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Foreword

As part of its on-going activities to foster research in undergraduate mathematics education and the dissemination of such research, the Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education (SIGMAA on RUME) held its nineteenth annual Conference on Research in Undergraduate Mathematics Education in Pittsburgh, Pennsylvania from February 25 - 27, 2016. The conference is a forum for researchers in collegiate mathematics education to share results of research addressing issues pertinent to the learning and teaching of undergraduate mathematics. The conference is organized around the following themes: results of current research, contemporary theoretical perspectives and research paradigms, and innovative methodologies and analytic approaches as they pertain to the study of undergraduate mathematics education. The program included plenary addresses by Dr. Sean Larsen, Dr. Peg Smith, and Dr. David Stinson and the presentation of over 160 contributed, preliminary, and theoretical research reports and posters.

The Proceedings of the 19th Annual Conference on Research in Undergraduate Mathematics Education are our record of the presentations given and it is our hope that they will serve both as a resource for future research, as our field continues to expand in its areas of interest, methodological approaches, theoretical frameworks, and analytical paradigms, and as a resource for faculty in mathematics departments, who wish to use research to inform mathematics instruction in the university classroom. The RUME COnference Proceedings include conference papers hat underwent a rigorous review by two or more reviewers. These papers represent current work in the field of undergraduate mathematics education and are elaborations of selected RUME Conference Reports.

The proceedings begin with the winner of the best paper award and the papers receiving honorable mention and the pre-journal award winner. These awards are bestowed upon papers that make a substantial contribution to the field in terms of raising new questions or providing significant or unique insights into existing research programs. RUME Conference Reports, includes the Poster Abstracts and the Contributed, Preliminary and Theoretical Research Reports that were presented at the conference and that underwent a rigorous review by at least three reviewers prior to the conference. Contributed Research Reports discuss completed research studies on undergraduate mathematics education and address findings from these studies, contemporary theoretical perspectives, and research paradigms. Preliminary Research Reports discuss ongoing and exploratory research studies of undergraduate mathematics education. Theoretical Research Reports describe new theo- retical perspectives and frameworks for research on undergraduate mathematics education. Poster Reports were quite varied and described curriculum, research and theoretical contributions.

The conference was hosted by West Virginia Univerisity. Their faculty and student provided many hours of volunteer work that made the conference possible and pleasurable, we greatly thank the faculty, students and institution for their support.

Last but not least, we wish to acknowledge the conference program committee and reviewers for their substantial contributions to RUME and our institutions, for their support, the conference would not exist without you.

Sincerely,

Tim Fukawa-Connelly, RUME Conference Chairperson

Nicole Engelke Infante, RUME Conference Local Organizer

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September 2, 2016

Philadelphia

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Students' Meanings of a (Potentially) Powerful Tool for Generalizing in Combinatorics

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In this paper we provide two contrasting cases of student work on a series of combinatorial tasks that were designed to facilitate generalizing activity. These contrasting cases offer two different meanings (Thompson, 2013) that students had about what might externally appear to be the same tool – a general outcome structure that both students spontaneously developed. By examining the students' meanings, we see what made the tool more powerful for one student than the other and what aspects of his combinatorial reasoning and his ability to generalize prior work were efficacious. We conclude with implications and directions for further research.

Key Words: Combinatorics, Generalization, Counting problems, Mathematical meanings

Introduction and Motivation

Generalization is a fundamental mathematical activity with which students at all levels engage (Amit & Klass-Tsirulnikov, 2005; Lannin, 2005; Peirce, 1902), and yet there is still much to learn about ways to foster productive generalizing activity. In particular, most of the work on generalization has been with younger children, commonly in algebraic settings (Amit & Neria, 2008; Becker & Rivera, 2006; Cooper & Warren, 2008; Ellis, 2007b; Mulligan & Mitchelmore, 2009; Radford, 2006; Rivera, 2010; Steele, 2008). In the context of a larger study, we sought to better understand students' generalization in the domain of combinatorics which involves the solving of counting problems and provides students with opportunities to engage with accessible yet challenging tasks (e.g., Kapur, 1970; Tucker, 2002). In this paper, we compare and contrast two students' work on a series of combinatorial tasks, during which they each spontaneously introduced a potentially powerful tool for generalization in the combinatorial setting. Each of these students used this new tool, but they varied in the meaning they seemed to make it. As a result, they differed in how effective they were able to be in using the tool generally and solving combinatorial tasks. We seek to answer the following research question: *What meaning do students make of the same spontaneously generated tool (which we refer to as the 11xx structure), and what do these meanings suggest about students' generalization in combinatorial contexts?* The results in this paper help to inform research on students' meanings in the context of both their generalizing activity and their combinatorial thinking.

Literature Review and Theoretical Perspective

Generalization. The act of generalizing is a key aspect of students' mathematical development, and both mathematics education researchers (e.g., Amit & Klass-Tsirulnikov, 2005; Davydov, 1972/1990; Ellis, 2007b; Vygotsky, 1986) and policy makers emphasize its importance (the Common Core State Standards highlight generalization in both the content and the practice standards; Council of Chief state School Officers, 2010). We seek to extend the current work on generalization by focusing on undergraduate students in the context of combinatorics. The tasks were designed with the overall aim of facilitating students' generalizing activity, and for this purpose we follow Ellis (2007a) (who drew on Kaput, 1999) in defining generalization as "engaging in at least one of three activities: a) identifying commonality across cases, b) extending one's reasoning beyond the range in which it originated, or c) deriving broader results about new relationships from particular cases" (p. 444).

We also think of generalization as being closely tied to notions of abstraction laid out by Piaget, especially reflecting abstraction, which "projects and reorganizes, on another conceptual level, a coordination or pattern of the subject's own activities or operations" (von Glasersfeld, 1995, p. 105). We additionally consider generalization in terms of reflected abstraction, which "also involves patterns of activities or operations, but it includes the subject's awareness of what has been abstracted" (p. 105). Broadly, these terms and these notions of reflective abstraction frame the students' generalizing activity we describe in this study.

Combinatorial Reasoning. We chose the context of combinatorics in part to examine generalization in a novel context, but we were also motivated to contribute to previous work on students' combinatorial thinking. There is evidence that students struggle with solving counting problems correctly (e.g., Batanero, Navarro-Pelayo, & Godino, 1997; Hadar & Hadass, 1981), and we hope to contribute to the existing literature by providing instances of meaningful combinatorial reasoning that might ultimately inform instruction. More specifically, we use the idea of a set-oriented perspective (Lockwood, 2014), in which we view attending to sets of outcomes as an intrinsic component of solving counting problems. Other prior research has shown the value of focusing on sets of outcomes, including demonstrating that listing outcomes is positively correlated with correctly solving counting problems (Lockwood & Gibson, 2016) and that outcomes may help students avoid problematic issues in counting such as order and overcounting (e.g., Lockwood, 2013). In our current study, adopting such a perspective means that we value students' work with outcomes and feel that focusing students' attention on outcomes can be a productive way to help foster rich combinatorial meaning.

Students' Meanings. We draw on Thompson's (2013) notions of meanings in this paper. He argues for the importance of developing meaning of the idea of *meaning* (p. 57) and that a greater emphasis on mathematical meaning could contribute to a more coherent educational experience for students overall. Thompson surveys different meanings of "meaning," and we adopt his Piagetian view of meaning as assimilating a scheme (p. 60). Thompson notes that, "From a Piagetian viewpoint, to construct a meaning is to construct an understanding – a scheme, and to construct a scheme requires applying the same operation of thought repeatedly to understand situations being made meaningful by that scheme" (p. 61). Also, importantly, Thompson emphasizes meaning from the students' perspective:

> "The meanings that matter at the moment of interacting with the students are the meanings that students have, for it is their current meanings that constitute the framework within they operate and it is their personal meanings that we hope students will transform" (p. 62).

For this reason, in this paper we seek to understand and interpret students' meanings in order to gain insight about what made their work particularly productive (or unproductive) in the contrasting cases. We use this notion of meanings in this paper because we have a situation in which two students introduce and use a tool that externally seems very similar, but their respective meanings of that tool cause them to use it differently. We thus find it useful to discuss the variety of meanings students had about what appears to be a very similar mathematical phenomenon.

Methods

In this study we conducted a set of single, individual, hour-long interviews with nine calculus students as they worked through a series of tasks we call the Passwords Activity, and in this paper we report on two contrasting cases of students' work. We report on these students who had not been taught combinatorics formally at the college level, who we chose with the hope that we could capture their initial reasoning about accessible yet novel tasks. The main goal of the tasks was to put students in a situation in which we could evaluate their generalizing activity as well as gain insight into their combinatorial reasoning. We begin this section with a mathematical discussion of the tasks, and then we discuss the data analysis.

The Passwords Activity. By offering the trajectory of the Passwords Activity, we hope to highlight how the tasks serve to facilitate generalization in a combinatorial setting, and we also hope to prepare the reader for subsequent discussion of student work.

First, we had students solve the problem, *How many 3-character passwords can be made using the letters A and B?*, and we explicitly directed them to organize their work by completing tables according to the number of As in the password (Table 1).

Number of As	Number of Passwords
0	1
1	3
2	3
3	1

Table 1 – The 3-character A,B table

Students may fill out the entries in the table in a couple of ways. One possibility is simply to list the passwords for each row (or to read the respective numbers of passwords from a previously generated list). The patterns of the respective tables come up in subsequent work, (1, 3, 3, 1 for 3-character passwords, 1, 4, 6, 4, 1 for 4-character passwords, 1, 5, 10, 10, 5, 1 for 5-character passwords, etc.[1]), and students can use previously created tables in subsequent work to engage in both combinatorial reasoning and generalization. Although students need not be familiar with binomial coefficients to fill out these tables, another possible way of generating the table is to recognize that the entries in the rows of the table are binomial coefficients ($\binom{3}{k}$ for k = 0, 1, 2, 3). This is true because for a given number of As in a password, we may choose the positions in which the As will go. The placement of the As determines the password since there are only Bs remaining to fill the empty slots. Once students have completed the 3-character passwords problem, we have them repeat this process for passwords of length 4 and 5, generating Tables 2 and 3 below. There were some opportunities to observe generalization in building up these cases, as students could observe relationships and similarities among the tables or could make combinatorial observations that held across cases. We wanted the students to build (typically through partial or complete listing) the tables to see how they would use them as we progressed to the next part of the tasks.

[1] Again, we recognize that these numbers are rows in Pascal's Triangle, but pursuing the relationship with Pascal's Triangle is not our goal.

[2] We did not push him on this or investigate his understanding of the multiplication principle. This is a potentially important aspect of his combinatorial reasoning and his generalizing activity, but it is tangential to the focus of this

Number of As	Number of Passwords
0	1
1	4
2	6
3	4
4	1

Table 2 – The 4-character A,B table

Number of As	Number of Passwords
0	1
1	5
2	10
3	10
4	5
5	1

Table 3 – The 5-character A,B table

Then, we can have students move on to passwords involving the number 1, and the letters A and B. We had students make tables for 3-character and 4-character passwords, organized according to the number of 1s in the passwords (the 4-character A,B,1 table is in Table 5). Note that we can use the previous tables in the following way: we can think first of determining positions for the 1s (which the previous A,B table provides), and then the problem is reduced to counting passwords involving only As and Bs. For example, in making a table for the number of 4-character passwords with 1, A, and B, for each respective row of the table (0, 1, 2, 3, or 4 1s), we can first think of counting the number of ways of placing the 1s. There are 1, 4, 6, 4, 1 respective ways of doing this, which is reflected in the previous 4-character A,B table. The reason these numbers are the same is that there are the same number of placing, say, two As in a 4-character password as there are to place two 1s. Once this is established, note that for each row in the table, once the ones are placed it is just a matter of counting passwords of length 3 using only As and Bs, reducing the problem to a previous problem (specifically, there are 2^3 such passwords).

Number of 1s	Number of Passwords
0	$1 \cdot 2^4$
1	$4 \cdot 2^3$
2	$6 \cdot 2^2$
3	$4 \cdot 2^1$
4	$1 \cdot 2^0$

Table 5 – The 4-character A,B,1 table

The point is that it is possible, with some combinatorial insight and understanding of the outcomes' structure, to leverage the previous work from the A,B passwords in the more complicated A,B,1 passwords case. Again, in this case if the students do not yet have a formula for binomial coefficients, they can still engage with the task and engage in generalization. They can look back at tables created for 3, 4, or 5-character A,B passwords and use those numerical results for the first stage in the counting process. They can also recognize that for any of the positions that are not 1s, they are simply creating A,B passwords, and again they can leverage previous results to complete current tables.

In order to see the motivation for this set of tasks, and to see where the tasks could ultimately lead, we describe further stages of the task that involve passwords with multiple letters and multiple numbers (although most students did not proceed beyond A,B,1 passwords). We consider one particular example of 5-character passwords consisting of the letters A, B, or C, and the numbers 1 or 2, which we call 5-character A,B,C,1,2 passwords. As before, students could count the total number of such passwords in two ways – first by simply computing the total by arguing about the number of choices for each position, and second by making a table, this time according to number of numbers. There are 5^5 total passwords, because there are five choices (3 letters and 2 numbers) for each of the five positions. The table can be filled out as in Table 6.

Number of Numbers	Number of Passwords
0	$1 \cdot 2^5 \cdot 3^0$
1	$5 \cdot 2^4 \cdot 3^1$
2	$10 \cdot 2^3 \cdot 3^2$
3	$10 \cdot 2^2 \cdot 3^3$
4	$5 \cdot 2^1 \cdot 3^4$
5	$1 \cdot 2^0 \cdot 3^5$

Table 6 - A 5-character A,B,C,1,2 table

To justify the table entries, let us consider one of the rows – the fourth row that counts the 5-character A,B,C,1,2 passwords that have exactly 3 numbers. We can first select places that will be numbers (there are $\binom{5}{3}$ ways to do this, which is 10), and then we can fill in each of those number places with either 1 or 2, giving us 2^3. Then we know that the remaining two positions must be letters, and there are 3^2 ways to filling those positions with A, B, and/or C. The same line of reasoning holds for any of the rows, and summing the rows (which count disjoint cases of how many numbers are in the passwords) yields the total number of passwords.

This line of reasoning can be extended to a general case of counting n-length passwords consisting of x numbers and y letters. In this way, we achieve a general statement of the binomial theorem, which is the potential culmination of the Passwords Activity:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} .$$

We had only one student made it to the end of this sequence of tasks (he was a more advanced student is not discussed in this paper), but engagement with the activity still served to facilitate combinatorial reasoning and generalizing activity among all of the students.

Data Analysis. The interviews were videotaped and transcribed, and overall the videos and transcripts were analyzed so as to construct a narrative about the teaching experiment (Auerbach & Silverstein, 2003). We discussed the two contrasting cases with the entire research team and together formulated hypotheses about the students' meaning in each case via repeated viewings of video and reading of enhanced transcripts. In particular, we looked for instances in which students' generated the certain tool (the *11xx* structure described below) and discussed and analyzed what caused students to interact with and use the tool in different ways.

Results

In presenting our results, we describe two different students' meanings of the same phenomenon. We highlight these results both to show an interesting phenomenon that emphasizes a potentially powerful tool toward meaningful combinatorial generalization and to suggest that ascribing certain meanings to such tools may help students leverage them in impactful ways. In this section we describe their work and present the major findings, and we compare and contrast the students' work in the Discussion section.

Example 1 – Tyler. We begin with Tyler, who demonstrated an ability to reason comfortably and easily with outcomes. His method of solving the tasks typically involved some organized listing. For example, in trying to determine the number of 4-character A,B passwords with exactly two As, Tyler made the list in Figure 1 and gave the following explanation:

Tyler: Um. Yeah I guess I started with the first one being A um, and then I did like 2 A's consecutively and then B's, and then moved the B over one, and then, um, moved the next B over one… And then, after that I just start with the B and kind of did the same thing.

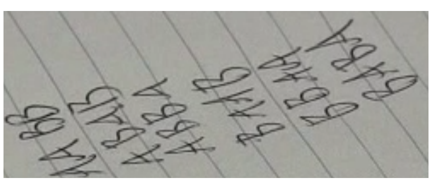

Figure 1

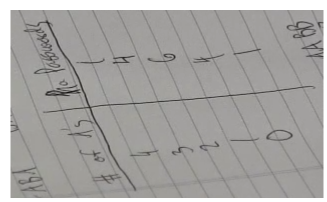

Figure 2

He ultimately correctly created the table for 3, 4 (Figure 2), and 5-character AB passwords. Here we highlight Tyler's willingness and ability to engage with organized listing activity in creating the tables, which suggests to us that he attended to outcomes.

Early in the interview, Tyler had established that there were a total of 2^n n-length passwords using only As and Bs. He established this primarily through noticing a numerical correlation after giving the totals for 3, 4, and 5-length passwords, read from his empirical tables (noticing the 3-character AB passwords table had 1+3+3+1 = 8 total passwords). He went on to write the relationship "*n length = 2^n combos*," but we suspect that he did not meaningfully understand the multiplication principle as a combinatorial way of explaining the expression 2^n.[2]

We then moved on to counting passwords that consist of characters 1, A, or B. Tyler felt that there was more to keep track of, but he persisted with listing outcomes and filling out the table as he had in the previous situations. He managed to list the entire table for the 3-character A,B,1 passwords, and again he used systematic listing used to do so, and he seemed to maintain a combinatorial understanding of the entries in his tables.

[2] We did not push him on this or investigate his understanding of the multiplication principle. This is a potentially important aspect of his combinatorial reasoning and his generalizing activity, but it is tangential to the focus of this paper and thus we do not address it here.

Next, we asked him with to fill out a table with the four-character A,B,1 passwords (organized according to number of 1s). He got started but paused and said, "I can't really think of any pattern," and he seemed to realize that this table would be more difficult to work out than the previous case. We directed him to perhaps start thinking about the rows for zero and one 1's (starting from that end of the table). Tyler then did something unexpected – he introduced a way of describing a general outcome involving 1s and *x*s. Specifically, he wrote out four general outcomes 1*xxx, x*1*xx, xx*1*x*, and *xxx*1 (Figure 3), and he used that reasoning to fill out the table he was working on. He discussed his reasoning in the following exchange, and we highlight that he referred back to his 4-character A,B table in Figure 4.

Tyler: *Yeah. Ok so the 0 [row] was gonna be, what did I come up with here [refers back to the 4-character AB table] 10, 15, 16 if uses, um. And then the 1, so what I was thinking, what I was saying earlier. How there is only a certain amount of spots for it, like it has to be, like I'm just gonna use x cause, um, has to be in one of these spots... [draws Figure 3]*

Int.: *Great.*

Tyler: *So there's, now there's just 3 x's um, and I know that for…3 spots with 2 different letters there's going to be 8 different ways to do it [points back to the previous 3-character AB table, see Figure 4]…Um so I guess 8…there's 8 different of each of those just using this same table umm, there's just 32 so I want to say there's gonna be, um, 32 for just the 1.*

Int.: *Okay and you got, you're thinking of that as kind of the 4 times 8?*

Tyler: *Yeah I, just adding them all up.*

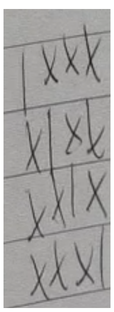

Figure 3

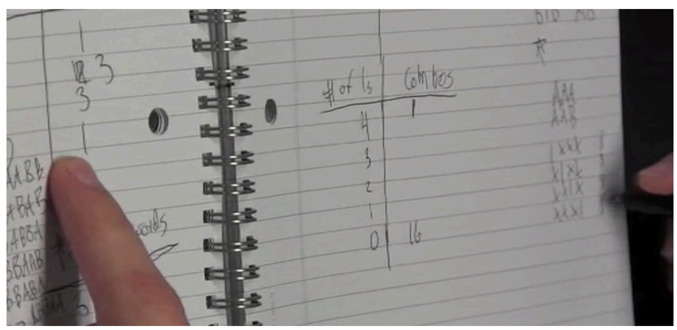

Figure 4

This was a key moment in Tyler's work. He spontaneously introduced a very powerful tool for how to count desirable passwords in the form of a general structure consisting of *x*s and 1s. (For ease of communication, we hereafter refer to the tool as "the *11xx* structure," which is meant to suggest the introduction of the variable of x as a means of representing a more general outcome.) We contend that this was a general representation of an outcome (a password), perhaps a product of his rich facility with listing. He realized that in each case where there was a 1 with three *x*s, there would be 8 such possibilities (because there were 8 total 3-character A,B passwords), and his total would be 32. He was thus able to recognize that he could use his previous case as a part of the more complicated new situation. We can further explore this moment of insight as he continued to use the *11xx* structure in filling out the rest of the 40-character A,B,1 table. Figure 5 shows his listing of x's and 1's in the four-character A,B,1 case, with exactly two 1's. There are exactly 6 of them, and the following exchange demonstrates Tyler's meaning of those six general outcomes as they relate back to his previous work. Specifically, note that he understands why 6 such outcomes would make sense, because he can understand that he is in a situation of arranging two distinct objects, which is what his previous work involving A,B passwords also entailed. He continued to work in a similar fashion for the case of three 1s, and he ultimately arrived at the correct table for 4-character A,B,1 passwords (Figure 6).

Tyler: *Yeah there you go. Is that all of them? Yeah so 6, 'cause that would make sense...*
Int.: *Does that 6 make sense?*
Tyler: *Does it? Uh, well that would that's um, 2 variables like instead of doing 3 things there's 2 umm, with the 4 combo, so 2, was 6 over here [points back to the 6 in the correct entry of the 4 character AB password table], so that's why I thought it made sense.*

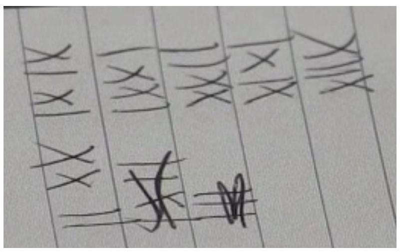

Figure 5

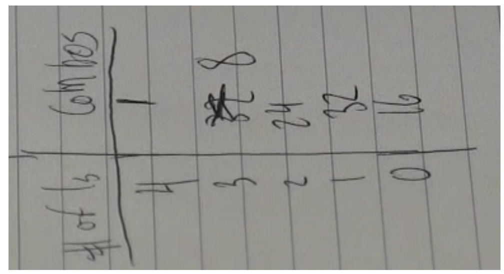

Figure 6

Tyler went on to use the same tool in subsequent cases involving 5-character A,B,1 passwords. Although we do not have space to detail all of his subsequent work, we provide one excerpt to highlight another important aspect of his activity. Specifically, we asked Tyler about creating another table of A,B,1 passwords that are 5 characters long. In the following excerpt he indicates that he would use the previous table that involved the coefficients 1, 5, 10, 10, 5, 1. In the underlined portion of the following excerpt we see that Tyler can think of the A,B,1 and the A,B situations as fundamentally similar because they are both involving combinations of two "things that are changing." We will argue that this is an important aspect of his productive generalizing activity. Tyler seemed to have a robust understanding of how he could use the tool to solve password problems involving more characters and more letters. And, more importantly, he was able to recognize two situations as being "the same."

Int.: *Um, what if I asked, so this same question like one 1, 10 letters but now a length 5 password. Like, what would the total number be?*
Tyler: *Um, I would use 1, 5, 10, 10, 5, 1.*
…
Int.: *And why did you go to that 1, 5, 10, 10, 5, 1…*
Tyler: *Um well these are all the number of combinations I can do, um, with 2 different, 2 things that are changing and this number of letters.*

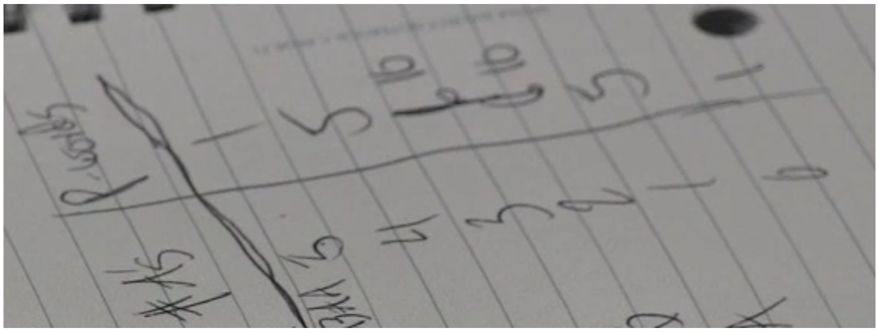

Figure 8

We point out a couple of important features of Tyler's work. First, it is noteworthy that Tyler spontaneously introduced a new, general structure that appropriately represented the situation and the outcomes he was trying to count. This is in and of itself impressive, and his work demonstrates an existence proof of the kind of thinking the Password Activity fostered in terms of combinatorial generalization. Second, Tyler was able to relate that new structure to his combinatorial activity to that point, and this relationship to prior work played a key role in him ultimately being able to solve more problems correctly. Importantly, he seemed to preserve the combinatorial meaning of the tasks and the situations as he related the *11xx* structure with his previous work. In terms of Tyler's meanings, we interpret that he understood the *11xx* structure as a general structure of the outcomes he had been working with, allowing him to relate back to a previous combinatorial situation involving just two objects (specifically, As and Bs). Although he did not demonstrate a deep combinatorial, multiplicative meaning of 2^n, he could recognize the 2^n as being numerically equivalent to a previous case, which he used effectively.

Example 2 – Richie. We now contrast Tyler's work with another calculus student, Richie. Richie, too, spontaneously introduced the *11xx* structure, but we highlight a key difference in that Richie was less successful in leveraging the new tool by relating it to previous circumstances. When making the tables for the A,B passwords tasks, Richie correctly filled out the tables, often using some listing, but it seemed as though he was more attuned to the numerical patterns he observed than in finding a combinatorial explanation that made sense. For example, when making a table for the 5-characer A,B passwords, we had the following exchange. Notice that his justification for why certain entries were in the table was based on the patterns he'd observed. This is not in and of itself problematic, but it shows perhaps that he was not establishing a robust combinatorial meaning but that his meaning was based on observed numerical regularity.

Richie: *So when you get to like the -- the second one, or it's not even like the second one, it's more like the one in the middle of 0 and 5 is going to be the most possibilities. And in previous problems it's been like 2 more than the preceding one.*
Int.: *Okay. Sure.*
Richie: *And I'm just assuming that this is 5, because the previous pattern's like increasing by 1.*

When we moved on to the A,B,1 passwords case, Richie, like Tyler, spontaneously moved toward a new structure involving 1s and *x*s. Figure 9 shows his drawing of the six ways to arrange two 1s and two *x*s. Again, it was somewhat surprising and impressive to us that a student independently generated this general outcome structure.

Richie: *I'm just trying to think of all the different configurations – that 1s can [inaudible] – so like they can be starting with x, or just like that. Or like this or like this. Or like this. And then it could be 1 inside.*

Int.: *Perfect.*

Richie: *So, 1, 2, 3, 4, 5, 6. There's 6 different possibilities for that. And then each of these can have 4 different configurations.*

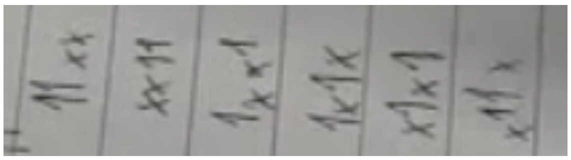

Figure 9

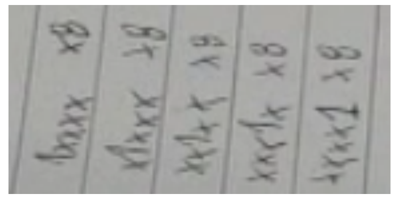

Figure 10

Richie then checked his work and reasoned that for each of those possibilities there would be four possibilities, running through BB, AA, AB, and BA. He concluded, "So 6 times 4 would be – it would be like 6 times all of these really [referring to the six configurations]…Yeah, okay. So I guess they would be 24. So it would be 24 possibilities." Richie then continued to work, and as he progressed to other rows in the table he made more diagrams involving 1s and *x*s.

We then had him move to the 5-character 1, A, B password, and I asked him to start making the table. Here again he made a similar diagram with 1's and x's, but here his work departed from Tyler's. Richie was able to think about there being a certain number of options for each case (each arrangement of the x's), and he knew there were two options for each *x* (A and B), but he added instead of multiplied the number of options, yielding 8 rather than 16 possibilities.

Richie: *So for 5 it would be 32. Same thing. And then for 1 there would be – (writes Figure 8 without the *8's) and so those would each have – this could be A or B, so that would be 2 for that, 2 for that, 2 for that, so these would each have 8 different possibilities [writes the *8's in Figure 8]. So it would be 5 – 5 times – it would be 40 for 1.*

In asking Richie to explain this work, we gain insight into his meaning of the diagram. He made no explicit connection to the previous tables or situations as Tyler had.

Richie: *This, like I made I want to say like a diagram basically of a position so 1 can be. And then I put Xs in for the – where the As and Bs could be, because those are variables that can be either A or B. And then I noticed that for every X it has 2 possibilities, either being A or B, and there's 4 Xs…So then I just multiply that by 2 to get 8. So each – for each 1 position there's 8 different possibilities for the password. And that's how I got 40.*

Richie continued his work and listed out all 10 of the configurations of two ones in a 5-character password. He demonstrated a consistent meaning by again adding the options – saying there were 6 passwords for each configuration, which is 2*3 rather than $2^3 = 8$. At first blush it seems that perhaps Richie simply made a mistake, adding instead of multiplying the options, but we do not feel that he simply made a numerical error. Rather, the evidence seems to point to the fact that he did not make meaning of the new structure as being related to the previous case, at least not directly to the previous tables. Unlike Tyler, he did not recognize that the power of the

11xx structure is that it can be very clearly related to the previous situation. There are two potential points of connection to the prior A,B tables (relating the placement of the 's to the rows of the AB tables, such as 1, 4, 6, 4, 1, and realizing the totals in the previous tables represent the possible number of passwords of a given length), both of which Tyler recognized, and neither of which Richie recognized.

This is not to say that Richie's meanings were unreasonable or that they did not make sense to him – indeed they did. His introduction of the *11xx* structure seemed to serve as a way of simplifying and organizing the problem so he could better break it down, but not in a way that facilitated rich connections to the previous problem. Indeed, we asked him how and why he came up with the structure, and he suggested that he was motivated by efficiency: "I started to write here different configurations for where the 1s could be and where the Bs and As could be, and I noticed that basically the As and Bs were just switching places for wherever the amount of 1s were. So I started putting *x*s there just so I wouldn't have to write as much." Thus, while Richie was motivated by the efficiency, which is justifiable, he did not realize that he could leverage his previous work in order to take full advantage of the tool he had developed.

Discussion

In this section we compare and contrast Tyler and Richie's work, highlighting some salient similarities and differences. We feel that our results offer a theoretical contribution to the research on generalization (by introducing a notion of *recursive embedding*), and we also connect these findings to students' combinatorial reasoning.

First, we highlight what was similar in Tyler and Richie's work. It is noteworthy that they both came up with the *11xx* structure, spontaneously and independently. This tool seemed to be motivated by efficiency for both of them, which suggests that efficiency may be an effective pedagogical idea for facilitating generalization. Also, their generation of this tool suggests a solid fluency with outcomes, which affirms combinatorial research that emphasizes the importance of students' set-oriented perspectives toward counting (e.g., Lockwood, 2013; 2014). Finally, both Tyler and Richie seemed to understand that there were a certain number of passwords "for each" *11xx* outcome they could write. While they used this idea differently (with Tyler using it correctly and Richie incorrectly), this suggests perhaps that they had at least some informal but accurate notion of the multiplication principle in this particular instance.

What, then, set these students apart in their abilities to more (and less) effectively leverage their *11xx* tool? We argue that there are two key differences that explain this phenomenon, and they each have implications in terms of generalization and in terms of combinatorial reasoning. First, Tyler engaged in what we call *recursive embedding*, and second Tyler was able to view AABB and 11xx as essentially "the same" – simply arrangements of two kinds of objects. We describe each of these below.

Recursive embedding. By recursive embedding, we mean that Tyler reflected on prior activity and applied that activity (not just the results of that activity) into a new situation. Specifically, recognized the *x*s in the *11xx* structure in the A,B,1 situation as representing placeholders into which he could embed a previous situation (A,B passwords, for example). Figure 11 demonstrates what we interpret to be involved in recursive embedding in Tyler's work. Note that the *x*s in the problem are replaced with AB passwords, and the prior work is used productively in a new situation.

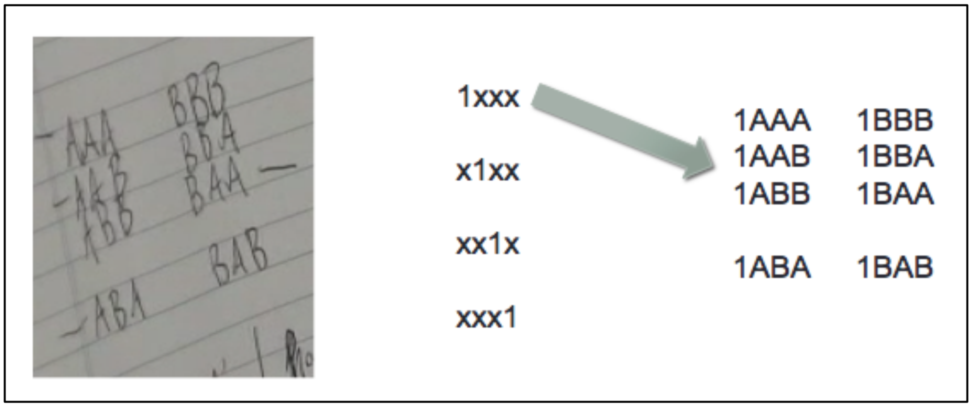

Figure 11

Viewing two combinatorial situations as similar. Another important aspect of Tyler's work is that he was able to recognize similarities between two situations in a way that Richie did not seem to observe. Specifically, Tyler extrapolated a similarity between arranging AABB and 11*xx*, as he recognized that ultimately he was just rearranging two kinds of things – it didn't matter if they are As and Bs or 1s and *x*s. This was a key element of him being able to connect back to prior work and to successfully leverage the *11xx* structure.

In sum, Tyler's activity of recursive embedding and of recognizing key situations as similar has important implications for generalization. In particular, his activity suggests an instance of generalization that is based in reflective abstraction. By embedding results of his prior activity into a new situation, Tyler is projecting and reorganizing, "on another conceptual level, a coordination or pattern of [his] own activities or operations" (von Glasersfelt, 1995, p. 105). This underscores that Tyler's powerful generalization is rooted in reflective abstraction, and it also suggests that fostering reflective abstraction may be a key aspect of helping students to engage in meaningful and effective generalizing activity.

Conclusion and Implications

By examining two students' meanings of the same tool that they each spontaneously developed, we gain insight both into students' generalizing activity and their combinatorial reasoning. Ultimately, we want to help students be more effective in creating productive generalizations, and we want to learn more about how students might effectively solve counting problems. We feel that Tyler's work – not only his production of the *11xx* structure but also his ability to make meaning of it in light of prior activity – is a powerful example of a student-generated general structure that led to progress in challenging combinatorial tasks. Set in contrast to Richie's work (which was also impressive in that he generated the *11xx* structure, but was limited in its lack of combinatorial meaning and connection to the previous situations), we can examine what aspects of Tyler's work and meanings were so efficacious.

The contrasting cases of *11xx* structure shed some light on generalization. We see that the desire for efficiency may motivate generalization, and that reflection on prior activity (and the ability to use that prior activity) was a distinguishing feature of Tyler's work. Specifically, recursive embedding and viewing AABB and 11*xx* as the same were aspects of his activity that facilitated effective generalization. Tyler's case provides an encouraging example of what might be attained through reflective abstraction.

Combinatorially, this is an example of the kind of thinking we want to foster. Tyler's association of AABB and 11*xx* as being the same is hugely important for productive combinatorial reasoning, as we want students to be able to draw out some structure that is not dependent on particulars. The *structure* of the outcome seems to be important to facilitate this association and connection. Indeed, one aspect of his work that was powerful was that he remained grounded in his prior activity, and he had a rich combinatorial meaning of those prior situations. The AB tables Tyler made were combinatorially meaningful for him, in the sense that he reasoned about outcomes and did not lose sight of the combinatorial context. This is in line with previous work that emphasizes the importance of outcomes (e.g., Lockwood, 2013; 2014). We suspect that because Tyler had such a strong sense of outcomes (as seen through his listing activity in his creation of AB tables), the *11xx* structure really did represent to him a more general form of an outcome. It resembled a password (still a sequence of characters on the page), and we posit that this enabled him to maintain his reasoning about the structure of his outcomes and thus a connection to the previous combinatorial situation.

In terms of implications, our findings suggest that students can, on their own, produce potentially powerful tools involving general structures. However, this alone is not sufficient for productive generalizing or counting activity, and these contrasting cases show some of the other reasoning necessary to make full use of such tools. A pedagogical implication is that teachers may need to be vigilant in helping students maintain contact their with their prior activity, as reflecting on one's prior activity seems to be a potentially productive way to facilitate generalization. To engender productive generalizations, we may direct students to reflect on their prior work by having them make explicit connections and statements of similarity or by answering meta-cognitive questions about their activity.

Specifically, in combinatorics this might mean that even as students notice patterns, teachers should help them to connect those patterns to the combinatorial context and not simply to numerical regularity. Combinatorially, another implication of the work is that this sequence of tasks does seem to be potentially useful in helping students to reason about the binomial theorem (or at least its initial stages). Tasks like these could be leveraged to introduce and teach combinatorial identities, which is a building block toward the learning of combinatorial proof.

Our findings show an example of rich generalizing activity in a combinatorial context. These findings emerged in a single interview, but we hope to extend this work through teaching experiments in which students' meanings can be developed and examined over time. Next research steps also include an investigation into more specific instructional interventions that might foster the kind of meanings that proved beneficial for students like Tyler.

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Graphing habits: "I just don't like that"

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We report on clinical interviews to describe U.S. undergraduate students' ways of thinking for graphs as representations of measurable attributes of dynamic situations (e.g., a road trip or an amusement park ride). In particular, we describe students' actions that we interpret to be habitual to their uses of graphs. By habits, we refer to specific schemes and actions constituting students' ways of thinking through which they routinely assimilate their graphing activity. Some habits were problematic in that they inhibited the students' abilities to represent covariational relationships they had conceived to constitute some dynamic situation. For example, we illustrate that some students' ways of thinking for graphs resulted in their experiencing perturbations if neither quantity's value increased or decreased monotonically.

Keywords: Graphing, Covariational Reasoning, Quantitative Reasoning, Cognition, Function

Researchers have persistently suggested that educators take seriously students' graphing activities given the difficulties students have with topics (e.g., function and rate of change) that involve the significant use of graphs (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Johnson, 2012; Thompson, 1994). More pointedly, researchers have illustrated that student difficulties in mathematics relate to students having impoverished experiences reasoning covariationally (Carlson, 1998; Oehrtman, Carlson, & Thompson, 2008; Thompson & Carlson, in press). We respond to the need to better understand students' covariational reasoning by describing undergraduate students' ways of thinking for graphs relative to the extent that they were productive when constructing graphs to represent measurable attributes of a dynamic situation. Specifically, we characterize two categories of students' ways of thinking for graphs. We contend that some students' ways of thinking for graphs entailed habits based in *figurative thought* (Piaget, 2001) that often inhibited their representing a covariational relationship that they conceived to constitute some situation. In contrast, we describe students' ways of thinking for graphs based in *operative thought* (Piaget, 2001), in which students exhibited a sustained focus on coordinating quantities' magnitudes when constructing a graph with figurative aspects of the trace being subordinate to this coordination.

Covariational Reasoning, Magnitudes, and Shape Thinking

Covariational reasoning is "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 354). Whereas some researchers have described covariational reasoning in terms of patterns in successive, corresponding *numerical values* of two sets (see Confrey & Smith, 1995), our approach to covariation adopts Thompson's (2011) description of *quantity*. Thompson (2011) described quantity as an attribute a person conceives to constitute some situation such that the person understands and anticipates the attribute as having a measurable *magnitude* (i.e., amount-ness).[1] We draw attention to Thompson's distinction between a quantity's magnitude

[1] We point the reader to Steffe and Olive (2010) and Thompson, Carlson, Byerley, and Hatfield (2014) for detailed characterizations of different conceptualizations of magnitudes.

and a quantity's measure (or value) because it enables us to describe a person's covariational reasoning in terms of her simultaneously coordinating (continuous) magnitudes in flux (Saldanha & Thompson, 1998). For instance, in a situation involving two corresponding lengths, a person can imagine two lengths in flux while *anticipating* that these lengths have specific measures (with an associated unit) at any instantiation of the covariation.

We do not intend our focus on the covariation of magnitudes to diminish the importance of reasoning with specified measures and patterns. Such reasoning is important for the construction of function classes, quantification of rate of change, and reasoning about limits (see Confrey & Smith, 1995; Ellis, Özgür, Kulow, Williams, & Amidon, 2015; Johnson, 2015; Oehrtman, 2008). Approaching covariation in terms of coordinating magnitudes, however, provides a complementary focus that researchers have found useful in characterizing how students might construct images of covariation that are productive for their reasoning about relationships between quantities in ways not confined to, but instead working in tandem with, reasoning about specified measures (see Carlson et al., 2002; Thompson, 1994, 2011).

Extending the aforementioned research on covariational reasoning, Moore and Thompson (2015) characterized students' *static shape thinking* and *emergent shape thinking*. The authors described that a student engaging in static shape thinking operates on a graph as an object in and of itself (i.e., graph-as-wire), basing her actions on perceptual cues and the physical features of a graph. Moore and Thompson (2015) provided an example of a student's static shape thinking in the form of a student associating rate of change or slope with properties of direction (e.g., a student reasoning that a graph of $y = 3x$ unquestionably implies a line sloping upward left-to-right regardless of coordinate system or orientation). In contrast to a student assimilating a graph as an object in and of itself, Moore and Thompson (2015) characterized that a student engaging in emergent shape thinking conceptualizes a graph as a locus or trace that is produced by coordinating two quantities' magnitudes simultaneously. They explained, "emergent shape thinking entails assimilating a graph as a trace in progress (or envisioning an already produced graph in terms of replaying its emergence), with the trace being a record of the relationship between covarying quantities" (Moore & Thompson, 2015, p. 785). Although this way of thinking requires conveying magnitudes in flux, we provide instantiations of covariation in Figure 1 that are associated with Part II of the task in Figure 2 to illustrate this way of thinking.

A person thinking emergently in response to *GAG Part II* [2] might proceed as follows:

(1) The person conceives that the car starts at a magnitude of zero from Atlanta ($||Y|| = 0$) and some non-zero magnitude from Gainesville ($||X|| > 0$). The person represents these attributes by plotting a point that simultaneously represents both magnitudes (Figure 1a).
(2) The person conceives that, over the first portion of the trip, for any particular magnitude increase in the distance from Atlanta, the car's distance from Gainesville simultaneously decreases by that magnitude (i.e. $|\Delta||X||| = |\Delta||Y|||$). The person conceives the distance from Atlanta increases at a constant rate as the car's distance from Gainesville decreases.
(3) The person represents the relationship constructed in (2) by constructing and imagining two magnitudes covarying along the axes in a way that maintains (2), with a point moving correspondingly to represent simultaneously both magnitudes (Figure 1b-1d).
(4) The person imagines the point leaving a trace representing all instantiated pairs of covarying magnitudes (Figure 1e).
(5) And so on (Figure 1f).

[2] *GAG* is a modification (see Task Design) of Saldanha and Thompson's (1998) task.

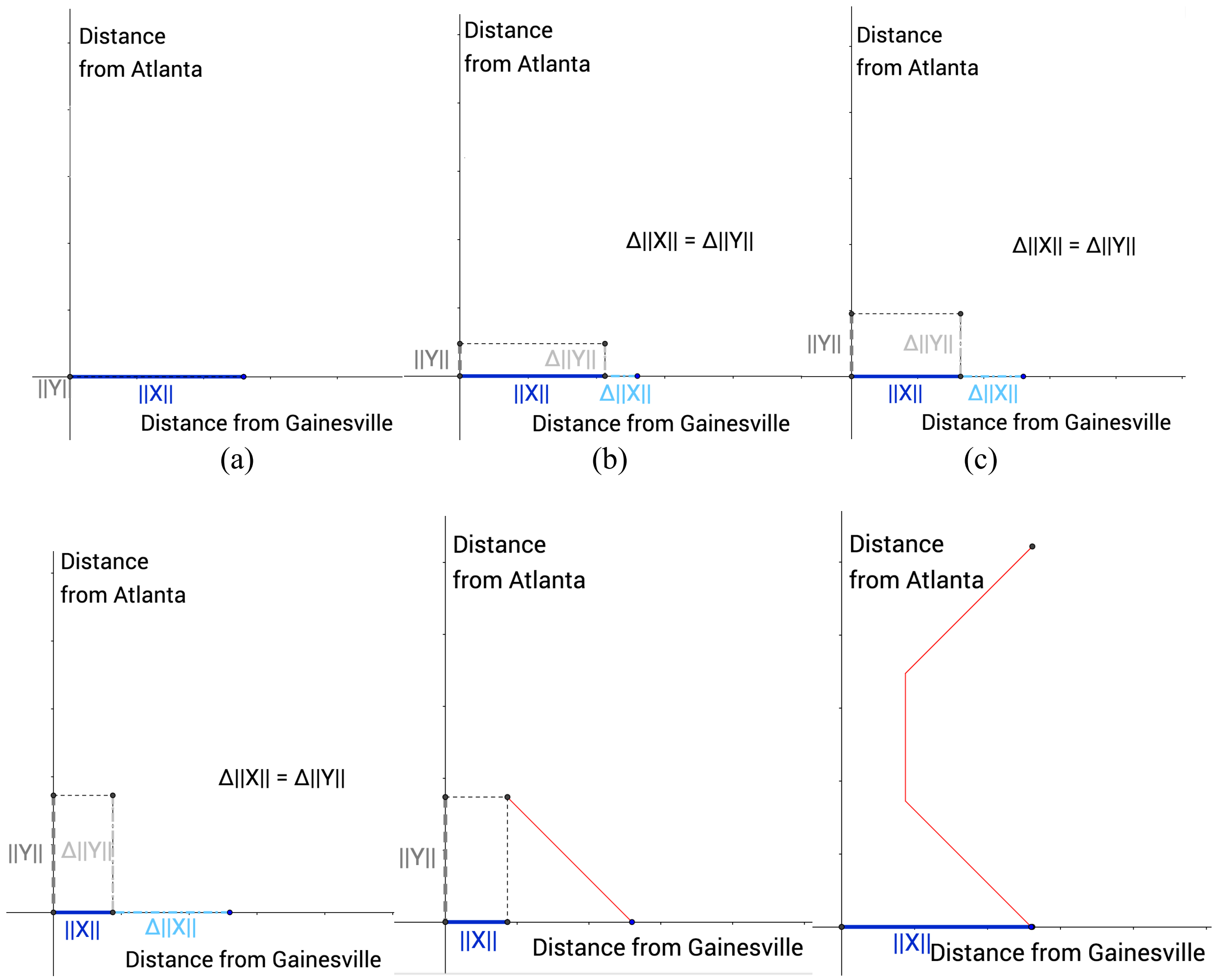

Figure 1a-f. A graph as a coordination of two magnitudes for the trip *there and back*.

Going Around Gainesville Part I

Some Georgia students have decided to road trip to Tampa Bay for Spring Break. Of course, this means traveling around Gainesville on their way down and back. The animation represents a simplification of their trip there and back. Create a graph that relates their total distance traveled and their distance from Gainesville during the trip.

Going Around Gainesville Part II

Create a graph that relates the students' distance from Gainesville and their distance from Atlanta during the trip.

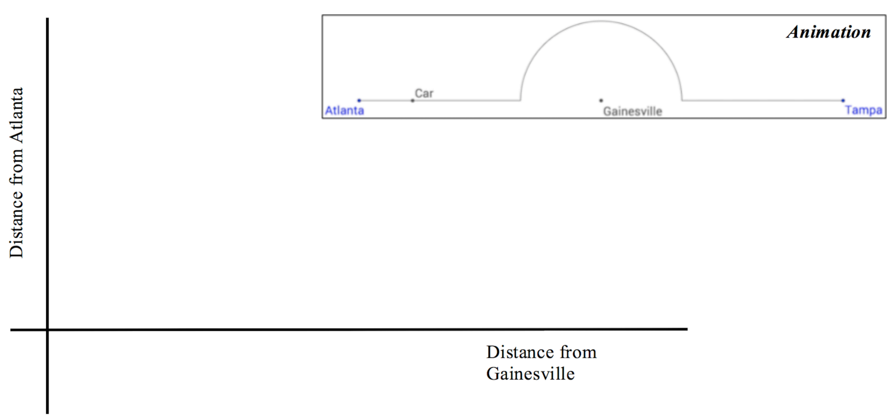

Figure 2. The *Going Around Gainesville* (*GAG*) task.

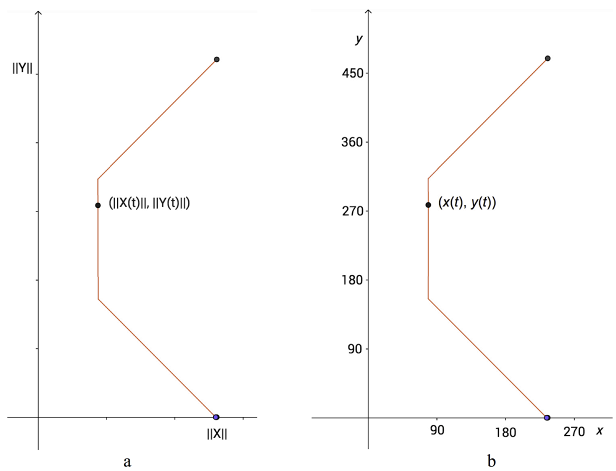

Figure 3a-b. A magnitude graph (a) and a values graph (b) representing a multiplicative object.

A person's actions are not likely to proceed in such a linear progressions, yet this example illustrates that thinking emergently involves sustaining an image of a graph as a uniting of two coordinated magnitudes. Hence, the graph emerges as a locus of points representing a multiplicative object—an "operation of having in mind two attributes of an object simultaneously" (Thompson, 2011, p. 48)—whose magnitudes and properties are equivalent to those of the multiplicative object the person conceives to constitute the situation.

Adopting the notation Thompson (2011) used to represent covariation as a multiplicative object, we can model a person's conception of a graph as an emergent trace with ($\|X(t)\|$, $\|Y(t)\|$), which represents the uniting of two quantities' magnitudes, $\|X\|$ and $\|Y\|$, so that these magnitudes are understood as varying simultaneously with respect to conceptual or experiential time, t (Figure 3a).[3] Upon choosing unit magnitudes for each quantity, a person can determine specified values that represent relative comparisons between these unit magnitudes and each quantity's magnitude at every instantiation of covariation. The result of comparisons between magnitudes and unit magnitudes enables the person to represent the attributes of the multiplicative object in terms of corresponding *values*, x and y (with associated unit magnitudes), thus representing the multiplicative object in terms of the points ($x(t)$, $y(t)$) (Figure 3b, with an associated unit of miles). We emphasize that a person with sophisticated ways of thinking for magnitudes (see Thompson et al., 2014) understands the covariational relationship, and hence the multiplicative object, as remaining invariant among unit choices; that person understands that the choice of unit influences how one partitions the respective axes, the numerical values defining coordinate pairs, and the measures of particular relationships between quantities' *values* (e.g., $\Delta y/\Delta x$), but this understanding is subordinate to the person's understanding that the displayed graph conveys a relationship between quantities' magnitudes invariant among unit magnitudes.

Ways of Thinking and Habits

We are interested in characterizing students' ways of thinking for graphs with sensitivity to their covariational reasoning, and thus it is necessary that we describe our use of *ways of thinking* and the associated terms *meaning* and *habit*. We follow Thompson and Harel's system of knowing, as reported by Thompson et al. (2014), which has its basis in Piagetian notions

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[3] See Thompson and Carlson (in press) and Stalvey and Vidakovic (2015) for discussions on conceptual and experiential time that are beyond the scope of this work.

including schemes, operations, images, assimilation, and accommodation. Thompson and Harel define *meaning* as the space of implications that results from assimilation (Thompson et al., 2014); it is the system of schemes and operations brought forth (which may be carried out or merely anticipated) in assimilation. For example, in a case that a student assimilates two marks as lines defining coordinate axes, this moment of assimilation might correspond to the student anticipating drawing a number of memorized shapes, plotting points, making a map, etc. Moreover, in assimilating a trace on a page as a displayed graph, the student might anticipate a particular coordinate system, making tables of values, coordinating covarying magnitudes, etc.

Thompson and Harel use *ways of thinking* to refer to a student's meanings that have been repeatedly constructed to organize her experiences to the extent that the meanings have become habitual. In many cases, students' ways of thinking are habitual to the extent that they are essentially a subconscious response or pattern of actions brought forth in the moment of assimilation. Moreover, a student might bring to bear, without perturbation, a way of thinking that *an observer* perceives as incompatible with the situation at hand. If a student perceives this incompatibility, she experiences a perturbation that might require significant effort (e.g., an accommodation) on the student's part to reconcile. In these ways we describe a student's way of thinking as entailing particular *habits*: (mental or physical) actions and schemes that a student essentially subconsciously enacts, or actions and schemes that a student might find difficult not to bring to bear on associated situations despite experienced perturbations. As an example detailed in our results, a student might develop ways of thinking for graphs that entail particular habits with respect to how a graph is drawn (e.g., a graph is drawn from left-to-right) so that she has difficulty accommodating situations she perceives as incompatible with these actions.

We find Piaget's (2001) distinctions between figurative and operative thought useful when characterizing particular actions and schemes (i.e., habits) that constitute a student's ways of thinking for graphs. Piaget distinguished thought based in and constrained to perception and sensorimotor activity from thought based in operational intelligence in which figurative elements are subordinate to mental operations. That is, figurative meanings or ways of thinking are dominated by re-presentations of perceptual material or sensorimotor experience (von Glasersfeld, 1995). In contrast, operative meanings or ways of thinking are not constrained to specific perceptual material or sensorimotor experience; they are meanings dominated by the coordination of mental actions (von Glasersfeld, 1995).

Returning to students' static or emergent shape thinking (Moore & Thompson, 2015), we interpret students' static shape thinking as an example of figurative thought, as such thinking is subordinate to perceptual (figurative) properties of shape. On the other hand, students' emergent shape thinking foregrounds the coordination of actions—specifically that of covarying magnitudes, which Thompson (1994) described as operative—so that figurative elements of their activity are subordinate to that coordination. Further illustrating the distinction between figurative and operative thought in the context of static and emergent shape thinking, Moore and Thompson (2015) argued that in the former case, mathematical objects (e.g., rate of change) become properties of or subordinate to perceptual features (e.g., direction of a line—the graphs in Figure 4 unquestionably imply a positive and negative rate of change, respectively, because the lines slope upward left-to-right and downward left-to-right, respectively). In the latter case, mathematical objects become properties of or subordinate to the coordination of actions (e.g., rate of change as a measure of how quantities change—the graphs in Figure 4 unquestionably imply $\Delta y = 3\Delta x$ because x and y are simultaneously increasing such that the change in y is three times as large as the change of x). We expand on these notions to describe the extent students' actions are subordinate to figurative or operative elements of thought.

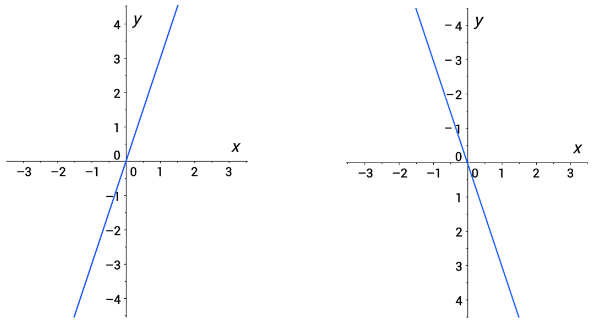

Figure 4. Graphs of $y = 3x$.

Participants, Setting, and Methods

Our participants were 10 prospective secondary mathematics teachers (hereafter referred to as students) enrolled in an undergraduate secondary mathematics education program in the southeastern U.S. The students were juniors to seniors in credits taken and had completed at least one mathematics course past an undergraduate calculus sequence. We chose the students on a volunteer basis and selected volunteers based on their availability. We conducted three approximately 75-minute semi-structured clinical interviews (Ginsburg, 1997) with each student facilitated by the lead author and another member of the research team. Each interview included tasks designed using the principles we describe below. Successive interviews occurred approximately 1.5 months apart over the course of a semester. The time between the interviews enabled us to design subsequent interviews based on retrospective analyses of prior interviews.

We video- and audio-recorded all interviews and digitized student work after each interview. The lead author and fellow interviewer recorded observation notes after each interview. We analyzed the data with selective open and axial methods (Corbin & Strauss, 2008) and *conceptual analyses*–building models of students' mental actions that explain their observable activity and interactions (von Glasersfeld, 1995). Members from the research team (the author team and additional mathematics educators) identified instances that provided insights into each student's thinking. The research team then viewed these selected instances in order to build models of the student's thinking, which they also compared to the observation notes captured after each interview. As the research team developed these characterizations, they continually returned to previously identified instances (across all students) to revise or provide alternative characterizations based on interpretations of latter instances. This iterative process generated themes among characterizations of students' ways of thinking—those meanings that we interpreted to be habitual as defined above—several of which we report in this paper.

