traditional word problems, emphasizing communication over traditional assessment. Data from a 2010 AMS survey suggests his vision has yet to manifest. Of the undergraduate programs surveyed, only 16% of college algebra sections required writing assignments, and 65% used a "traditional" approach, meaning the course content and delivery methods were essentially the same as those in 1990 (Blair, Kirkman, & Maxwell, 2010).

A digital learning environment allows students to engage in activity conducive to QL. The purpose of the present study is to examine the efficacy of an online adaptation of Small's vision. In light of the fact that 32% of students in higher-education institutions enrolled in at least one online course in 2011 (up from 18% in 2005), such research is overdue (Allen & Seaman, 2013). The researcher has created an online college algebra course that will be piloted in the fall 2014 semester. The primary assessment in the course is weekly discussion (often news-related) and data-driven projects. The pedagogy underlying the structure is problem-based learning (PBL); research provides support for this method in both face-to-face and online courses (Strobel & Van Barnevald, 2009; Cheaney & Ingebritsen, 2005; Sendag & Odabasi, 2009).

The researcher's hypothesis is that the online course provides a significantly better means of increasing QL outcomes in comparison to traditional teaching of the course. The researcher will use the QLRA, developed by the National Numeracy Network; more than 25 institutions used the assessment in 2013 (Gaze, et. al, 2014). It includes 20 multiple-choice questions and five affective questions. As shown in Figure 1 below, students in both delivery modes will take the QLRA at the beginning and end of the semester. The online students will receive a grade for taking it, while face-to-face students will receive compensation (as the researcher is not teaching those sections). While there are limitations in this approach due to departmental logistics, potential implications include the placement of more QL-designated courses online. This poster will include statistical and qualitative analyses of the results of the study. The researcher looks forward to receiving feedback concerning future directions for research in online courses, QL, and college algebra.

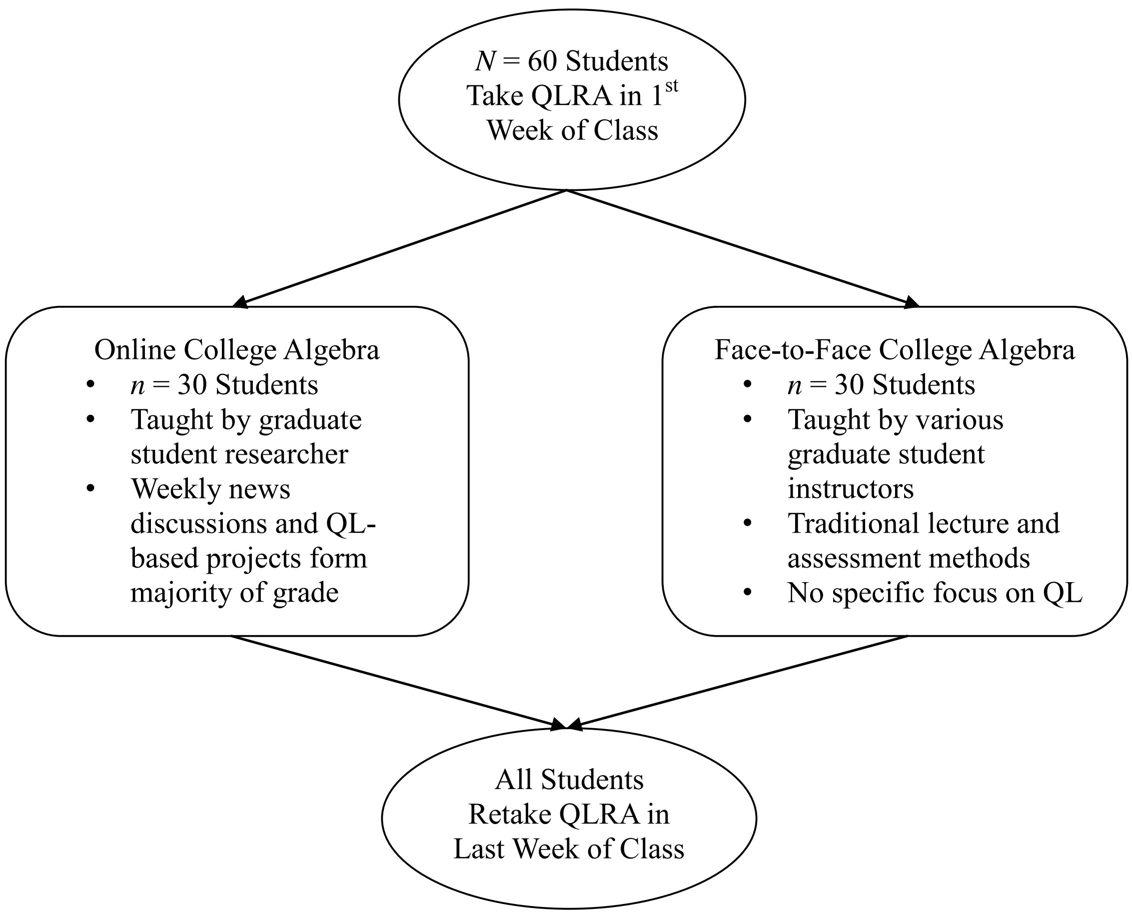


Figure 1 – Setup of Study

References

Allen, I.E., & Seaman, J. (2013). *Changing course: Ten years of tracking online education in the United States*. Babson Research Survey Group & Quahog Research Group, LLC. Retrieved from http://www.onlinelearningsurvey.com/reports/changingcourse.pdf

Blair, R., Kirkman, E., & Maxwell, J. (2010). *Statistical abstract of undergraduate programs in the mathematical sciences in the United States: Fall 2010 CBMS survey*. American Mathematical Society. Retrieved from http://www.ams.org/cbms/cbms2010-Report.pdf

Cheaney, J., & Ingebritsen, T. (2006). Problem-based learning in an online course: A case study. *The International Review of Research in Open and Distance Learning, 6*(3). Retrieved from http://www.irrodl.org/index.php/irrodl/article/view/267/433

Gaze, E.C., Montgomery, A., Kilic-Bahi, S., Leoni, D., Misener, L., & Taylor, C. (2014). Towards developing a quantitative literacy/reasoning assessment instrument. *Numeracy*, *7*(2). doi: 10.5038/1936-4660.7.2.4

Hallett, D.H. (2003). The role of mathematics courses in the development of quantitative literacy. In B. L. Madison (Ed.), *Quantitative literacy: Why numeracy matters for schools and colleges* (pp. 91-8). Princeton, NJ: National Council on Education and the Disciplines. Retrieved from http://www.maa.org/sites/default/files/pdf/QL/WhyNumeracyMatters.pdf

Madison, B. L., & Steen, L.A. (2008). Evolution of *Numeracy* and the National Numeracy Network. *Numeracy, 1*(1). doi: 10.5038/1936-4660.1.1.2

Sendag, S., & Ferhan, O. H. (2009). Effects of an online problem based learning course on content knowledge acquisition and critical thinking skills. *Computers & Education, 53*(1), 132-141. Retrieved from http://0-search.ebscohost.com.wncln.wncln.org/login.aspx?direct=true&db=eric&AN=EJ836739&site=ehost-live

SIGMAA in Quantitative Literacy. (2004). *Charter for SIGMAA on QL*. Retrieved from http://sigmaa.maa.org/ql/_charters/2004.php

Small, D (2006). College algebra: a course in crisis. In N.B. Hastings, F.S. Gordon, S.P. Gordon, J. Narayan (Eds.), *A fresh start for collegiate mathematics: Rethinking the courses below calculus*. Washington, DC: Mathematical Association of America. Retrieved from http://0-search.ebscohost.com.wncln.wncln.org/login.aspx?direct=true&db=e000xna&AN=454236&site=ehost-live

Steen, L. A. (2006). *Supporting assessment in undergraduate mathematics*. Washington, D.C.: Published and distributed by Mathematical Association of America.

Strobel, J., & van Barneveld, A. (2009). When is PBL more effective? A meta-synthesis of meta-analyses comparing PBL to conventional classrooms. *Interdisciplinary Journal of Problem-Based Learning*, *3*(1), 44-58. Retrieved from http://0-search.ebscohost.com.wncln.wncln.org/login.aspx?direct=true&db=eue&AN=508034001&site=ehost-live

Connecting Research on Students' Common Misconceptions about Tangent Lines to Instructors' Choice of Graphical Examples in a First Semester Calculus Course

Brittany Vincent — West Virginia University

Vicki Sealey — West Virginia University

Common misconceptions that students have about tangent lines are well documented in the research literature. This study seeks to understand the efforts that instructors make to address these common misconceptions in their classroom instruction. Specifically, we looked at video data from classroom sessions of five instructors when they covered the graphical representation of derivative. Language and gestures instructors used as well as the graphical examples they provided to the students were analyzed.

Key words: Tangent Lines, Calculus, Instruction, Derivative

Introduction

When constructing tangent lines in first-semester calculus, students may try to apply geometric properties of lines tangent to a circle when attempting to sketch a line tangent to a point on a function. One common misconception is that students believe that a tangent line cannot cross the function at more than one point (Figure 1). While this is true of tangents to circles, it is not true of tangents to all functions. This study examines how undergraduate calculus instructors introduce the notion of tangent line. In particular, what language, gestures, and example spaces do instructors use in the classroom when introducing tangent lines? Furthermore, what efforts are made to address common misconceptions about tangent lines to a function graph?

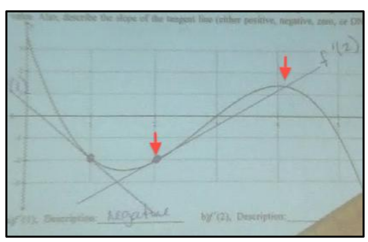


Figure 1.

Literature Review

Students' prior knowledge about a line tangent to a circle influences their understanding of the more general tangent to a curve (Tall, 1987). Early experiences with tangents influence the way students think about tangents in subsequent settings, and consequently, many students have a concept image of tangent involving "circle-like pictures" (Biza, Souyoul, & Zachariades, 2005). Biza et al., (2008) identified three basic perspectives on tangent lines. Students holding *a Geometrical Global* perspective often do not accept tangent lines that coincided with the graph, cross through the graph, or have more than one point in common with the curve. Essentially, their definition of a tangent line is still the one they learned in geometry class, and they just apply those properties globally when constructing tangent lines to a function graph (Biza et al., 2008). Students holding an *Intermediate Local* perspective often believe the tangent line can have only one point in common with the curve *locally* and must stay in the same semi-plane as the curve locally. While this is more accurate, it does not address the case of a tangent line drawn at an inflection point, in which the tangent line cuts “through” the curve. In general, students characterized by an *Analytical Local* perspective hold a more sophisticated concept image of tangent lines. Their concept image of tangent lines involves cases where the tangent line has more than one common point with the curve, intersects the curve at an inflection point, or coincides with the curve (Biza et al., 2008).

Theoretical Perspectives

Ball and Bass (2000) referred to pedagogical content knowledge as a “special form of knowledge,” and pedagogical content knowledge specific to mathematics as mathematical knowledge for teaching. It is a blending of mathematical knowledge with knowledge of learners, learning, and pedagogy (Ball & Bass, 2000). This special form of knowledge enables mathematics educators to anticipate areas where students will have difficulty and prepare alternative explanations and representations of the concept to mediate those difficulties.

Additionally, we draw on the work of example space in our theoretical perspective. Watson and Mason (2005) defined the notion of example space as a “collection of examples that fulfill a specific function” and distinguished between various types of example spaces: *personal example space*- can be triggered by a task, *personal potential example space*- consisting of a person's past experience (though not explicitly remembered), which may be structured in a way that is difficult to access, *conventional example space*- understood by mathematicians and as displayed in textbooks, and *collective example space*- particular to a classroom or group.

Methodology

The participants of this study are five Calculus I instructors from a large research university. Due to the small sample size, limited background information will be provided on the participants, in order for the instructors to remain anonymous. Each section consisted of approximately the same class size of twenty-five to thirty students, and all but one of the five instructors, in addition to meeting in their specific classrooms, also met with their students in the computer lab at least one of the two days of class.

The data analysis is still preliminary, but we have begun by comparing the data to what we know about students' conceptions of tangent lines. In particular, students’ misconceptions that a tangent line cannot have more than one common point with the curve, cut through the curve, or coincide with the curve. The analysis focused on the instructors’ use of language and examples that addressed these issues, either explicitly or implicitly. The data was organized in a chart format to identify key phrases and classify examples. The instructional example spaces discussed in this paper refer specifically to the examples the instructor placed on the board or discussed with the students and does not include examples from in-class worksheets. It is important to make this distinction, although there does not seem to be evidence that the worksheets add any significant variety to the example space of the instructor.

Results

Of the five participants (A, B, C, D, and E), four used phrases that could be interpreted by students as support for a misconception of a tangent line only having one common point with the curve. It is possible that the instructors were intending to emphasize “one-point” in order to distinguish the tangent line from the secant lines used to approximate an instantaneous rate of change. However, such phrases may serve to further establish students’ wrong beliefs (Table 1).

Instructor	Quote
A	Tangent lines only touch the graph at one point.
B	...the tangent line is a line that touches the function at a specific point and has the same

	direction of the function at that point.
C	Tangent lines are at one point and... it’s like two points become one
E	If I have one point, I don’t have a secant line. For a secant line I have two points. If I have one point, what kind of line do I have? [*A student answers, “tangent line.”*]

Table 1.

Similar to instructor E, instructor A also asked the class what kind of line only goes through one point, to which a student replied, “tangent line.” This is interesting because in both cases the question was confidently and quickly answered by a student. This could be interpreted that there is a strong association between the phrases “one-point” and “tangent line.” Two of the five instructors directly addressed the fact that their students may believe a tangent line can only intersect the graph at one point (Table 2).

Instructor	Quote
A	Sometimes a very bad definition of tangent line is 'ah it just crosses the graph once.' That's a very bad definition.
C	Is it ok that it intersects over here [pointing to a second intersection point]? Is that ok or not?... It *is* ok if it intersects some place way far away.

Table 2.

Of the five instructors, only A and C included graphs of tangent lines with more than one common point with the curve in their instructional example space, one and two examples, respectively. The remaining instructors did not include this type of example during the two days of instruction we observed. The phrasing “some place way far away” in C's quote above, may reinforce the wrong belief that a tangent line cannot coincide with the graph near the point of tangency. Two of the five instructors, C and E, included an example of a tangent line that coincided with the graph. For both instructors, the context of the example was sketching $f'(x)$ given the graph of $f(x)$. One instructor used $f(x)=|x|$ at a point other than $x = 0$, and the other used a function that was not defined symbolically but had a portion of the graph that looked like $f(x)= -|x|$. The misconception that a tangent line cannot coincide with the graph was implicitly addressed in both cases through this example. However, neither instructor paid particular attention to the example or demonstrated evidence that this example may be troublesome to the students. In fact, instructor C did not sketch a tangent line. He/she just discussed, in so many words, that when the graph of $f(x)$ is linear, the graph of $f'(x)$ is horizontal.

Three of the five instructors modeled an incorrect tangent line through a point on the curve and discussed why this tangent line was incorrect. Instructor C sketched two tangent lines on the board and asked the class why the first represented a tangent and the second did not (Figure 2). One student responded quickly that the second was not a tangent line because it passed through two points on the curve. The instructor reiterated that it is ok for a tangent line to cross a graph more than once and provided a quick sketch of a tangent line with two common points (one at the point of tangency and one “far away”). Another student responded, “Would it be because that's on the outside of the curve [referring to the tangent line on the left].” To this comment, instructor C agreed and explained that tangent lines “skim” or “float” along the “outside” of the curve. Instructor B used similar language by suggesting that the students think of the tangent line as a “surf board, surfing along the curve.” While these explanations are meaningful and can be helpful for sketching tangent lines, such language may reinforce the misconception that a tangent line can't cut through a curve, such as at an inflection point

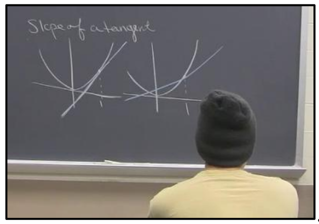


Figure 2.

Concerning this misconception, two of the five instructors, C and E, included two examples of a tangent line that cuts through the curve at the point of tangency. They used the same examples, sketching the tangent line at $x = 0$ for $f(x) = x^3$ and $f(x) = \sqrt[3]{x}$. The data does not show evidence of the five instructors directly addressing that a tangent line can "cut through" the curve during the two days of classroom video that we observed, when the definition of the derivative and the graphical representation of the derivative were being taught.

In our previous work (blinded for review), we found that, despite working with tangent lines in their classes, some students still sketched the graph of $y = \tan(x)$ when asked to sketch a tangent line on a graph. In reviewing the classroom data, instructor D made a point to clarify that, although the tangent line and the tangent function share a common name, they are indeed different. We also found in our previous work that for several students, horizontal tangent lines dominated their concept images of tangent lines. We noticed from the data that a majority of the instructors sketched a horizontal tangent line more than any other tangent line. Though these two results may not be related, it is worth noting and looking into further as we continue our analysis.

Implications for Teaching and Directions for Future Research

From informal conversations with the instructors in this study, we know that they often felt pressured for time. They often mentioned that they would have liked to have spent more time on certain topics, but often needed to move on to the next section in the textbook. Although we do not have specific advice for instructors at this point, we hope that our future research as well as our continued analysis of the data discussed in this preliminary report will help to shape curriculum materials to allow instructors to choose powerful examples to use in their classes with the limited time they have available.

Questions for the Audience

1. Do you have examples of other literature that discussed undergraduate instructors' mathematical knowledge for teaching?
2. Do you have recommendations for us on how we can capture all of the examples the student is using (in class, on homework assignments, or in the textbook)? Or, if we are unable to capture them all, how much is enough?
3. For the next phase of the study, we plan to interview instructors prior to teaching the material as well as after the lessons are completed. How can we determine whether or not the instructor is aware of specific misconceptions that are common to students without teaching the instructor something he/she may not already know?

References

Authors. (2014). [Title omitted for blind review]. Under review.

Ball, D. L. and Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (ed.), *Multiple perspectives on the teaching and learning of mathematics* (pp. 83-104). Westport: Ablex.

Biza, I., Christou, C., & Zachariades, T. (2008). Students perspectives on the relationship between a curve and its tangent in the transition from Euclidean Geometry to Analysis. *Research in Mathematics Education*, 10(1), 53-70

Biza, I., Souyoul A., & Zachariades, T. (2005).Conceptual change in advanced mathematical thinking discussion paper. *Fourth Congress of ERME*, Sant Feliu de Guixols, Spain.

Tall, D.O. (1987). Constructing the concept image of a tangent. In J.C. Bergerom, N. Herscovics & C. Kieran (Eds.), *Proceedings of the 11th conference of the international group for the psychology of mathematics education* (Vol. 3, pp. 69-75). Montreal, Canada.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity. Learners generating examples*. Manwah, NJ: Lawrence Erlbaum.

Student Understanding of Solution

Rebecca K. Walker
Guttman Community College

Student understanding of solution is central to success in much of mathematics, from basic algebra through linear algebra and differential equations. This research explores college algebra students' understanding of solution. In particular, it explores student definitions of what it means for a number to be a solution to an equation and whether students can determine if a number is a solution to an equation. Results show that most of the students could determine if a given number is a solution to an equation but that fewer than half of the students could write a reasonable definition of solution. Categories of student responses are identified along with possible reasons for the misconceptions. These results have implications for teaching all levels of mathematics.

Key words: College Algebra, Solution

At the heart of much of algebra is solving equations. However, it is not clear that students really understand what it means to find a solution to an equation. DeLima and Tall conclude that, "There is no evidence that the students are looking for a value of the unknown that satisfies the given equation" (p. 10). A 2010 study by Stigler, Givven, and Thompson, explored what community college developmental math students understood about solutions to equations. They found that very few students were able to provide good explanations for why equations such as $x + 1 = 1$ or $x^2 = -9$ do not have real solutions. A study by DeVries and Arnon (2004) looked at student understanding of solution in the context of undergraduate linear algebra. They found that students confused the concept of solution with the process of solving an equations.

This research is related to the research in both of these studies and explores two questions:

1. Are College Algebra students able to determine if a given value is a solutions to an algebraic equation? If so, how do they do it?
2. What does it mean to College Algebra students when they are told that a number is a solution to an equation?

The data for this work was gathered from 347 students as part of a final exam in an undergraduate College Algebra course. The two questions that were included on the final exam were:

1. What does it mean for a number to be a solution to an equation?
2. Is 3 a solution to $2^x + 7 = 5x$?

The responses were then coded individually by two researchers and any discrepancies were discussed until agreement was reached on the most appropriate code for each response.

The results given in Table 1 indicate that almost all (93.1%) of the students were able to determine that 3 was a solution to $2^x + 7 = 5x$.

Table 1
Responses to, "Is 3 a solution to $2^x + 7 = 5x$? Explain your reasoning."

Student Response (n = 347)	Percent (number) of students

Yes 3 is a solution and student provides appropriate reasoning	93.1% (323)
No, 3 is not a solution. But student answers no because of arithmetic error that indicates the two sides are not equal when $x = 3$	1.7% (6)
Student shows that two sides are equal when $x = 3$ but does not state whether 3 is or is not a solution.	1.2% (4)
No, 3 is not a solution.	3.2% (11)
Other response	0.9 % (3)

This indicates that for the most part students know how to check whether or not a given value is a solution to an equation. We can conclude that the instruction that students have received in this area has been effective and students can determine whether a given number is a solution, even for equations that cannot be solved through symbolic manipulation.

However their knowledge of how to check if a given value is a solution does not seem to transfer to being able to write a good response to the question about what it means for a number to be a solution to an equation. The student responses to this question are summarized in Table 2.

Table 2
Responses to, "What does it mean if a number is a solution to an equation?"

Student response	Percent (number) of students
The solution is the value of the variable that makes the equation true or makes the sides equal	41.2% (143)
It is the answer to the problem or the result when you solve the equation	24.8% (86)
It is the input that gives the correct output	12.7% (44)
The solution is the output, what comes after the equal sign, or the result of calculations	5.5% (19)
The student stated something about the number or nature of the solutions	3.7% (13)
The solution makes the equation equal to zero	2.6% (9)
Student response said something related to the definition or domain of a function	2.0% (7)
The solution gives point on the graph or line	1.4% (5)
Other	6.1% (21)

The data shows that far fewer of the students, only 41.2%, were able to say that the solution is the value of the variable that makes the equation true or makes the sides equal to each other. Other responses were quite varied and included that the solution was the answer to the problem, that it was the output or what comes after the equals sign, or the result of calculations.

The incorrect responses seem to reflect several possible difficulties in the student understanding. The fact that many saw "answer" and "solution" as interchangeable terms may be a reflection of sloppy vocabulary usage on the part of students and instructors of mathematics. It is possible that more careful use of vocabulary could help alleviate this confusion. Another set of responses seem to be related to the lack of a robust understanding of the equals sign. Some of the students may be thinking of the equals sign as an operator rather than a relational symbol. This is related to research by Knuth et al (2006). Further research, likely including student interviews and possible classroom observations would allow us to better pinpoint the reasons why these students were not able to provide an appropriate definition of solution and to explore what types of learning opportunities might help students develop a more robust understanding of solution.

References

de Lima, R. Nogueira, & Tall, D. (2008). Procedural embodiment and magic in linear equations. *Educational Studies in Mathematics, 67*(1), 3-18.

DeVries, D. & Arnon, I (2004). Solution--What does it mean? Helping linear algebra students develop the concept while improving research tools. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, Vol 2 p. 55–62.

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education*, 37, 297–312.

Stigler, J., Givvin, K., & Thompson, B. (2010). What community college developmental mathematics students understand about mathematics. *MathAMATYC Educator*, 1(3):4-16.

Knowledge for teaching: Horizons and mathematical structures

Nicholas H. Wasserman
Teachers College, Columbia University

Ami Mamolo
University of Ontario Institute of Technology

In this study we use the lens of knowledge at the mathematical horizon to shed light on the underlying structural components of school mathematics content. We attend specifically to algebraic structures, identifying ways in which awareness of such structure may be transformative to mathematical knowledge for teaching. In particular, we analyze curricular content from elementary, middle, and secondary school mathematics with respect to the inherent structures that encompass and connect that material, as well as with specific attention to the opportunities afforded to teachers by their horizon.

Key words: Mathematical Knowledge for Teaching, Mathematical Horizon, Advanced Mathematics, Algebraic Structures

The mathematical knowledge for teaching (MKT) framework (e.g., Ball, Thames, & Phelps, 2008) frames a professional knowledge base rooted in the work of teaching, and builds on Shulman's (1986) work, establishing sub-domains for both subject matter and pedagogical content knowledge, specifically in the context of teaching mathematics. Of the sub-domains for subject matter knowledge, Horizon Content Knowledge (HCK) was less developed within the MKT framework. Various scholars (e.g., Fernandez & Figueiras, 2014; Jakobsen, Thames, & Ribeiro, 2013; Wasserman & Stockton, 2013; Zazkis & Mamolo, 2011) have begun to further conceptualize and describe horizon knowledge by providing and analyzing examples from the profession. This preliminary study draws on some of the budding notions of horizon knowledge to conceptualize some of the ways that knowledge of advanced mathematics might inform and support middle and secondary school teaching. In particular, we investigate underlying mathematical structures within the scope of school mathematics, focusing on the potential utility of teachers' awareness of such structures as transformative for more elementary ideas.

Literature/Theoretical Perspective

In developing the MKT framework, Ball, Thames, and Phelps (2008) operate under a practice-based approach to teacher knowledge: that knowledge for teachers must be linked to the actual work of teaching. As part of their delineation of content knowledge, they describe horizon content knowledge as "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (p.403). Ball & Bass (2009) further described four elements of a teacher's knowledge of her students' horizons: a sense of the mathematical environment surrounding the current location in instruction; major disciplinary ideas and structures; key mathematical practices; and core values and sensibilities. Zazkis and Mamolo (2011) extended this conception to focus on teachers' horizons and provided several examples of how teachers' Advanced Mathematical Knowledge [AMK] (Zazkis & Leikin, 2010) informed their interactions with pupils. Paralleling Husserl's philosophical constructs of the inner and outer horizons of a conceptual object, they interpret horizon knowledge as awareness of a mathematical object's "periphery" – that is, the specific features or properties of the object which are not currently at the individual's focus of attention (inner horizon), as well as the underlying mathematical structure, generalities, and connections which embed the object within a "greater mathematical world" (outer horizon) – as it relates to, and is accessed in, teaching situations.

Wasserman and Stockton (2013) argued for broadening the impact of horizon knowledge on the work of teaching. Their conceptualization of the horizon similarly relates to more advanced mathematics but with a focus on *transformation*. In particular, that horizon knowledge of advanced mathematics for teachers transforms their own perception of elementary, middle, or secondary content, in the sense that the content is seen in a new light, that the meaning or understanding of ideas is shifted, or that the content is re-organized, re-ordered, or re-structured in the teachers' mind. This notion of transformation relates to Simon's (2006) description of key developmental understandings (KDUs), which changes one's thoughts about and perceptions of mathematical ideas and their relationships; however, in the context of the horizon this is specific to advanced mathematical knowledge serving as a KDU for more elementary mathematics. Such a transformation in understanding may inform teachers' choices for sequencing content, impact what concepts they emphasize, alter their exposition of ideas, or shape ways they transition and prepare students for future ideas – all of which are connected to the work of teaching.