Task Design

We designed a series of six tasks (two per interview) that: (1) provided a dynamic, albeit often simplified, situation through video; (2) did not include numerical values for attributes of the situation; (3) prompted the student to graph a relationship between two quantities; and (a majority of which) (4) prompted the student to create a second graph, either between similar quantities or the same quantities under different axes orientations. To illustrate, we used *GAG* (Figure 2) during the second interview with each student. *GAG* entails a video depicting a point representing a car traveling back and forth on a simplified path between Atlanta and Tampa. Reflecting (1) and (2), the task involves a dynamic situation depicted by a video without numerical information. Part I prompts the student to graph a particular relationship between two quantities (i.e., (3)). Part II, which we presented after the student completed Part I, uses the same video and the student graphs a similar relationship with an imposed axes orientation (i.e., (4)).

In general, (1)-(3) reflects Saldanha and Thompson's (1998) description of covariational reasoning with particular attention to students (potentially) simultaneously coordinating magnitudes in the context of both graphs and situations. Slightly modifying Saldanha and Thompson's task, we designed several of our tasks (including *GAG*) to more likely afford students coordinating *amounts of change* (see Carlson et al., 2002, mental actions) between the quantities' magnitudes. Our decision to design tasks based on principle (4) stems from findings from our previous work (Moore, Silverman, Paoletti, & LaForest, 2014) where we identified that students' ways of thinking for functions and their graphs led to perturbations when graphing equivalent or related relationships in different axes orientations, or when graphing relationships that are non-canonical (e.g., neither quantity monotonically increasing or decreasing) with respect to U.S. curricular approaches to functions and their graphs.

Results

We structure the results around several interrelated habits that constituted students' ways of thinking for graphs. When describing a particular habit, we do not imply that the student brought to bear those actions on all graphing situations. Rather, we remind the reader that habits are (mental or physical) actions and schemes that a student subconsciously enacts in some (and possibly related) situations, or actions and schemes that a student might find difficult not to bring to bear on associated situations despite experienced perturbations.

Graphs 'starting' along the vertical axis

Several students constructed a graph by first determining a point along the vertical axis and drawing a graph emanating from that point. Often, upon reflecting on their drawn graphs, *the students* conceived their graphs as incompatible with the relationships they intended the graph to represent. However, *in the moment* of drawing the graph, a student 'starting' her graph along the vertical axis often described her graphing activity in ways that contradicted the relationship *we perceived* the drawn graph to represent (e.g., the student describing that some quantity decreases while drawing a graph that we perceived to represent that quantity increasing).

As an example, we return to *GAG Part II* (Figure 2).[4] Some students immediately marked a point on the vertical axis and anticipated drawing a graph from that point (Excerpts 1).

Excerpts 1. Two students 'starting' graphs along the vertical axis.

[*Both Patricia and Andrea have constructed normative graphical solutions to GAG Part I.*]

Patricia: Your distance from Athens starts at zero [*plots point at origin*] because you're in Athens. Um, so as you get. Mmm, no, you're gonna start up here [*plots point on vertical axis but not at origin to represent a non-zero distance from Gainesville as Excerpts 2 indicates*]. Ignore that [*covering origin*]. 'Cause, oh wait, no, stop [*crosses out second plotted point*]. No, you're here [*points to origin*].

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[4] One version of the task used Athens in place of Atlanta as a starting city.

Andrea: We're in Athens [*moves to paper and motions along the vertical axis, marks point at origin*], as we're moving away from Athens we're getting closer to Gainesville [*draws segment starting at the origin moving up and to the right,* Figure 5*, and explains that the quantities change at a constant rate with respect to each other*].

A notable feature of both students' actions is that they plotted an initial point by focusing on *one* quantity's initial magnitude and identifying a point on the vertical axis based on that magnitude. Patricia alternated which quantity she considered when plotting the point while maintaining the initial point along the vertical axis. Andrea maintained an explicit focus on the distance from Athens in determining her initial point. As both students continued, and although they both had in mind a (correct) relationship with respect to the situation, they experienced perturbations due to their action of 'starting' the graph along the vertical axis. We explain Patricia's sustained perturbation in the subsequent section (Excerpts 2).

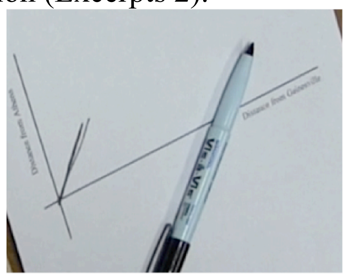


Figure 5. Andrea's drawn graph.

Graphs drawn or read left-to-right

Some students' ways of thinking entailed the habit of drawing or reading graphs left-to-right. When constructing a graph, students drew or imagined drawing graphs by 'starting' at a point (predominantly along the vertical axis) and exclusively moving their pen to the right while allowing for vertical movements. The students' vertical pen movements were either to connect previously plotted points (regardless of the order that these points were plotted) or to capture variation in one quantity. To illustrate, we present two students' activities in Excerpts 2. Karen was working the task presented by Saldanha and Thompson (1998), which includes graphing how a traveler's distances from two cities covary (see Figure 6 for animation and a sketch of a normative graph). Patricia's condensed work is a continuation of that in Excerpts 1.

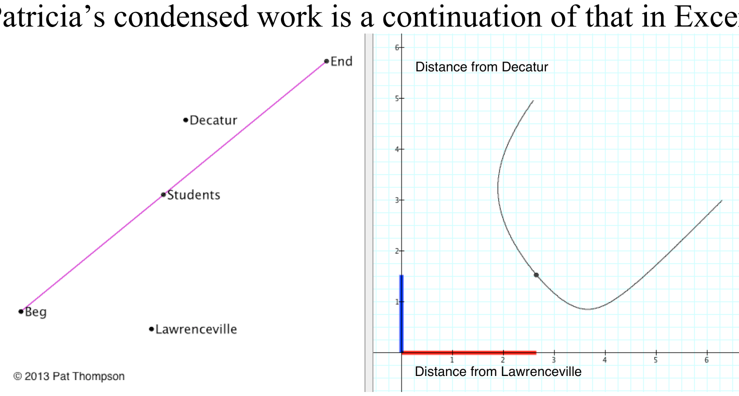


Figure 6. *City Travels* animation and sketched graph (modified from Saldanha and Thompson, 1998), an alternative of which used Dekalb in place of Decatur.

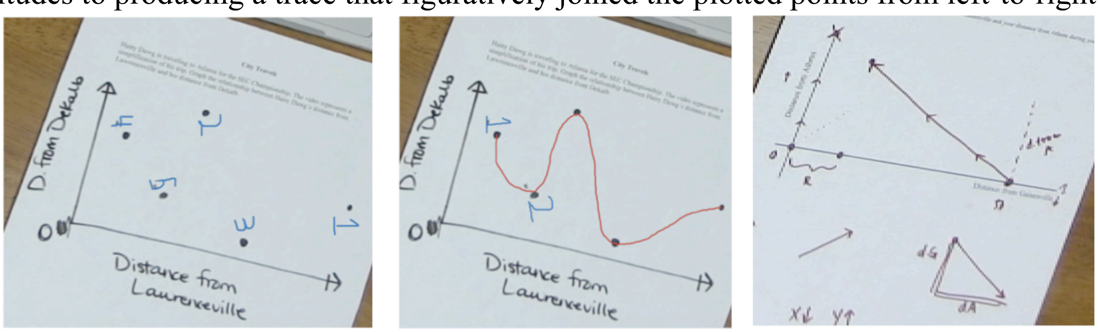
Excerpts 2. Two students drawing graphs left-to-right.

[*Karen has plotted five points corresponding to locations during the trip. She plotted the points in the order we have annotated in* Figure 7a.]

Karen: Okay, wait. This one [*pointing at the leftmost point she plotted*] was when he's closest to Lawrenceville, which happens first [*labels the point '1'*], then this one [*labels the next leftmost point '2', moves pen to the third leftmost point*] so it's something like that [*making a sweeping motion indicating a curve connecting the points from left-to-right in the manner we have annotated in* Figure 7b].

Patricia: [*Patricia has now determined an initial point that is not at the origin but is along the vertical axis. She motions as if drawing a segment sloping downward left to right from this point that she later crosses out—see the crossed out point on vertical axis in* Figure 7c] I wanted to show that the distance was decreasing [*motioning down and to the right from the point plotted on the vertical axis*], but that means that your distance from Athens is decreasing [*tracing vertical axis from the initial point to the origin*]…But your distance from Athens is growing. But your distance from Gainesville is decreasing. So, if that's growing and that's decreasing, so [*draws an arrow pointing downward beside horizontal axis label and then an arrow pointing upwards beside the vertical axis label*, *Patricia then works for six additional minutes without making progress before having an insight*]…Oh, what if I started it like here [*plots point on the right end of the horizontal axis*]. Okay…but I don't want to start like, like I don't like starting graphs. You know, I don't know. Work backwards. That's weird…[*in the next minute and a half Patricia draws in what we perceive to be a correct initial segment of the graph*]… my graph is from right-to-left, which is probably not right…Backwards is traveling from right-to-left. But I think my graph is just, I think I'm just not clicking. I think I'm missing something.

We highlight Karen's immediate move to imagine connecting the points from left to right after ordering *two* points from left to right (Figure 7b), an order that contradicts the order she created the points (Figure 7a) and the order with respect to the animation. Karen did not show signs of considering how the quantities' magnitudes vary between those paired magnitudes (within the situation or graph). Upon plotting the points, Karen's activity shifted from representing paired magnitudes to producing a trace that figuratively joined the plotted points from left-to-right.



(a) (b) (c)

Figure 7a-c. Karen's annotated work (a-b) on the *City Travels* task and Patricia's work (c) for the first portion of *GAG Part II*.

Whereas Karen did not appear to hold in mind a covariational relationship that she intended her graph to represent, Patricia had a developed image of how the two relevant quantities' magnitudes covaried with respect to the situation. However, she experienced a sustained state of perturbation in creating her graph. On one hand, after nearly ten minutes of careful activity (which is notable due to her graphing a similar relationship in an alternative orientation on Part I that she recalled and reproduced in the lower right of Figure 7c), Patricia produced what we perceived to be a correct graph by thinking of the graph emergently (Figure

7c). On the other hand, she reluctantly drew her final graph and then remained perturbed by her completed graph due to figurative aspects (e.g., where the graph "started" and having to "work backward" or "right-to-left") that were not compatible with her ways of thinking for graphs.

Graphs pass the 'vertical line test'

Related to researchers' (Carlson, 1998; Leinhardt, Zaslavsky, & Stein, 1990) observations that students' predominant meaning for function in the context of graphs is the *vertical line test*, some students' ways of thinking for graphs involved students anticipating that their drawn graph must *pass the vertical line test* (i.e., a graph such that each abscissa value or magnitude only corresponds to only one ordinate value or magnitude). In some cases, the students' anticipations were related to drawing graphs exclusively left-to-right. The students understood that drawing a graph exclusively from left-to-right necessarily produces a graph that passes the vertical line test. However, anticipating a graph as satisfying the vertical line test also emerged when students could anticipate graphs as being drawn left-to-right or right-to-left. To illustrate, consider Tara's and Alisha's work on *GAG, Part II* (Excerpts 3).

Excerpts 3. Two students considering graphs that do not pass the vertical line test.

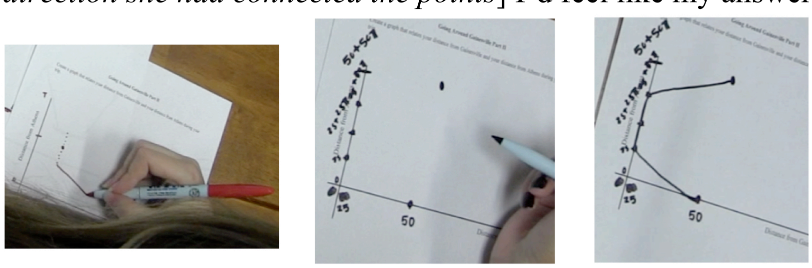
Tara: [*Referring to traveling on the semicircular path*] Your distance from Gainesville isn't changing but your distance from Athens is [*silence for 10 seconds*]. That makes me uncomfortable [*laughs*].

Int.: What makes you uncomfortable?

Tara: 'Cause right here [*motioning at the car as it moves around the semicircular arc*] your distance from Gainesville isn't changing but your distance from Athens is changing…I'm gonna have to like [*making vertical motion representing having to draw a vertical segment*]. I know it's not supposed to be a function and it doesn't matter but it still makes me uncomfortable [*laughs*] to graph things all weird on the vertical axis. [*Hesitantly draws dots aligned vertically, but not a continuous segment, see* Figure 8a]

[*Alisha has plotted five points with three points corresponding to positions on the semicircular path, see* Figure 8b.]

Alisha: So, that's weird [*motions pen indicating segment that would be drawn if she connected the three points*]. I don't wanna connect those dots, but, [*laughs softly*] I really don't like that…I just don't like that [*connects all the points of the graph in order from start of trip to end of trip, see* Figure 8c] my graph looks like this…I dunno. If I was taking a test and I drew that [*quickly motions the pen over the graph in the direction she had connected the points*] I'd feel like my answer was wrong.



(a) (b) (c)

Figure 8a-c. Tara (a) and Alisha (b-c) creating graphs with a vertical segment.

Both students had in mind relationships they intended to represent (i.e., the distance from Gainesville remaining constant as the distance from Athens increases), but their ways of thinking for graphs entailed some habit that they found difficult not to bring to bear when constructing a graph containing a vertical segment. Both students ended the task unable to reconcile the conflict they experienced between thinking that the graphs represented the correct relationship and thinking that the graph entailed figurative aspects that they considered to be incorrect.

Graph features representing some perceived aspect or phenomenon

Iconic interpretation refers to a student associating the physical appearance of a situation with the physical appearance of a graph (Leinhardt et al., 1990). More generally, students in our study frequently sought to represent some perceptual aspect or physical experience associated with the situation using figurative aspects of their graphs. At times, students' associations were iconic in nature (e.g., drawing a graph curved because a Ferris wheel is curved). At other times, students' associations were based in some perceived phenomenon or attribute of the situation that often involved attributes of motion (but not based in the explicit coordination of varying quantities). Thompson (2015) referred to these as thematic associations. A notable example is that many students drew a graph composed of linear segments to represent an object traveling at a constant speed despite their identifying that neither axes represents elapsed time (see *GAG*); these students argued that they needed to change drawn graphs (e.g., either the slope or curvature) if the speed of the car changed. A second example entailed students creating graphs that included perceptual records (e.g., a horizontal segment) of an object pausing in motion despite neither axis representing elapsed time. Patricia explained, "I wanna say kinda like the curved one, kinda like that one [*her graph corresponding to the situation of the ride without pauses*], but I just want like, breaks, like breaks in the graph." Patricia then sought to create a graph with visual "breaks" in order to convey the "breaks" in motion.

Graphs as emergent traces of covariation

Another way of thinking that some students brought to bear repeatedly (although not exclusively; see Tara, Excerpts 3) entailed conceptualizing graphs in terms of emergent traces of covariation; their ways of thinking for graphs entailed the habit of coordinating and uniting two magnitudes represented along the axes so that figurative elements entailed in drawing a graph were secondary to their reasoning covariationally. Excerpts 4 provide two examples of students envisioning graphs as emergent traces. Amy and Tara are both constructing a graph after conceiving how the surface area and height of a growing and shrinking cone covary (Figure 9).

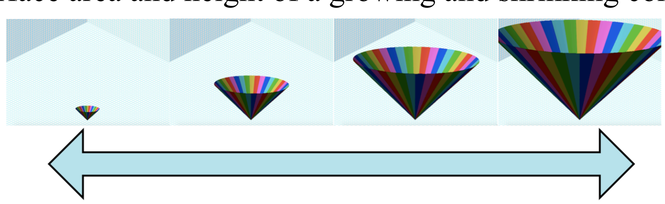
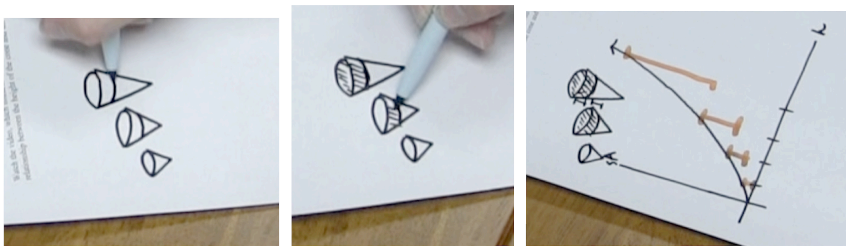


Figure 9. The changing cone.

Excerpts 4. Students thinking of graphs as traces of covariation.

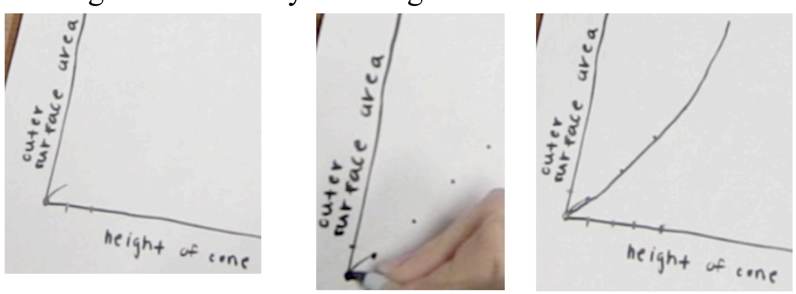
Amy: [*Amy has draw diagrams illustrating how the surface area changes for successive equal changes in height,* Figure 10a-b]. So that means for equal changes in height, the change in surface area increases. [*Amy draws axes labeled as depicted in* Figure 10c]. So for equal changes in height [*marks equal changes along horizontal axis and draws a curve*], I want to see [*draws in vertical segments to verify changes in the vertical quantity increasing for corresponding successive changes along the horizontal axis,* Figure 10c] increased changes in surface area.

Tara: So, for some change of height [*marks point on horizontal axis; see* Figure 11a, it changes like, let's make that smaller [*marks point closer to the origin on horizontal axis*]. It changes some. And then we have like another change in height, and it's gonna grow a lot more. So, there's where it is [*draws a point on her graph and connects the point with line from the origin as in* Figure 11a]. And then for another change in height [*adds a tick mark along horizontal axis*], it's gonna grow a lot more [*plots second point on graph*] and a lot more [*adding another mark along horizontal axis and then another point above it in her graph,* Figure 11b]. So I think it would look something like concave up [*connects her points and makes sweeping concave up motion with hand,* Figure 11c].



(a) (b) (c)

Figure 10a-c. Amy's emergent traces of covariation.



(a) (b) (c)

Figure 11a-c. Tara's emergent traces of covariation.

We note a few important features of each student's actions that we take to indicate her thinking about her graphs in terms of covarying magnitudes. First, Amy and Tara focused on coordinating magnitudes among different instantiations of the situation to draw conclusions about how the quantities change in tandem. They *then* drew a graph *while* focusing on how their graph represented a relationship within its respective coordinate system equivalent to that they conceived as constituting the situation. Second, neither student's understanding of her graph was mediated by reasoning about specified values or formulas. Rather, they both exhibited actions that suggest their reasoning about quantities' magnitudes within the respective representational contexts (e.g., the situation and the graph). Lastly, any indication to a figurative aspect of their graphs only occurred in reference to representing some identified relationship; the figurative aspects of their graphs were implications of and subordinate to their conceived relationships.

Discussion

Notably, the students' difficulties rarely stemmed from underdeveloped images of situations as some researchers have reported elsewhere (see Carlson et al., 2002; Moore & Carlson, 2012). Instead, the students' difficulties more frequently stemmed from their ways of thinking for graphs limiting their ability to represent relationships they conceived to constitute some situation. Carlson et al. (2002) alluded to a similar observation relative to calculus students' covariational reasoning, explaining, "We have provided examples of students who appeared to be able to apply covariational reasoning…in a kinesthetic context but who were unable to use the same reasoning patterns when attempting to construct a graph…for these situations" (p. 376). Our results provide insights into particular ways of thinking that can lead to students experiencing difficulties graphing relationships they have conceived constituting some situation; students' ways of thinking for graphs can entail problematic habits rooted in figurative elements of thought.

In some cases, students' figurative elements of thought entailed issues of perceptual shape (e.g., a graph is not composed of a vertical segment). In other cases, students' elements of thought were dominated by the sensorimotor experiences of drawing a graph (e.g., a graph is drawn or read from left-to-right). Regardless, we interpret these instances to be compatible with Moore and Thompson's (2015) description of students' static shape thinking. That is, a student thinking of a graph figuratively entails her assimilating a graph as an object in and of itself, thus foregrounding perceptual properties and sensorimotor experience. This thinking stands in contrast with a student maintaining an image of a graph as an emergent trace constituted simultaneously by two attributes with properties of shape and sensorimotor experience merely being a product of uniting those attributes. Our results corroborate researchers' (Carlson et al., 2002; Moore & Thompson, 2015) conjecture that ways of thinking that foreground understanding graphs as emergent traces of covarying quantities are more productive for accommodating novel situations and relationships than those ways of thinking that foreground figurative thought. The mental operations that constitute thinking emergently are, at their most fundamental bases, akin regardless of the produced trace and properties of its shape (Thompson, 2011). On the other hand, ways of thinking for graphs that foreground recalling a repertoire of memorized shapes and properties of figurative thought (e.g., graphs passing the vertical line test or graphs being traced left-to-right) are constrained to those situations that are compatible with these shapes and properties.

Another notable finding that underlies several of the habits provided above is students' propensity to focus on one quantity's magnitude without a persistent and explicit focus on a second quantity's magnitude. For instance, a student 'starting' her graph along the vertical axis typically entailed her considering one quantity's magnitude when determining the 'starting' point (see Patricia and Andrea, Excerpts 1), with Patricia alternating between which quantity she considered when determining the point. In the case of drawing or reading a graph left-to-right, several students' activities implied their having an explicit focus on variation in one quantity's magnitude (i.e., the quantity represented along the vertical axis) without attending to explicit variations in the other quantity's magnitude (i.e., the quantity represented along the horizontal axis). Covariational reasoning entails the formation of a multiplicative object through the explicit and persistent tracking of two quantities' magnitudes so that these magnitudes are understood as varying *simultaneously* (Carlson et al., 2002; Saldanha & Thompson, 1998; Thompson, 2011). Our observations of the aforementioned habits suggest these students did not maintain an explicit and persistent image that a change in one quantity necessitated a change in another quantity, or that a particular magnitude in one quantity necessarily implied a particular (or possibly several) magnitude(s) in another quantity.

Lastly, we find the results notable given the extent that some students were unable to reconcile perturbations that stemmed from their constructing a graph they conceived to represent an intended relationship that simultaneously contradicted ways of thinking for graphs based in figurative thought. In these cases, the students questioned the correctness of their graphs (see Patricia, Excerpts 2; Tara and Alisha, Excerpts 3). Piaget (2001) described that a key feature of operational intelligence is that it dominates and transforms elements of sensorimotor experience. Thus, we find the students' difficulties in reconciling their states of perturbation significant and illustrative of the extent that particular perceptual features and sensorimotor experiences were habitual (i.e., durable and somewhat implicit) to their use of graphs.

Closing Remarks

A limitation of the present study is that it was conducted with 10 undergraduate students. We suggest that researchers investigate other populations' responses to tasks designed like the ones in this manuscript. The results of such investigations will not only provide insights into those populations' ways of thinking for graphs, but they will also provide points of comparison for our reported findings. Second, we suggest that researchers both within the U.S. and internationally investigate similar populations to the one reported on here. Researchers who extend this work will provide insights into ways of thinking including comparisons to and extensions of those ways of thinking and habits reported here. One interesting point of comparison would include characterizing differences and similarities among students in various countries, demographics, and academic experience.

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Reinventing the Multiplication Principle

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Counting problems offer opportunities for rich mathematical thinking, yet students struggle to solve such problems correctly. In an effort to better understand students' understanding of a fundamental aspect of combinatorial enumeration, we had two undergraduate students reinvent a statement of the multiplication principle during an eight-session teaching experiment. In this presentation, we report on the students' progression from a nascent to a sophisticated statement of the multiplication principle, and we highlight two key mathematical issues that emerged for the students through this process. We additionally present potential implications and directions for further research.

Key Words: Combinatorics, Reinvention, Counting problems, Teaching experiment

Introduction and Motivation

The multiplication principle (MP), called by some "The Fundamental Principle of Counting" (e.g., Richmond & Richmond, 2009), is a fundamental aspect of combinatorial enumeration. Broadly, it is the idea that for independent stages in a counting process, the number of options at each stage can be multiplied together to yield the total number of outcomes of the entire process. It is foundational to many of the counting formulas students learn and is also a much-needed source of justification for why these counting formulas work as they do. In spite of its importance, little has been studied about the MP in and of itself. In order to better understand student thinking about the MP, we had two undergraduate students reinvent a statement of the MP over the course of an eight-session teaching experiment (Steffe & Thompson, 2000). In this paper, we describe their overall reinvention process, presenting their progression of statements. We also introduce and discuss a handful of mathematical issues that are entailed in the MP and that arose for the students (specifically, the independence of stages in a counting process and the need to count distinct composite outcomes). We seek to address the following research goals:

1. *Describe a pair of students' trajectory as they reinvent a statement of the MP, and, in so doing,*
2. *Present mathematical issues in the MP to which the students attended as they reinvented the statement.*

Research about the MP in Combinatorics Education Literature

Previous work has demonstrated the importance of the MP in counting (e.g., Lockwood & Caughman, 2015; Lockwood, Swinyard, & Caughman, 2015), and the lack of a well-developed understanding of the MP appears to be a significant problem and hurdle for students, particularly in terms of their ability to justify or explain formulas. We have found that students can easily assume that they understand the MP in counting because multiplication is a familiar operation for them. As a result, they use the operation frequently but without careful analysis, and they tend not to realize when simple applications of the operation are problematic. Kavousian (2008) suggested that students may conflate operations like addition and multiplication. Although her work does not speak directly to the multiplication principle, it gives evidence that students perhaps do not necessarily understand appropriate operations to use, and this suggests that their understanding of the multiplication principle is not robust. In addition, a number of authors have documented issues that students face in determining the appropriate combinatorial operation, such as permutations or combinations, to use in a given situation (e.g., Batanero, Navarro-Pelayo, & Godino, 1997; Dubois, 1994; Fischbein & Gazit, 1988). Given that the multiplication principle is an underlying component of each of these formulas again suggests, even obliquely, that the lack of a solid understanding of the MP could negatively affect understanding of these broader formulas.

Lockwood, Swinyard, & Caughman (2015) conducted a study in which students reinvented basic counting formulas, and the students in that study did not appear to have a solid understanding of the MP. They worked with outcomes empirically but lacked the understanding of how those outcomes related to the underlying counting process involved with the MP. These authors concluded that their findings indicate that, "the students' work was, surprisingly, not based on the multiplication principle, but instead it was almost entirely based on empirical patterning" (p. 56). In addition, Lockwood, Reed, and Caughman (2015) recently conducted a textbook analysis that examined statements of the MP in university combinatorics and discrete mathematics textbooks (relevant details are outlined in the Mathematical Discussion section below). This study revealed a variety of statement types, raising questions about necessary components of the MP and what kind of statement might be most effective pedagogically for students. All of the studies mentioned above emphasize that the MP is an important aspect of counting and suggest the need for more research that targets students' understanding of the MP as a fundamental counting process.

Theoretical Perspectives

In this section we first discuss reinvention and our general adherence to a view that values students' mathematics. Then, we explore mathematical issues related to the MP in order to provide motivation for our study and to facilitate subsequent discussions of student reasoning about the MP.

Reinvention. Gravemeijer, Cobb, Bowers, and Whitenack (2000) describe the heuristic of *guided reinvention* as "a process by which students formalize their informal understandings and intuitions" (p. 237). From this perspective, students can formalize ideas through generalization of their previous mathematical activity. We had students reinvent statements of the MP because we felt this would allow students to meaningfully understand and articulate a statement, giving us insight into how students come to understand the MP. This is a different approach than first teaching them about the MP and then asking them questions about it. We adopt this approach because we value how students reason about mathematical ideas, and we feel we can learn about students' reasoning by observing how they come to reason initially about an idea. This is in line with other researchers who have used principles of reinvention to gain insight into students' reasoning about a particular concept or definition (e.g., Larsen, 2013; Oehrtman, Swinyard, & Martin, 2014; Swinyard, 2011).

Mathematical Discussion. Lockwood, Reed, and Caughman (2015) studied 64 different textbooks and found a wide variety of statements of the MP (Figures 1, 2, and 3 below all represent differing formulations of the MP), and this variety is surprising given how fundamental the MP is to counting. The wide range of statements raises questions about what mathematical issues should be handled by a statement of the MP, whether various statements are mathematically equivalent, and whether or not there is a particular MP statement (or kind of statement) that would be most appropriate for students to be taught.

Generalized Product Principle: *Let* $X_1, X_2, \ldots, X_k$ *be finite sets. Then the number of k-tuples* $(x_1, x_2, \ldots, x_k)$ *satisfying* $x_i \in X_i$ *is* $|X_1| \times |X_2| \times \ldots \times |X_k|$.

Figure 1 – Bona's (2007) statement of the MP

The Product Rule: *Suppose that a procedure can be broken down into tasks. If there are* n_1 *ways to do the first task and* n_2 *ways to do the second task after the first task has been done, then there are* $n_1 n_2$ *ways to do the procedure.*

Figure 2 – One of Rosen's (2007) statements of the MP

The Multiplication Principle*: Suppose a procedure can be broken down into m successive (ordered) stages, with* r_1 *different outcomes in the first stage,* r_2 *different outcomes in the second stage, …, and* r_m *different outcomes in the mth stage. If the number of outcomes at each stage is independent of the choices in the previous stages, and if the composite outcomes are all distinct, then the total procedure has* $r_1 \times r_2 \times \ldots \times r_m$ *different composite outcomes.*

Figure 3 – Tucker's (2002) statement of the MP

Statement types. Lockwood, Reed, and Caughman (2015) identified three different kinds of statements: *structural*, *operational*, and *bridge* statements. This categorization is based on Sfard's (1991) distinction between structural and operational conceptions. In terms of the MP, Lockwood, Reed, et al. (2015) defined a structural statement as a statement that "characterizes the MP as involving counting structural objects (such as lists or *k*-tuples)" (p. 20). Bona (2007), Figure 1, is an example of a structural statement, because k-tuples are being counted and there is no explicit connection to a process that generates the k-tuples. An operational statement is defined as a statement that "characterizes the MP as determining the number of ways of completing a counting process" (p. 20), because the things that are being counted are operational, like ways of completing a process or procedure. Rosen (2007), Figure 2, is an example of an operational statement, because the nature of what is being counted in this statement is the "ways of completing a procedure." Finally, a bridge statement is a statement in which the nature of what is being counted list structural, but there is a clear link in the statement to a process that generates those objects. Specifically, such statements simultaneously characterize the MP "as counting structural objects and specifies a process by which those objects are counted" (p. 20). Tucker (2002), Figure 3, is a bridge statement because it counts outcomes and yet clearly connects those outcomes to a procedure that generates them.

The distinction between these three statement types is relevant because there are mathematical implications for statements that are strictly structural or operational statements. While we do not delve into the details of these implications, here we emphasize that bridge statements are our preferred statement type. Indeed, although bridge statements may initially seem overly complicated or wordy, they can be appropriately applied to a wide variety of problems while avoiding issues of overcounting. Part of the motivation for the current study, then, is to build upon the textbook analysis by actually studying how students think about mathematical issues that arose in the textbook statements of the MP. The findings from Lockwood, Reed, and Caughman (2015) framed and informed the mathematical issues we pursued with the students, and the following section we discuss these key mathematical issues.

*Key mathematical issues in the MP***.** In the textbook study, there were also a handful of mathematical issues that emerged in the statements of the MP. Here we briefly describe two mathematical issues in the MP. In the Results section we will describe the students' reasoning about these key ideas, and so we briefly introduce them here to facilitate subsequent discussion. First, there is the notion of *independence* of stages in the counting process, which captures the idea that a choice of options at a given stage does not affect the number of outcomes in any subsequent stage. Independence is a necessary condition in order to apply the MP, or else overcounting may occur. To see this, we can consider the Language Books problem: *You have 4 different Russian books, 5 different French books, and 6 different Spanish books on your desk. In how many ways can you take two of those books with you, if the two books are not in the same language?*. It may be tempting to solve the problem in two stages: first pick a first book, and then pick a second book. Note, however, that in this problem, one cannot simply consider that there are just 15 options for the first book in the pair (from 4+5+6 = 15 total books), because subsequent books might depend on what language the first book was. To handle this, a correct solution may break down the problem into cases according to which pair of languages are being chosen. In order to fix this, one can break the problem into cases according to which language book was first, arriving at the correct answer or 4*5 + 4*6 + 5*6 = 74).

Another mathematical issue that arose from the textbook study is that the MP must yield *distinct composite outcomes*, which means that when applying the MP we want to ensure that there are no duplicate outcomes. This qualification, too, prevents instances of overcounting. Specifically, we need to make sure that there are not multiple counting processes that actually yield the same outcome. A problem that highlight this issue is the 3-letter sequences problem (found in Tucker, 2002): *How many 3-letter sequences made of the letters a, b, c, d, e, f contain the letter e, where repetition of letters is allowed?*. Note that a tempting (but incorrect) answer is to specify a 3-stage process – first, pick where the *e* can go (3 options), second, since now we are guaranteed to have an *e*, fill in the next available position with any of the 6 letters (6 options), and third, fill in the last available position with 6 letters (6 options). There are 3*6*6 = 108 total ways to complete the process. However, there is not a 1-1 correspondence between ways to complete the process and the number of desirable outcomes. The process overcounts the total number of desirable outcomes. For example, consider the sequence *eae*. It was counted once when an *e* is placed in the first position in the first stage, then the password is filled out with *a* then *e*. However, it is also counted when *e* is placed in the last position in the first stage, then the password is filled out with *e* then *a*. Thus, overcounting can occur if there is not care to have distinct composite outcomes of the counting process generated by the MP. These examples point to the mathematical subtleties in statements of the MP.

Methods

Data Collection. We conducted a teaching experiment (Steffe & Thompson, 2000) in which a pair of undergraduate students solved counting problems over eight hour-long sessions. The sessions were video and audio-recorded. The students were enrolled in vector calculus in a large university in the western United States, and they were chosen because they had not been explicitly taught about the MP in their university coursework (and thus would not simply try to recall a statement of the MP). The interviews took place outside of class time over a period of four weeks, with approximately two sessions per week. The students worked together at a chalkboard while the researcher and a witness posed problems and, at times, asked clarifying questions. The students were encouraged to talk and work together, and they quickly established a rapport where they could ask each other questions and probe each other's thinking.

The overall structure of the teaching experiment was first to have the students solve a series of counting problems. They were then asked to write down and characterize when they were using multiplication as they solved these problems, and they wrote down several iterations of statements of the MP throughout the teaching experiment. In the last session they were also asked to evaluate textbook statements and compare them with their final statement.

Broadly, the students engaged in four kinds of activities throughout the teaching experiment (summarized in Table 1). First, students solved a set of initial counting problems that involve multiplication. The aim was to have them engage in some joint solving of counting problems that involved multiplication so they could reflect on when and why they multiplied This occurred during Sessions 1 and 2. Second, at the end of Session 2, students were asked to articulate an initial statement of the MP – specifically, how they might characterize when to multiply when solving counting problems. This resulted in an initial statement of the MP. Then, in Sessions 3 through 7, the students solved more counting problems and engaged in iteratively refining their statements of the MP. During these sessions, the overall aim was to have the students develop more sophisticated and rigorous statements of the MP. To facilitate this, the researchers gave students new problems in order to target mathematical issues, they asked students to explain their thinking, and they encouraged students to use more general language. Then, finally, during Session 8, the students were asked to evaluate given textbook statements of the MP.

Session	Emphasis of Session
1 – 2	Solving initial set of counting problems that involve multiplication
End of Session 2	Articulating an initial statement of the MP
3 – 7	Iteratively refining statement of the MP
8	Evaluating given textbook statements

Table 1 – Overall structure of the teaching experiment

Data Analysis. The research team transcribed the sessions, and we made enhanced transcripts in which images from the video are embedded into the text. The videos and transcripts were analyzed so as to construct a narrative about the teaching experiment (Auerbach & Silverstein, 2003). Specifically, we read back through the enhanced transcripts multiple times, recorded the progression of the actual statements themselves, and documented how they had evolved over the sessions. Then, we used prior understanding of the MP that had emerged from the textbook analysis to guide our focus of particular mathematical issues, including independence and distinct composite outcomes. Key episodes involving mathematical issues were flagged and reviewed, and we scrutinized the students' statements of the MP and their explanations for insights about their reasoning. We also examined students' initial work in the sessions prior to articulating their first statement of the MP, with the aim of better understanding how they thought about multiplication prior to being asked to characterize when they multiply.

Results

We organize the results chronologically. We begin by characterizing students' initial conceptions of multiplication during initial problem solving sessions. We then discuss the students' articulation and iterative refinement of statements of the MP in Sessions 3 through 7 (due to space, we emphasize Sessions 4 and 6). In so doing, we highlight the development of key mathematical issues of independence and being attuned to overcounting. Due to space we do not share results from Session 8 of evaluating existing textbook statements.

Students' initial conceptions of multiplication (Sessions 1 and 2). In Sessions 1 and 2, the students were given a number of counting problems that involved multiplication in a variety of contexts. During these sessions, we were able to observe the students using multiplication and describing why they were multiplying. In this section, then, we provide evidence of a couple of conceptions about multiplication that they respectively possessed.

The very first problem states, *A student is going to complete a True/False quiz with 5 questions. How many different possible outcomes are there for how the quiz could be answered?* In answering this question the students almost immediately recognized the answer to be 2^5 = 32, and we asked how they got the answer. Pat used branching language and a tree diagram to reason through why they should multiply. Pat said, "Yeah so you branch, you take the first one and you can have two possibilities and each of those possibilities will have two possibilities, each of those will have two possibilities," and he drew out the tree in Figure 4. For Pat, a tree representation and the notion of branching was a natural to think about multiplication, and he would describe having "options" for different stages in the counting process.

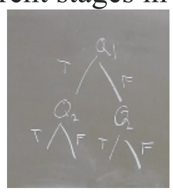

Figure 4 – Pat's initial branching idea

Caleb seemed to focus on a different perspective of multiplication, and he brought up the notion of multiplication as involving groups, which elicits ideas related to multiplication as repeated addition. As evidence of this, we consider the following exchange about the problem *How many ways are there to arrange 4 people in a line?*

Caleb: I think it's multiplied because they're like groups. So you have your first group and then it's like it could be one and two and then that'd be a group (in the first two positions), and then one and two here (in the first and third positions) would be another group. So each time you go like into another level (he made a gesture like he was alluding to the tree diagram) you are grouping them a different way.

Int.: Ok, and so why does multiplication get at that grouping? Or, why when you're having those groups, why multiply?

Caleb: I mean multiplication, isn't that just putting things together in groups?

Thus, we see that options and tree branches were key for Pat, while the idea of groups was important for Caleb. These ideas were important for them as they considered and used multiplication. They seemed open to each other's ideas and language, however, and there were instances in which the students merged their two ideas together.

Initial ideas about independence and overcounting. In the first two sessions, there were also three issues that started to emerge involving independence and overcounting. These issues are central to their refinement of statements of the MP, and we discuss them in more detail throughout the paper. However, it is important to highlight some of their initial ways of thinking about these mathematical issues early on to get a sense of how they were developed.

Independence. As noted above in the mathematical discussion section, independence of stages in a counting process is a key aspect of a statement of the MP (especially an operational or a bridge statement). This came up for students in the Language Book problem discussed above. The students started to reason about whether they could just consider 15 options for which book is first in the pair, but they soon realized that they could not simply multiply 15 by anything to get the answer. As the exchange below shows, they became aware of the issue of dependence. In order to fix this, they broke the problem into cases according to which language book was first, arriving at the correct answer or 4*5 + 4*6 + 5*6 = 74).

Int.: *Um so why... so I think I don't, I mean I think your idea of like the fifteen options for the first one, why did that break, which I think is a good idea, but what, what about the problem made it so you couldn't just do like fifteen times something?*

Pat: *Um because the, whatever you select for the first one, then determines what kind of book you can select for the next one.*

Int.: *Okay.*

Pat: *So there are, technically there are fifteen options for the first book, but you have no way of knowing if they selected a Russian, French, or Spanish. And so then you don't know if you have four times five, or five times six, or six times four as your next option.*

Caleb: *Yeah.*

We will continue to revisit their reasoning of independence and their treatment of it in their statements of the MP. The point here is that they had a meaningful experience early on in a problem in which independence was introduced, and they seemed to realize the importance of accounting for independence in their statement of the MP.

Overcounting. A second mathematical issue that arose even within the first session was overcounting (also described above). In particular, the students were made aware of an overcounting error on a problem that states, *How many ways are there to place two different-colored rooks in a common row or column of an 8x8 chessboard?* To solve this problem, there are 64 options for where to place the white rook, and then there are 14 other spaces for the black rook to go.[1] Note that in this problem, the fact that the rooks are distinct is important, but the choice for whether to start with the white or the black rook does not need to be taken into account in the final solution. We can decide either to place the white rook first and then the black rook, or vice versa, and in either case the answer will be the same. It might be tempting to multiply the answer by 2 to account for whether or not the white or the black rook is first, but that is not necessary and in fact will overcount. As we will see in Pat's discussion below, this is difficult to see. Specifically, in this problem, they had correctly considered options for placing the rooks, but Pat had multiplied by an additional factor of two for which colored rook was placed first. However, the additional multiplication by 2 is not necessary and thus would overcount. We brought this to their attention fairly explicitly, and after quite a bit of reasoning they did have the following realization:

Pat: *That's a good point. Cause if you're saying you put white rook here, and then selecting to put the black rook here, but that would be the same as if saying I selected to put the black rook first here and put the white rook there. So it is, everything is being counted twice this way.*

There is further evidence that this experience of overcounting was meaningful for them, as they subsequently referred to it on other problems. For example, another problem says *How many different numbers can be formed by various arrangements of the six digits 1, 1, 1, 2, 3?*. The students referred back to the Rooks problem, which suggests that they were being careful about overcounting.

Pat: *I feel like that's a way to go. Just thinking about it. Cause the 1's are gonna be the same...*

Caleb: *Yeah exactly. So we have how many spots to put a 2 and a 3 that are different?*

Pat: *Yeah, uh…*

Caleb: *Five spots originally then uh four. So five times four?*

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[1] An equivalent is to first select one of 8 rows and then to select one of 8 columns in which to place the white rook. Then, there are 14 remaining positions in that row and column for the black rook to go.

Pat: *That equals twenty. Yeah, yeah, that sounds right. That's, uh, will we get the counting twice thing like we did with the chess pieces?*

Caleb: *Uh no. Because if we put whatever number it is, so like say we put a 2 here, here, here, here, here, then we have to, we have five.*

In sum, the students had a formidable experience in which they were exposed to the issue of overcounting. This proved to be a salient point for them to which they returned throughout the teaching experiment. We introduce it here to show the genesis of their reasoning about overcounting.

Articulating and refining statements of the MP – An iterative process involving key mathematical ideas. We now present work that occurred between the end of Session 2 through the end of Session 7. In these sessions the students articulated and then iteratively refined a statement of the MP. We describe this process, and we highlight the progression of statements and the mathematical issues that arose for students. For the sake of space, we focus primarily on Sessions 4 and 6, as these are where most of the development occurred.

Initial Articulation of a Statement of the MP (Session 2). At the end of Session 2, we asked them the following: “So, I'm curious if you can take a stab at characterizing for me, like when you use multiplication when you're solving these problems? Um and actually maybe a more precise way to ask that is let's say you had to write a rule like you're writing a textbook and you're going write a rule for when you're going use multiplication to solve counting problems like this.” We encouraged the students to write down a statement, and they each independently wrote statements on the board. After some time, Caleb wrote an initial statement (Figure 5): *Use multiplication in counting problems when…there is a certain statement shown to exist and what follows has to be true as well.* He explained his work by saying, “Basically I thought it would be useful to write big words, I was like you know I'll try that. So I'm saying basically that if you set like parameters or something that's shown to exist, like you have something and then you want to add on to that. You have to multiply them if you're saying the first one is true.” We observe that this statement is a reasonable first step but is far from a rigorous statement of the MP.

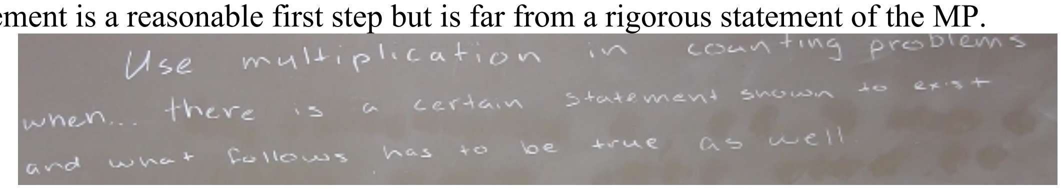

Figure 5 – Caleb Statement 1

After the students had written statements in Session 2, we decided to have them solve problems involving addition/case breakdowns in Session 3. The goal was that the students would be able to make a contrast between problems involving addition and problems involving multiplication. The major reason that we wanted to highlight this, though, was because we were anticipating that independence would be an important mathematical issue that should be incorporated into their statement, and these addition problems highlight the need for independence. While we do not provide examples due to space, the students were able to think about adding and how it related to an operation like multiplication, and they were able to realize that a case breakdown was an effective means by which one could handle issues of independence.

Refining the statement (Session 4). In Session 4, Caleb and Pat spent a good deal of time on a particular problem, and they drew heavily on a representation of tree diagrams to reason through the problems and to articulate a statement at the end of the session. This enabled them, for the first time, to articulate a mathematically correct insight about why multiplication works. As we will see, however, although they made progress toward a rigorous statement there were a couple of problematic issues with their statement. Thus, Session 4 represents some significant progress wherein they clearly articulated when it is appropriate to multiply, but they struggled to identify language that would be meaningful for them. Specifically, Session 4 involved three aspects of their reinvention – understanding multiplication as involving equal sized groups, struggling with language, and addressing overcounting.

Understanding multiplication as involving equal sized groups. We began Session 4 by giving them the following problem: *How many ways are there to flip a coin, roll a die, and select a card from a standard deck?* Pat connected a solution to his initial views of multiplication, which involved options.

Pat: I feel like, I feel as if options leading to other options is the way I'm thinking about it.

Int.: Ok. So say more about options leading to other options.

In response, Pat wrote a tree diagram in Figure 6, which shows branching and options leading to other options. He and Caleb both seemed to understand that the branching in the tree diagram represented the different options for coins, dice, and cards.

Pat: Well I think of, you have your initial condition of flipping a coin, which has two outcomes, you know heads or tails. And for each time you do that, you then have a dice roll no matter what, for each of those dice rolls you have six options. And off of those six possible options there will be 52 options for what cards you can pull from a deck. Uh we won't do those, but there are options there.

Int.: Ok, and 52 for each of those?

Pat: Yeah.

Caleb: Yes.

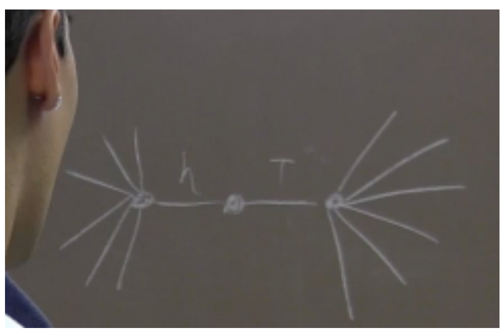

Figure 6 – Part of a tree diagram that Pat drew

We interpret that this tree diagram and their discussion surrounding it established a powerful, shared representation of multiplication, and they seemed to agree on how this representation could be useful in describing when to use multiplication when counting. An interesting discussion emerged about the nature of multiplication and how and why it worked on that problem. As seen in the exchange below, Caleb noticed an important distinction between addition and multiplication. He drew a tree diagram (Figure 7) that has one branch with 5 and the other with 6. He noted that this situation would not by symmetric and would thus entail addition instead of multiplication.

Caleb: The way you did this, I don't know this might be towards addition, I was thinking like if you have two outcomes that are different, like say so this side there's one and then say you have a five sided dice and a six sided dice or something, if it doesn't have symmetry then you do addition.

Pat: *Ok that's fair.*
Caleb: *So like 1, 2, 3, 4, 5, 6. And then there's, it's not symmetrical on either side, so you have to end up doing this one plus this one.*
Pat: *Yeah okay.*
Caleb: *If that makes sense.*
Pat: *I like that.*
Caleb: *So it might have to do with like reflecting or like…(trails off)*
Int.: *Ok awesome.*
Pat: <u>*Yeah I like that. So multiplication is when all groups...*</u>
Caleb: <u>*Are equal.*</u>
Pat: <u>*When all the groups you're talking about are equivalent. Whereas addition is when you have you know non equal groups. Different groups.*</u>

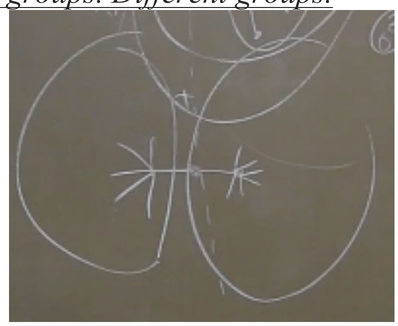

Figure 7 – A diagram showing non-symmetric branching

This was an important moment for them because it established for them mathematically when (and why) it might make sense to multiply versus add (noted in the bold sections of the previous excerpt). We then asked them to articulate their ideas into a refined statement, and this proved to be challenging for them.

Struggling with language. The students recognized that they would want to incorporate this idea of equal groups into their statement. Notice again they reiterate that they associate equal groups with multiplying (in the first bold section). Then, notice that they try to determine if groups are equal or not, and they try to articulate some particular language. The second underlined passage highlights that they found this to be a difficult task.

Caleb: *So let's, we've definitely come to the conclusion that if their groups are equal we multiply.*
Pat: <u>*If we're, if we're combing equal, if we're combing equal groups we're multiplying.*</u>
Caleb: <u>*Yes.*</u>
Pat: <u>*And if we're combing non equal groups it's mult...*</u>
Caleb: <u>*It's addition.*</u>
Pat: <u>*Yeah addition.*</u>
Caleb: <u>*So for multiplication. How would we decide if they're equal or not?*</u>
Pat: *Ok um. If, if for every possible selection, or for every possible outcome there's the same choices after that, for that.*
Caleb: *For every time?*
Pat: *For every possible outcome. Like for, like for the die. For every possible outcome of the die there is the same number of cards to select. And the same cards themselves.* <u>*So like, I can, I can figure out how to say the first part.*</u>
Caleb: <u>*Yeah that's exactly, that's how I feel.*</u>

The students then proceeded to engage in more discussion related to refining of their statements. Articulating an appropriate statement for when to multiply continued to be very difficult for them, and Caleb noted at one point, "It's so hard to think about. It's just something we naturally do." They struggled to articulate language and could not come up with a statement they were happy with after several stops and starts. Given this difficulty with the language, it is perhaps not surprising that they turned to language involving pathways, which was tied to a meaningful representation for them. Caleb said, "Um. What about, let's incorporate the pathway in this one. So for every possible outcome there's a pathway leading to it." Caleb then worked on writing down a particular statement, and he wrote what is written in Figure 8.

Caleb: Alright that's kind of what I'm thinking now. So for each possible pathway to an outcome, there's an equal number of options leading to that path.

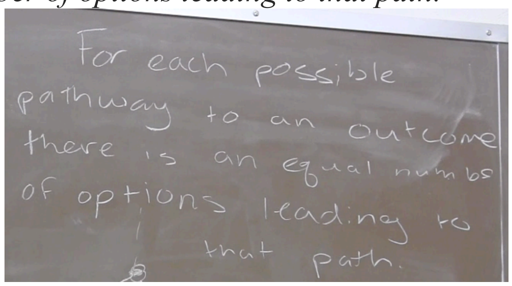

Figure 8 – The students' statement involving pathways

At this point, the students had come to an understanding of when multiplication might be appropriate in terms of groups, and although they struggled to articulate language, they arrived at a statement involving "pathways" language. However, there was still more for them to consider in refining their statement.

Addressing overcounting. The statement in Figure 8 addresses some issues but was still susceptible to overcounting, as there is nothing in the statement that would prevent them from overcounting on the Language Books problem, for example. To address this, I asked them to revisit the Language Books problem. This had the effect of highlighting overcounting, and the following exchange suggests that they did not want to allow for outcomes to be counted more than once.

Caleb: Alright, well I mean that would be without repeating the same pathway. But we still have to talk about how the pathway is multiple different outcomes. But.
Int.: [to Pat] Do you see what he's saying there?
Pat: Yeah I see what's going on.

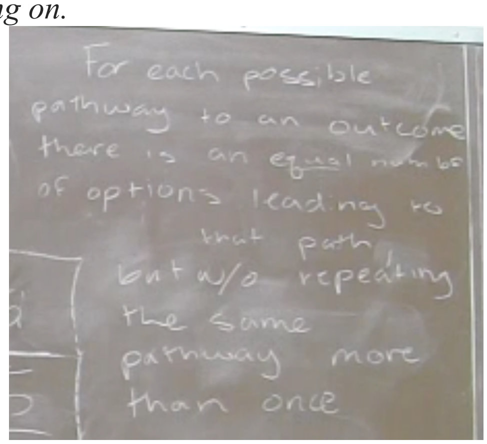

Figure 9 – The students' pathways statement after considering overcounting

There are a couple of conclusions to draw from Session 4. First, in terms of them actually really reasoning about multiplication and why it works, they made good progress due to the tree representation. The idea that multiplication could be thought of as entailing equal groups seemed to be solidified for them, and they wanted to consider an equal number of options as they progressed through. We would argue that the representation itself was an important part of them coming to reason about multiplication in this way. However, in spite of the progress they made during this session their struggle to come up with particular language or terminology seemed to hinder their progress. So they emerged with a statement that was understandably specifically tied to the given representation, but one with which we were not ultimately satisfied.[2]

Developing a mathematically rigorous statement (Sessions 6 and 7). In Session 6, the students made substantial progress toward a correct and rigorous statement of the MP. There are two things that happened in Session 6 that we feel contributed to their refinement of their statement. First, we explicitly asked the students to be more general in their language (moving away from "pathways" language of Session 4). This led them to almost immediately come up with language of options, selections, and outcomes, which they were able to clearly define and use. We pushed them to articulate what they meant by these terms at each step along the way, and this difficult yet valuable refinement of language proved to be helpful for them. Second, the students wrestled through the mathematical idea of independence, in particular a distinction between independence of the *number* of options and not necessarily the options themselves. This engagement with powerful and nuanced mathematical ideas seemed to help solidify their understanding of the statement of the MP and how and why it might need to be refined. Through, they produced the following statement at the end of Session 6 (Figure 10): "If for every selection towards a specific outcome, there is no difference in the number outcome, regardless of the previous selections, then you multiply the number of all the options in each selection together to get the total number of possible outcomes."

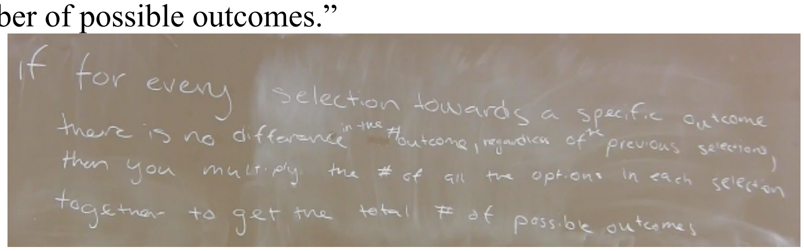

Figure 10 – The students' statement at the end of Session 6

Then, in Session 7 we asked the students for further clarification of language. This led them to offer definitions for "selection," "option" and "outcome," which was very productive. Specifically, they defined selection as "when a choice has to be made"; option as "one of the possible choices for a selection"; and outcome as "one unique combination of all chosen options." This language, which they continued to use consistently throughout the rest of the teaching experiment, was helpful as they continued to reason about their statement. Figure 11 shows how they see the relationship between selections, options, and outcomes.

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[2] We do not discuss Session 5 in depth, in part because we are restricted by space, and also because it is tangential to the story of the students' the reinvention of the MP. After the students had made significant progress in Session 4, we expected that we could give them a couple of additional tasks that would help to generalize their language (beyond the pathways language). However, we went down an unexpected rabbit trail in this session, as the students focused on order and introduced a less productive representation of the MP. Session 5 came and went with no change to their statement of the MP.

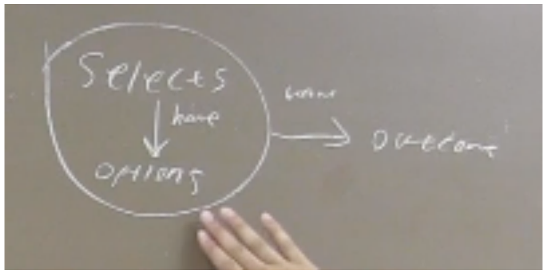

Figure 11 – The interaction between selections, options, and outcomes

Finally, in Session 7 they arrived at the last statement, given in Figure 12. They added the word "unique" after we prompted them again to consider how overcounting might occur. Specifically, we had them revisit a problem that involved overcounting (the 3-letter sequences problem), and they realized that they wanted their statement to explicitly account for potential overcounting. We had the following exchange, and they added the word "unique" to their previous statement. This is a way of taking care of the overcounting issues.