This work focuses specifically on mathematical structures, drawing both inner- and outer-horizon connections to advanced mathematics through a lens of transformation. Our broader work attends to structures from various areas of mathematics (e.g., measures, orders, sets); however, we focus here on algebraic structures and address the following research questions:

(i) *What mathematical structures underpin and connect the content areas across elementary, middle, and secondary school mathematics, and how can these structures provide avenues for transforming individuals' perceptions of said content? In particular, what are the transformative possibilities of algebraic structures on more elementary content?*

(ii) *In what ways can awareness of a mathematical object's periphery provide opportunities for connecting elementary, middle, and secondary mathematics content to their underlying mathematical structures?*

Methodology

To address these questions, we began by analyzing the US-based Common Core State Standards for Mathematics (CCSS-M, 2010), focusing on standards that have similar trajectories and emphases in international curricula. Based on both independent and collaborative coding, we identified some of the primary standards for which knowledge of algebraic structures might be particularly transformative for teaching. The 65 standards identified were then analyzed using a grounded theory approach (Strauss & Corbin, 1990) to conceptualize primary content areas and their progression across K-12 mathematics. (Although projects to develop learning progressions for CCSS-M standards exist (e.g., http://commoncoretools.me/), which track the vertical progression of a mathematical topic (e.g., fractions or functions), the analysis for this work was independent of strand and complements existing work by allowing for cross-integration across different mathematical domains, but simultaneously related to more advanced algebraic structures.) In particular, we present a preliminary analysis of some of these content areas and their related conceptualizations "at the horizon", providing both examples of the potential impact on teaching practice and a larger global progression of K-12 mathematics in relation to algebraic structures. Further synthesis work was done for conceptualizing and understanding the role that horizon knowledge might play in school mathematics teaching. On-going work will attend to how undergraduate mathematics programs may foster horizon knowledge for teaching.

Findings/Implications

Through our preliminary analysis, four content areas emerged within which knowledge of abstract algebra was potentially transformative to instruction (Table 1). For example, the notion of inverse is evident throughout K-12 mathematics, and is one for which knowledge of the structures of abstract algebra has positively influenced teaching decisions (e.g., Zazkis & Mamolo, 2011; Zazkis & Zazkis, 2013). The concept plays an extremely important role for students understanding operations on sets and relations between them. For a teacher, understanding the general notion of inverse, where additive, multiplicative, functional, etc., inverses become examples of the same concept, unified within some algebraic structure, is insightful and may help provide consistency in developing and discussing these ideas. This awareness of structure characterizes outer-horizon knowledge and may provide a teacher with a better sense of the experiences and ways of reasoning that can support students' mathematical growth (Mamolo & Pali, 2014). In the context of inverse functions, this transformation may impact the types of questions a teacher poses. For example, "*Compare the following: –sin(x), csc(x),* $sin^{-1}(x)$*. Are any the same? Are they all different? Rewrite –sin(x) and csc(x) using a "-1" somewhere; why do all of these use "-1"?*" This line of questioning draws attention away from the particular uses of "-1" as disparate entities (e.g., coefficients, powers) toward its general conceptualization. The notion of inverse from school mathematics to the advanced content in abstract algebra develops along the Action-Process-Object-Schema (APOS) framework (Asiala, et al., 1996), where inverses initially associated with an "opposite" action or process turn into "objects" themselves within a specified group. In fact, the development in secondary school of inverse functions also serves as the set for which inverse operations (elementary school) become inverse elements; the ontological shift required for viewing inverse operations as set elements exemplifies a KDU that transforms an understanding of inverse operations (inner horizon) along with the broader mathematical world to which those operations are connected (outer horizon).

Awareness of these connections and structures requires a broad view of the periphery of the mathematical object of inverses, which is influenced by an individual's focus of attention. As such, we have begun analyzing how the CCSS-M may direct a teacher's attention, illuminating or obscuring the horizon. Though our analysis is still on-going, we have begun by attending to organizational aspects (e.g., where and how content is presented) as well as semantic aspects (e.g., process- vs. object-based language, syntax), and we note a few preliminary observations here. With respect to inverses, the word itself is not introduced by the CCSS-M until grade 7, although the concept appears implicitly as early as kindergarten. While the standards identify relationships between common arithmetic operations, early descriptions emphasize process-based understandings (e.g., find the answer to a division problem by finding the answer to a multiplication problem). The relationships emphasized attend to operations that may be "inverted" but obscure ideas of inverse as a relationship between two numbers. When fractions are introduced (grade 3) and arithmetic properties developed (grade 4), no explicit connections to inverse relationships are made. A footnote in the grade 5 standards hints at inverse relationships between multiplication and division and their use in fraction arithmetic, however the relationship between b and $1/b$, and their status as inverse elements, goes unaddressed. The first mention of inverse as an object appears in grade 7 ("Understand subtraction of rational numbers as adding the additive inverse") and marks a distinct shift in focus of attention. Whereas elementary content focuses on processes applied to numbers as a way to understand (inverse) relationships between operations, middle and secondary content focuses on relationships between numbers (or matrices or functions) as a way to understand the relationship between operations. We note that these observations serve as a description (not judgement) of how the standards potentially direct or do not direct a teacher's attention toward mathematical structure; indeed, developing action/process notions of inverse early on may be beneficial, although whether teachers connect these ideas to broader structures in the horizon is unclear.

By isolating common mathematical structures (conceptualized as elements of a "mathematical outer-horizon") across school standards, we have highlighted some of the areas in which algebraic structures may become transformative for teaching. Attention to the particulars of the standards underscores how different conceptualizations and relationships are emphasized and, as such, influence what may be accessible in the periphery, both in terms of the specific properties of objects (inner horizon) and their structural connections (outer horizon). Additionally, how these structures are connected to the context of school mathematics can inform approaches in post-secondary mathematics courses for developing horizon knowledge and supporting the associated pedagogical sensitivities of prospective teachers. As part of the presentation, we propose the following questions for audience discussion: 1) Is knowledge of mathematical structure useful for school teaching?; 2) Should developing explicit links amongst school content and its underlying structure form part of prospective teachers' mathematical preparation?; 3) What experiences at the undergraduate level could be relevant for developing teachers' horizon knowledge, with a focus on transforming perceptions of elementary content?

Table 1: Four content areas and their progression across school mathematics

<table>
<tr><th>Content Area</th><th>Elementary School</th><th>Middle School</th><th>High School</th></tr>
<tr><td rowspan="3">Arithmetic Properties</td><td>Properties of Addition, Multiplication on N, Q+</td><td>Properties of Addition, Multiplication on Z, Q, R; Algebraic expressions</td><td>Properties of Addition, Multiplication on R, C; Polynomials, Rationals
Properties of Matrix multiplication; Properties of Function composition</td></tr>
<tr><td colspan="3">Fields / Rings ⟷ Groups</td></tr>
<tr><td colspan="3">Concrete ⟷ Abstract</td></tr>
<tr><td rowspan="2">Inverses</td><td>Inverse Operations</td><td>Transitioning to Inverse Elements</td><td>Inverse Elements and Inverse Functions</td></tr>
<tr><td colspan="3">⟶ Comprehensive Framework for Inverses</td></tr>
<tr><td rowspan="2">Structure of Sets</td><td>Equivalence within number sets (identity)</td><td>Equivalence within algebraic expressions (arithmetic properties); Expansion on number sets to Z, Q (closure)</td><td>Expansion to C (closure); Parallel structures across number sets and algebraic expressions</td></tr>
<tr><td colspan="3">Within Sets ⟷ Across Sets</td></tr>
<tr><td rowspan="2">Solving Equations</td><td>(Guess and check)</td><td>Systematic solving (arithmetic operations)</td><td>Systematic solving (complex functions); Systems of equations</td></tr>
<tr><td colspan="3">Arithmetic Operations ⟷ Complex Functions</td></tr>
</table>

References

Asiala, M., Brown, A., DeVries, D., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education. *Research in collegiate mathematics education II, Conference Board of the Mathematical Sciences (CBMS) Issues in mathematics education, 6,* 1-32.

Ball, D.L., & Bass, H. (2009). With an eye on the mathematical horizon: knowing mathematics for teaching to learners' mathematical futures. Paper presented at the 43rd *Jahrestagung der Gelleschaft fur Didaktic der Mathematik*, Oldenburg, Germany. Retrieved 15 May 2011 from www.mathematik.uni-dortmund.de/ieem/BzMU/BzMU2009/BzMU2009- Inhalt-fuer-Homepage.htm.

Ball, D., Thames, H. M., & Phelps, G. (2008). Content knowledge for teaching. *Journal of Teacher Education, 59*(5), 389-407.

Common Core State Standards in Mathematics (CCSS-M). (2010). Retrieved from: http://www.corestandards.org/the-standards/mathematics

Fernandez, S., & Figueiras, L. (2014). Horizon content knowledge: Shaping MKT for a continuous mathematical education. *REDIMAT, 3(1), 7-29. Doi:10.4471/redimat.2014.38*

Jakobsen, A., Thames, M.H., Ribeiro, C.M., & Delaney, S. (2012). Delineating issues related to horizon content knowledge for mathematics teaching. Paper presented at the Eighth Congress of European Research in Mathematics Education (CERME-8). Retrieved from: cerme8.metu.edu.tr/wgpapers/WG17/WG17_Jakobsen_Thames_Ribeiro.pdf

Mamolo, A. & Pali, R. (2014). Factors influencing prospective teachers' recommendations to students: Horizons, hexagons, and heed. *Mathematical Thinking and Learning, 16*(1), 32-50.

Shulman, L.S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Simon, M. (2006). Key developmental understandings in mathematics: A direction for investigating and establishing learning goals. *Mathematical Thinking and Learning, 8*(4), 359-371.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. Newbury Park, CA: Sage.

Wasserman, N.H., & Stockton, J.C. (2013). Horizon content knowledge in the work of teaching: a focus on planning. *For the Learning of Mathematics, 33*(3), 20-22.

Zazkis, R., & Leikin, R. (2010). Advanced mathematical knowledge in teaching practice: Perceptions of secondary mathematics teachers. *Mathematical Thinking and Learning, 12*(4), 263-281.

Zazkis, R. & Mamolo, A. (2011). Reconceptualizing knowledge at the mathematical horizon. *For the Learning of Mathematics, 31*(2), 8 – 13.

Zazkis, R. & Zazkis, D. (2013). Exploring mathematics via imagined role-playing. In A. Lindmeier & A. Heinze (Eds.) *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4. Kiel, Germany: PME.

Secondary mathematics teachers' perceptions of real analysis in relation to their teaching practice

Nicholas Wasserman	Matthew Villanueva	Juan Pablo Mejia-Ramos	Keith Weber
Columbia University	Rutgers University	Rutgers University	Rutgers University

Fourteen secondary mathematics teachers were given a task-based interview in which they were presented with four mathematical tasks from secondary school mathematics that a real analysis course might prepare them to handle. We found that most participants could not answer these questions correctly. Some participants believed the answers to some of these questions could inform their teaching, but felt these topics were not included in their real analysis course. We suggest that explicitly discussing the corollaries relevant to secondary school mathematics in real analysis courses for prospective teachers might be a useful first step in helping future teachers see real analysis as relevant to their instruction.

Key words: Mathematical Knowledge for Teaching; Real Analysis; Teacher development.

Teaching high school mathematics requires having a deep and flexible understanding of the content that is being taught. To build this knowledge, many future teachers are required to complete a substantial number of courses in advanced mathematics, sometimes even obtaining an undergraduate degree in mathematics (e.g., CBMS, 2001, 2012; Stacey, 2008). However, the effectiveness of these policies has been questioned. There is little relationship between the number of university mathematics courses that teachers complete and their students' mathematical achievement (e.g., Darling-Hammond, 2000; Monk, 1994). Further, courses such as real analysis, which representatives of mathematical professional societies and some mathematics educators often consider necessary for truly understanding and being able to teach secondary mathematics (e.g., CBMS, 2012), are frequently viewed by teachers as unnecessary and unrelated to their teaching (Goulding, Hatch, & Rodd, 2003).

In this paper, we focus on how secondary mathematics teachers perceive the value of their real analysis course. We chose to focus on real analysis because of the ostensibly strong links between the content of a real analysis course and the secondary mathematics curriculum. Many topics covered in most real analysis courses overlap with the high school curriculum, including the real number line, functions, inverses, the intermediate value theorem, limits, continuity, derivatives, and integrals. In principle, completing a real analysis course provides secondary teachers with the opportunity to address gaps in their knowledge (e.g., what does $5^{\sqrt{2}}$ mean?), address misconceptions (e.g., thinking that functions with cusps are not continuous), and illuminate why some statements about functions are true (e.g., the Intermediate Value Theorem). Hence, while we would not expect the mathematical content to be learned in this course to provide *all* the mathematical content needed to teach algebra and calculus effectively – the mathematical knowledge needed for teaching extends beyond a mastery of the content (e.g., Hill et al., 2007) – we might predict that this would help future teachers to develop *some* of the requisite content knowledge. The goal of this paper is to explore the extent to which this is the case. In this paper, we interviewed 14 secondary teachers about their experiences in a real analysis course and their perceptions on how this informed their teaching. We use this to shed light on the following questions:

1. To what extent can teachers apply what they learned in real analysis to topics in high school mathematics?
2. To what extent do these teachers think that real analysis informs their teaching? If they do not see value in learning real analysis, why do they feel this way?
3. How might we change the way that real analysis courses are taught to teachers to increase their relevance to teachers' classroom practice?

Literature Review

Several studies have found that pre-service high school mathematics teachers have substantial gaps in their understanding of secondary mathematics, even after completing an undergraduate degree in mathematics. For instance, Bryan (1999) interviewed nine pre-service secondary mathematics teachers near the end of their undergraduate degree. These teachers had difficulty explaining the meaning of important concepts such as function and exponents; in some cases, the teachers could not even complete procedural tasks regarding these concepts. These findings are consistent with other studies (e.g., Cankoy, 2010; Even, 1990; Sanchez & Llinares, 2003) that find that completing a mathematics degree does not necessarily afford students with an understanding of and an ability to explain ideas from high school mathematics.

Other studies have found that it is not clear to teachers how their experiences in advanced mathematics can be used to inform their instruction. Zazkis and Leikin (2010) surveyed 52 practicing secondary teachers on how advanced mathematical knowledge impacted their pedagogical practice. Roughly half the participants indicated that they rarely used this knowledge in their teaching. Those that did find this information relevant generally could not provide specific ways in which it was used. Such results are similar to other studies where teachers viewed advanced mathematics courses as unimportant in their development as teachers (e.g., Goulding et al, 2003; Rhoads, 2014).

Theoretical Perspective

We view applying mathematical content knowledge learned in advanced mathematical coursework to topics in secondary mathematics as a particular case of *transfer*. In traditional theories of transfer, a learner is thought to form an abstract representation of some initial learning event, recognize similarities in the structure of a target learning event, and then apply similar reasoning in the targeted learning event as he or she did in the initial learning event (e.g., Gentner et al, 2003). For instance, in real analysis, students learn about the definition of inverse functions and theorems that specify conditions for when inverse functions will exist (e.g., the original function is injective) or cannot exist (e.g., a continuous function that is not strictly monotonic). If a secondary school student asks whether $\sqrt{x}$ is an inverse function for x^2, the hope is that a teacher who has completed a course in real analysis can draw on his or her knowledge of theorems learned in that course and explain that x^2 has no inverse function because it is not injective, but that it can have an inverse function on a restricted domain in which it is strictly monotonic, such as the non-negative numbers. There are many factors that may inhibit this transfer, including time (the gap between real analysis and student teaching may be several years), representation systems (the formal epsilon-delta notation in real analysis differs from the diagrams and more informal notation of algebra and calculus), and the desired product (a formal proof vs. an explanation comprehensible to a high school student). In this paper, we examine a somewhat simpler transfer task. Can teachers apply what they used in real analysis to answer questions about high school math?

Methods

Fourteen secondary mathematics teachers participated in this study. Eight participants had recently completed a five-year masters and certification program in mathematics education. In completing their degree, these participants completed an undergraduate mathematics degree as well as a semester of student teaching in which they taught five courses. The other six participants had completed the same degree and had one to five years experience working as a high school math teacher.

During the interviews, participants were first asked if they found real analysis useful in their teaching and then to elaborate on their response. Next they were given four tasks:

a. Explain why 0.999… = 1.
b. What is $5^{\sqrt{2}}$?
c. Is $f(x) = \sqrt{x}$ an inverse function for $g(x) = x^2$ and is $h(x) = \arcsin x$ an inverse function for $i(x) = \sin x$?
d. Is $f(x) = \sqrt[3]{x}$ differentiable on all of the real numbers?

Tasks (a) and (b) were selected after mathematicians whom we interviewed cited them as examples of knowledge from real analysis that was relevant to the teaching of high school mathematics. Tasks (c) and (d) were developed by our research team as related to knowledge from real analysis that can potentially inform the pedagogical practice of teachers.

For each one of the four tasks above, participants were (i) asked to answer the question, and then (ii) given a normatively correct response to the task by using ideas from real analysis (e.g., a proof of why 0.999… = 1). After this, participants were (iii) asked if they understood the given explanation, and (iv) told that this topic was related to common student difficulties in mathematics (e.g., the belief that 0.999…=1 gets arbitrarily close to 1 but does not reach it) and asked if the response from (ii) could inform their teaching. Finally, participants were (v) asked if there was anything that could have been done in their real analysis class to help them relate this content to their teaching. Interviews were audiotaped and lasted about one hour.

Preliminary Results and Significance

Analysis of our data is ongoing, but we can report on four interesting trends that we have observed in our data.

1. For each task, the majority of our participants were unable to solve the task correctly. That is, despite completing a real analysis course where this content was ostensibly covered, participants were generally unable to use this content to answer mathematical questions. This result is consistent the findings of Bryan (1999) and Cankoy (2010) that a degree in advanced mathematics may not prepare students with a mastery of the content in secondary mathematics.

2. For tasks (a) and (b), participants generally did not think this mathematical knowledge was relevant to their teaching. This is notable because teachers' inability to understand why 0.999…=1 was viewed by some mathematicians as an important hole in their knowledge. The participants in our study did not agree with this. They argued that this topic did not come up during their teaching, suggesting that some assumed benefits of a real analysis course might not be that important for pedagogy, at least not from the perspective of high school teachers.

3. For tasks (c) and (d), most participants thought this could inform their teaching, but they felt this was not covered in their real analysis course. From our study of the textbook used in the analysis course where this study took place, we observed that the content needed to address these questions was covered, but specific questions of this type were not. These are, in effect, corollaries to theorems covered in the book, although corollaries of this type (i.e., relating the real analysis content to secondary mathematics) were rarely given.

4. Some participants thought teachers might benefit from a special section of real analysis that emphasized relationships to teaching. As mentioned above, connections to secondary mathematics and to pedagogy were notably absent from the textbook (Fitzpatrick, 2006) we reviewed. This is perhaps because these topics are less important to the typical mathematics major who completes a real analysis course, who may be interested in practical applications or connections to other domains of advanced mathematics. Such connections are, however, of interest to teachers (as point (3) illustrates), but are not being made by the teachers (as point (1) illustrates). Having a class targeted for this population of students might be helpful.

Questions for the Audience

1. What further questions could we ask about our data?
2. How can we use different lenses of transfer (e.g., traditional vs. Actor-Oriented) to interpret our data? What different insights might these provide?
3. How would we design a real analysis class that might be useful to teachers?

References

Bryan, T. J. (1999). The conceptual knowledge of preservice secondary mathematics teachers: How well do they know the subject matter they will teach? *Issues in the Undergraduate Mathematics Preparation of School Teachers: The Journal*, *1*, 1–12.

Cankoy, O. (2010). Mathematics teachers' topic-specific pedagogical content knowledge in the context of teaching a^0, 0! and a/0. *Educational Sciences: Theory & Practice*, *10*, 749-769.

Conference Board of the Mathematical Sciences. (2001). *The mathematical education of teachers.* Providence RI and Washington DC: American Mathematical Society and Mathematical Association of America.

Conference Board of the Mathematical Sciences (2012). The mathematical education of teachers II. Retrieved from http://www.cbmsweb.org/MET2/MET2Draft.pdf

Darling-Hammond, L. (2000). Teacher quality and student achievement: A review of state policy evidence. *Educational Policy Analysis Archives, 8*(1).

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics*, *21*, 521–544. doi:10.1007/BF00315943

Fitzpatrick, P. (2006). *Advanced calculus* (2nd ed.). Providence, RI: American Mathematical Society.

Gentner, D. Lowenstein, J., & Thompson, L. (2003). Learning and transfer: A general role for analogical encoding. *Journal of Educational Psychology*, *95*, 393-405.

Goulding, M., Hatch, G., & Rodd, M. (2003). Undergraduate mathematics experience: its significance in secondary mathematics teacher preparation. *Journal of Mathematics Teacher Education*, *6*, 361–393.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*, 372-400.

Monk, D.H. (1994). Subject area preparation of secondary mathematics and science teachers and student achievement. *Economics of Education Review, 13*(2), pp. 125-145.

Rhoads, K. (2014). High school mathematics teachers' use of beliefs and knowledge in high quality instruction. Unpublished doctoral dissertation.

Sánchez, V., & Llinares, S. (2003). Four student teachers' pedagogical reasoning on functions. *Journal of Mathematics Teacher Education, 6*(1), 5–25.

Stacey, K. (2008). Mathematics for secondary teaching: Four components of discipline knowledge for a changing teacher workforce. In P. Sullivan & T. Wood (Eds.), *The international handbook of mathematics teacher education: Knowledge and beliefs in mathematics teaching and teaching development* (Vol. 1, pp. 87-113). Rotterdam, The Netherlands: Sense.

Zazkis, R., & Leikin, R. (2010). Advanced mathematical knowledge in teaching practice: Perceptions of secondary mathematics teachers. *Mathematical Thinking and Learning*, *12*, 263-281.

Adding explanatory power to descriptive power: Combining Zandieh's derivative framework with analogical reasoning

Kevin L. Watson
Brigham Young University

Steven R. Jones
Brigham Young University

The derivative is an important foundational concept in calculus that has applications in many fields of study. Existing frameworks for student understanding of the derivative are largely descriptive in nature, and there is little by way of theoretical frameworks that can explain or predict student difficulties in working with the derivative concept. In this paper we combine Zandieh's framework for understanding the derivative with "analogical reasoning" from psychology into the "merged derivative-analog framework." This framework allows us to take the useful descriptive capabilities of Zandieh's framework and add a layer of explanatory power for student difficulties in applying the derivative to novel situations.

Key words: calculus, derivative, theoretical framework, analogical reasoning

The derivative is a widely used concept in many fields of study, both inside and outside of mathematics. Given its importance, we might hope that calculus students would develop a deep understanding of the derivative. However, many researchers have documented that students struggle to fully understand and use the derivative concept (e.g., Aspinwall, Shaw, & Presmeg, 1997; Byerley, Hatfield, & Thompson, 2012; Orton, 1983; Park, 2013; Siyepu, 2013). While some frameworks have been developed to analyze student understanding of the derivative (García, Llinares, & Sánchez-Matamoros, 2011; Zandieh, 2000; Zandieh & Knapp, 2006), most current frameworks focus on *describing* student understanding rather than *explaining* or *predicting* potential student difficulties. In this paper we present a blending of one such framework (Zandieh, 2000) with a theory of analogical reasoning from psychology (Holyoak, 2012), the result of which may help explain, in part, some of the difficulties students might face as they attempt to apply their derivative knowledge. In the following sections we provide brief descriptions of Zandieh's derivative framework and Holyoak's approach to analogical reasoning, discuss the merging of these two frameworks into the "merged derivative-analog framework," and provide examples of the merged frameworks' usefulness.

Zandieh's Derivative Framework

Zandieh (2000) provided a useful way to map out students' understanding of the derivative in order to "clarify, describe and organize the facets that we as a mathematical community consider to be part of the understanding of the concept of derivative" (p. 103). Her framework consists of two main components: multiple representations, or *contexts*, and *layers* of process-object pairs.

Contexts. Based on existing categories of concept image from the mathematics education literature (see Hart, 1991; Tall & Vinner, 1981; Vinner & Dreyfus, 1989), Zandieh posits that the concept of the derivative can be represented (a) *symbolically* as the limit of a difference quotient, (b) *graphically* as the slope of the tangent line to a curve at a point, (c) *verbally* as the instantaneous rate of change, and (d) *physically* as speed or velocity. Each of these representations is referred to by Zandieh as a "context." Other contexts are possible, but all have underlying commonalities that cause us to call all of them by the same name: the derivative.

Process-object layers. The commonalities among the different contexts are what Zandieh refers to as the "layers" of the framework, consisting of the ideas of ratio, limit, and function. Each layer can be viewed as both *processes* and *objects*. That is, the division between two quantities can be reified as a ratio. Then, “infinitely many of the ratios” (p. 107) are passed through to get the limit, a process that can also be reified. Finally, the derivative function can be thought of as the result of doing the limit process at every point in the domain.

Pseudo-objects. Note that, at times, students will not grasp the underlying process of a layer, and fail to develop a deep, structural conception of the object on that layer, but will still be able to consider the next process in the derivative structure by using what Zandieh (2000) abbreviates as *pseudo-objects*. To define a pseudo-object, Zandieh uses what Sfard (1992) calls a *pseudostructural* conception, which “may be thought of as an object with no internal structure” (Zandieh, 2000, p. 107). For example, a student might consider the value of a limit as an object, without having an idea of the limiting process that gave rise to that value. Or a student might see a ratio as a fraction, such as ½, without perceiving that it can also mean a comparison of two quantities. While pseudo-objects are not inherently negative, as they can be used in efficiently simplifying thinking about later processes, students who only have a pseudo-object understanding may not fully grasp the concept of the derivative.

Matrix of derivative understanding. The four *contexts* and the three *layers* mentioned in this section are compiled by Zandieh into a matrix (see Figure 1) that can map out a student’s understanding of the derivative and can be used to visually compare the student’s understanding with that of the mathematics community or other students. Zandieh recommends using an open circle in a box to represent that a student has demonstrated a pseudo-object understanding of that context and layer, and a filled in circle if the student also shows an understanding of the process involved.

Contexts: / Layers:	Graphical (Slope)	Verbal (Rate)	Physical (Velocity)	Symbolic (Diff. Quot.)	Other
Ratio					
Limit					
Function					

Figure 1: Zandieh’s (2000) matrix for student understanding of the derivative concept

While this framework is useful in analyzing student understanding of derivatives, Zandieh mentioned that her framework “is not meant to explain how or why students learn as they do, nor to predict a learning trajectory” (2000, p. 103). That is, her framework is largely descriptive and comparative in nature. In order to add some explanatory power to this framework, we propose combining it with analogical reasoning.

Holyoak’s Analogical Reasoning

Analogy and relational reasoning (for a more complete overview, see Holyoak, 2012) is the process of identifying common patterns of relationships between two (or more) situations through comparison, and using these commonalities to make inferences about the lesser known of the two situations (see Figure 2). This usually happens by first encountering a problematic situation, or *target analog*, that serves as a retrieval cue for a potentially useful *source analog*, or well-known situation in a person’s memory. The person then creates a *mapping*, or set of systematic correspondences between the two analogs, that seeks to align the elements of the source and target. Based on this mapping, and the relevant relationships within the source analog, new inferences can be made about the target that the “reasoner” hopes will help solve the problem. After this analogical reasoning has taken place, some form of relational generalization may take place, helping the reasoner develop a more abstract schema for a category of situations.

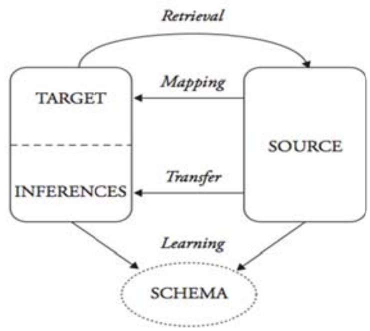


Figure 2: Analogical reasoning process (Holyoak, 2012, p. 236)

As an example, consider the scientific analogies between sound and water waves (see Pask, 2003). Historically, scientists encountered the problem of trying to understand how sound works. This *target analog* led scientists to think about other phenomena in the world that might be useful. Noticing that there seemed to be common patterns between sound and water waves, water waves became a *source analog* for making inferences about why sound acts as it does. These two analogs for waves then, in turn, helped refine a schema for waves that was later used in understanding other phenomena, such as the behavior of light.