Caleb: We could say without any repeated outcomes.
Int.: Ok. Say more about that.
Caleb: So our problem here is where we're getting like a repeated outcome. If we say um.
Pat: Oh hey we already have specific outcome in there.
Caleb: Yeah.
Pat: So how about we say specific unique outcome?
Caleb: Yeah.

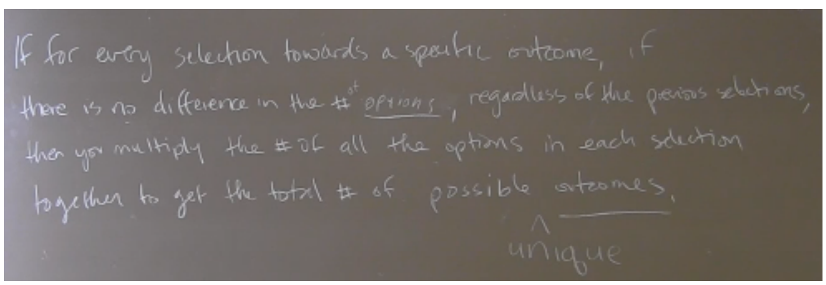

Figure 12 – The students' final statement

With this work, Pat and Caleb arrived at a final statement of the MP. In terms of the statement types identified in Lockwood, Reed, and Caughman (2015) this is a *bridge* statement, because they are counting something structural ("the total number of possible unique outcomes"), and yet those outcomes are tied to a process of making selections. Thus, we note that the students' statement aligned with a bridge statement. And, although we do not have time to provide details, the students were able to make sense of a number of statements we gave to them. In particular, they interpreted Tucker's (2002) bridge statement, from Figure 3 above. They were able to make sense of the statement and relate it to theirs. Caleb said, "That kinda gets back to ours. It addresses the independent choices and the unique outcomes." And said " So each R1 would be the selection and then the sub N M and yeah that's what they used, are like the sub things, would be their options. So this stage would be their selections, and the outcomes in this stage are the options." He concluded by saying "I think if we broke down each of ours we could basically reword them to be the same."

Conclusion and Implications

To summarize our findings, we present the students' broad progression of statements of the MP (Table 2). They key sessions in their reinvention were 2, 4, and 6-7.

Session	Statement
2	#1 – Use multiplication in counting problems when… there is a certain statement shown to exist and what follows has to be true as well.
4	#2 – For each possible pathway to an outcome there is an equal number of options leading to that path but without repeating the same pathway more than once.
6-7	#3 – If for every selection towards a specific outcome, there is no difference in the number outcome, regardless of the previous selections, then you multiply the number of all the options in each selection together to get the total number of possible unique outcomes.

Table 2 – The students' progression toward a statement of the MP

By having students reinvent a statement of the MP, and by closely analyzing aspects of multiplication to which they attend, we gain insight both into how students reason about the MP, and also how productive reasoning about the MP might be developed. In particular, by engaging with particular tasks, the students we worked with were able to come to reason about key mathematical aspects of the MP (such as independence and unique outcomes) that they wanted to include in their statement of the MP. In addition to insights about how they come to understand particular mathematical ideas, we can draw a couple of key conclusions from their overall progression from to a final statement.

First, we have an existence proof that it is possible for students to develop, through guided reinvention, a mathematically rigorous statement of the MP. It is not trivial to characterize many of the subtle mathematical details of the MP, and it is impressive that the students were able to do so. Second, we see that although they were able to accomplish this task, characterizing when to use multiplication in solving counting problems was not a straightforward activity. This is demonstrated most clearly in their first statement, which shows that even after they had successfully used multiplication in counting problems for two sessions, they still struggled with articulating a formal/general statement about it.

Our findings suggest a couple of implications. First, as a fundamental aspect of counting, the MP is invaluable, yet potentially challenging, for students to understand well. Although it deals with a familiar operation, it entails subtle mathematical features, which might take time and effort for students to learn. In terms of research, we plan to continue to explore what might be entailed in a robust understanding of the MP, which includes interviews with more students and also with mathematicians. Based on our findings from this study, especially insights about understanding independence and distinct composite outcomes, we can look to design instructional interventions that might draw students' attention toward such ideas.

Pedagogically, more work is needed to more carefully evaluate how best to teach the MP to students in a classroom setting, but our findings suggest that it may be worthwhile to unpack some key mathematical issues of the MP with students. Instructors should appreciate and seek to understand the mathematical details in the MP and should help students think carefully about when multiplication properly applies in counting situations.

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Developing mathematical knowledge for teaching in content courses for pre-service elementary teachers

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Abstract. The article reviews efforts of three iterations over the course of three semesters in developing a written assessment of the mathematical knowledge for teaching (MKT) of pre-service elementary teachers enrolled in a course on number and operations. Content addressed in the items discussed in the article include discourse knowledge (DK) coupled with specialized content knowledge (SCK). Results show that the pre-service teachers can engage in reasoning and justification up to a point and use their discourse knowledge in novel ways and situations, though there appear to be limits in their constructive process.

Key Words: Mathematical Knowledge for Teaching, Mathematical Discourse, Reasoning, Justification

Background and Research Questions

A central tenet of teacher education research has long been identifying the types of knowledge that teachers need to know in order to teach mathematics. Such attempts date back to Shulman's (1986) original proposal of a new type of knowledge that he called *pedagogical content knowledge* (PCK), defined as "the particular form of content knowledge that embodies the aspects most germane to its teachability" (p. 9). Since then, research teams such as Hill, Ball and Schilling (2008) and Hauk, Toney, Jackson, Nair, and Tsay (2014) have worked to conceptualize PCK. Ball and company have developed typologies for the much broader realm of *mathematical knowledge for teaching* (MKT), shown in figure 1, for which PCK is a subconstruct. Note that the left half of the oval consists of subject matter knowledge (SMK) which they claim requires no knowledge of students, thereby distinguishing it from the right half which is PCK. It is worth noting that the Ball model is specifically designed for the K-8 setting; this is important because as Speer, King, and Howell (2014) note, generalizability comes into question when trying to apply the model outside of the K-8 context.

Within the Hill, Ball, and Schilling (2008) model, *common content knowledge* (CCK) is defined as "knowledge that is used in the work of teaching in ways in common with how it is used in many other professions or occupations that also use mathematics"(p.6). In contrast, *specialized content knowledge* (SCK) is specialized in the sense that it is specific to the task of teaching. SCK includes various ways to represent mathematical ideas, provide mathematical explanations for rules and procedures, and examine and understand innovative solution strategies (Hill et al., 2008, p.377). As an example, consider fraction division. Most middle school graduates can readily use the invert-and-multiply algorithm to divide fractions. Thus, this piece of knowledge is CCK. Yet, few can explain to a novice learner *why* the algorithm exists in school mathematics nor why it is justified, thereby making this particular piece of knowledge SCK. Within the realm of PCK are *knowledge of content and students* (KCS) and *knowledge of content and teaching* (KCT). KCS is

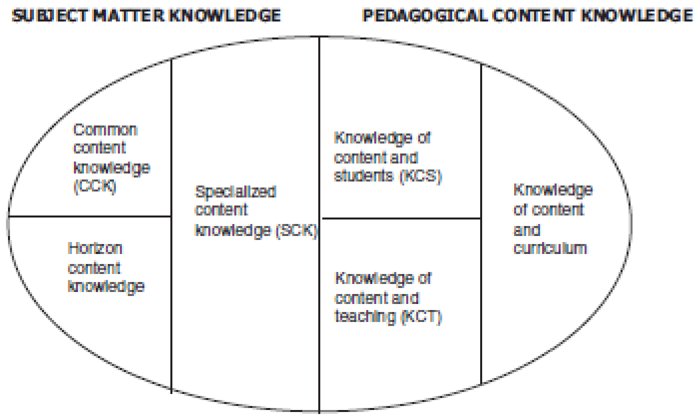


Figure 1. Domain Map for MKT (Hill et al, 2008)

"content knowledge intertwined with knowledge about how students think about, learn, or know this particular content" (p. 375), while they define KCT as a knowledge of teaching moves. So, using our division of fractions example again, a teacher who is aware that students often invert the dividend instead of the divisor is demonstrating KCS, and might use fraction diagrams as a way of scaffolding student understanding of division of fractions by using her KCT.

Implicit in the use of KCS and KCT is an awareness of the words, grammar, syntax, and forms of standard mathematical language in use – what Gee (1996) calls the "little d" discourse of mathematics. Also at work in the teaching of mathematics are nuances about what is valued in mathematical discourse in a mathematics class (as opposed to mathematics in a physics or biology class), the socio-mathematical norms for questions and answering, and myriad other interactions that make a mathematics lesson recognizable in an instant (e.g., by someone listening in or looking through the window of a classroom for just a few seconds). This kind of *situated* "little d" discourse is what Gee called "big D" Discourse. Hauk, Toney, Jackson, Nair, and Tsay (2014) have brought these ideas into a further unpacking of the components of PCK. The extended model, shown in Figure 2, adds a fourth dimension to PCK called *Knowledge of Discourse* (KD). Hauk et al. argue that effective teaching of mathematics includes facilitating student learning of *mathematical discourse* (along with other discourses), which we define as Discourse about mathematics that is enacted in the classroom when students and teacher engage in mathematically appropriate, accurate, and effective communication situated in the context of reasoning and justification of mathematical ideas. Clearly, a rich and textured Knowledge of Discourse is required for teachers to use and promote the valued mathematical skill of justification: engaging in reasoning about and explaining how one knows something is true (Cioe, King, Ostien, Pansa, and Staples, 2015).

Here, we use *accuracy* to denote that the statements in a declaration are in fact mathematically true, while *effectiveness* concerns the degree to which an explanation can fully demonstrate the necessary mathematical ideas and reach its intended audience (e.g. elementary students in the case of K-8 pre-service teachers). *Appropriateness* takes into consideration the level of mathematical sophistication of the argument's audience. As an example, consider the case of a 3rd grade teacher Alicia who is working with her students on even and odd numbers. The students have noticed a pattern: when you add two even numbers, you get another even number. To help the students understand why this is true, Alicia is considering the following two explanations:

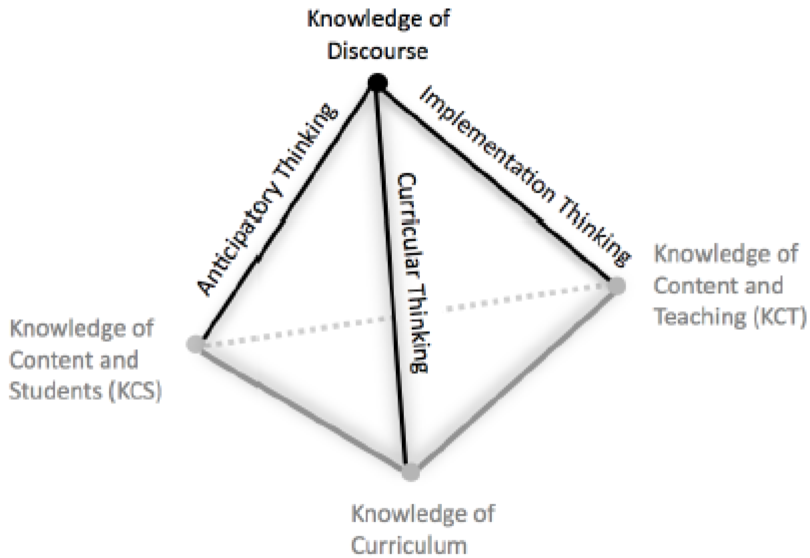


Figure 2. Tetrahedron Model of PCK (Hauk et al., 2014)

Explanation 1: “If you think about it, every even number is 2 times another number: 6=2x3, 8=2x4, and so on. So let’s let the first even number we are adding be 2xm where m is some number. Let’s let our second number be 2xn where n is some other number. When we add them, we get 2 x m + 2 x n , but by the distributive property, we can say that 2 x m + 2 x m is the same thing as 2 x (m+n), so that the new number we get is also 2 x some number, so it must be even as well.’’

Explanation 2: “If you think about numbers as being sticks, then I can always put even numbers into bundles with 2 sticks in each bundle: for example 6 can be put into 3 bundles of 2 sticks each, 8 can be placed into 4 bundles of 2 sticks each, and so on. So if I am adding two even numbers, then the first number can be placed in bundles of 2 sticks each and so can the 2nd number. But when I add up all the bundles of 2, I get a number that still is placed into bundles of 2, so the number I get must also be even as well.”

In the first explanation, we see that Alicia’s reasoning is certainly accurate. However, for most 3rd graders, her explanation will neither be effective nor level appropriate as her audience has not developed facility with the concept of variable. Contrast this to say an 8th grade audience where her first explanation is now accurate, effective, and appropriate as variable quantities arise in the course of solving linear equations, which is a 6th grade standard in the Common Core. As for the second explanation, if Alicia is speaking to 3rd graders, then she is now mathematically accurate, effective, and appropriate. However, if she is teaching an 8th grade audience, the second argument will be accurate and effective, but **not** level appropriate as most of her students are able to reason with the concept of variable (i.e. her argument is now below the level of her students).

It is worth noting that our notions of accuracy, effectiveness, and appropriateness are similar and related to the three components of a *proof* as expounded upon by Styliandes,

Styliandes, and Schilling-Traina (2013). Though we distinguish here between a proof and justification with the latter being on a different level of rigor and generalization than that of the former, the two structures do serve the same purpose in teaching. This distinction between the two and their common goal in instruction is delineated in the third Standard of Mathematical Practice of the Common Core State Standards which states that students should be able to construct viable arguments and critique the reasoning of others. Indeed, the authors note that

> ...Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments (CCSSO, p. 6-7).

For Styliandes et al., proof does each of the following:

- it uses *a set of accepted statements* in use by the classroom community that are true and available without further justification (our *accuracy)*,
- it employs *modes of argumentation* which are ways of reasoning that are valid, known to, and within the conceptual reach of the classroom community (our *appropriateness),* and
- it utilizes *modes of argument representation* which are forms of communicated expressions that are appropriate, known to, and within the conceptual reach of the classroom community (i.e. our *effectiveness).*

The available justifications a person could provide for an item of mathematical knowledge will be limited by the person's ways of knowing that knowledge. Balacheff and Gaudin (2010) use the term *conception* to model different ways of knowing. A conception has four primary components: a set of problems, a set of operators, a representation system, and a control structure. The set of problems is a collection for which a conception can be utilized to determine their solution (which they call the *sphere of practice* of the conception), while the set of operators consists of those objects needed to transform and/or manipulate linguistic, symbolic, or graphical representations. The representation system contains the linguistic, graphical, and symbolic means by which the person who holds the knowledge interacts with that particular piece of knowledge, while the control structure consists of all the means needed to make choices, to take decisions, and express judgment. Of particular interest to us is the representation system because of its relevance to discourse knowledge. Indeed, Balacheff and Gaudin (2010) note that "Whatever it is, depending on the state of the subject/milieu system, the representation system must be adequate to give account of the problems and to allow operators to perform" (p. 215). In essence, a lack of a sufficient representation system, which includes an appropriate mathematical discourse that acts as a referent for the knower, will fail to allow the operators to perform. For example, if a teacher tells her student that to solve the equation $x^2 = 25$, it is necessary to perform the inverse operation of taking the square root, such an action would have no meaning for the student if they lack an awareness and understanding of what is meant by "inverse," and hence the student is hindered in her ability to take the action of performing the square root operation needed to solve the problem.

To measure PCK and MKT more generally, both the Ball and Hauk research teams developed multiple choice assessments designed for administration to in-service teachers receiving professional development for Ball's team and completing a master's degree for mathematics teachers in the case of Hauk's team. It should be noted that in the case of Hauk's team, the focus was on the PCK development of teachers at the 7-12 level unlike Ball's team. While items in the instrument developed by Hauk's team measured in large extent the syntactic structure of KD and did attempt to measure the teachers' ability to engage in proof validation, neither their instrument nor the items developed by Ball's team measure the larger components of discourse required to engage in reasoning and justification: i.e., neither team tried to specifically measure mathematical discourse more generally. Hence, the current project is designed to address three key missing ideas in the existing literature: (1) How might discourse knowledge development be characterized for pre-service elementary teachers (PSETs) during a semester long course on number and operations? (2) How can discourse skills support reasoning and justification among PSETs? (3) What conceptions about number and operations do PSETs have as they enter a course on number and operations?, How prevalent are these notions among this group of PSETs?, How have these conceptions evolved by the end of the course?

Research Methods

Beginning in the summer of 2011, an instrument was developed to begin measuring different aspects of MKT for PSETs with particular emphasis on items that require some combination of SCK, KCS, and KD. Our emphasis for the current work is on the subset of items requiring a combination of SCK and KD to answer. The teachers must engage in mathematical discourse to justify certain mathematical facts or procedures to answer these items. Examples of the items from the most recent administration of the instrument are given below:

- You have given the multiplication exercise in part (a) [i.e., compute 2.74 x 2.2] to your 4th grade class. Shonte, one of your students, says that she knows where the decimal goes, but she does not know why it goes there. Give Shonte an explanation that she can understand for why the decimal is placed where it is.

- John asks you in math class one day why $4^0 = 1$. Give John an explanation that he can understand for why this is true.

- Nancy, a student in your 5th grade math class, asks you day why she cannot divide 5 by 0. That is, why she cannot do 5 ÷ 0. Give Nancy an explanation that she can understand for why she cannot do this.

- Give an explanation for your 7th grade students that they can understand for why it is that $\frac{5}{6} \div \frac{2}{3} = \frac{5}{6} \times \frac{3}{2}$. That is, why the invert-and-multiply algorithm works for division of fractions.

These items were chosen for inclusion in the instrument because it is well documented in the teacher education literature that the ideas needed for the items' completion are ones with which both in-service and PSETs commonly struggle. For instance, Levenson (2012) discusses in-service elementary teachers' lack of ability to distinguish between definitions and theorems when considering the zero exponent. Wheeler and Feghali (1983) and Russell and Chernoff

(2011) found that both PSETs and in-service teachers tend to have the same conceptualizations of zero as many children do, such as the notion that zero is not a number and/or it means nothing. Even and Tirosh (1995) found that many of their in-service teachers answered that 4 divided by zero is undefined, although none could explain why: most simply stated it is a mathematical rule. However, other researchers (e.g. Crespo & Nicol, 2006; Ball, 1990; Wheeler & Feghali, 1983) have found that many PSETs do not even know that the expression is undefined, let alone know how to explain why it is undefined. Some PSETs remembered a rule of anything divided by 0 being 0, while others reasoned that the answer should be 0 because you are dividing by nothing. Ma (1999) found that among the American in-service teachers she interviewed, only about half could perform the division of two fractions, while most could not create a word or story problem modeled by fraction division. The ones who did perform the division correctly relied almost solely on the invert-and-multiply algorithm and none could explain why the algorithm works. Tirosh (2000) discusses the differences in "knowing that" something is true and "knowing why" something is true; the PSETs in her study also failed to explain why division of fractions behaves in the manner it does algorithmically. Stacey et al. (2001) report an over reliance on rules and facts among PSETs when performing operations with decimals such as occurs in decimal placement in multiplication.

PSETs enrolled in a course on number and operations at a large public state university in the northeastern US were given the instrument upon entering the course as well as upon exiting in a standard pre-post format. During the course, PSETs are expected to consistently engage in mathematical discourse. This discourse generally occurs through reasoning and justification that is deemed as acceptable and appropriate for elementary aged children as this is a socio-mathematical norm exercised in class and group discussions, online homework exercises, and on exams. The instrument contains 13 items (including the four mentioned previously), and the current report focuses on data collected from 6 sections of the course between Fall 2014 and Fall 2015 semesters, with N=113 teachers. In addition to administering the instrument, several teachers were interviewed during this time concerning their answers to 3 of the items to discern their abilities to communicate effectively orally in addition to written formats. Participants were also presented with novel (to them) tasks during the interviews that gauged their abilities to engage in reasoning and justification more generally through proof validation by determining the accuracy of different explanations. For instance, one of the items in the instrument asks for a justification of the invert-and-multiply algorithm. During the interviews, the teachers discussed their own justifications for the algorithm and then were presented with justifications that had not been discussed during the course and were asked to discuss the appropriateness of the justification for an elementary classroom.

Each of the four items for all teachers were coded using a grounded theory approach by a research team consisting of a mathematician, a mathematics educator, and a graduate student enrolled in an education Master's of Science program. Responses were assigned a triple of the form (T, E, A), where the T measures mathematical accuracy on a scale of 0-4, the E measures effectiveness on a scale of 0-2, and the A measures appropriateness also on a scale of 0-2. Our rubric is summarized in Figure 3: note that Yumus (2001) provides a scale similar to ours in the accuracy construct.

Data Analysis and Results

	0	1	2	3	4
Mathematical Accuracy (0-4 scale)	No Response	Restates Item or relies on rule	Attempts to explain but fails	Accurate but lacking details	Accurate with sufficient details
Effectiveness (0-2 scale)	Low Impact	Students not able to follow	Most students could follow		
Appropriateness (0-2 scale)	Low Impact	Is above or below grade level	Is at grade level		

Figure 3. Rubric for the discourse items

After all items for each of the 113 PSETs who took both the pre and post assessment were coded, the intraclass correlation coefficient (ICC) for the items (excluding those coded (0,0,0): i.e. the ones for which responses were not given) was calculated for all three constructs in each item to determine the strength of inter-rater reliability. These coefficients ranged from .842 to .924 in appropriateness, .845 to .947 in effectiveness, and .885 to .940 in accuracy. Hence, good to excellent agreement in the rubric use between the three researchers existed on all four items and all three constructs. The lowest coefficients among the three constructs all came from the item on fraction division. Items were assigned a final code when at least two of the three team members agreed in their initial coding: for those responses for which none of the three researchers agreed, the item was discussed and the team came to a final consensus. This occurred about 10% of the time.

A reliability analysis was also performed on the pre and post assessments for this group of PSETs. Before stating the results, it is worth discussing the choices that the authors have made in their current analysis. It has been well documented in the psychometric literature that Cronbach's alpha, a favorite unit of analysis among social scientists, tends to be a biased estimator of reliability (Dunn, Baguley, and Brunsden, 2014; Green et al., 1977; Green & Hershberger, 2000; Green & Yang, 2009; Huysamen, 2006; Raykov, 1998; Sitsma, 2009;Zimmerman et al., 1993; Zinbarg et al., 2005). Even Cronbach (2004) himself, together with Shavelson, stated that "The numerous citations to my paper by no means indicate that the person who cited it had read it, and does not even demonstrate that he had looked at it" (p. 392). The use of alpha is appropriate when the assumptions of *tau-equivalence* are met, which requires that the items in a measurement essentially have standard deviations that are at least relatively close to one another, an assumption that is rarely met in most psychological measurements (Dunn, Baguley, and Brunsden, 2014). Alpha also tends to be sensitive to normality assumptions in the data, with skewness affecting the resulting statistic (Sheng & Sheng, 2013). Further complicating our current analysis is the fact that our data are ordinal, something which can further bias estimates of reliability by severely underestimating them, especially in the presence of skewness (Gadermann, Guhn, & Zumbo, 2012).

In light of this, many psychometricians favor the use of McDonald's omega (Dunn, Baguley, and Brunsden, 2014; Graham, 2006; McDonald, 1999; Revelle & Zinbarg, 2005; Zinbarg et al., 2005, 2006, 2007) over the use of Cronbach's alpha, especially when data are congeneric rather than tau equivalent. Given that our data are ordinal, skew (as is to be expected at least for the pre assessment), and not tau equivalent (i.e. the standard deviations of the items are significantly different from each other), we calculated McDonald's omega for the current data set and found on the pre-assessment, $\omega_{pre} = .62$, while on the post-assessment, $\omega_{post} = .73$. As is the case with Chronbach's alpha in small N studies, values of .6 or higher are considered acceptable, and so we can be reasonably confident that the items are demonstrating unidimensionality.

As we coded the responses for the four discourse items discussed earlier, several themes about PSET conceptions emerged among them for each item. These themes are summarized in Figures 4-7. As expected, many PSETs entered the course with limited conceptions about the number 0, some thought that the decimal should be placed in the hundredths place in the multiplication exercise, and the few who attempted the fraction division item generally responded with a rule. Thus, we see connections with the existing literature: however, we are also now getting a sense of just how pervasive these conceptions are, at least for this population of PSETs. Note that each of the tables summarizes the frequency counts pre and post of PSETs who held those particular conceptions: this then means that the post counts will differ from the pre counts as many of the PSETs have changed their conceptions, usually to a conception more in line with what is presented in elementary curricula.

Theme	Pre Count	Post Count
Count the # of places	47	21
Reference to place value	7	40
Estimation	4	2
Line the numbers up	8	2
Other (to test the children, pizza/apples, as part of the algorithm)	3	1

Figure 4. Summary of themes for conceptualizations for the decimal item

Each item has its own interesting and intriguing conceptions. For decimals, many PSETs generally entered the course by stating the rule for counting the number of places after the decimals in the problem and then placing the decimal that many places from the right in the answer. Others made a vague reference to place value in saying that the decimal distinguishes the ones place from the tenths place. Note that many of the 40 PSETs who mentioned place value in the post test also gave answers that were conceptually sound and therefore considered accurate, appropriate, and effective. A few PSETs gave size arguments (i.e., the idea that the decimal must follow the 6 because 60.28 is too large since the answer should be close to 6), while others thought that the numbers had to be lined up as they are in addition and subtraction, and then the decimal would be placed after the 0 since the answer will have hundredths.

Theme	Pre Count	Post Count
0 power means you aren't multiplying by anything	15	1
0 is not a real number	3	2
How many fours do we have?	4	1
Answer is 0/4/There is no answer	3	1
Pattern is Divided by 2	2	2

Figure 5. Summary of themes for the zero power item

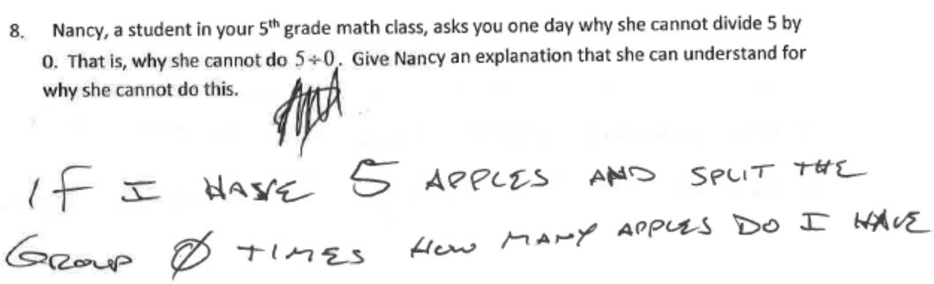
In the zero power item, some PSETs exhibited 'nothing; or 'nonnumeric' conceptions of zero. This is of particular interest here because in terms of Balacheff and Gaudin's (2010) framework, the PSETs' representation systems seem to be impacting their understandings significantly. If, because of linguistic constraints, I say that because 0 means the absence of something, or likewise that zero means nothing, then my cognition is going to be stable when I think of exponents as repeated multiplications: i.e., in Piaget's terms, cognitive disequilibration does not automatically occur as I can give meaning to a mathematical idea that occurs *on the boundary* or *at an extreme* of applying a definition for exponents that seems to fit with my current understanding of what zero is. Students in calculus courses face a similar issue in trying to understand infinite limits: in linguistic terms, if one says that a limit *is* infinite, then should not that limit exist? Interestingly enough, many PSETs are faced with a challenge to this line of thinking about the zero exponent for the first time in these courses: it is not uncommon for one or more of their classmates to hear this line of reasoning and ask why the answer is not 0 then, which of course accounts for a few of the other responses given in the pre-assessment. These are precisely the moments when cognitive disequilibrium first occurs for many of the PSETs, and they begin to realize that their representation system for the number 0 does not suffice to respond to such a challenge. Blake and Verhille (1985) agree that the "zero is nothing" interpretation "effectively prevents the teaching of the deep, complex structure of zero" (p. 37) : this is due in large part, they claim, to the linguistic structure of the representation system for 0 (i.e. in particular the use of common language creates a superficial understanding of what zero actually is).

The division by zero item saw reinforcements of the notions that the number 0 is not a real number or that it means nothing, as expected. However, we now have an idea of just how pervasive such conceptions are among this student population: of the N=113 PSETs who took the pre and post assessments, almost half (48.67%) mentioned these ideas in their responses. Such a finding clearly has implications for the teacher educator as it shows how important it is to help teachers redefine their representation system for the number zero as mentioned earlier. Several PSETs tried to use what Levenson (2009) calls *practically based explanations* (PBEs),

Theme	Pre Count	Post Count
Apples/Pizza	25	2
Can't divide something by nothing	40	7
0 is not a real number/undefined number	15	5
Dividing by zero doesn't make sense	2	1
The answer is 0 or 5	9	4
Multiplication and division are inverses	0	8
There are infinite or 2 answers so undefined	0	9

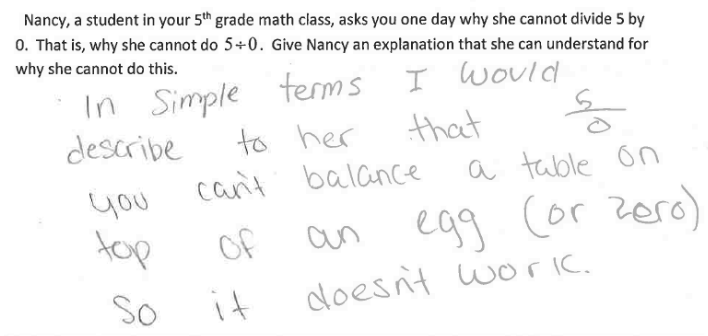
Figure 6. Summary of the conceptional themes for the division by zero item

where daily contexts and/or manipulatives "give meaning" to mathematical expressions. PBEs are in contrast to *mathematically based explanations* (MBEs), which are based on mathematical definitions or previously learned mathematical properties and hence often use mathematical reasoning. Levenson (2009) found that although in some instances 5th graders preferred PBEs, almost all unequivocally were capable of and actually produced MBEs. Now, like Levenson, we do not argue that PBEs are not to be used in all cases, since as Dreyfus (1999) notes, "for mathematics educators there appears to be a continuum reaching from explanation via argument and justification to proof" (p. 102). However, in the case of division by zero, PBEs can lead to problematic conceptions because many of these explanations often are based on the representation of zero as meaning nothing, which then leads to the belief that the answer should be 0 or 5, as we saw occurring in our data. For example, consider the following explanation offered by one of our PSETs on the pre-assessment:


8. Nancy, a student in your 5th grade math class, asks you one day why she cannot divide 5 by 0. That is, why she cannot do $5 \div 0$. Give Nancy an explanation that she can understand for why she cannot do this.

IF I HAVE 5 APPLES AND SPLIT THE GROUP Ø TIMES HOW MANY APPLES DO I HAVE

This example brings out several intriguing points. First, whereas Ma (1999) discusses the tendency of American elementary teachers to use circular foods when discussing fractions, similarly we found almost a fifth (22.12%) of our PSETs used a PBE that included some reference to apples, such as this student (who actually tended to be quite capable mathematically during the course interestingly enough). Second, as Levenson (2009) and Levenson, Tsamir, and Tirosh (2007) note, a common response from elementary students could be that you did not actually divide in this case by this line of thinking so the answer should be 5, or some others might say that you end up with 0 groups and so the answer should be 0. Again, it is likely this line of reasoning that nine of the PSETs used when they indicated initially that the answer is 0 or 5. The division by zero item, unlike the other three items, also brought out new conceptions in the post assessments. Some PSETs realized that understanding the division by zero item has something to do with connecting multiplication and division as inverses but failed to explain this connection in this particular instance, while others tried to use the notion that for an arithmetic expression to be defined, there should be one and only one value that can be assigned to that expression for consistency: they of course failed to see that while true, this idea does not apply in this particular case. The team also found the following pre-response particularly intriguing:


Nancy, a student in your 5th grade math class, asks you one day why she cannot divide 5 by 0. That is, why she cannot do $5 \div 0$. Give Nancy an explanation that she can understand for why she cannot do this.

In Simple terms I would describe to her that $\frac{5}{0}$ you can't balance a table on top of an egg (or zero) So it doesn't work.

This is obviously a PBE that is essentially rule bound: however there is an attempt to explain and so we coded this response (2,0,0). This PSET has a rather fascinating representation system for the problem: the division symbol represents the table top and the 0 is the egg. For her, her conceptualization is adequate to convince a 5th grader of the inability to divide by zero. She has not been challenged with the thought of the egg being extremely large and the table extremely small (say the size of a table in a dollhouse), so that one could indeed balance the table on top of the egg. Thus again, cognitive disequilibration does not occur for her because her conceptualization makes sense to her and is hence very stable. It is worth noting that as division by zero has become a rule for her, this rule is now an operator for her conceptualization.

Theme	Pre Count	Post Count
Multiplication and division are inverses/ opposites	8	5
It's the same as cross multiplying	8	6
Keep/switch/flip	4	3
Use decimals/double negative	2	0
It's easier than dividing	0	2
Division is repeated subtraction	0	1

Figure 7. Summary of conceptualization themes for the fraction division item

Table 7 summarizes the different conceptions for the fraction division item. Not surprisingly, this item proved to be the most challenging of the four items for the PSETs, both pre and post. Many of the PSETs chose not to complete the item, although about a third made an attempt in the post test. Again, we see that some of the PSETs have rule bound conceptualizations in which the rules have become operators. Some mentioned that multiplication and division are inverses, though they could not explain how this fact helps to reason about the invert-and-multiply algorithm.

Each of the four items were discussed during the course at some point, and so the PSETs were exposed to arguments for them that are commonly seen in elementary curricula. The instructor at one point in the Fall 2015 semester decided to give an item as a bonus on the first exam that had not been discussed in class to see how many PSETs could use their discourse knowledge to reason in a novel (for them) situation. To this end, the students were presented with the following:

> You are introducing decimals for the first time to your 4th grade class. As the class talks about the place values of digits after the decimal, Tomeka, one of your students, asks you the following:
>
> "So we go from the 100s place to the 10s place to the 1s place to the 10ths place to the 100ths place. Why is there no 1ths place? I mean, to the left of the decimal we went from 10s to 1s, so why do we go from 1s to 10ths; why not go from the 1s place to the 1ths place?"
>
> Give Tomeka a mathematically accurate, effective, and level appropriate answer for why there is no 1ths place.

Almost 20% of students gave explanations that were considered acceptable. Two of the representative responses are shown below:

Because we go from the 10^2 to the 10^1 to the 10^0 to the 10^{-1} which is the tenths place. There is not place in between 10^0 and 10^{-1}. Plus, if we have oneths place, it would be the same as the ones place Because any whole number over one is in the ones place – i.e. $\frac{8}{1} = 8$.

There is no 1^{ths} place because we can represent a decimal as $\frac{1}{10}$ for the tenths, or $\frac{1}{100}$ for the hundreths. We would therefore represent the 1^{ths} as $\frac{1}{1}$, which would give us 1.

Of course, these responses only show that there is a possibility that these PSETs are reasoning in novel situations because it is quite possible that they had previously been exposed to this very situation and so were already aware of an appropriate, effective, and accurate response. However, at the very least the responses demonstrate that some PSETs are developing discourse knowledge and using it in commanding ways to communicate intricate ideas.

A paired samples t-test was performed on all four items for each of the three constructs to gauge changes in means. Statistically significant learning gains occurred in all four items for each construct with all p-values of .000. The largest gain in accuracy was in the zero power item, while the smallest gain in accuracy from pre to post was in the decimal item. The largest gains in effectiveness and appropriateness were in the zero power item, while the smallest gain for these two constructs was in the fraction division item. Hence as a whole group the 113 PSETs showed growth in their ability to engage in mathematical discourse and use it to engage in reasoning and justification.

Implications and Conclusions

PSETs have a myriad of conceptions when it comes to number and operations, especially about the number zero. Our data suggest that these conceptions (which are well documented in the literature) are very prevalent among PSETs. Courses for PSETs must include attention to helping PSETs redefine their representation systems as often times such courses will be the last chance many of these PSETs have to construct conceptions that are useful for the classroom before they actually begin teaching. There appears to be some hope that such efforts can be successful as we saw that at least some of the PSETs moved away from conceptualizations in which rules acted as operators for them to conceptualizations in which a rich mathematical discourse knowledge allows them to justify claims, perhaps even in novel situations such as the oneths place example. The data on decimals suggests an even more problematic concern in that it shows that some PSETs not only lack the SCK necessary to teach this topic, but they also lack the necessary CCK as they cannot actually do the multiplication upon entering the course, often lacking the awareness of where the decimal is placed in the answer, let alone why it goes there.

While our instrument is one of the first to try and measure discourse knowledge for PSETs, the items we included only measure one aspect of discourse knowledge: in particular, that part of knowledge that teachers "have" which is very stable and is very much in the sense of MKT that Hill, Ball, and Schilling (2008) discuss. According to Hauk et al. (2014), there is also a more dynamic side to discourse knowledge that actually comes from attending to and being responsive to student productions and discourse that occurs in problem solving and justification. Hence, future item development should include attending to measuring this dynamic side of discourse knowledge in robust ways. The importance of developing and measuring both facets of discourse knowledge becomes evident when one considers the vision of the mathematics classroom set forth in the Common Core Standards of Mathematical Practice.

Further analysis and consideration of PSETs' conceptions based upon Balacheff and Gaudin's (2010) framework is also needed. Understanding the nature of these conceptions will be crucial for aiding teacher educators in creating learning environments that are conducive to helping PSETs reinvent those conceptualizations.

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Eliciting mathematicians' pedagogical reasoning

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Given the prevalence of work in the RUME community to examine student thinking and develop instructional materials based on this research, we argue it is important to document the ways in which undergraduate mathematics instructors make sense of this research to inform their own teaching. We draw on Horn's notion of pedagogical reasoning in order to analyze video-recorded conversations of over twenty mathematicians who elected to attend a workshop on inquiry-oriented instruction at a large national mathematics conference. In this context, we examine the questions: (1) How do undergraduate mathematics instructors engage in efforts to make sense of inquiry-oriented instruction? (2) How does variation in facilitation relate to instructors' reasoning about these issues? Our findings suggest that the nature of participants' engagement with the mathematics was related to their subsequent pedagogical reasoning, and that differences in facilitation appear to have played a role in how participants engaged in the mathematics.

Key words: mathematicians, pedagogical reasoning, instructional change

National organizations have called for instructional change in undergraduate Science, Technology, Engineering, and Mathematics (STEM) courses, relating poor instructional quality to a lack of student interest and persistence (e.g., Fairweather, 2008; PCAST, 2012; Rasmussen & Ellis, 2013). In response to this need, undergraduate mathematics education researchers have developed and documented student-centered instructional approaches that are related to greater conceptual learning gains, as well as the development of more productive and equitable student attitudes and dispositions when compared with classes in which lecture is the dominant form of instruction (e.g. Kogan & Laursen, 2013; Kwon, Rasmussen, & Allen, 2005; Larsen, Johnson, & Bartlo, 2013). A recent meta-analysis of 225 studies in undergraduate STEM identified drastically different student outcomes between lecture-based courses and courses that actively involve students as learners; students in lecture-based classes were 1.5 times as likely to fail as students in classes with active learning, and students in active learning classes outperformed those in lecture-based classes on concept inventories by almost half a standard deviation (Freeman et. al., 2014).

While these findings certainly create impetus for instructional change, the fact remains that instructional change is incredibly difficult to achieve at scale. In fact, we posit that the question of how to achieve instructional change at scale is one of the most central, pressing questions facing mathematics education researchers today in both K-12 and undergraduate contexts. The undergraduate context has the potential to provide fresh insights into this issue, as many of the challenges faced in the K-12 context (e.g. lack of instructor autonomy, weaknesses in content knowledge) are likely to be less of an issue when dealing with mathematics instructors at the undergraduate level. Given the prevalence of work in the RUME community to examine student thinking and develop instructional materials based on this research (e.g. Wawro et. al., 2013; Larsen, Johnson & Bartlo, 2013; Rasmussen & Kwon, 2007), we argue it is essential that our community consider issues related to the dissemination and use of findings from our research – with an eye toward engaging this broader question of how to achieve instructional change at scale. This work aims to serve that goal by examining our efforts to engage practitioners (instructors of undergraduate mathematics) in thinking about research-based, inquiry-oriented instructional materials for undergraduate mathematics courses.

Literature

It is well documented that instructional change is difficult to achieve at scale. In particular, after reviewing 191 relevant articles, Henderson, Beach & Finkelstein (2011) highlight the fact that the development of research-based instructional materials is a common but ineffective way to support instructional change at scale, and argue that their review of the literature suggests that effective change strategies should be sustained over time and seek to align with or change individuals' beliefs while taking into account their institutional setting. We put these findings in conversation with the work of Nardi (2007). In the context of mathematics, Nardi (2007) suggests that one possible explanation for the disappointing adoption of such curricular materials and pedagogical techniques is the fragile relationship between the mathematics education researchers who develop the approaches and the mathematicians who might implement them. On one hand, mathematics education researchers frequently draw on qualitative research methodologies, which can seem both methodologically and epistemologically distant from traditional mathematics research. On the other hand, Ralston (2004) argues that mathematicians sometimes display negative attitudes regarding the mathematical background of mathematics education researchers. Artigue (1998) contended that this tension can be heightened in that research in mathematics education "often exposes [mathematicians] to the weaknesses of [their] teaching and [their] complicity in the malfunctioning and ineffectiveness of the educational system" (p. 483).

Fortunately, in the years since Artigue's claim, there is evidence (Hurtado, et al., 2012) that suggests a sizable number of faculty in undergraduate STEM fields *are* making efforts to offer their students the kinds of student-centered learning experiences supported by mathematics education research studies. Indeed, though 61% of STEM faculty report they use extensive lecturing when they teach, a full 49% of STEM faculty report they incorporate cooperative learning into their courses (Hurtado et. al., 2012). Brown (1998) proposed that a way to keep these doors of communication open was to talk to mathematicians about mathematics: "Apart from gaining credibility with mathematicians, mathematics education researchers who stay in touch with the subject are likely to maintain a more vivid sense of what the encounter with mathematics feels like and are thus in a good position to develop empathy with the learners their work is intended to support" (as quoted in Nardi, 2007, p.270).

Considering the rates at which STEM faculty now report use of cooperative learning in their instruction, we argue that there is a pressing need to document and leverage the pedagogical reasoning of those faculty who are working to implement these kinds of instructional approaches. The work described in this paper aligns with Henderson et. al's (2011) recommendations by aiming to gain insights into the beliefs and reasoning of those who might implement research-based curricular materials in undergraduate mathematics – in order to inform sustained efforts at providing a system of supports for instructional change organized around such content-specific instructional materials. This work also aligns with Brown's (1998) recommendation by engaging mathematicians in the curricular tasks and the mathematics underlying them; in this way workshop facilitators attempted to leverage mathematical conversations into pedagogical ones. In this paper, we explore two research questions: (1) How do instructors of undergraduate mathematics (who are interested in inquiry-oriented instruction) reason about instructional issues, particularly in the context of inquiry-oriented mathematics instruction? (2) How does variation in facilitation relate to the ways in which instructors engage in reasoning about these instructional issues?

Theoretical Perspective

To answer these questions, we follow Rasmussen & Kwon's (2007) characterization of inquiry-oriented instruction in which students are actively inquiring into the mathematics (e.g., by developing, justifying, and generalizing their own solution methods to open ended problems) and instructors are actively inquiring into students' thinking about the mathematics so as to build on students' informal and intuitive ideas to help them make sense of and engage in more formal and conventional forms of mathematical reasoning.

We take a situated perspective, in which we view knowledge and learning to be evidenced in the interactions among members of a community (Lave & Wenger, 1991) – in this case, the community of instructors of undergraduate mathematics. As such, we look to document mathematicians' pedagogical reasoning by examining their conversations about instruction. We follow Horn's (2007) characterization of pedagogical reasoning, considering it to be instructors' reasoning about issues or questions about teaching "that are accompanied by some elaboration of reasons, explanations, or justifications" (p. 46). Analytically, we draw on the vertices of the instructional triangle (teaching, students, and mathematics) as a conceptual tool for organizing our analysis of these conversations about instructional issues.

Data Sources and Methods of Analysis

The data under consideration in this study were collected from a workshop conducted as part of a national mathematics conference, and these data are part of a broader project that is developing and analyzing a set of ongoing instructional supports for undergraduate mathematics instructors interested in inquiry-oriented (IO) instruction. The workshop was organized around research-based curricula that have been developed in the areas of linear algebra, abstract algebra, and differential equations. The workshop lasted a total of four hours, which was split across two 2-hour sessions on consecutive days. On each day, about half of the time was devoted to content-specific work in breakout groups (self-selected by the participants), and the other half of the time was spent discussing issues of inquiry-oriented instruction that cut across all three curricula. On Day 1, facilitators planned to engage participants with an overview of inquiry-oriented instruction, followed by time to engage in mathematical tasks from the curricula in the area of participants' selected breakout group. On Day 2, the focus was to be on student thinking related to the Day 1 tasks and instructional moves intended to help instructors implement IO curricula.

The workshop included 25 participants, 21 of which responded to a workshop pre-survey that provided us with information about their background and home institutions. All participants except one were housed in mathematics departments, and the group represented a diverse set of institutions and positions (see Figure 1). Less than a third of survey respondents reported that they prefer to lecture most of the time, more than 70% reported that they like to have students work in groups on problems in class, and more than 60% report they frequently ask students to explain their thinking to the whole class when they teach. This was significant to our research because it suggests our sample is part of the sizable subset of undergraduate STEM faculty working to teach in student-centered ways, and the choice to attend the workshop also points to an interest, outside the RUME community, in research-based instructional approaches.

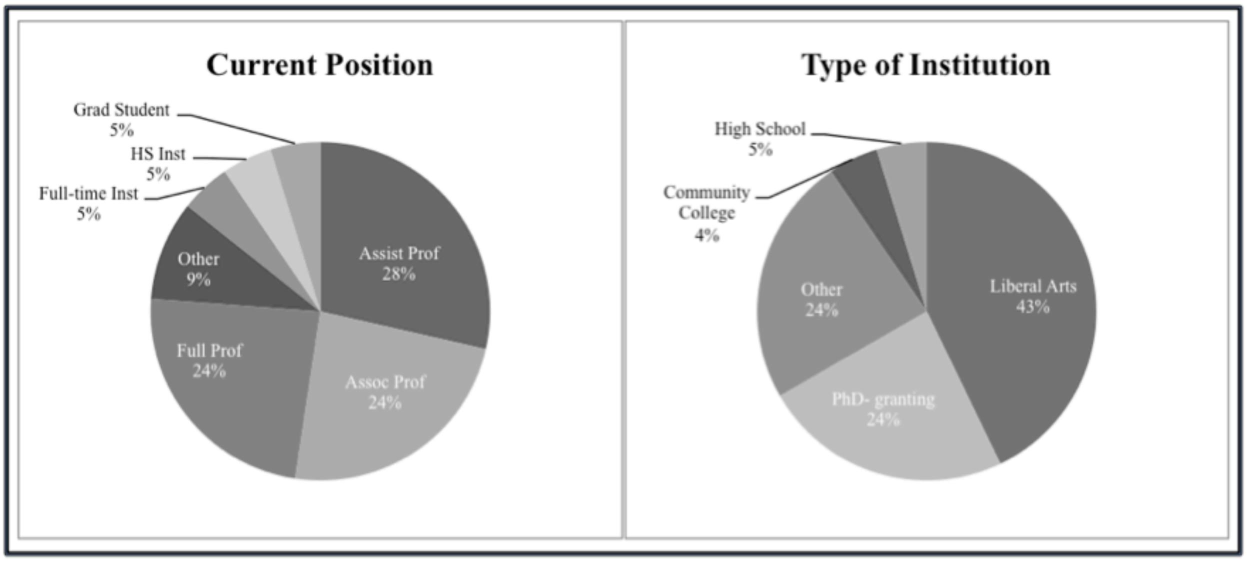

Figure 1. Position and Institution Types of Survey Respondents

In each of the two 2-hour workshop sessions, all whole-group and breakout segments were video- and/or audio-recorded for subsequent analysis. Our first phase of analysis was to analyze recorded data by generating content logs to document the sequence of events in each segment of the workshop. Each content log was organized in a table with four columns: timestamp, description of events, focus of talk, and other comments. The 'focus of talk' column aimed to help us track whether the focus of talk was on the mathematics (M), the teacher (T), or students (S), and whether it was the facilitator or participants who were doing that talking. From these content logs, we generated summaries of each session to describe the focus and use of the time along with initial characterizations of the participants' pedagogical reasoning. We noted stark differences in the conversations of the linear algebra breakout group as compared to those in the abstract algebra group, so we decided to conduct our analysis as a comparative case study of these two groups (Yin, 2003).

In our second phase of analysis, we drew on our content logs to code talk into six categories: logistics (e.g., "Does everyone have a handout?"), participant introductions, implementation questions (e.g., "How many times a week does your class meet?"), pedagogical moves (e.g., "I have them present their work as soon as we finish a task"), discussing mathematics (e.g., "They [students] came to different conclusions based on whether or not they considered all linear combinations"), and doing mathematics (e.g., "We want to show the additive inverse we'd expect from the big group stays in the small group"). In this way, we quantified the conversational focus of each group in terms of the amount of time spent.

In our third phase of analysis, we drew on content logs to identify conversational moments in which participants talk focused on the vertices of the instructional triangle: mathematics, students, or teaching. These moments were transcribed for closer analysis in order to examine participants' pedagogical reasoning, identify differences between the two groups. In our fourth and final phase of analysis, we identified facilitator moves that may have contributed to differences observed between the two groups.

Mathematical Tasks in our Study Context

The mathematical activities that were the focus of participants' time during content-specific breakout groups are research-based task sequences taken from the IOLA (Inquiry-Oriented Linear Algebra; http://iola.math.vt.edu) and TAAFU (Teaching Abstract Algebra for Understanding; http://www.web.pdx.edu/~slarsen/TAAFU/home.php) materials. More specifically, the abstract algebra group drew on the definition of "subgroup" (a subset of a group that is itself a group) in order to (1) show 5Z is a subgroup of Z (under addition), (2) find a simpler way than checking all the group criteria to check if a subset of a group is a subgroup, and (3) show the identity of a subgroup is the identity of the group.

The linear algebra group worked on activities from a sequence of tasks set in a context intended to help students develop and coordinate geometric and algebraic ways of symbolizing linear combinations of vectors in order to support their learning about span and linear independence. The context for this work is shown in Figure 2. The sequence of four tasks entails (1) determining if and how a given pair of vectors in $\mathbb{R}^2$ can be weighted and combined to "reach" a particular location in the plane, (2) determining if there is any location in the plane that *can't* be reached using that pair of vectors (to set up formalization of span), (3) determining if three given vectors in $\mathbb{R}^3$ can be combined to represent a journey that starts and ends at the origin (to set up formalization of linear independence), and (4) generating examples of sets of vectors that satisfy particular properties (e.g. give an example of a set of two vectors in $\mathbb{R}^2$ that form a linearly dependent set), as well as conjectures about generalizations that emerge from this example generation work.

You are a young traveler, leaving home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the *hover board's* movement by the vector $\begin{bmatrix}3\\1\end{bmatrix}$.
By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 miles East and 1 mile North of its starting location.

We denote the restriction on the *magic carpet's* movement by the vector $\begin{bmatrix}1\\2\end{bmatrix}$.
By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 mile East and 2 miles North of its starting location.

Figure 2: Context for the linear algebra group's task sequence

Findings

We found that there were marked differences in the conversational foci of the two groups – both in terms of how much time was spent focusing on particular topics and issues, and in the nature of talk. In this section, we first provide examples of the way in which participants in each group talked about mathematics (M), students (S), and teaching (T). We then describe facilitator moves that may have contributed to these differences. Overall, we contend that the nature of participants' engagement with the mathematics was related to their subsequent pedagogical reasoning about student thinking and possible instructional moves, and that differences in facilitation appear to have played a role in how participants engaged in the mathematics.

As previously mentioned, the facilitators of both the linear algebra and the abstract algebra group intended for the first day's breakout group to be focused on working through the mathematics in the respective task sequences. However, our content logs revealed the Abstract Algebra group spent a much larger portion of their hour-long breakout session on the this day of the workshop working through the math (86% of the time spent working through the math) than did the Linear Algebra group (30% of the time spent working through the math). Figure 3 summarizes each breakout group's conversational focus on during the first day's breakout groups.

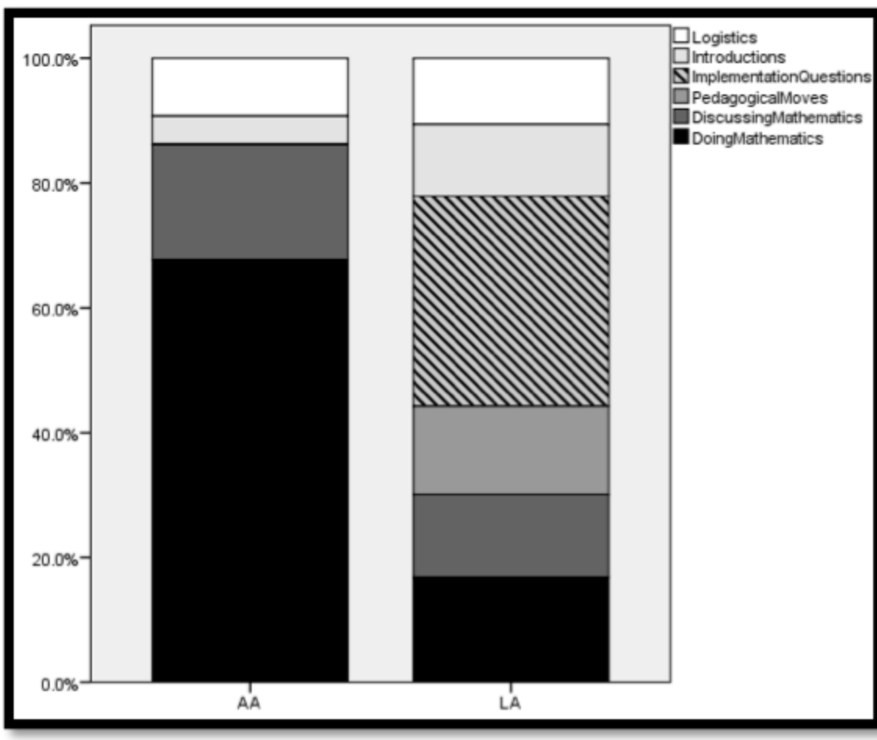

Figure 3: Conversational focus during Day 1 Breakout Groups

Trajectory of Participant Engagement: Abstract Algebra group

In this section, we highlight *how* participants engaged in the mathematics, and argue that this initial, deeply focused mathematical engagement is related to participants' subsequent reasoning about student reasoning and the role of the instructor. In particular, we highlight a trajectory of talk in which participants first engage in the mathematics (single vertex M), then discuss students' reasoning about that mathematics (two vertices M+S), and finally discuss moves they might make as an instructor in order to advance students' thinking beyond how they are currently reasoning about the mathematics (all three vertices M+S+T).

On the first day of the workshop, abstract algebra participants spent nearly the entire one-hour breakout session discussing how to apply the definition of a 'subgroup' (i.e., a subset of a group that is itself a group) in various contexts. First, they used the definition of 'subgroup' to demonstrate that 5Z is a subgroup of Z under addition. They then drew on this definition in order to develop a different, more efficient method (i.e., the subgroup criterion) for deciding when a subset of a group is a subgroup. An example of a quote that is typical of how participants engaged in reasoning through this math is, "I combined closure and inverses. We need to check for all *a* in *H* that negative *a* is in *H*. Negative *a* is in *G* because *G* is a group and it is unique, so just need to check it is in *H*." Note that this highlights our previous distinction between *'doing* the math' and '*talking about* the math' -- the participant is actually reasoning through the math directly from his/her own point of view.

On the second day of the workshop, the abstract algebra breakout group was shown a video of students explaining their partial progress in thinking about ways to develop a more efficient way to check if a subset of a group is a subgroup. Following this video, participants reasoned about how students in the video were making sense of the mathematics. For example, one participant argued:

> The group was concerned about, is the identity the same? Are the inverses the same? … Partial arguments were also generated for both including the inverse and closure gives you identity, and the identity of the subgroup has to be the identity of the group. They weren't full arguments, the students were just generating reasons that they thought that had to be.

This comment highlights participant reasoning that coordinates two vertices of the instructional triangle: *students* reasoning about the *mathematics*. While the comment doesn't fully characterize the nature of these partial arguments that were generated, it does synthesize the participant's assessment of students' mathematical contributions and reasoning. We note that the mathematics that is the subject of students' reasoning is the same as the mathematics the participants had engaged in during the previous day of the workshop.

After discussing students' reasoning about ways to develop a more efficient way to check if a subset of a group is a subgroup, workshop participants brainstormed what they would do next if they were the instructor and the whole class discussion they had just watched on video had just happened in their own classroom. The comments made by several participants evidenced a layering of reasoning involving all three vertices of the instructional triangle: the mathematics, students' reasoning about the mathematics, and what the teacher can do to help students advance their reasoning about the mathematics. The exchange below highlights this layering of reasoning:

Participant: I'm assuming they have already shown the identity is unique.
Facilitator: They have shown the identity is unique.
Participant: So, and I'm, if they haven't shown that the inverse is unique, then I would ask can an element have more than one inverse. And then I would ask what can you say about a subset of a group if it is closed under inverses.