The strongest and most powerful analogies are those whose mappings are based on the essential *causal relations*, or cause-and-effect relationships, within analogs (Holyoak, 2012). For example, with waves, the farther the wave travels the more diminished it becomes. That is, distance *causes* diminishment. When two analogs are generalized, it is these underlying causal relations that are abstracted to form a schema (Gick & Holyoak, 1983). For example, Holyoak (2012) describes the "wave schema" as one in which a physical phenomenon or quantity "exhibits a pattern of behavior corresponding to that of water waves: propagating across space with diminishing intensity, passing around small barriers, rebounding off large barriers, and so on" (p. 234). The fundamental causal relations are applicable across all contexts that match the underlying schema.

Analogies based on surface similarities hinder the formulation of a schema, and often lead to unhelpful and erroneous inferences from the source to the target (Holyoak & Richland, 2014). When a reasoner has a schema developed and is able to use it in analogical reasoning, analogical inference to other contexts with similar causal relations is greatly facilitated (Gick & Holyoak, 1983). Gick and Holyoak further found that schemas are hardly ever formed from a single analog, and that they are more likely to be formed when the common, underlying causal relations in two or more analogs are explicitly pointed out.

The Merged Derivative-Analog Framework

We will now explain how these two frameworks can be merged. The different contexts within Zandieh's derivative framework (2000) are equivalent to analogs. The layers of Zandieh's framework are the schema that underlies all derivative contexts, which we will refer to as the ratio-limit-function schema. When a student encounters a new context where derivatives are meant to be useful, a student recalls from familiar contexts, or source analogs, what they understand about derivatives, and attempts to create a mapping from the familiar contexts to the new context, or target analog.

The key causal relations within this ratio-limit-function schema are the processes within the layers mentioned by Zandieh. That is, in the ratio layer, we consider the reification (see Sfard, 1992) of division into the object "ratio" as a causal relation. In the limit layer, the reification of taking repeated ratios over smaller intervals into a "limit" object is a causal relation. And, finally, in the function layer, the reification of taking a limit at each point into an entire function of such limits is a causal relation. These three "causal relations" form the derivative schema, which is applicable across all the contexts mentioned by Zandieh, including the graphical, symbolic, verbal, and physical contexts, as well as any other context which can be considered another representation of the derivative. Like with the wave schema, where the analogy only works if all aspects of the schema are applicable (propagation, reflection, etc.), any context that can be considered a representation of the derivative will necessarily have all three of these causal relations.

If a student only has pseudo-object understandings of the different layers and contexts, we say "the reasoner fails to understand (or misunderstands) the causal [relations] of the source" (Holyoak & Richland, 2014, p. 227). If a student with a pseudo-object understanding attempts to use analogical reasoning with a new derivative context, their inferences from their source analog(s) to the target analog will most likely be problematic, as they will be based only on surface similarities, or simply on the fact that they were told to find the derivative in the new context (Gick & Holyoak, 1983; Holyoak & Richland, 2014). Furthermore, as suggested by Gick and Holyoak, a student who heavily relies on a single context/analog when thinking about derivatives is not likely to have formulated the underlying ratio-limit-function schema in their cognitive structure of derivatives, and will probably struggle to see the analogous mapping and inferences that will be useful in making sense of new derivative situations.

What does this merged framework afford us? We offer three applications of the framework. First, while previous frameworks are largely *descriptive* of student understanding, our framework gives *explanatory* power for making sense of student difficulties in applying the derivative concept to novel situations. Second, our framework may provide some ability to *predict* how well a student might do in a new context, such as a physics or engineering context, based off of their understanding about the derivative in previous contexts. Third, our framework provides teaching implications, since analogical reasoning happens best when students have more than one analog, coupled with explicit extraction of the underlying schema.

Examples of the Merged Derivative-Analog Framework's Usefulness

To illustrate the potential usefulness of this merged framework, we show an analysis of two students who participated in a derivative-focused interview at the end of a first-semester calculus course. In the interview, the students were first given two tasks in which they calculated and discussed derivatives within a pure-mathematics context. Next, they were given three subsequent tasks that asked them to calculate and discuss derivatives from real-world contexts. We analyzed the interviews by first looking at the student data from the first two (mathematics) items and filling out a matrix of derivative understanding for each student. Each of our analyses was done independently so we could compare our matrices and ensure we were properly categorizing the students. Next, we used our framework to provide reasons for the difficulties one student had, and the facility in working with the derivative the other student had. While common sense would indicate that one of the students would struggle, our framework offers the ability to explain *why* these difficulties may occur.

In Figure 3 and Figure 4 we show the results of our matrices for Don and Jay. Don's understanding of the derivative seemed to heavily reside in the single, graphical context/analog. In addition, it is clear that he often operated with pseudo-object understandings, meaning he likely did not have an extracted ratio-limit-function schema from this single analog, as suggested by analogical reasoning (Gick & Holyoak, 1983). By contrast, Jay's understanding cut across multiple contexts/analogs with strong, process-based understandings. Consequently, as analogical reasoning also suggests, we expect he has cognitively extracted and compiled the ratio-limit-function schema (Gick & Holyoak, 1983).

	Graphical	Verbal	Physical	Symbolic	Other
Ratio	○				
Limit	○				
Function	○				

Figure 3: Matrix of derivative understanding for Don

	Graphical	Verbal	Physical	Symbolic	Other
Ratio	●	●		●	
Limit	●	●		●	
Function	●	●			

Figure 4: Matrix of derivative understanding for Jay

Our merged framework would predict that Don would not perceive the causal relations that make up the ratio-limit-function schema and would therefore not be able to map from the graphical derivative context to the new physics derivative context (Holyoak & Richland, 2014). Consider the following excerpts from an interview item dealing with the gravitational force between an object and the earth, $F = \frac{GmM}{r^2}$. In this item, the students were asked to calculate dF/dr and discuss its meaning. Both Don and Jay correctly used differentiation rules to find $dF/dr = -2\frac{GmM}{r^3}$, but differed significantly in their discussion of its meaning. Don began by reasserting that the derivative indicates the slope, or steepness at a given point.

> *Don*: I think it should mean that whenever you put in a value for *r*, you know whatever that value is, it should give you the slope, or I guess, you know, like, how steep the graph is at that point that you put in for *r*.

When pushed to explain what the "slope" of the graph would mean in this context, he seemed to mostly focus on the *direction* of force, much like how slope can be reduced to thinking of a line pointing up or down.

> *Don*: I'm not really sure. But, I think, if I were to guess, then, so my guess would be, the force of gravity would be, I guess, positive if you were being pulled toward the earth, because gravity pulls you, you know, toward it. And then, I guess, if it was negative, it would be pulling you away. But, I don't know, that's just my best guess.

In this excerpt, we can see that Don did attempt to use analogical reasoning to map from the graphical context to the force of gravity context. But because he did not grasp the causal relations that make up the ratio-limit-function schema, his inferences "suffered" (see Holyoak & Richland, 2014). Thus, our framework may be able to explain the underlying mechanisms regarding *why* a pseudo-object understanding of the derivative layers creates problems for applying derivative understanding to novel contexts.

By contrast, our merged framework would predict that Jay would be able to take in a new analogical context (the derivative of force) and map his developed schema onto it, and use analogical inference in order to make sense of the meaning of the derivative. In fact, the data show that this is the case, as Jay was able to use the ratio-limit-function schema he had extracted from the graphical, verbal, and symbolic analogs to make sense of this new derivative context.

> *Jay*: It tells me the rate at which the force of the attraction, in between the earth and whatever object it is, is changing at any given height above the earth… As *r* changes, it tells me the rate at which the force is changing.

Interviewer: What does that negative sign mean?
Jay: As the height gets higher, it means that the force is decreasing.
…
Jay: I mean, it's negative, so the, the negative derivative, it's basically the definition of what that means, is that the actual function, so F, uh, force, is decreasing at that given point.
…
Interviewer: What's true about how quickly gravity is decreasing? For every mile you go, does it decrease the same amount of gravitational force for every mile?
Jay: Looking at this [the derivative], it decreases less and less the farther you go. 'Cause the larger the radius gets, the smaller this number gets, until eventually it begins to approach zero… So, in general, the, the force of gravity on an object is decreasing by less and less the farther away you get.

Here we see Jay flexibly drawing on the ratio, limit, and function layers to make sense of the new context/analog. In particular, he described (a) how the derivative depicted the covariational relationship between distance and force (ratio), (b) how that rate of change could happen at a single point (limit), and (c) how the rate of change itself changes (function). The ratio-limit-function schema had been extracted from the various contexts/analogs and Jay had it cognitively "ready" to apply to a novel analog.

Conclusion

Our merged derivative-analog framework adds explanatory power to Zandieh's (2000) derivative framework, allowing us to account for the reasons why students struggle with applying the derivative in novel contexts. Furthermore, it enables us to make reasonable predictions about how well a student will be able to apply their existing knowledge about derivatives to applications of the derivative they have never seen before. This framework we have presented can be used by researchers to better understand and explain why students struggle with the derivative concept, particularly those students who seem to understand the derivative in mathematical contexts, but struggle significantly when presented with different derivative applications. Additionally, this framework and analogical reasoning suggest that we, as calculus instructors, need to take time in class to discuss multiple derivative analogs (see Gick & Holyoak, 1983), and make sure that students recognize the key causal relations, or the processes of the ratio, limit, and function layers of the derivative, that underlie those analogs. By doing so, students will be more likely to extract the ratio-limit-function schema, and will be better prepared to apply their understanding of derivatives to unfamiliar situations.

References

Aspinwall, L., Shaw, K. L., & Presmeg, N. C. (1997). Uncontrollable mental imagery: Graphical connections between a function and its derivative. *Educational Studies in Mathematics, 33*(3), 301-317.

Byerley, C., Hatfield, N., & Thompson, P. W. (2012). Calculus students' understandings of division and rate. In S. Brown, S. Larsen, K. A. Marrongelle & M. Oehrtman (Eds.), *Proceedings of the 15th special interest group of the Mathematical Association of America on research in undergraduate mathematics education* (pp. 358-363). Portland, OR.

García, M., Llinares, S., & Sánchez-Matamoros, G. (2011). Characterizing thematized derivative schema by the underlying emergent structures. *International Journal of Science and Mathematics Education, 9*(5), 1023-1045.

Gick, M. L., & Holyoak, K. J. (1983). Schema induction and analogical transfer. *Cognitive Psychology, 15*, 1-38.

Hart, D. K. (1991). *Building concept images: Supercalculators and students' use of multiple representations in calculus.* Unpublished doctoral dissertation, Oregon State University.

Holyoak, K. J. (2012). Analogy and relational reasoning. In K. J. Holyoak & R. G. Morrison (Eds.), *The Oxford handbook of thinking and reasoning* (pp. 234-259). New York: Oxford University Press.

Holyoak, K. J., & Richland, L. E. (2014). Using analogies as a basis for teaching cognitive readiness. In H. O'Neil, R. Perez & E. Baker (Eds.), *Teaching and measuring cognitive readiness*. New York, NY: Springer.

Orton, A. (1983). Students' understanding of differentiation. *Educational Studies in Mathematics, 14*(3), 235-250.

Park, J. (2013). Is the derivative a function? If so, how do students talk about it? *International Journal of Mathematical Education in Science and Technology, 44*(5), 624-640.

Pask, C. (2003). Mathematics and the science of analogies. *American Journal of Physics, 71*(6), 526-534.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification--The case of function. In G. Harel & E. Dubinksy (Eds.), *The concept of function: Aspects of epistemology and pedagogy (MAA notes, no. 25)*. Washington, DC: Mathematical Association of America.

Siyepu, S. W. (2013). An exploration of students' errors in derivatives in a university of technology. *The Journal of Mathematical Behavior, 32*(3), 577-592.

Tall, D. O., & Vinner, S. (1981). Concept image and concept definition in mathematics, with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151-169.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*(4), 356-366.

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. In E. Dubinksy, A. Schoenfeld & J. Kaput (Eds.), *Research in collegiate mathematics education IV* (pp. 103-127). Providence, RI: American Mathematical Society.

Zandieh, M., & Knapp, J. (2006). Exploring the role of metonymy in mathematical understanding and reasoning: The concept of derivative as an example. *The Journal of Mathematical Behavior, 25*(1), 1-17.

An RME-Based Instructional Sequence For Change Of Basis And Eigentheory

Megan Wawro
Virginia Tech

Michelle Zandieh
Arizona State University

Chris Rasmussen
San Diego State University

Christine Andrews-Larson
Florida State University

Linear algebra is widely viewed as pivotal yet difficult for university students, and hence innovative instructional materials are essential. The goals of this NSF-funded research project include producing: (a) student materials composed of challenging and coherent task sequences that facilitate an inquiry-oriented approach to the teaching and learning of linear algebra; and (b) instructional support materials for implementing the student materials. This poster will highlight the third unit of the IOLA (Inquiry Oriented Linear Algebra) materials that focus on a research-based approach to introducing eigentheory, change of basis, and diagonalization.

Keywords: Linear algebra; eigentheory, change of basis, curriculum design; inquiry- oriented instructional materials

Linear algebra is widely viewed as pivotal yet difficult for university students, and hence innovative instructional materials are essential. The goals of this NSF-funded research project include producing: (a) student materials composed of challenging and coherent task sequences that facilitate an inquiry-oriented approach to the teaching and learning of linear algebra; and (b) instructional support materials for implementing the student materials. This poster will highlight the third unit of the IOLA (Inquiry Oriented Linear Algebra) materials that focus on a research-based approach to introducing eigentheory, change of basis, and diagonalization.

Prior Work and Theoretical Framing

Our current research program builds from a previously NSF-funded project focused on student learning of basic ideas in linear algebra as students transitioned from intuitive to more formal ways of reasoning. Through conducting interviews and watching classroom video data, we analyzed and reported extensively on student thinking about particular mathematical ideas (e.g., Larson & Zandieh, 2013; Wawro & Plaxco, 2013; Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012). This design research consisted of a cyclical process of ongoing analysis of student reasoning and simultaneous task design and conjecture modification regarding the possible paths that students' learning might take (Wawro, Rasmussen, Zandieh, & Larson, 2013; Cobb, 2000; Gravemeijer, 1994).

Our theoretical framework for designing instructional materials draws on heuristics of Realistic Mathematics Education (summarized by Cobb, 2011). First, a task sequence should be based on experientially real starting points. Second, the task sequence should be designed to support students in making progress toward a set of associated mathematical learning goals. Third, classroom activity should be structured so as to support students in developing models-of their mathematical activity that can then be used as models-for subsequent mathematical activity. Finally, with instructor guidance, students' activity evolves toward the reinvention of formal notions and ways of reasoning about the mathematics initially investigated.

Purpose of the Poster

Initial versions of the sequence were used in classroom teaching experiments in 2009-2010, during which we collected written and video data of small group and whole class discussions. We present an innovative instructional sequence for an introductory linear algebra course that supports students' reinvention of change of basis, eigentheory, and how they are related through diagonalization. Task 1 builds from students' experience with linear transformations in $\mathbb{R}^2$ to introduce them to the idea of stretch factors and stretch directions and how these create a nonstandard coordinate system for $\mathbb{R}^2$. In Task 2 students create matrices that convert between the standard and non-standard coordinate systems and work toward reinventing the equation $PDP^{-1}\boldsymbol{x} = \boldsymbol{Ax}$. In Tasks 3 and 4, students build from their experience with stretch factors and directions to create for themselves ways to determine eigenvalues and eigenvectors, to develop the characteristic equation as a solution technique, and to connect ideas about eigentheory to their earlier work with change of basis through the idea of diagonalization. We will share information about our project website which contains instructor resources such as examples of student thinking, implementation notes, and homework suggestions for this task sequence.

References

Cobb, P. (2000). Conducting teaching experiments in collaboration with teachers. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 307–330). Mahwah, NJ: Lawrence Erlbaum Associates.

Cobb. P. (2011). Learning from distributed theories of intelligence. In E. Yackel, K. Gravemeijer, & A. Sfard (Eds.), *A journey into mathematics education research: Insights from the work of Paul Cobb* (pp. 85-105). New York: Springer.

Gravemeijer, K. (1994). Educational development and developmental research. *Journal for Research in Mathematics Education, 25(*5), 443-471.

Larson, C. & Zandieh, M. (2013). Three interpretations of the matrix equation Ax=b. *For the Learning of Mathematics, 33*(2)*,* 11-17.

Wawro, M., Larson, C., Zandieh, M., & Rasmussen, C. (2012). A hypothetical collective progression for conceptualizing matrices as linear transformations. In S. Brown, S. Larsen, K. Marrongelle, and M. Oehrtman (Eds.), *Proceedings of the 15th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1-465 – 1-479), Portland, OR.

Wawro, M., & Plaxco, P. (2013). Utilizing types of mathematical activities to facilitate characterizing student understanding of span and linear independence. In S. Brown, G. Karakok, K. H. Roh, and M. Oehrtman (Eds.), *Proceedings of the 16th Annual Conference on Research in Undergraduate Mathematics Education, Volume I* (pp. 1-15), Denver, Colorado.

Wawro, M., Rasmussen, C., Zandieh, M., & Larson, C. (2013). Design research within undergraduate mathematics education: An example from introductory linear algebra. In T. Plomp, & N. Nieveen (Eds.), *Educational design research – Part B: Illustrative cases* (pp. 905-925). Enschede, the Netherlands: SLO.

Wawro, M., Rasmussen, C., Zandieh, M., Sweeney, G., & Larson, C. (2012). An inquiry-oriented approach to span and linear independence: The case of the magic carpet ride sequence. *PRIMUS, 22*(8), 577-599.

The simple life: An exploration of student reasoning in verifying trigonometric identities

Ben Wescoatt
Valdosta State University

Reasoning used by students as they verify trigonometric identities has not been investigated. Through analyzing students' spoken explanations of their thought processes in clinical interviews, this current study explores why students choose certain expressions to begin manipulating and why certain manipulations or substitutions are performed as they verify the purported identity. Preliminary findings suggest that students may use beliefs about mathematics to inform and monitor their decisions, namely, the belief that mathematical answers are results of a simplification process. Thus, students spoke of the flow of verification problems in terms of a simplifying process. This belief may be imported from a cultural belief that being simpler is better and more desired; thus, mathematical tasks could validate the imported belief and in turn strengthen the belief about mathematics as simplification. To support this possibility, this paper will share student comments. Implications for instructional practice will also be suggested.

Key words: Trigonometric identities, Problem solving, Mathematical reasoning, Simplifying

Explorations into student understanding of trigonometric concepts remain sparse, yet, the field is rich with areas for exploration. Specifically, the concept of verifying trigonometric identities allows for investigations of students' algebraic reasoning applied to more advanced structures, trigonometric functions, and students' nascent concepts of what it means to mathematically prove something to be true or false. The intent of this current study is to continue an ongoing exploration into how students verify trigonometric identities, focusing on the verification process as an instance of problem-solving.

In order to verify a trigonometric identity, an individual uses a series of steps to demonstrate that one expression is equivalent to another expression under a suitable domain. At each step, the individual replaces one expression with a more suitable equivalent expression. The first decision typically made is to choose which expression to begin manipulating. Generally, as individuals learn how to verify identities, advice is provided to choose the "most complicated" side. The specific goal of this study is to explore the reasoning individuals use when determining which expression to begin manipulating and which following substitutions to perform and to better understand the meaning individuals attach to the advice given to choose the most "complicated side."

Related Literature

In substituting equivalent expressions at each step of the verification process, individuals either use a known identity or they develop an equivalent expression through algebraic manipulation. Delice (2002) investigated these steps as students manipulated trigonometric expressions. He found that students followed a certain pattern as they "simplified" the given expression. The first stage was the "recognition" stage. They began by reading the problem. Next, they recognized a certain visual form, or cue (Mamona-Downs & Downs, 2005), within the expression to be simplified. Finally, they recalled the particular identity, a known identity or one created through algebraic manipulation, that they could use. As suggested in Delice and Roper (2005), students may have relied on their knowledge of manipulating form-equivalent algebraic expressions, a structure sense for trigonometric expressions, in order to successfully manipulate the trigonometric expression. Once they recognized what to do, they transitioned to the "doing" stage. The students rewrote the expression by performing the proper substitution. Then, they cycled back through the recognize, recall, and rewrite phases until they believed they had fully "simplified" the given expression.

While the tasks in Delice's study requested that students "simplify" the given expression, Delice admitted that what the "simplest" expression was to be was not always apparent. Wescoatt (2014) suggested that when students spoke of simplifying trigonometric suggestions, they implied "a process of taking an expression to its most basic state in order to reduce the perceived size (physical or cognitive) of the expression" (p. 6). Viewing the simplification process in this way conforms to beliefs students may hold about mathematics and tendencies students display while working with mathematical expressions. For example, many studies have explored the difficulties students have in accepting a "lack of closure" for an expression. In an exploration of teachers' awareness of such a tendencies (Tirosh, Even, & Robinson, 1998), one of the teacher participants commented, "Students tend to make it as simple as possible. They tend to 'finish' it [the expression]" (p. 56).

Theoretical Framework

The manipulative actions that students performed while verifying the trigonometric identities were viewed through the lens of the Multidimensional Problem-Solving (MPS) Framework (Carlson & Bloom, 2005). The MPS Framework characterizes problem solving in terms of four behavioral phases: orientation, planning, executing, and checking. The framework also describes four attributes of problem solving (resources, heuristics, affect, and monitoring) and explains their roles during the phases. Thus, the decisions students make while manipulating the trigonometric expressions were assumed in part to be guided by conceptual knowledge, problem-solving strategies, and mathematical beliefs.

Methodology

The data for this study were collected from a college trigonometry course at a large research university as part of a larger case study of a class unit on verifying trigonometric identities. Thirty-three students participated, responding to prompts involving verifying identities and solving verification problems. Of these thirty-three students, eight agreed to participate in individual task-based interviews. Each interviewee solved verification problems while speaking aloud his or her thought processes. The audio from the interviews was captured and transcribed.

The transcripts were analyzed using an open-coding process. Each transcript was read with an emphasis on analyzing participants' reasoning as they began the verification problem and then the reasoning in subsequent actions. From this initial analysis, an initial framework of student reasoning evolved. Then, each interview underwent an in-depth analysis through the lens of the evolving framework. As needed, the framework was adjusted to better capture the essence of the student reasoning. This analysis is currently ongoing; interviews will continually be reanalyzed through the framework, and this cycle will continue to improve the explanatory power of the framework.

Preliminary Results

In order to describe how student verified identities, a discussion of the "flow" of a verification should occur. Students describe the verification process using "breaking down" metaphors. Helen stated, "Well, you're breaking, you're either breaking it down or condensing it to something smaller," while Alan commented, "You're just trying to, uh, strip apart, piece by piece, to get to where they're equal." For the problems to flow in this manner was a natural thing for students to believe. As Alan clarified, "It's easier to condense things down than to try to build them up." Amber concurred, saying, "Normally it's easier to go to, from complicated to simple than simple to complicated."

The flow of verification in a sense controlled the actions that students took. To begin with, it directed which expression they should start manipulating. Alan furthered his discussion of the flow of verification. "That's why I start with the most complicated side to begin with," he stated. "So you're kind of taking off the layers I guess you could say, unraveling them, to eventually get to its, to the more s-, to the simpler of the two sides." So, choosing the perceived more complicated side, related to the structure, went beyond just blindly following advice given to him by an authority; Alan formulated his own interpretation and meaning for the strategy.

Beyond the picking of the initial expression, the flow still exerted an influence. Some students used it to monitor their actions. For example, students were asked to write an expression equivalent to $\sin(2x)/\sin x$. After Cooper wrote $2\cos x$, he was asked if any other equivalent expressions existed. Cooper responded, "I'm just kind of leaving it as is. And it's in a pretty simplified form." In another problem, Maria had to choose which identity for $\cos(2x)$ to utilize; she explained, "I just try to find the one that looks like it could, um, help me simplify it, I guess. I mean, just the one that has the terms closest to the ones I already have, um, to try to condense it a little bit." Amber described her verification process in a blunt manner: "If I can simplify stuff, I'll simplify it. And then I'll keep simplifying it until I can't do anything more with it." Thus, students appeared to use their beliefs about how verification problems should flow in order to monitor their progress.

For many students, the flow of the problem matched their beliefs about mathematics. Present in many of the students' explanations was the notion of "simplify." Students explained this phenomenon as a natural result of math. Explaining her process, Bella said, "I do it like, you start with the big equation, you get it smaller and smaller and smaller. … That's just math in general." This belief about mathematics could be reinforced by the types of problems students encounter, as Helen explained, "That's what we're taught to do is something times something, you know, a small number is outside of a big parentheses number, you automatically distribute it. So that seems like it breaks it down and makes it simpler."

Discussion and Implications

Overall, students' action while verifying identities appear to be influenced by beliefs about how the problems should flow which in turn may have been influenced by their general beliefs about what mathematics problems are supposed to do. Thus, students chose the most complicated expression to begin manipulating and used actions that "simplified" it since that is how math supposedly works. If the students truly have these beliefs, then one may ask questions as to how they came by these beliefs and if these beliefs are acceptable.

Students offered clues about the origins. In explaining her desire to work with simple expressions, Amber quoted, "Keep it simple, stupid." Cooper shared a similar sentiment when questioned about simplifying expressions: "Uh, from being a kid I think, you know. From sitting there and looking at it and going, why can't life just be mud puddles? … I think it's something that's just, uh, bred into people. I think they're born with it honestly." Thus, perhaps students enter the mathematical culture with either natural or nurtured views concerning how life should be. Then, as early mathematics problems begin with procedural, condensing tasks (3 is found to be the simpler version of $1 + 2$), the beliefs about life are found to be true in the mathematical culture as well.

These beliefs could be helpful. A typical strategy in solving problems is to find a simpler problem to solve. In mathematics, many times this is achieved by representing the problem with a simpler, equivalent representation. However, these beliefs could be harmful, stunting the learning opportunities for students. For example, rather than recognizing that in verification, expressions are being replaced with *equivalent* expressions, a conceptual issue, students focus on a procedure aspect, simplification. Additionally, the feeling that the answer should be simplified could relate to the process-object duality inherent in expressions, with students being stuck in the lower-level process conception. Furthermore, if students view mathematics in terms of simplification, does this belief of the flow of a problem affect other processes that counter the flow, such as a partial fraction decomposition (e.g., Herscovics & Linchevski, 1994)?

If this belief about simplification exists, as it appears to, how could negative affects be attenuated? One suggestion would be to banish overuse of the word "simplify" and just say what is meant. A typical textbook homework problem starts off by saying "Simplify the following expressions." Saying "simplify" is lazy and plays to and reinforces the trope that the simpler, the "better", meaning, the sought after "correct" answer is the simpler form. Instead, questions should be phrased to emphasize conceptual relations: "For each given expression, find an equivalent expression by," and then list the particular results desired, e.g., "no negative exponents."

1. What theories or studies could better situate the results in the literature?
2. How does the context of a problem influence student actions and reasoning?

References

Carlson, M. P., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational Studies in Mathematics, 58*(1), 45-75.

Delice, A. (2002). A model of students' simplification of trigonometric expressions. In S. Goodchild (Ed.), *Proceedings of the British Society for Research into Learning Mathematics, 22*(1&2), 19-24.

Delice, A., & Roper, T. (2006). Implications of a comparative study for mathematics education in the English education system. *Teaching Mathematics and Its Applications: an International Journal of the IMA, 25*(2), 64-72.

Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics, 27*(1), 59-78.

Mamona-Downs, J. & Mamona, M. (2005). The identity of problem solving. *The Journal of Mathematical Behavior, 24*(3-4), 385-401.

Tirosh, D., Even, R., & Robinson, N. (1998). Simplifying algebraic expressions: Teacher awareness and teaching approaches. *Educational Studies in Mathematics, 35*(1), 51-64.

Wescoatt, B. (2014). What is simplifying?: Using word clouds as a research tool. In (Eds.) T. Fukawa-Connelly, G. Karakok, K. Keene, and M. Zandieh, *Proceedings of the 17th Annual Conference on Research in Undergraduate Mathematics Education*, *2014, Denver, Colorado*, (pp. 1159-1165). Retrieved from http://sigmaa.maa.org/rume/RUME17.pdf

Painter's paradox: Epistemological and didactical obstacle

Chanakya Wijeratne
Simon Fraser University

Rina Zazkis
Simon Fraser University

In mathematics education research paradoxes of infinity have been used in the investigation of students' conceptions of infinity. We analyze one such paradox – the Painter's Paradox – and examine the struggles of a group of Calculus students in an attempt to resolve it. This study shows that contextual considerations hinder students' ability to resolve the paradox mathematically. We suggest that the conventional approach to introducing area and volume concepts in Calculus presents a didactical obstacle. A possible alternative is considered.

Keywords – Infinity, Paradoxes, Cognitive conflict, Gabriel's horn, Calculus

In mathematics education research paradoxes have been used as a lens on student learning. (e.g., Movshovitz-Hadar & Hadass, 1990). In particular, paradoxes of infinity have attracted attention of mathematics education researchers and played an important role in investigating students' conception of infinity (e.g., Mamolo & Zazkis, 2008, Núñez, 1994). We extend this research by attending to a particular paradox, the Painter's Paradox, which was not yet examined in mathematics education research. Painter's Paradox is different from other paradoxes of infinity used in mathematics education research to explore conceptions of infinity, as it does not involve infinite subdivision of space or time.

Gabriel horn and Painter's Paradox

The surface of revolution formed by rotating the curve $y = \frac{1}{x}$ for $x \geq 1$ about the x-axis is known as the Gabriel's horn (Stewart, 2012), though the origin of the name is unclear. This surface and the resulting solid were discovered and studied by Evangelista Torricelli in 1641. Torricelli showed that a certain solid of infinite length, now known as the Gabriel's horn, has a finite volume. Torricelli's infinitely long solid gave rise to epistemological and ontological issues at the time of its discovery. It provided a non-trivial knowledge about infinity and generated lengthy debates between mathematicians and philosophers of the 17th century (Mancosu & Vailati, 1991)

Torricelli's solid raised the issue of the ontological status of geometrical entities and stretched some of the basic intuitive geometrical notions. In addition, it is exploited in the Painter's Paradox given below:

> *The inner surface of the Gabriel's horn is infinite; therefore an infinite amount of paint is needed to paint the inner surface. But the volume of the horn is finite (π), so the inner surface can be painted by pouring a π amount of paint into the horn and then emptying it.*

In Calculus textbooks the paradox is mentioned occasionally, but the resolution is not elaborated upon explicitly. We were interested in how Calculus students react to the situation.

Generalized area and volume in Painter's Paradox

In Calculus textbooks the definite integral is generalized to include integration over infinite intervals and of unbounded integrands. These integrals are called improper integrals. However, the textbooks usually do not explain that this kind of volume calculation through an improper integral generalizes the volume concept to an infinite object. In fact, in calculating the area by a definite integral the underlying idea is the sum of infinite series in the guise of Riemann sums.

Infinite series are often discussed in a separate section, with no apparent connection to area and volume calculations. Such connection is introduced in the integral test, which links convergence of improper integrals of certain functions over infinite intervals with convergence of infinite series. The integral test can be stated as: $\int_N^{\infty} f(x)dx$ is convergent if and only if $\sum_{n=N}^{\infty} a_n$ is convergent, where $a_n = f(n)$. In Gabriel's horn the surface area is associated with the divergent infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$, and the volume is associated with the convergent infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. This contradicts the intuitive expectation that for a horn generated by rotating a curve around the x-axis has either infinite surface area and infinite volume or has finite surface area and finite volume.

In summary, the Painter's Paradox is based on the fact that Gabriel's horn has infinite surface area and finite volume. The paradox emerges when we attribute finite contextual interpretations of area and volume to an intangible object of Gabriel's horn. Linking the volume and the area of Gabriel's horn to infinite series may settle the paradox, or at least connect the perceived paradox to the counterintuitive fact that some infinite increasing series are converging, while others are diverging.

THE STUDY

We begin this section by introducing the participants, their background relevant to our study, and the Task. We then introduce theoretical considerations that guided our analysis and present our research questions. The subsequent data analysis is structured according to the themes identified in the data.

Participants, Setting and Task

Participants in our study were 12 undergraduate students enrolled in a Calculus course. At the time of the study they were familiar with integral calculus techniques in calculating volumes and surface areas of surfaces of revolution, including unbound regions, like Gabriel's horn.

The participants were presented the Painter's Paradox with detailed mathematical justifications of computing the volume and surface area of the Gabriel's horn and showing that its volume is finite while the surface area is infinite. The Task, in which participants were asked to respond to the paradox, is presented in Figure 1. Shortly after completing the Task the participants were interviewed by the first author. The interviews were aimed at probing the participants' written responses and seeking additional articulation of their explanations. The interviews were audio recorded and transcribed.

Theoretical considerations

We rely on several theoretical frameworks in our data analysis – epistemological obstacles by Brousseau (1983), and platonic and contextual distinction by Chernoff (2011).

Epistemological obstacles. Brousseau's theoretical construct of epistemological obstacles is based on the assumption that knowledge is an optimal solution in a system of constraints. In his view, knowledge is a solution to a problem independent of the solver. He characterized epistemological obstacles as “those [obstacles] that cannot and should not be avoided, precisely because of their constitutive role in the knowledge aimed at. One can recognize them in the history of the concepts themselves” (Brousseau, as quoted in Radford, Boero, & Vasco, 2000, p. 163). Brousseau (1983) classified sources behind students’ recurrent and non-aleatorical mistakes in learning mathematics as follows:

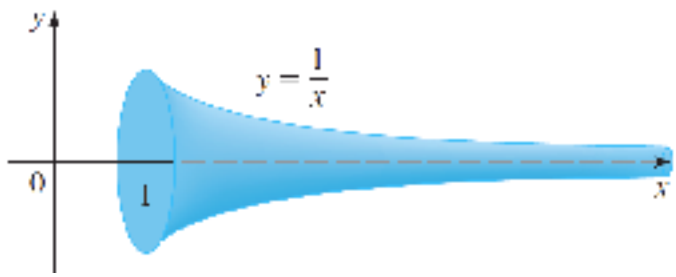
(1) an ontogenetic source (related to the students' own cognitive capacities);
(2) a didactic source (related to the teaching choices);
(3) an epistemological source (related to the knowledge itself).

So, epistemological obstacles arise from the third source. Brousseau suggests that they can be detected through a confrontation of the history of mathematics and today’s students’ learning mistakes. We consider the area-volume relationship of the Gabriel’s horn as an epistemological obstacle, as it created a considerable debate among mathematicians at the time.

Gabriel’s Horn Task

Look at the following problem in Calculus by James Stewart.

If the region $\{(x, y) \mid x \geq 1, o \leq y \leq 1/x\}$ is rotated around x-axis, the volume of the resulting solid is finite. Show that the surface area is infinite. (The surface is shown in the figure and is known as Gabriel’s horn.)



The volume of the solid is given by

$$\lim_{a\to\infty}\int_1^a \pi(\tfrac{1}{x})^2 dx = \lim_{a\to\infty}(\pi\int_1^a \tfrac{1}{x^2}dx) = \lim_{a\to\infty}(\pi\left[-\tfrac{1}{x}\right]_1^a) = \lim_{a\to\infty}(\pi\left[1-\tfrac{1}{a}\right]) = \pi .$$

And the surface area is given by $\lim_{a\to\infty}\int_1^a 2\pi(\tfrac{1}{x})(\sqrt{1+(-\tfrac{1}{x^2})^2}\,dx$.

But $\lim_{a\to\infty}\int_1^a 2\pi(\tfrac{1}{x})(\sqrt{1+(-\tfrac{1}{x^2})^2}\,dx > \lim_{a\to\infty}\int_1^a 2\pi(\tfrac{1}{x})dx = \lim_{a\to\infty} 2\pi\left[\ln x\right]_1^a = \lim_{a\to\infty} 2\pi\left[\ln a - \ln 1\right] = \infty.$

So the volume of the Gabriel’s horn is finite but its surface area is infinite. So to paint the inner surface of the horn we need an infinite amount of paint. But we could pour a π amount of paint into the horn and then empty the horn so that the inner side is painted. What do you think of the paradoxical situation here? Please write down your thoughts. Please note that the above calculations of volume and surface area of the horn are correct.

Figure 1: Gabriel’s Horn Task

Contextualization. We also rely on the theoretical constructs introduced by Chernoff (2011) in distinguishing between platonic and contextualized situations or objects. Chernoff distinguished between platonic and contextualized sequences in the relative likelihood tasks in probability. A platonic sequence is characterized by its idealism. “For example, a sequence of coin flips derived from an ideal experiment (where an infinitely thin coin, which has the same probability of success as failure, is tossed repeatedly in perfect, independent, identical trials) would represent a platonic sequence” (p. 4). But, a contextualized sequence is characterized by its pragmatism. For example, “the sequence of six numbers obtained when buying a lottery ticket (e.g., 4, 8, 15, 16, 23, 42)” (p. 4) would represent a contextualized sequence.

Gabriel’s horn is a platonic object. It is formed by rotating a breathless and infinitely long curve. But Painter’s Paradox is presented in a ‘realistic’ context and its resolution requires de-contextualization from the physical reality, which is assuming that the paint can reach every part of the Gabriel’s horn and time is not a factor.

Research Questions. Our study explores the specific challenges faced by students in resolving the Painter’s Paradox. We address the following interrelated research questions:

> *How do undergraduate calculus students attempt to resolve the Painter’s Paradox? What challenges do they face? What mathematical and what contextual considerations do students rely on in dealing with the challenges?*

Data analysis

We examined all students’ written responses to the task as well as their further explanations, elaborations or clarifications in the interviews. Several interwoven themes were identified; their frequency of occurrence is presented in Table 1. Note that more than one theme was present is some of the responses.

Table 1: Themes is participants’ responses

Theme	**Subthemes**	**# of Responses**	
Epistemological obstacles		6	
Inadequacy of mathematics	Calculus will be developed further	1	3
	Paradoxes are part of mathematics	2	
Contextualization	Horn cannot be filled in a finite time	3	8
	Paint will get stuck	4	
	Cannot paint as the horn cannot be seen	1	

We now present analysis of the data according to the above themes.

<u>Epistemological obstacles</u>. In identifying the theme of epistemological obstacles theme we looked for any firmly held notions that indicate that the infinitely long Gabriel’s horn should have infinite volume. Half of the students had trouble dealing with Gabriel’s horn having a finite volume. Like the seventeenth century mathematicians and philosophers they reacted in disbelief. For example, Sean wrote:

> *if it is infinitely long it’s going to have infinite volume. So I don’t know how we are getting a finite volume.*

Kevin’s resistance to accepting Gabriel’s horn having finite volume is captured in the following interview excerpt:

Interviewer: Why is it paradoxical?

Kevin: Because we can't imagine this geometrical figure it does not make sense to us […] I can not imagine this situation where I keep adding this area and this area gets very very large but my volume stays the same.

Kevin seems to think that if you keep adding to something it has to get very large. In fact, he has difficulty in seeing that a positive increasing infinite series can converge.

Inadequacy of mathematics. Rather surprisingly, three participants treated this paradoxical situation as a consequence of the inadequacy of mathematics. Kevin expressed a belief that in a "more complicated" Calculus we will not have this paradox:

> *I feel like this part of Calculus is not perfected yet. I mean, people always tend to think that their contemporary science is very good, but then new generations of scientists prove them wrong. […] I believe that in future someone can come up with a new more complicated Calculus, that will not have such paradoxes in it.*

Unlike Kevin, Bryan was not seeking a different Calculus, but rather referred to a similar situation within mathematics. Bryan's response seems to suggest that the paradox is in the mathematics itself, as a similar paradoxical situation exists in infinite series. Bryan wrote that "*the paradoxical situation is correct as defined by the solutions*" and that it reminded him of the paradoxical situation that arises in the summation of an infinite geometric series:

> *a geometrically infinite series never ends in the amount of terms in the series, however, the sum of all of all these terms is a finite number.*

We note that these ideas resonate with Wallis's, who shared Leibniz' opinion, that Torricelli's solid having a finite volume was as surprising as the infinite series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+...$ being equal to 1.

Contextualization. The paradox itself is presented in a contextual setting of painting the horn. However, several responses indicate further contextualization of the paradox, that is, adding contextual consideration that are not present in the description of the paradox. Three students suggested that the horn cannot be filled in a finite time. David wrote and further reiterated that *"you can never fill the horn with paint since the horn is ∞ long. Time must be a factor"*. The time factor is an additional contextualization to demonstrate that the set task is impossible to carry out in reality.

Alysa suggested that the horn cannot be painted *"because paint molecules will get stuck in the horn"*. And while Alysa referred to molecules in her explanation, Peter mentioned "atomic level". For Peter, the horn cannot be painted because at the atomic level you cannot see the surface:

> *that surface.., when it goes to that atomic level…,you can't see it anymore.., how can you paint something you can't see? […] It goes smaller than the size of some microscopic.., we can't see it.., we can't paint it!*

These responses highlight the difficulty in decontextualizing the Painter's Paradox.

A possible resolution? Bruce was the only participant who wrote about the connection between area and volume, and Riemann sums. He noted that the area and volume are defined using Riemann sums. To say that the volume is finite, he wrote, is to say that the [Riemann] sum converges. And to say that the surface area is infinite is to say that the [Riemann] sum diverges. His thinking is captured well in the following excerpt:

> *But not all Riemann sums are well behaved – some diverge and our intuition stops us from accepting so called paradoxes, like Gabriel's Horn, as reality in the world of abstract mathematics.*

Bruce's written response is in accord with mathematical decontextualized resolution of the Painter's Paradox presented above. During the interview, when asked to elaborate on his ideas, Bruce took a different approach. He suggested that there could be two kinds of paint, one that occupies volume and one that covers surface area. This was his apparent attempt to achieve an equilibrium focusing on the dimensional difference between the surface area and volume.

Bruce considered the paradoxical finite-volume-infinite-surface area situation by connecting it to something tangible:

> *Some finite things can occupy infinite surface area, like let's say I have a cube let's say I squash with some kind of really strong plate, so that no matter how thick it is it can always become thinner then the surface area… Yeah, it would be finite volume but it would spread over an infinite surface area if you think of painting that way [...] what is thickness of an area it would have to be infinitely thin…*

This new realization and his reference to "infinitely thin paint" is in accord with an additional resolution, suggested in Gethner (2005). Gethner (2005) explains that the Painter's Paradox is based on the "realistic" consideration that the thickness of paint is the same on the whole painted area, as is reasonable to assume in painting a wall. The alternative resolution does not reject contextualization, but includes an imaginary context, a "mathematical" paint of infinitely thin layers. The inner surface of the Gabriel's horn can be painted with π amount of paint (or less) if the thickness of the paint layer at x is smaller than or equal to $\frac{1}{x}$. With this assumption the paint, when poured into the Gabriel's horn, can reach every point of the infinite surface area.

Discussion and Conclusion

Painter's Paradox, as well as any other mathematical paradox, presents a cognitive conflict to a learner. In what follows we summarize and discuss the participants' reactions to the paradox in their attempt to resolve the cognitive conflict. We then suggest a possible emphasis in Calculus instruction that could deepen students' understanding of frequently used approaches to determine surface area and volume of objects resulting from rotating graphs of functions.

Cognitive conflict and different ways to deal with tension

Movshovitz-Hadar and Hadass (1991) indicate that "as long as a person can not resolve a paradox, he or she is in a state of cognitive conflict" (p. 80). All of the participants seemed to be experiencing a cognitive conflict dealing with the Painter's Paradox. They faced the seemingly paradoxical claims that an infinitely long solid has a finite volume and an infinite surface area, and the counterintuitive nature of this realization was reinforced by the story of paint.

Movshovitz-Hadar and Hadass (1991) point out that the state of cognitive conflict and the tension it creates stimulates an attempt to get out of it and achieve a new equilibrium with a more advanced mental structure. While all students attempted to achieve equilibrium, their chosen approaches to deal with the situation differed. Some participants referred to the impossibility of the presented situation and in such echoed epistemological obstacles experiences by mathematicians of the 17th century. Others acknowledged the paradox, but accepted it as a part of "not yet fully developed mathematics". A majority of participants added contextual considerations, suggesting that it was impossible to paint the horn due to time limits, thickness of paint, or the painter's ability to see the object. So rather than resolving the paradox mathematically, these participants rejected the viability of the presented "story". Bruce appeared to be the only participant who generated ideas in accord with decontextualized resolution that attends to convergent and divergent series, as well as with the alternative resolution that introduces imaginary-magic context of infinitely thin paint applied at not uniform paint layers (thickness of the paint layer at x is less than or equal to $\frac{1}{x}$) .

De-contextualization of Painter's Paradox seemed to be difficult as the majority of participants found various ways of contextualizing the paradox further and in such avoiding its resolution. But the ability to consider something platonically is an important skill to acquire in mathematics. In fact, Mamolo and Zazkis (2008) in their study of several paradoxes of infinity argued for an instructional approach that helps students separate their realistic and intuitive considerations from conventional mathematical ones.

Obstacle, not only epistemological

Associating a finite attribute with something infinite, as in the case of Gabriel's horn, is clearly an epistemological obstacle as the difficulty of our participants replicates the development of the mathematical knowledge through history. But this also can be seen as a didactic obstacle, as Calculus textbooks – and consequently many instructors – do not emphasize that calculating areas and volumes of unbounded regions through improper integrals generalizes the concepts of area and volume to unbounded regions. The connection of integral calculations to converging series is a key to understanding why Gabriel's' horn has a finite volume. Explicit didactic attention to the connection between the generalizations of area and volume and infinite series, converging and diverging, is essential for students to be able to strive for and derive the paradox resolution, rather than accept paradoxes as inadequacy of mathematics.

References

Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics*. Dordrecht, Netherlands: Kluwer.

Chernoff, E. (2011). Investigating relative likelihood comparisons of multinomial contextual sequences. *Proceedings of CERME 7*.

Gethner, R. M. (2005). Can you paint a can of paint?. *College Mathematics Journal*, *36*(5), 400-402.

Mamolo, A. & Zazkis, R. (2008). Paradoxes as a window to infinity. *Research in Mathematics Education, 10*(2), 167-182.

Mancosu, P., & Vailati, E. (1991). Torricelli's infinitely long solid and its philosophical reception in the seventeenth century. *Isis*, *82*(1), 50-70.

Movshovitz-Hadar, N., and Hadass, R. (1990). Preservice education of math teachers using paradoxes. *Educational Studies in Mathematics, 21*, 265 87.

Núñez, R. (1994). Cognitive development and infinity in the small: paradoxes and consensus. In Proceedings of the Sixteenth Annual Conference of the Cognitive Science Society (pp. 670-674). Hillsdale, NJ: Lawrence Erlbaum Associates.

Radford, L., Boero, P., & Vasco, C. (2000). Epistemological assumptions framing interpretations of students understanding of mathematics. In Fauvel, J., Maanen, J. A., & van Maanen, J. A. (Eds.), *History in mathematics education: An ICMI study*. (pp. 162-167). Dordrecht, Netherlands: Kluwer Academic Publishers.

Stewart, J. (2011). *Single variable calculus: Early transcendentals*. Brooks/Cole Publishing Company.

An exploration of students' conceptions of rational functions while working in a CAS-enriched dynamic environment

Derek Williams
North Carolina State University

Studying families of functions is important for developing an understanding of the concept of function. This study utilized a computer algebra system (CAS) learning environment as it explored how students conceptualize rational functions. Using APOS theory as a theoretical framework, student artifacts from video, audio, and screen recordings were coded as either: action, process, or object conceptions of rational functions. The results of this study describe how students portray different conceptions of rational functions. This list is not exhaustive, but provides a starting point for identifying how students perceive of rational functions in a technological environment. Developing an understanding of how students conceptualize rational functions could lead to useful implications for teaching.

Keywords: Rational functions; APOS; conceptions; functions; technology

The purpose for this study was to begin gathering information about how community college students understand rational functions when technology is an integral part of the learning experience. Research on technology and computer algebra systems (CAS) has indicated that learning environments with dynamic representations and CAS foster conceptual understandings without harming skills (Bostic & Pape, 2010; Heid, Blume, Hollebrands, & Piez, 2002). Specifically, the study aimed to identify characteristics of students' conceptions of rational functions as an action, process, or object using APOS theory (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dubinsky & Harel, 1992; Dubinsky, 1991) as a theoretical framework.

To address the research focus above, a group of three community college students participated in four teaching sessions in which they used pre-constructed *GeoGebra* (Hohenwarter, 2002) files to explore properties of rational functions. Students were given a handout to guide their exploration and discussion. Each session lasted roughly one hour ending with an assessment activity which challenged students to formalize what they had discovered during their investigation. Each session was video-recorded and screen capturing software was used to record student interactions with the computer. The APOS theory was used to analyze artifacts obtained from video, audio, and screen recordings, and students' actions, discussions, and justifications were coded as displaying either: action, process, or object conceptions of rational functions.

Results of this study indicate that an action conception of rational functions manifest primarily in three ways: 1) students are very questioning, and rarely make direct statements about their investigation, 2) students rely heavily on actions completed on the computer to complete their thoughts, and 3) students utilize a trial-and-error method of interaction with the computer while making and testing conjectures. As a process, students are able to perform actions on rational functions unassisted by the computer. Students can now mentally account for changes in parameters and they no longer rely on trial-and-error approaches. Finally, as an object, students now consider rational functions as a single entity composed of numerator and denominator. They are able to decompose this object into numerator and denominator and use those as objects to complete processes, and then reconstruct the original function - understanding that the properties described by the numerator and denominator are properties of the entire function.

Results of this study include descriptions of student behaviors that are indicative of action, process, or object understandings of rational functions. However, a larger study of this kind is necessary to corroborate these findings as well as to provide implications for teaching.

References

Bostic, J., & Pape, S. (2010). Examining students' perceptions of two graphing technologies and their impact on problem solving. *The Journal of Computers in Mathematics and Science Teaching*, *29*(2), 139–154.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics*, *23*(3), 247–285.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinkning* (pp. 95–123). Dordrecht, The Netherlands: Kluwer.

Dubinsky, E., & Harel, G. (1992). The nature of the process conception of function. In E. Dubinsky & G. Harel (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 85–106). Washington DC: Mathematical Association of America.

Heid, M. K., Blume, G. W., Hollebrands, K., & Piez, C. (2002). Computer algebra systems in mathematics instruction : Implications from research. *Mathematics Teacher*, *95*(8), 586–591.

Hohenwarter, M. (2002). *GeoGebra: A software system for dynamic geometry and algebra in the plain*. University of Salzburg.

Investigating backward transfer effects in calculus students

Siobahn Young
University of Delaware

One of the most important concepts in calculus is the derivative. Unfortunately, many studies have shown that students have trouble understanding derivatives, possibly due to deficiencies in knowledge about concepts learned prior to calculus. However, few studies have investigated how learning about other calculus topics affects students' understandings of the derivative. In this study, qualitative methods were used to investigate the influence, or backward transfer effect, that learning about integration has on students' prior understandings of derivatives. Semi-structured task-based interviews were conducted with four high school students before and after students received instruction on integration. Interview tasks involved finding derivatives and antiderivatives algebraically and graphically. Results from this study may show that learning about integration is another potential reason for why students have trouble understanding derivatives. Alternately, results may show that it is possible for teachers to help reinforce students' understandings of derivatives through instruction on other calculus topics.

Keywords: Calculus, Backward Transfer, Derivatives, High school students, Qualitative methods

The derivative is an important topic in calculus because the derivative is used to describe rates of change, which in turn is an important topic in almost any field studying science, technology, engineering, or mathematics. Unfortunately, many studies have shown that students have trouble understanding the derivative (e.g. Baker & Cooley, 2000; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Orton, 1983; Selden, Selden, & Mason, 1994; Teuscher & Reys, 2012; Thompson, 1994; Ubuz, 2007). Students may have difficulty understanding the derivative because their knowledge about rates (Teuscher & Reys, 2012; Thompson, 1994), slopes (Teuscher & Reys, 2012), and functions (Asiala, Cottrill, & Dubinsky, 1997; Orton, 1983; Selden et al., 1994) is not well developed. These three concepts are usually learned *before* calculus. However, few studies (if any) have investigated how learning about other *calculus* topics affects students' understandings of the derivative.

The purpose of this project is to investigate the influence that learning about integration has on students' prior understandings of derivatives. This influence is conceptualized as a backward transfer (BT) effect. From the actor-oriented perspective of transfer, Lobato (2008) defines (forward) transfer as "the influence of a learner's prior activities on his or her activity in novel situations" (p.169). Therefore, forward transfer is defined as prior knowledge influencing a learner's way of thinking in a new situation. Hohensee (2014) proposed expanding the actor-oriented transfer perspective to include BT, which is defined as "the influence that constructing…new knowledge has on one's ways of reasoning about…topics that one has encountered previously" (p.4). In other words, BT is defined as how learning something new influences one's prior knowledge. For example, learning about adjectives in a new language, such as Spanish, may influence one's understanding of adjectives in English, since adjectives are placed in a different location in English sentences than in Spanish sentences. This describes a BT effect because Spanish was learned *after* English and the subject's understanding of English adjectives was influenced by the act of learning a new language. Because integration is typically taught after derivatives, this study is situated within a context in which BT effects might be present. The following research question guides this study: In what ways does learning about integration influence the ways in which students reason and think about derivatives?

This study was conducted with four high school calculus students. Semi-structured task-based interviews were conducted after students received two months of instruction on derivatives but before they received instruction on integration. Four tasks were given during the interview. The first two tasks involved algebraic functions (as opposed to graphical functions); the first task asked the students to find the derivative when given an explicit function, while the second task asked students to find the original function when given an explicit derivative function. The last two tasks were graphical tasks and asked students to sketch the derivative function when given a function graphically (without being provided with the explicit equation for the function) and to sketch an original function when given a derivative function. After being introduced to integration, the students will be interviewed again with similar tasks.

In addition, classroom observations are being conducted to be able to relate to the influence that students may experience from instruction on integration. For example, if the participants begin solving a derivative problem during the second interview in a way that seems similar to something they did in class involving integration, I will be able to see some of the ways that instruction has influenced their thinking about derivatives. This will allow me to question and probe them further about what happened during class that they believe affected their way of thinking about derivatives, and I will have some knowledge as to what happened in class to be able to understand their response. From two classroom observations, I noticed that the class atmosphere is open, where students are allowed to openly debate with each other with little interference from the teacher. Unexpectedly, due to this nature of the classroom, the observations are also being used to further explore the ways the participants reason and think about derivatives. The interviews will be analyzed for ways in which students' thinking about derivatives changed after learning about integration.

Results from this study may show that learning about integration is another potential reason for why students have trouble understanding the derivative. Alternately, results may show that it is possible for teachers to help reinforce students' understandings of derivatives through instruction on other topics in calculus, such as integration. Either finding would be important information for teachers and curriculum developers.

References

Asiala, M., Cottrill, J., & Dubinsky, E. (1997). The Development of Students ' Graphical Understanding of the Derivative. *Journal of Mathematical Behavior*, *16*(4), 399–431.