This participant's comment shows evidence of pedagogical reasoning that attends to the mathematics (with a great degree of specificity), what students know, and what the instructor might do next to advance students' reasoning about that mathematics. We posit that participants' own deep engagement with this mathematics themselves on the first day of the workshop supported this nuanced coordination of the vertices of the instructional triangle.

Trajectory of Participant Engagement: Linear Algebra group

In addition to the fact that far less of the time of the Day 1 linear algebra breakout session was spent doing mathematics than the abstract algebra session, we observed that the nature and quality of mathematical and pedagogical conversations differed between these groups. In abstract algebra conversations, the talk followed a clearly identifiable trajectory of discourse relative to the instructional triangle, beginning with a thorough exploration of the mathematics, followed by a structured discussion about how students reasoned about the mathematics in a video clip, and finally a discussion of possible in-the-moment instructional moves that might be productive based on how students were thinking about the mathematics in the video clip. In contrast, Day 1 conversations in the linear algebra group did not follow this clear trajectory of topics, and repeatedly turned toward issues related to implementation. Due to frequent shifts in the focus of conversation of the linear algebra group, our initial impression was that participants' talk about mathematics (and consequently their subsequent talk about student reasoning about the math and possible instructional moves) was not at the same level of depth as that of the linear algebra group. A more careful analysis revealed there were examples of participants engaging in mathematical talk, but this talk was less sustained and of a somewhat different nature. We also identified distinct examples of linear algebra group members discussing student reasoning and identifying potential instructional moves in the context of the mathematical tasks posed. In this section, we highlight examples of talk from the linear algebra group that parallel examples presented for the abstract group – by first offering examples of the way in which participants talked about the mathematics, and then presenting examples of how participants reasoned about student thinking and possible instructional choices after watching a video clip of a classroom where students explained their approaches. We then consider the facilitation moves that help explain these observed differences.

Day 1: As previously mentioned, facilitators of the breakout groups collaborated to structure workshop activities so as to focus on each vertex of the instructional triangle in turn, starting with mathematics and building to consider student thinking and the role of the teacher, a structure that was reflected in the trajectory of conversation in the abstract algebra group. As in the abstract group, linear algebra participants were asked to engage in specific tasks from the curricular materials and work through the math together on the first day of the workshop. The first mathematical task the linear algebra participants were asked to engage in was whether it was possible to use two specific vectors (that were linearly independent) to "get everywhere" in the plane by "travelling" on each for different amounts of "time." Here is how one participant spoke about possible solutions to the task:

> So what they can do is, once they get a parallelogram, they can generate a new vector that is on one of the opposite sides of the parallelogram, so they can generate all of these very easily because all they have to do is take this and add one of these, or they can do the same here...

This participant's strategy for finding a solution is valid, but it is interesting that he frames his response in terms of a student's perspective without first discussing how he would himself solve the problem. Though this demonstrates a connection between mathematics and students' reasoning, it is framed hypothetically. This is in contrast with the abstract algebra participant's description of students' mathematical contributions (partial arguments for including the inverse and closure) were framed in terms of the core mathematical issue with which students were wrestling (is the identity the same) – though that comment was made after watching video of student reasoning, is perhaps unsurprising that the framing was not hypothetical.

Immediately following the parallelogram comment made by a participant in the linear algebra group, another participant shifted the focus somewhat, asked "Do they have problems dealing with the fractional? Because it might be the case that they have to ride one for a fractional amount of time…." This seemed to move the group away from doing mathematics and toward a discussion of common student responses. One participant who had previously used the materials described common student approaches, which led to a conversation of issues related to the different prerequisites for linear algebra in various departments. In an attempt to redirect the conversation and have people work through the mathematics, the facilitator once again invited participants to consider the mathematical task, this time explicitly from the perspective of a student. The first response to this, however, was "I can't think of any other way than how I would do it."

Eventually, most members of the linear algebra group worked through the task more or less completely on Day 1. They also came together for a more focused discussion about participants' own mathematical strategies, and again the conversation turned toward issues of implementation as they related to student thinking. The following excerpt highlights one of the more extensive math-focused exchanges from the Day 1 linear algebra transcript. It is noteworthy that the prompt by the facilitator at the outset of this conversation is the third such invitation for participants to share their mathematical thinking.

Facilitator: Let's start by discussing strategies because they were actually different when people solved it for themselves.

Participant 2: I immediately put the numbers in a matrix and solved for an arbitrary vector (x, y)

Participant 3: So, I am rusty on my linear algebra so I went back to the analogy, if I am given a location (x, y) that I need to get to, how long using each mode of transportation. And so I built a linear combination and then turned it into a matrix and row-reduced by hand. That way, given x and given y, I knew how long to ride this and how long to ride that. [...]

Facilitator: Were there other approaches when people solved it for themselves? [...]

Participant 4: Since the pre-req for my class is precalc and we go over systems of equations, I turned it into a system of equations, but I don't think my students would do that because it is a system with two unknowns and that would probably freak them out.

These quotes are indicative of the conversation and the types of comments heard across both days of the linear algebra breakout sessions in that when participants did speak about their own solutions to the mathematical tasks (which was infrequent compared with the abstract algebra group), their statements about math were often tied either to student thinking or to comments and questions about instructional decision-making. In other words, participants in the linear algebra group rarely made comments that were located solely on the 'math' vertex of the instructional triangle. Importantly, LA participants' engagement with the mathematics tended to be framed in terms of what "they" (students) might do, rather than in terms of their own mathematical reasoning. This stands in contrast to mathematical comments made by abstract algebra participants, who talk directly about their own solutions to the tasks, rather than referencing hypothetical student solutions and course prerequisites. While this may not in and of itself be a bad thing, it certainly seems to be a pattern that impeded the linear algebra group's deep, sustained engagement in the mathematics of the task sequence.

Day 2: During the linear algebra breakout group on the second day of the workshop, participants examined artifacts of actual student work and watched video in which students explained their reasoning. Participants were encouraged to consider pedagogical strategies in the context of teaching with these materials. For example, a participant discussed students' reasoning about three vectors in $\mathbb{R}^2$:

> If they were relating to systems of equations, then they would recognize that they would have two unknowns and three equations and if they relate it back to that then they will think, 'ok, if, assuming these aren't all scalar multiples of each other, then I am already over-determined.' Since we did systems of equations first, then they might already have some intuition about that if they relate it back to that.

We interpret this quote as evidence that this participant is coordinating two vertices of the instructional triangle (mathematics and students' reasoning). This quote was made shortly after the group viewed video clips of students talking about their solutions to this task, so it is interesting to note that the language used in this quote is hypothetical ("*If* they…, then they *might*…"). One possible explanation for this disconnect is that, having not worked carefully through the mathematics from their own perspective, it may have been difficult for participants to make sense of students' reasoning in the video if they were still thinking through the mathematics themselves as they watched.

After watching this video, participants are asked to describe how they would introduce the concept of span if students were reasoning in the way shown in the video they had just watched. One participant responded in terms of what conclusions the class had made about what locations in the plane you can reach with (linear combinations of) the given pair of vectors:

> If the consensus becomes 'everything' then you set the tone, you can go into the animation, and the definition can be using linear combinations and… all possible ways of scaling the first one and adding the second, and then scaling the second one and adding the first one… then you're primed to talk about… all possible linear combinations and it spans $\mathbb{R}^2$ in this case.

This participant's comment involves all three vertices of the instructional triangle. Similar to the "next pedagogical move" quote provided the abstract algebra group, this participant frames the next pedagogical move in terms of a relevant mathematical consensus of the class. This comment differs however, in that it is framed more in terms of what the instructor would tell the students rather than what the instructor would ask students to think about to move their reasoning forward.

Ultimately, then, there were two main points of contrast between abstract algebra and linear algebra participants' contributions during the breakout sessions. First is the clear difference in how breakout groups spent their time during sessions, with far more mathematical engagement evidenced from the abstract algebra participants. Second, while abstract algebra participants' conversation across the workshop cleanly mirrored the trajectory of the workshop activities themselves (mathematics, students' reasoning, the role of the instructor in building on students' reasoning), linear algebra participants' talk during the breakout sessions was more scattered. In the linear algebra group, multiple vertices of the instructional triangle were present throughout their conversations in a way that suggested this group consistently reasoned from an instructional perspective. We posit that this lens may have functioned to impede deep, sustained mathematical engagement among the linear algebra group and ultimately constrained an exploration of more nuanced mathematical issues as they relate to student thinking and pedagogical moves in the context of the content-specific instructional materials.

Differences in Facilitation

There were a number of differences in facilitation between the linear algebra group and the abstract algebra group, in particular with regard to the setup of participants' initial engagement with the mathematics. In this section, we detail differences we noted that might help explain differences in the nature of conversations (and thus pedagogical reasoning elicited) between the two groups. A summary of these differences is shown below in Table 1.

Linear Algebra Group	**Abstract Algebra Group**
Asked participants to introduce themselves and their teaching contexts	No whole-group introductions; participants put in pairs to work
Overview of instructional sequence given	No overview of instructional sequence
Began with second task in sequence	Began with first task in sequence
Engaged pedagogical questions throughout	Postponed pedagogical questions 'til end

Table 1: Differences in set-up of mathematics

Of particular note are differences in the way facilitators of the two groups asked participants to engage in the mathematics. For instance, the linear algebra facilitator initiated the breakout group by asking participants to sit in a circle and introduce themselves and their teaching context, pointing out that "linear algebra courses differ widely by institution… if you have one that serves mostly engineers then the goal is different than one serving math majors." After these introductions (which lasted about 6 ½ minutes), the facilitator offered an overview of the sequence of tasks (which lasted about 9 minutes), highlighting how the mathematics was intended to emerge from the sequence of tasks in the magic carpet ride context, before asking participants to begin by working through the second task in the instructional sequence:

> So, one of the things I wanted us to do was to dive in and solve Scenario 2 – not because it is particularly mathematically difficult, but so you could think about solving it and then think about it from the perspective of a student who has seen one day of linear algebra and seen linear combinations and turned it into a system of equations and solved it and that is the extent of his linear algebra – try to solve it from that perspective.

This facilitator prompt marks a clear request to consider what students might do as they engage in the mathematics – which suggests that this trend in participant engagement is clearly tied to the facilitator's framing of their work with the mathematics. Additionally, we argue that the facilitator fostered a setting that had the potential to create professional risk for participants by trivializing the mathematics involved in the activity. This professional risk might have been amplified by the fact that participants were not provided with the opportunity to work through the mathematics in the first task of the sequence, which might have been important for participants to become familiar with the task setting.

In contrast, the abstract algebra facilitator initiated that breakout group by asking participants to sit with a partner and work through the mathematics, giving them a handout with the sequence of task statements. Interestingly, the first question in the packet in some ways paralleled the introductions done in the linear algebra group. This question asked participants how they normally teach subgroups or were taught subgroups -- but this question was only answered individually or between partners (rather than discussed amongst the larger breakout group) and those captured on camera spent less than 3 minutes on this question. The first interruption to this partner work came from the facilitator about 11 minutes into the breakout session:

> This is how we start the class. A subgroup is a subset of a group that is itself a group. Use this to prove 5Z is a subgroup of Z. I get to assume 5Z is a subset. Fill in the proof.

In this way, the facilitator clearly framed how participants were asked to engage in the mathematics – namely, from a mathematical perspective.

One might conjecture that differences in participants' engagement with the mathematics arises from differences in participants' personalities, preferences, or experience with the mathematics. However, we argue that the facilitator played an important role in framing the way participants were asked to engage in the mathematics. In addition to the differences in how the two facilitators framed participants' initial engagement with the mathematics, there is additional evidence that suggests differences in participant engagement was tightly linked to facilitator moves. For example, after participants in the linear algebra group noted that they weren't sure how their students would approach the mathematics in the given task, the facilitator suggested they solve the problem from their own point of view -- a suggestion that proved fruitful for participants' mathematical engagement:

Facilitator: First why don't we hear the strategies because they were actually different when people solved it for themselves…

Participant 1: I immediately put the numbers in a matrix and solved for an arbitrary vector xy

Participant 2: I am rusty on my linear algebra so I went back to the analogy, if I am given a location (x,y) that I want to get to, I want to know how long using each mode of transportation and so I built a linear combination and then turned it into a matrix and row-reduced by hand, that way given an x and given a y, I knew how long to ride this and how long to ride that…

In contrast to the shifting expectation seen in the linear algebra group with regard to how participants should engage in the mathematics (e.g. as mathematicians or as instructors thinking from the perspective of students), the facilitator of the abstract algebra group maintained a consistent press on the first day of the workshop with regard to how participants were expected to engage in the mathematics. Examples of facilitator press that support this claim are shown below.

Facilitator: It doesn't just have to be from a student's perspective. Why does a subgroup have to have the same identity as the one in the group? [Day 1, 48:03]

[…]

Facilitator: I want to ask you as mathematicians… is your doubt removed? [Day 1, 61:10]

This sustained press to approach the tasks in the instructional materials from a mathematical perspective appears to have been a fruitful one, given the amount of time the abstract algebra group spent engaging deeply in the mathematics and the rich connections these participants were subsequently able to make with regard to student reasoning and pedagogical moves.

Discussion

Our findings suggest that when mathematicians' pedagogical reasoning is engaged through a content-specific mathematical lens, rich and layered connections among their own mathematical reasoning, students' mathematical reasoning, and possible instructional moves can be forged in a relatively short period of time. In the context of our study, these forms of reasoning were most prominent when participants engaged deeply in particular mathematical tasks before watching video of students reasoning about those same tasks, and finally discussing possible in-the-moment instructional moves that could help students move forward their mathematical reasoning. This is not to say that these conversations will lead to immediate or sustained shifts in practice, but it does suggest that mathematicians' pedagogical reasoning is a complex, understudied resource that has great potential to advance efforts aimed at instructional change in both undergraduate and K-12 mathematics – as mathematicians are the people who most fundamentally shape students' mathematical learning experiences in post-secondary educational settings.

However, the differences we noted between the abstract algebra and linear algebra groups also suggest that the way in which mathematicians are positioned to engage in this pedagogical reasoning plays an important role in the kinds of reasoning that are developed, even within the context of a short, four-hour workshop spread across two days. Importantly, we argue that engaging mathematicians' pedagogical reasoning *through the lens of mathematics*, rather than through the lens of instruction, draws on the unique strengths that mathematicians bring to the work of teaching – in particular, the resource of deep disciplinary knowledge. As noted in our findings, the facilitator plays an important role in framing mathematicians' engagement with the mathematics in a context such as the workshop in which our data were collected. Additionally, the facilitator plays an important role in managing the professional risk that these settings like these workshops can entail for mathematicians who may not often work on unfamiliar mathematics problems in public spaces without first thinking about such problems privately.

Nardi (2007) quoted Sfard's call for this kind of work in mathematics education research: "Math ed research needs to study more extensively and more systematically the mathematician's ways of thinking as they can be substantial and illuminating" (p. 265). The data presented in this paper offers a glimpse into the thinking of mathematicians interested in instructional change, and points toward what is possible when mathematics education researchers work with such mathematicians. We further argue that a way to move toward achieving instructional change at scale is by forging sustainable alliances between mathematicians and mathematics education researchers, and that these alliances should be structured so as to leverage mathematicians' deep disciplinary knowledge and teaching experience in ways that support robust instructional change. If we can learn to productively build these kinds of relationships, we can reorganize our notion of unidirectional dissemination of findings from mathematics education research into a vision of a dialogue that has the potential to mutually inform the work of those who do the bulk of undergraduate mathematics teaching as part of their broader work as mathematicians and those who research issues of teaching and learning at the undergraduate level. Alliances with mathematicians have the potential to be drivers of instructional change, and if we as a field can learn to leverage the unique resources and insights brought to these alliances by both mathematicians and mathematics education researchers, we will be far better equipped to implement changes that will more effectively support the mathematical learning of students.

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Exploring student understanding of negative quantity in introductory physics contexts

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Recent studies in physics education research demonstrate that although physics students are generally successful executing mathematical procedures, they struggle with the use of mathematical concepts for sense making. In this paper we investigate student reasoning about negative numbers in contexts commonly encountered in calculus-based introductory physics through the lens of a cognitive blending framework. We describe a cross-sectional study (N > 700) involving two introductory physics courses: calculus-based mechanics and calculus-based electricity and magnetism (E&M). We present data from assessment items that probe student understanding of negative numbers in physics contexts. Results suggest that even mathematically well-prepared students struggle with symbolizing in physics, and that the varied uses of the negative sign can present an obstacle to understanding that persists throughout the introductory sequence. We discuss the implications for instruction and directions for future work.

Keywords: quantity, negativity, minus sign, integers, physics, mechanics, electricity, magnetism

Introduction and Background

In physics, signed numbers have varied meanings and carry rich information about physical context. For students, the operations of addition and subtraction (represented by the symbols "+" and "−") can easily be confused with the descriptors, positive and negative, that characterize the opposite natures of a quantity (e.g. charge) or the value of a quantity relative to a reference (e.g., potential energy). Furthermore, mathematical representations of the intricate relationships between quantities can involve multiple meanings of the negative sign in a single expression.

Developing flexibility with negative numbers is a known challenge in math education. Vlassis used written diagnostic questions and interviews to investigate the understanding of negative numbers of Belgian algebra students (Vlassis, 2004). She found that full understanding of the concept of a negative number required that students develop flexibility with the various ways negative numbers are used in context, i.e., with the "negativity" of the number. Vlassis created a "map" in tabular form that describes these different uses (reproduced in Table I.) She based her map on one developed earlier (Gallardo & Rojano, 1994), and enhanced it by including additional signifiers from others' work (Thompson & Dreyfus, 1988; Nunes, 1993).

Table I: Negativity: A map of the different uses of the negative sign in elementary algebra; the triple nature of the minus sign. (Vlassis, 2004)

Unary	**Binary**	**Symmetrical**
Structural signifier	*Operational signifier*	*Operational signifier*
Subtrahend Relative number Isolated number Formal concept of negative number	Completing Taking away Difference between two numbers Movements on the number line	Taking opposite of, or inverting, the operation

Few studies have been published that focus on student understanding of negativity outside of mathematics courses. We describe here the most relevant related studies that were done in the context of physics.

Sherin observed patterns in student reasoning by studying successful problem solving behaviors of 3rd semester engineering students taking physics at a highly selective university (Sherin, 2001). Regarding negativity, Sherin uses the symbolic form "competing terms cluster" to describe the quantification of opposites in physics. This form includes the notion of zero to represent balance, and positive and negative quantities as competing terms in an expression. He observes that flexibility with the competing terms cluster is a feature of expert problem solving. We interpret use of this symbolic form as demonstrating flexibility with the symmetrical nature of the minus sign.

Bajracharya and colleagues (Bajracharya, Wemyss, and Thompson, 2012) investigated student understanding of integration in the context of *P-V* diagrams in introductory physics. Their results suggest difficulties with the criteria that determine the sign of a definite integral. Students struggle with the concept of a negative area, and with positive and negative directions of single-variable integration. We interpret this struggle to be rooted in an incomplete understanding of the symmetrical nature of the minus sign in the calculus context.

These observations of advanced undergraduate physics students struggling with the nuances of negativity resonated with our own informal classroom observations, and led us to pose the research questions below. The remainder of this paper describes our preliminary work in addressing these questions.

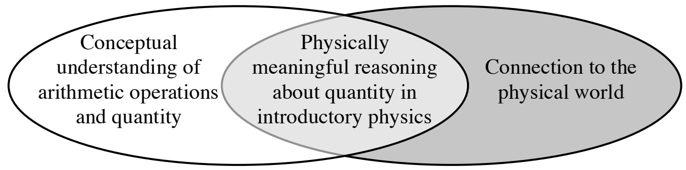
1. *After instruction, what themes are evident in student understanding of the unary, binary and symmetrical ways that negative signs are used in the context of physics?*
2. *Are some contexts (e.g., nature of physical quantity, nature of the minus sign) more challenging for students than others?*

Theoretical Framework: Cognitive Blending

The theoretical framework of cognitive blending (Fauconnier & Turner, 2008; Bing & Redish 2007) supports our view that an interdependence of thinking about the mathematical and physical worlds is necessary for quantifying effectively with negative quantity in physics. Figure 1 illustrates a *double scope negative quantity reasoning blend*, in which two distinct domains of thinking are merged to form a new cognitive space that is optimally suited for productive work.

We draw attention to prior work by Czocher (Czocher, 2013) who conducted a study of several engineering students enrolled in a differential equations course and observed them solving a variety of physics problems over the course of the semester. She reports that successful students functioned most of the time in a "mathematically structured real-world" in which they moved back and forth fluidly between physics ideas and mathematical concepts. Czocher describes this space as being between the "real world" and the "math world",

Figure 1: Double scope negative quantity reasoning blend



We suggest that deep understanding in introductory physics is best supported through a completely homogeneous blend (as observed by Czocher), such that there is no distinction between the physics and the arithmetic worlds. We propose a thinking space in which physical sensemaking is essential for, and integrated with, mathematical reasoning. Our experimental design and assessment item development emerge from a cognitive blend framework. Correct responses depend on students reasoning about negativity correctly, with the further constraint that their response is also consistent with the real world. We analyze our results using this framework and draw conclusions that can inform both instruction and curriculum development.

Methods

As part of a larger assessment project, we had access to two large-enrollment, introductory, calculus-based physics courses taught during Fall 2015. To examine trends in student reasoning about the negative sign in physics contexts, we designed a set of six written questions and administered them at the end of the mechanics course, which students take as their first semester of physics, and the E&M (electricity and magnetism) course, which students take as their third semester of physics. (No students saw the questions twice.) The questions were ungraded, and bundled with concept inventories that are routinely given as part of end-of-semester assessment. A portion of the sample reported on here from each class received multiple-choice versions of the items (n=210 in Mechanics, n=402 in E&M), while the remainder received free-response versions (n=84 in mechanics, n=138 in E&M.) The free-response versions allowed us to evaluate the extent to which the multiple-choice distractors are clear and aligned with student reasoning.

The six items cluster into two sets of three, one set involving mechanics contexts and one involving E&M contexts. Within each set, the first item probes student understanding of the unary nature of the negative sign, the second probes the symmetrical nature, and the third, the binary nature (see Table I). On the first item of each set, students must interpret the sign of a vector component (either acceleration, for the mechanics context, or electric field, for the E&M context). On the second item, which assesses the ability to quantify opposite actions, students must interpret the sign of either the work done by an agent (for the mechanics context) or the transfer of electric charge (for the E&M context). Finally, on the third item, designed to probe the ability to coordinate a measured quantity with its reference, students must interpret the sign of either position (mechanics) or potential difference (E&M). Table II presents all six items. We note on EM2 that the only physically correct answer is (a); only negatively charged particles transfer easily by rubbing matter - (II) makes mathematical sense, but not physical sense.

Table II: Multiple choice assessment items and the percentage of students who selected each answer. The correct answers are in boldface type.

	Unary structural signifier	Symmetrical operational signifier	Binary operational signifier
MECHANICS	*direction of a vector component*	*signifying that work is done by the block as opposed to being done on the block*	*position relative to an origin*
MECHANICS	***Mech 1:*** An object moves along the x-axis, and the acceleration is measured to be $a_x = -8\ \text{m/s}^2$. Consider the following statements about the "–" sign in "$a_x = -8\ \text{m/s}^2$". Pick the statement that best describes the information this negative sign conveys about the situation. a. The object moves in the negative direction. *(8%)* b. The object is slowing down*(26%)* c. **The object accelerates in the –x-direction (26%)** d. Both a and b *(6%)* e. Both b and c *(34%)*	***Mech2:*** A hand exerts a force on a block as the block moves along a frictionless, horizontal surface. For a particular interval of the motion, the hand does W = – 2.7 J of work. Consider the following statements about the "–" sign in the statement W = –2.7 J. The negative sign means: I. *the work done by the hand is in the negative direction* II. *the force exerted by the hand is in the negative direction* III. *the work done by the hand decreases the mechanical energy associated with the block* Which statements are true? a. I only *(17%)* b. II only *(17%)* **c. III only (23%)** d. I and II only *(29%)* e. II and III only *(14%)*	***Mech 3:*** A cart is moving along the x-axis. At a specific instant of time the cart is at a position x = –7 m. Consider the following statements about the "–" sign in "x = –7 m". Pick the statement that best describes the information this negative sign conveys about the situation. a. The cart moves in the negative direction *(6%)* b. **The cart is to the negative direction from the origin (67%)** c. The cart is slowing down *(6%)* d. Both a and b (*19%*) e. Both a and c *(2%)*
ELECTRICITY AND MAGNETISM	*direction of a vector component*	*signifying a specific type of charge that is opposite of positive charge*	*potential at one location relative to another*
ELECTRICITY AND MAGNETISM	***EM 1:*** At a location along the x-axis, the electric field is measured to be $E_x = -10$ N/C. Consider the following statements about the "–" sign in "$E_x = -10$ N/C". Pick the statement that best describes the information this negative sign conveys about the situation. a. The test charge is negative *(16%)* b. The field is being created by negative charge *(21%)* c. **The field points in the –*x*-direction (36%)** d. Both a and b *(12%)* e. Both b and c *(14%)*	***EM 2:*** Valeria combs her hair in the winter and there is a transfer of charge such that $\Delta Q_{comb} = -5$ mC. Consider the following statements about the "–" sign in the mathematical statement $\Delta Q_{comb} = -5$ mC. The negative sign means: I. negative charge was added to the comb. II. charge was taken away from the comb. III. all of the electric charge in the comb is negative Which statements could be true? a. **I only (33%)** b. II only *(28%)* c. III only *(18%)* d. I and III only *(15%)* e. II and III only *(5%)*	***EM 3:*** In physics lab, a student uses a voltmeter to measure the voltage across the terminals of a battery. The voltmeter reads –5V. Consider the following statements about the "–" sign in the voltmeter reading "– 5V". Pick the statement that best describes the information this negative sign conveys about the situation. a. the voltage is in the opposite direction as the current (*32%)* b. there are 5V of negative charge in the battery *(14%)* c. the voltage is in the negative direction *(18%)* d. **the voltage at one terminal is 5V less than the voltage at the other terminal (33%)** e. this battery has negative voltage *(3%)*

Results

Table II presents the fraction of students selecting each distractor on the multiple-choice versions of the questions. Table III presents examples of student reasoning for each item, drawn from responses to the free-response versions. Finally, Figure 2 summarizes the results reported in Table II by comparing the correct response rates on the multiple-choice versions.

Table III: Sample incorrect student reasoning from the open-ended version of the assessment items.

	Unary structural signifier	Symmetrical operational signifier	Binary operational signifier
MECHANICS	*direction of a vector component*	*signifying that work is done by the block as opposed to being done on the block*	*position relative to an origin*
	Mech 1: *"It means that the acceleration of magnitude 8 m/s^2 is acting in the opposite direction to the direction of the motion the object currently has"* *"This negative sign means the opposite of an increasing speed so a decreasing speed"*	***Mech2:*** *"A negative simply means that the block is going in the negative direction"* *"the hand is applying … a force to the block towards the left"* *"the work was done using negative force - a force that goes against positive force"*	***Mech 3:*** *"It is moving in the opposite direction. If moving right indicates +7 m, then this cart is moving left."* *"That means it is traveling left or the cart is starting from a distance > -7 and is going right (-7 away from origin)"*
E & M	*direction of a vector component*	*signifying a specific type of charge that is opposite of positive charge*	*potential at one location relative to another*
	EM 1: *"The electric field is moving in the negative x-direction"* *"every Coulomb of charge experiences 10 N of force in negative direction"*	***EM 2:*** *"The charge from the electrons that transferred from the comb gives Valeria's hair a negative charge"* *"charge has been transferred from comb to hair. Comb lost 5 uC of charge."*	***EM 3:*** *"The voltage moves from the negative terminal to the positive"* *"The negative signifies the direction of the voltage"* *"The voltmeter is attached backwards"*

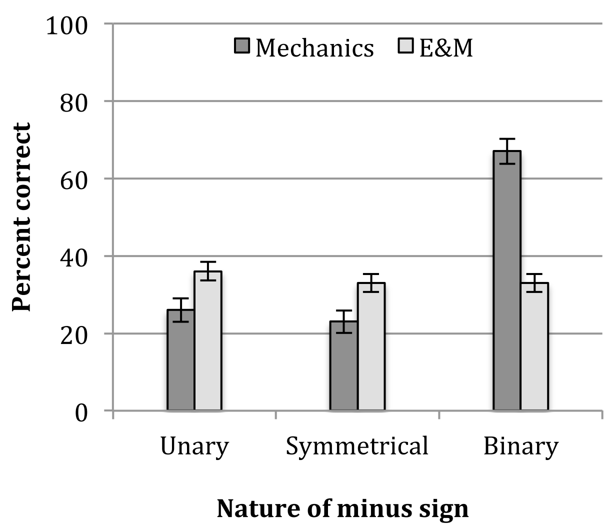


Figure 2: The percentage correct for each multiple-choice item, grouped by the nature of the minus sign and by course. Error bars represent standard error of the mean for binomial variables.

Discussion

We discuss the results in the context of the research questions:

RQ1: After instruction, what themes are evident in student understanding of the unary, binary and symmetrical ways that negative signs are used in the context of physics? We observe from the free response answers that students seemed to struggle less with unary natures than with binary or symmetrical. Their answers suggest intuition on unary items, but the reasoning is not robust. Strong multiple-choice distractors can derail reasoning, which we believe explains the low unary correct response rates seen in Figure 2. We consider students' cognitively blended conceptual understanding on unary items to be emerging, but tenuous, as can be seen in the fact that 3^{rd} semester students are statistically better than their first semester counterparts in the face of strong distractors at recognizing the meaning of the negative sign that can be part of a vector component. We observe both in the multiple choice and free response versions that students struggle especially with the symmetrical uses in physics and note that this is the use that Sherin (Sherin, 2001) observed as being a hallmark of successful problem solvers in physics.

RQ2: Are some contexts more challenging than others? We see three important trends that emerge relating to the cognitive blend of negativity and physics from our results. We observe that, even after instruction, introductory physics students often erroneously associate negativity with an action (e.g. moving, pushing.) They also struggle to make sense of negative scalar quantities (e.g., work, voltage.) They commonly assign a direction to a negative scalar quantity (e.g. the work is acting in the negative direction.) In the context of work they typically associate negativity with the direction of one or both of the vectors that combine to create a scalar product (e.g. negative work is done using a negative force.) In the context of voltage, they commonly treat voltage as having a "direction", confounding it with current or electric field. And lastly, we observed significant struggle to combine more than one nature of negativity in a single calculation (e.g., removing negative charge.) The students were prone to associate the negative sign with "taking away " in that context.

In general, the students struggle with negativity in the ways that it is typically and commonly used in physics for quantification. We observe that student interpretations of negative numbers are often associated with negative position. We note that by comparison to the other physics contexts, students did anomalously well on item Mech3 that asks about one-dimensional position – which is essentially the number line context seen in pre-algebra. In the next section we discuss the need for a cognitively blended learning space in which negativity, and its varied uses, continues to be part of the discussions both in mathematics courses beyond pre-algebra and in physical science courses. We also discuss, through example, some implications of the challenges posed by combining multiple natures of negativity in a typical physics context.

Implications for instruction

We interpret student responses to physics questions through a lens of the three natures of the minus sign, emphasizing that sensemaking with the physical quantities involved is neither entirely physics cognition nor entirely mathematics cognition. A deep understanding requires thinking in a blended cognitive space in which students use conceptual understanding in each domain to fully understand the nature of the physical quantity.

Seen through the lens of the three natures of the minus sign, we can see the complexity of the symbolizing that is a normal part of physics instruction and the additional cognitive load it places on students. Negativity is deeply embedded in the meaning of physical quantities, and the negative sign takes on varied meanings in context. Oftentimes physical quantities are constructed using multiple negativities in the same statement, and we've seen that this practice is not readily assimilated or appreciated by the students. Take for example electric potential energy, *U(r),* of two particles separated by distance *r*. Electric potential energy is an extremely important quantity in E&M:

$$U(r) = \frac{\text{(a physical constant)} \bullet \text{(charge 1)} \bullet \text{(charge 2)}}{r^2} + \textit{any constant}$$

- As *r* increases, the interaction force does *negative* work if the interaction is *attractive* (*opposite* electric charges), and does *positive* work if the interaction is *repulsive* (*like* charges)
- Thus *U increases* with *r* for *attractive* interactions and *decreases* with *r* for *repulsive* interactions
- By convention, $U(r) = 0$ when $r = \infty$ for both types of interaction, implying that if *U(r)* is non-zero it is negative for attracting particles and positive for repelling particles

Note the cognitive blend associated with the distinct uses of the negative sign: a negative scalar quantity (energy) measured relative to a reference point (binary operational signifier), involving other negative quantities (charges) associated with the quantification of opposites (symmetric operational signifier)

We find that at the end of calculus-based physics courses, students commonly both appropriately and inappropriately associate the negative sign with *spatial direction* or *opposition*, but do so without the nuance and flexibility characteristic of expert practice exemplified above – they appear to be associating meaning to the negative sign perhaps based on contexts learned in their pre-algebra courses. We observe that, in general, they struggle to interpret signed quantities in new physics contexts, *even after* relevant physics instruction that is focused on learning the quantities.

While experts interpret *negative* flexibly while symbolizing, many students are unable to do so, even after taking introductory physics. Instruction that explicitly targets student ability to distinguish the three natures of the negative sign, and to make symbolizing decisions using signs, could enhance students' quantification in physical science. Such instruction would be most likely to succeed if it is intentionally distributed and coordinated between mathematics courses and physics courses. Without such explicit attention to negativity, our results suggest that students may not spontaneously adopt an expert perspective. Instruction should challenge students to make sense of varied uses of the negative sign in a variety of physical contexts, and to blend multiple natures of the negative sign in the same overall mathematical context.

This paper describes a preliminary investigation. We intend to conduct individual student interviews and classroom observations to further explore the associations students seem to make between some sort of action or movement and negativity. We plan to repeat the current study using positive quantities to determine the extent to which student difficulties are associated with negativity, *per se*, or with signed numbers in general. Ultimately, we hope to explore, develop, and classroom test instructional interventions, both in the context of mathematics courses and physics courses, that can help students develop the blended thinking space necessary to make sense of signed physical quantities.

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Why research on proof-oriented mathematical behavior should attend to the role of particular mathematical content

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Because proving characterizes much mathematical practice, it continues to be a prominent focus of mathematics education research. Aspects of proving, such as definition use, example use, and logic, act as subdomains for this area of research. To yield content-general claims about these subdomains, studies often downplay or try to control for the influence of particular mathematical content (analysis, algebra, number theory etc.) and students' mathematical meanings for this content. In this paper, we consider the possible negative consequences for mathematics education research of adopting such a content-general characterization of proving behavior. We do so by comparing content-general and content-specific analyses of two proving episodes taken from prior research of the two authors and by re-analyzing the data and results presented in one instance of research from the field. We intend to sensitize the research community to the role particular mathematical content can and should play in research on mathematical proving.

Keywords: Proving, mathematical meanings, comparative analyses

Since at least the time of Euclid's geometry, proving has been understood to characterize mathematics as a discipline. Inasmuch as mathematics educators endeavor to engage students in authentic mathematical activity, they have expended much effort to provide students with meaningful proving experiences and document the emergence of proving as a mathematical practice among novices. While we certainly endorse this agenda for instruction and research, we are concerned that framing mathematical proving as a single, content-general practice may inappropriately downplay the role particular mathematics content plays therein. We observe two trends in the research literature on mathematical proving: 1) making content-independent claims about mathematical proving using data from a particular mathematical context (i.e. analysis, algebra, number theory, geometry) or 2) eliciting the proving behavior of the same students in multiple mathematical contexts in order to make content-independent claims about proving. In this paper, we consider the possible consequences for research on mathematical proving of downplaying the role of particular mathematical content. We do not at all intend to deny the validity or value of prior research framed in a content-independent manner (some of which we authored), but rather seek to sensitize the community to possible blind spots induced by common lenses applied to research data and to endorse a research agenda focused on the interplay between proving and particular mathematical content. Furthermore, if we are to advance the agenda of proof as a process through which students develop key mathematical understandings (Reid, 2011; Stylianides, Stylianides, & Weber, in press), our research lenses for mathematical proving must accommodate the specific mathematics being learned.

To portray such blind spots induced by a research lens, this paper presents dual analysis of three proving episodes taken from prior studies. One episode presents our analysis of another researchers' data and the other two episodes each appear in one of the two authors' prior research, respectively. In each case, we compare 1) a content-general analysis focused on common constructs from proof-oriented mathematics education research – example use, definition use, proof production, logic – with 2) a content-specific analysis focused on explaining students' proving behavior in situ. We use these comparative analyses to reflect on researchers' interpretive process itself and how the framing of research questions non-trivially influences the nature of the phenomena observed and the results of such research on proof-oriented behavior.

Motivating trends and questions

It is common to frame both the research questions and findings using these content-independent constructs such that they form informal subdomains of proof-oriented research. One can find numerous examples of studies on proof-oriented mathematical activity that make content-independent claims about

- example use – Alcock & Inglis, 2008; Karunakaran, 2014; Sandefur, Mason, Stylianides, & Watson, 2013,
- definition use – Alcock & Simpson, 2002; Ouvrier-Buffett, 2011,
- proof production – Dawkins, 2012; Raman, et al., 2009; Stylianides & Stylianides, 2009,
- logic – Epp, 2003; Selden & Selden, 1995, and
- understanding of proof – Sowder & Harel, 2003; Stylianou, Blanton, & Rotou, 2015.

It is not our goal to critique these studies per se, but rather to sensitize mathematics education researchers to the consequences of consistently investigating proving while downplaying the mathematical meanings that populate the arguments that students produce.

Why do many proof-oriented studies downplay mathematics content? Even if this question had one answer, no available evidence reveals it. Nevertheless, we proffer some possible explanations. One explanation is psychological. Proof's role in mathematics as a discipline and the mathematics education community's emphasis on mathematical process both lead researchers themselves to conceptualize proving in real analysis as one instantiation of a broader phenomenon. Because we as experts see consistencies across our broad experiences with proving, we assimilate instances of proving into our general understanding. Dewey (1902) similarly described the asymmetry between experts' and novices' perspectives of academic disciplines. While adults recognize the internal structure of and distinctions between disciplines, children may learn them in an integrated way. Dewey emphasized that these structures are the results of learning, but may not appropriately describe the process of learning itself.

A second explanation involves empirical findings. The growing body of evidence of students' difficulties interpreting, producing, and assessing proofs compels mathematics educators to improve proof-oriented instruction. Students perceive the transition into proof-oriented courses as a difficult transition, so it seems natural to partition such courses apart from other aspects of the curriculum (though we agree with Reid's, 2011, argument that proving should become and is becoming integrated as a ubiquitous means of mathematics teaching and learning). Because students' patterns of proving behavior that diverge from mathematical practice can be documented within multiple mathematical domains (i.e. the problems are content-general), we may falsely assume that these challenges can and should be addressed independent of particular content (the solutions are content-general).

A third explanation relates to the analytic process itself. Mathematics educators frequently use localized data to make analytic generalizations (Firestone, 1993) by constructing frameworks and in-depth characterizations of relatively few cases. While such studies rarely make explicit claims to sample-to-population generalizations (or even claims that the same student would exhibit similar patterns of proving behavior on a different proving task), it remains unclear how to situate the resulting empirical claims about student behavior. Our first dual analysis of a proving episode exemplifies this tension. Case studies are instances, but it is up to researchers and readers to determine what they instantiate.

Comparative analysis 1: Indirect proof in planar geometry

Antonini (2003) presented findings suggesting conditions under which students may spontaneously produce proofs by contradiction (or indirect proofs more generally), which previous studies report that students find particularly challenging. He draws upon the Cognitive Unity (Mariotti et al., 1997) theoretical framework, which characterizes students' proving by attending to interplay between informal arguments and proofs. Antonini (ibid.) further highlights the form of the task as an influential factor in the emergence of indirect proof ("*given an hypothesis A what can you deduce?*", p. 50). The data presented in the paper are limited to students' exploration of a geometric conjecture involving transversal configurations (two lines intersected by a third line), but the author nevertheless frames his research hypothesis in the following way:

> In task like "given A what can you deduce?" the conjecture can be produced via the analysis of a non-example. The argumentation that justifies the fact that the generated example is a non-example can be re-elaborated and become part of the argumentation of the conjecture. In this case, the argumentation takes an indirect form. (Antonini, 2003, p. 50)

We find this hypothesis striking in its generality. While we do not intend to devalue Antonini's characterization of a phenomenon of proving behavior, we question how researchers should interpret the scope and purpose of such characterizations. Is Antonini's hypothesis merely a description of an (possibly general) indirect proving phenomenon that others may call upon to characterize other episodes, or does it purport to be an *explanation* of why the argumentation took an indirect form? Over the course of this paper, we shall elaborate this dichotomy between general(izable) *descriptions* of proving phenomena and *explanations* for particular instances of those phenomena. To motivate our distinction in the case of Antonini's data, we propose an alternative, content-specific explanation for the reasoning pattern he observed.

Episode 1: Classifying relationships between pairs of lines

The task presented to Antonini's (2003) research subjects was, "Two lines *r* and *s* on a plane have the following property: each line *t* intersecting *r,* intersect *s* too. Is there anything you can say about the reciprocal position of *r* and *s*? Why?" (p. 51). The given property (which the author abbreviates as "A") is equivalent to saying that lines *r* and *s* are parallel. The two students featured in the paper (Valerio and Christina, age 13) justified this by showing that the lines *r* and *s* cannot be *perpendicular* or *intersecting*, because in those cases there would be a line *t* that intersected *r* but not *s*. Valerio and Christina seem convinced of the generality of their claims regarding perpendicular and intersecting lines and justified that the lines *r* and *s* must be parallel because they cannot be intersecting.

Content-specific analysis of Episode 1

How might the geometric context, and transversal configurations in particular, influence the emergence of this indirect line of reasoning? Certainly it was easy for the pair to generate and test cases by visualizing and rotating lines, but we notice another aspect of this context that we hypothesize as relevant to the emergence of indirect proof. In school mathematics, there are three basic relationships between two non-identical coplanar lines: parallel, intersecting, and perpendicular. Two of these properties are implicitly negations of one another. To be non-intersecting is to be parallel and vise versa. Valero and Christina may have been taught that parallel means "non-intersecting," or they may have a less formal meaning thereof. Inasmuch as indirect proof entails negating mathematical conditions, we hypothesize that it was important in

Antonini's (2003) proving episode that Christina and Valerio were familiar with properties that exactly characterized the pairs of lines exhibiting property A and those not exhibiting A.

In recent studies on students' reinvention of truth-functional logic, Dawkins and Cook (2015) documented some students' reluctance to reason about negative properties (not [X]); such students instead exhibited preference for familiar categories. This was especially true in geometric contexts where students often paraphrased "[a triangle] is not acute" by saying "it is obtuse" or "[a quadrilateral] is not a rectangle" with "it is a parallelogram." In the former case, *obtuse* is a familiar category of triangles that has empty intersection with but is not the complement of the class of *acute* triangles (i.e. *obtuse* is not a proper negation of *acute*, but all obtuse triangles are "not acute"). In the latter case, the set of *parallelograms* is neither mutually exclusive to the set of rectangles nor equivalent to the complement of the set of rectangles. It is important to note that in neither case is there a familiar (non-negative) category that corresponds to the given negative property. Dawkins and Cook (2015) hypothesized that this pattern of behavior revealed 1) an aversion to describe geometric shapes according to properties they lacked and 2) an implicit reliance on the familiar organization of these shapes according to categories taught in school ("acute, right, and obtuse" rather than "acute and not-acute").

Both of the patterns of students' reasoning about geometry Dawkins and Cook (2015) hypothesized are relevant to Antonini's (2003) proving task. First, students' preference for familiar categories (if applied to Antonini's subjects) suggests that Valerio and Christina explored perpendicular and intersecting lines because they are two of the three familiar relations between two lines. Furthermore, we have no evidence whether their initial investigation of these two properties, rather than parallel, was strategic or idiosyncratic. This suggests that they may not have been asking the question Antonini (2003) cites as central to indirect argumentation: "...if it were not so, it would happen that..." (p. 49). It seems implausible that Christina and Valerio first investigated perpendicular lines as non-examples of property A because they were unfamiliarity with property A. We hypothesize they merely explored the familiar properties of pairs of lines – perpendicular, intersecting, and parallel – in that order.

Second, the availability of two properties that corresponded perfectly with A and not-A allowed the students to use indirect argumentation without using negative properties. If these or other students were working on a similar task whose solution depended upon the properties rectangle and non-rectangle, would indirect proof have emerged as easily or at all? For instance, consider the similarly framed task "Given that a quadrilateral has congruent diagonals that bisect each other, what can you conclude? Why?" Students may begin the task by exploring the diagonals of familiar shapes, but none of those classes of shapes would correspond to non-rectangles to afford a contrapositive argument similar to the one developed by Christina and Valerio. We acknowledge that this analysis of Antonini's data relies on a generalization of reasoning phenomena from very different data (Dawkins & Cook's interviews with college students), but it is warranted for our purposes because 1) this generalization is comparable to Antonini's own citation of other studies on students' production of indirect proofs and 2) we provide this *plausible* explanation to justify our contention that the specific mathematics in the task may be essential to explaining Valerio and Christina's proving behavior. Whether this pair or other students would exhibit similar patterns of proving behavior on alternative tasks of a similar form is an empirical question, and researchers must make use of Antonini's (2003) analysis with this in mind.

These observations about the particular affordances of Antonini's (2003) chosen mathematical content do not necessarily undermine the utility of his characterization of a phenomenon of proving behavior, but we posit that they demonstrate how his characterization likely falls short of fully explaining the students' reasoning behavior and why it emerged. We could sympathetically interpret Antonini's hypothesis regarding tasks of the form "given an hypothesis A what can you deduce?" by inferring an implicit comparison to the alternative proof task prompt: "given the hypothesis A, prove that B must be true." In this case, Antonini (2003) suggested that the former, open task prompt may invite more indirect reasoning since the latter prompt would focus students' attention on A and B (rather than not-A and not-B). Our intention is not to dispute this content-general claim. Rather, we want to emphasize to the research community the relative importance of the mathematical content of A and B if we want to document and explain these type of phenomena of students' proving behavior. Furthermore, we argue that the operative research framing (indirect proof, Cognitive Unity, the sense in which two tasks are "like" one another) can non-trivially influence 1) the ways we interpret students' behavior, 2) those aspects of the situation to which we do not attend, and 3) which types of phenomena we take observed behavior to instantiate.

Comparative analysis 2: Logic and inference regarding transversal configurations

This and the following sections set forth two empirical episodes, each taken from one of the authors' prior research, and dual analyses thereof. Episode 2 occurred during a sequence of task-based interviews as part of the first author's investigation of student learning of neutral, axiomatic geometry. It features two undergraduate mathematics majors trying to prove the equivalence of Euclid's Fifth Postulate (EFP) and Playfair's Parallel Postulate (PPP). Analysis of Episode 2 also appeared in Dawkins (2012). The second episode appeared during a sequence of task-based interviews with expert and novice mathematics students conducted by the second author. For the sake of brevity and clarity in this theoretical paper, we omit presenting the full methodology of the study, which is available in the cited reference.

Episode 2: Proving the equivalence of geometric postulates

For reference, the students' statements and diagrams for EFP and PPP appear in Figure 1. As part of a homework assignment prior to the interview, Kirk and Oren had each proven the equivalence of the two postulates using the auxiliary claim we shall call Theorem *, which states "Given two lines cut by a transversal, if the same side interior angles sum is 180, then the two lines do not meet on that side of the transversal." When asked to explain the postulates, the pair found themselves using language from each of the two statements to explain the other. Oren noted this circularity and attributed it to the fact that the statements implied one another. Kirk expressed a similar sentiment, but did so by simply claiming the postulates "are the same." Oren explained the meaning of each postulate by extending his forearms to represent parallel lines and noting that any amount of rotation from the parallel position would cause the lines to intersect. While EFP characterized that rotation in terms of angle sums and PPP did so in terms of deviation from the one parallel line through *P*, Kirk and Oren seemed convinced of the unity between the claims. Interestingly, this arm gesture also provided Oren's initial response about what it *meant* for two lines to be parallel. It appeared that he took other properties such as intersecting and equidistant to be consequences of that arrangement, such that the term *parallel* was, for him, defined in an inherently spatial way.

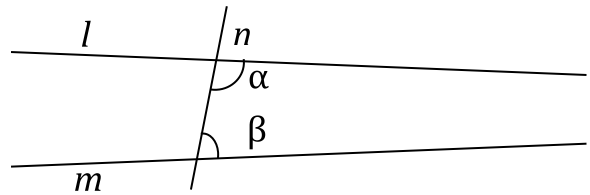
Euclid's Fifth Postulate (EFP): "Given two lines cut by a transversal, if the two interior angles on one side of the transversal sum to less than 180°, then the lines intersect on that side of the transversal."	
Playfair's Parallel Postulate (PPP): "Given any line and a point not on that line, there exists only one line through the given point that does not intersect the given line."	

Figure 1: Kirk and Oren's statements and diagram for EFP and PPP

The students began the task intending to prove that EFP ⇒ PPP. The students' argument depended upon dividing the line arrangements into three cases, depending upon the angle sum $\alpha + \beta$. They successfully argued, using EFP and Theorem *, that:

- if $\alpha + \beta < 180°$, lines *l* and *m* meet on that side of line *n*,
- if $\alpha + \beta = 180°$, lines *l* and *m* do not meet,
- if $\alpha + \beta > 180°$, lines *l* and *m* meet on the other side of *n*.

Upon completing their three-case argument, Kirk and Oren disagreed about their relative progress toward constructing a proof. Kirk considered this argument sufficient to prove PPP because it guaranteed that there was only one instance in which the lines *l* and *m* are parallel. He said, "Playfair's Postulate basically states that there's only one instance or case where the lines will not meet." Oren disagreed because he was concerned about how the choice of lines through point *P* (in PPP) corresponded to the angle sums (in EFP). Through their discussion, Kirk adopted some of Oren's concern about the sufficiency of their argument saying, "It's just hard because Playfair's doesn't include this line *n*, so you are trying to find a way to go from having this line *n* to not having this line *n* in Playfair's." Once it appeared they had reached an impasse, the interviewer invited the students to begin their argument with the diagram for PPP. The pair then was able to use their three cases argument to complete the proof. Oren showed great attention to the warrants necessary for constructing the transversal line and for relating each line through the point *P,* to a unique angle sum. Despite their work prior to the interview, Kirk and Oren's proof production took over 40 minutes.

Analysis 1 of Episode 2. The study in which this episode occurred sought to investigate students' interpretation and use of conditional ("if…then…") statements in proof-oriented mathematics. The first author used this task because it contained at least three conditional claims:

- EFP: "Given two lines cut by a transversal, if the two interior angles on one side of the transversal sum to less than 180°, then the lines intersect on that side of the transversal."
- PPP: "If *P* is a point not on a given line *l*, then there exists only one line through *P* that does not intersect *l*."
- If EFP, then PPP (EFP ⇒ PPP).

Kirk and Oren's initial difficulties in proving EFP ⇒ PPP can be reasonably attributed to the logical structure of their argument, specifically the proof frame (Selden & Selden, 1995). Zandieh, Knapp, and Roh (2008) also reported on students' difficulties with this proof and others with similar logical form. They explain that students do not adopt a Conditional-Implies-Conditional (CIC) proof frame in which the proof proceeds from the hypotheses of the consequent statement (in this case the point and line arrangement of PPP) to the conclusions of that statement (exactly one parallel through *P*). Kirk and Oren displayed difficulties similar to those reported in Zandieh et al. (2008) because they adopted the standard conditional proof frame that begins with hypotheses (EFP) and ends with the conclusion (PPP). Kirk's was convinced they had produced an adequate proof with this proof frame, which could be modeled by the valid syllogism: EFP (and Theorem *) ⇒ Three Cases, Three Cases ⇒ PPP, therefore EFP ⇒ PPP. However, this argument failed to prove that the conclusions of PPP are implied by its hypotheses, as the CIC proof does. In Raman et al.'s (2009) language, Kirk understood the *key idea* of the proof (the Three Cases argument), but lacked the *technical handle* (the proof frame) to construct a valid proof. Ultimately, the interviewer had to prompt the pair to begin with the diagram from PPP, which implicitly introduced the CIC proof frame. This modification allowed the students to produce a valid and normative proof.

Analysis 2 of Episode 2. Several aspects of Kirk's behavior in the episode are not explained by the absence of an appropriate proof frame. For instance, why was Kirk convinced by his 3 Cases argument while Oren was not? Also, when Kirk described their intention to prove PPP from EFP, he appeared to metonymize (Zandieh & Knapp, 2006) the two statements by their diagrams. To get from EFP to PPP, one diagram needed to be transformed into the other, which required removing a transversal. We posit that a viable explanation for these phenomena requires attention to the geometric nature of Kirk's reasoning (in a visuo-spatial sense). Much like Oren's explanation using his forearms to observe the possible arrangements of two lines, Kirk seemed to interpret the postulates as describing geometric possibilities. For him, these conditional statements were not so much warrants for possible inferences in a chain of deduction as they were articulations of occurrences among lines in a plane. This explains why Kirk so closely linked the postulates to their diagrams and said the two postulates were "the same": the two statements pointed to geometric phenomena and described the same set of three possibilities. Oren provided a stark contrast to Kirk's reasoning inasmuch as he attended closely to how warrants in the body of theory afforded individual steps in proof construction. Oren's hypothetico-deductive mode of activity led him to recognize the insufficiency of their three cases argument when Kirk could not within his visio-spatial mode of reasoning.

This account of Kirk's reasoning suggests that a more honest rendering of his understanding into the language of logic would be the invalid syllogism: "EFP (and Theorem *) ⇒ Only One Instance, PPP ⇒ Only One Instance, therefore EFP ⇒ PPP." The deeper implication of our second analysis is that propositional logic is a poor tool for modeling the nature of Kirk's reasoning. This can be seen in the way that each "implication" in this supposed syllogism has a distinct meaning. Kirk perceived the claim that there was Only One Instance (of a parallel through *P*) as an implication of EFP via the Three Cases argument. He viewed Only One Instance as a paraphrase of PPP rather than a consequence of it. Finally, his reasoning did not display clear directionality in the relationship between EFP and PPP (Dawkins, 2012). It appears that Kirk's empirical reasoning convinced him that the 3 Cases argument proved that EFP and PPP were equivalent because they entailed the same geometric possibilities. Framing his reasoning thus as a quasi-empirical report on the set of geometric possibilities described by the two postulates provides a better explanation for various aspects of his behavior. We thus conclude that this description is more faithful to the nature of the underlying reasoning process.

Comparative analysis 3: Examples and expertise in real analysis

Episode 3 occurred during a comparative study of the proving behaviors of expert and novice mathematicians. This episode features a graduate student in mathematics, designated an expert prover, attempting a novel analysis task about sequences. Analysis of Episode 3 also appeared in Karunakaran (2014), which presents the fully methodology of the study.

Episode 3: Proving and disproving conjectures about sequences of real numbers.

Upon being asked to validate or refute the claim given in Figure 2, Zander quickly stated, "So, the first thing that I would do is to see if [the series] obviously doesn't converge." When asked to elaborate his goals, Zander stated that he would search for a counterexample to the statement. That is, he would look for a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers satisfying the condition that $0 < a_n \leq a_{2n} + a_{2n+1}$, such that the series $\Sigma_{n=1}^{\infty} a_n$ does not converge.

Task 1
Validate or refute the following statement: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $0 < a_n \leq a_{2n} + a_{2n+1}$ $\forall n \in \mathbb{Z}$ & $n \geq 1$. Then the series $\sum_{n=1}^{\infty} a_n$ converges.

Figure 2. The statement of the original Task 1 statement as presented to Zander.

Zander quickly generated the valid counterexample sequence $a_n = 1\ \forall\, n$. At this juncture, the interviewer asked Zander to prove a slightly modified version of statement in Task 1. The modified version negated the conclusion to state, "Then the series $\Sigma_{n=1}^{\infty} a_n$ ***diverges***." As before, immediately after being given the modified task statement, Zander stated, "Ok. Uh well … right so then I would have to find an example where it converges." Upon being asked, Zander confirmed that he was looking for a counterexample to the modified statement. Also, Zander quickly considered and discarded the use of various convergence and divergence tests (e.g. ratio test; comparison test) because he anticipated that none of the tests would "guarantee divergence."

Then, Zander recalled a convergent series with which he seemed familiar: $\Sigma_{n=1}^{\infty}\left(\frac{1}{2^n}\right)$. He stated his intentions for choosing this example saying, "maybe we can find a way to make a sequence where a_n [the sequence in the task] is equal to $\frac{1}{2^n}$ or smaller than or something like that. Cause then that would converge as well." However, he noted that the sequence $\left\{\frac{1}{2^n}\right\}$ does not satisfy the inequality condition $0 < a_n \leq a_{2n} + a_{2n+1}$. To work around this, he attempted to generate a counterexample by modifying his example sequence such that $a_0 = a_1 = a_2 = a_3 = 1$ and then each of the terms for $n > 3$ were constructed using the rule $a_{2n} = \frac{1}{2} a_n$ and $a_{2n+1} = \frac{1}{2} a_n$. At this point he argued that "halving" the terms was the "best–case scenario" for satisfying the inequality since "it's sort of the cutoff I mean because if we take it to be any smaller a half, say like a tenth of a tenth and then it no longer fulfills this second inequality."