Baker, B., & Cooley, L. (2000). A Calculus Graphing Schema. *Journal for Research in Mathematics Education*, *31*, 557–578.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352–378.

Hohensee, C. (2014). Backward transfer: An investigation of the influence of quadratic functions instruction on students' prior ways of reasoning about linear functions. *Mathematical Thinking and Learning*, *16*(2).

Lobato, J. (2008). Research methods for alternative approaches to transfer: Implications for design experiments. In A. E. Kelly, R. A. Lesh, & J. Y. Baek (Eds.), *Handbook of Design Research Methods in Education: Innovations in Science, Technology, Engineering, and Mathematics Learning and Teaching* (pp. 167–194). New York: Routledge.

Orton, A. (1983). Students' understanding of differentiation. *Educational Studies in Mathematics*, *14*(3), 235–250.

Selden, J., Selden, A., & Mason, A. (1994). Even good calculus students can't solve nonroutine problems. *MAA NOTES*.

Teuscher, D., & Reys, R. (2012). Rate of Change: AP Calculus Students' Understandings and Misconceptions After Completing Different Curricular Paths. *School Science and Mathematics*, *112*(6), 359–376.

Thompson, P. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Learning Mathematics*, *26*(2), 229–274.

Ubuz, B. (2007). Interpreting a graph and constructing its derivative graph: stability and change in students' conceptions. *International Journal of Mathematical Education in Science and Technology*, *38*(5), 609–637.

Code-switching and mathematics assessment: Some anecdotal evidence of persistence of first language

Balarabe Yushau
Preparatory Year Math Program,
King Fahd University of Petroleum and Minerals,
Dhahran 31261, Saudi Arabia

byushau@kfupm.edu.sa

One of the most difficult things for a monolingual teacher to decide is if errors by a student who is acquiring English language reflect a lack of mathematical understanding or some problems with English language. This paper buttresses this point with some anecdotal evidences from mathematical works of Preparatory Year University students in which students might be right, but the chances are high that the teacher, especially monolingual will wrong him. The result shows the dominance of students' first language (Arabic) over the language of teaching and learning (English) while students are doing mathematics. This show how the language conflict can affect students' assessment in mathematics.

Key words: mathematics, Assessment, code-switching, Bilingual Arabs, Preparatory Year Program

Language is what teachers use largely to facilitate learning. Also, students use language to demonstrate knowledge and understanding of mathematical ideas (Bell, 1978). With English becoming the language of science, and a lingua franca in this era of globalization, many of the mathematics students are learning mathematics in their second or third language (Ellerton and Clarkson, 1996). As a result, researchers have highlighted the need for taking language factors into consideration in the mathematics classroom. In particular, the attention of teachers is called upon to consciously take into account a range of linguistic backgrounds in their classrooms. This will facilitate learning as well as give the students the opportunity to effectively participate in mathematics classroom discourse.

Although the role of language proficiency on students performance in mathematics has been investigated (Barton, and Neville-Barton, 2003; Neville-Barton and Barton, 2005; Roardria, 2010; Yushau and Omar, 2015), not much is known on how code-switching can affect the mathematics assessment of bilingual Arabs students.

Motivation for the study

Students answering questions depend completely on their comprehension of the questions. In a way, it depends largely on the students' reading and writing skills. A good number of students cannot do well in examinations simply because they cannot understand the questions. This is more common among bilingual students who are acquiring the language of assessment. Students sometimes have no option but to skip the questions, and in the case of multiple choice questions, students may resort to guessing. As a non Arab teacher, who has been teaching mathematics in English for students who are learning English as a second language, I appreciate how difficult it is for me to decide sometime if the students' wrong answer in an exam script or during classroom discourse, is as result of lack of mathematical understanding or is due to lack of proficiency in the language of instruction. This tension was aptly observed by Secada & Cruz (2000). I have gathered many anecdotal evidences that show the case where the students might be right, but the chances are high that the teacher, especially monolingual will wrong him.

In addition, some students do most of their calculation in Arabic and then translate back into English. This process is not only cumbersome, but students often make silly mistakes due to a conflict between the Arabic and English languages. These calculation errors are often algebraically correct but notationally wrong due to conflict between Arabic and English languages. All these have some effects on the students assessment, and hence the need to be highlighted.

Methodology

Data were collected qualitatively from students' exams scripts, as well as from the classroom discourse where the researcher saw in many occasions the dominance of Arabic in students work while doing mathematical activities using English language.

Some Anecdotal Evidence of Code-Switching specific with Bilingual Arabs

1. *Wrong Reading of Numbers:* Reading numbers from right to left in place of left to right is very common in classroom. For instance, reading 65 as 56.
2. *Confusion in writing Five and Zero:* In Arabic, five is written as English zero, and zero is dot. A students asked why is $\sqrt{0}=0$? Thinking that we are saying square root of five is equal to five. And also the possibility of reading of writing 5.7 as five hundred and fifty seven.
3. *Comma and Dot:* In Arabic, Comma is used as a Dot. Now the confusion between $2,3$ and 2.3. And the possibility of student writing 2.3 and $2,3$?
4. *Distributive law:* $(x-3)-(x+3)=-x+3+x+3=6$; thinking that the middle minus sign is for the first bracket as it is in Arabic.
5. *Exponential Notation and factorization:* $a^2b+ab=ab(b-1)$; thinking that the exponent 2 is for b - as in Arabic.
6. *Graphs related information:* In getting information from graphs some students do commit the mistakes that are clearly due to the influence of their first language. For example, anything that reads from left to right or right to left, often cause confusion for Arabic speaking students. *Interval notation; interval in which a graph is increasing, decreasing, constant; interval in which a graph is continuous; Domain and Range of the function from the graph* are some of the mathematical notions that cause confusion for the students. Some students write them wrongly by inverting the answers.

Implication for the Assessment

The anecdotal evidences highlighted above, are found in many students' mathematical activities at the preparatory year. This certainly has ramification on the students' assessment in mathematics. Therefore, there is the need for the teachers, especially those teaching bilingual to be aware of these language conflict.

Conclusion

In this note, we have highlighted some anecdotal evidences that indicate the persistence of students' first language in the students' cognitive mathematical activities while doing mathematics using a second language other than their mother tongues. Result show how code–switching can affect students' assessment in mathematics. Certainly this is an area that needs further investigation.

Acknowledgments

The authors acknowledge with thank King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia.

References

1. Bell, F. H. (1978). Teaching and Learning Mathematics. Wm C Brown Company Dubuque ,

2. Barton, B. and Neville-Barton, P. (2003). Language issues in undergraduate mathematics: A report of two studies. *New Zealand Journal of Mathematics*, **32** (Supplementary Issue), 19-28.

3. Ellerton, N. F. and Clarkson, P. C., 1996, Language Factors in Mathematics Teaching. In: Bishop, A. J. et al, 1996. International Handbook of Mathematics Education (Kluwer Academic Publishers, Netherlands).

4. Neville-Barton, P. and Barton, B. (2005). The relationship between English language and mathematics learning for non-native speakers. Teaching and Learning Research Initiative final report). Retrieved August.

5. Secada, W. & Cruz, Y., (2000). Teaching Mathematics for Understanding To Bilingual Students. http://eric-web.tc.columbia.edu/ncbe/immigration/mathematics.htm.

6. Yushau B and Omar, M.H. (in press) Performance in Mathematics as it relates to the English Language Proficiency Level of Bilingual Arab University Students.

Challenges and resources of learning mathematics in English for a "mathematically intelligent" student with weak English background

Balarabe Yushau
Preparatory Year Math Program,
King Fahd University of Petroleum and Minerals,
Dhahran 31261, Saudi Arabia

byushau@kfupm.edu.sa

Lack of Proficiency in English is one of the major obstacles in students learning of mathematics in English medium universities in Saudi Arabia. Despite, this, some students are found doing exceptionally good in mathematics examinations conducted in English – a language that the students are not proficient in. In this paper, we share some of the challenges of learning mathematics in English language by this class of students, and the learning resources that help them to overcome the problems.

Key words: Mathematics learning, Bilingual Arabs, Challenges of learning, Resources of learning

Public education in both primary and secondary school in Saudi Arabia is mediated through Arabic language. As a result, most of the high school graduates in Saudi Arabia have limited proficiency in English. It is then not surprising that students admitted into English medium universities face challenges related to the new language of instruction. A good number of these students face difficulties in following the classroom lectures and understanding the textbook both of which are in English language. Some of them find it difficult to ask or respond to questions in classroom due to lack of proficiency in the new language of instruction. Consequently, some of these students tend to get confused, lose confidence and question their intellectual capabilities. Despite, you come across some students that are classified as "very weak" in English, but are performing excellently in mathematics exams that were conducted in English.

In the literature, bilingualism and multilingualism are no more considered as a setback (Cummins, 2000; Clarkson, 2006 Moschkovich, 2010). Rather, studies have shown that students use multiple resources in learning mathematics, and hence bilingualism and multilingualism can be advantage or disadvantage for the cognitive development of the students (Clarkson, 1992). Cummins (1976) speculatively hypothesized that there might be a threshold in which bilingualism or multilingualism can be advantageous or disadvantageous to the cognitive development of a student. However, the Threshold Hypothesis did not tell us the relationship between the students' first language (L1) and second language (L2). Perhaps to address this, Cummins (1979) suggested another hypothesis known as Developmental Interdependence Hypothesis (DIH). The Interdependence Hypothesis proposed that the level of proficiency already achieved by a student in their first language would have an influence on the development of the student's proficiency in their second language (Baker, 2001).

Motivation and methodology of the study

Participants in this study were students classified as very weak in English language, but they turn out to be among the outstanding performers in mathematics examination conducted in English. To shed light on this, series of open ended questions were asked to this class of students. The questions border on the challenges they face due to the language switch from Arabic to English language as well as the resources that help them to overcome the challenges. In response to these questions, participants freely self reported among other things: their challenges in learning mathematics in English, and the resources that help them in the survival of the fittest. The findings in this study are reported below.

Findings

1. *Challenges*

The major challenges faced by this class of students include:

- Language barrier
- Feeling shy for not understanding English language
- Feeling shy to ask questions in class.
- Knowing the answer but not knowing how to say it in English.
- Demanding course contents

Others challenges with minor rating include:

- Adaptation with new environment
- Personal problem
- Boring class

In particular, the linguistic challenges in order of difficulty are:

- Vocabulary (41%)
- Semantics (32%)
- Syntax (17%)
- Mathematical Concepts (10%)
- Mathematical Symbols (none)

2. *Resources*

It was found that one major thing that is common with these students is their strong background in both Arabic and mathematics. On the scale of 10, their average proficiency level in Arabic and mathematics are 8.8 and 8.55, respectively.

Cummins' Interdependence Hypothesis suggests that the students' levels of proficiency already achieved in their first language (Arabic in this case), have an influence on the development of their proficiency in a second language (English in this case). The hypothesis also states that the greater the level of proficiency achieved by the students in their first language will allow for a better transfer of skills to the second language. This might be the major resources this class of students is utilizing to quickly adjust to the new language of instruction.

It is interesting to see that top among the factors that respondents attributed to their success in learning mathematics in English, is their *"commitment to their studies"*. This coincides with the necessary conditions stated by Cummins (1982a) for a proper transfer to take place from L1 to L2, which is "adequate motivation to learn in Ly" (p.29). The other factors mentioned by the respondents that contribute to their success include *"competent instructors and sufficient reading material"*. This is also inline with the necessary condition of the transfer that was stated by Cummins (1982a) as "provided there is adequate exposure to Ly" (p.29). From the response of the students, it appears that this class of students considers teachers as one of the major resources for their success. It is interesting that the respondents consider teachers support as second to their commitments among the top on the list of factors that contribute to their success. It has been noted that many students acquiring English receive little encouragement to speak about their ideas, in part due to the belief that they will find it too difficult to express themselves (Secada & Crux, 2000).

Conclusion

The findings of this study show that despite their impressive performance, this class of students still considers their lack of English proficiency as a major challenge of learning mathematics in the preparatory year. In line with the Interdependence Hypothesis theory, the strong Arabic and mathematics background appears to be the major resource of this class of students. This is followed by their commitment to work, and teachers' supports.

Other class of students should equally be investigated, and coherent pedagogical theories should be developed on how to minimize the challenges of these students as well as foster their learning resources.

Acknowledgments

The authors acknowledge with thank King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia.

References

1. Baker, C. (2006). Foundations of bilingual education and bilingualism (4th ed.). Clevedon ; Buffalo: Multilingual Matters.
2. Clarkson, P C (1992). Language and mathematics: A comparison of bilingual and monolingual students of mathematics Educational Studies in Mathematics, 23(4), 417-430.
3. Clarkson, P. C. (2007). Australian Vietnamese students learning mathematics: High ability bilinguals and their use of their languages. Educational Studies in Mathematics, 64(2), 191-215.
4. Cummins, J. (2000). Language, power and pedagogy: Bilingual children in the crossfire. Clevedon, England: Multilingual Matters.
5. Cummins, James. (1976). The influence of bilingualism on cognitive growth: A synthesis of research findings and explanatory hypotheses. Working Papers on Bilingualism, 9, 1-43.
6. Cummins, James. (1979). Linguistic interdependence and the educational development of bilingual children. Review of Educational R
7. Moschkovich, J. N. (2007d). Using two languages while learning mathematics. Educational Studies in Mathematics, 64(2), 121-144.
8. Moschkovich, J. Editor (2010). Language and Mathematics Education: Multiple Perspectives and Directions for Research. Charlotte, NC: IAP.

   Secada, W. & Cruz, Y., (2000). Teaching Mathematics for Understanding To Bilingual Students.

Solving Linear Systems: Augmented Matrices and the Reconstruction of x

Michelle Zandieh
Arizona State University

Christine Andrews-Larson
Florida State University

The origins of linear algebra lie in efforts to solve systems of linear equations and understand the nature of their solution sets. In our experience, instructors of linear algebra see the work of teaching students to solve linear systems as the more straightforward and procedural portion of the course. We speculate that solving linear systems and interpreting their solution sets in fact entails hidden and significant challenges for students that are important for their later success in linear algebra, as well as their work in related STEM courses. In this paper, we examine final exam data from 69 students in an introductory undergraduate linear algebra course at a large public university in the southwestern US. Our analysis suggests that students are largely successful in representing systems of linear equations using augmented matrices, but that interpreting the row reduced echelon form of these matrices is a common source of difficulty.

Key words: Linear algebra, systems of equations, augmented matrices, student reasoning

The origins of linear algebra lie in efforts to solve systems of linear equations and understand the nature of their solution sets. In our experience, instructors of linear algebra tend to see the work of teaching students to solve linear systems as the more straightforward and procedural portion of the course. We speculate that solving linear systems and interpreting their solution sets in fact entails hidden and significant challenges for students that are important for their later success in linear algebra, as well as their work in related STEM courses. In particular, students will encounter and need to make sense of systems of linear equations that have infinitely many solutions throughout an introductory linear algebra course: for instance when making sense of linearly dependent sets of vectors, when dealing with linear transformations whose null spaces are non-trivial, and when making sense of eigenvectors.

In this paper we focus on the following research question: How are students reasoning when solving systems of equations in which the number of unknowns differs from the number of equations?

Theoretical Framing and Literature

The existence of student struggles in linear algebra is well-documented (e.g., Dreyfus, Hillel, & Sierpinska, 1999; Dorier, Robert, Robinet, & Rogalski, 2000; Harel, 2000; Stewart & Thomas, 2009; Trigueros & Possani, 2013; Larson & Zandieh, 2013). Researchers have speculated that the formalization of ideas such as span, linear independence, null spaces, basis, and eigenvectors is problematic for students for a variety of reasons ranging including their preference for practical rather than theoretical thinking (Dorier & Sierpinska, 2001) and struggles shifting among modes of representation (e.g., Hillel, 2000; Sierpinska, 2000).

More recently, Larson and Zandieh (2013) have developed a framework for making sense of student thinking by identifying three important interpretations of the matrix equation $A\mathbf{x}=\mathbf{b}$ where A is an nxm matrix, $\mathbf{x}$ is in $\mathbf{R}^{\mathbf{m}}$ and $\mathbf{b}$ is in $\mathbf{R}^{\mathbf{n}}$. We especially note how the role of the vector $\mathbf{x}$ shifts across those interpretations. Namely, $A\mathbf{x}=\mathbf{b}$ can be interpreted as a system of equations (where $\mathbf{x}$ is a point of intersection), a linear combination of column vectors (where $\mathbf{x}$ is a set of weights on the column vectors of A), or as a transformation from $\mathbf{R}^{\mathbf{m}}$ to $\mathbf{R}^{\mathbf{n}}$ (where $\mathbf{x}$ is an input vector corresponding to the output vector **b**). This framework has been useful for making sense of ways in which students blend ideas from these three interpretations. In this paper, we work to expand this framework to the context of augmented matrices – where the literal symbol **x** disappears completely from the algebraic representation [A|**b**].

Data Sources and Methods of Analysis

In this work, we aim to identify student reasoning strategies and sources of difficulty when dealing with systems of equations, particularly when the number of equations differs from the number of unknowns. In order to explore this issue, we draw on data taken from the final exams of 69 students enrolled in two introductory linear algebra classes at a large public university in the southwestern United States. Both sections were taught by the same instructor, who was a seasoned linear algebra instructor at the institution. Most students in the class were engineering majors in their junior or senior year of college, and had completed Calculus III prior to enrolling in this linear algebra course. About half of the students in the classes were non-traditional students (e.g. not continuously enrolled in post-secondary education since graduating from high school), about twenty percent of the students spoke a native language other than English, and 8 of the 69 students were female. Course topics included systems of linear equations, span and linear independence, linear transformations, determinants, eigenvectors, eigenvalues, and diagonalization.

In this preliminary report, we examine student responses to two versions of a similar question, asked on two different versions of the final exam. Each question asked students to solve a system of 4 equations (which had a unique solution) and give the geometric interpretation of the solution set corresponding to each. The questions analyzed are shown below in Figure X. Version A can be interpreted as equations of planes (3 unknowns), whereas version B can be interpreted as equations of lines (2 unknowns). Note that 32 students took version A of the exam, and 37 students took version B.

Consider the system of equations below. Show your work. Explain any work that you are using technology for. a. Find the intersection of the following four planes or show that there is no intersection. $x-2y+z=0$ $x+y+z=3$ $-x+2y-z=0$ $3x-y+2z=4$ b. Circle the correct answers: The intersection of the **four** planes is … a point / a line / a plane / a 3-space	Consider the system of equations below. Show your work. Explain any work that you are using technology for. a. Find the intersection of the following four lines or show that there is no intersection. $-2x+y=-2$ $6x-3y=6$ $4x-2y=4$ $x-2y=0$ b. Circle the correct answers: The intersection of the **four** lines is … a point / a line / a plane / no intersection
Exam Version A	Exam Version B

Figure 1: Exam questions to be analyzed

In our analysis, we first examine the number of students who got each item correct. We then categorize student approaches, which tended to take either a systems approach or an augmented matrix approach. We examine in particular the ways in which students draw on RREF to produce their solution to the system and its geometric interpretation.

Findings

An initial look at the data shows that a higher percentage of students found the correct solution in the line context than in the plane context (see Figure 2 above "Solution"). In the line context, when students were asked to circle whether they thought the intersection was a point, line or a plane, *all* of the 23 students who got the algebraic solution correct and 3 additional students answered correctly. However, in the plane context, only 11 of the 16 with a correct algebraic interpretation (and 4 others) answered the point, line or plane question accurately. Thus we see in Figure 2 above "Geom Interp" an even greater difference in the percent correct than when comparing the algebraic solution. From this we can see that the two dimensional context (lines) was easier for students to interpret than the context of planes. Our next step was to examine in more detail how students went about determining their algebraic solution and their geometric interpretation.

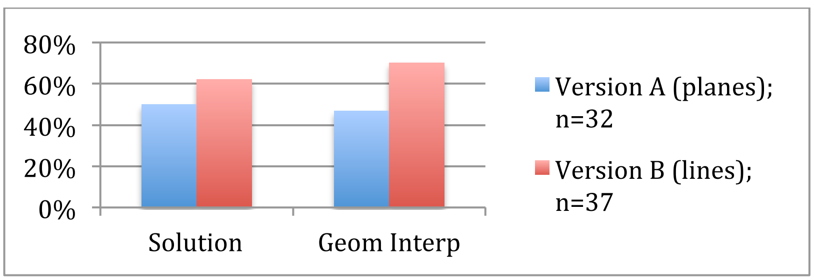


Figure 2: Correct responses by version

On both versions of the exam, most students attempted to set up an augmented matrix to try to solve the system of equations (81% of those in the plane context and 78% of those in the line context, see Figure 3). However, while *all* of the students who set up the augmented matrix in the plane context went on to find the correct RREF (81% of the class), only 17 students (46% of the class) found the correct RREF in the line context. Below we first explain the two main solution strategies of the students who found the correct RREF in the case of four planes. Second, we contrast the students in the line context who found the correct RREF with those who abandoned that approach.

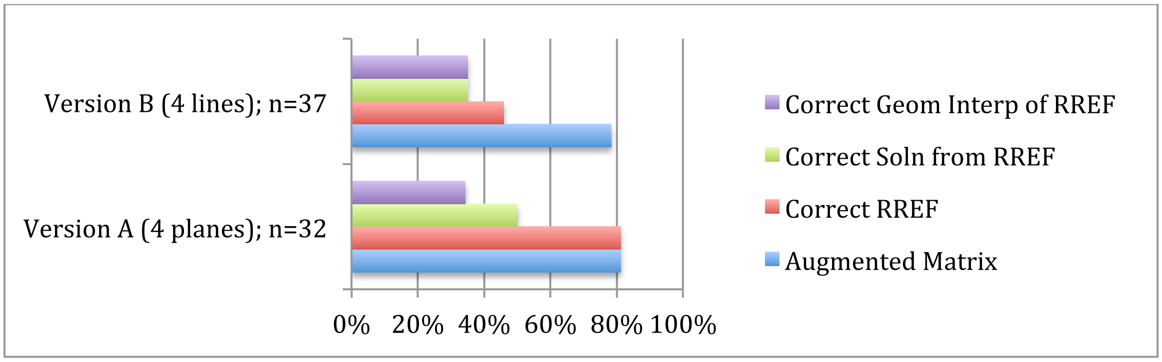


Figure 3: Solution accuracy using an augmented matrix approach

The intersection of four planes (Exam Version A)

In Figure 3 we see that 50% of the students answering the question about the intersection of the planes were able to interpret a correct RREF to find a correct algebraic solution. Figure 4 is an example of student work of this type.

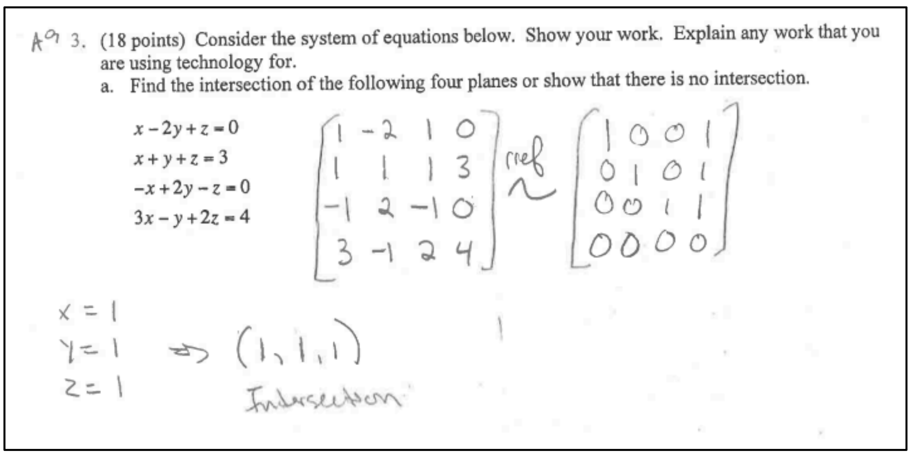
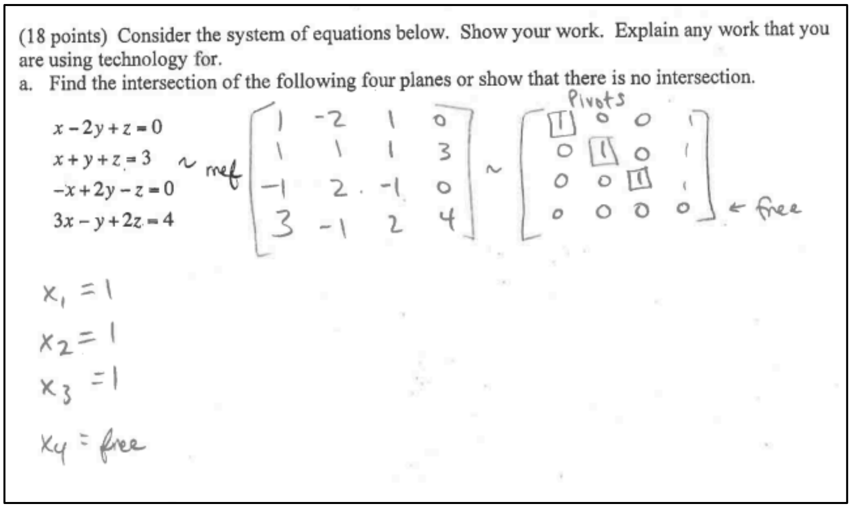


Figure 4: Typical correct student approach using RREF (Version A)

The other 10 students who had a correct RREF but could not interpret it correctly struggled with what we refer to in our title as reconstructing the vector **x** or reconstructing the values for each of the three variables. Once one has correctly row reduced, the resulting matrix for this problem has three pivots and a bottom row of zeros (Figure 4 and Figure 5). A student must then reconstruct information about x, y and z from this matrix.



(18 points) Consider the system of equations below. Show your work. Explain any work that you are using technology for.

a. Find the intersection of the following four planes or show that there is no intersection.

$$x - 2y + z = 0$$
$$x + y + z = 3$$
$$-x + 2y - z = 0$$
$$3x - y + 2z = 4$$

$$\sim_{\text{rref}} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & 1 & 1 & 3 \\ -1 & 2 & -1 & 0 \\ 3 & -1 & 2 & 4 \end{bmatrix} \sim \begin{array}{c} \text{Pivots} \\ \begin{bmatrix} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \text{free} \end{array}$$

$x_1 = 1$

$x_2 = 1$

$x_3 = 1$

$x_4 =$ free

Figure 5: Student introduces a fourth variable (Version A)

The student work in Figure 4 illustrates a correct reconstruction and a further interpretation that this is a single point of intersection. The student work in Figure 5 seems to ignore the variables of x, y and z that were stated in the problem and creates *four* new variables, seemingly one for each row of the matrix. The student writes the word "free" next to the row of zeros and next to x_4 to indicate the presence of a free variable. In this case the student then interpreted that there were multiple solutions instead of just one.

The intersection of four lines (Exam Version B)

In contrast to student work on Exam Version A, students working in the context of lines had a different set of issues at play. To begin with, the augmented matrix in this case has 4 rows and 3 columns. Since the number of rows is larger than the number of columns, the TI-83 and TI-84 calculators used by many students will give an error message when asked to row reduce such a matrix. Several students wrote comments on their paper that the matrix could not be row reduced and stopped trying to solve the problem guessing that this meant their was no solution. Other students chose to abandon their augmented matrix but continued on with a systems of equations approach. The student work in Figure 6 illustrates a student abandoning a correct augmented matrix and correctly solving and interpreting the system of equations using their knowledge of the algebra and geometry of lines.

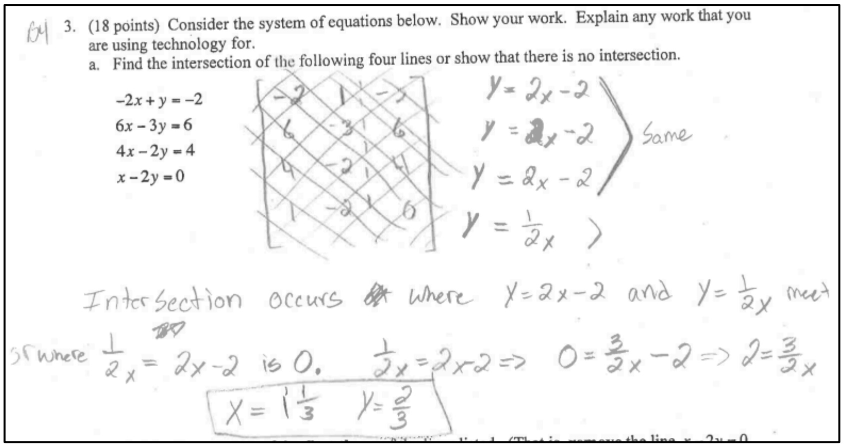


Figure 6: Student's correct systems approach (version B)

In the case of working with a system, a student never loses sight of the variables and no reconstruction is needed. In addition, for students who did try to interpret a correct RREF, the context of lines intersecting may have been more familiar and thus made it easier to reconstruct the appropriate variables than in the case of the four planes. In fact, only a few students tried to add an additional variable in the context of lines even though row reducing the matrix creates two rows of zeros.

Future work

These two exam questions were chosen for study in this preliminary report because they have the following three properties (1) the number of unknowns is different than (less than) the number of equations, (2) the solution is unique, and (3) a two or three dimensional geometric context is part of the problem statement. In the future we intend to expand this work to (1) other relationships between the number of unknowns and the number of solutions, (2) situations where the solution does not exist or there are infinitely many solutions and (3) contexts that are not geometric or involve more than three variables.

Questions

By the time of the conference we expect to have analyzed data from some of the other situations described above in future work. We look forward to discussing with the audience such questions as, What are the range of student strategies when solving problems that involve interpreting solutions using augmented matrices? Does the answer differ when students come to the algebraic representation [A|**b**] from a question about systems versus a question about a vector equations (e.g., a question about span or linear independence) or a question involving a matrix equation (e.g., a questions about null-space or when solving for eigenvectors)?

References

Dorier, J.L., Robert, A., Robinet, J., & Rogalski, M. (2000). The obstacles of formalism in linear algebra. In J.L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 85-124). Dordrecht: Kluwer.

Dorier, J.-L., & Sierpinska, A. (2001). Research into the teaching and learning of linear algebra. In D. Holton (Ed.), *The teaching and learning of mathematics at university level: An ICMI study* (pp. 255–273). Dordrecht, the Netherlands: Kluwer Academic Publishers.

Dreyfus, T., Hillel, J., & Sierpinska, A. (1999). Cabri-based linear algebra: Transformations. In I. Schwank (Ed.), *Proceedings of the First Conference on European Society for Research in Mathematics Education* (Vol. 1, pp. 209–221). Osnabrück, Germany. Retrieved from http://www.fmd.uni-osnabrueck.de/ebooks/erme/cerme1-proceedings/papers/g2-dreyfus-et-al.pdf

Harel, G. (2000). Three principles of learning and teaching mathematics: Particular reference to linear algebra—Old and new observations. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 177–189). Dordrecht, the Netherlands: Kluwer Academic Publishers.

Hillel, J. (2000). Modes of description and the problem of representation in linear algebra. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 191–207). Dordrecht, the Netherlands: Kluwer Academic Publishers.

Larson, C., & Zandieh, M. (2013). Three interpretations of the matrix equation Ax= b. *For the Learning of Mathematics*, *33*(2), 11-17.

Sierpinska, A. (2000). On some aspects of students' thinking in linear algebra. In J.-L. Dorier (Ed.), *On* the *teaching of linear algebra* (pp. 209–246). Dordrecht, the Netherlands: Kluwer Academic Publishers.

Stewart, S., & Thomas, M. O. J. (2009). A framework for mathematical thinking: The case of linear algebra. *International Journal of Mathematical Education in Science and Technology, 40*(7), 951–961. doi:10.1080/00207390903200984

Trigueros, M., & Possani, E. (2013). Using an economics model for teaching linear algebra. *Linear Algebra and Its Applications, 438*(4), 1779–1792. doi:10.1016/j.laa.2011.04.009

Extending Multiple Choice Format To Document Student Thinking

Michelle Zandieh
Arizona State University

David Plaxco
Virginia Tech

Megan Wawro
Virginia Tech

Chris Rasmussen
San Diego State University

Hayley Milbourne
San Diego State University

Katherine Czeranko
Arizona State University

The purpose of this preliminary report is to introduce a new type of assessment instrument to the mathematics education research community and to reflect with our colleagues about the possible affordances and constraints of this instrument. The questions that comprise the instrument consist of a multiple choice (MC) stem followed by a series of options from which students choose explanations (E) that support their multiple choice response. We call this style of question multiple-choice with explanation (MCE). Our decision to use MCE style questions is informed by cutting edge work in physics education research (Wilcox & Pollock, 2013) and introduces an innovative idea for assessing student thinking in mathematics. Our work is part of a larger project in linear algebra; as such, the mathematical context of the assessment instrument is linear algebra. The format of the questions, however, could be used for other subject matter as well.

Key words: Assessment instrument, span, linear independence, student thinking

The purpose of this preliminary report is to introduce a new type of assessment instrument to the mathematics education research community and to reflect with our colleagues about the possible affordances and constraints of this instrument. The questions that comprise the instrument consist of a multiple choice (MC) stem followed by a series of options from which students choose explanations (E) that support their multiple choice response (see Appendix A for examples). We call this style of question multiple-choice with explanation (MCE). Our decision to use MCE style questions is informed by cutting edge work in physics education research (Wilcox & Pollock, 2013) and introduces an innovative idea for assessing student thinking in mathematics. Our work is part of a larger project in linear algebra; as such, the mathematical context of the assessment instrument is linear algebra. The format of the questions, however, could be used for other subject matter as well.

Literature Review and Theoretical Perspective

We conducted a two-part literature review, investigating both frameworks for characterizing student understanding in linear algebra, as well as conceptually oriented assessment instruments in undergraduate mathematics and physics. Within linear algebra, we consulted well-known works that had potential to inform our work in characterizing what it means to understand linear algebra, such as Hillel's (2000) modes of description, Sierpinska's (2000) modes of reasoning, Stewart and Thomas's (2009) dual framework utilizing Tall's 3 Worlds and APOS Theory, Larson and Zandieh's (2013) interpretations of the matrix equation $A\mathbf{x} = \mathbf{b}$, Wawro and Plaxco's (2013) concept images through modes of mathematical activity, and Selinski, Rasmussen, Wawro, and Zandieh's (2014) within- and between-concept connections in linear algebra. As we move forward with the assessment we are developing and evaluating it by considering both the structure of the linear algebra concepts themselves and the framing of connections within and between these concepts.

Our review of the literature focusing on conceptually-oriented assessment instruments in undergraduate mathematics and physics revealed five main relevant sources: the Precalculus Concept Assessment (Carlson, Oehrtman, & Engelke, 2010), the Force Concept Inventory (Hestenes, Wells, & Swackhamer, 1992), the Calculus Concept Inventory (Epstein, 2013), the multiple choice format of the Quantum Mechanics Assessment Tool (Sadaghiani, Miller, Pollock, & Rehn, 2013), and the multiple choice adaptation of the Colorado Upper-division Electrostatics (CUE) diagnostic (Wilcox & Pollock, 2013). In particular, Wilcox and Pollock discussed their methods for adapting a valid and reliable free-response assessment instrument into a multiple choice style exam (with question format similar to our MCE format) that preserved the instrument's validity and reliability. We found this work especially useful because of its efforts to bridge the gap between open- and closed-ended questions yet still illuminate students' conceptual understanding.

Methods

We developed the assessment instrument in four phases: (a) reviewing literature and compiling possible questions, (b) developing MCE style questions, (c) piloting the assessment instrument, and (d) analyzing and refining the MCE questions. We began the first phase by exploring all questions used by our team over six separate studies in the previous four years as well as possible questions that existed in the research corpus, consulting various web resources (e.g., http://mathquest.carroll.edu/) and published papers (e.g., Britton & Henderson, 2009; Hamdan, 2005; Rensaa, 2007; Stewart & Thomas, 2009). We focused on questions that specifically addressed the concepts of span and linear independence. The team then iteratively reviewed and trimmed the question list, focusing on questions thought to elicit key aspects of student understanding. We then developed a taxonomy of possible representational systems through which the assessment questions might be asked or across which conceptual understandings might be related.

In the second phase, the research team used this taxonomy and question list to develop thirteen pilot MCE questions. This adaptation focused on developing (E) responses most likely to reflect students' understanding of relationships between concepts and different representations of the same concept. In the third phase, eight of the questions were used in individual interviews (Bernard, 1988) with seven students at two different universities; the other five questions were piloted with four students at one of these universities. We then used four of these MCE questions to conduct a larger pilot implementation with 124 students in five classrooms at three different universities. Student responses were collected, digitally scanned, and coded in a spreadsheet format, the organization of which we describe in the next section.

Results and Discussion

In this section we present three types of analyses that are possible with data from MCE style questions: (1) grid of mathematical relationships, (2) student-focused matrices, and (3) coincidence matrices (and associated Venn diagrams). Type 1 focuses on relationships between the MC and (E) parts of each question, type 2 focuses on patterns in students' responses across questions, and type 3 focuses on a group's responses to a particular question. As this is a preliminary report, we illustrate the beginnings of each analysis type and indicate what we think are interesting points to consider.

Grid of mathematical relationships

Figure 1 illustrates the grid of relationships between the multiple choice (MC) stem of the question and the explanation (E) section. Some questions have 2-4 MC options (See Appendix A), but Figure 1 only includes the correct answer for each. These are listed across the top as column headings. Along the left side, the rows are the possible (E) responses. Within the grid the appropriate column and row intersection is blank if the (E) response was not part of the question in that column. If it was listed as a possible (E) response, then the intersection box indicates the (E) response number. For example, in Question 1, "Vectors span all of $\mathbb{R}^m$" was (E) response (v). The color of each indicates the relationship of the response to the multiple choice answer. Green indicates that the answer is true and relevant and thus should have been chosen. Red indicates that the response is false and therefore should not have been chosen. Black indicates that the response is true but should not have been chosen because it is not relevant to the problem. For example, "Vectors span all of $\mathbb{R}^m$" was not relevant to whether the set of three vectors in R^2 was linearly dependent, so "v" is in black font for the corresponding cell in Figure 1.

At minimum the grid gives us an indication of what types of relationships we are testing for with these questions. This should allow us to note if there is a relationship that we wished to test for and have not or if there is something that we have tested for more than once. In the latter case, we may choose to eliminate the extra to be efficient or use that opportunity to triangulate information about a student's responses. We would like to expand our use of the gridding tool to indicate more about individual student thinking. For example, we could indicate whether a student chose certain (E) responses in coordination with certain MC answers to get a snapshot of an individual student's understanding. One constraint we have with this method is that when a student does not choose a particular (E) response, we cannot be certain whether they did not choose it because they think it is false or because they think it is irrelevant.

	Question 1	Question 2		Question 3	Question 4
	Set S is linearly dependent	Matrix as a transformation is onto $\mathbb{R}^2$	Matrix as a transformation is not 1-1	A is invertible when $n \neq 10$	Vectors span all of $\mathbb{R}^m$
S includes the *0* vector	i				i
# vectors > # entries	ii				ii
No vectors is a scalar multiple of the others	iii				iii
The only soln to homogeneous eqn. is trivial soln	iv	iii	iii		iv
Vectors span all of R^m (m = # entries in vector)	v				
Matrix made from a subset of S row reduces to I	vi			i, ii, iii	vi
Set S is linearly dependent		i	i	iv, v, vi	
T: $R^m \rightarrow R^n$, $m > n$		ii	ii		
Row-reduced M has a pivot for each row		iv	iv		
The range of T is all of the codomain		v	v		
Set S is linearly independent					v
Two vectors form a linearly independent subset					vii
Two vectors do not lie on the same line					viii
Two vectors lie on the same line					ix

Figure 1. Grid indicating relationships between the MC and (E) parts for each question

Student-focused matrices

We focus on individual student thinking by using a matrix such as that in Figure 2. Across the top are the four question numbers and the (E) response numbers for each. Each row represents one of 29 different students. Under each question column, the student MC response is listed first followed by a “1” in each column of an (E) response that the student chose and a “0” in each column of an (E) response that the student did not choose for that particular question.

Through displaying the data in this way, we examine the variety of responses for an individual student. The data in one row indicates all responses for one student and could be re-gridded into the format of Figure 1 to examine some aspects of that student’s response. In addition, through Figure 2 we can see how a student’s response corresponds to other students’ responses across questions. For example, in Question 1 we see that all students who answered B also chose (i), whereas only about half of the students who incorrectly chose A also chose (i). Because (i) states that the set contains the zero vector, it is likely that those who (incorrectly) answered A (the set is linearly independent) noticed that the zero vector was included in the set but thought it irrelevant to their claim of linear independence. Conversely, only two students who chose B chose (vi), whereas about half who chose A chose (vi). This is sensible because students who thought the set was linearly independent may associate independence with row reducing to the identity matrix.

Another noteworthy aspect is whether students who chose one of the (E) responses chose another related response. For example, in Figure 2, all students who chose B for Question 1 chose the (E) response (i) and half of those also chose (ii), much more than any other (E) choice for Question 1. We consider these types of paired responses in the next section.

	Question																													
	1							2						3							4									
Student		i	ii	iii	iv	v	vi		i	ii	iii	iv	v		i	ii	iii	iv	v	vi		i	ii	iii	iv	v	vi	vii	viii	ix
1	A	0	0	1	0	1	1	A	1	0	0	1	1	B	0	0	0	0	1	0	A	0	0	1	1	0	1	1	1	0
2	A	1	0	1	0	1	0	B	0	1	0	1	0	B	0	1	0	1	0	0	A	0	0	0	1	0	1	0	1	1
6	A	0	0	0	1	1	1	B	0	1	0	1	1	B	0	1	0	0	1	0	C	0	0	0	1	1	1	1	1	0
8	A	1	0	0	0	0	1	B	0	1	0	1	1	B	0	1	0	0	1	0	C	0	0	0	0	1	0	0	1	0
7	A	0	0	0	0	1	1	B	0	1	0	0	0	B	0	1	0	0	0	0	C	0	0	1	0	0	1	1	1	0
3	A	1	1	1	0	1	0	B	0	0	0	1	1	B	0	1	0	1	0	0	C	0	0	1	0	0	0	1	1	0
4	A	1	0	1	0	1	1	B	1	0	0	1	0	B	0	1	0	1	0	0	C	0	0	1	0	1	1	1	1	0
5	A	0	0	1	1	0	1	B	0	1	0	1	1	B	0	1	0	1	0	0	C	0	0	1	1	1	1	1	1	0
9	A	1	0	0	0	1	0	B	0	0	0	1	1	C	1	0	0	0	0	1	C	0	0	0	0	0	0	1	1	0
10	A	1	1	1	0	1	1	C	0	0	0	1	1	B	0	1	0	0	0	0	C	0	0	1	0	1	1	1	1	0
11	A	0	0	1	0	1	1	D	0	1	0	0	0	B	0	1	0	1	0	0	C	0	0	1	1	1	1	0	0	0
12	A	0	0	0	1	0	1	D	1	0	0	1	0	B	0	1	0	1	0	0	D	0	0	1	0	0	0	0	1	0
13	A	0	0	1	1	0	1	D	1	1	0	1	1	NR	0	1	0	1	0	0	C	0	0	0	1	1	1	1	1	0
14	B	1	0	1	0	1	1	A	0	1	0	1	1	B	0	1	0	0	0	1	C	0	0	1	0	0	0	1	1	0
15	B	1	1	0	0	0	0	B	0	0	0	1	1	B	0	1	0	0	0	0	B	0	0	0	0	0	0	1	1	0
16	B	1	0	0	0	1	1	B	0	0	0	1	0	B	0	1	0	0	1	0	C	1	0	0	0	0	1	1	1	1
18	B	1	0	0	0	0	0	B	0	1	0	1	1	B	0	1	0	1	0	0	C	0	0	0	0	0	1	1	1	0
19	B	1	1	0	0	0	0	B	0	0	0	1	1	B	0	1	0	0	1	0	C	0	0	0	0	0	1	1	1	0
21	B	1	0	0	0	0	0	B	0	0	0	1	1	B	0	1	0	0	0	0	C	0	0	0	0	0	0	1	1	0
22	B	1	0	0	0	1	0	B	0	0	0	1	0	B	0	1	0	0	1	0	C	1	1	0	0	0	0	0	0	1
17	B	1	0	1	1	0	0	B	0	0	1	1	0	B	0	0	1	0	0	0	C	0	0	1	0	1	0	0	0	0
20	B	1	1	1	0	0	0	B	0	1	0	0	1	B	0	1	0	0	0	0	C	0	0	1	0	0	0	0	0	0
23	B	1	1	0	0	0	0	B	0	1	0	1	1	B	0	1	0	1	0	0	D	0	1	0	0	0	1	1	1	1
24	B	1	1	0	0	0	0	B	1	1	0	0	0	B	0	1	0	1	0	0	D	0	0	1	0	0	1	1	1	0
25	B	1	1	1	0	1	0	B	1	1	0	1	0	C	0	0	0	0	0	1	D	1	1	0	0	0	0	1	1	1
26	B	1	1	0	0	0	0	B	1	0	0	1	0	C	1	0	0	0	0	0	NR	0	0	1	0	0	1	1	1	1
27	B	1	0	0	0	1	0	D	0	1	0	1	0	A	0	0	1	1	0	0	C	0	0	1	0	0	1	0	1	0
28	B	1	0	1	0	0	0	D	1	0	0	1	1	B	0	1	0	0	0	0	A	1	1	1	0	0	1	1	1	1
29	B	1	1	0	0	0	0	D	1	1	0	0	0	B	0	1	0	1	0	0	C	0	0	0	0	0	1	1	0	0

Figure 2. Student responses Fall 2013

Coincidence matrices

To check for pairs of student (E) responses we have constructed coincidence matrices. For instance, Figure 3 shows the participants’ selection of (E) responses for Question 1, sorting the (E) responses by MC response. In these matrices, each entry gives the number of participants who chose both the (E) response in a given row and also the (E) response in a given column. The diagonal entries show the number of participants who chose the particular (E) response. For instance, in the matrix on the left, the entry in Row i, Column i shows that only 19 of the 49 students who incorrectly chose MC response A (that the set of vectors $\left\{\begin{bmatrix}4\\5\end{bmatrix},\begin{bmatrix}2\\3\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}\right\}$ is linearly independent) also correctly chose (E) response (i) (that the set contains the zero vector). As discussed above these students likely thought that the zero vector was irrelevant to determining that the set is linearly independent. On the other hand, 73 of the 75 participants who correctly chose MC response B (the aforementioned set is linearly dependent) also correctly chose (E) response (i). In addition, 54 of the 75 participants who chose MC response B also chose (E) response (ii) (that the set is 3 vectors in $\mathbb{R}^2$). All but one of these participants also chose (E) response (i) (note the 53 in the Row i, Column ii cell).

Another example of a coincidence matrix shows participants who have correctly answered the MC part of Question 4. We see that these students have widely varying responses to the (E) part. The three correct (E) responses are (vi), (vii), and (viii). The students who chose these responses and pairs of these responses are highlighted in green in the right of Figure 4. The box in the lower right corner shows the total number of participants (32) who correctly responded to both the MC and (E) parts of this question. This example shows one of the limitations of the coincidence matrix organization of the data. Specifically, the coincidence matrix only provides pair-wise counts of coincidental responses, whereas it may be advantageous to consider whether more than two (E) responses are in common for a given student. This matrix can be supplemented with a Venn Diagram showing how the responses intersect. On the left of Figure 4 is a Venn diagram that breaks down the 93 students who answered C into the 7 who chose none of the correct (E) responses, the 32 who chose all three of the correct (E) responses and other categories of pairs or singular responses from the three (E) correct responses.

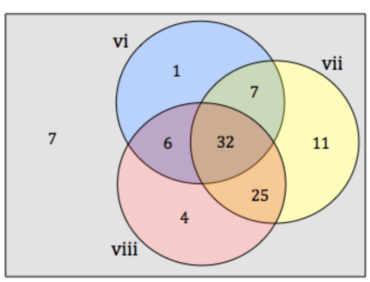
MC Selection: A

49	i	ii	iii	iv	v	vi
i	19	7	11	6	12	7
ii		10	3	5	5	4
iii			29	10	15	12
iv				24	10	9
v					24	11
vi						21

MC Selection: B

75	i	ii	iii	iv	v	vi
i	73	53	10	8	20	6
ii		54	6	6	14	4
iii			11	4	5	3
iv				9	4	2
v					20	6
vi						6

Figure 3. Coincidence matrices for Question 1, separated by MC response



C

93	i	ii	iii	iv	v	vi	vii	viii	ix	
i	15	2	6	3	4	7	10	10	4	
ii		10	6	1	1	5	8	7	4	
iii			37	7	16	20	26	28	4	
iv				12	6	9	9	9	3	
v					24	12	14	16	2	
vi						46	39	38	5	
vii							75	57	10	
viii								67	4	
ix									12	32

Figure 4. Venn diagram and coincidence matrix for Question 4, Response C

We would like the audience to think with us about the affordances and constraints of the MCE style question format, as well as these three ways of organizing the data for analysis. In particular, what types of student thinking can be measured with the MCE style questions? How can we best leverage the different data organization methods to get at student thinking? In what ways do the various analyses help us study individuals versus classrooms?

References

Bernard R. H. (1988). *Research methods in cultural anthropology*. Newbury Park, CA: Sage Publications.

Britton, S. & Henderson, J. (2009). Linear algebra revisited: An attempt to understand students' conceptual difficulties. *International Journal of Mathematical Education in Science and Technology, 40*(7)*,* 963-974.

Carlson, M., Oehrtman, M., & Engelke, N. (2010). The precalculus concept assessment: A tool for assessing students' reasoning abilities and understandings. *Cognition and Instruction, 28*(2), 113–145.

Epstein, J. (2013). The calculus concept inventory - Measurement of the effect of teaching methodology in mathematics. *Notices of the AMS, 60*(8), 1018–1026.

Hamdan, M. (2005). Nonlinear learning of linear algebra: Active learning through journal writing. *International Journal of Mathematical Education in Science and Technology, 36*(6), 607-615.

Hestenes, D., Wells, M., & Swackhamer, G. (1992). Force concept inventory. *The Physics Teacher, 30,* 141–158.

Hillel, J. (2000). Modes of description and the problem of representation in linear algebra. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 191-207). Dordrecht: Kluwer Academic Publisher.

Larson, C. & Zandieh, M. (2013). Three interpretations of the matrix equation Ax=b. *For the Learning of Mathematics, 33*(2)*,* 11-17.

Rensaa, R.J. (2007). A choice option between proofs in linear algebra. *International Journal of Mathematical Education in Science and Technology, 38*(6), 729-738.

Sadaghiani, H., Miller, J., Pollock, S., & Rehn, D. (2013). Constructing a multiple-choice assessment for upper-division quantum physics from an open-ended tool. Retrieved from http://arxiv.org/abs/1308.4226

Selinski, N., Rasmussen, C., Wawro, M., & Zandieh, M. (2014). A methodology for using adjacency matrices to analyze the connections students make between concepts: The case of linear algebra. *Journal for Research in Mathematics Education, 45*(5), 550-583.

Sierpinska, A. (2000). On some aspects of students' thinking in linear algebra. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 209-246). Dordrecht: Kluwer Academic Publisher.

Stewart, S., & Thomas, M.O.J. (2009). A framework for mathematical thinking: The case of linear algebra. *International Journal of Mathematical Education in Science and Technology, 40*(7), 951-961.

Wawro, M., & Plaxco, P. (2013). Utilizing types of mathematical activities to facilitate characterizing student understanding of span and linear independence. In S. Brown, G. Karakok, K. H. Roh, and M. Oehrtman (Eds.), *Proceedings of the 16th Annual Conference on Research in Undergraduate Mathematics Education, Volume I* (pp. 1-15), Denver, Colorado.

Wilcox, B., & Pollock, S. (2013). Multiple-choice assessment for upper-division electricity and magnetism. In Engelhardt, Churukian, and Jones (Eds.), *2013 PERC Proceedings,* pp. 365-368.

Appendix A: MCE questions as implemented in 2013 and 2014

1) The set of vectors $\left\{\begin{bmatrix}4\\5\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right\}$ is
   - A) Linearly independent
   - B) Linearly dependent

   Because … (Select ALL that support you choice)
   - i) the set includes the 0 vector.
   - ii) the set has 3 vectors in $\mathbb{R}^2$.
   - iii) none of the vectors are multiples of each other.
   - iv) the only solution to $c_1\begin{bmatrix}4\\5\end{bmatrix} + c_2\begin{bmatrix}2\\3\end{bmatrix} + c_3\begin{bmatrix}0\\0\end{bmatrix} = 0$ is the trivial solution.
   - v) the vectors span all of $\mathbb{R}^2$.
   - vi) $\begin{bmatrix}4 & 2\\5 & 3\end{bmatrix}$ is row equivalent to the identity matrix

2) The transformation defined by $T(x) = Mx$, where the matrix is $M = \begin{bmatrix}2 & 4 & 6\\1 & 3 & 3\end{bmatrix}$
   - A) One-to-one
   - B) Onto
   - C) Both
   - D) Neither

   Because … (Select ALL that support you choice)
   - i) the columns of the matrix are linearly dependent.
   - ii) the matrix maps $\mathbb{R}^3$ to $\mathbb{R}^2$.
   - iii) the only solution to $c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}4\\3\end{bmatrix} + c_3\begin{bmatrix}6\\3\end{bmatrix} = 0$ is the trivial solution.
   - iv) the row-reduced echelon form of M has two pivots.
   - v) the range of this transformation is all of $\mathbb{R}^2$

3) Let $A = \begin{bmatrix}1 & -2 & 5\\2 & -4 & n\\0 & 1 & 1\end{bmatrix}$. For what value(s) of n is the matrix A invertible?
   - A) $n = 10$
   - B) $n \neq 10$
   - C) $n \in \mathbb{R}$

   Because … (Select ALL that support you choice)
   - i) $\begin{bmatrix}1 & -2 & 5\\2 & -4 & n\\0 & 1 & 1\end{bmatrix}$, for all $n \in \mathbb{R}$, row-reduces to the identity matrix.
   - ii) $\begin{bmatrix}1 & -2 & 5\\2 & -4 & n\\0 & 1 & 1\end{bmatrix}$, for all $n \neq 10$, row-reduces to the identity matrix.
   - iii) $\begin{bmatrix}1 & -2 & 5\\2 & -4 & n\\0 & 1 & 1\end{bmatrix}$, for $n = 10$, row-reduces to the identity matrix.
   - iv) the set of vectors $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}-2\\-4\\1\end{bmatrix}, \begin{bmatrix}5\\n\\1\end{bmatrix}\right\}$ is linearly dependent for $n = 10$.
   - v) the set of vectors $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}-2\\-4\\1\end{bmatrix}, \begin{bmatrix}5\\n\\1\end{bmatrix}\right\}$ is linearly dependent for $n \neq 10$.
   - vi) the set of vectors $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}-2\\-4\\1\end{bmatrix}, \begin{bmatrix}5\\n\\1\end{bmatrix}\right\}$ is linearly dependent for $n \in \mathbb{R}$.