Zander stated that he now believed the modified task statement to be true. Zander attempted to use the harmonic series to prove that the modified statement is true. However, he realized that the harmonic series does not satisfy the inequality condition, and he explained that he would like to show that the terms of the harmonic series (or some variant of it) would be a necessary lower-bound to the corresponding terms of the series in the task and thereby the series in the task would also have to diverge (using the comparison test).

Analysis 1 of Episode 3. The study in which this episode occurred focused on finding similarities and differences between expert and novice's proving behaviors. As such, the original analysis characterized Zander's proving behaviors across the various real analysis tasks provided. Zander repeatedly used the strategy of searching for a counterexample on this and other tasks. When asked about why he did so, he replied, "Because the counterexample might tell you why it always diverges … or rather the inability to find a counterexample might tell you why it always converges." So, on multiple tasks Zander used this strategy of searching for a counterexample to either successfully find a counterexample invalidating the statement or to gain knowledge about why the statement is valid through the inability to find a counterexample. However, Zander's choice of sequences/series appears strange since neither was a counterexample of the original or modified claim. This is because neither satisfied the inequality condition in the hypothesis. The interviewer asked Zander why he called on the series $\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$ and the harmonic series, even though neither one satisfies the inequality condition. He explained that he routinely looked for examples that were relevant to the task and provided him with "a picture" or a "prototypical" example that helped him understand the task better. Zander's use of the term "prototype" seems inconsistent with its use in mathematics education as an example central to a category. Since Zander engaged in a process of example generation similar to Antonini's (2011) *example transformation*, he may use the term "prototype" to refer to his starting example that he modifies to satisfy the conditions in a task.

Thus, this episode supports the general claim that Zander's proving strategy often included searching for counterexamples (regardless of whether he believes one exists). He was aware of this strategy and valued it because it aided him either in disproving the claim (by counterexample) or proving the claim (using some insight from the example search). Furthermore, Zander's work within this episode also supported the claim that he routinely used what he considered "prototypical" examples or visualized "pictures" to gain insight into why a particular claim is true, consistent with previous finding associating visualization and examples with conviction and insight (e.g. Alcock & Simpson, 2004).

Analysis 2 of Episode 3. Even though we can make the content–general claims present in Analysis 1, this may not account for Zander's "expertise" or his relative success on this task. We observe nuances within Zander's search of counterexample and his choice of sequence/series ($\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$ and the harmonic series) that provide insights about his use of his content-specific knowledge about series. Throughout the task, Zander paid particular attention to the growth patterns of various series, which can rightly be considered a conceptual link between the inequality condition and the convergence of monotone increasing series. To find a counterexample to the modified task, Zander called on the series $\Sigma_{n=1}^{\infty}\left(\frac{1}{2^n}\right)$ because he knew this to be a series that converged. A minimal way to satisfy the inequality condition is for $a_{2n} = a_{2n+1} = \frac{1}{2}a_n$, and $\left\{\frac{1}{2^n}\right\}$ similarly halved its adjacent terms. Zander noted that $\left\{\frac{1}{2^n}\right\}$ did not satisfy the inequality condition, but examining the rate at which its terms decreased prompted him to search for "a way to make a sequence where a_n is equal to $\frac{1}{2^n}$ or smaller than or something like that … then that would converge as well." Zander deduced that "halving" the sequence terms would be the "best–case scenario" since,

> "if we take it to be any smaller a half, say like a tenth of a tenth and then it no longer fulfills this second inequality [and] if we take something that was bigger than a half then that's only more problematic because you're just throwing in bigger numbers into the sequence … I think this if I'm right in saying that this sequence always diverges this actually might be a key to the reason why."

In what ways was this scenario "best?" Zander wanted to find a series that converged, so the added terms must decrease, but the inequality limited the rate at which they decreased. Zander's modified example was his "best" possibility to have a minimal growth rate (so as to converge) while satisfying the inequality condition in the task. We infer that Zander's analysis of the series $\Sigma_{n=1}^{\infty}\left(\frac{1}{2^n}\right)$ convinced him that the modified task statement was true because he perceived that no sequence satisfying the inequality condition could decrease fast enough for the associated series to converge. He called upon the harmonic series (even though it is not a series that satisfies the inequality condition) as a "prototypical" example of a divergent series with a small growth rate. Part of what made Zander's proving successful (his "expertise") was his ability to interpret the conditions in the task as constraints on the growth rate of the sequence/series and call upon canonical examples that displayed particular growth behaviors. Both his knowledge and use of the prototypical examples point to his analysis-specific knowledge of series, growth rates, and comparison proof methods.

Discussion and Conclusions

We present dual analyses of these three proving episodes to portray the alternative insights gained by content-general analysis (of direct and indirect argumentation, logic, example use, etc.) versus content-specific analysis (of exploring the reciprocal relations between lines, the possible arrangements of transversal configurations, growth rates of sequences and series, etc.). These three studies reflect common research paradigms within mathematics education: 1) documenting the emergence of proving from student argumentation while solving well-designed tasks, 2) task-based interviews intended to elicit instances of mathematical behavior related to a general topic of interest and 3) comparing and contrasting expert/novice mathematical behavior. While the latter two studies employed grounded theory methods, affording these various analyses, they both began with guiding questions and theoretical framings (as no investigation can avoid being, on some level, theory-laden). Regarding Episode 2, it was only after attempts to generally characterize Kirk and Oren's interpretations of conditional statements failed that the author attended to the broader differences between the ways they interpreted the statements and the task at hand. This helped explain Kirk and Oren's very different proving behaviors and their assessments of their progress on the task. Regarding Episode 3, the second author designed the study to include tasks in various mathematical contexts, but later refined the study tasks to only include real analysis tasks. While the content-general claims about Zander's proving expertise are supported by Zander's proving practice and his self-reflection, they may also hide the role and value of Zander's extensive experience with real analysis. In other words, content-general claims will necessarily fail to characterize what it means to be an expert in real analysis and how that expertise plays a role in the observed proving. We may conclude from cases like Zander that experts use examples more and differently than novices, but we must focus on content-specific ideas such as growth-rates and convergence to truly characterize how Zander used those examples productively. One must specify what category an example instantiates for the learner, as Zander productively used series that did not satisfy the task constraints, but exemplified various growth rates.

In what follows, we present our arguments about why research on proof-oriented mathematical behavior should attend to the role of particular mathematical content. We organize our observations around two main points each of which highlight the role of content in proving behavior: 1) distinguishing when research *describes* from when it *explains* the emergence of proving behavior and 2) attending to the role researchers play in identifying and framing observed phenomena. We then provide some recommendations for future research regarding the interplay between proving and mathematical meaning.

Description versus explanation

As we stated before, our goal is not to deny the value of content-general proof research, but rather to sensitize the mathematics education research community to the limitations and possible liabilities of content-general research lenses (i.e. questions and theoretical frameworks). As was discussed previously in the paper, we find it helpful to distinguish whether a study's analysis provides *descriptions* of student proving phenomena or *explanations* for the emergence of those phenomena. We contend that one may *describe* many recurrent phenomena of student proving in a content-general way, but *explaining* the emergence of the observed reasoning in most cases requires attending to students' understanding of the mathematical content at hand. This is because, generally, students reason about mathematics while they solve mathematical tasks!

Many frameworks in mathematics education set forth categories that *describe* patterns in student proving behavior. These categories are useful inasmuch as they provide language for further investigation and tools for the formulation of testable hypotheses and inferences regarding student learning. Indeed, one does not necessarily need underlying mechanisms and explanations to identify and document recurrent phenomena in students' proving behavior. However, such descriptions of recurrent student behaviors also should not be confused for explanations of the reasoning processes that lead to their emergence. It seems obvious to claim that students are thinking about the mathematics at hand when proving. This suggests that any explanation of their proving behavior must account in some way for students' meanings (Thompson, 2013) for that mathematics. Accounts of proving phenomena that are not content-specific are likely to capture epiphenomena that emerge at the chosen level of analysis, but whose underlying causes or mechanisms remain as-of-yet unspecified.

How can we observe this distinction between description and explanation in each of our three episodes? Regarding Antonini's (2003) account of the emergence of indirect proof, we provided an alternative account of how the particular geometric task afforded the students' line of reasoning. Our account suggests that any use of Antonini's hypothesis that open-ended proof tasks invite indirect reasoning may require careful attention to students' familiarity with the properties in the task and their negations. Based on our analysis, we hypothesize that even traditionally stated proof tasks that involve negative properties may invite indirect proof, such as "Prove that any number that is not a multiple of 4 is not a multiple of 12." In cases such as Antonini's (2003) where both direct and indirect argumentation are viable, open-ended proof prompts may elicit indirect argumentation. The task frame alone falls short of explaining why it does so. Regarding Episode 2, analyzing proof frames or imposing a hypothetico-deductive structure on Kirk's arguments both reveal deficiencies in his proving. One can describe his reasoning to show that it was non-normative, but this fails to reveal the subjective rationality of Kirk's behavior. These forms of analysis do not provide insight into how his engagement in the proving process differed qualitatively from Oren's. Because Kirk's reasoning was intrinsically tied to the visuo-spatial representations, one cannot explain the emergence of his reasoning without attending to the geometric nature of the task. Regarding Episode 3, the strategic use of examples may provide an observable, qualitative difference between experts' and novices' reasoning, but that does not mean differences in example use constitute the nature of expertise in this proving arena. Our second analysis of Zander's reasoning demonstrated how he used examples as implicit measures of rates of convergence, which we argue may be closer to an explanation of the nature of his expertise in sequence and series convergence.

Our distinction between description and explanation is of greatest import when using and applying the results of prior research. We acknowledge the appeal of reporting local findings in a content-general way because it maximizes the possibility that other researchers may use the results of a study. However, comparing phenomena across studies involves (at least implicitly) generalizing those phenomena across individuals and often generalizing across age, learning background, and mathematical content. We hypothesize that our field's implicit invitations to overgeneralize empirical findings are partly to blame for the confusing and seemingly contradictory claims available in the literature on proof (e.g. Reid & Knipping, 2010). We offer our caution about *explaining* mathematical proving behavior in terms of mathematical meaning as a call to respect how situated students' mathematical knowledge is.

Distinguishing signal from noise

Our second point focuses on how researchers' framing of their investigations influences data analysis. A brief survey of mathematics education literature attests to the fact that students' mathematical reasoning is an incredibly multi-faceted and complex forum for investigation. No single study can account for all of the dimensions of variation at play, and thus researchers must choose how to separate signal from noise in observed data. The broad prevalence of qualitative methods renders this difficulty particularly acute in mathematics education. How do researchers decide which dimensions of observed behavior to attend to and which to ignore? In most study designs the theoretical frameworks, research questions, observation protocols, etc. explicitly funnel the complexity of student behavior into interpretable streams of data. In studies using grounded theory methods, researchers cast a wide net in initial observation and allow trends to emerge along more natural contours in the data. In all cases, though, we contend that researcher questions, interests, and prior theory play a necessary role in helping distinguish signal from noise. There are two possible pitfalls within this process, each of which is exemplified in our three episodes: 1) researchers filter as noise some aspect of student behavior that is in some way essential to understanding the emergent phenomenon or 2) researchers' framing and theory impose structures on student behavior that are not native to it.

Episodes 1 and 3 exemplify the first issue. The studies purported to be about indirect argumentation and example use, respectively, which implicitly framed the observed behavior as instances of content-general phenomena. The research questions themselves may warrant a researcher's choice to filter the mathematical content of tasks as noise, or to characterize the observed behavior in content-general language. This creates the illusion that the research subjects reasoned about "examples" and "non-examples" rather than transversal configurations or series. This is one instance of how mathematics educators use the language of meta-theory to discuss student behavior. Mathematics educators have rightly called upon mathematical meta-theory to explicate the normative mathematical processes into which we intend to apprentice students. Such theory provides useful categories, terms, and structures for modeling student behavior, but we must carefully distinguish researcher models from observed phenomenon. We find that describing student behavior using meta-theory and assessing whether their behavior is mathematically normative (both of which are appropriate) can subtly lead researchers to impose the structures of meta-theory upon student behavior itself. This suggests how the first issue cited in the last paragraph can often lead to the second. This is probably nowhere more likely or prevalent than in logic (Dawkins, 2014), which is why Episode 2 serves as a cautionary tale.

Proving involves drawing inferences and, somewhat controversially, logic can be used as a meta-theory for describing human inference. The relationship between formalized logic and reasoning has been widely disputed (e.g. Oaksford & Chater, 2002; Toulmin, 1958). It is beyond the scope of the current discussion to elaborate on this debate and its relevance to mathematics education research on proving, but we find Piaget's use of logic an insightful case. Piaget was fond of using formal mathematics as a descriptive language for children's reasoning (e.g. Piaget, 1950). Even though he clearly argued that he was recasting logic to reflect student reasoning (rather than the other way round), reviewers of Piaget's work often interpreted his models of student reasoning as beholden to the structure of formal logic in problematic ways.

Episode 2 demonstrates how content-general models such as logic may be applied to a broad range of students' proving activity, but may also be misleading or dishonest to the underlying reasoning process. We maintain that the two syllogisms we used to model Kirk's arguments are, on some level, representations that express some aspect of his reasoning process. However, each one likely hides other aspects of that process as well. These descriptive models reduce "implication" to mere links between claims with the effect that it fails to distinguish what may have been distinct types of inferences. The logical frame also imposes structure such as directionality that further analysis suggests was not native to Kirk's reasoning (Dawkins, 2012). We perceive that several of the connections Kirk drew between the given claims were distinct in meaning. As such, using logic to model these inferences unintentionally imposes foreign structure upon them. Oren's reasoning also provides evidence for the non-trivial influence of researcher models on data analysis. Oren's reasoning in that interview was far easier to analyze because it was more consistent with a hypothetico-deductive frame of inference. This is to say that modeling student reasoning is easier when the data fits the model on a structural level.

Many researchers avoid using formal logic to model students' argumentation and proving, but this does not circumvent the danger that by applying content-general models mathematics educators will recast student behavior after the image of their analytical lens. Toulmin (1958), dissatisfied with the implicit assumptions entailed in using formal logic as a modeling tool, set forth an alternative scheme for modeling argumentation that has been increasingly popular in mathematics education research. However, this scheme does not alleviate the burden upon researchers to attend to the interpretive process, because the same argument can be modeled in different ways using Toulmin schemes and there is some debate about which dimensions of his framework must be used in mathematics education research (e.g. Inglis, Mejia-Ramos, & Simpson, 2007). How and when do content-general frames such as formal logic or Toulmin schemes obscure the role of mathematical meaning in proving? Researchers should apply these tools with some caution because they can focus analysis on the "signal" of warrants, claims, and data leaving particular mathematical claims as background or "noise." How can mathematics educators adapt these frames or couple them with other tools to integrate mathematical meaning in the analyses of the emergence of mathematical proving? Researchers must be cautious applying content-general models to student reasoning lest the resulting characterizations reflect *researcher's questions* more than *students' mathematical behavior*. Furthermore, if a key goal of proof-oriented instruction is for students to experience proving as a means of developing mathematical understanding, our research lenses for mathematical proving must accommodate the particular mathematical understandings that we intend students to develop.

Future directions

While the primary goal of this theoretical paper was to sensitize the mathematics education research community to the role of mathematical meaning in research on proving, we also see viable avenues for future research to directly address the issues discussed here. There already exist many studies that approach proving phenomena in a contextual way, but often such studies are classified as studies of real analysis learning, geometry learning, etc. One promising research approach is to document student's apprenticeship into mathematical proving across content domains to document characterize content-independent proof skills if and when they emerge. Empirical study is required to determine how and when students develop a sense of mathematical proving as a unified practice with universal techniques and governing principles. Other studies might instead compare students' proving in various mathematical contexts to learn when and how proving knowledge and know-how are situated within a mathematical domain (e.g. Mejia-Ramos, Weber, & Fuller, 2015). Perhaps our highest recommendation is that researchers attend to the co-emergence of proving competencies and conceptual understanding of key mathematical content. We prioritize this because we view it most hopeful for facilitating the integration of proving as a *means* of mathematical learning across the curriculum.

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CLEARING THE WAY FOR STEM MAJORS IN INTRODUCTORY CALCULUS

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One of the reasons for the exodus in STEM majors is the introductory calculus curriculum. Although there is evidence that curricula like CLEAR calculus support students' adopting a more growth-orientated mindset, it is unclear how this curriculum promotes mindset changes. The purpose of this case study was to investigate which features of CLEAR Calculus students notice and attribution to their success. After administering the Patterns of Adaptive Learning Scale to assess students' initial mindset in one section of calculus, four students were selected for interviews. At the end of the semester, students were re-administered the PALS and all students who took an additional mathematics class were given the survey at the end of their next course. Students cited that the labs CLEAR Calculus curriculum challenge them in ways that facilitate deeper comprehensive learning than that of prior courses, and the positive changes in approaches to mathematics continued in the next course. Students did not notice the formative assessments as part of the course, but the actions taken by the instructor as part of the course was taken as evidence as instructor caring by the students.

Keywords: Calculus, formative assessment, mindsets

Prospective STEM majors who declare a non-STEM major are most likely to do so after introductory calculus (Bressoud, Rasmussen, Carlson, & Mesa, 2014); students cite their lack of a perceived relationship with their instructor and the inability to seek help as primary reasons for switching (Ellis & Rasmussen, 2014). One possible solution is the use of formative assessments such as exit tickets; such assignments show promise in helping students to perceive their instructor as more approachable and caring about their success (Black & Wiliam 1998, 2009; Dibbs, 2014).

However, the number formative assessments completed are a far stronger predictor of students' success than their weight in the course grade would indicate (Author 1, 2015). One possible explanation for this effect was that students who completed more post-labs had different mindsets about learning mathematics than those that did not. It has been noticed that mindsets play a significant role in the overall success of calculus students. Dweck (2006) defines mindset in two different ways: fixed mindset and growth mindset. Students classified under the fixed mindset, if not immediately successful in introductory calculus often leave the STEM field. However, growth mindset students can persist and succeed, even after failures as severe as failing a course (Dweck, 2007).

We examined how CLEAR Calculus supports positive mindset changes in students through a case study of four students enrolled in an introductory calculus class taught using CLEAR Calculus. This research will be guided by the question: What are the features of CLEAR Calculus that students notice and value? By understanding what makes this curriculum effective, an interested practitioner who is not implementing CLEAR Calculus can learn what components to add to their classes if they would like to see a positive increase in their students' mindsets. We argue CLEAR Calculus supports positive changes in students' mindsets because the labs make a challenge and conceptual understanding central components of the course; although students did not notice the formative assessments, the actions were taken by the instructor as the result of formative assessments helped create a positive perception of their relationship with the instructor.

Theoretical Perspective and Methods

Every week followed the same general schedule. On Monday, there was a new section of material introduced and students were given a prelab. The prelab asked students to complete the Unknown Value of the approximation framework (Figure 1) and to identify a quantity that could be used to approximate the unknown value. Students were asked to complete the prelab before class on Tuesday; the prelab was graded on completion at the beginning of class. During class on Tuesday, students worked in groups of three or four on their assigned lab context. After class, students completed a post-lab using the online course management software. Each post-lab asked students to summarize what their groups did, evaluate how well they understood the material, perform a computation similar to the ones expected on the lab, and identify which portions of the lab they still needed help on. This information was summarized and sent to the instructors that night and used to launch a classroom discussion on Wednesday. The remainder of the week was spent on concepts from the textbook. For the derivatives and definite integral labs, the next week would be a repeat of the first; all of the other labs proceeded directly to the regrouping described next. On the third week, students would be placed in new groups, where they were responsible for teaching their context to their new group members; this type of presentation is called a Jigsaw presentation because each student is responsible for one piece of a larger idea. After this Jigsaw presentation, students were expected to write up their individual answers to the 20 parts of the approximation framework. Each lab had one formative prelab and two or three post labs associate with the activity.

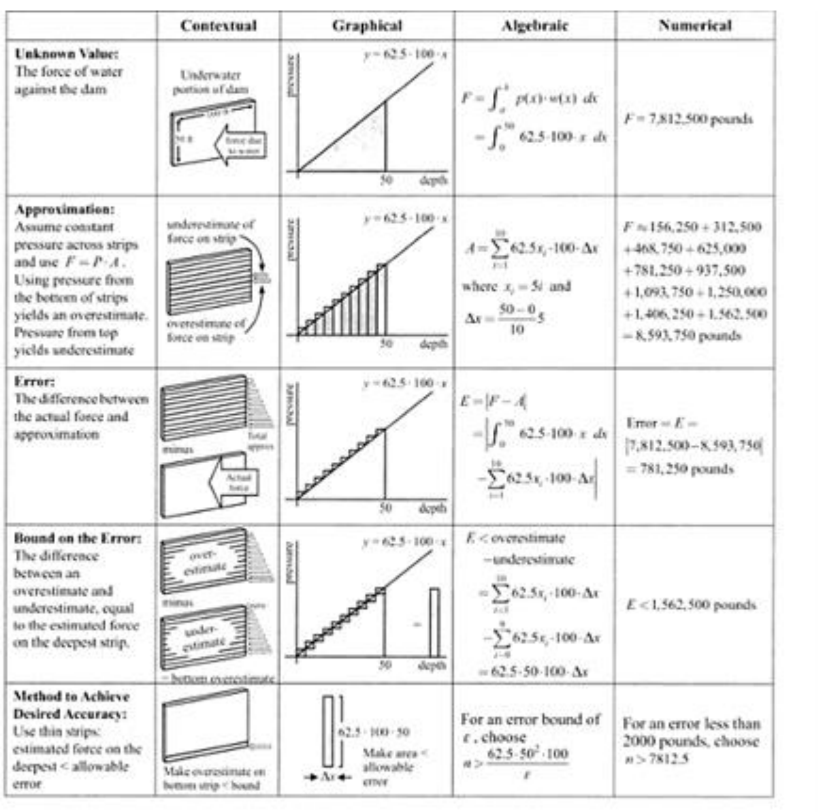


Figure 1: Approximation framework (Oehrtman, 2008)

The approximation framework is built upon developing systematic reasoning about conceptually accessible approximations and error analyses but mirroring the rigorous structure of formal limit definitions and arguments (Oehrtman, 2008, 2009). This study focused on the three multi-week labs developing the most central topics in the course: limits, derivatives, and definite integrals. Each approximation lab consists of 20 questions designed to help students understand their context in terms of approximating a limit. For each calculus concept, students are asked to identify the unknown value that cannot be solved with algebra, an algebraic technique for approximating the unknown, quantify the error, bound the error, and describe how an approximation can be computed to any desired accuracy. Students are asked to represent these five components of the approximation framework contextually (words and pictures), graphically, algebraically, and numerically. The figure above illustrates the 20 components of the approximation framework for one context of the definite integral lab, where students have been asked to compute the water pressure on a dam (Figure 1). Each lab had three or four different contexts; one of which was more challenging and intended for students that had seen calculus before.

The theoretical perspective for this case study (Patton, 2002) was Dweck's (2006) mindsets (Figure 2). Participants attended a midsized rural regional research university in the South, and were recruited from an introductory calculus course taught using CLEAR Calculus labs. These labs are built upon developing systematic reasoning about conceptually accessible approximations and error analyses but mirroring the rigorous structure of formal limit definitions and arguments (Oehrtman, 2008, 2009).

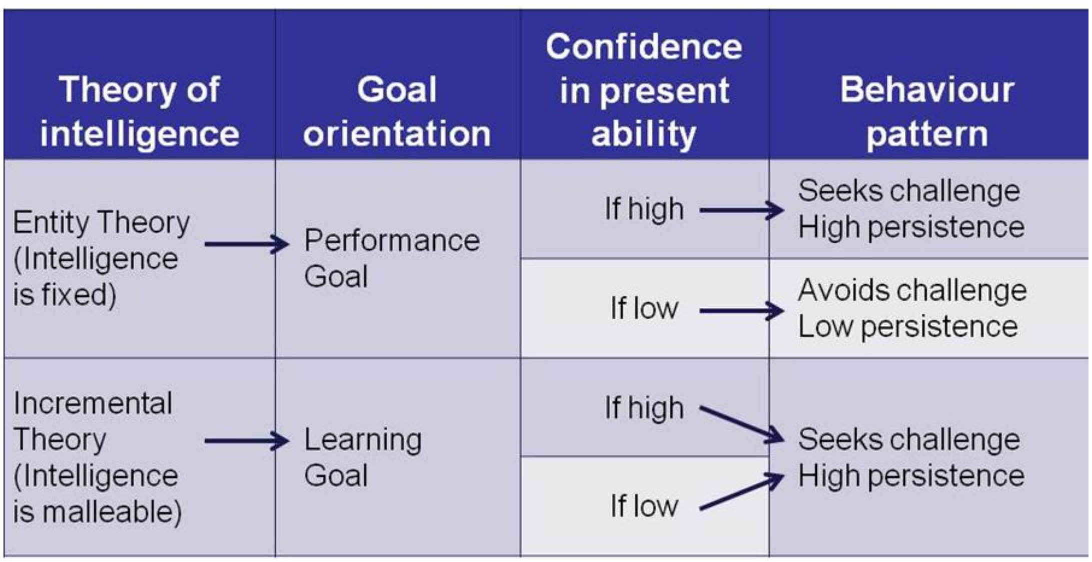


Figure 2. Summary of Dweck's mindset theory

Students in the course who consented to participate in the research (10/12 students in the course) took the Patterns of Adaptive Learning Scale (PALS) during the second week of the semester. Eight of the participants were male, and two were female. All students were math minors, math majors, or double majoring in math and another STEM field. Three participants were upperclassmen, and the rest were freshmen. Six had been exposed to calculus topics before this course. Four participants participated in semi-structured interviews (Patton, 2002) to obtain a sample with maximum variation according to their mindset (Table 1). Author 1, who instructed the course kept a journal of observations of each class and of students' frequency and reasons for office hour visits throughout the semester. Author 2 observed the class and consulted with the instructor of the course for triangulation of the interview data.

Table 1
Interview participants

Pseudonym	Year	Major	Mindset	Final Grade
Ian	Junior	Math	Strong Growth	C
Roland	Freshmen	Biology	Weak Growth	A
Penelope	Sophomore	Biology	Weak Fixed	B
Steven	Freshmen	Math	Strong Fixed	A
Quentin	Freshmen	Engineering	Weak Fixed	D

Participants were interviewed in a semi-structured interview at the end of the semester. The interview questions in the core script are listed below, though each participant had somewhat different probing questions. Interviews lasted approximately 20 minutes. After each interview, the authors discussed initial impressions that were recorded in research journals.

- Tell me what kind of student you've been in previous math classes
- Tell me what this class is like.
- If I had never seen one of the labs, tell me about them. What are they like? What do you do in them?
- Why do you think you do labs?
- Have labs changed you? How?
- What do you do for calculus when you're not in class?
- What will you remember about this class?
- What should I have asked you but didn't?
- Do you think your experience in this class will make you approach your next class differently?
- What have you learned in the class?
- What are the features of the class that stood out to you?
- Have you changed as a student as a result of this class?
- A lot of people mentioned instructor caring?
- What made you decide caring was a thing?
- Would you recommend this class to another student?

After transcribing the interviews, Dibbs and Patterson coded the transcripts separately using the coding scheme developed from the literature and theoretical perspective (Table 3). The interrater agreement on the separate coding was 80%. After reconciling the codes where the researchers differed, a one page summary of each participants' analysis was typed up and given to the participants as a member check. These member checks led to the challenge code to be broken into two separate codes. After the interview analysis was completed, the second PALS survey was scored, and the PALS results were analyzed using Mann-Whitney due to the small sample size.

To if this behavior persisted in future mathematics classes, particularly those that were more traditionally taught, we observed Ian, Steven, and Quentin (who chose to repeat calculus in the one section of CLEAR calculus offered) in their next mathematics classes. Penelope and Roland were not required to take additional mathematics courses. Ian, Quentin, and Steven also took the PALS at the end of their next semester and had brief, informal interviews with Author 1 during the next two semesters. These interviews were coded using the same coding scheme given in Figure 3. To if this behavior persisted in future mathematics classes, particularly those that were more traditionally taught, we observed Ian, Steven, and Quentin (who chose to repeat calculus in the one section of CLEAR calculus offered) in their next mathematics classes. Penelope and Roland were not required to take additional mathematics courses. Ian, Quentin and Steven also took the PALS at the end of their next semester, and had brief, informal interviews with Author 1 during the next two semesters. These interviews were coded using the same coding scheme given in Figure 3.

Table 3
Standards of Evidence

Code	Definition	Standard of Evidence	Example
Challenge is a feature	Students' perception of the class as having difficult problems as the course of the course and seeing this as a positive thing	• Direct statements made by students that (challenging/real life/difficult problems are important in class • Direct statements made by students indicating this is a good thing • Tone of students' response indicated bullet point #2	"This class us challenging, but that is the point" -Penelope
Things that are challenging	What parts of the class students fins to take the most time outside of class/require the most work	• Direct statements or complaints about the things that are the most difficult	"Labs…they've been the most difficult challenges so far. Even more difficult than the test" -Penelope
How labs changed me	Students' perceptions of the work habits and beliefs that participating in this curriculum changed them this semester &/or in the future	• Direct statements made by students about homework or study habits that changed as a result of the class • Direct statements made by students about any beliefs that changed as a result of this class • Evaluative statements about whether this belief will persist into the next class	"Taught me to think through the problems more carefully…to figure out what is wanted" -Roland "I'll think more in depth about [future mathematics courses]" -Ian
Student noticing	Features of CLEAR Calculus labs and course design students recognize as different from other mathematics classes	• Direct statements made by students about the parts of the course that were different from previous math courses	"The main difference between this class and [my last calculus class] is that there are these weekly labs" -Quentin
Perceived course goals	Student's beliefs about the purpose for the components of CLEAR calculus they have not experienced in prior courses	• Direct statements made by students about what they believed the goals of the features they noticed about CLEAR calculus were	"So that we can get almost real world experience so we can retain what we learn" -Steven

Findings

There were four major themes in the interview data: students do not notice the post-labs, students attribute all post-lab based instruction to instructor caring, there was an increase, though not a statistically significant one, in growth mindset tendencies, and participants' challenged in introductory calculus became more growth mindset oriented in the next course.

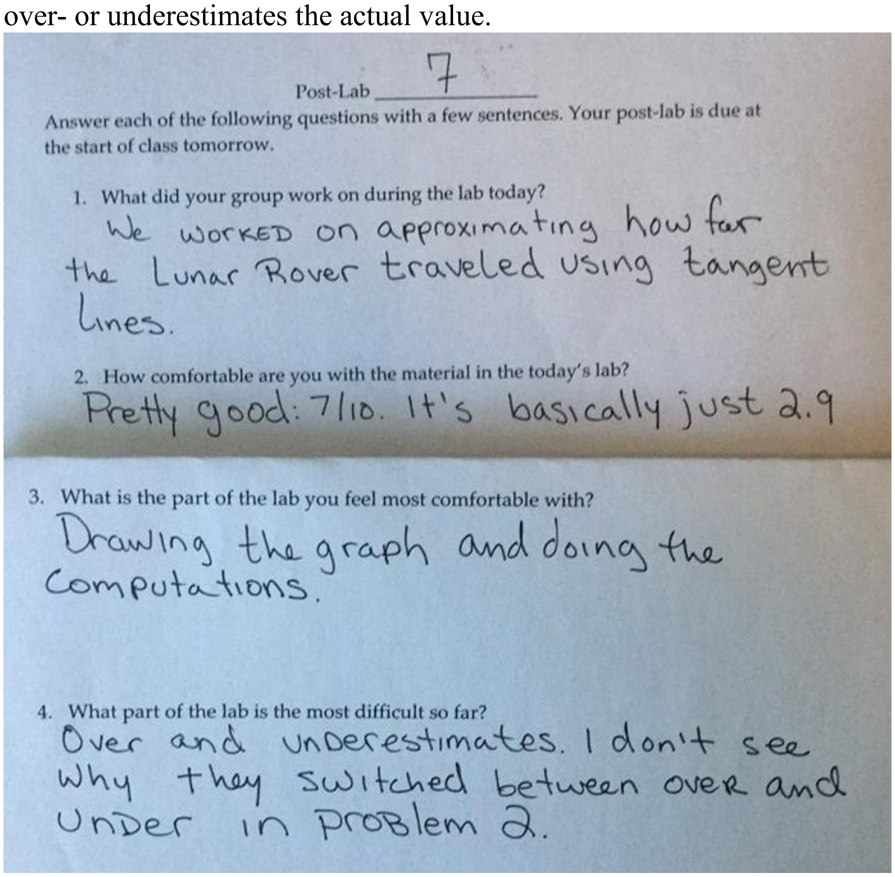
The interview participants all agreed that the labs were the central component of the course. Steven, like the other interview participants, saw the labs as central to the class, challenging, but useful:

> [This class] is harder and easier than you'd expect. As long as you can keep up with your work, you can keep your grades up… [Labs] are great because I'm not necessarily testing as well as I do on my labs. This is my first class with a hard lab so that it was new for me. They're [the labs] kind of like complicated word problems… [I think we do labs] probably so that we can get almost real world experience so we retain what we learn.

The main difference between CLEAR calculus and prior experience with calculus classes was the weekly labs, as Ian explained in his interview:

> Why do we do labs? I think it's to see how the math will be used in real world situations…The labs, probably [are what I will remember about this course]. I've taken calculus before and we didn't do any labs. This is much different than any math class I've ever taken.

As a part of the CLEAR Calculus curriculum, students completed a post-lab in the form of formative assessment after each lab. On this assignment, summarized what their group accomplished, their confidence in the material, the part in which they excelled the most in the lab, and the part(s) of the lab where they struggled. Figure 3 below gives an example of a typical student post-lab. Here, the student is comfortable with the basics of linear approximation, but is not noticing that the first problem is approximating an input value and the second is approximating an output value, which changed whether the computed value over- or underestimates the actual value.


Post-Lab 7

Answer each of the following questions with a few sentences. Your post-lab is due at the start of class tomorrow.

1. What did your group work on during the lab today?

We worked on approximating how far the Lunar Rover traveled using tangent lines.

2. How comfortable are you with the material in the today's lab?

Pretty good: 7/10. It's basically just 2.9

3. What is the part of the lab you feel most comfortable with?

Drawing the graph and doing the Computations.

4. What part of the lab is the most difficult so far?

Over and underestimates. I don't see why they switched between over and under in problem 2.

Figure 3. Typical student post-lab

The post-labs are a significant reflection piece of CLEAR Calculus labs that students rarely appeared to notice. These labs provided the instructor with the knowledge of the areas of the content that students have mastered and the areas that students have struggled. But students did not perceive the post-labs, a brief assignment at the end of class, to be the source of this knowledge. When students finished the in-class lab early, they were usually able to finish the post-lab in the last 5 minutes of class. In the interviews, none of the participants brought up post-labs until the interviewer did. Roland was the only student that mentioned post-labs at all and he only did so in a way that summarized the lab process, "…we do a pre-lab, a lab, and then a post-lab. Overall, I don't really like labs."

Students display a hurried approach to post-labs. They seem to try to finish the post-lab before they leave class on lab day so as to not leave it to the rest of the week to be forgotten. The following was a common episode at the end of every lab. One to three students said something similar at the end of each lab about the post lab starting in the third week of classes:

> Quentin smiled as he left Lab 6 today. "Here is my post-lab," he told me as he put the paper in the folder, "This week, at least, I didn't forget to do it." –Author 1 fieldnotes, Week 6 (Tuesday)

In class, post-labs were only mentioned explicitly by students as something they were glad that they did not forget to complete them. Combined with the students neglecting to mention the post-labs during the interviews, this suggests students approached post-labs as a low-priority afterthought.

Students are not making the connection that the instructor uses the post-labs to alter the course of the content presented in class in any way. When asked, no student in the interview could identify the post lab as something other than a part of the lab. The assignment students most frequently asked to be able to turn in late (late work was not accepted in the course) were post-labs, because they often forgot to complete them. This indicates students did not notice the post-labs as an important component of the lab.

Although students did not see the informal reflections they did on the post-lab as central to the course or important to their learning, students did see the higher stakes test corrections and reflections to be hugely important to their understanding of the material. Students are allowed to correct their tests to receive a token amount of points added to their original test grade; however, in order to receive any credit, the corrected test must be wholly correct and each problem must include at least a one sentence reflection on the error in the original test solution. When talking about correcting the test problems that he got wrong during the test, Steven saw this as an opportunity to learn:

> ....Which is great because I'm not necessarily testing as well as I do on my labs…I try to figure out what's going on again and again and watch YouTube. I like something my grandpa said, 'You're your own best teacher. So you only learn things if you want to learn them. So I'll go back and try to figure out exactly what was going on to see if I can correct it.

While students do not notice the post-labs, they all believe that there was value in the labs and that their instructor probably cared about their learning. However, the actions that were seen by the students as evidence of instructor caring were based upon the reflections in the post-labs, and so the unacknowledged post-labs did help to create a perception of a positive relationship between students and instructor.

Students all saw labs as the most important and memorable part of the course. In three of the four interviews, students made statements comparing the relative importance of labs and tests. While students' lab grades were 15% of their course grade, the same as one test, students saw the labs as being a much bigger part of the course. Although Steven felt that the labs were less difficult than the test, the other three interview participants who did not earn A's in the course disagreed with this sentiment:

> What I'll remember most is probably doing all the labs. I'm taking chemistry as well, and we have a lab every week. So I'm used to doing the lab. I'm used to doing lab assignments, but like the lab assignments she gives us in this class are more challenging because you have a more complex pre-lab. Like lab assignments in chemistry, you don't have to do a whole write up usually. She makes us like write paragraph answers to one-word answers. I guess that way we're answering it more in depth. So we get a better understanding of it. I guess I'll remember the labs the most. They've been the most difficult challenges so far. Even more difficult than the test.
> –Penelope

Participants believed that the labs were valuable, even though the labs were difficult and frustrating. When asked about what he will remember about the labs, Steven responded,

> Probably [I will remember] all the labs, in a little of both [a good way and a bad way]. I'll remember because they were so hard. But it's also nice when you understand one and you get it done. It's a nice feeling of accomplishment. I'll be a little more scared to take Calculus II, but that might be a good thing too.

Since Steven was generally successful at the labs (he actually earned the highest grade in the course) he enjoyed the challenge of the labs and did not necessarily have an experience that would challenge his strongly fixed mindset.

Quentin, who also had a strongly fixed mindset, did not get the same sense of accomplishment from the labs; he actually failed the class because he only turned in two of the 14 labs and only one post-lab. "I don't turn in anything that isn't fit to be seen," he explained, "and my labs are not good enough. I care too much to turn in bad work." When he was asked about not doing the post-labs, which are graded on completion, Quentin's justification was that the other people who did post-labs were asking his questions already: "I don't really need to do them. All my questions get answered on Thursday [when labs are discussed in class], so I don't add anything by doing a post-lab." Overall, the labs were seen as a positive thing, and even though Quentin did not feel post-labs were something he needed to do personally, he did feel that he benefitted from the discussion based on everyone else's post labs later in the week.

After the semester ended, the pre- and post- PALS surveys were analyzed using the Mann-Whitney U test since there were only 10 students who completed the survey. The Mann-Whitney U test was not significant ($p = 0.15$), but the small sample size makes it difficult to rule out mindset changes in general. Although we did not see significant mindset changes, all of the interview participants made actions during class throughout the semester and statements in their interview that suggested that there were qualitative, if not quantitative shifts towards a growth mindset.

In the first month of the course, before the first test, participants' behavior indicated an unwillingness to seek help or admit they did not understand. Two-thirds of the labs and homework were turned in a pattern of completely correct solutions or blank problems. When students were asked if they needed help during the labs and recitation by the instructor or teaching assistant, students responded in the negative or that they 'didn't get it' and could not elaborate (Dibbs fieldnotes, weeks 1-4). During lectures, wait times of over a minute were required before students would participate in class. Only one of the ten students-Ian- visited office hours before the first test. However, on the final question of the post-lab, seven participants did note specific topics in the lab or on the homework that they did not understand on at least three of the four post-labs during the first unit.

During the derivatives unit, students began to be more confident and more willing to seek assistance when they ran into trouble. Students' reported confidence on post-labs improved; 8

out of 10 students reported feeling confident they understood the material, and seven consistently earned passing grades on the labs. Six students, including all three male interview participants, began attending office hours or the optional Friday recitation on at least a weekly basis. The average wait time per question dropped to 20 seconds, and the total number of hours the class spent in the math lab increased from two to 10. For the seven students that eventually passed the course, there was a marked reduction of blank responses; students were more willing to give partial responses and admit they had gaps in their knowledge (Dibbs field notes, Weeks 5-10).

Students were interviewed in the final unit. By the end of the semester, students sought help when needed: in the math lab, on YouTube, from a classmate, via email, in office hours. Roland described his out of class work as a process for seeking help:

> [Outside of class I] do the [optional] problems that she assigns me to do. Try to do them or get someone to help me when I can't understand them. I watch a lot of YouTube videos. I find the book [Stewart 6th Edition] hard to – hard to understand things from the actual book so I usually have to use other things for the harder problems. Usually, I'll do all the ones I can do, and save the ones I can't do for later. I'll use additional resources or get help from the TA.

There were only three comments on the post labs on this unit, all of them asking for help with sigma notation. After midterms, students visited the office at least every other week.

Overall, even though there was not a statistically significant shift in students' mindset, their behaviors and interview statements indicate that participants have at least adopted some more growth-oriented learning strategies. However, grades were still seen as a primary motivator for these students since the average goal orientation sub-score on the PALS was almost a full point below the average response on any of the other sub-score of the PALS results, so in essence students learned to emulate a growth mindset because such characteristics were required to satisfy their performance goals which were more aligned with their actual mindset.

In the next course, two of the three participants became more growth oriented. Steven, who easily earned an A in the second-semester calculus course and the introduction to proof course, actually became more fixed, but Steven is unlikely to experience a change in his mindset until he begins to struggle with the material. Ian's mindset became slightly more growth oriented after second-semester calculus, but his growth score is so high that there is a real chance of a ceiling effect. Quentin became much more growth-oriented after successfully repeating introductory calculus with the CLEAR calculus curriculum; by the end of the semester, his PALS score had become significantly more growth. Figure 4 shows the changes in participants' scores over time; a score of 140 marks the boundary between a more fixed mindset and a more growth mindset.

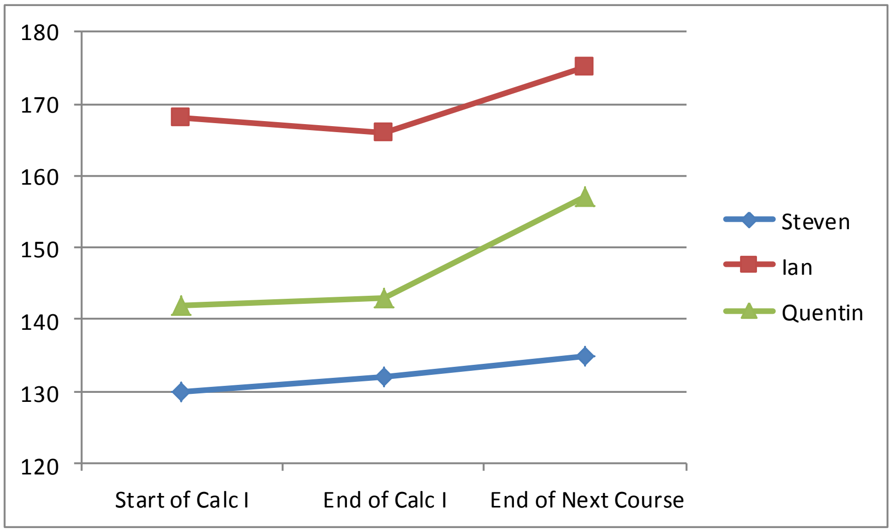


Figure 4. Participants' mindset scores over two semesters

Discussion

The feature of CLEAR Calculus that students notice and value the most are the labs. The post-labs, which were valued most by the instructor were worth a nominal percentage of students' final grades and were neither noticed nor valued by participants. However, students did believe that the instructor cared about them and became increasingly willing to seek help throughout the semester.

There were two components of the course that led students to believe that this course was different and that their instructor cared about them. The first and most important were the labs; the prominence and difficulty of the labs convinced students that this introductory calculus was different from their prior high school or college courses. Without some even causing disequilibration, students are unlikely to make any changes to their learning patterns and make the successful transition to college (Craig, Graesser, Sullins & Gholson, 2004; Zaretskii, 2009). The labs, with the decreasing amount of scaffolding throughout the semester, provided a challenge to their idea of their role as a student, but the test corrections gave students the perception of a margin for error in the course that there was room to struggle without risking their grades. The second component course the students noticed was the test corrections. Test corrections, which could raise students' final grade by a maximum of 2.5% at an institution with no +/- grades did not actually make a difference in the final grade of any student enrolled in the course, but the perception of a safety margin was of central importance to the students. Steven, Penelope, and Roland were especially focused on maintaining their high grades in the course and maintained that the test corrections were very important.

Although the more formative (1% of final grade) post-labs were not noticed by participants, this is likely a combination of the students' strong performance goals and their general success of these participants in the course. The participants that earned A's and B's in the course mastered the limit concepts in the labs around the eighth lab and had few questions on their post-labs after that; for them, the post-labs had limited utility for months be the time they were interviewed. The only participant that mentioned these assignments in any depth was Quentin, who never did post-labs because he felt all of his questions were already answered. Quentin did choose to repeat calculus with the same instructor rather than change majors after failing the course. Given the importance of a perception of their instructor caring about their success is for Persisters (Ellis & Rasmussen, 2014), Quentin's belief that his instructor intuited and answered his questions likely helped him to continue with his intended major. Given that low stakes reflection assignments like these post-labs scale well onto free online platforms with semi-automated feedback like the Just in Time Teaching software (Marrs, Blake & Garvin, 2003; Urban-Lurain et al., 2013), these types of assignments are likely to be the most efficient way to create a perception of instructor caring with some positive effect on student persistence in their STEM major after introductory calculus.

Although there were no significant gains in students' mindsets, we believe that this area is worthier further exploration. The sample size in this study was small and the p-value was relatively low. We also did not collect baseline data from traditional calculus courses or other entry-level mathematics courses. Since students tend to adopt a more fixed mindset in times of stress, like transitioning to college (Murphy & Thomas, 2008), it is worth investigating whether maintaining students' mindsets through their first semester or college is better than a typical curriculum. Regardless, post-class reflections do appear to help students build positive perceptions of their instructor for minimal additional grading.

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What do students attend to when first graphing $y = 3$ in R^3?

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I interviewed 11 differential calculus students as they graphed for the first time in R^3. This paper considers how students generalise as they graph $y = 3$. Some students drew a line, while others drew a plane. In creating their graphs, students attended to equidistance, parallelism, specific (x, y), (y, z), (x, y, z) tuples, and the role of x and z. Students' use of these ideas was often generalised from thinking about the graphs of $y = b$ equations in R^2. A key finding is that the students who thought the graph was a plane always attended to the z variable as free. I discuss this specific result using Harel and Tall's expansive, reconstructive, and disjunctive generalisations framework. I propose that the expansive and reconstructive categories provide a powerful way to explain and understand the two main ways students generalise from R^2 to R^3, and why multivariable topics are difficult for learners.

Key words: multivariable functions, graphing, generalisation

Introduction

Most real-world uses of mathematics involve reasoning about situations with many variables. Being able to reason about multivariable ideas is so important that the Mathematical Association of America recently recommended that high school mathematics curricula include multivariable topics (Shaughnessy, 2011). This recommendation came as the result of the *Curriculum Renewal Across the First Two Years* report (Ganter & Haver, 2011), which examined the mathematical needs of biology, chemistry, economics, engineering, physics, and other STEM disciplines. Hence research about how students understand multivariable topics has the potential to affect learning outcomes for a large group of students of a variety of ages and in variety of STEM fields.

Because multivariable topics share many similarities with their single variable counterparts, many researchers studying student learning of multivariable topics focus on how students generalise from the single- to multivariable context (e.g., Dorko & Weber, 2013; Fisher, 2008; Jones & Dorko, 2015; Kabael, 2011; Martinez-Planell, Trigueros, & McGee, 2015; Yerushalmy, 1997). This research tells us not only how students generalise specific ideas, but also allows us to take a broad, cross-context look at how students transition from the single- to multivariable setting. The goal of this paper is to contribute to both knowledge of specific multivariable topics and knowledge about students' single- to-multivariable transition. I focus on the following questions:

(1) What do students attend to as they first think about graphing $y = 3$ in R^3?
(2) How do students generalise as they first think about graphing $y = 3$ in R^3, and how are these ways of generalising similar to or different from students' generalisations of other multivariable topics?

Theoretical Perspective

I take an actor-oriented transfer perspective in which generalisation is "the influence of a learner's prior activities on his or her activity in novel situations" (Ellis, 2007, p. 225). The actor-oriented perspective privileges what students see as similar across situations, even if their perceived similarities are not normatively correct (Lobato, 2003; Ellis, 2007). The idea is that a student may do something in a multivariable setting that is not normatively correct (e.g., draw $y = 3$ in R^3 as a line instead of a plane), but in doing so the student is making sense of a situation based on past knowledge and experience (e.g., $y = 3$ in R^2 is a line). Hence the student's work counts as a generalisation, even if she has drawn an incorrect graph.

Under the actor-oriented umbrella, I use Harel and Tall's (1991) expansive, reconstructive, and disjunctive generalisation categories to describe the particular ways students generalise (Table 1). This framework categorises generalisation in terms of how students use existing schemas or develop new ones as they move from one context to another.

Table 1. Harel and Tall's generalization framework (Harel & Tall, 1991, p. 1)

Type	Definition
Expansive	• Expansive generalisation occurs when the subject expands the applicability range of an existing schema without reconstructing it. • Earlier schemas are included directly as special cases in the final schema.
Reconstructive	• Reconstructive generalisation occurs when the subject reconstructs an existing schema in order to widen its applicability range. • Reconstructive generalisation differs from expansive in that the existing schema is changed and enriched before being encompassed in the more general schema.
Disjunctive	• Disjunctive generalisation occurs when, on moving from a familiar context to a new one, the subject constructs a new, disjoint schema to deal with the new context and adds it to the array of schemas available.

These categories fit into the actor-oriented focus on what students see as similar by providing language for instances when students perceive similarity (expansive, reconstructive) and when they do not (disjunctive). In the next section, I identify examples of these three types of generalisation in existing research about students' understanding of multivariable functions' graphs.

Background Literature

Thinking about functions and graphs in three dimensions requires students to coordinate three quantities, as well as shift from thinking of y as a dependent variable to considering z as dependent on x and y. Evidence from a variety of contexts indicates that this is a difficult generalisation for students to make. For instance, some students give the domain of a function $f(x, y)$ as x and the range as y (Dorko & Weber, 2013). Others may offer an (x, y ,z)-tuple as an element of the domain or range (Kabael, 2011). These responses indicate that students have not conceptualised y as independent.

Students' difficulties with multivariable graphing tasks also provide evidence that coordinating three quantities is hard, particularly when a function has a free variable. For instance, Martinez-Planell and Gaisman (2013) described how some students graph $f(x, y) = x^2$ as a parabola instead of a parabolic surface. Additionally, students may draw $f(x, y) = x^2 + y^2$ as a cylinder or a sphere because they are accustomed to $x^2 + y^2$ representing a circle in R^2 (Martinez-Planell & Gaisman, 2013). I characterise drawing $f(x, y) = x^2$ as a parabola as an expansive generalisation because the students seem to apply the idea that x^2 is a parabola to the multivariable context. In contrast, I believe the cylindrical- and spherical-shaped graphs represent reconstructive or disjunctive generalising because students have attended to z. Martinez-Planell and Gaisman (2013) write only that "upon seeing the expression '$x^2 + y^2$', some students think of 'circle' and end up interpreting the graph as a cylinder or a sphere" (p.671) so it is impossible to know if students change how they think of $x^2 + y^2$ (reconstructive) or if they think something like '$x^2 + y^2$ is a circle in R^2 but it is a sphere in R^3' (disjunctive). The latter would represent a disjunctive generalisation because the student has developed different, context-dependent meanings for the symbols $x^2 + y^2$.

In addition to coordinating three quantities, students must also develop a schema for representing points in space. Students sometimes struggle with conceptualising $f(x, y)$ as and output or the height of the graph at a particular (x, y)-tuple (Martinez-Planell & Trigueros, 2013), which may explain Kabael's (2011) finding that students have difficulty projecting a graph to the xy plane to determine its domain and range. Students also have trouble determining the intersection of a surface with fundamental planes (a plane of the form $x = a$, $y = b$, $z = c$ for a constant c). Trigueros and Martinez-Planell (2010) found that students who had taken multivariable calculus knew that these were planes, but weaker students had trouble placing such planes in a set of manipulatives and drawing the planes on a 2D image of a multivariable graph. Stronger students could place the planes, but had difficulty determining the intersection of such planes with a multivariable surface. My research, by focusing on a particular fundamental plane, may provide instructional implications that help students overcome these issues. In particular, my focus on what students attend to when they first graph in R^3 may lend insight into how students build a schema for R^3.

Martinez-Planell and Trigueros' research has been in the context of developing a set of activities to help students learn how to graph multivariable functions. They concluded that students' difficulties with multivariable graphing could due to familiar symbols such as x^2 in $f(x, y) = x^2$ and $x^2 + y^2$ in $f(x, y) = x^2 + y^2$. They subsequently altered the activity sets to avoid familiar notations, so it is unknown if students are able to use such notations productively. My work builds on this research by paying explicit attention to how students' conceptions of single-variable functions' graphs interact with their conceptions of graphs of multivariable functions. Findings from other studies about various multivariable topics indicate that students can often successfully leverage their single-variable knowledge to make sense of multivariable topics (e.g., Dorko & Weber, 2013; Jones & Dorko, 2015; Kabael, 2011; Yerushalmy, 1997), and I wanted to study whether this was also the case when students graph in three dimensions.

Data Collection and Analysis

I interviewed 11 differential calculus students about multivariable functions so that I could observe the initial sense making of students who had not yet received instruction regarding these functions. I thought this would let me observe students' generalisations in real time. Additionally, given the actor-oriented perspective, I wanted to focus on students' reasoning and how that was being constructed. This paper focuses on data from two tasks. The first task directed students to graph $y = 2$ in R^2 which was the intent. The second task directed students to graph $y = 3$ on R^3 axes. After students read the task, I showed them an image of xyz axes and explained that the xy plane was flat with the z-axis perpendicular to it. I used a tabletop (xy plane) and a pen (z-axis) to show students what these axes looked like in 3D[1]. I asked follow-up questions such as "why did you draw a [line, plane] here?"

[1] It is important to note that such a demonstration does not guarantee that students understood how such a coordinate system works; in fact, researchers have found that students often must

I chose to focus on these problems because of reported difficulty students experience with multivariable functions' graphs, and also because fundamental planes can help students complete graphing and other tasks in calculus. That is, thinking of some $f(x, y)$ equation for $x = c$ or $y = c$ results in a cross-section that can then be used to help visualise the function's shape. Parts of fundamental planes also often form the boundaries of solids for multiple integration. These examples provide two good reasons to explore how students might think about equations of the form $y = c$ in R^3. Finally, Gaisman and Martinez-Planell (2011) point out that graphs of fundamental planes are the first R^3 graph students meet in the widely-used Stewart (2006) text. Hence this seemed an appropriate task for students graphing in R^3 for the first time.

I recorded the interviews with audio, video, and LiveScribe technology. I transcribed the interviews and used the transcripts in data analysis. I used the constant comparative method (Strauss & Corbin, 1998) for data analysis, reading the transcripts to identify students' answers to the task (that is, what did they draw for graphs?) and then re-reading to identify how students arrived at their answers.

I observed that some students had drawn $y = 3$ in R^3 as a line, others had drawn it as a plane, and two drew a line but then thought the graph might be a plane. I hence coded students' work as belonging to one of two categories: plane or line. Students' reasoning involved words/phrases like parallel, equidistant, "all x points," "z can be any value," "x can be any value," "I don't think that x and z really have like any effect", all values of x and z, variables as "not mattering," x and z being "any value," and "no matter what x or z is." I also observed students considering about specific points, such as "if you say $x = 2$ and $z = 2$, it's going to be 3." I noticed that these utterances fit into three broader, non-mutually-exclusive categories: reasoning using equidistance and parallelism, reasoning about x and z, and considering specific (x, y), (y, z), and (x, y, z) tuples.

Finally, I looked for patterns in how students had thought about the graphs, and whether their graphs were planes or lines. That is, I looked specifically to see if there were something common to all of the students who drew the graph as a plane, and all the students who drew the graph as a line. I noticed that the difference between the graphs seemed related to whether or not the students explicitly attended to z as a free variable.

Results

Of the 11 students, three drew $y = 3$ in R^3 as a line and 8 drew a plane. Figure 1 groups students' work first by answer, then how they had arrived at that answer. For instance, four of 11 students drew a plane by thinking about x and z as free variables. One student thought about parallelism and drew a line, and one student thought about parallelism and equidistance and drew a plane. The plane and line categories are mutually exclusive. The subcategories are not. The (n/11) parentheses next to each subcategory in the diagram represents the main way students reasoned. In some cases, students reasoned in multiple was (e.g., S3 both thought about particular coordinate points and free variables, but is in the coordinate point category because, in my opinion, this played a bigger role in her reasoning than did the idea of free variables).