4) The set of vectors $\left\{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}2\\5\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right\}$ spans
   - A) a point in $\mathbb{R}^2$
   - B) a line in $\mathbb{R}^2$
   - C) all of $\mathbb{R}^2$
   - D) a plane in $\mathbb{R}^3$
   - E) all of $\mathbb{R}^3$

   Because … (Select ALL that support you choice)
   - i) the set includes $0 \in \mathbb{R}^2$.
   - ii) the set has 3 vectors.
   - iii) none of the vectors are multiples of each other.
   - iv) the only solution to $c_1\begin{bmatrix}1\\3\end{bmatrix} + c_2\begin{bmatrix}2\\5\end{bmatrix} + c_3\begin{bmatrix}0\\0\end{bmatrix} = 0$ is the trivial solution.
   - v) the set of vectors is linearly independent.
   - vi) $\begin{bmatrix}1 & 2\\3 & 5\end{bmatrix}$ is row equivalent to the identity matrix.
   - vii) two of the vectors in the set are linearly independent.
   - viii) the two vectors $\begin{bmatrix}1\\3\end{bmatrix}$ and $\begin{bmatrix}2\\5\end{bmatrix}$ do not lie on the same line.
   - ix) the two vectors $\begin{bmatrix}1\\3\end{bmatrix}$ and $\begin{bmatrix}0\\0\end{bmatrix}$ lie on the same line.

Variation in successful mathematics majors proving

Dov Zazkis, Oklahoma State University
Keith Weber, Rutgers University
Juan Pablo Mejía-Ramos, Rutgers University

We examined the proof-writing behaviors of six highly successful mathematics majors on proving tasks in calculus. We found that these students approached the proof writing tasks in two different ways. Three students, who we labeled as drillers, would develop a strong understanding of the statement they were proving, choose a plan based on this understanding, develop a graphical argument for why the statement is true, and formalize this graphical argument into a proof. The other three students, who we labeled as probers, would begin trying different proving approaches immediately after reading the statement and would abandon an approach at the first sign of difficulty. Despite being inconsistent with theories of effective problem solving in the mathematics literature, the probers were highly successful in their advanced mathematics courses and on the proving tasks in this study.

Key words: problem solving; proof; metacognition.

Educational research on proof writing at the undergraduate level is largely comprised of highlighting specific difficulties that prevent mathematics majors from successfully writing proofs (e.g. Harel & Sowder, 1998; Hart, 1994; Selden & Selden, 1995; Zandieh, Roh & Knapp, 2014). The research literature documents that mathematics majors struggle with proof and highlights broad competencies that these students lack. However, there has been comparatively little work on how students *can* or *should* construct proofs. Indeed, aside from suggesting that instructors pay more attention to areas in which students are deficient, the literature cited above does not offer guidance for how mathematics majors' proof-writing performance might be improved.

One approach to identifying the competencies needed to construct proofs is to carefully study the behavior of those who are proficient at proof writing. There has been limited work in this area. A first body of work has examined the proof-writing behavior of mathematicians using an expert-novice research paradigm (Lockwood, et al., 2012; Samkoff, Lai & Weber, 2012, Weber, 2001). This focus on mathematicians provides valuable insight into how proofs may be successfully written. However, mathematicians' proving strategies might rely on experiences and understandings that most undergraduates may lack. If so, teaching students to apply these strategies might be counterproductive (cf., Reif, 2008). A second set of studies has examined the behavior of students who wrote proofs successfully (Gibson, 1998; Sandefur et al. 2013). The goal of this paper is to contribute to this second set of studies.

Theoretical framing.

Schoenfeld (1985) described four competencies of problem solving expertise-- resources, heuristics, metacognition, and beliefs. He used these competencies to explain why mathematicians were more effective at solving mathematical problems than undergraduates, even when both populations possessed the requisite knowledge to solve these problems. Two aspects of Schoenfeld's (1985) framework, resources and metacognition, were especially useful in helping us describe the successful mathematics majors' proof-writing behavior.

Homogeneity and heterogeneity in mathematical expertise

There is an often implicit assumption present when mathematics educators investigate those who are successful in some aspect of mathematical practice-- namely that there are a core set of behaviors or competencies that are utilized by most who are successful at this practice. We refer to this as the homogeneity assumption. Schoenfeld (1985) was explicit about this assumption in his discussion of heuristics for mathematical problem solving:

> "*There is a substantial degree of homogeneity in the way that expert problem-solvers approach new problems [...] if two experts grapple with an extended series of unfamiliar problems, there will be substantial overlap in the problem-solving strategies that they try"* (p. 71).

Although others are not as explicit in endorsing the homogeneity assumption, we find much of the research in mathematical problem solving as accepting this viewpoint. This can be evidenced by the focus on commonalities in experts' problem solving behaviors (e.g., Carlson, 1999; Schoenfeld, 1985; Weber, 2001) and models that aim to capture the essence of mathematical problem solving (e.g., Carlson & Bloom, 2005; Polya, 1957).

On the other hand, there is both anecdotal and empirical evidence that successful mathematical performance and mathematicians' practice might not be homogeneous (Burton, 2004; Pinto & Tall, 1999; Weber, Inglis & Mejía-Ramos, 2014). Recent work has also pointed to heterogeneity in undergraduate practice (e.g., Alcock & Simpson, 2004, 2005).

If mathematical expertise in proof writing is indeed heterogeneous, then this complicates the goals of instruction. If we use a homogeneous assumption to analyze mathematical practice that is heterogeneous, we risk encouraging students to do mathematics in a certain way when there are other viable approaches to doing mathematics. In the analysis in this paper, we searched both for commonalities and differences in the successful mathematics majors' behaviors on the proving tasks.

Methods

This study was conducted at a large state university in the northeastern United States with one of the top 25 mathematics departments in the country (US News and World report, 2014). We recruited mathematics majors who had completed an advance calculus course, a second proof-oriented course in linear algebra and a transition-to-proof course. From this solicitation, 73 students volunteered to participate in our study in exchange for payment.

We posed challenging non-routine proving tasks to our participants, but the content knowledge required to complete these proving tasks did not extent beyond a first course in calculus. These tasks are included in the Appendix. Participants met individually with an interviewer and were video-recorded as they were asked to "think aloud" as they completed seven proving tasks. They were told to write up their final proofs as if they were handing them in for a final exam. Participants were given ten minutes to complete each task and were allowed to stop working on a task at any time if they felt that they could make no more productive progress.

We used two measures to identify the most successful mathematics majors in our sample-- from hereon, the *stars*. A student was designated as a *star* if (i) they answered at least four of the seven proving tasks correctly and (ii) their GPA in the three math courses we considered was a 4.0. Six participants met these criteria. Answering only four of seven tasks correctly might not seem like exceptional performance. However, two of the tasks in the study were successfully answered by only one of the 73 participants in the study (both were amongst the six stars). These extremely difficult problems provided insight into what the stars did when they reached an impasse.

Results

In our analysis, we identified two qualitatively different approaches that the stars students used to generate proofs, which we termed probing and drilling. The probers exhibited the following characteristics: (i) After reading the problem statement, the probers expend little time or effort in understanding the statement to be proven. (ii) Probers also do not spend much time in choosing or evaluating a plan to address the problem. Rather, they usually implemented the first plan that comes to mind without considering its likelihood of success. (iii) When implementing a plan, if they stop making progress for a short time or they perceive further progress to be difficult, they quickly either modified the plan or abandoned it. In the latter case, they chose another plan and implemented that.

In contrast to probers, drillers exhibited the following characteristics on the proving tasks: (i) When reading the problem statement, the drillers invested considerable effort in understanding what the statement was asserting. (ii) After representing the hypotheses of the problem statement (often graphically), the drillers would try to understand why the conclusion of the statement would be true. (iii) The proof that the drillers wrote often consisted of trying to formalize an intuitive (often graphical) argument. (iv) The drillers were reluctant to abandon the plan that they were implementing. They would only do so if they reached an impasse that they could not resolve for some time.

Probers and Drillers Unsuccessful proving

In this section, we contrast excerpts that illustrate probing and drilling approaches to the same task. Here we focused on episodes where participants were unable to successfully construct a proof, as these allowed us to focus on how probers and drillers addressed impasses. We contrast the work of Wolfe, a prober, and Sarah, a driller, working on task 2 (Prove that the only real solution to the equation $x^3+5x=3x^2+\sin(x)$ is $x=0$). We begin with Wolfe's approach.

[1] [00:00] Prove that the only real solution to the equation $x^3 + 5x = 3x^2 + sin(x)$ is $x=0$.
[2] [00:18] Well it's pretty obvious that zero is a solution, because if you just plug it in you get zero on both sides. Hurray.
[3] [00:29] So why is that the only real solution? Well let's see.
[4] [00:44] I'm just going to write down. I'll write down the thing, the equation but leave the *sin(x)* by itself.
[5] [00:50] So $sin(x) = x^3 + 5x - 3x^2$ which equals $x(x^2 - 3x + 5)$.
[6] [01:03] That doesn't factor nicely. And this is just tricky because of the *sin(x)* term.
[7] [01:13] So is there anything I can do here to make the problem easier?
[8] [01:20] Well squaring both sides wouldn't help.
[9] [01:28] Dividing by *sin(x)* wouldn't help.
[10] [01:32] Why is x the only real solution? Well usually with an x^3 term that means that you will have three solutions. Sometimes it means that they are complex.
[11] [01:49] So let me think of a way to prove those terms. Well I can't really think of any other x's that would work. Even complex. So I just have to work with what we got.
[12] [2:14] So zero obviously works. And then so if some sort of a polynomial I want that to equal *sin(x)* which is some periodic term. [sketching the graph of *sin(x)*] That was a terrible *sin(x)* graph. But that's okay.
[13] [2:47] So *sin(x)* is periodic and how about I take the derivative of the part without the *sin(x)*. So I get $3x^2$-$6x^2$+5. Does that even factor? 15 could become 3 and 5.. only 5 is prime so it's 1 and 5. No, that doesn't really help. So taking the derivative doesn't really help.
[14] [03:18] So why would there be no other solutions other than the fact that x is trivially a solution?
[15] [03:30] Alright *sin(x)* just oscillates and I showed that the other stuff can factor into x and $(x^2 - 3x + 5)$. Which well I know that that this only has a root at zero and this only has complex... or no, not complex roots.
[16] [04:12] So what can I use with sin. Are there properties of *sin(x)* that can help with this? Well *sin(x)* is bounded between ... well $|\sin(x)| \leq 1$. So does putting a bound on it do anything? Well it does. Like I could try to make another equation that uses one or minus one and see if that goes anywhere I don't want to just be stuck on a problem. Um but it doesn't look like it would help either way.
[17] [04:59] So proving that the only solution is zero... what could I do.
[18] [05:10] Interviewer: So it's been about five minutes. Do you want to keep going on this one? Or go onto the next one?
[19] [05:13] [stares at the problem silently for 30 sec] I'm pretty sure I don't know what to do with this one so... we'll give up on this one.

We observe several aspects of this transcript that are representative of Wolfe and probers in general. First, note that little effort is made trying to understand the problem. Immediately after reading the problem in [1], Wolfe begins doing calculations in [2] to establish the trivial part of the problem statement, that x=0 is a solution. After that, Wolfe spends only 15 seconds reflecting before attempting to factor the expression in [4-5]. He begins his attempts at proving the statement with little time spent understanding the statement or formulating a plan. Second, Wolfe proposes numerous strategies, such as factoring the polynomial [4], squaring both sides [8], dividing by sine [9], showing that all but one of the roots to the cubic equation will be complex [10], and using the fact that sine is bounded [16]. Most of these plans are dismissed quickly. Third, Wolfe twice asked "why" the statement would be true, but he does not appear to seek an intuitive explanation for why the proven statement is true; rather he responds by proposing techniques that he could plausibly use to establish the theorem. Finally, Wolfe generated many plausible approaches by thinking about the mathematical concepts in the statement to be proven and lists proving techniques and facts that he knows about these concepts. Generating ideas in this way did not result in a proof, but as we will show later, they often provided probers with insights that led to clever proofs. We compare Wolfe's proof attempt with the work of Sarah, a driller, on the same task.

[1] [00:00] Prove that the only real solution to the equation $x^3 + 5x = 3x^2 + sin(x)$ is $x=0$. Oh great [sarcastic]

[2] [00:15] Interviewer: What are you thinking about?

[3] [00:16] How do I do this? I don't like this. So we have $x^3 -3x^2 + 5x = sin(x)$. So what does this function look like? Hey I might actually use the calculator thing.

[4] [00:52] Interviewer: Alright. Yeah here. You want to graph a new function?

[5] [01:10] [Sarah and the interviewer work together to use the graphing software to plot both the graph of the polynomial part of the equation, $f(x)=x^3+5x-3x^2$ and $sin(x)$]

[6] [01:58] So what is there to show here? Both functions are increasing there but one is increasing faster than the other. So they intersect right there and that function is increasing faster.

[7] [02:43] Alright. So if you take the derivative $3x^2-6x+5$ and then the derivative of $sin(x)$ is $-cos(x)$. So what does this mean at $x=0$? At zero it's going to be 5 and the derivative of this thing is going to be between zero and one. So and after this this function is going to continue increase. Why does it continue to increase? Alright, so we have $3x^2-6x+5$. Um.... So right now I'm trying to show why it never intersects again after x is equal to zero. And I think that's because it intersects once and then it can't intersect again because the polynomial increases so much faster than the $sin(x)$ function. I'm a bit confused as to why it continues to increase afterwards though. I mean this is given by f '(x). And this is always going to.. well this is $3x$. When is this thing going to be equal to zero?

[8] [04:53] $6x+5$. Ohhh $-b\pm \sqrt{b^2 -4ac}$. 3 times 5 so that's going to be 15 times something so this is going to be negative. So this is never going to be equal to zero. Which explains why the function is always increasing. I guess. Instead of switching around. Okay so.

[9] [05:39][Sarah begins writing a proof] $x^3 + 5x = 3x^2 + sin(x)$ is the same problem as $x^3 -3x^2 + 5x=sin(x)$. Clearly $x=0$ is a solution because we can just plug it in. Solution. Um lets call this thing $f(x)$. So $f'(x)$ is actually always increasing. Right? Yeah. Can I clear this and graph something instead? [Sarah graphs the derivative, $f'(x)=3x^2+5-6x$]

[10] [06:45] Thanks, $3x$, oh, $3x^2+5$, hold on. $3x^2+5-6x$. That's hopefully the derivative of this thing. It's starts decreasing after that. $3x^2-6x+5$. Oh wait no it doesn't have any x intercepts. But...

[11] [07:51] Anyway that's the original function oh yeah here it... looks like it's negative. Oh this function is always increasing. [Sarah works unsuccessfully to show that $f(x)=x^3+5x-3x^2$ and $sin(x)$ only intersect at zero by showing that $f'(x)>cos(x)$. She deals with x>0 and x<0 as separate cases. She ends up running out of time before she is able to complete a rigorous proof.]

There are several aspects of this transcript worth noting that are representative of Sarah's behavior. First, Sarah does not begin doing calculations until 2:43 had elapsed when she computed the derivative of f(x) in [7]. Prior to that point, Sarah had been working to represent the problem that she was asked to solve to see why its statement might be true. Second, Sarah developed graphical arguments for why the task statement was true, which she then set as a goal to formalize. In [6], she notes that f(x) is increasing faster than sin x, which leads to her taking the derivative in [7]. The calculation that she undertook was not aimless or exploratory, but based on how she understood the task situation.

In Figure 1, we compare how Wolfe and Sarah spent their time trying to write this proof. As can be seen, Sarah spent substantially more time than Wolfe in trying to understand the problem statement and formulate a plan. Wolfe suggested multiple plans in quick succession, while Sarah adopted only a single plan.

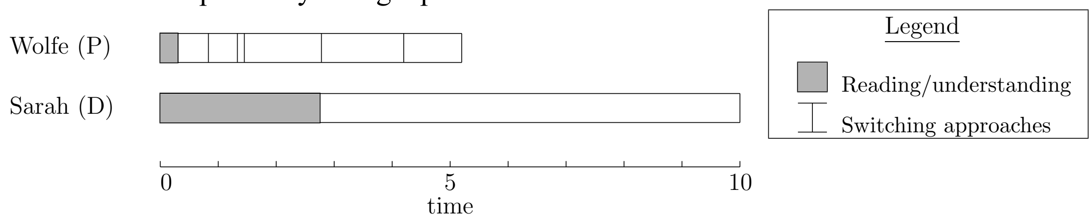


Figure 1: Sarah and Wolfe work on task 2

Stars successfully writing proofs

In the previous section we focused on episodes where star students were unable to produce proofs. This highlighted what star students did when they reached impasses and illustrated the differences between drilling and probing. However, this focus on unsuccessful proofs may give the reader a false impression regarding the mathematical abilities of these students. In addition to showing that the star students are often successful on proof writing tasks, this analysis allows the reader to see how these approaches could yield valid proofs and the resources that the probers drew upon to implement these strategies successfully.

We begin by discussing an excerpt of Ronald working on task 4 (Prove that $A^2+AB+B^2 \geq 0$ for all real numbers A and B).

[1] [00:00] Alright. Prove that $A^2+AB+B^2 \geq 0$ for all real numbers A and B.
[2] [00:07] Alright well um... hmmm is greater than or equal to zero.
[3] [00:13] Well this is a sum of three terms so one thing we could use is maybe the arithmetic mean geometric mean thing.
[4] [00:24] Lets see if that works out. Well we have let see, what does the arithmetic mean thing say. $(x+y+z)/3 \geq \sqrt[3]{xyz}$. And if we can show that maybe $\sqrt[3]{xyz} \geq 0$ then we are done.
[5] [00:42] So lets see A^2+AB+B^2 is equal to, not equal to, greater than or equal to, three times the cube root of their product. So that would be oh wow A^3 B^3, [writes $\sqrt{A^2ABB^2}=\sqrt{A^3B^3}=AB$], which is equal to [writes $(A^2+AB+B^2)/3$].
[6] [00:59] Wait, wait wait a second. Um....oh. The cubed root of A cubed B. That's 3AB. hmmm. [$3AB \leq A^2+AB+B^2$]. Alright well this is not necessarily equal to zero for all A and B so.
[7] [01:09] I guess that doesn't really work.
[8] [01:20] Or maybe I'm doing this wrong. Anyways well.
[9] [01:22] The other thing I noticed was that this term appears in the factorization of $A^3 - B^3$.
[10] [01:36] So $A^3-B^3=(A-B)(A^2+AB+B^2)$. Alright lets see.
[11] [01:49] Alright well if we assume that $A \geq B$ then.... we should be able to..
[12] [02:08] Alright. So I guess since this term A^2+AB+B^2 is symmetric. So I guess I'll write this down here too. Because of the symmetry of, I don't even know how to write this, of the above term. We can assume without loss of generality, which I will abbreviate here as WLOG. That $A \geq B$. Then A-B is greater than or equal to zero. So, So $A-B \geq 0$ and $A^3-B^3 \geq 0$. And well that implies that well that. Well lets see, we'll put this on a separate line, but, $A^3-B^3 \geq 0$ implies that it's factorization $(A-B)(A^2+AB+B^2) \geq 0$. And I guess I'll have to be a little careful here. We'll split this into two cases.
[Ronald proceeds to write a proof with two cases. Case 1: A=B. $A^2+AB+B^2= A^2+ A^2+ A^2=3A^2 \geq 0$. Case 2: $A>B$=>Both A^3-B^3=A-B are positive. Since $A^3-B^3=(A-B)(A^2+AB+B^2)$, A^2+AB+B^2 is also positive. The entire proof takes 4:37]

In the above, after reading the task in [1], Ronald spends only six seconds reflecting on it before choosing his first strategy in [3], the arithmetic mean geometric mean inequality. Much like the other prober excerpts discussed Ronald spent little time reflecting on the task before choosing an initial strategy. Next notice that Ronald abandons the arithmetic mean/geometric mean strategy after encountering difficulty in [9]. He chooses to switch to a new strategy in [9], using the fact that A^2+AB+B^2 appears in the factorization of A^3-B^3. This strategy yields a clever solution, which Ronald proceeds to write up.

Now we shift to discussing successful drilling. Below is Theodore's work on task 3 (Suppose f(x) is a differentiable even function. Prove that f'(x) is an odd function.):

[1] [00:00] Okay, suppose *f(x)* is differentiable and even. Prove that f prime is odd.

[2] [00:12] Alright, so *f(x)* is differentiable so it's like continuous and smooth. What's the definition? I need the definition of differentiability, which is an annoying one, $\lim_{h\to 0} (f(x)-f(x+h))/h$ exists as h→0 from the left or right.

[3] [00:43] *f(x)* is even. So let me just draw what an even function might look like. So across the origin. [Draws an arbitrary function with reflectional symmetry about the y-axis].

[4] [01:16] So basically, if I look at a slope on this function [draws a tangent on the negative side of the previously drawn function], then I look on the other side [draws a mirror tangent on the positive side], it's the same slope but negative. So that's going to show that f prime is odd.

[5] [01:41] So if I look at the definition of derivative as a slope and then I find the derivative on the negative side using the fact that it's even I should get the negative of the derivative, showing that f prime is odd.

[6] [01:56] [Theodore writes up a formal proof showing that- $f'(x)=-f'(-x)$ by manipulating the limit definition of $f'(-x)$ until it looks like $-f'(x)$.]

After reading the problem in [1] Theodore spends time recalling the limit definition of derivative in [2] and constructing a graph of a generic even function in [3]. This takes over a minute. Spending significant amounts of time getting to grips with a task is typical of drillers. The graph Theodore produced while working to understand the task is then use to construct an informal argument regarding why the result holds in [4]. Translating this argument is the basis of his proof strategy. Theodore works to formalize his intuitive argument by relating it to notation in [5] and then using this notation to write out a rigorous proof.

Metacognition.

A common assumption in the problem solving literature is that good problem solvers will devote substantial time to understanding the problem and formulating a plan to solve it (Polya, 1957; Schoenfeld, 1985). The probers in this paper did not do this. They began implementing a plan very early in the proof construction process. It is interesting to observe how the probers avoided what Schoenfeld (1985) referred to as "chasing wild mathematical geese" (i.e., failing to solve a problem since they spent all their time pursuing an unproductive approach). The probers avoided this by switching approaches frequently. They often switched the first time progress implementing an approach ceased to progress smoothly.

In contrast the drillers' proof construction efforts were consistent with recommendations in the literature. They followed the first three stages of Polya's (1957) problem solving model. These students would understand the problem, choose a plan, and implement a plan[1]. The drillers were generally able to avoid "chasing wild mathematical geese" by carefully choosing productive plans that they believed were likely to succeed from the outset.

Resources

Since probers abandon a plan at the first sign of difficulty, in order to excel as a prober, a student would need to be able to quickly generate many plausible plans. One way that the probers did this was associating concepts with approaches used to write proofs related to those concepts. This type of "symbol sense"—immediately associating mathematical terms and expressions with other mathematical concepts and proving techniques—provide the probers with considerable power in quickly formulating many plausible plans.

---

[1] These participants were less consistent in implementing Polya's fourth stage of problem solving, looking back. Theodore did so-- in fact, after writing a correct proof, Theodore sometimes put in considerable effort to make the proof more elegant. Frank and Sarah did not engage in looking back, although this may have been due to the time constraints imposed on them in this study.

In order to excel as a driller, a student would need to consistently generate informal arguments and work effectively to translate these into rigorous proofs. We note that the drillers in this study were quickly able to translate the assumptions in the to-be-proven statement into a graphical representation. They were also comfortable with formalizing a graphical argument into a deductive proof within the conventional verbal-symbolic representation system in which proofs were written (cf., Weber & Alcock, 2009). This translation ability is an important resource possessed by the drillers in this study. Although discussion of the mechanism behind such translation is beyond the scope of this paper, in related work we have established that ability to translate effectively is a non-trivial skill involving multiple interrelated abilities (Zazkis, Weber, Mejía-Ramos, 2014).

Discussion

This study explored the proving behaviors of star mathematics students. There was substantial variation in how stars generated proofs. The prober/driller distinction was introduced in this work to account for this variation. Probing involves moving between multiple approaches in an attempt to find an efficient solution. Drilling involves understanding the problem well, using this understanding to see why the result holds and then formalizing this reasoning. We are not suggesting that every successful mathematics student can be classified in this way, or that most students will reside on one extreme of the driller-prober continuum. We are only claiming this data illustrates the existence of two types of successful mathematics majors. Further, we note that we only discussed participants' behavior with tasks in calculus. We would need to observe these participants in other domains to see if their proof writing approaches varied.

The drillers are consistent with several theories about how problem solving and proof writing should ideally proceed (e.g., Garuti, Boero, & Lemut, 1998; Gibson, 1998; Sandefur et al., 2013; Raman, 2003; Weber & Alcock, 2004). Consequently, the data in this study expand the scope of these theories by illustrating how they can account for the behavior of some highly successful mathematics majors, which to our knowledge has not yet been done.

In contrast to the drillers, the probers were not consistent with the problem solving models of Polya (1957) and Schoenfeld (1985) and they did not base their proofs on informal arguments. Consequently, the prober/driller distinction contributes to theory in two ways. First, the probers provide interesting counterexamples to claims that success in proof writing requires one to understand problems deeply and to form intuitive arguments for why theorems are true. Second, the probers/driller distinction suggests that successful mathematics majors are not homogeneous in their proof writing behavior.

A possible explanation of prober's success might be the types of proofs that students are asked to produce in undergraduate mathematics. Most of these have proofs that are fairly short and we expect that mathematics majors can write these proofs in a relatively short period of time (e.g., write four or five proofs on a 50 minute exam). Hence, if probers find an approach that looks computationally or conceptually intimidating, they can abandon this approach with the confidence that an easier approach can inevitably be found.

Since a probing approach enabled some mathematics majors to earn A's and solve challenging problems, perhaps this is a high enough expectation for mathematics majors. That being said, one of the probers, who is now enrolled in a mathematics PhD program, indicated that the nature of graduate mathematics has encouraged him to shift to a driller approach. This data point hints that probing maybe less effective for graduate mathematics.

References

Alcock, L. J. & Simpson, A. P. (2004). Convergence of sequences and series: Interactions between visual reasoning and the learner's beliefs about their own role. *Educational Studies in Mathematics, 57*(1),1-32.

Alcock, L. & Simpson, A. (2005). Convergence of sequences and series 2: Interactions between non-visual reasoning and the learner's beliefs about their own role. *Educational Studies in Mathematics, 58*, 77-110.

Alcock, L. and Weber, K. (2010). Undergraduates' example use in proof production: Purposes and effectiveness. *Investigations in Mathematical Learning*, 3(1), 1-22.

Burton, L. L. (2004). *Mathematicians as enquirers: Learning about learning mathematics*. Springer: Dordrecht.

Carlson, M. P. (1999). The mathematical behavior of six successful mathematics graduate students: Influences leading to mathematical success.*Educational Studies in Mathematics*, *40*(3), 237-258.

Carlson, M. P., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational Studies in Mathematics*, *58*(1), 45-75.

Garuti R., Boero P. & Lemut E. (1998). Cognitive unity of theorems and difficulty of proof. *Proceedings of the 22nd PME Conference*, Stellenbosh, South Africa, 2, 345 – 352.

Gibson, D. (1998). Students' use of diagrams to develop proofs in an introductory real analysis. *Research in Collegiate Mathematics Education,* 2, 284-307.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education III* (pp. 234–283). Providence, RI: American Mathematical Society.

Hart, E. (1994). A conceptual analysis of the proof writing performance of expert and novice students in elementary group theory. In J. Kaput, and Dubinsky, E. (Ed.), *Research issues in mathematics learning: Preliminary analyses and results* (pp. 49-62). Washington: Mathematical Association of America.

Lockwood, E., Ellis, A. B., Dogan, M. F., Williams, C., & Knuth, E. (2012). A framework for mathematicians' example-related activity when exploring and proving mathematical conjectures. In *Proceedings of the 34th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 151-158).

Pinto, M., & Tall, D. (1999). Student constructions of formal theory: Giving and extracting meaning. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 65–73). Haifa, Israel: PME.

Polya, G. (1957). *How to Solve It: a new aspect of mathematical method, ed*. London: Penguin.

Raman, M. (2003). Key ideas: What are they and how can they help us understand how people view proof? *Educational Studies in Mathematics, 52,* 319-325.

Reif, F. (2008). *Applying cognitive science to education: Thinking and learning in scientific and other complex domains.* MIT Press: Cambridge, MA.

Samkoff, A., Lai, Y., and Weber, K. (2012). On the different ways that mathematicians use diagrams in proof construction. *Research in Mathematics Education*, 14(1), 49-67.

Sandefur, J., Mason, J., Stylianides, G. J., & Watson, A. (2013). Generating and using examples in the proving process. *Educational Studies in Mathematics*, *83*(3), 323-340.

Schoenfeld, A. H. (1985). *Mathematical Problem Solving*. Orlando, FL: Academic Press.

Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*, *29*(2), 123-151.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, *48*(1), 101-119.

Weber, K., Inglis, M., & Mejía-Ramos, J.P. (2014). How mathematicians obtain conviction: Implications for mathematics instruction and research on epistemic cognition. *Educational Psychologist*, *49*, 36-58.

Zandieh, M., Roh, K. H., & Knapp, J. (2014). Conceptual blending: Student reasoning when proving "conditional implies conditional" statements. *The Journal of Mathematical Behavior*, *33*, 209-229.

Zazkis, D., Weber, K., & Mejía-Ramos, J.P. (2014). Activities that mathematics majors use to bridge the gap between informal arguments and proofs. *Proceedings of the Conference for Psychology of Mathematics Education.* Vancouver, Canada.

APPENDIX: Tasks

1: Suppose $f(0)=f'(0)=1$. Suppose $f''(x)>0$ for all positive *x*. Prove that $f(2)>2$.

2: Prove that the only real solution to the equation $x^3 + 5x = 3x^2 + \sin x$ is x=0.

3: Suppose $f(x)$ is a differentiable even function. Prove that $f'(x)$ is an odd function.

4: Prove that $a^2+ab+b^2 \geq 0$ for all real numbers *a* and *b*.

5: Suppose $f''(x)>0$ for all real numbers *x*. Suppose *a* and *b* are real numbers with $a<b$. Define $g(x)$ as the line through the points $(a, f(a))$ and $(b, f(b))$. Prove that for all $x \in [a,b]$, $f(x) \leq g(x)$.

6: Prove that $\int_{-a}^{a} \sin^3(x)\,dx = 0$ for any real number *a*.

7: Let *f* be differentiable on $[0,1]$, and suppose that $f(0)=0$ and f' is increasing on $[0,1]$. Prove that $g(x)=\dfrac{f(x)}{x}$ is increasing on $(0,1)$.

Task		1	2	3	4	5	6	7	Total Correct
Star % Correct		83%	67%	83%	83%	17%	100%	17%	64%
Non-star % Correct		12%	7%	49%	39%	0%	43%	0%	22%
Probers	Chase	√	√	√	√		√		5
	Ronald		√	√	√		√		4
	Wolfe	√		√	√		√		4
Drillers	Sarah	√			√		√	√	4
	Frank	√	√	√			√		4
	Theodore	√	√	√	√	√	√		6

Table 1: Questions stars answered correctly.

Application of multiple integrals: From a physical to a virtual model[1]

Ivanete Zuchi Siple
Elisandra Bar de Figueiredo
State University of Santa Catarina – BRAZIL

This paper describes the evolution of a teaching technique for calculating volume using multiple integrals - from a physical to virtual model - in a Differential and Integral Calculus (CDI) course involving students in math and engineering programs at a public university in Brazil. This transition was made possible by the use of e-learning tools on the Moodle platform, which is used to support classroom activities. As a result of this technique, we saw the deepening of the concepts of parameterization taught in analytic geometry that were necessary to build the model using graphics software (virtual) as well as a change in the dynamic of sharing the technique among teachers and students through discussion forums in a continuous process.

Key words: Multiple integrals. Technology. Model. Teaching technique.

Introduction

In this paper, we will report the evolution of a teaching technique that began as a manual technique in the classroom, developed over ten years in a teaching project in the CDI course, with students studying exact sciences at a public university. The course was then moved to a blended learning modality. In both modalities, the work involved the application of multiple integrals in calculating the volume of a given solid taking into account its different representations - analytical, graphical and algebraic. In our experience as teachers, we have found that the teaching and learning process of this course is not an easy task. It is a major challenge for most teachers and students involved, particularly in the teaching and learning of multiple integrals. One of the major difficulties is the three-dimensional visualization that is difficult to achieve with pencil and paper, and generates many difficulties in the graphical representation of the surfaces.

In order to overcome such difficulties, it is essential to create mechanisms to enable students transitioning between different registers of representation of multiple integrals (DUVAL, 2003). Thus, we believe that the three-dimensional features of a computing environment, such as Winplot, can be instrumental in coping with this problem.

So initially we will make some considerations on how the technology supported the classroom teaching system and how it contributed to the learning of mathematical content and continuous evaluation. We will describe the methodology developed, and then present parts of the discussions of the work in a virtual forum and inferences derived from this technique.

Calculation of volume: a methodology mediated by technology

With the goal of enriching the construction of the teaching and learning processes, we can speak of the combination of two educational modalities - classroom and virtual (Araújo & Paneirai, 2012; Oliveira, 2013). In this context, blended learning emerges. This term combines the two types of learning: classroom learning and online learning (e-learning). The blended learning modality seeks to enhance the best of classroom and online as the authors Owston, Garrison and Cook (2006, p 348, cited OLIVEIRA, 2013, p.26.) describe: "Blended learning has the potential to integrate immediate, spontaneous and rich verbal communication

[1] The researchers wish to thank the Research and Innovation Support Foundation of Santa Catarina (FAPESC) for financial support of the PEMSA research group.

with reflective, rigorous, and precise written communication, as well as visually rich media and simulations."

According to Araújo and Paneirai (2012), in blended learning there is a convergence of experiences from the classroom to the virtual classroom, allowing teachers and students to build knowledge in a more pluralistic and participatory way. In this paper, the Moodle learning management system (LMS) and Winplot graphic software were used, both free. The choice of these technologies was due to the potential convergence of the extension of the activities done in the classroom and complemented in the virtual environment.

The activity consisted in giving students a volume calculation problem for a solid defined by the intersection of two or more surfaces. The problems were given by the teacher and each team (randomly selected) had the task of solving that problem, by following these steps:

- Identification of the surfaces that delimit the solid;
- Graphical representation of the projections of the intersections of surfaces in the coordinate planes;
- Construction of the solid (model);
- Determination of the volume of the solid using triple integrals;
- Sharing.

These steps were always present in both modalities. The differences occurred in the model construction stage, which was changed from physical to virtual, and in the form of sharing, which went from the classic final presentation in class to an ongoing discussion mediated by technology. To guide this discussion, each team had to:

- Start a thread in the discussion forum on the choice of technological tool for building the solid, and the difficulties related both to the construction of the solid and the resolution of the given problem;
- Discuss and comment on the work presented by the other teams in the virtual environment;
- Answer questions regarding their work in the Moodle discussion forum.

The virtual model activity included three CDI classes, with an average of 40 students per class in the first semester of 2014.

From a physical to a virtual model: extracts from the projects

Below, we present a parallel between extracts of the work done by students in the classroom to that of the students in the blended learning modality.

In the classroom modality, the use of the graphics capabilities of computer technology to aid visualization of the projections, the intersection of the surfaces and the resulting solid was encouraged. So in many studies the students used Winplot to make sketches of the projections, the graphical representation of surfaces and the graphical visualization of the intersection of these surfaces. In some cases, they used the parameterization of the surfaces to obtain the resulting solid, as shown in Figure 1, and in other cases they did not parameterize the resulting solid, as shown in Figure 2.

In Figure 1, Team A identified the surfaces given by their algebraic representations doing a conversion of this representation to the graphic register and also doing it for the algebraic register[2]. In the graphic register, each cylinder is plotted first, then the overlap of the two and finally the solid without the excess parts, which highlights the importance of the use of parameterization. The team built the model found in Figure 3a.

In Figure 2, Team B also used the computational resources to plot the projection on the coordinate plane, as well as the graphical representations of the surfaces that delimit the solid.

[2] Due to the restriction on the number of pages, several steps of Team A's presentation were omitted, including the algebraic procedure in the identification of surfaces, projections and resolution.

However, parameterization that would allow the viewing of the resulting solid was not performed. The delimitation of the resulting solid was explored in the presentation of the physical model, as shown in Figure 3b.

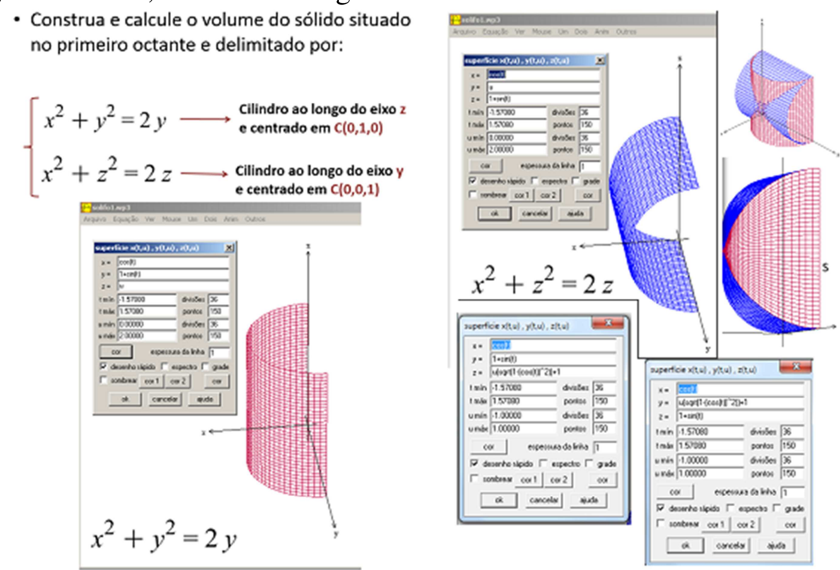


Figure 1: Identification of the surfaces, three-dimensional graphics and parameterizations of Team A.

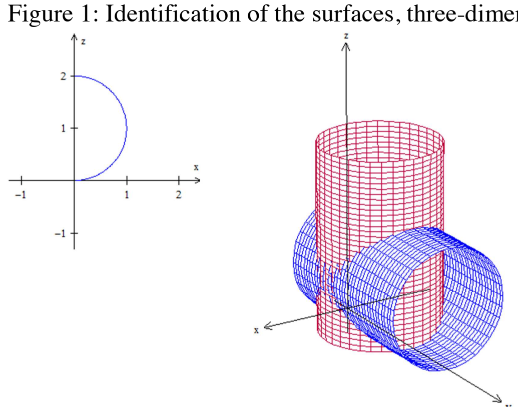


Figure 2: Three-dimensional graphical representation and projections of Team B

All teams shared the work among their classmates in a classroom presentation at a scheduled time, using the physical model (see Figures 3a and 3b), the technological resources for 2D and 3D graphical representations of the activity developed and a justification of the choice of the type of integration that made it possible to determine the numerical value of the volume of the solid.

Figure 3a: Team A's model

Figure 3b: Team B's model

In the blended learning modality, all steps described above were also present: the identification of the surfaces, the graphical representation of the projections of the intersections of the surfaces in coordinate planes, the choice of the technological tool and determining the volume of the solid using triple integrals. What changed in this modality was the virtual construction of the model, and therefore knowledge of parameterization became essential. So there was a discussion among the students in Moodle about how to do this parameterization based on the tools chosen. There was an interaction between the teams to parameterize their respective solids. However, the main difference was the way that the work was shared, not being restricted to a single presentation of the teams on a predetermined date. From the time the work was assigned, the students had to post to the Moodle forum, providing information about the progress of their work to their classmates. In this environment, the evaluation of the work was performed continuously since with every step posted by the team, there could be an interaction from the teacher or a classmate, either to ask for clarification, ask for help or facilitate a situation where the identification of the surfaces or the solution of the integral was not correct.

Figure 4 shows the posting of the work of Team C, which used parameterization of curves to make the virtual model. The problem was to determine the volume of the solid delimited by $z = 0$; $x = 0$; y = 3; $x = \sqrt{3y}$ and $x + z = 3$. Initially, they did a graphical representation of the surfaces that delimited the solid, identifying the surfaces. However, they represented and identified only planes z = 0; $x = 0$; y = 3. After the intervention of the professor, about the graphical representation of the other surfaces, they presented these ones too, as shown in Figure 4a. An interesting idea of this team was that they used the knowledge of parameterization of curves in space to delimit the solid (Figure 4b), including posting the way to do it in their tool of choice - Winplot (Figure 4c), after the question from the professor. In addition, they also made the parameterization of the surfaces, as illustrated in the virtual model (Figure 4d).

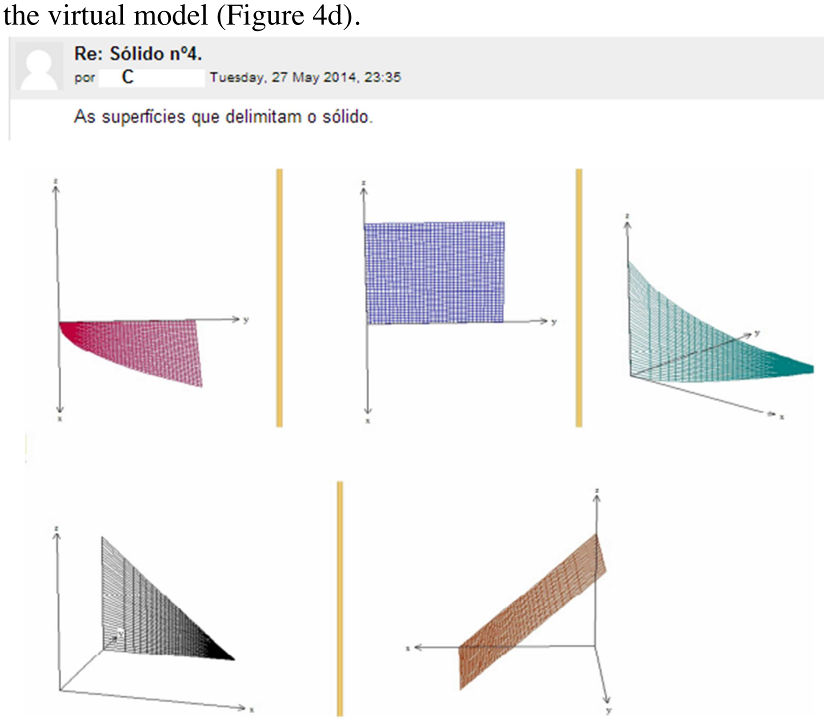


Figure 4a: Graphical identification of the surfaces

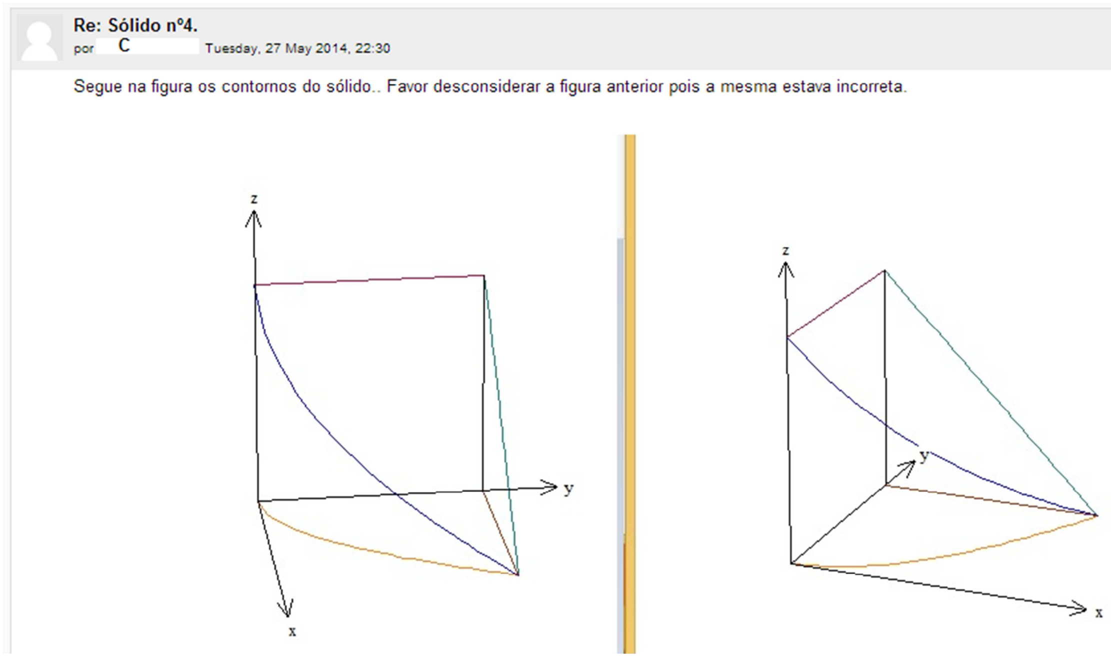
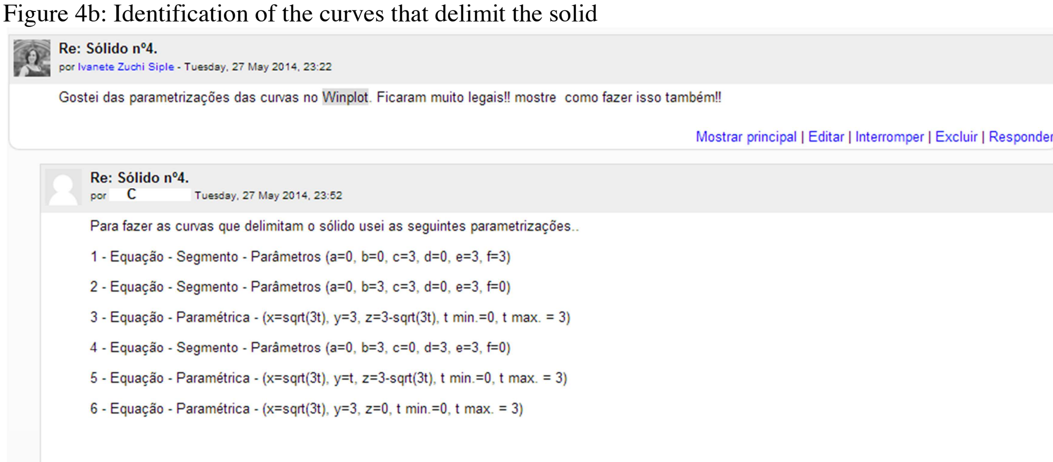


Figure 4b: Identification of the curves that delimit the solid


Re: Sólido nº4.
por Ivanete Zuchi Siple - Tuesday, 27 May 2014, 23:22

Gostei das parametrizações das curvas no Winplot. Ficaram muito legais!! mostre como fazer isso também!!

Mostrar principal | Editar | Interromper | Excluir | Responder

Re: Sólido nº4.
por C Tuesday, 27 May 2014, 23:52

Para fazer as curvas que delimitam o sólido usei as seguintes parametrizações..

1 - Equação - Segmento - Parâmetros (a=0, b=0, c=3, d=0, e=3, f=3)

2 - Equação - Segmento - Parâmetros (a=0, b=3, c=3, d=0, e=3, f=0)

3 - Equação - Paramétrica - (x=sqrt(3t), y=3, z=3-sqrt(3t), t min.=0, t max. = 3)

4 - Equação - Segmento - Parâmetros (a=0, b=3, c=0, d=3, e=3, f=0)

5 - Equação - Paramétrica - (x=sqrt(3t), y=t, z=3-sqrt(3t), t min.=0, t max. = 3)

6 - Equação - Paramétrica - (x=sqrt(3t), y=3, z=0, t min.=0, t max. = 3)

Figure 4c: Explanation of the parameterization of the curves in Winplot

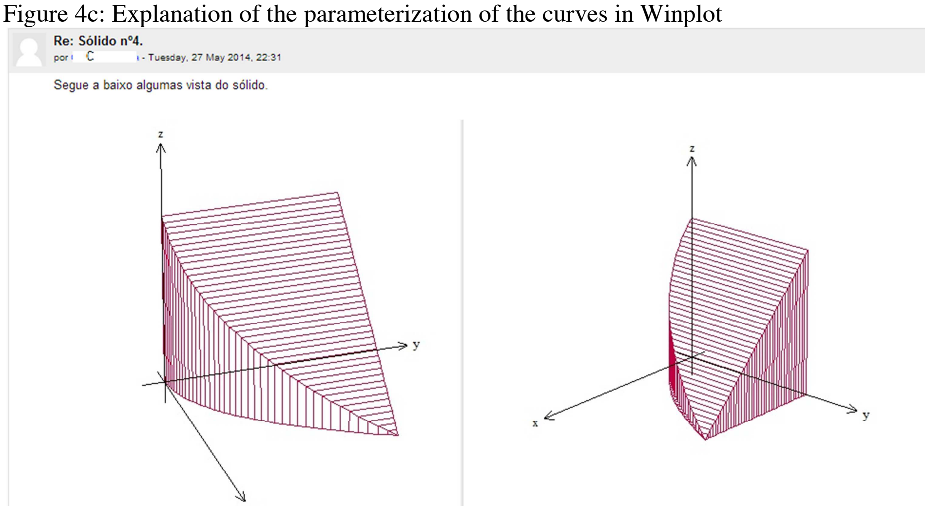


Figure 4d: Virtual model

Team D constructed and determined the volume of the solid delimited by $2z = x^2 + y^2 - 2$ and $z = \frac{2}{1+x^2+y^2}$. In their first post, the team posted the outline of the solid with pencil and paper and in Winplot, without using parameterization. After a discussion among Team D, classmates and the professor, the team constructed their virtual model, as shown in Figure 5.

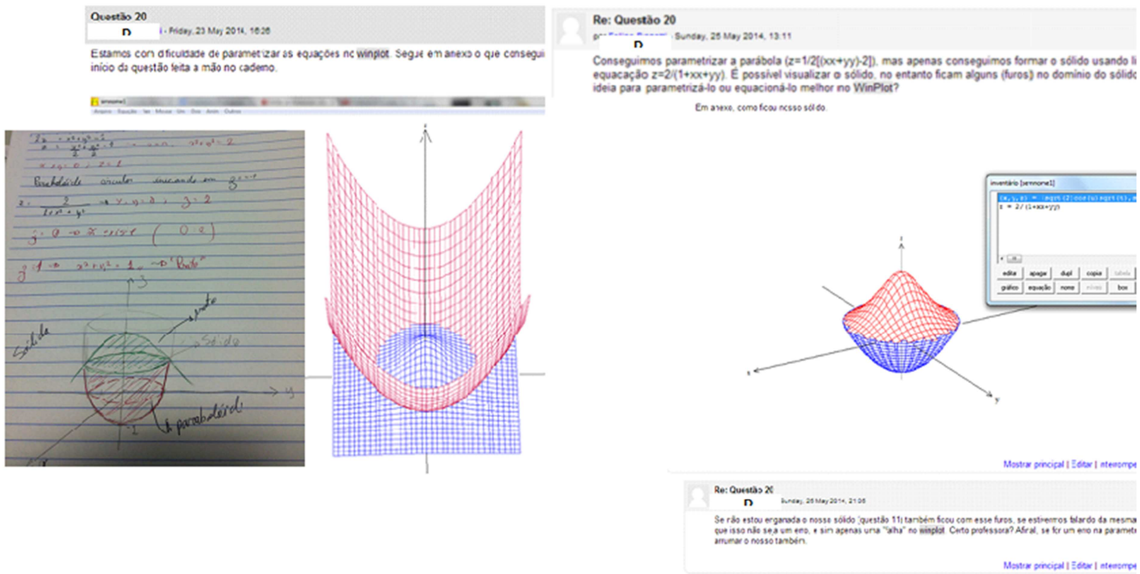


Figure 5: Steps to construct the solid - from pencil and paper to parameterization in Winplot

It is important to reiterate that discussions also occurred in solving the integrals, especially regarding the choice of the domain of the integration region as well as in the choice of coordinate system - Cartesian, cylindrical or spherical.

Conclusion

Exploring the link between the classroom and distance modalities can enhance important elements in learning mathematics. The construction of the model in the virtual environment allowed for student involvement in active and continuous learning. Specifically addressing the use of technology in education, it is essential that the issues related to the use of virtual spaces for teaching and learning have their place in the program curriculum, especially for teacher training.

The exploration of the relationship between the different registers of representation of a surface, mediated by a technological tool, can enhance important elements in learning mathematics. The movement between these different types of registers was a challenge for both students and teachers, because in traditional classes generally we observe that there are students with excellent algebraic skills and weak three-dimensional visualization or vice versa. This challenge is present in mathematical language and the meaning of concepts, as in the analytical, algebraic and graphical form of surfaces. It may be possible to use this research to verify the ability of students in treatment transformations occurring within the same system of representation, e.g. algebraically manipulating the equations of the surfaces; and conversions consisting of a system change conserving the selected objects, for example moving from the algebraic representation of a surface to its graphical representation.

In the move from the physical model to the virtual model, it was possible to explore the important connections of the CDI course with courses such as analytical geometry and vector calculus. In the surface and graphical projection identification stage in the pencil and paper environment, students needed to have knowledge of analytical geometry and, in general, used the Cartesian coordinate system that is not always the simplest graphical representation for a computing environment. Therefore, the student had to recognize the surfaces in the coordinate systems (Cartesian, cylindrical and spherical), since this facilitated the graphical representation as well as the resolution of the integral. In order to build the virtual model, it was necessary to have knowledge of surface parameterization, part of the content taught in vector calculus (next phase of the program).

It is also important to discuss the challenges, both for the professor and the student teacher, of using LMS environments such as Moodle in classroom activities. One of the great challenges of an LMS is the time required for mediation, as well as the number of students. It is important to note that an activity such as the one reported in this paper requires time and dedication from the professor, in addition to technical training. The professor has an important role in classroom teaching and in distance education, usually requiring different approaches. For these approaches, he should propose interaction methods that enable the student to perform multiple connections, enabling cooperation and exchange of ideas.

As professors, we can observe that the purpose of this activity was not only to acquire specific knowledge, but also the possibility of both students and professors to participate at all stages of its development. The activity was not restricted to the team, since the others could ask, assist and review the solutions in each post. So the evaluation was ongoing, and based not only on the resolution of the task on which the team worked as an author, but also as a participant in the threads of other teams in the discussion forum.

We know that the issue of the use of technology in education is not new, but the potential of tools enables a privileged space for the interaction between students and teachers, exchanges of experiences between researchers and the extension and evolution of classroom resources.

References

Araújo, R. Panerai,T. (2012). Relato de Experiência de Blended Learning: O Moodle e o Facebook Como Ambientes de Extensão da Sala de Aula Presencial. In: *XVIII Congresso Brasileiro de Informática na Educação*. Anais do XVIII WIE.

Duval, R. (2003). Registros de representação semióticas e funcionamento cognitivo da compreensão em matemática. In: Machado, S.D.A. et al. *Aprendizagem em matemática: registros de representação semiótica*. Campinas, SP: Papirus, p.11-33.

Oliveira,C.M.S. (2013). *Aprendizagem em Ambientes de Blended-Learning - uma Abordagem na Formação Contínua de Professores*. Instituto de Educação Faculdade de Ciências: Universidade de Lisboa, 130p. Dissertação de mestrado.