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develop a schema for R^3 through specific actions like plotting (x, y, z)-tuples and working with fundamental planes (Martinez-Planell, & Trigueros, 2012; Trigueros & Martínez-Planell, 2010).

I present the results by giving examples of students in each category (parallelism, *x* and *z* as not changing, tuples, and free variables), focusing on what they attended to and how they generalised.

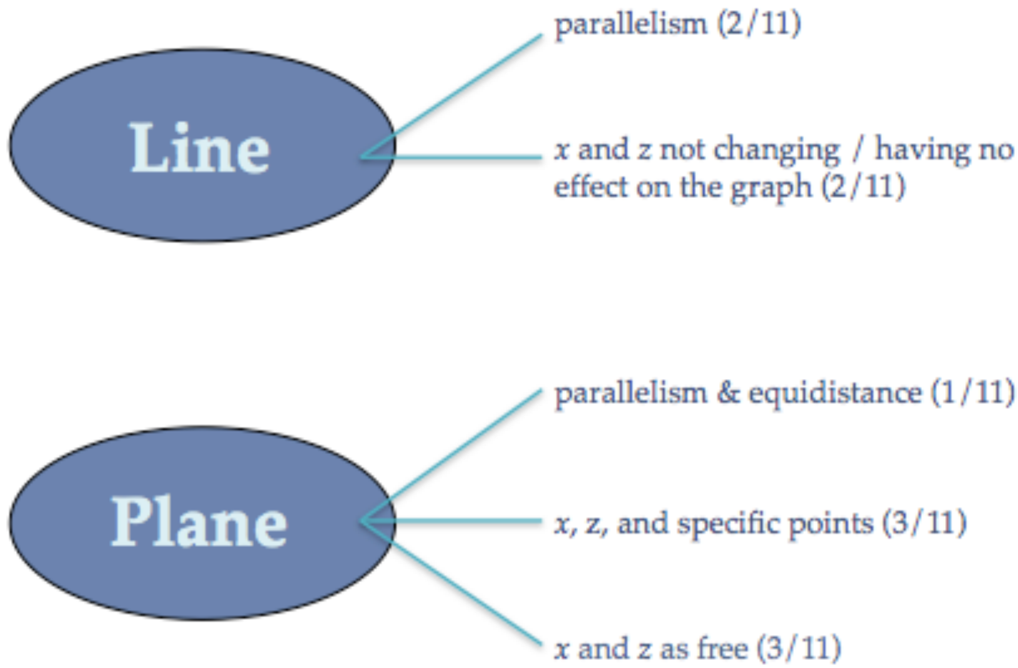

Figure 1. Results

Attending to and generalising parallelism

S6, S9, and S8 both generalised parallelism, but S6 and S9 drew lines while S8 drew a plane:

S6 So $y = 3$. So on an *xy* graph [draws R^2 axes] at 3, would be going this way. So on the *y*, following the *x*. So [switches to R^3 axes] this would be on the *y*, this is the 3 point on the *y*, and it's following the *x* axis [Figure 2].

S9 So if this is, if this is *y* is equal to 3, then I'm wondering if it will just go parallel with the *x* axis…but just at the *y* is equal to 3 point… just kind of like looking here [$y = 2$] it's like parallel to the *x* axis at whatever *y* point because it's just that straight line… I'd assume just for all *x* points that you know *y* would equal to 3. [I think the graph is a line] just cuz … your *y* is only 3. It's not like interacting with, with anything else.

S8 $y = 3$ would be something like this, where this distance right here between each, between *y* and each of these axes would be 3, I think. I'm thinking that because if you take like this thing, and that would be everything except for *y* [shades *xz* plane]... I'm thinking of this plane in relation to *y* and having *y* be every distance that is 3 away from that plane... it would be an entire plane... it has to be parallel to *x*, and this has to be parallel to *z*, so it would be this plane right here that is 3 away from the plane that *x* and *z* creates... like for the last question when *y* is equal to 2, that is every value that is 2 away from $y = 0$, right? So I'm thinking that like $y = 0$ would be the same as this [xz plane; Figure 3]. So it's 3, it's 3 in the positive [y] direction it's going to be parallel to x in the same way that this line right here [draws $y = 2$ in R^2] is parallel …to the x axis.

S6 began by drawing R^2 axes and graphing $y = 3$, describing the line as "following the x." He then drew R^3 axes and said "this is the 3 point on the y, and it's following the x axis." I infer his use of the word 'following' as attending to the parallelism between the x axis and $y = 3$. Moreover, S6 generalised what $y = 3$ in R^3 would look like by thinking about what it looked like in R^2, then drawing the same picture in a new orientation on the R^3 axes (Figure 2). This is an expansive generalisation because S6 expanded his existing schema to a new context without reconstructing it. In particular, he generalised that $y = 3$ in R^3 would be parallel to the x axis, just like in R^2, and he generalised that it would be a line. S9 made a similar generalisation, explicitly generalising between tasks when he said "I'm wondering if it will just go parallel with the x axis…but just at the y is equal to 3 point… just kind of like looking here [$y = 2$] it's like parallel to the x axis at whatever y point". This is an expansive generalisation because he applied his schema for $y = b$ equations as parallel to the x axis to the new R^3 context.

S8 also generalised parallelism. His reference to the R^2 task, "it's going to be parallel to x in the same way that this line right here [draws $y = 2$ in R^2] is parallel …to the x axis" is evidence that saw the two contexts as similar and generalised between them. In contrast to S6, however, S8 generalised by taking properties he knew to be true in R^2 (equidistance; parallelism) and reconstructed them for R^3. He first noted that $y = 3$ would be parallel to the x and z axes. He then shaded the xz plane and described $y = 3$ as always being "3 away from that plane." I take his reference to $y = 2$ as "every value that is 2 away from the x axis" as evidence that he generalised the equidistance property from R^2 to R^3. Because he talked about $y = 3$ as being three away from the xy plane, I consider this a reconstructive generalisation. S8 had to modify his equidistance and parallelism schema from the equidistance and parallelism of two lines to the equidistance and parallelism from two planes. There is evidence that he engaged in modifying these ideas in his statement "this distance right here between each, between y and each of these axes would be 3, I think." S8 first thought about equidistance and parallelism between $y = 3$ and *axes*. He then thought about where $y = 0$ would be in R^3, and having "y be every distance that is 3 away from that plane... it would be an entire plane... it has to be parallel to x, and this has to be parallel to z". That is, S8 modified his equidistance and parallelism schema to thinking about the equidistance and parallelism between two planes.

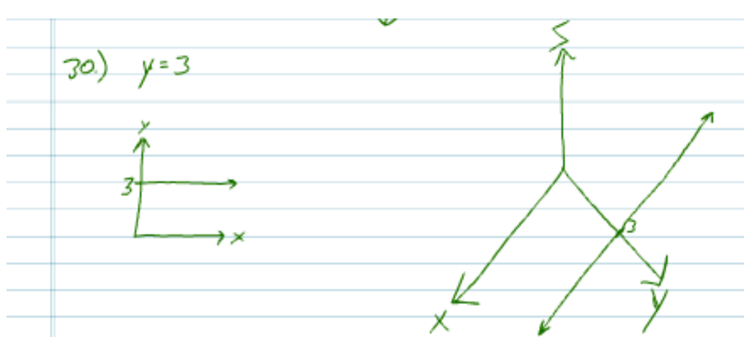

Figure 2. S6's graph of $y = 3$

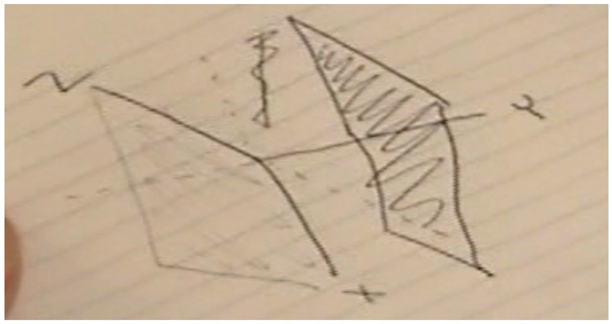

Figure 3. S8's graph of $y = 3$

I believe that although both students generalised parallelism, S6 drew a line because expansive generalisation did not force him to attend to z. Expansive generalisations, however, do not always result in non-normatively correct answers. The next section provides an example of how S3 and S7 engaged in expansive generalisations that supported their drawing planes. I say more about normatively correct and non-normatively correct expansive generalisations in the discussion section.

Attending to and generalising *x* and *z* as not changing

S1 and S11 drew lines for $y = 3$ in R^3, reasoning that x and z did not effect the graph. This was generalised from how they thought about $y = 2$:

S1 [$y = 2$] So, x and y, and at this point y will equal 2, and the graph would just go through 2 all the way, for all values of x.

S1 [$y = 3$] So the z direction, the z can't change and the x can't change… z is the vertical here… I guess I would just use this line for y always equal to 3.

S11 [$y = 2$] So this would be x and this is $y = 2$, and since it's just 2 all around, and it would go that way too [draws $y = 2$ for the negative x values]. Because usually like with the, it's just kind of giving you like a number I guess, so like no matter like what number like input, there's not really like, there's no x, so whatever number it is it doesn't matter, that the output is always going to be 2.

S11 [$y = 3$] It stays on like that like xy plane because there's no like z and there's no x… no matter like what the other ones are, it's just going to be that one number, which is like a straight line across the thing… I don't think that x and z really have like any effect to $y = 3$.

S1 talked about y equaling two "for all values of x" in the R^2 task but said "x can't change" in the R^3 task. From an observer-oriented perspective, we might have expected her to say that y equaled three for all x in R^3. Her statement that x could not change hence seems like the opposite. I wonder if from an actor-oriented perspective, both phrases 'for all x' and 'x can't change' mean the same thing to S1. This seems reasonable given that S11 did the same thing when she said "no matter what the input" and "there's no x" in the same sentence. I infer that S1 generalised that x did not affect the graph of $y = 2$ in R^2, and that x and z would not affect the graph of $y = 3$ in R^3.

S11 also generalised the idea of x having no effect to the idea of x and z having no effect. The evidence for this is that she used the same phrases in both problems: "no matter like what number the input…there's no x… the output is always going to be 2" to "there's no like z and no x… no matter like what the other ones are". I consider both students to have engaged in expansive generalisation because they applied the idea that a variable was not present in the equation, and hence did not matter, to the new context.

Attending to and generalising coordinate tuples

S7 and S3 and graphed $y = 3$ as a plane. Both thought about particular coordinate points, though S7 thought about a three-tuple and S3 thought about (x, y) and (y, z) pairs:

S7 So here's 3, we've got x, or the y, but then it's also going to be for all the z values, since z evaluated at any point will be $y = 3$. So I guess it would come out to be a plane… I kind of just thought since x evaluated[2] at any point on the graph equals 3, since the function is basically saying all, it's saying $y = 3$ at all points on the graph, any point you evaluate, so if you say $z = 2$ and $x = 2$, it's going to be 3.

S3 [$y = 2$] Whatever you plug in for x, at all, ever [makes dashes along the x-axis], y is just going to equal 2. So it's a horizontal line.

[$y = 3$] No matter what x or z is, y is always going to equal 3… I want to draw a line like this [indicates a line following the gridline for $y = 3$ on the xy plane] and a line like that [indicates a gridline parallel to the z axis and going through (0, 3, 0); Figure 4]. So what it's saying here is if x were 1, y equals 3 [Figure 5], or if $x = 2$, $y = 3$. And here too if z were 1, y is always going to equal 3 here [Figure 6]... I guess I drew a plane... [a plane makes sense] when it is drawn out like that.

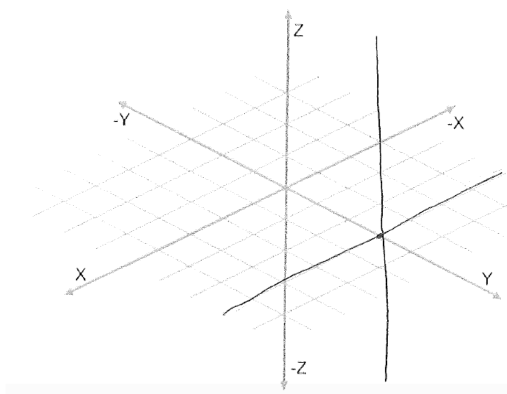

Figure 4. S3's plane

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[2] I believe that S7 meant to say 'y evaluated at 3', particularly because she followed this phrase with 'it's saying $y = 3$ at all points on the graph' and later used the example of $x = 2$. Consequently, I have analysed this excerpt assuming that she did indeed mean to say y.

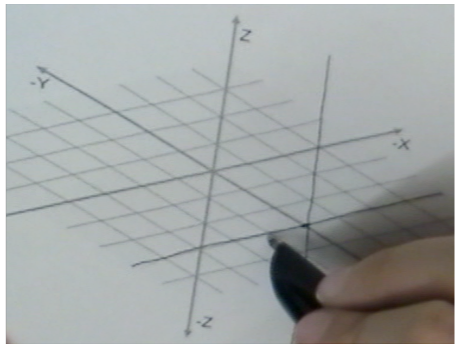

Figure 5. S3 considering (x, y) = (1, 3)

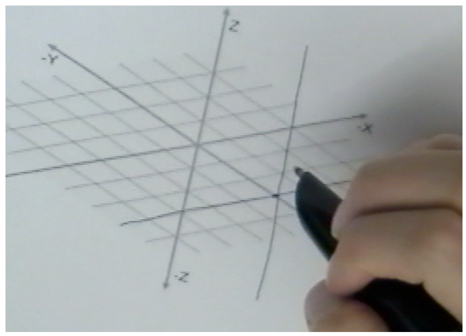

Figure 6. S3 considering (y, z) = (1, 3)

Both S7 and S3 generalised that if a coordinate point satisfies a particular condition (in this case, y = 3), it lies on the graph. S7 identified this condition ("it's saying y = 3 at all points on the graph") and then explained that this meant (2, 3, 2) was on the graph because y equaled 3 in that tuple. S3 identified the condition that "y is always going to equal 3," and used this to draw two perpendicular lines that intersected at (0, 3, 0). She then considered the points *(x, y)* = (1, 3), *(x, y)* = (2, 3), and *(y, z)* = (1, 3) and said "I guess I drew a plane". I interpret the phrase "I guess" as an in-the-moment realisation that the graph was a plane, afforded by considering particular points. I emphasise that S3's gestures (Figures 5 and 6) seemed to play a role in her thinking. Moving the pen from (0, 3, 0) to (1, 3, 0) and then to (2, 3, 0), and then from (0, 3, 0) to (0, 3, 1) seemed to help her see that the graph was a plane. S3 did not gesture toward particular points, but like S3, she said "I guess" the graph is a plane, and explained that she thought it was so because any point in which y = 3, such as (2, 3, 2), would be on the graph.

For both students, considering specific points was an expansive generalisation of the scheme 'If a coordinate point satisfies a particular condition (in this case, y = 3), it lies on the graph'. This is expansive because the students implemented a scheme that they likely had from experience with R^2 graphs, and they did not have to modify the scheme for R^3.

S3's particular points contain examples of both expansive and reconstructive generalisation. She considered two variables at a time, which I infer she generalised from being accustomed to considering *(x, y)*-tuples in R^2. Because she did not consider an *(x, y, z)*-tuple (which would be a reconstruction), I considered this an expansive generalisation. S3's use of a *(z, y)*-tuple, however, is a reconstructive generalisation because she expanded the notion of using *(x, y)*-tuples to using tuples of other variables.

S3 also generalised the notion of a free variable. I detail this generalisation in the next section, along with examples of other students who made similar generalisations.

Attending to and generalising free variables

Three students determined that the graph was a plane by focusing on x and z as able to take on any value. These students differ from the students who thought of x and z as "not changing" because the students here thought of x and z as free, and this afforded their graphing planes.

For example, in the $y = 3$ task, S3 said, "no matter what x or z is, y is always going to equal 3." I interpreted these statements as generalising the idea of a free variable because it seemed similar to her comment in the R^2 task that "whatever you plug in for x… y is just going to equal 2." S3's generalisation of the notion of a free variable seemed to be of the reconstructive variety because she realised that both x and z were free. S12 also appeared to engage in reconstructive generalisations:

S12 So maybe, maybe it would be like this [draws and shades plane]. Well this would kind of just be like just a flat sheet of paper on the $y = 3$, because all x values are 3, and then I guess you assume that all z values, since it's only, the only variable in the equation is $y = 3$, then it would have to be $y = 3$ for all x and z values. It's kind of just like a, I think it's supposed to be like a flat sheet kind of, like a piece of paper, and it's on $y = 3$, so it's supposed to encompass all the x values for negative and positive, and all the z values for z, positive z and negative z. They're all on $y = 3$... Well, I just thought like since $y = 2$ it should be like this, so if it's $y = 3$ it's like that, like all x values are $y = 3$. And z is going this way, so it must be, since there's no z in the equation, then it must be covering all this area.

I believe that the fact that S12 talked about x and then talked about z provides evidence that he engaged in reconstructive generalisations. S12 said "all the x values are 3" (from which I interpreted him as meaning 'for all the x values, y equals 3') followed by "I guess you assume that all z values since… the only variable in the equation is $y = 3$, then it would have to be $y = 3$ for all x and z values." He seemed to generalise that $y = b$ in R^2 would mean $y = b$ for all x, and hence $y = b$ in R^2 would mean $y = b$ for all x and z. The reconstruction is $y = b$ for all z. I take the phrase "I guess" here to indicate changing something in his schema.

Discussion

I observed students engage in expansive and reconstructive generalisations, but not in disjunctive generalisations. The particular generalisations are summarised in Table 2. This table also shows what students attend to when they first think about first graphing $y = 3$ in R^3: they attend to parallelism, equidistance, particular coordinate points, and free variables.

Table 2. Summary of generalisations

Student(s)	Generalisation	Type
S6	$y = 3$ is a line in R^3 just like it is in R^2	Expansive
S6, S8, S9	$y = 3$ in R^3 is parallel to the x axis	Expansive
S8	$y = 3$ in R^3 is parallel to the x axis and the z axis	Reconstructive
S8	$y = 3$ in R^3 is equidistant from the xy plane, just like $y = 2$ in R^2 is equidistant from the x axis	Reconstructive
S1, S11	x has no effect on $y = 2$ in R^2, so x and z have no effect on $y = 3$ in R^3	Expansive
S3, S12	Notion of a free variable: x is free in $y = 2$ in R^2; x and z are free in $y = 3$ in R^3	Reconstructive

S3, S7	If a coordinate point satisfies a particular condition (in this case, $y = 3$), it lies on the graph	Expansive
S3	A coordinate point has two elements	Expansive
S3	The elements of a coordinate point need not always be *(x, y)*	Reconstructive

Trigueros and Martinez-Planell (2010) and Kabael (2011) found students' difficulties with multivariable functions are often related to impoverished schemas for three-space. Our findings lend insight into how students begin to develop an R^3 schema. One way students construct an R^3 schema seems to be treating it like R^2. S5, who drew $y = 3$ in R^2 and then copied that image to the R^3 axes, provides an example of this, as do the students who draw $f(x,y) = x^2$ as a parabola (Martinez-Planell & Gaisman, 2013). Students seem to treat R^3 like R^2 in other function topics as well. Dorko and Weber's (2013) observation of students who gave the domain and range of $f(x, y)$ as x and y (respectively) provides a good example of this. Such treatment of R^3 may be a temporary stage for students. For instance, Yerushalmy's (1997) seventh-graders, engaged in their first multivariable graphing task and not shown R^3 axes like the students in this study, initially drew two graphs, one of independent quantity 1 vs. dependent quantity and one of independent quantity 2 vs. dependent quantity. The students struggled with, but eventually found, a way to draw a single graph. In this study, S3 also considered quantities two at a time. More research is needed to see if S3, S5, the student from Martinez-Planell and Trigueros' (2013) study, and others initially treat R^3 like R^2 but move out of this stage like Yerushalmy's students did.

Instructional Implications

My findings indicate that instructors can help students graph correctly by teaching them to attend to z. Instructors could begin by proposing that students think of a graph as the set of all points that satisfy a particular relationship. This might lead to teaching students to consider specific (x, y, z)-tuples and whether or not those tuples satisfy the given relationship. An instructor could ask a student to give a particular tuple that was on the graph and explain why that particular point is part of the graph, or could ask a student "is (1,3,7) on the graph? Why or why not?"

Instructors should also emphasise that variables vary. I say this because some of the students in this study (S1, S11) thought of x in $y = 2$ (in R^2) as not changing, which led them to conclude that for $y = 3$, x and z 'did not matter'. If we remind students that variables vary, they might consider how z varies in $y = 3$. One way we could reinforce this might be to write free variables as the variable times 0. For instance, perhaps writing $f(x, y) = x^2$ as $f(x, y) = x^2 + 0y$ would help students attend to y as free.

Finally, I suggest instructors use the ideas of equidistance and parallelism to help students think about fundamental planes. This was a powerful strategy for S8, and equidistance and parallelism are ideas that should be familiar to students.

Theoretical contributions

I wish to offer three comments about the expansive, reconstructive, and disjunctive generalisation categories. These are (1) reconstructive generalisations as first requiring expansive generalisation; (2) the idea of reconstruction as explanatory for why multivariable calculus topics are so difficult for students; and (3) expansive and reconstructive generalisations may be normatively correct or non-normatively correct, but in either case the Harel and Tall's (1991) framework helps us see what students are attending to.

I believe that any reconstructive generalisation involves first an expansive generalisation in which the student tries to apply an existing schema to a new context, followed by a reconstruction of the schema to account for differences between the two contexts. The theoretical support for this is the definition of reconstructive generalisation as “ocurr[ing] when the subject reconstructs an existing schema in order to widen its applicability range” (Harel & Tall, 1991, p.1) and expansive generalisation as “occur[ing] when the subject expands the applicability range of an existing schema without reconstructing it” (Harel & Tall, 1991, p.1). That is, if a student is going to reconstruct a schema, she must first see it as potentially applicable to the context. Hence she expands it to the context, then reconstructs as needed. One might envision Harel and Tall’s categories like shown in Figure 7, in which expansive generalisation can occur in its own right, and as part of reconstructive generalisation.

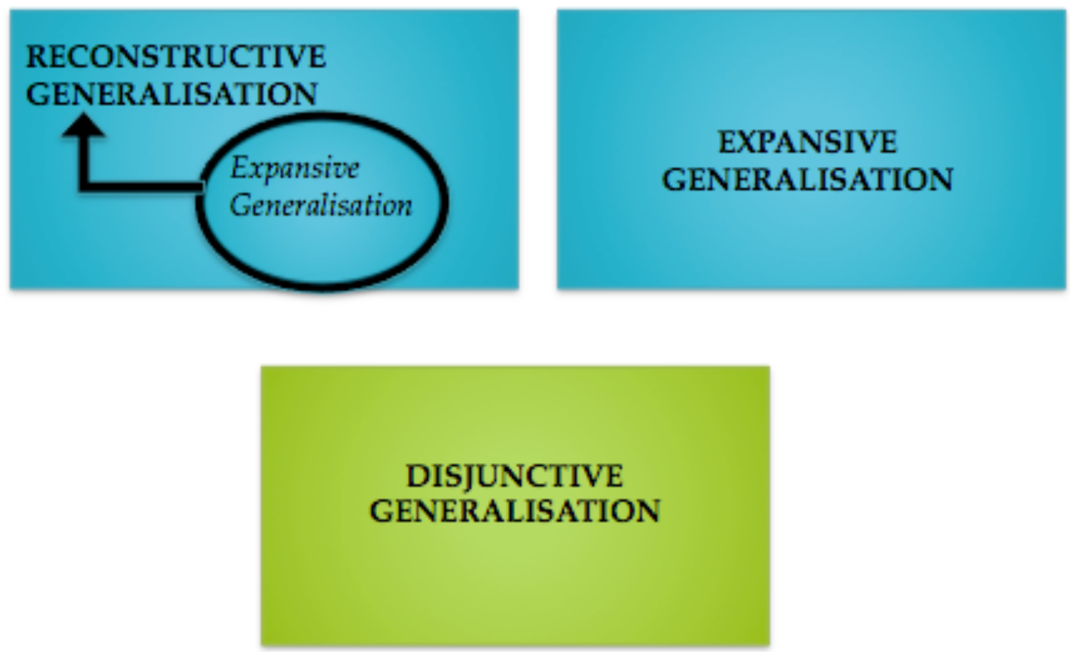

Figure 7. Proposed reconceptualisation of Harel and Tall’s framework

S8’s thinking provides an example for empirical support for expansive generalisation as a necessary condition for reconstructive generalisation. S8 explained his plane as three units away from the *xz* plane, “like for the last question when y is equal to 2, that is every value that is two units away from $y = 0$, right? So I'm thinking that like $y = 0$ would be the same as this [*xz* plane; Figure 3]”. He expanded the idea of y = b as equidistant from $y = 0$ to R^3, reconceptualising $y = 0$ from a line in R^2 to the *xz* plane in R^3. The reconstruction could not have occurred in absence of the expansion. Hence I believe that reconstructive generalisation first involves expansive generalisation.

Secondly, I believe that thinking about expansive and reconstructive generalisation provides explanatory power for why students often struggle with multivariable topics. Understanding that multivariable functions depend on multiple quantities, that domain is a set of *(x,y)*-tuples instead of a set of numbers, that derivatives in R^3 depending on direction, and so on depends on students reconceptualising ideas. Moreover, students need to *realise* that they have to reconstruct their ideas. Expansive generalisation does not always allow for this (e.g., S6). Perhaps students who struggle with multivariable topics tend to engage primarily in expansive generalisation when they need to engage in reconstructive generalisation. More research is needed to determine if this is the case.

Lastly, although I believe many multivariable calculus topics require students to reconstruct their schemas for function and graphing, expansive generalisations may still be productive. They may be productive *for students* when students are able to expand overarching notions such as points being on a graph as long as they satisfy a particular relationship (e.g., S3) or the notion of function as input-output (Dorko & Weber, 2013; Kabael, 2011). These examples illustrate that a reconstructive generalisation is not necessarily better or more sophisticated than an expansive one.

Categorising generalisations as expansive can be productive *for researchers* because doing so highlights what students are attending to. For example, we see from Table 2 that S6 attended to the shape of the graph of $y = 2$ in R^2, and that S3 and S7 attended to specific points and the broader idea that if a point satisfies a particular relationship, it is on the graph. S6's generalisation was non-normatively correct, while S3 and S7 both gave correct responses. However, in both cases, these expansive generalisations lend insight toward what students attended to. Similarly, the reconstructive category allows us to identify what facets of ideas students reconstruct. In summary, the expansive and reconstructive categories provide a powerful way to explain and understand the two main ways students generalise from R^2 to R^3.

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Gender, switching, and student perceptions of Calculus I

We analyze survey data to explore how students' reported perceptions of their Calculus I experiences relate to their gender and persistence in calculus. We draw from student free-responses from several universities involved in a comprehensive US national study of Calculus I. We use social cognitive career theory to help inform a thematic analysis on the data. Our analyses indicate that there are significant differences in affective statements among four student groups determined by gender and persistence. Teacher behavior and teaching practices are highly correlated with affective statements, and thus may be very influential in the quality of students' Calculus I experiences.

Keywords: Calculus, gender, persistence, affect, thematic analysis, mixed methods

Stemming from national need to increase persistence in Science, Technology, Engineering, and Mathematics (STEM), Ellis, Fosdick, and Rasmussen conducted a study focused on student persistence in calculus and investigated factors which may impact the likelihood of a student switching out of a STEM major (2016). They identified a striking relationship between gender, switching, and mathematical confidence. Specifically, females were significantly more likely to decrease their intentions to take Calculus II after taking Calculus I. When given a list of potential reasons for not continuing, female students cited that they, "do not believe [they] understand the ideas of Calculus I well enough to take Calculus II," with significantly greater frequency than their male counterparts. However, there did not exist significant differences in the number of A's and B's between men and women. These results highlight the role that calculus is playing in students' decisions to leave their STEM pursuits, and may help to explain the larger issue of the STEM Gender Gap (Eagan, Lozano, Hurtado, & Case, 2013; Seymour & Hewitt, 1997). This work motivated us to delve more deeply into student reports of their experiences in Calculus I. Specifically, we examine the relationships between students' description of their experience in Calculus I, their gender, and their decisions to persist in calculus.

Educators have long been interested in identifying factors that may contribute to the disparity in gender representation in STEM (Fennema & Sherman, 1976 & 1978; Griffith, 2010; Good, Rattan, & Dweck, 2012; Ellis, Fosdick, & Rasmussen, 2015). While there is consistent evidence against gender-based differences in mathematical ability (Fennema & Sherman, 1978; Islam, & Al-Ghassani, 2015; Lindberg, Hyde, & Peterson, 2010), there are clear distinctions between men and women in their persistence in STEM fields (Cunnigham, Hover, & Sparks, 2015; Eagan et al., 2013), and their self-reports of success in these fields (Griffith, 2010; Good, Rattan, & Dweck, 2012). To better understand these differences, we draw on social cognitive career theory (Lent, Brown, & Hackett, 1994), with an emphasis on self-efficacy and affect (Phillip, 2007). Social cognitive career theory is a theoretical framework that maps a student's career/academic choice and integrates demographics as a key component impacting the learning experience through psychosocial processes that contribute to the development of career-related self-efficacy and outcome expectations. Lent, Brown, and Hackett (1994) hypothesize that gender (as well as race and ethnicity) differences in career interest, career goals, and actions arise largely through differential access to opportunities, supports, and socialization processes, rather than due to ability. More recently, researchers have begun to articulate factors related to persistence and the representation of females and other minorities in STEM majors (Ellis, Fosdick, & Rasmussen, 2016; Fennema & Sherman, 1976; Graham, Frederick, Byars-Winston, Hunter, & Handelsman, 2013; Griffith, 2010; Wolniak, Mayhew, & Engberg, 2012). Griffith found that certain environmental factors (such as the representation of females and minorities in graduate programs) can increase STEM participation and success by minorities (2010). Good, Rattan, and Dweck (2012) found that a sense of belonging was related to student persistence in math and that women who exhibited a fixed intelligence mindset coupled with gender stereotyping in the classroom experienced reduced sense of belonging (2012). Gender differences in confidence have also been identified as a possible factor to explain why women discontinue pursuing mathematics at a higher rate than men (Ellis, Fosdick, & Rasmussen, 2016; Fennema & Sherman, 1978).

Our research contributes to this literature by offering an inductive, qualitative analysis of student statements pertaining to their experiences in Calculus I. We draw on students' responses to an open-ended survey question from the *Characteristics of Successful Programs in College Calculus (CSPCC)* project. In this report we address the following research question: *How do student characterizations of their experience in Calculus I relate to student gender and persistence in calculus?*

Methods

This work is embedded within a larger project aimed at investigating college calculus at a national level – the CSPCC project. The first phase of this work involved a survey of "mainstream" Calculus I students from a stratified random sample of colleges and universities. Two surveys were sent to students at the beginning and the end of the fall term. On the beginning-of-term survey, students were asked questions related to their demographics, previous mathematical experiences, affect towards mathematics, and career plans. On the end-of-term survey, students were asked questions related to their experience in Calculus I, affect towards mathematics, and career plans, as well as the open-ended question: "Is there anything else you want to tell us about your experience in Calculus I?" We analyze students' responses to this question in this report. The surveys provide us with information to distinguish students based on gender and whether they continued in calculus. Continuation to Calculus II is a proxy of whether students will persist in STEM majors because it is required of most STEM majors. Students who indicated an intention to take Calculus II at the beginning-of-the-term survey and still reported this intention at the end-of-the-term survey were coded as *Persisters*. Students who first intended on taking Calculus II but reported differently on the end-of-the-term survey were coded as *Switchers.* Students were divided into four student groups, Male Persisters, Female Persisters, Male Switchers, and Female Switchers, based on reported gender and intention to take Calculus II.

In order to answer our research question, we sought to identify themes in the data and determine how those differ among the four student groups. There were 2,266 students who responded to the beginning-of-term and end-of-term surveys and were part of the analysis conducted by Ellis, Fosdick, and Rasmussen (2016). Of these students, there were 522 students who provided a response to the open-ended question. As shown in Table 1, the subset of 522 students was composed of more male Persisters than in the original data set (51.3% compared to 45.6%), less female Persisters (30.7% compared to 34.7%), slightly more male Persisters and slightly fewer female Switchers. Further, this subset of students reported slightly lower standardized test scores, the same level of Instructor Quality, and slightly higher levels of Student-Centered Practices as compared to the larger data set. *Instructor Quality* and *Student-Centered Practices* were variables ranging from 1-6, based on student reports of sixteen instructional practices and behaviors (see Tables S3 and S4 for detailed information on the derivation of these variables). Instructor Quality characterizes the level of conventional quality teaching, including availability outside of office hours, listening to questions, and encouraging students mathematically. Low values on this scale indicate low perceived instructional quality, and high values correspond to high instructional quality. Student-Centered Practices characterizes the frequency of classroom practices such as whole-class discussion, students giving presentations, and group work. Low values coincide with reported traditional, instructor-centered instructional practices, and high values correspond to more innovative, student-centered teaching.

Table 1. Comparison of larger data set to data set with open-ended responses

	Larger data set *(n=2588)*	Open-ended responses *(n=522)*
Number of Institutions	142	102
Number of Instructors	80	70
Student Group		
1 – Male Persister	1181 (45.6%)	268 (51.3%)
2 – Female Persister	897 (34.7%)	160 (30.7%)
3 – Male Switcher	203 (7.8%)	43 (8.2%)
4 – Female Switcher	307 (11.9%)	51 (9.8%)
Standardized Test average (percentile)	86.37	84.80
Instructor Quality	4.61	4.61
Student-Centered Practices	3.29	3.35

To characterize the ways these students discussed their experiences in Calculus I, and to relate these characterizations to student gender and persistence, we employed thematic analysis (Braun & Clarke, 2006). Social cognitive career theory guided this work, which emphasizes the role of demographics, self-efficacy, and learning experiences in a person's career decisions (Lent, Brown, & Hackett, 1994). During the thematic analysis we familiarized ourselves with the student responses, blind to the gender and persistence of the students, though aware of the literature related to the STEM gender gap and, more specifically, aware of the relationship in this data set between gender, reported mathematical confidence, and persistence in calculus. We took an inductive approach, deriving themes from the data, but we brought to bear our knowledge of the literature in organizing these themes. The two authors each coded subsets of 50 student responses to develop and refine codes. The final codes, reported in Table 2, were finalized after multiple iterations of comparing codes and once 85% reliability was consistently achieved between researchers. One researcher then coded all responses, with a small percentage of questionable responses coded by both researchers. We weighted the codes on a scale of -1 to 1 to indicate a negative, neutral, or positive connotation. We coded each student response with as many codes as appropriate. We did not repeat codes within a student comment unless the same code carried different weights (positive, negative, or neutral) in the same student response. For instance, "*I was pretty nervous about calculus because I was never strong at math … but so far it is going well*" was coded twice for affect since this comment includes affective statements with different weights. The student communicated negative feelings about self with regards to mathematical ability, but made a positive statement about the calculus course. The NA code was only used if the entire student response was not relevant directly to the student's Calculus I experience. Responses were regarded as not relevant if the student stated that they had no other comments about the course or if the statement was too vague to apply to the calculus course.

To frame this work we draw on literature surrounding self-efficacy and affect. We define and understand affect according to Phillip's summary of research done on mathematical belief and affect from the years 1992 to 2007. By consolidating definitions from research, Phillip defines affect as "a disposition or tendency or an emotion or feeling attached to an idea or object. Affect is comprised of emotions, attitudes, and beliefs" (Phillip, 2007, p. 259). In our examination of students' open-ended responses about Calculus I, we analyse students' reported affect. Based on Phillip's definition, nearly all student responses could be viewed as affective statements. Thus, we narrowed our use of the "Affect" code to only capture statements about a student's emotions, attitudes, or beliefs towards the calculus course, oneself as a learner, or mathematics in general. For instance, "*This professor is pretty good at explaining the concepts*," is an example of a response that was coded as being about the teacher but not as a report of the student's affect. By contrast, "*I feel that I am loving math because my professor loves to teach it. She makes class so much fun and she believes in us*," is an example of a response that was coded with both the "Teacher" and "Affect" codes. The former comment is merely an evaluation of the teacher, while the latter comment discusses the teacher as well as the student's attitude towards math.

Table 2. Codes, code descriptions, and examples.

Code	Includes statements about…
Affect	Student's emotions, attitudes, and beliefs about (a) the calculus course, (b) mathematics, (c) themselves as learners. *For the first time in my life I really struggled in a math class.*
Assignments and assessments	Assignments, and both formative and summative assessments. *The exams tended to be stressful/time consuming.*
Pacing	The pacing of the course in general and of class sessions. *The length of class … didn't really allow for … anything rather than the 'spewing' of material*
Preparation	Preparation coming into the course and preparation going into the next course. *Taking calculus in high school helped me succeed in this class!*
TA	The TA and his/her aspects such as communication, availability, helpfulness, etc. *The help desk hours with the T.A. were great.*
Teacher	The teacher and his/her aspects such as communication, availability, helpfulness, etc. *I thoroughly enjoyed my professors teaching style and presentation of material.*
Teaching	Specific teaching practices and teacher-controlled aspects of class room environment. *My instructor …had no passion for learning and lectured instead of taught.*
Other	Other reasons and resources that may have impacted the student's

	success *Without the math tutoring lab there is no way I would do well in this class or even pass.*
Not applicable	Anything irrelevant to the calculus course. *If a teacher truly loves the subject, students can tell and learn to love it too.*

Once the responses were coded, we investigated patterns within this data, drawing on descriptive and correlational analysis. These analyses provide a rich understanding of the overall aspects of student's Calculus I experience that they find important, and that may be related to their decisions to persist (or not) through the calculus sequence, and eventually their STEM career decisions. After identifying relationships between students' comments and their gender and/or persistence we took a deeper look into the qualitative nature of the comments that emerged as related to career decisions.

Results

To understand the relationships between students' responses, their gender, and their calculus persistence, we provide an overview of the distribution of the codes among the four categories of students in Figure 1. Of the 522 original student responses, 68 were coded as not applicable and were filtered out, leaving 454 relevant comments. Half of these comments came from Male Persisters, 9% from Male Switchers, 32% from Female Persisters, and 10% from Female Switchers. Among all students, the most frequent responses were related to Affect, the Teacher, Assignments and Assessments, and Preparation. However, the frequency of these responses within each student group varies; for instance, 37% of Male Persisters' responses were coded as Affect while 63% of Male Switchers' responses were coded this way. There are statistically significant differences among the distribution of the responses related to Affect, the Teacher, and Preparation.

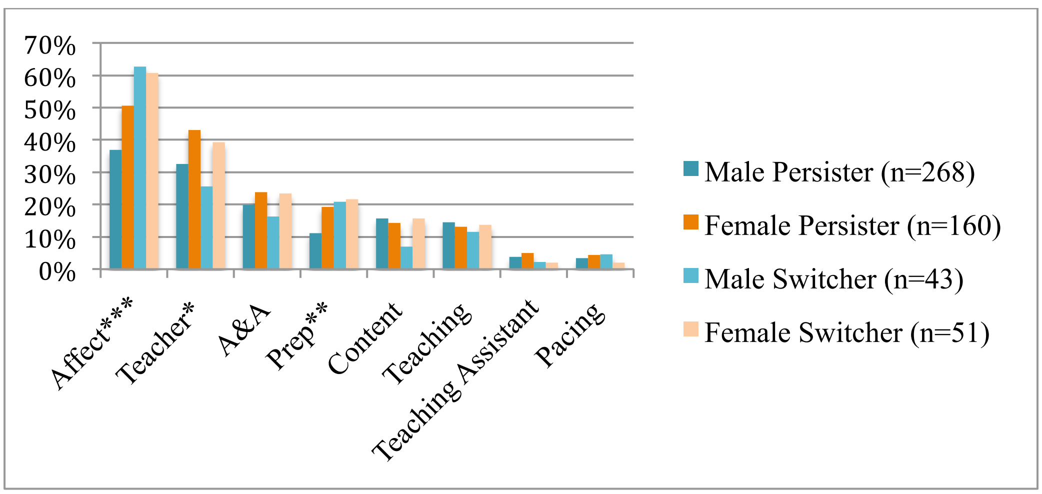

Figure 1. Prevalence of codes among four student groups

The most significant differences are in affective comments, where around 60% of both male and female Switchers' comments were coded as Affect, while these percentages are significantly lower for male and female Persisters. Much research has been done on mathematical affect and its role in student persistence (Ellis, Fosdick, & Rasmussen, 2016; Fennema & Sherman, 1978; Good, Rattan, & Dweck, 2012), with clear links identified between students' affect and their career decisions. To better understand the nature of students' affective statements, we conduct two secondary analyses: first, we compare the weight distribution of Affect statements across the four groups of students. As shown in Figure 2, the majority of male and female Persisters' Affect statements were positive, while male Switchers' Affect comments were nearly evenly split between positive, negative, and naturally, and the majority of female Switchers' Affect statements were negative. These findings harken back to decades of research on mathematical self-efficacy and gender. Bandura (1986) defined self-efficacy as "people's judgments of their capabilities to organize and execute courses of action required to attain designated types of performances" (Bandura, 1986, p. 391). In a meta-analysis of research on mathematical self-efficacy and gender, Pajares (2005) concludes that women consistently have lower mathematical self-efficacy compared to men at all ages. This literature may help explain the differences in the weights of Affect statements among the four student groups. Men's stronger mathematical self-efficacy may explain why the males, even those who choose to leave their STEM pursuits after Calculus I, tend to report more positive affective statements. Comparing the proportion of comments coded with positive Affect, the male Switchers in our data set report positive affect much more frequently than do female Switchers. Although our Affect code is not identical to self-efficacy as defined by Bandura, the two are certainly related. Thus, the gender disparity in mathematical self-efficacy may help explain the gender differences in the weight distribution of responses coded with Affect.

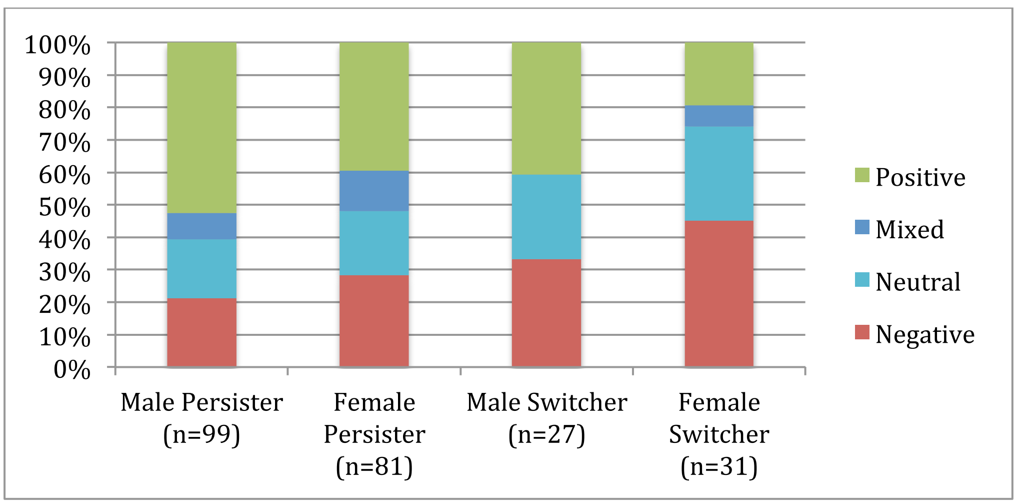

Figure 2. Weight distribution of Affect code

The second ancillary analysis we conduct to further understand the relationships between gender, STEM persistence, and students' calculus experience is a correlational analysis among the codes to understand what codes are most associated with Affect statements. As shown in Table 3, Affect statements are significantly correlated with Teacher statements [$r = 0.601$, $n = 85$, $p < .001$] and Teaching statements, [$r = 0.819$, $n = 31$, $p < .001$]. Thus, it seems that of the many aspects of the calculus learning experience, the instructors' behaviour and teaching approach can have the most impact on the students' emotions, attitudes, and beliefs about mathematics, the calculus course, and themselves as learners.

Table 3. Correlations between Affect code and other codes

	Aff	AA	C	P	Pr	TA	Teaching	Teacher	O
Pearson Correlation	1	.038	.284	-.232	-.090	-.577	.819**	.601**	-.092
Sig. (2-tailed)		.794	.115	.547	.557	.423	.000	.000	.640
N	238	49	32	9	45	4	31	85	28

This analysis indicates that the overlap between the codes Teacher and Teaching with Affect are especially important. Thus, we investigate these overlaps more deeply – first with Teacher and then with Teaching. We initially do this qualitatively and look into the nature of the comments in these overlaps. At this point in the analysis, the number of student responses under examination is small and so our findings cannot be taken as representative of a pattern in the larger data set. The purpose of this analysis is to understand the specifics aspects of the teacher or teaching that were related to students' affect. We then look at the prevalence of these overlaps between the four groups of students (again, here the numbers are quite small and thus cannot be representative of larger trends), and revisit the analysis conducted by Ellis, Fosdick, and Rasmussen (2016) to compare the trends in our smaller sample of data to the trends in the data set of 2,266 students.

Affect and Teacher

There were 85 student responses coded with both Affect and Teacher. Through thematic analysis, we refined the analysis further to uncover two subthemes related to affect and the teacher that were most prevalent in this data set: comments related to the teacher's approachability, and comments related to the teachers' communication. Comments in the approachability subtheme included comments about how the teacher's (lack of) approachability made students feel about themselves as learners, calculus, or mathematics in general. Comments in the communication subtheme included comments about how the teacher's communication, including both potential language barriers and their general communication style, made students feel about themselves as learners, calculus, or mathematics in general.

Table 4. Prevalence of Teacher and Affect subthemes among four student groups.

	Approachability	Communication
Male Persisters *(n=27)*	4	7
Male Switchers *(n=7)*	2	2
Female Persisters *(n=33)*	11	8
Female Switchers *(n=14)*	5	0

Ellis, Fosdick, and Rasmussen (2016) constructed an aggregate variable, called *Instructor Quality,* from end-of-term survey questions related to various aspects of the quality of the instructor, including questions related to approachability and communication. Here we extend their analysis to shed light on how instructor quality may be related to student gender and/ or persistence. As shown in Table 5, Persisters report significantly higher Instructor Quality than Switchers [$F(1, 4111) = 64.981$, $p < .001$], but there were not significant differences in reports of Instructor Quality between men and women. However, when we compare across the four groups it appears that the difference between female Persisters and Switchers is larger than that between male Persisters and Switchers.

Table 5. Comparison of means of Instructor Quality among student groups.

	Switcher Code***		Gender		Combined Student Groups***			
	Persisters	Switchers	Male	Female	Male Persister	Female Persister	Male Switcher	Female Switcher
Mean	4.66	4.33	4.66	4.64	4.69	4.72	4.47	4.36
n	3474	638	1468	1211	1284	930	184	281
Std. Dev.	.927	1.053	.896	.931	.882	.882	.962	1.034

Affect and Teaching

There were 31 student responses coded with both Affect and Teaching. These responses addressed the instructional approach, the technology used in the class, and the connection between material taught in class and material assessed.

Table 6. Prevalence of Teaching and Affect among four student groups.

	Teaching and Affect
Male Persisters	13
Male Switchers	3
Female Persisters	12
Female Switchers	3

Ellis, Fosdick, and Rasmussen (2016) constructed an aggregate variable, called *Student-Centered Instruction,* from end-of-term survey questions related to various aspects of the format of the instruction, including questions related to the level of student-centered instruction such as group work and whole class discussion versus lecture. Here, again, we extend their analysis to shed light on how instructional approach may be related to student gender and/ or persistence. As shown in Table 5, Persisters report slightly significantly higher Student-Centered Instruction than Switchers [F(1, 4117) = 3.702, p = .054], significantly higher report by men than women [F(1, 2682) = 3.918, p = .048]. However, when we compare across the four groups these differences diminish, though it does show that female Persisters report slightly lower levels of Student-Centered Instruction than the other student groups.

Table 7. Comparison of means of Student-Centered Instruction among student groups.

	Switcher Code*		Gender**		Combined Student Groups			
	Persisters	Switchers	Male	Female	MP	FP	MS	FS
Mean	3.31	3.22	3.32	3.24	3.33	3.23	3.30	3.30
n	3479	639	1471	1213	1287	932	184	281
Std. Dev.	1.078	1.096	1.036	1.086	1.035	1.084	1.046	1.094

Discussion

This work was motivated by work that clearly linked gender to persistence in calculus, with a lack of confidence in mathematical ability as a major contributing factor for women's decisions to leave calculus but not for men's. In this report, we further investigated aspects of male and female Calculus I students' reports of their experience in calculus to try to better understand the link between gender and persistence in calculus. Our analyses identified a number of aspects of the Calculus I experience as related to gender and persistence.

After coding 454 relevant student responses, we found that the differences in distribution among the four coding groups were statistically significant for responses coded with Affect, Teacher, and Preparation. In particular, the proportion of affective statements made by male and female Switchers was significantly greater than the proportion for Persisters. This indicates that students who did not continue to Calculus II more frequently made statements expressing their emotions, attitudes, or beliefs about the calculus course, math, or themselves as learners than did those who persisted with their intention to pursue a STEM major. It is possible that the students who left the calculus sequence made affective statements more often than Persisters because they may have had more impactful experiences that they felt should be reported.

There were also differences among the student groups in the distribution of weight of the Affect code. As would be expected, Persisters largely made affective statements which held positive connotation. The female Switchers' affective responses were primarily negative in connotation, which also would be expected. However, the weight distribution among the male Switchers' Affect comments was very even between positive, negative, and neutral statements. This disparity in the proportion of negative affective statements made among male and female Switchers may be explained in part by the literature surrounding mathematical self-efficacy. The literature indicates that women consistently have lower self-efficacy than men, where women tend to judge that they have less mathematical capability than do men. This could be linked to the pattern in our data set, that among students who no longer pursue a STEM major, men tend to make positive statements about the course, mathematics, or themselves much more often than do women. Self-efficacy related to STEM careers plays a large role in actually pursuing that career, which may explain the underrepresentation of women in STEM fields.

Student responses that were coded with Teacher or with Teaching were significantly correlated with responses coded with Affect. Thus, teachers' behaviors and practices may be a very impactful factor in shaping students' affect towards mathematics, the calculus course, and themselves as learners. We found that in a small subset of student responses coded with both Teacher and Affect, female students made proportionally more comments about teacher approachability than about teacher communication. Although this data set was too small to be representative of the entire body of student responses, it prompted us to extend the analyses of Ellis, Fosdick, and Rasmussen (2016) to explore differences among the four student groups in various aspects of teaching. Two aggregate variables we examined from this past study were Instructor Quality, which relates to our Teacher code, and Student-Centered-Instruction, which is relevant to our Teaching code.

When examining these two variables, we were working with a data set of 2,266 students who provided responses to the beginning and end-of-term survey. As would be expected, Persisters reported significantly higher Instructor Quality than did Switchers. No significant gender differences were seen with regards to this variable. When considering differences among the four student groups, there is a greater difference between female Persisters and Switchers than between male Persisters and Switchers, and female Switchers have the lowest number for

Instructor Quality among the four groups. This may indicate that female Switchers are having encounters with their teachers that are not as positive as for other students. This may be linked to our finding that the women in a small sample reported teacher approachability more often than men. Perhaps female Switchers are having more negative experiences with their calculus teacher, which may dissuade them from continuing to Calculus II. Again, the smaller difference between male Persisters and Switchers may indicate that men generally have better math self-efficacy and thus make similar reports regardless of switching.

There are statistically significant differences in Student-Centered Instruction, with Persisters reporting a higher score than Switchers, and with men reporting higher Student-Centered Instruction than women. The differences decrease across the four student groups, although female Persisters report a somewhat lower level of Student-Centered Instruction. It is expected that Persisters would report beneficial teaching practices more often than Switchers. Women do not report student-centered teaching practices as often as men, which may play a role in deterring women from taking more STEM courses. Although the differences are small among the student groups, it is interesting that female Persisters reported the lowest levels of Student-Centered Instruction. Perhaps a lack of student-centered practices is not enough to prevent some women from taking Calculus II. Yet, it is still possible that these students may switch out of STEM major later on in their educational career. These small differences may be influential over the course of time, and could be factors that contribute to the shortage of women in STEM careers. The Teacher and Teaching codes were highly correlated to Affect, and affect and self-efficacy are very influential factors in a student's pursuit of STEM and eventual career choice. Thus, teachers may help with student retention in STEM fields by striving to better their communication and perceived approachability and by shifting teaching practices to be more student-centered. These actions may improve student affect towards mathematics and STEM in general, and may also reduce the disparity in gender representation in STEM.

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Learning to think, talk, and act like an instructor: A framework for novice tertiary instructor teaching preparation programs

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In this report I present a framework to characterize novice tertiary instructor teaching preparation programs. This framework was developed through case study analyses of four graduate student teaching assistant professional development (GTA PD) programs at institutions identified as having more successful calculus programs compared to other institutions. The components of the framework are the structure of the program, the departmental and institutional culture and context that the program is situated within, and the types of knowledge and practices emphasized in the program. In this report I characterize one of the programs involved in the development of the framework as an example of how it is used. In addition to characterizing existing programs, this framework can be used to evaluate programs and aid in the development of new novice tertiary instructor teaching preparation programs.

Keywords: Graduate student teaching assistant, professional development, pedagogies of practice, mathematical knowledge for teaching, framework

Theoretically driven research centered on the teaching preparation of tertiary instructors pales in comparison to the research related to the professional development of K-12 teachers. While there are aspects of K-12 professional development (PD) programs that can be highly relevant and informative to the tertiary level, there are also many ways in which tertiary-level teaching preparation should be examined as its own field. In this paper, I articulate ways that K-12 PD literature can inform tertiary level teaching preparation and the components that are more idiosyncratic to the tertiary context. I then introduce a theoretical framework that draws on K-12 PD literature and responds to the particular needs at the tertiary level, and use this framework to characterize two novice instructor teaching preparation programs as an example of its use as a way to characterize programs as well as to compare programs.

The National Science Board (NSB) uses the term professional development to refer both to teacher preparations (i.e. the teaching of pre-service teachers, prospective teachers, and teacher candidates) and to the development of practicing teachers (i.e. in-service teachers and practicing teachers) (National Science Board 2012). Novice tertiary instructors, especially graduate students, have commonalities with both categories of teachers: the training they receive for these roles is typically the first training to teach they will have received, however often they receive a large portion of this training while they are teaching. For many practicing tertiary instructors, any professional development related to teaching they may have received as graduate students or post-doctoral fellow is likely to be their only formal training as mathematics educators, rather than as mathematics researchers, and can help enculturate graduate students into academia (Austin 2002). Thus, the literature on professional development programs designed both for pre-service and in-service teachers at the K-12 level is relevant to tertiary teaching preparation. While there is extensive research into the professional development of teachers at the K-12 level, there is substantially less literature focusing on tertiary instructor teaching preparation, especially research that is theoretically driven. A large portion of the studies focused on tertiary instructor teaching preparation report on the success of existing programs or needs (often unmet) of novice instructors (e.g. Hauk et al. 2009; Kung and Speer 2009; Speer, Gutmann, and Murphy 2005). However, the body of research that connects aspects of these programs to identify commonalities and key features to consider when creating a new program is lacking.

Ten years ago, Speer and her colleagues (2005) initiated the conversation among mathematics education researchers interested in novice tertiary instructor teaching preparation, calling attention to what we could learn from K-12 PD, and identified a number of research directions to pursue. Some of these directions have been pursued directly by Speer and others since this call, and as a result there are more productive models of novice tertiary instructor teaching preparation programs in existence. In this paper, I develop a theoretically driven model that connects such productive programs. This framework may be used to better understand (and make improvements to) existing programs as well as to influence the development of a new program geared at preparing novice tertiary instructors.

Background

As part of a large, national study focused on identifying elements present in successful calculus programs, *Characteristics of Successful Programs in College Calculus (CSPCC),* (Bressoud, Mesa, and Rasmussen 2015), I studied the novice instructor teaching preparation programs at four institutions with successful calculus programs where graduate students and post-doctoral fellows were involved in the teaching of calculus. Through analyses of survey data, the project team identified institutions that were more successful than comparable institutions, where success was viewed as a combination of retaining students' positive dispositions towards mathematics, retaining students' intentions to take Calculus II, and having a reasonable pass rate. We then conducted case studies (Stake 1995) at these institutions to learn what they were doing in calculus that may be contributing to students' success, and how this success could be translated to other institutions. Robust novice instructor teaching preparation programs were one such element, and were then studied in depth in the national sample and at the case study institutions (Author). It is important to emphasize that the novice instructor teaching preparation programs themselves were not identified as successful; rather, these programs existed at institutions (a) whose calculus programs were identified as successful, and (b) where novice instructors were responsible for a considerable amount of calculus instruction.

Methods

Analyses of the case study data at the selected institutions led to development of the framework for novice instructor teaching preparation programs that I introduce in this paper. While I primarily attended to the ways in which these institutions prepared graduate students in their roles as instructors, these programs can be informative for preparing other novice tertiary instructors, such as post-doctoral fellows, lecturers, and new tenure-track faculty. As part of the CSPCC study, an abundance of data was collected surrounding each of four PhD-granting institution's novice tertiary instructor teaching preparation programs. At each institution, the data set includes the collection of all documents related to the novice instructor teaching preparation programs (e.g. handbooks for graduate students and post-doctoral fellows, observation protocol for observations, etc.), observations of the training when possible, observations of instructor meetings, observations of novice instructors teaching and leading recitation sections, and interviews with novice instructors, administrators, teaching preparation program facilitators, and students. The purpose of this data collection was to gain an in depth understanding of the ways that novice instructors (predominantly graduate student teaching assistants) were prepared and supported for their roles in the calculus programs at each of the institutions selected as having a more successful calculus program. The goal of this data collection was not to evaluate the novice instructor teaching preparation, but rather to understand the relationship between the novice instructor teaching preparation and the success of the calculus program at each institution.

I analyzed the data with an eye towards understanding the components of the teaching preparation program at each institution, and to understand how this related to the success of the calculus program. To do so, I conducted inductive thematic analysis of the case study data. Thematic analysis involves "identifying, analyzing and reporting patterns (themes) within data. It minimally organizes and describes your data set in (rich) detail. However, frequently it goes further than this, and interprets various aspects of the research topic" (Braun and Clarke 2006, p.79). Inductive thematic analysis is a bottom-up approach, where the themes are data-driven, though may also be informed by research literature. This process involves first becoming familiar with the data, then generating initial codes, searching for themes, reviewing these themes, defining and naming these themes, and lastly producing the report. The themes that were developed through this process both emerged from the data set and were influenced by the literature. For each theme, I used multiple components from each data set to complement and triangulate the information obtained through the interviews, specifically to fact check information and to add details when needed.

After identifying the main themes related to the novice instructor teaching preparation programs, I revisited the data to identify the ways that these themes were related to one another and how they were related to the success of the calculus programs. Together, the themes developed through the inductive thematic analysis and the relationships between them comprise the framework for novice tertiary teaching preparation.

Components of framework

The central dimension of the framework is the *structure* of a program; when it occurs, for how long, who participates, what is discussed, and how. The structure of a program includes objective information about the formal and informal structural components of the novice instructor teaching preparation programs program. This includes the five components identified by Belnap and Allred (2009): (a) timing, (b) frequency, (c) duration, (d) topics covered, and (e) overall design. The structure of a teaching preparation program is the aspect that is typically used to characterize a program, much like the specifications of a house (number of bedrooms and bathrooms, square footage, architectural design, etc.) are typically used to characterize it. However, like these specifications are shaped and constrained by the environment in which one builds a house (including the lot size, zoning laws, and builder and/or designers' preferences), the structure of a teaching preparation program is constrained, determined, and enabled by the environment within which it is situated: the institution and the department. The *institutional and departmental context* and *culture* together comprise the environment within which the teaching preparation program exists. The institutional and departmental context guides the needs and capabilities of a teaching preparation program. For instance, the responsibilities of novice instructors are determined by (a) the number of graduate students, post-docs, and other novice instructors in the department in relation to the number of other faculty and in relation to the number of undergraduates served by the department, (b) the types of classrooms available (large lecture halls versus small classrooms), and other components of the context of the institution and department. The institutional and departmental culture shapes how the department responds to these needs and capabilities. For instance, whether graduate students serve as discussion section/recitation leaders or course instructors will be shaped by (a) the institution and departments' views on class size, (b) their orientation toward optimal learning environments, (c) their aspirations for undergraduate instruction, and other components of the culture of the institution and department.

Within the structure of the program, different *knowledge* and *practices* are emphasized and in different ways. Once a structure has been developed, various knowledge and practices can be emphasized and fostered in different ways. As part of becoming an instructor, one develops knowledge and practices surrounding instruction. Thus, the tertiary teaching preparation programs emphasize different types of knowledge and practices depending on the community within than institution.

To characterize the types of knowledge needed to teach I draw on the classic distinction by Shulman (1986), who differentiated between pedagogical knowledge (PK), content knowledge (CK), and pedagogical content knowledge (PCK). Pedagogical content knowledge is distinct from a blend of basic pedagogical knowledge and basic content knowledge and was introduced by Shulman in response to the wide-held belief that content knowledge alone was sufficient to teach. PCK is the particular form of content knowledge related to the aspects of content knowledge "most germane to its teachability", including ways of representing content so that it is understandable to others (Shulman 1986, p. 9).

Each of the three types of knowledge can be emphasized to varying degrees though a teaching preparation program. To illustrate the level of emphasis I use shading in the visual representation of the framework. No shading illustrates that this type of knowledge was not emphasized at all during the program. Light shading represents that this type of knowledge was emphasized to a small degree, but developing this type of knowledge was not the main focus of the teaching preparation program. Dark shading represents that developing this type of knowledge was the main focus of the preparation program. For instance, consider teaching preparation programs aimed at mathematics graduate student teaching assistants (GTAs) that does not provide any opportunities for participants to increase their content knowledge related to the area they are going to be teaching, is heavily focused on developing basic pedagogical knowledge, such as how to write on the board clearly, how to organize a lecture, and how to prepare a syllabus and an exam, talks briefly about common students misunderstandings in calculus and how to present ideas to help students grapple with these misunderstandings. Such a program would be characterized with no shading for content knowledge, dark shading for pedagogical knowledge, and light shading for pedagogical knowledge, as shown in Figure 1.

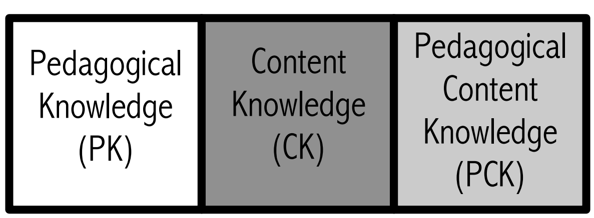

Fig. 1 Shading of three types of knowledge to represent three levels of emphasis of each type of knowledge through the teaching preparation program

To characterize the practices graduate students can legitimately and peripherally engage in as they learn how to be tertiary instructors, I draw on Grossman et al.s' (2009) *pedagogies of practice.* Grossman and her colleagues (2009) identified three concepts for describing ways to teach practices in professional education: representations of practice, decompositions of practice, and approximations of practice. *Representations of practice* comprise different ways practice can be represented for novices. In teacher education, one may represent the practices of teaching through written case studies, Video cases, photographs of the classroom, narratives, lesson plans, technological reproductions, among many others. The authors note that "the nature of the representation determines to a large extent the visibility of certain facets of practice" (p. 2066) and thus different representations of the same practices have different affordances for the learner. *Decompositions of practice* break down a complex practice into its multiple parts, which has affordances as well as limitations. By decomposing a practice, it may remove the practice from the actual context within which it is situated (for an elaboration on this point see Putnam and Borko 2004) however it also enables the novice to focus on specific aspects of a practice without the complications of the actual context. *Approximations of practice* are activities that allow novices to engage in legitimate practices of a community in a peripheral way, meaning that they are "more or less proximal to the practices of a profession." (p. 2058) These approximations may take the learner directly to the practice, as is done during student teaching, or bring the practice to the learner through various representations, such as Video cases or role-playing.

Teaching preparation programs provide many examples of representations, decompositions, and approximations of the practices of teaching with varying levels of authenticity. For instance, by watching Videocases, novice teachers are able to "enter" the classroom, observe student behavior and imagine how they would react as the teacher, without the actual responsibility of being in the classroom. This approximation of teaching has a low level of authenticity because real teachers do not have the opportunity to pause or rewind classroom activity in order to decide how to react or how to interpret the situation. Practice teaching is an example of an approximation of teaching with much higher authenticity. During practice teaching, novice teachers have limited responsibility in the classroom, but are able to experience it in real time and in a much more authentic way than by watching a video. Grossman and her colleagues (2009) highlight the benefits of representations, decompositions, and approximations of practice with varying levels of authenticity, which "quiet the background noise so that they can tune in to one facet of practice at a time" (p. 2083). As novices participate in the practices of a community (through approximations of practice, representations of practice, and/or decompositions of practice) they do not just develop the skills of the community, but also develop (to varying degrees) a shared knowledge base and shared dispositions.

A teaching preparation program can involve each pedagogy of practice to a varying degree. To illustrate the level of involvement of each pedagogy of practice I use shading in the visual representation of the framework. No shading illustrates that the teaching preparation program did not involve this type of pedagogy of practice. Light shading represents that this pedagogy of practice was involved in the program to a small degree, but was not the main feature of the teaching preparation program. Dark shading represents that this type of pedagogy of practice was the main feature of the teaching preparation program. For instance, consider a teaching preparation program held the week before graduate students teach for the first time that involves a short session where graduate students prepare a five-minute lecture and present it to other graduate students, video presentations about how to foster whole class discussion and group work and a lecture presentation about how to organize notes well and write on the board clearly. Such a program would be characterized with light shading for approximations of practice, dark shading for representations of practice, and light shading for decompositions of practice, as shown in Figure 2.

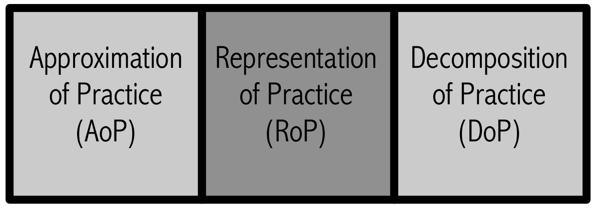

Fig. 2 Shading of three pedagogies of practice to represent three levels of involvement of each through the teaching preparation program

These components are summarized in Table 1. Figure 3 illustrates the relationships between them, and provides a visualize representation of the framework for novice tertiary teaching preparation.

Table 1 Dimensions of the tertiary teaching preparation programs

Dimension	Description
Institutional and Departmental Context	Objective information about the institution and department that is relevant to Calculus 1 instruction and the graduate student professional development program. This includes details about the current state of the institution, department, calculus program, and novice instructor teaching preparation program, and the history of each of these elements.
Institutional and Departmental Culture	Objective and/or subjective information about the views, beliefs, objectives, goals, and aspirations of the institution and department that are relevant to Calculus 1 instruction and the graduate student professional development program. This includes the views, beliefs, objectives, goals, and aspirations of (a) the institution regarding undergraduate education, (b) the department regarding Calculus 1 instruction, (c) the department regarding novice instructors' roles in Calculus 1 instruction, and (d) the department regarding novice instructor preparation for their role in Calculus 1 instruction. These views, beliefs, objectives, goals, and aspirations may or may not be explicitly stated. The priorities surrounding the Calculus I program and/or the novice instructor preparation program and the department's general orientation to teaching both fall somewhere in between the institutional and department culture and context.
Structure	Objective information about the formal and informal structural components of the novice instructor teaching preparation programs program. This includes the five components identified by Belnap and Allred (2009): (a) timing, (b) frequency, (c) duration, (d) topics covered, and (e) overall design.
Development of knowledge	Objective and subjective information about the types of knowledge emphasized through the structure of the program. This includes the three main types of knowledge identified by Schulman (1986): (a) pedagogical knowledge (PK), (b) content knowledge (CK), and (c) pedagogical content knowledge (PCK).
Development of practices	Objective and subjective information about the pedagogies of practice that are involved in the structure of the program. This includes the three pedagogies of

practice identified by Grossman et al. (2009): (a) representations of practice, (b) decompositions of practice, and (c) approximations of practice.

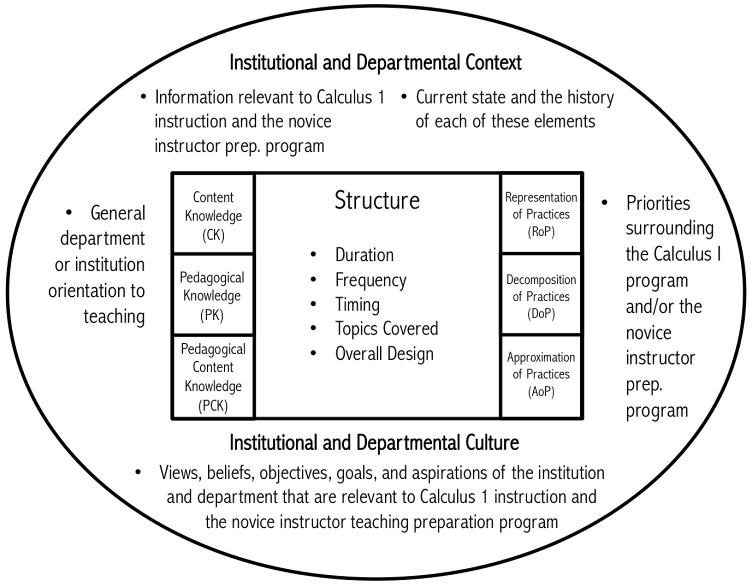

Fig. 3 Framework of instructor teaching preparation programs

Two Examples

Here I use the framework to visually represent two models of novice instructor teaching preparation program models that were observed through the case studies at institutions with successful calculus programs and where novice instructors (including graduate students, post-doctoral fellows, an undergraduate students) were heavily involved in Calculus I instruction. I call these two models the Apprenticeship Model and the Coordinated Innovation Model. These examples serve to illustrate how the framework can be used to characterize programs as well as to aid in comparison of programs.

The Apprenticeship Model. *The Apprenticeship Model* of novice instructor teaching preparation was enacted at a small university with approximately 5,000 undergraduate students, where fall enrollment in Calculus 1 is around 270 and class sizes are around 45. Graduate students, both Master's and Doctoral students, are involved in the teaching of Calculus I as teaching assistants, tutors, and course instructors. Post-docs are not involved in the teaching of Calculus I at this university.

The primary guiding philosophy behind the Apprenticeship model is the desire to transition graduate students into the role of instructor, both as part of their immediate role as GTAs and as their (potential) future role as undergraduate mathematics instructors. Embedded within this philosophy is the belief that people learning a new profession (who will develop a professional identity surrounding it) must participate in the practices of the profession with growing responsibility. This belief is in line with a perspective in which learning is viewed as the process of engaging a novice in the practices of the profession with legitimate but peripheral participation (Lave and Wenger 1991). The term "peripheral" indicates that the practices novices are involved in are less central versions of the authentic practices, or are central practices with limited responsibility. As one clinical psychology professor involved in the Grossman et al. (2009) study said when describing how clinical psychologists are prepared, "if you're learning to paddle, you wouldn't practice kayaking down the rapids. You would paddle on a smooth lake to learn your strokes" (p. 2026). The main components of the Apprenticeship model are:

- A three-unit class, inspired by Lesson Study (Lewis, Perry, and Hurd 2009), that takes place during the semester before the graduate student is placed as a course instructor.
- A mentor instructor for whom the mentee acts as a teaching assistant in the class they will be teaching during the semester before the graduate student is placed as a course instructor.
- Weekly course meetings once the graduate student is placed as a course instructor.
- Observations and feedback once the graduate student is placed as a course instructor.

Graduate students are required to participate in a number of teaching development activities, both prior to teaching and while they teach. All new GTAs must attend a one-day seminar led by the mathematics department, with some of this time spent doing practice teaching presentations. During the seminar faculty conduct workshops on topics including pedagogical basics, such as how to write well on the board, as well as more advanced pedagogical topics, such as how to implement cooperative learning. Additionally, all first-year GTAs are assigned a faculty mentor during the orientation session.

As shown in Figure 4, the framework representation of the Apprenticeship Model gives a clear overview of the structure and encompassing environment of the novice instructor teaching preparation program.

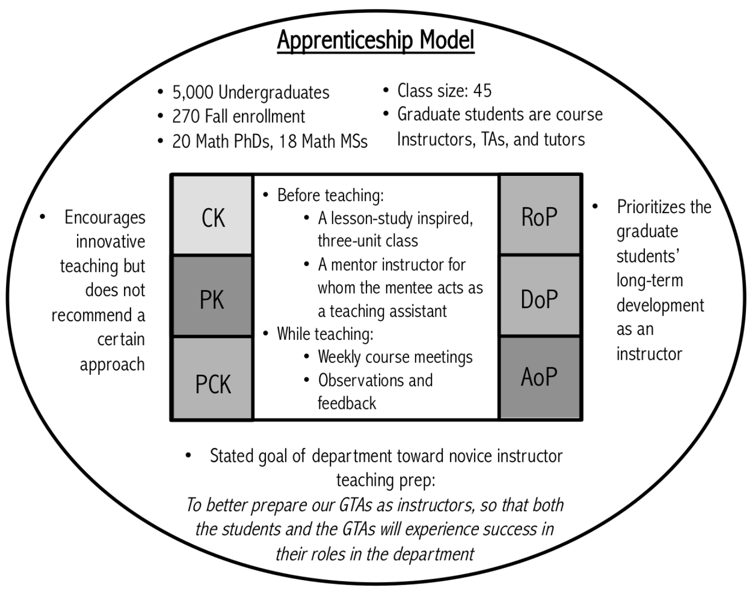

Fig. 4 Apprenticeship model

The main structural components of the program are a lesson-study (Lewis, Perry, and Hurd 2009) inspired course and mentoring that occur before the GTA is placed as an instructor, and ongoing meetings and observations once the GTA is placed as an instructor. The shading provides a visual representation for the level of emphasis of the knowledge and the level of authenticity of the practices involved in the programs. Within this structure, pedagogical knowledge is emphasized more deeply than PCK or content knowledge, though PCK is emphasized through both the lesson-study inspired course and the mentoring. Content knowledge is potentially emphasized through the mentoring, although it is not a primary focus. During the lesson-study course, novice instructors participate in a number of pedagogies of practice to varying degrees of authenticity. Through the lesson-study-like iterations of developing, presenting, and refining lessons, graduate students engage in approximations of the practice of teaching to increasing degrees of authenticity. The practice of teaching is decomposed into planning, presenting, and refining through the lesson-study course. Through both the lesson-study course and the mentoring, graduate students have multiple opportunities for teaching to be represented, by other graduate students, their mentor instructor, and by reading and watching cases. This program is situated within a small department that prioritized graduate students' long-term development as instructors and encourages innovative teaching but does not require a certain pedagogical approach.

Coordinated Innovation Model. *The Coordinated Innovation Model* of novice instructor teaching preparation was enacted at a large university with around 25,000 undergraduate students, where fall enrollment in Calculus I is around 2,000 and class sizes are around 30. Novice GTAs teach the majority of these sections. The remaining instructors are experienced GTAs, post-docs, and occasionally faculty. All courses are coordinated by a team of three permanent faculty, and all Calculus 1 courses are taught using an Inquiry Based Learning (IBL) inspired instructional method, which emphasizes student discovery, group work, and conceptual understanding (see http://www.inquirybasedlearning.org/ for more information.

The primary guiding philosophy behind the Coordinated Innovation Model is the

that Calculus I should be taught in an innovative and student-centered way, in small, highly coordinated classes. This innovation addresses the approach to the content, which is conceptually oriented and application driven, as well as the pedagogical approach, which includes group work and whole class discussions surrounding students' mathematical activity (rather than the teacher's). The coordination of these classes ensures that students have similar experiences. Further, this coordination helps to support the secondary guiding philosophy: that graduate students can be prepared and supported to successfully implement innovative instruction. This model is also motivated by a third, underlying philosophy: that graduate students can be, as Seymour (2005) termed it, "partners in innovation" and that graduate students who are effectively prepared to implement innovative instruction will likely carry these innovative practices into their future roles as undergraduate mathematics instructors. The main components of the Coordinated Innovation Model are:

- An intensive four-day training seminar that takes place the week before GTAs are placed as course instructors.
- Weekly course meetings once the GTA is placed as a course instructor.
- Observations and feedback once the GTA is placed as a course instructor.

The Coordinated Innovation model prepares and supports GTAs to teach coordinated sections of Calculus I with a conceptually oriented and student centered approach. The main component of this novice instructor teaching preparation program is a five-day seminar that takes place the week before the semester begins. It provides multiple opportunities for graduate students to present a prepared lesson and get feedback, and a series of presentations aimed to introduce graduate students to the department's approach to calculus, to explain the rationale for the approach, and to share evidence of its success. Many of the materials are reused year after year, with small additions or changes based on facilitators' experiences and feedback from the GTAs. All first-time GTAs participate in the seminar. After the third day of the seminar, the GTA supervisors make course assignments based on availability and graduate student participation in the seminar, specifically their performance in their practice lessons. Most graduate students are placed as course instructors for Calculus I, while some are placed as instructors for precalculus, Calculus II, or are assigned to be tutors in the calculus tutoring center.

Part of the support that graduate students receive once placed in the role of course instructor comes through the coordination of the course. This coordination encompasses common homework, quizzes, exams, and schedule. Because there are many aspects of teaching that are new to graduate students, having these aspects of the course coordinated by an expert instructor allows them to focus their energy on other aspects of instruction. This can be especially helpful when implementing more innovative instruction.

Part of the coordination that is especially helpful for supporting novice instructors is weekly meetings. These meetings involve all instructors for Calculus I and the course Coordinator. These meetings serve as a place to address class management issues including use of group work or how to address specific content, and function not only as an opportunity to cover the logistics of the week, but also as a venue for discussions about student thinking and difficulties.

An experienced graduate student or faculty member observes all GTAs (new and experienced) at least once each term. The observers give feedback to the graduate student. If issues were noted, these are communicated to the GTA, along with concrete ways to address these concerns. In these cases, additional observations are done. As shown in Figure 5, the framework representation of the Coordinated Innovation Model gives a clear overview of the structure and encompassing environment of the novice instructor teaching preparation program.

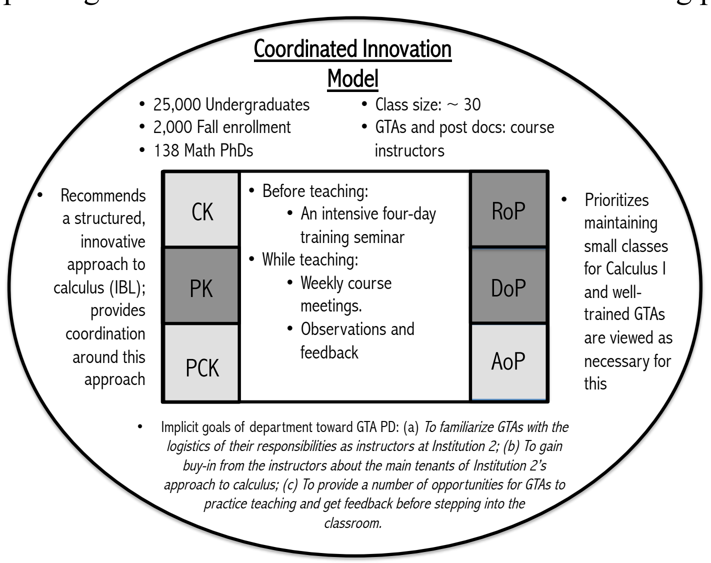

Fig. 5 Apprenticeship model

The main structural components of the program are an intensive four-day seminar that takes place the week before graduate students are places as course instructors, and ongoing meetings and observations once the GTA is placed as an instructor. The shading provides a visual representation for the level of emphasis of the knowledge and the level of involvement of the pedagogies of practices in the programs. Within this structure, pedagogical knowledge is emphasized more deeply than both PCK or content knowledge. The majority of the focus is on preparing graduate students to enact a specific pedagogical approach, and helping them understand why this approach is taken. During the summer seminar, there is some discussion surrounding student thinking related to difficult topics, and this may also occur through out the year during meetings – especially during exam time. During the summer seminar, teaching is decomposed into lecture, asking questions, discussions, and group work. Novice GTAs discuss each aspect with experts, and then have opportunities to practice teaching these practice, decomposed into a practice lecture and a practice student-centered activity, where GTAs are expected to engage their students in group work and discussion. Both the practice lecture and the practice student-centered activity are fairly inauthentic as the audience members are fellow graduate students. Thus, these activities serve both as an opportunity for the GTAs to approximate the practices (in an inauthentic environment) as well as watch many of their peers represent the practices for them, while also benefitting from the feedback given to their peers. This program is situated within a large department that has a team dedicated to the calculus program and the preparation of the graduate students who enable the calculus program. In this program, innovative, student-centered instruction is prioritized, and is able to be enacted on a large scale due to the high levels of coordination within the calculus instruction.

Comparison

By representing the Apprenticeship Model and the Coordinated Innovation Model using the framework, a number of similarities and differences become salient. One major similarity between the two programs is that they rely on substantial resources from their surrounding environment (department and institution). The Apprenticeship Model relies on the department supporting graduate students for a semester before they are placed in the classroom, as well as on mentor instructors in the department. The Department Chair at the institution where this model was enacted is a proponent of supporting graduate students in their long term development as instructors, and believes that one semester of support is worth it to almost guarantee quality graduate instructors for their remaining semesters. The Coordinated Innovation Model relies on the institution to secure small classes for the large numbers of Calculus I students, and relies on the department to bring in large numbers of graduate students to support the abundance of Calculus I sections.

While there are clear differences in the structure of the programs within both models, the visual representation of the framework highlights the more subtle differences in the emphases and goals of these programs. The Apprenticeship Model focuses on developing graduate students as instructors, not just in their current roles as GTAs but as future mathematics faculty. Influenced by this goal, the structure of this program emphasizes both pedagogical knowledge as well as pedagogical content knowledge, to support graduate students in beginning to think like instructors. Throughout the program, graduate students participate in an array of pedagogies of practice, and with increasing authenticity. This supports the graduate students in developing a professional identity related to being an instructor, and the ongoing meetings and observations are geared towards helping the graduate students internalize their role as instructor. In contrast, the Coordinated Innovation Model focuses on preparing their GTAs to successfully implement a specific type of Calculus I instruction, rather than supporting them as long-term instructors. Certainly, graduate students will benefit as instructors in the long-term by being well-prepared to teach Calculus I in a specific way, but this is not the focus of the preparation, rather a side benefit. Influenced by this goal, the structure of this program strongly emphasizes pedagogical knowledge related to teaching using the IBL approach. Throughout the program, this approach is decomposed and represented to novice instructors, and they get some practice in approximating this practice. This supports the graduate students to implement IBL instruction, and the ongoing meeting and observations are geared towards solving problems, answering questions, and improving instruction related to the IBL approach.

Both models are influenced and constrained by their surrounding environments, and thus it would not make sense to pick up the Coordinated Innovation Model and use it in place of the Apprenticeship Model or vice versa. However, if an institution were looking to develop a new novice instructor preparation program, and their goals of this preparation were in line with those related to the Apprenticeship Model, but the institutional characteristics were more in line with the institution that enacted the Coordinated Innovation Model, this institution could identify aspects of each model that may work within their constraints and help them achieve their goals.

Conclusion

While the framework representation does not give the rich detail of the program on its own, it provides information useful in comparing across models, and can be used to ask and answer questions regarding the evaluation or implementation of an individual model. The framework can also be used to evaluate a program or to help with the creation or improvement of a teaching preparation program. To aid in the evaluation of a program, a mathematics department may determine that their GTAs and post-doctoral fellows seem to know very little about how their students may think about mathematics, their difficulties, and how to explain problems so that they will better understand them. They could use this framework to describe their current program and identify that they are not, in fact, spending time during the teaching preparation discussing PCK. To aid in the development of a program, this framework can help direct attention to important components to consider, as well as provide a visual representation of the many important components of a novice instructor preparation program. In many mathematics departments, a more robust novice teacher preparation program is developed based on the initiative of one or two motivated individuals – the change agents. Often, these change agents are not necessarily mathematics education experts, or may have good ideas about what the novice instructors need at their institution but do not know how to go about setting up a new program. The framework introduced here provides an organized and systematic way to think about the components of a tertiary teaching preparation program.

The primary intention of this framework is to characterize existing novice instructor teaching preparation programs situated within a mathematics department at a tertiary institution. Often such programs are mainly intended for graduate students works as teaching assistants or classroom instructors, but such programs may also be used to support post-doctoral fellows, instructors, visiting faculty, and junior faculty in their teaching. The framework draws on theoretical underpinnings from the K-12 education literature, some of which are specific to mathematics and some of which are not. Thus, a number of questions arise about the relevance of this framework in thinking about teaching preparation programs outside of tertiary mathematics departments.

For instance, one may wonder about how the ideas presented here can help one think about a graduate student teaching assistant training program in a Philosophy Department. There are a number of ways that this framework can be leveraged in this context, however there are also a number of reasons why it is specific to mathematics. Mathematics is a unique content area in that it is taught throughout K-12 education, rather than primarily (or exclusively) at the tertiary level, such as with Philosophy. This results in students coming in with a wide range of previous experiences in mathematics, which adds complexity to the teaching of tertiary mathematics, especially in the first or second years of study (such as with Calculus). Mathematics graduate students, who are often tasked with teaching lower level courses rather than upper level tertiary courses, then must be prepared not just to teach the content, but also to identify student (mis)understanding in the moment and bridge this understanding with the understanding needed in their class.

Mathematics is also largely a service department, and first and second year tertiary mathematics courses are widely required service courses. Thus, mathematics graduate students are tasked with teaching a course that a large percentage of the students may not be particularly interested in taking. Further, first and second year mathematics courses have been identified as greatly contributing to students losing interest in pursuing STEM fields – thus the instructors of these course should also be aware of the role their teaching may play in students' career goals. While the same may be true to some extant in introductory Philosophy courses, it is to a much lesser degree. Thus, there are aspects of this framework that may be applicable when thinking about the development of a novice instructor preparation program for non-mathematics departments, but there are also important contextual differences that must be considered.

There are also aspects of this framework that may be applicable to the teaching preparation of a specific subset of K-12 teachers: alternatively places teachers in high-need schools. Teach for America is the most widely known of these programs, but others include New York City's Teaching Fellows Program, Teach Kentucky, and the Mississippi Teacher Corps programs (Heilig and Jez 2010). Teach for America (TFA) is a non-profit organization that recruits recent graduates from elite colleges to teach in low-income schools for two years. TFA teachers often do not have an education background and thus do not participate in the extensive teaching preparation that teachers receive over four-years (or more) in undergraduate credentialing programs. Instead, TFA teachers participate in a five-week summer program between graduating from college and beginning their teaching assignments. This training often includes a brief stint of student teaching, short lessons on pedagogy, content, and classroom management. In addition to the summer training, TFA teachers must continue coursework in local colleges to pursue full teaching credentials while they teach. Research into the efficacy of TFA teachers is mixed, with some indicating that students taught by non-TFA teachers outperform students taught by TFA teachers on reading and mathematics (Darling-Hammond, Holtzman, Gatlin, and Heilig 2005), though there are large differences between first-year TFA teachers and second-year teachers (Boyd, Grossman, Lankford, Loeb, and Wyckoff 2006).

TFA teachers, and other alternatively-certified teachers, share a number of commonalities with mathematics graduate student teaching assistants: they are often strong with respect to their content knowledge, receive relatively little pedagogical preparation compared to other teachers, are younger than most other teachers, are not as interested in a long-term career in teaching as other teachers, teach students who are mathematically weaker than they were as students, and are often thought of as not as effective as traditional teachers (Boyd, Grossman, Lankford, Loeb, and Wyckoff 2006; Heilig and Jez 2010). However, both GTAs and TFA teachers are cost effective as teachers and serve a need. Thus, thinking about how to best prepare TFA teachers within the confines of the public school system is similar to thinking about the preparation of GTAs within the confines of tertiary mathematics departments, and as such the framework presented here can contribute to both situations.

In this paper, I have provided two examples of how this framework can be used to characterize existing programs. In addition to being used in this way, the framework can also be used to help develop a novice teaching preparation program and to evaluate an existing program. The framework can be used to help develop a new program by narrowing down specifics aspects of the desired program. The individuals tasked with creating a new novice instructor teaching preparation program should think about what types of knowledge their novice instructors need to develop, and what types of activities they want novice teachers to engage in during the teaching preparation. This will result in their own shading of the types of knowledge and the types of pedagogies of practice desired in the teaching preparation program. This information can be used then to help organize the program develops and bring ideas together, or the program developers could compare what they are hoping to develop to characterizations of existing programs to "find a match." The institutional and department culture and context will help the program developers determine how applicable and adaptable existing programs would be to their department.

The framework can also aid in the evaluation of existing programs. Those familiar with the needs a novice instructors could identify types of knowledge their novice instructors need to develop, and what types of activities they believe the novice teachers need to engage in to develop this knowledge. This could be then contrasted with a characterization of their existing program using the framework, which would illuminate discrepancies between the needs of novice instructors and the existing program. If changes are then implemented, a post-implementation of the program could then be characterized using the framework and compared to the original characterization to identify how the changes were in line with the needs of the novice instructors.

Due to recent increased attention to tertiary mathematics instruction, coupled with continued reports of the role of poor teaching in introductory mathematics courses in the leaking STEM-Pipeline, many tertiary mathematics departments are focusing more attention on the teaching preparation of instructors who teach introductory courses. The framework presented here aims to aid these improvement efforts by providing organization to the abundance of considerations related to teaching preparation.

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Student responses to instruction in rational trigonometry

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In this paper I discuss an investigation on students' responses to lessons in Wildberger's (2005a) rational trigonometry. First I detail background information on students' struggles with trigonometry and its roots in the history of trigonometry. After detailing what rational trigonometry is and what other mathematicians think of it I describe a pre-interview, intervention, post interview experiment. In this study two students go through clinical interviews pertaining to solving triangles before and after instruction in rational trigonometry. The findings of this study show potential benefits of students studying rational trigonometry but also highlight potential detriments to the material.

Key words: [Rational Trigonometry, Undergraduate Mathematics, Interviews]

Introduction

Students struggle with trigonometry. This struggle is a contributing factor to students not pursuing studies in the STEM fields. Students struggle with trigonometry at many points during their mathematical studies. While many pedagogical changes to trigonometry instruction have been tried (Bressoud, 2010; Kendal & Stacey, 1996; Weber, 2005) little has been done looking at replacing or augmenting trigonometry instruction with a mathematical alternative.

Rational trigonometry is a system for studying triangles using different units to measure length and the separation between two lines instead of using distance and angle (Barker, 2008; Campell 2007, Franklin, 2006; Henle, 2007; Wildberger, 2005a, 2005b). The use of a different unit necessitates different formulas than traditional trigonometry. Wildberger (2005a, 2005b) claims that rational trigonometry is simpler to learn, understand, and use than its traditional counterpart. He believes this based on the formulas for rational trigonometry lacking the sine, cosine, tangent or other transcendental functions. Little if any research has been conducted looking into educational benefits of rational trigonometry.

To investigate his claims, I conducted task-based interviews before and after lessons in rational trigonometry to explore the following: How do mathematics majors approaches to solving problems pertaining to triangles change after studying rational trigonometry?

Traditional Trigonometry

Trigonometry as we know and teach it causes many difficulties for students. Previous research on students' difficulties with trigonometry include studies using quantitative methods (Brown, 2005), teaching experiments (Moore, 2009, 2013; Weber, 2005, 2008), and theoretical pieces (Bressoud, 2010; Gilsdorf, Moore, 2012; Wildberger, 2005a, 2005b, 2007).

What is Trigonometry?

This is a question that is rarely answered explicitly in mathematics texts (Wildberger, 2005a). One method to defining words is the etymological approach. "Tri" being the prefix for three, "gon" referring to a polygon (e.g. pentagon, hexagon etc.) and "metry" referring to measure. Putting these together yields trigonometry as the study of the measure of three sided polygons.

A second way to define a word is to look at its use throughout history. The predecessor of the sine function was developed in the second century BCE (Bressoud, 2010). This was a relationship between central angles and chords of a circle (Bressoud, 2010; Gilsdorf, 2006).

Using these techniques for triangles started in the 11th century CE and was formalized as sine and cosine in the 16th century (Bressoud, 2010). Introducing students to the trigonometric functions through the use of triangles began in the 19th century (Bressoud, 2010).

A third approach to defining trigonometry is to see how the word is currently used in the literature. Looking at texts yields the following list of topics: triangles, trigonometric functions, trigonometric identities, trigonometric equations, trigonometric graphs, imaginary numbers, polar coordinates, De Moivre's theorem, McClaurin Series, integral substitutions, waves, Fourier Analysis and more (Hirsch, Fey, Hart, Schoen, & Watkins, 2009a, 2009b; Larson & Edwards 2014, Liebeck, 2005). This would lead us to defining trigonometry as the study of anything pertaining to angles, triangles, or the functions sine, cosine, and tangent.

Based on these three perspectives, trigonometry is the study of everything pertaining to the functions, which resulted from applying the study of circles, to the study of triangles. For this study I am going to focus on the mathematics of triangles.

Student difficulties with trigonometry. Many difficulties pertaining to trigonometry are well documented (e.g., Akkoc, 2008; Blackett & Tall, 1991; Bressoud, 2010; Brown, 2005; Marchi, 2012; Moore, 2009, 2012, 2013; Weber, 2005, 2008, Wildberger, 2005b). Most of the documented difficulties can be sorted into two categories: 1) difficulties pertaining to the concept of angle (Akkoc, 2008; Bressoud, 2010; Moore, 2009, 2012, 2013; Wildberger, 2005b), and 2) difficulties pertaining to the sine, cosine, and tangent functions (Bressoud, 2010; Brown, 2005; Marchi, 2012; Moore, 2012; Weber, 2005, 2008; Wildberger, 2005b).

Student difficulties with angles. Akkoc (2008) claims that student difficulties with angles stem from gaps in their teachers' understanding of angles. Moore (2012) attributes this difficulty to the current common approaches to teaching trigonometry. Bressoud (2010) attributes difficulties with angles to incompatibilities between the ratio and the unit circle approaches to understanding trigonometry. These approaches are associated with degrees and radians respectively. Students are then taught that they are interchangeable yet certain problems are to be done in terms of one and other problems in terms of the other without any justification for the decisions made (Akkoc, 2008; Bressoud, 2010). Wildberger (2005b) takes these views to an extreme by claiming that the unit itself is overly complicated and that with the exception of a few values cannot be calculated without a background in calculus.

Student difficulties with trigonometric functions. Moore (2012) attributes flawed understandings of the trigonometric functions on the volume of inconsistent definitions used for them. Brown (2005) found that students compartmentalize two different definitions for sine and cosine. These two definitions for sine and cosine are as the ordinate and abscissa respectively of points on the unit circle and as ratios of side lengths of a right triangle. Some authors have found that the meanings of the trigonometric functions are obscured by the use of the unit circle instead of the use of ratios of side lengths of right triangles (Kendal & Stacey, 1996; Markel, 1982). Markel (1982) argues that the unit circle includes angles above 180° which are unnecessary and does nothing to help students differentiate sine and cosine. Kendal (1996) found that the unit circle approach gave students more opportunities to make mistakes. However, Weber (2005) states that the unit circle was a more effective pedagogical tool than right triangles. He found that students were more likely to recognize sine and cosine as functions if taught using a unit circle approach. Students have problems viewing sine, cosine, and tangent as functions due to their non-algebraic nature and as such are unsure about how to perform algebraic operations with them (Weber, 2005). This could be due to the pedagogy straying away from beginning with the study of circles and chords (Bressoud, 2010; Gilsdorf, 2006) or it may be due to the transcendental nature of the functions (Weber, 2005; Wildberger 2005b).

Need for trigonometry. One debated topic is the importance of studying traditional trigonometry. While the importance of many mathematical topics is debated in the K-16 curriculum the inclusion and exclusion of trigonometry can be seen in multiple scenarios. Multiple groups believe that high school students are not being taught enough trigonometry and that it should be the penultimate high school course instead of calculus (Bressoud, 2012; Markel, 1982). While many college calculus courses expect a prior knowledge of trigonometry many colleges now offer variants of their calculus courses that attend to the same topics with the exception of omitting trigonometry-based problems.

Rational Trigonometry

Rational trigonometry is a reformulation of trigonometry based on replacing the units of distance and angle, with the units of quadrance and spread (Wildberger, 2005a, 2007). Quadrance is distance squared. The spread between lines l_1 and l_2 is the quadrance of $\overline{BC}$ divided by the quadrance of $\overline{AB}$ shown in Figure 1.

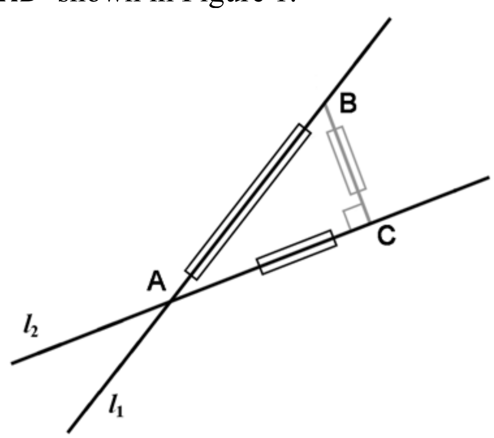

Fig. 1

Replacing the concept of angle with the concept of spread, results with the main formulas in trigonometry needing to be reformulated. The result is that the traditional trigonometry laws are replaced with the laws of rational trigonometry. They are analogous to the tradition trigonometric laws but the trigonometric functions are replaced with algebraic operations shown in Table 1 (Barker, 2008; Franklin, 2006; Henle, 2007; Wildberger, 2005a).

Table 1. Analogous Formulas in Tradition Trigonometry and Rational Trigonometry

Traditional	Rational
$c^2 = a^2 + b^2 - 2ab\cos C$	$(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2(1 - S_3)$
$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$	$\frac{S_1}{Q_1} = \frac{S_2}{Q_2} = \frac{S_3}{Q_3}$
$A + B + C = 2\pi$	$(S_1 + S_2 + S_3)^2 = 2(S_1^2 + S_2^2 + S_3^2) + 4S_1S_2S_3$

Curricular Change

For something new to be adopted by the mathematics community it needs one of two things. It needs to either be able to do old tasks better than older approaches or it must be able to do new things. In the next three subsections I will discuss arguments supporting Wildberger's (2005a, 2005c, 2007) claim and opposing his claim that Rational Trigonometry is simpler than its traditional counterpart.

Arguments in favor of rational trigonometry. Arguments in favor of rational trigonometry being simpler than traditional trigonometry are that it gets rid of the difficulties caused by the angle and the trigonometric functions by replacing them. With Rational Trigonometry, sine, cosine and tangent are no longer needed to study triangles (Barker, 2008; Franklin, 2006; Henle, 2007; Wildberger, 2005a, 2005b). Wildberger (2005a, 2005b) claims that the most complex operation needed for trigonometry becomes the square root function and that a student who has learned the quadratic formula has the prerequisite skills needed to study rational trigonometry.

Arguments against rational trigonometry. Three arguments have been made against rational trigonometry. One of these is that the units are less intuitive (Campell, 2007; Gilsdorf, 2006). Adding consecutive angles of 30° and 30° yielding 60° is more intuitive than adding two consecutive spreads of 1/4 and 1/4 and getting 3/4 as your result. Another is that many triangle problems would have irrational solutions when solved with rational trigonometry and that the irrational solutions from rational trigonometry are no more useful than the transcendental solutions from traditional trigonometry (Gilsdorf, 2006). A third is the inflexibility of the educational system (Campbell, 2007). Educational sequencing rarely changes and it pushes students to study traditional trigonometry before higher mathematics.

Questions from the arguments. The two sides of this argument bring up some interesting points in comparing the systems. Is the benefit of avoiding trigonometric functions worth a unit that is less visually intuitive? Should simpler be defined in how one uses the material or in how one learns the material? Is there any benefit to rational trigonometry when you have to study traditional trigonometry anyway? While all of these are interesting my research question only addresses aspects of the first two. This study shows a glimpse at students working with quadrance and spread instead of the trigonometric functions. It also lets us see how two students use both trigonometric systems to address the same problems.

Mathematical Research

As stated earlier there are two reasons for the mathematics community to adopt alternative mathematics. The second of these mentioned was that if it does something that has not been done before. There is a small yet existent body of literature in higher mathematics that makes use of rational trigonometry. Authors have applied the concepts of rational trigonometry to geometry (Alkhaldi, 2014; Le & Wildberger, 2013; Vinh, 2006, 2013; Wildberger, 2010), computer programming (Kosheleva, 2008), and robotics (Almeida, 2007).

Factors Influencing Students Pursuing Mathematics

One of the factors that determines students' course taking patterns in college mathematics is their overall confidence with mathematics.

Students who expressed confidence in their mathematical abilities are more likely to take additional mathematics courses (Fennema & Sherman, 1977; Else-Quest, Hyde & Linn, 2010, Oakes, 1990). Those courses tend to be at a higher-level than the ones taken by their less confident peers (Fennema & Sherman, 1977; Else-Quest et al., 2010; Laursen, Hassi, Kogan, Hunter, & Weston, 2011; Stodolsky, Salk, & Glaessner, 1991). Typically, a loss in confidence is caused by performing lower than one's expectations (Ahmed, van der Werf, Kuyper & Minnaert, 2013). Improving students' performance in trigonometry would help their confidence and positively influence their future studies.

Students' Problem Solving Strategies.

Students tend to use the strategies and techniques they are most recently familiar with when approaching problems (Chi, Bassok, Lewis, Reimann, & Glaser, 1989; Owen & Sweller, 1985). This explains why students might solve a quadratic by formula instead of factoring or use the law of aines when solving a right triangle. This phenomenon is stronger in weaker students who are less likely to stray from the patterns established in examples (Chi et al., 1989). Situation and context also influence how students attempt to solve problems (Moore, 2012). A student is most likely going to use the formulas they think an instructor or exam wants them to use.

As it pertains to trigonometric problems the strategies are the same in both rational and traditional trigonometry but the techniques are different. For example, consider a problem where a student is given the measurements for two sides of a triangle and the vertex (Note: In this paper I use vertex to refer to the corner points of a triangle to avoid referring to angle as both an object and a unit of measurement) between them and asked for the third side. A strategy would be to use a formula that relates those four quantities. In traditional trigonometry the technique would be to use $c^2 = a^2 + b^2 - 2ab\cos C$ while in rational trigonometry the technique would be to use $(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2(1 - S_3)$. Based on the previous paragraph I expected their techniques to change after studying rational trigonometry. Nothing I found in the literature leads me to believe their strategies will or will not change.

Methodology

Decisions need to be made when designing a study. These decisions impact the result of a study. Giving an assessment before an intervention may influence the intervention. Omitting such an assessment limits before and after comparisons in data analysis. Using students who struggle in mathematics increases the chance of benefit but also increases the potential unforeseen issues in intervention design. Using students who do not struggle in mathematics may decrease the potential for benefit but also decreases the likelihood of unforeseen issues in design. Some decisions are not made because of something being definitively better than an alternative but are made because they fit the study that researcher has decided to conduct.

The comparative nature of this study influences many design decisions. Only distances are given and asked for in these tasks. To give or ask for spread or angle would inherently design the questions towards the use of a particular approach. A second outcome of this is the triangles presented in both interviews are geometrically similar. Without similarity is it possible that one interview task was inherently simpler due to the triangles used. A third result is the tasks asking for an altitude, median, and vertex bisector. These three concepts have been studied since antiquity (Heath, 1956) and as such are not dependent upon rational trigonometry for analysis.

Research Design

The inquiry approach for this study is case study design. Case study is the study of a case across a timespan (Hatch, 2002; Yin 2009). Case studies can be exploratory or explanatory in nature (Yin, 2009). For this study the cases are the two participants and the timespan is five days (the pre-interview, three days of lessons, and the post-interview). Combining the need for a before and after and the exploratory affordances of task-based interviews (Confrey, 1981; Maher & Sigley, 2014; Schoenfeld, 2002) leads to the design of pre task based interview, lessons, post task based interview. The first interview is being used to look at strategies and techniques used by participants without a background in rational trigonometry. The lessons are used to create a background in rational trigonometry. The post interview is being used to see how a participant's behavior and/or reasoning when approaching the same task is altered after studying rational trigonometry. Three video lessons on rational trigonometry were given to the participants. I designed these lessons to give familiarity with the units and formulas for rational trigonometry. The first lesson focused on the units of quadrance and spread. Quadrance was described as distance squared and spread was first defined geometrically. After that I detailed arithmetical properties of spread such as the range of spread (0 to 1) and the spread of parallel lines (0) perpendicular lines (1) and showed examples of how to calculate the spread for lines given in both slope-intercept and standard forms. Spread bisection was also shown. The second lesson focused on the formulas $(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2(1 - S_3)$ (the cross law) and $\frac{S_1}{Q_1} = \frac{S_2}{Q_2} = \frac{S_3}{Q_3}$ (the spread law). Examples were shown of using the cross law with three given quadrances to get one spread and then use the spread law to find the remaining spreads. The last lesson focused on the triple spread law. The third lesson focused on $(S_1 + S_2 + S_3)^2 = 2(S_1^2 + S_2^2 + S_3^2) + 4S_1S_2S_3$ (the triple spread law). The lesson was an example of using the triple spread formula to find the third spread in a triangle if only two spreads were known. Each lesson was accompanied with a worksheet that acted as practice for the participant, additional data for myself, and verification that they watched the videos.

Participants

Due to the comparative nature of this study, participants with a strong background in mathematics in general, trigonometry in particular, and with no background in rational trigonometry were recruited. To ensure this, mathematics students with a 4.0 in their first year mathematics courses including Euclidian trigonometry were chosen.

Data Collection

Data was collected through a pre-interview, three worksheets and a post-interview.

Interviews. Task based interviews were used to gather information about the participants' approaches to solving problems pertaining to triangles. The two interviews were audio recorded and occurred four days apart. Between the pre and post interviews the participants watched all three lessons and completed all three worksheets. Participants were supplied with pencil, paper, and a selection of traditional trigonometric formulas. During the second interview they were also given the rational trigonometry formulas from the lessons. From the interviews both their spoken word and written work were collected. The three tasks chosen for the interviews were chosen to have no inherent bias towards traditional or rational trigonometry. The first task was to find the length of an altitude of a triangle. This task is commonly shown in the high school curriculum and is often done with and without the use of the sine, cosine, and tangent functions (Keenan & Gantert, 1989; Hirsch, et al., 2009b). The second task was to find the length of a median of a triangle. The last task was to find the length of a vertex bisector. The triangle from the pre-interview is shown in Figure 2.

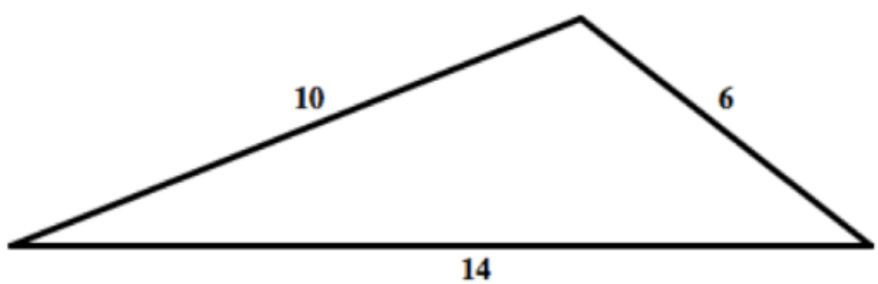

Fig. 2

Worksheets. The primary purpose of the worksheets was to ensure that the participants watched the videos. The work was analyzed with respect to the findings from the interviews for triangulation purposes. All three worksheets were collected at the second interview.

Data Analysis

I began my data analysis by transcribing the interviews. At this point I was already making decisions about what data had the potential to show interesting findings. After this my next step was coding the data. That data was separated and regrouped for organizational purposes (Creswell, 2014; Maxwell, 2013; Saldana, 2009; Seidman, 2012). My coding efforts were focused on the written work and verbal statements given during the interviews. Once this was done I focused on the findings that were most abundant and different between both interviews. The strongest examples are highlighted here.

Findings

After my analysis three themes emerged. These themes were strategies, numerical properties of triangles, and confidence. The strategies used involved the Pythagorean theorem and the relationships represented by the laws of sine and cosine and the spread and cross laws. Numerical properties were that distances must be positive, the triangle inequality, and that the longest sides of a triangle are across from the largest angles / spreads.

Maureen

Maureen is a mathematics major with the goal of becoming a high school mathematics teacher. Her undergraduate course on trigonometry ended four months prior to the study.

Strategies. Maureen started the pre-interview using the Pythagorean theorem in an attempt to find the value of an altitude. After multiple iterations gave her more unknowns than equations or values that did not make sense to her she abandoned this strategy. Her next attempt was to use the law of cosines to find one of the angles. Her goal was to use that angle in the law of sines to find the altitude. Once she found $\cos\theta = \frac{13}{14}$ she abandoned that approach as well. During the second interview Maureen used the cross and spread laws in the manner she intended to use the laws of cosines and sines in the first interview. In this attempt she successfully used both formulas. Though her use of the spread law gave her the quadrance of the altitude she did not turn that value into a length as the question was asking for. When questioned she said that the answer she gave was the length of the altitude.

Numerical properties. During the pre-interview Maureen made ample use of numerical properties of triangles. In particular, she made use of the fact that side lengths cannot be negative and she made use of the triangle inequality. She used these to check her computational results. The triangle inequality was also used to determine ranges for the answers to the interview tasks. Since she did not compute an angle there was no opportunity to observe if she would have used that the longest side is opposite the largest angle. In the post interview there was no use of the triangle inequality. This could have been used to alert her to not having the right answer in the first task. She did however use the property that the largest spread has to be across from the largest side of a triangle.

Confidence. Maureen's confidence in approaching these tasks appeared to increase after the lessons in rational trigonometry. In the first interview she spent a lot of time staring at the tasks without performing any calculations. After a particularly long silence she said:

> As much as I hate to admit that I can not remember how to solve for altitude, I'm just going to spend 20 minutes staring at this, because I'm not liking what I'm getting. I feel very bad saying that and admitting that, but it's not gonna happen.

In the second interview the gaps in work and expressions of frustration lessened. After the interview she gave the following two statements: "That was really really cool the whole quadrance and [spread]" and "if I had more time to practice I think I could have gotten all 3." These statements point towards a higher confidence level using rational trigonometry.

Tom

Tom is a mathematics major aiming towards graduate studies in applied mathematics. He took his trigonometry course approximately three years before the study.

Strategies. In the first interview Tom's strategy was to solve for anything he could find in hopes that he would come up with pieces he needed to solve the tasks. When he found $\cos\theta = \frac{13}{14}$ he used that value in another law of cosines equation in order to solve one of the tasks. In the second interview his strategies were nearly identical. The biggest change between the two interviews was he was using the rational trigonometry formulas instead of the traditional trigonometry formulas.

Numerical relationships. There was no evidence in either interview that Tom used the numerical properties listed above. He submitted answers to all three tasks and he could have but did not find two of them to be impossible due to the triangle inequality. In both interviews he was confident in his strategies (which would have worked) and his computations (which contained errors).

Confidence. Tom showed no notable change in confidence.

Discussion

If nothing else was accomplished this paper serves as another data point for the belief that some difficulties in trigonometry stem from difficulties related to the teaching, learning and use of functions. This aligns with the previous findings of many authors and studies (Bressoud, 2010; Brown, 2005; Marchi, 2012; Moore, 2012; Weber, 2005, 2008; Wildberger, 2005b).

Based on the findings I believe it is safe to say there may be some benefits to students studying rational trigonometry. The strongest evidence for benefits come from Maureen's case. Maureen falls into the category of students who are weaker with their algebraic manipulation of functions, which hindered her mastery of trigonometry (Weber, 2005). She seemed to increase in confidence after studying rational trigonometry and appeared more capable of solving problems when using the rational trigonometry formulas. Tom showed a strong mastery of the algebra of functions and little change in performance using the rational formulas. This may point to potential benefits being more likely for students with a weaker skill set pertaining to functions. Potential weaknesses also need to be mentioned. Maureen did not apply the numerical properties that she showed earlier evidence of using. She also at one point equated quadrance and distance. Quadrance being less intuitive than distance (Campbell, 2007; Gilsdorf, 2006) is likely a contributing factor of this.

I would also like to take this moment to point out a missed opportunity in this study. Since I did not notice it until data analysis I did not ask Maureen why she did not use the triangle inequality. Did she just not think of it? Did she not know how to work with it while using rational trigonometry? Did she know how to work with it but decide it wasn't worth the effort? Did she not feel the need for it? The reasoning for this change may be useful for later lesson design or research.

Future Research

There are many possibilities for future research. While it would be impossible to list them all, the following are the ones that stand out between my experiences with this study and with the mathematics.

Similar Studies

As stated earlier there were decisions with regard to this study that had both pros and cons. Redesigning this study as a longitudinal study or with a larger participant count may allow for more insights. Omitting the pre-interview, inserting lessons on traditional trigonometry before the pre-interview, or picking students with different backgrounds and experiences may also offer different results.

The Esperanto Effect

In the early twentieth century linguists were considering the idea of teaching Esperanto as a second language before teaching a third language to students. The idea being that a less complicated yet similar language would ease the transition from one language to the next. Eaton (1927) brought up multiple studies that supported this idea.

A similar principle could be studied in regards to traditional and rational trigonometry. Students could develop their problem solving strategies with regards to triangles without the burden of the trigonometric functions. Then later on be introduced to traditional trigonometry to see how they respond.

Other Geometric Structures

Lastly students and rational trigonometry can be studied in non-Cartesian / non-Euclidian contexts. Student difficulties in studying geometries outside of the Cartesian plane are not unheard of. Wildberger has published mathematical works applying rational trigonometry to both polar coordinates (Wildberger, 2005a) and hyperbolic geometry (Wildberger, 2009).

Conclusion

Will rational trigonometry ever replace traditional trigonometry in the mathematics curriculum? Most likely not. Very large portions of most curriculums and branches of mathematics are dependent on traditional trigonometry. However it does have potential as supplemental material. To quote Almeida (2007) and his work on robotics "The main role of R[ational] T[rigonometry] in this work was that it forced a different approach than the standard, due to difficulties that had to be resolved, that ended up forcing the paving of a new path of study." (p. 44). Using an alternative system gave him the opportunity to look at a problem differently. The potential gains from this type of opportunity merit further investigation.

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Using Student Reasoning to Inform Assessment Development in Linear Algebra

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The primary goal of this study was to design and validate an assessment of student reasoning in undergraduate linear algebra. We worked toward this goal by conducting semi-structured clinical interviews with 8 undergraduate students who were currently enrolled or had previously taken linear algebra. We identified the variety of ways students reasoned about the items in order to determine the extent to which the assessment measured or failed to measure students' reasoning about the intended topics. Students were interviewed using a think-aloud protocol while they completed the assessment, and interview data was analyzed by using the theoretical framing of concept image and concept definition (Tall & Vinner, 1981). In this paper, we describe how students' reasoning on items relating to span and linear independence informed adjustments to our assessment.

Key words: assessment, linear algebra, student reasoning

Students from a variety of science, technology, engineering, and mathematics (STEM) disciplines are required to take linear algebra as part of their undergraduate mathematics coursework. Students typically struggle with the theoretical nature of linear algebra as it is often their first time grappling with abstract mathematical concepts. Carlson (1993) argued that students' mathematical background up to this point is often primarily computational in nature; this often creates a barrier for students to overcome when they reach linear algebra. Linear algebra is a pivotal course that includes the mathematical underpinnings of different STEM fields, but it is filled with challenges for students. According to Wawro (2011), "The content of linear algebra, however, can be highly abstract and formal, in stark contrast to students' previous computationally-oriented coursework. This shift in the nature of the mathematical content being taught can be rather difficult for students to handle smoothly" (p. 2). The abstract concepts of linear algebra are often taught in such a way that students do not find any connections between new linear algebra topics and their previous knowledge of computational mathematics (Carlson 1993).

Researchers have worked to address this issue by developing inquiry-oriented instructional materials that help instructors to bridge students' informal and intuitive ideas with more formal and conventional understandings (Wawro, Rasmussen, Zandieh, & Larson, 2013). Our work aims to move towards documenting the effectiveness of these inquiry-oriented instructional materials in supporting students' conceptual understanding of central topics in an introductory undergraduate linear algebra course. In this study we have designed an assessment that aligns with four focal topics typically covered in an introductory linear algebra course: (1) span and linear independence, (2) systems of linear equations, (3) linear transformations, and (4) eigenvalues and eigenvectors. We aimed to create two questions for each of these four topics in order to develop an 8-item written assessment that could be completed by students in less than one hour. Based on findings from similar studies, we anticipate the results of the assessment to show greater conceptual learning gains from students who learned in inquiry-oriented classrooms along with similar procedural learning gains (Rasmussen & Kwon, 2007).

The central goal of this work is to develop a valid assessment to measure students' conceptual understanding of linear algebra. In order to develop an assessment that is sensitive to students' reasoning in linear algebra, we found it necessary to delve into the various ways that students reasoned about the items on our assessment. In particular we wanted to consider the kinds of reasoning elicited by our items, both to determine whether students interpreted items in ways that we intended and, importantly, to identify aspects of students reasoning to which we need our assessment to be sensitive. Our process of item refinement was informed by the kinds of student reasoning we observed during the interviews. With these ideas in mind, our work was guided by the following research questions:

- What is the nature of student reasoning elicited by the items on our assessment?
- To what extent do the items on our assessment accurately measure student reasoning?

Literature

In this section, we lay out our literature review in two parts. First, we report on the process of assessment development for other undergraduate STEM courses and how others have established validity and reliability. Then, we discuss research on student thinking and common difficulties faced in learning linear algebra concepts, particularly with regard to our four previously mentioned focal topic areas. Examining the research on student thinking in linear algebra is important for contextualizing our work examining student reasoning.

Literature on Assessment Development

Assessments can serve a number of purposes – they can be designed to inform teachers' ongoing instructional decisions (formative assessments), to evaluate students' learning (summative assessments), to establish the effectiveness of an instructional approach or compare multiple instructional approaches, or to inform policy decisions (among other reasons). Our assessment is intended to measure students' conceptual understanding related to the four previously mentioned focal topic areas in introductory undergraduate linear algebra.

The processes used to develop assessments that measure conceptual understanding in multiple STEM topic areas have been well-documented (e.g. Sadaghiani, Miller, Pollock, & Rehn, 2013; Carlson, Oehrtman, & Engelke, 2010; Barniol and Zavala, 2014; Melhuish, 2015). For example, Carlson et al.'s (2010) approach to developing an assessment for Pre-Calculus was by first identifying foundational content areas within the course, creating open-ended questions to comprehensively cover those content areas, interviewing students to determine how students approached each question, refining the questions and converting them to multiple-choice format based on interview data, and finally administering the instrument to multiple samples. In contrast, Barniol and Zavala (2014) followed a simpler process in which they collected their questions from existing instruments and had the content validated by experts.

An important aspect of assessment development relates to the ways in which researchers validate these assessments. Examining how students reason about particular items and using this to inform the work of creating appropriate distractor items for multiple choice questions was an important phase in the process for several researchers (e.g. Hestenes, Wells, & Swackhamer, 1992; Carlson, Oehrtman, & Engelke, 2010; Wilcox & Pollock, 2013; Sadaghiani, Miller, Pollock, & Rehn, 2013). Their interview component helped establish construct validity by giving researchers a glimpse into students' interpretations of the questions and the kinds of reasoning that the items elicited. Appropriate probing and occasional direction from the interviewer also provided insight on student reactions to potential distractor items (Sadaghiani, et al., 2013).

Literature on Student Thinking in Linear Algebra

Broadly, linear algebra can be considered to encompass theory about systems of linear equations and their solution sets as well as theory about vector spaces and mappings between them (Andrews-Larson, 2015). Difficulty in the teaching and learning of linear algebra during students' first year of undergraduate study is well documented (Hillel, 2000; Sierpinski, 2000; Stewart & Thomas, 2009). The literature reveals that students also struggle with formal mathematical language in linear algebra courses (Stewart & Thomas, 2010; Sierpinska, 2000; Carlson, 1993). Additionally, the need to learn and coordinate modes of the description and representation of abstract concepts of linear algebra can function as a source of difficulty for students in their early learning (e.g. Carlson, 1993; Hillel, 2000). For example, Larson & Zandieh (2013) developed a framework that highlights geometric and algebraic interpretations of three views of the matrix equation A**x**=**b** that students need to develop: a linear combinations view, a system of equations view, and a transformation view. While these three views are organized around the mathematics that students need to learn, the framework highlights the ways in which the literal symbols A, **x**, and **b** are coordinated in each view – providing a useful lens for identifying the (sometimes idiosyncratic) ways in which students often blend these views. Flexible, coordinated use of these three views lies at the core of many ideas central to an introductory linear algebra class. For instance, understanding a matrix times a vector as a linear combination of the column vectors of the matrix is importantly related to ideas about span and linear independence (properties of sets of vectors), and understanding a matrix times a vector as a representation of a linear transformation is important for understanding ideas about mapping between vector spaces.

Span and linear independence: The literature has shown that students generally rely on procedural methods for approaching questions about linear dependence and independence, and often do not leverage geometric intuition (Bogomolny, 2007; Aydin, 2014). Furthermore, students have been found to perform better on questions about linear independence that require algebraic interpretation than on questions that require geometric interpretation, and that performance on both kinds of items was largely unrelated to students' performance on questions that require understanding of the formal definition of linear independence (Ertekin, Erhan, Solak, & Yazici, 2010). Stewart & Thomas (2010) laid a similar argument with regard to students' understanding of span. To ameliorate the problem of this disconnect between algebraic and geometric interpretations and their connection to formal definitions, Wawro et. al. (2012) have developed instructional materials intended to support students in developing well-connected understandings of span and linear independence.

Systems of linear equations: There is extensive literature on student thinking in K-12 settings documenting the importance of students' interpretations of the equals sign as they learn to solve single equations in one variable (e.g. Star & Rittle-Johnson, 2007; Li, Ding, Capraro & Capraro 2008). In the previously mentioned set of views of the matrix equation A**x**=**b**, Larson & Zandieh (2013) highlight the flexible set of interpretations for the vector **x** that students need to develop and ways in which students coordinate these views. Zandieh and Andrews-Larson (2015) have subsequently expanded this framework to consider ways in which the augmented matrix [A|**b**] relates to this set of interpretations, and commented on the mischief created by ***x***'s disappearing act. Students often struggle to interpret row-reduced forms of this augmented matrix to make sense of the solution sets of systems in which the number of equations differs from the number of unknowns, or in which the solution is unique; whereas in R^2, students are better able to leverage their geometric intuition to interpret the solution than in higher dimensional settings (Zandieh & Andrews-Larson, 2015).

Linear transformations: Research on students' thinking about linear transformations has identified a variety of metaphors students draw on to reason about how their understandings of functions might relate to linear transformations: input/output correspondence, morphing or mapping of vectors, and thinking in terms of a function 'machine' (Zandieh, Ellis, & Rasmussen, 2012). Bagley, Rasmussen, and Zandieh (2015) identified two productive ways of student reasoning about matrix invertibility: thinking about an inverse as a transformation that undoes another, and conceiving of an inverse as a transformation that, when composed with another, acts as the "do nothing" function – the latter being more closely consistent with typical definitions of invertibility. Others have considered students' thinking about invertibility as it relates to the set of equivalent statements included in the invertible matrix theorem (e.g. Wawro, 2015; Selinski, Rasmussen, Wawro, and Zandieh, 2014).

Eigenvectors and eigenvalues: Sinclair & Tabahgi (2010) have noted the highly geometric ways in which mathematicians talk and gesture as they reason about ideas related to eigenvectors and eigenvalues. In contrast, students often primarily draw on algebraic ways of reasoning about these ideas, and can struggle to make sense of the reasons behind symbolic shifts such as shifting from $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to $(A - \lambda I)\boldsymbol{x} = 0$ (Thomas & Stewart, 2011). It can be challenging for students to coordinate algebraic interpretations with geometric ones (e.g. Stewart & Thomas, 2010; Larson & Zandieh, 2013), and students' ideas about eigenvectors are often not well connected to other conceptual aspects of linear algebra (Lapp, Nyman, & Berry, 2010). In light of the value of these findings, researchers have developed interventions to support students in developing geometrically motivated ways of reasoning about eigenvectors and eigenvalues (Tabaghi & Sinclair, 2013; Zandieh, Wawro, & Rasmussen, 2016).

Theoretical Framing

A theoretical construct that has been useful in many areas of mathematics education for making sense of students' reasoning is the notion of concept image and concept definition (Tall & Vinner, 1981). According to Tall and Vinner (1981), concept image is the "total cognitive structure that is associated with the concepts, which include all the mental pictures and associated properties and process" (p. 152). For a given concept, every individual creates an image or structure in their mind that helps the individual understand and remember that concept. This concept image may or may not be similar to other individuals' images, and these images can be quite different from the formal definition of the concept. Tall and Vinner (1981) use the term "formal concept definition" to refer to the definition that is largely accepted by the mathematical community; they argue that this can be different from an individual's 'personal concept definition,' which may change over the time and with new knowledge as is the case with one's concept image.

The notion of concept image and concept definition has been used as a lens for examining student reasoning at the undergraduate level in a number of content areas, including linear algebra (e.g. Wawro et. al., 2011; Britton & Henderson, 2009). We draw on the notion of concept image and concept definition as an analytic tool for interpreting students' responses to assessment items. For our analysis, we look for alignment between a student's elicited concept image and the formal concept definition as evidence of understanding.

Data

In this study, we conducted semi-structured clinical interviews (Bernard, 1988) with eight university undergraduate students: six males and two females. One of the participants was taking linear algebra at the time of the interview, and the other participants had taken linear algebra within the last two years. The participants' majors covered fields that included mathematics, statistics, education and economics. At the time of the interview, participants had completed an average of four math classes after taking linear algebra.

Each participant was asked to work through eleven assessment questions using a think-aloud interview protocol in which the interviewer asked the student to read each item aloud and think aloud as he or she came to an answer. The interviewer then asked follow-up questions as needed to understand the student's reasoning in arriving at their answer. Each interview lasted for approximately one hour and was audio and video recorded.

We developed the assessment items used in this study by consulting past assessments prepared by five different mathematics faculty members at different institutions. After identifying a set of questions related to each of our four focal topics, three mathematics faculty members from three different institutions were consulted to identify which items these experts felt focused on key ideas and had the potential to assess students' conceptual understanding of these ideas. We modified our assessment according to experts' initial feedback, and the assessment items to be used in interviews were selected after receiving a second round of feedback from these experts.

In this paper, we provide an in-depth analysis of student reasoning on the first two items on our assessment. Figures 1 and 2 below show the wording of these two items as they appeared in the interview.

1. Answer the following questions regarding the set of vectors $V = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$.

a. Which of the following best describes the <u>span</u> of the set V?
   - i. A point
   - ii. Two points
   - iii. A line
   - iv. Two lines
   - v. A plane
   - vi. Two planes
   - vii. A 3-dimensional space

b. Give an example of one vector in the span of V, and show or explain how you found that vector.

c. Give an example of one vector in R^3 that is not in the span of V and explain how you know it is not in the span of V, OR explain why there is no such vector in R^3.

Figure 1: Assessment item focused on span

The items shown in Figures 1 and 2 focus on span and linear independence, which also entail conceptual connections to linear combinations and dimensionality. These concepts, along with their abstraction and notation, are often new ideas for students taking their first linear algebra course, and their position near the beginning of the course can also make these concepts the first hurdles to complete. They serve as core concepts in linear algebra as they provide foundation for thinking about solution sets to systems of linear equations as well as vector spaces and the mappings between them. In addition, the interview data was particularly rich for these items, possibly because they appeared at the beginning of the interview when students and interviewers were at their "freshest." For these reasons, as well as for the sake of time and space, we will only provide a closer look at students' reasoning on these two items.

2. Consider the set of vectors $\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 6 \\ -5 \\ 6 \end{bmatrix} \right\}$. To determine whether the set is linearly independent or dependent, a student did the following correct row reduction:

$$\begin{bmatrix} 1 & 4 & 3 & 6 & 0 \\ -2 & -1 & 1 & -5 & 0 \\ 3 & 0 & -3 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

a. The set of vectors is (circle one): linearly independent / linearly dependent.

b. Explain what it is about the solution that tells you whether or not the set is linearly independent or dependent.

Figure 2: Assessment item focused on linear dependence/independence

Methods of Analysis

In order to identify the kinds of student reasoning elicited by our assessment items and the extent to which these accurately assessed student understanding, we conducted our analysis in four phases: (I) characterizing individual students' concept images, (II) identifying themes across students, (III) identifying alignment between responses and reasoning. Phases I and II allowed us to identify the kinds of student thinking elicited by our assessment items. Phase III offered insight to the extent to which the items accurately assessed students understanding. Specifically, we looked to see whether the assessment items accurately documented alignment between students' concept images and the formal concept definition. The phases of analysis are described in greater detail below.

Phases

Phase I: Characterizing individual students' concept images: We developed a short description of what our data revealed about each student's concept image by first watching the video and, in some cases, transcribing each student's interview response to the question. We then developed a list of themes that characterized how he/she thought about the concept and collected quotes that exemplified characteristics of the student's thinking.

Phase II: Identifying themes across students in how students reason: We grouped students according to the nature of their concept images. This helped us document themes in how students reason about the items. These groupings of students' concept images were organized in a table to make it easier to identify trends in thinking.

Phase III: Documenting written responses: We identified what students stated their final answer would be (and other answers they offered if they changed their mind) as well as the justification they offered for their answer(s). This was accomplished by drawing on students' written work as well as using audio/video data as needed in cases when the response was given orally but not written on the student pages. Each response was then color-coded to indicate whether a correct response corresponded to correct or incorrect reasoning and whether an incorrect response corresponded to correct or incorrect reasoning. This was used to assess the extent to which the item accurately measured what we intended.

Findings

In this section, we will describe in detail trends observed in students' reasoning on items 1 and 2. We then offer an overview of how we coded alignment between correctness of responses and rationales offered for responses in order to synthesize the extent to which items accurately assessed students' understanding. In the discussion section, we address the ways in which our analysis informed revisions to the assessment.

Students' Reasoning about Item 1: Span

After interviewing students and transcribing their interviews we analyzed the first assessment item to document students' concept image of span. In part a of item 1, which asked students which best described the span of the given set of two (linearly independent) vectors in $\mathbb{R}^3$, four of eight students correctly chose "a plane," and the other four students were split with one each choosing a point, two lines, two planes, and a three-dimensional space. In examining students' justifications for these choices, we identified four themes in students' responses that offer insight into those aspects of students' concept images for span that are elicited by this item: (i) interpreted individual elements with notable effort, (ii) attempted to resolve the number of entries with the number of vectors (related to issue of dimensionality), (iii) reasoned about relationships between elements (related to linear dependence/independence), and (iv) interpreted procedurally.

Three students' responses were coded as entailing effortful interpretation of elements. For example, Barry reasoned "x is at 1, y is at 2, and then z is at 3, so that I would say that's a point." Kody argued "I'm pretty sure each of these represents a plane… could be two lines but I don't think it is because there's three coordinates." While these students selected different choices for what best represented the span of the set given in the question, both quotes highlight the students' focus on interpreting individual elements of the set.

Three students spent time working to resolve the number of entries in each vector (3 entries) and the number of vectors (2 vectors) as they came to their answer. Brenda, who ultimately came to incorrectly conclude the answer was a 3-dimensional space, explained her reasoning, "This [writes R^2] was x and y, and R^3 was x, y, and z. This might be 3-dimensional space, but there are two of them [vectors]." Carissa, who ultimately resolved the issue to correctly conclude the answer was a plane, said "So since there are two it will span just in the second dimension and in the second dimension would be a plane so then it might actually just be a plane."

Four students addressed relationships between elements of the set as they explained their reasoning on this question, in particular focusing on linear independence and dimension. Notably, all four of these students answered the question correctly. Carissa elaborated her earlier comment with the remark, "I think I need three linearly independent vectors to span a third dimension. So I think in this case it doesn't span a third dimension." Another student Lee argued "If they were linearly independent then one is either basically a scalar multiple of the other, so basically the same vector just different magnitude."

One student, Kenneth, described a procedural approach to answering the question. He put the two given vectors into a matrix, row reduced that matrix, and (incorrectly) concluded that the vectors in the row reduced matrix were in the span, and that the row reduced matrix would "span" the given set of vectors V. In explaining his choice that the span was best described as two lines, Kenneth explained, "I don't remember the graph, I don't think we went over the graphing. How to convert like the matrices and the variables to like graphs and stuff. I guess since I have two variables it would get two lines?" The comment about two variables is consistent with his work row reducing a matrix with two columns, and suggests he is drawing on a systems of equations interpretation to respond to the question – but he is unable to effectively use this interpretation to reason about the span of the two vectors given. It is also noteworthy that, while Kenneth's procedure was incorrect, it does correspond to a procedure that would be correct in a different context – for instance if one is given a vector and told to determine if it is in the span of a particular set of vectors, row reducing is a common method for making this determination. We posit that Kenneth was aware of some version of this procedure and attempted to adapt it to this context.

Looking across these themes in student responses as shown above in Table 1, we note that all students who reasoned about relationships between vectors in terms of linear dependence/independence responded to the item correctly. Students who exhibited effortful interpretation of elements gave incorrect responses, which suggests students need to be able to consider relationships among vectors to answer the item correctly.

Student	Response	Reasoning Categories Observed
Barry	Point (two?)	Effortful interpretation of elements (interpreted as points)
Kody	Two Planes	Effortful interpretation of elements (interpreted as planes) Attempting to resolve number of entries with number of vectors
Brenda	3d space	Effortful interpretation of elements; related to Attempting to resolve number of entries with number of vectors
Carissa	Plane	Attempting to resolve number of entries with number of vectors Evidence of reasoning about linear dependence
Lee	Plane	Evidence of reasoning about linear dependence (geometric)
Derrick	Plane	Evidence of reasoning about linear dependence (geometric)
Ronaldo	Plane	Evidence of reasoning about linear dependence (geometric) **ONLY student to draw on definition of span**
Kenneth	Two Lines	**Procedural approach** (row reduce matrix of given vectors; vectors in row reduced matrix are in the span of given set)

Table 1: Student responses to Question 1a and categories of reasoning observed

Students' Reasoning about Item 2: Linear Dependence and Independence

When analyzing the second item, we documented students' reasoning on the topic of linear independence and linear dependence. Seven out of eight students came to the correct conclusion that the vectors were linearly dependent. However, only four students reached this conclusion using sufficient, conceptually correct, generalizable reasoning. Amongst the eight students, five different ways of reasoning about linear dependence and independence were identified. Many of the students reasoned about the problem in more than one of these ways. The five ways of reasoning identified are: (i) dependence as a linear combination or one vector being a multiple of another, (ii) independence as a pivot in each row and/or column in the RREF augmented matrix, (iii) dependence as a row of zeros in the RREF augmented matrix, (iv) dependence as a relationship between variables (e.g. in the equation $x - z + 2t = 0$, x can be thought of as depending on z and t), and (v) dependence as the number of vectors exceeding the dimension space in which the vectors live. Below, we offer examples to illustrate these ways of reasoning. The first four ways of reasoning were used in ways that were sometimes correct and sometimes incorrect.

Three students talked about the concept of dependence as a linear combination (Ronaldo, Lee, Carissa). For example Ronaldo explained, "… so linear dependence also means that there is one of the vectors that can be written as a linear combination of the other ones, so there will have to be a solution to the system and that system has a solution because we don't have anything of the form all zeros…" We argue this aspect of these students' concept image of linear dependence was consistent with the concept definition of linear dependence. In contrast, Kenneth, the one student who arrived at the incorrect conclusion that the set was linearly independent argued that, "If the vectors are dependent that means they are like multiples of each other." While it is true that if one vector in a set is a multiple of another, the set will be linearly dependent, this is not the only way that a set can be linearly dependent as it doesn't include the important possibility that a set can be linearly dependent if one of the vectors is a linear combination of other vectors in the set. Because Kenneth's concept image of linear dependence only aligned partially with the definition of linear dependence, he was unable to answer this item correctly.

Two students argued the set would be linearly independent if there is a pivot in each row and/or column (Derek and Carissa). Carissa gave the explanation "… I think it has to be linearly independent because it has to have a pivot in every row or column I can't remember which one, but either way I don't have that here …" Carissa elaborated when she was asked about how the pivots relate to the linear dependence of the set, saying "we have two pivots so that means … since there is 4 vectors in the third dimension and we only have two [pivots], since we don't have the correct number of pivots they are linearly dependent". This suggests that Carissa was aware that there is a relationship between pivots and linear dependence, her uncertainty about whether the pivots need to be in rows or columns indicates that her understanding of the relationship between pivots and linear dependence is largely procedural and perhaps somewhat tenuous.

Two students reasoned that dependence can be determined by whether the augmented RREF matrix has a row of zeros (Brenda and Kody). Kody for example said "… it's linearly dependent because of that row of zeros … we can't have a row of zeros and have it be independent". While this reasoning was not considered incorrect, it is easy to give examples in which a row of zeros in the augmented RREF matrix does not imply that the corresponding set of vectors is linearly dependent. Therefore, although both students who cited this reasoning arrived at the correct conclusion (that the given set of vectors is linearly dependent), this particular reasoning was classified as conceptually insufficient or incomplete. In fact, the item was ultimately revised to remove the possibility that students could use this justification to argue that the set of vectors was linearly dependent.

Three students used the idea of dependence as a relationship among variables to give justification for the set being linearly dependent (Barry, Brenda and Ronaldo). Barry argued, "... whenever you look at these top two rows this is essentially like $x_1 - x_3 + 2x_4$ and similar to this one $x_2 + x_3 + x_4$ and so there is a kind of some dependency there because you can arrange this equation but you can see like x_1 depends on what x or … these solutions are kind dependent upon each other on x_1, x_3, x_4 and x_2, and x_3 and x_4 so if I am remembering things correctly I am gonna say it is linearly dependent." When the interviewer wanted to know more about what it is that he is saying it is linearly independent he replied “the variables, so I think abstractly it does not necessarily … it does not have to be necessarily x_1, but I am saying x_1 is linearly dependent on x_3 and x_4 but I feel like you could definitely switch around and say x_3 is linearly dependent on x_1, x_4 or x_4 linearly dependent on x_3 and x_1.” While Barry’s reasoning about the variables was correct, when probed by the interviewer, he said he did not know what this revealed about the original set of vectors. Note that Ronaldo’s previous comment shows how he was able to use ideas about linear combinations of vectors to relate the idea of dependence as relationships among variables to the idea of dependence as a property reflecting relationships among a set of vectors.

Finally, one student correctly noted that no computations were needed to solve this question, pointing out that because the number of vectors exceed the dimension space in which the vectors live, the set must necessarily be linearly dependent. Derek stated “It has to be linear dependent before even looking at the row reduction, because it is the 3-dimensional vectors so you know you are in R^3 and if you have 4 vectors in R^3 then you know that it is a linearly dependent, because at most the span of $\square^3$ can be all of R^3 but then only three vectors can do that.” This comment further highlights the interrelated nature of students’ understanding of span with their understanding of linear dependence and independence.

Looking across these ways of reasoning, we see that students who reasoned about linear dependence in terms of linear combinations were the most successful at producing correct and generalizable justifications for their responses. When students reasoned about linear dependence in terms of pivots and variables, there was more variation in evidence that they were able to link this reasoning to what this revealed about the original given set of vectors. This is perhaps unsurprising as these forms of reasoning are organized around representations that are more obscured from the original set of vectors when compared to reasoning organized around linear combinations of vectors.

Summary of Interview Analysis

Table 2 summarizes student responses to each assessment item. The “Correct%” column indicates which percent of students answered the item correctly, and the “Works%” column displays percentage of occurrences for which response and reasoning were aligned. We coded an item as “working properly” if a correct response was accompanied by correct reasoning or if an incorrect response was accompanied by incorrect reasoning. An example of misalignment between response and reasoning would be Barry’s response to item 1b; even though he provided a vector that was in the span of the set, he only picked it because he interpreted span to simply be the set, not as any vector that was a linear combination of the vectors in the set.

Table 2 also provides basic information about which items were changed, which will be addressed in more detail in the Discussion section. All changes were made after the eight interviews were completed with exception to 1a and 5. These items were modified after the first and sixth interview respectively to account for adjustments needed in light of these interviews. The additional distractors in 1a were added as a result of Barry’s response that the two elements of the vector set each represented a point. “Two points” as well as “two lines” and “two planes” were added to make the list of distractors comprehensive for all cases where students might use effortful interpretation of elements. The change in item 5 was made after six interviews because no student was able to correctly answer the question as it stood.

5. “Assume that T is a linear transformation where $T: \mathbb{R}^2 \to \mathbb{R}^2$ first rotates the plane clockwise by 90 degrees about the origin, and then stretches the plane horizontally by a

factor of 2. If A is the standard matrix for *T*, which of the following is true? Justify your choice."

Four answer options were provided, each of which expressed the transformation as the product of two matrices. Students had a difficult time interpreting the transformation resulting from each answer choice. The process was often time-consuming, and students usually started interpreting or computing the product of the two matrices given each answer choice, which offered limited insight into how students reasoned about the composition of two linear transformations. The revision version of the question gives students transformation described as product of two matrices, and asks them to decide which of four geometric descriptions best describes this transformation. While the two students who received the modified version still did not answer the question correctly, they reasoned about the item in ways more aligned with our intent.

Overview of Student Work										
Correct%	Q*	Barry	Brenda	Ronaldo	Derrick	Kody	Lee	Carissa	Kenneth	Works%
50.0%	1a	WA, WR	WA, WR	RA, RR	RA, RR	WA, WR	RA, RR	RA, RR	WA, WR	100.0%
75.0%	1b	RA, WR	WA, Cst.	RA, RR	RA, RR	RA, RR	RA, RR	RA, RR	WA, WR	75.0%
75.0%	1c	RA,WR	DK	RA, RR	RA, RR	RA, WR	RA, RR	RA, RR	WA, WR	71.4%
87.5%	2	RA, IC.	RA, RR	RA, RR	RA, RR	RA, IC.	RA, RR	RA, UC.	WA, WR	75.0%
87.5%	3a	RA, RR	RA, RR	RA, RR	RA, RR	RA, RR	RA, RR	RA, RR	WA, WR	100.0%
50.0%	3b	WA, WR	WA, WR	WA, WR	RA, RR	RA, RR	RA, RR	RA, RR	WA, WR	100.0%
37.5%	3c	WA, Cst.	WA, Cst.	WA, WR	RA, RR	RA, UC.	RA, UC.	WA, WR	WA, WR	75.0%
87.5%	4	RA,RR	RA,RR	WA,RR	RA,RR	RA,RR	RA,RR	RA,RR	RA,RR	87.5%
0.0%	5	Question modified, no one answered correctly								N/A
50.0%	6	DK	DK	RA,RR	RA,RR	WA,WR	RA,RR	RA, US.	DK	80.0%
75.0%	7	DK	RA, US.	RA,RR	RA,RR	RA,RR	RA,RR	WR, US.	RA,RR	85.7%
50.0%	8	RA,RR	DK	RA,RR	RA,RR	WA,WR	RA,WR	WA,WR	WA,WR	85.7%
75.0%	9	RA,RR	WA, WR	RA,RR	RA,RR	RA,RR	RA,RR	WA,WR	RA,WR	87.5%
50.0%	10	WA,WR	WA,WR	RA,RR	RA,RR	WA,WR	RA,RR	WA,WR	RA,RR	100.0%

Codes			
RA with RR, or WA with WR		No Change	
RA with WR, or WA with RR		Change in wording/presentation of Answers	
WA with Cst. reasoning from a prv. WA		Change in wording/presentation of Questions	
Unclear/Unsure/Incomplete/Don't Know		Question taken out	

Table 2: Alignment between correct reasoning and correct answers

**1: Span; 2: Linear Independence; 3&4: Liner Systems; 5&6: Transformations; 7&8: Invertibility; 9&10: Eigenvalues, Eigenvectors*

***RA: Right Answer; RR: Right Reasoning; WA: Wrong Answer; WR: Wrong Reasoning; Cst: Consistent; UC: Unclear; US: Unsure; IC: Incomplete; DK: Don't Know*

As Table 2 indicates with green shading, we can see that there was consistent alignment between answer and reasoning on many of the items. Some items were more problematic than others with respect to having alignment between responses as reasoning, such as items 1b and 1c. Other items elicited unclear or incomplete reasoning from students, making it difficult to judge whether the question was working properly. More careful future analysis of these items is needed. Item 6 was a more theoretical question that was written in a symbolically abstract way, and as a result, students had a very difficult time reasoning through the question and coming to an answer without some manner of interviewer scaffolding.

We can also begin to say something about the discriminatory power of many of the questions based on the percentage of correct responses. For example, several were answered correctly 50% of the time, while other items were answered correctly closer to 0% or 100%. This informed us about the strength of some of our questions to discriminate between students with strong and weak conceptual understanding. None of the eight students were able to correctly answer item 5, suggesting that this item was doing a poor job measuring any student understanding about transformations.

Discussion

Trends in Student Reasoning

Reflecting on our findings from item 1, we noted a key feature of students' reasoning that distinguished students who answered correctly from those who did not: students whose responses focused on making sense of the individual vectors answered the item incorrectly whereas students who focused on the relationship between vectors (typically framed in terms of linear dependence/independence) answered the item correctly. This finding has potential pedagogical implications, in that it suggests that students need opportunities to develop rich interpretations of vectors and linear combinations of vectors to help them develop ways of thinking about all possible linear combinations of a set of vectors in a coordinated way – the very definition of the span of a set of vectors. Additionally, our findings highlight the ways in which students' understanding of linear dependence and independence is interconnected with their understanding of span.

The trends we noted in students' responses to item one also have implications for assessment development in linear algebra. First, students' responses to item 1 also suggested that, in order to more fully students' understanding of span, we need to find a way to assess students' strategies for resolving differences between the number of vectors in a set and the number of entries in each vector – an idea related to dimensionality. Second, it might also be worth asking questions that target students' understanding of the linear combinations and span separately. Third, we also find it important to consider the relationship between students' understanding of span and linear independence and how we can assess students' recognition of that connection. We've added items to our assessment that aim to capture the first two of these implications; the third remains as an area for future exploration.

For item 2, students expressed a number of ways of reasoning about linear dependence and independence. For example, some students focused on the row of zeros in the RREF as providing justification to claim that the vectors were linearly dependent. Others drew on the presence of a pivot in each row or column of the RREF as the basis for their reasoning. One student also believed linear dependence was solely based on the idea that one vector should be a multiple of another vector. It was difficult to establish whether these students had a clear conceptual understanding of linear dependence of vectors or if they were simply trained to look for certain conditions to answer such questions. The prevalence of these relatively procedural ways of reasoning suggest it would be fruitful to be able to document conceptual connections students have with regard to those procedures.

Assessment Adjustments

We left the first item largely as-is because it had good discriminatory power, and this discriminatory power aligned with a clearly distinguishable aspect of students reasoning (e.g. whether they focus on individual elements in a set of vectors or relationships among elements when reasoning about span). However, we adjusted the second part of item 1 to more fully document students' concept image by requiring that they indicate all vectors among several options are in the span of a given sent of vectors; this accounted for a misalignment between correctness of reasoning and response documented on the second part of item 1 (which was not part of our fine-grained analysis). In addition, we added new item that aims to capture students' reasoning about the relationship between the number of entries in a vector and the space in which that vector lives.

The second item was adjusted so that students were still given a (linearly dependent) set of four vectors in $\mathbb{R}^3$, but these correspond to a row-reduced matrix that does not include a row of zeros. This removes the possibility that students would use the justification that there is a row of zeros to claim the set of vectors is linearly dependent. In fact, this change might increase the discriminatory power of the item, as if students think this is the only way for a set of vectors to be linearly dependent, they will now answer the item incorrectly.

Although, this paper focuses on the items related to span and linear dependence and independence, we also briefly describe the other kinds of modifications made in other items given the findings of the interviews. There were two main reasons for additional adjustments. One of these reasons was lack of discriminatory power (e.g. most students got question 4 correct and most students got question 5 correct) – so these items were removed or omitted. The other reasons were more logistically oriented, e.g. many students spent far more time than intended performing a row reduction on question three, so we elected to provide the row reduced matrix and ask students to interpret the result. We similarly removed item 7 because it had the potential to be computationally cumbersome and we wanted students to be equally likely to do well on the assessment regardless of their access to a calculator.

While this paper shares useful findings about the creation of an assessment to measure students' conceptual understanding of linear algebra, there is still more work to do. Our next step is to administer the revised assessment to several larger samples and conduct item and factor analysis based on this data. While the validation process for our assessment may not yet be complete, we believe this preliminary study reveals important insights into student reasoning that could be helpful to linear algebra instructors and assessment developers.

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Example construction in a transition-to-proof classroom

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Accurately constructing examples and counterexamples is an important component of transition-to-proof courses. This study investigates how one instructor of a transition-to-proof course taught students to construct examples, and compares this instruction to the examples constructed by the students.

Keywords: transition-to-proof; undergraduate instruction; example construction

Introduction

Learning to write proofs is a complicated process, and students develop a variety of beliefs about how to construct a proof (Harel & Sowder, 2007). Using examples is a well-established strategy in the proof writing process, and examples can be used for several purposes when developing and proving conjectures (Alcock & Inglis, 2008; Alcock & Weber, 2010; Lockwood, Ellis, & Knuth, 2013). Yet undergraduate students often use examples ineffectively (Iannone, Inglis, Mejia-Ramos, Simpson, & Weber, 2011).

The term example has many meanings in mathematics (Watson & Mason, 2002). Within this study, the term example is limited to a mathematical object which satisfies specific characteristics and illustrates a definition, concept or statement (Moore, 1994). This definition excludes sample proofs, such as demonstrations of the direct proof technique or proofs by induction. Alcock and Weber (2010) claim that this definition of example is "probably the most common intended meaning of the term when it is used by mathematicians and mathematics educators in the context of proof-oriented mathematics" (p. 2).

Research questions. In this study, the following questions are addressed:
1. In what ways do students construct examples effectively and ineffectively on tasks in their transition-to-proof course?
2. How did the instructor teach example construction?
3. What connections are found between the students' construction of examples and the instruction given?

Literature and Theory

Considerable literature is available on the proving abilities of students and mathematicians, and the use examples on proving tasks (Alcock & Weber, 2010; Buchbinder & Zaslavsky, 2011; Dahlberg & Housman, 1997; Iannone et al., 2011; Lockwood, Ellis, Dogan, Williams, & Knuth, 2012; Moore, 1994; Watson & Mason, 2002, 2005; Weber, Porter, & Housman, 2008). However, in the interest of space, much of this literature has been omitted. The review below focuses on the literature concerning the construction of examples, and the role of example in teaching advanced mathematics courses.

Example construction. Antonini (2006) sought to answer how examples are constructed, by conducting clinical interviews with seven mathematicians. From these interviews three distinct techniques emerged: *trial and error*, *transformation*, and *analysis*. *Trial and error* is characterized by constructing objects, and then testing whether the object has the desired characteristics. *Transformation* is characterized by taking a known object which has some of the necessary characteristics, and then performing adjustments until the object has all the required characteristics. *Analysis* is characterized by deducing additional properties of the object, until this list of properties evokes a known example or produces an algorithm for constructing an example.

Antonini (2006) observed that mathematicians often follow a process of starting with *trial and error* and then using *transformation*, if *trial and error* failed. The *analysis* technique was used after failing to construct an example with both the *trial and error* and *transformation* techniques. Antonini (2006) notes that the *analysis* technique is appropriate when it is possible that no example with the given properties exists, since the derivation of properties could lead to a proof by contradiction.

This classification of example constructions was applied to undergraduate students by Iannone et al. (2011), when they asked students to generate examples of a particular type of function and classified the students' construction technique. The students generated examples with a *trial and error* technique on 51 out of 62 constructions, and on the remaining 11 constructions the students used a *transformation* technique where they modified a known example.

Iannone et al. (2011) theorized that the *trial and error* strategy resulted in weaker conceptual gains than the other strategies. However, in a second study Iannone et al. (2011) found that there was no significant differences between the proof productions of students who generated their own examples and those who were provided examples. In fact, Iannone et al. (2011) found that the proof productions of the example reading group was slightly higher than the proof productions of the example generating group, although the difference was not significant.

The teaching and learning of mathematics. One of the primary goals of mathematics education is to develop and implement interventions that change mathematics teaching (Fukawa-Connelly, 2012a). At the undergraduate level, Speer, Smith, and Horvath (2010) criticized that "very little empirical research has yet described and analyzed the practices of teachers of mathematics" (p. 99), even though poor undergraduate mathematics teaching is often cited as a reason students change majors away from science, technology, engineering, and mathematics fields (Seymour & Hewitt, 1997). In fact, Mejia-Ramos and Inglis (2009) conducted a literature of 102 mathematics education research papers concerning undergraduate students' experience reading, writing and understanding proofs, yet none of these papers described the instruction the students received. Although some studies have investigated instruction in proof writing since the publication of these critiques (e.g. Fukawa-Connelly, 2012a, 2012b; Mills, 2014), there is still a need for additional studies in this area.

Instruction can influence the choices that students make and their preferences when solving problems, including proofs. Students need strategic knowledge in order to select appropriate strategies (Weber, 2001), but students typically do not learn these strategies without instruction (Lester, 1994). However, some instructors try to design their courses in order to explicitly teach students strategic knowledge (Weber, 2004, 2005).

Theoretical framework. This study draws on the emergent framework developed by Cobb and Yackel (1996), which links the social perspectives of classroom mathematical practices to the psychological perspectives of mathematical conceptions and activity. This theoretical perspective explains the design choice to look simultaneously at instructional practices and student responses to tasks.

In addition, this study draws on a theory of effective example use on proof-related task (Hanusch, 2015). This theory was developed by utilizing grounded theory, in the style of Glaser and Strauss (1967). When using examples effectively on proof-related tasks, an individual proceeds through four phases: deciding to use an example, establishing the intended purpose of the example, constructing the example, and finally drawing conclusions from the example (see Figure 1). In the first phase the individual decides to use an example, usually either because it was indicated by the task language or because the individual felt they lacked sufficient knowledge about a concept. In the second phase, the individual identifies their intended purpose for using the example, such as to gain improved understanding or to determine the truth of a statement. Then the individual moves into the third phase, constructing the example. Finally, in the fourth phase, the individual draws conclusions from their example, such as determining if they have fulfilled their purpose or if they need additional examples. This study focuses on the third phase of this framework, example construction, but the analysis of the constructions must consider the other phases of the process.

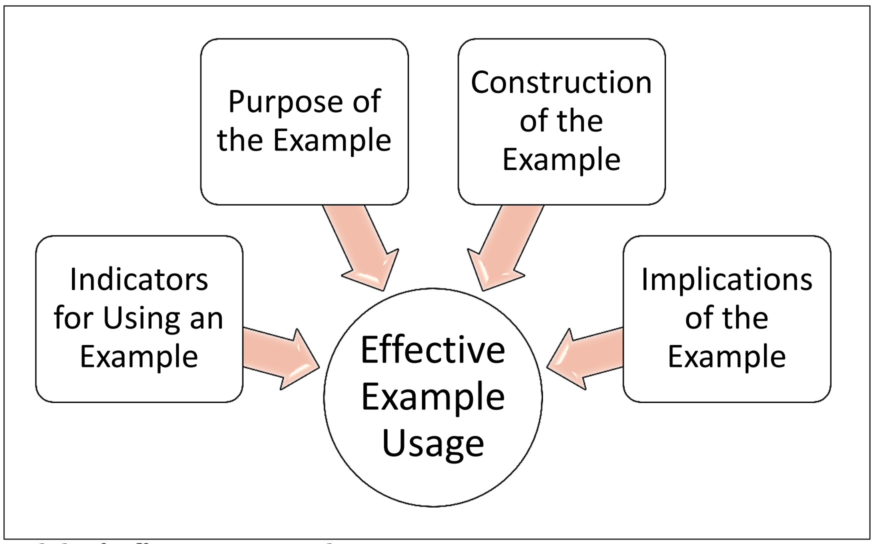

Figure 1. A model of effective example usage.

Method

This study is part of a larger study that focused on all example activity within a single section of a transition-to-proof course at a large university. The participants in this study are the instructor, Dr. S, and the 27 students enrolled in her course during the semester of the study.

Four students were selected for more detailed data collection during the fourth week of the semester using maximal variation sampling (Creswell, 2013). By varying the stu-

Table 1
The characteristics of the sampled students.

Name	Year	Major	GPA	Course Attempt
Amy	Sr.	Mathematics for Secondary Teaching	2.50-2.99	3rd
Carl	Soph.	Mathematics for Secondary Teaching	2.50-2.99	1st
Raul	Jr.	Applied Mathematics and Biochemistry	3.50-4.00	1st
Mike	Sr.	Mathematics and Spanish	3.00-3.49	2nd

dents' levels of academic success (indicated by a self-reported grade point average), mathematical preparation (indicated by self-reported grades in mathematics coursework), and specialization (pure, applied, secondary teaching), the findings have increased transferability (Merriam, 2009). The characteristics of the four students included in the sample are presented in Table 1. These students were purposefully selected because they frequently spoke during class, both by asking the professor questions and presenting their own work on the blackboards.

Data collection. Interviews were conducted with the four selected students, in order to observe each student's process on proof-related tasks while working independently. These interviews occurred three times during the semester: around the seventh week of the semester, the twelfth week of the semester, and the last week of the semester. Unfortunately, Mike was unavailable for the final interview, which explains why he used significantly fewer examples than the other participants.

Each interview with a student had three components: a semi-structured portion addressing proof strategies and goals for the course, a task-based portion where students attempted several proof-related tasks, and a reflection on the tasks. The semi-structured portion asked the students to talk about their impressions of the course, namely what they had learned and what they thought they should be learning. The tasks for the interviews were selected from the textbook, or other studies on undergraduate proof writing (Alcock & Weber, 2010; Dahlberg & Housman, 1997; Iannone et al., 2011). The mathematical content of the questions varied over the three interviews, corresponding to the recent content from the course. Additionally, the tasks asked for different types of products including, constructing examples, constructing counterexamples, making conjectures, validating proofs, and writing proofs. After a student completed all tasks, then the students were asked to reflect on their work. Sometimes the final reflection was omitted due to time constraints.

The classroom was observed daily to observe the examples used by the instructor during lectures and student presentations. The observations are supplemented by three interviews with the instructor during the semester, and a member checking interview the following year. These interviews focused on the motivation for the choices made during class and how those choices influenced the desired instructional goals.

Results

Construction of examples. When considering the construction of examples, two levels of analysis were needed: 1) the accuracy of the example, and 2) the construction technique used. The accuracy was determined by the author by checking that the hypotheses and conclusions were met by each construction, and the accuracy levels for each student can be found in Table 2.

Four categories were used to describe the construction technique: *blind trial and error*,

Table 2
This table summarizes the construction abilities of the students.

Construction	Amy	Carl	Raul	Mike	Total
Accurate Construction	30	16	18	4	68
Inaccurate	1	1	3	1	6
Incomplete	5	2	0	1	8
Authoritarian Source	3	2	2	0	7
Blind Trial and Error	2	1	7	3	13
Informed Trial and Error	14	17	4	1	33
Transformation	17	2	8	2	29

informed trial and error, *transformation* and *authoritarian*. An *authoritarian* example is retrieved from a source, instead of being constructed by the prover. The sources included the textbook, their notes, and the statement of the question. The terms *trial and error* and *transformation* came from the definitions of Antonini (2006). However, the students used two distinct version of the *trial and error* technique: *blind trial and error* where a student selected a potential example with no consideration for the hypotheses and then tests the hypotheses and conclusion, and *informed trial and error* where a student purposefully selected a potential example thinking about the hypotheses. Neither the students nor the professor discussed the *analysis* technique, so this category was not used. Table 2 indicates the frequency of each of these classifications by student.

Only six of the constructions were inaccurate. Four of these occurred during the first interview, when Raul and Mike appeared to misunderstand which conditions must hold for a construction to be an example or a counterexample. During the first interview, Raul wrote the constructions found in Figure 2, while attempting to prove if $a|(b-c)$ and $a|(c-d)$, then $a|(b-d)$. The construction that Raul labeled as a counterexample does not satisfy the hypotheses or the conclusion of the statement, which means it is not a counterexample. The remaining two inaccurate examples were due to the students making errors when verifying the hypotheses.

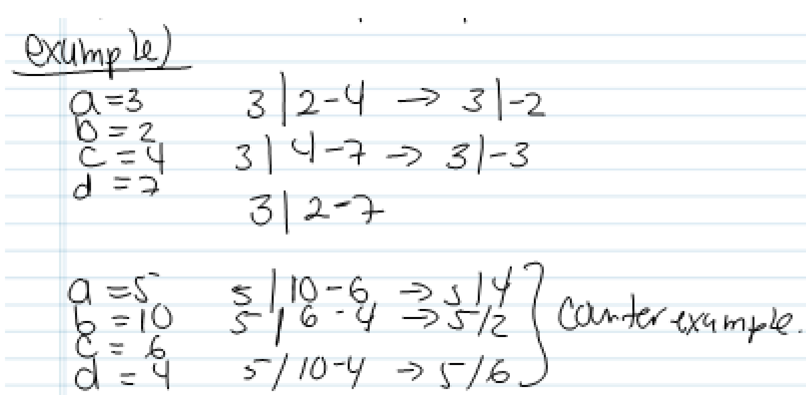

Figure 2. The constructions generated by Raul for if $a|(b-c)$ and $a|(c-d)$, then $a|(b-d)$.

Some of the accurate constructions were not useful for fulfilling the intended purpose. For instance, Mike was seeking a potential counterexample on a divisibility problem and chose $a = 1$ as the value for the divisor, stating that he chose this value because "1 divides

Table 3
The construction techniques used by the students during the interviews

Interview	1	2	3
Authoritarian Source	0	5	2
Blind Trial and Error	9	0	4
Informed Trial and Error	14	6	13
Transformation	1	5	22

everything." Mike did not realize that this choice for a meant that every possible construction would be an example of the statement. Although other students constructed examples that were not useful for their purpose, this was the only instance in which a student stated a fact that would directly indicate the lack of usefulness.

The students transitioned to more advanced construction techniques later in the semester, see Table 3. During the first interview, Mike was the only student to utilize the *transformation* construction technique, and he only did so once. By the final interview, the students were using the *transformation* technique more frequently than *trial and error*. For instance, when constructing an example of a fine function, Amy, Raul and Carl all began by modifying the graph of $\sin x$ so its zeros coincided with the integers satisfying the definition of a fine function. Raul made two additional transformations of the sine function to construct examples of fine functions, see Figure 3.

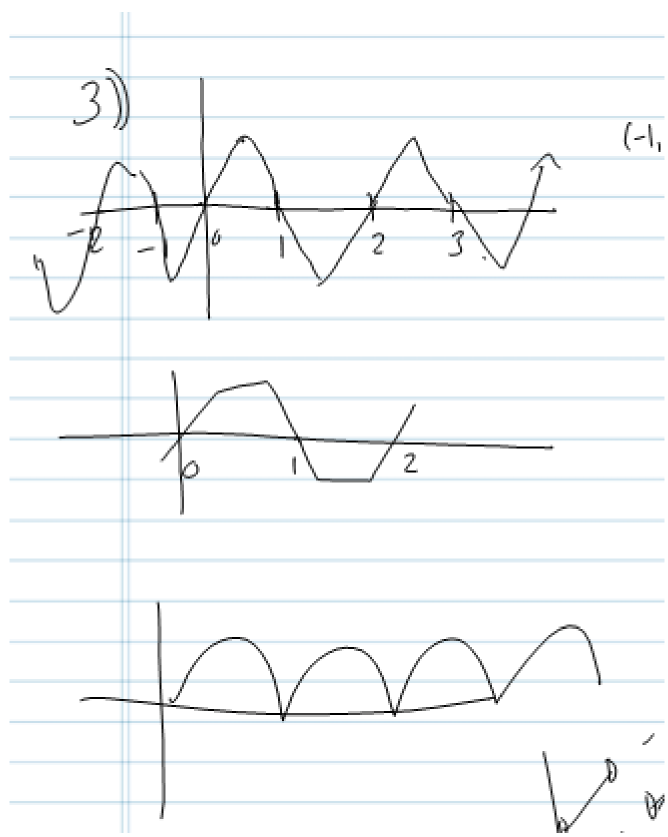

Figure 3. The graphs of fine functions constructed by Raul using the transformation technique.

The instruction. Dr. S rarely modeled example construction during the lecture. Although she presented many examples throughout the semester, she seldom talked about how these examples were constructed. Dr. S did model how to determine which properties an example or counterexample of a statement needs to satisfy, and how to go about verifying that a construction satisfies those properties. When asked what construction techniques she expected from her students Dr. S stated that "It depends on the problem, but to some extent trial and error is the first step." Dr. S continued by saying some students are "not always ready yet" for more sophisticated reasoning, and "I'm okay with them randomly trying at first." To provide students opportunities to practice example construction, Dr. S assigned homework that the student's were to write on the board prior to class. Dr. S began class everyday by reviewing and correcting this student work. Furthermore, Dr. S frequently reminded the students to "make mistakes on [their] homework, and we'll talk about them." She suspected that the students would often fail before they succeeded at example construction, and that the best way to help the students improve would be to review their constructions.

There were two episodes from the lecture where Dr. S emphasized example construction. The first instance occurred shortly after formally defining functions. Dr. S emphasized the importance of a function being well-defined, particularly when the domain is a partition. To do this, Dr. S presented three potential functions:

$$\begin{array}{ll} f:\mathbb{Z}_3\to\mathbb{Z}_6 & f\left([x]_3\right)=[3x+2]_6 \\ g:\mathbb{Z}_4\to\mathbb{Z}_2 & g\left([x]_4\right)=[3x]_2 \\ h:\mathbb{Q}\to\mathbb{Z} & h\left(\frac{a}{b}\right)=a+b \end{array}$$

The first example was generated using numbers suggested by the students, the last two were purposely chosen by Dr. S. Dr. S proved that f and h are not well-defined by producing counterexamples that show that two different representatives of the equivalence classes produce different outputs. For g, Dr. S provided the students with a proof that it was well-defined. Ultimately this episode demonstrated the meaning of a well-defined function, but also included instruction in constructing examples.

Dr. S intended that the students would move towards the *transformation* construction strategy by asking themselves questions such as "is the statement similar to one [I] know?" and then using that response to construct their example. During the an interview, Dr. S reiterated this by saying "I would like to move them toward more directed examples where they are intentionally trying to go certain places but I doubt that most of them are ready for that. Right now I'm happy if they try random examples to see what's going on, as long as they don't stop there." Perseverance was a frequent theme when discussing proof and example constructions in the lecture.

When the students wrote incorrect example constructions on the board, Dr. S would usually ask the student who presented (or sometimes the whole class) to help her revise the construction. In one instance, Carl presented a relation on $A = 1, 2, 3$ that should have the properties of symmetry and transitivity, but not reflexivity. Carl presented the relation $\{(1, 2)(2, 1)(1, 3)(2, 3)(3, 2)(3, 1)\}$, but this example is not transitive. Dr. S argued that if $(1, 2)$ and $(2, 1)$ are in the relation, then transitivity requires that $(1, 1)$ and $(2, 2)$ must also be included. As such, Dr. S changed the relation to $\{(1, 2), (2, 1), (1, 1), (2, 2)\}$, which is symmetric and transitive, but not reflexive because it is missing $(3,3)$. Through this discussion, Dr. S walked the students through using the *transformation* technique for example construction, since she transformed an existing example to satisfy the given criteria.

Comparing the instruction and the students. The students used the *trial and error* construction technique for all of the examples constructed during the first interview, with one exception. However, as the semester progressed the students used the *transformation* technique with increasing frequency, as seen in Table 3. Dr. S predicted this behavior of the students. In the first interview, Dr. S said

> It depends on the problem, but to some extent, *trial and error* is the very first step. You just try stuff. I've seen this even with advanced REU students, where there is a good strategy. They're not always ready yet. I'm okay with them randomly trying at first. Now, I want them to move toward more careful construction. As they go through this, they should be looking for things that are similar and using that to give them a hint.

Dr. S recognized that as beginning students, they would not have the mathematical experience to use the more advanced *informed trial and error*, *transformation* and *analysis* techniques, but she hoped they would grow to that point. The *analysis* technique was not demonstrated by the instructor or used by the students; however, during the member checking interview, Dr. S argued that the *analysis* technique was too advanced to be useful to the students at their current level of understanding.

During the first interview, Dr. S stated that although she expects the students to have some familiarity with using examples from their calculus classes, "they just never had to construct [examples] themselves before." As such, she expected some of the difficulties the students had with example construction.

Dr. S did not explicitly vocalize an expectation of the accuracy problems exhibited by some the students during the initial interviews. Both Raul and Mike had created examples that violated the statement hypotheses. Raul did not seem to realize that failing the hypotheses was a problem. During the member checking interview, Dr. S said students often make these types of construction errors at this point in their development. She furthered this by explaining that many students present counterexamples that are not actually counterexamples, especially on the first test of the course.

Dr. S usually did not talk about the construction technique when she presented examples to the class, focusing instead of how to verify the accuracy. Additionally, she designed the course so that most of the example construction tasks were assigned as student presentations, and that she would talk about example construction as she reviewed and corrected the examples in the presentations. However, the students tended to present problems asking for proofs rather than the problems asking for examples. Consequently, Dr. S did not have the opportunity to talk about construction techniques with the expected frequency.

Overall, Dr. S had the experience to know the capabilities of the students with respect to example construction. She recognized that *trial and error* would be the primary technique at the beginning of the semester, and that some of the students would not be able to move beyond that technique in this course. However, towards the end of the semester, she introduced the *transformation* construction technique for the benefit of the students who were ready for more advanced techniques. The students in the sample were able to apply the *transformation* technique in some circumstances, and likely will be able to utilize it more frequently in their subsequent courses.

Discussion

By the end of the semester, all of the students selected examples with more thought, and used the *transformation* construction technique with increased frequency. It is unclear exactly what caused this growth. Possible explanations include the students' individual development throughout the semester, the influence from the instruction, and the new content.

The mathematical content of the task-based interviews corresponded to the content of the course, which varied throughout the semester. Specifically, the first interview consisted entirely of number theory tasks, the second interview consisted for set theory and equivalence relation questions, and the final interview concerned real-valued functions. The students appeared to have a large sample space for real-valued functions to utilize for example constructions. This means one reason the students were able to use the *transformation* technique more frequently is because they had examples of real-valued functions to use as the starting point for the transformation process.

In particular, when asked to construct an example of a fine function on question 3 of interview 3, the first example constructed by each student was a transformation of $y = \sin x$. The students recognized that the pattern of the zeros in $y = \sin x$ could be adjusted to satisfy the conditions of a fine function. It is unlikely that the students could have constructed an example of a fine function via *trial and error* because of how difficult it would be to verify the conditions. However, it is equally difficult to imagine a students utilizing a *transformation* technique on $a|(bc)$ implied $a|b$ or $a|c$, especially for an initial example of the statement. Most students will not have a sufficient background in the formal language of divisibility to have such examples in their personal example space.

Previous research on undergraduate example construction showed that the students used *trial and error* techniques approximately 80% of the time (Iannone et al., 2011). This percentage is considerably higher than than the 57% *trial and error* observed in this study. It is unclear what accounts for this discrepancy, although the one cause may be the task selection. Additionally, both studies had small samples, this one had four participants and Iannone et al. (2011) had nine, so the individual characteristics of the participants strongly affected the percentages.

Implications for teaching transition-to-proof courses. One implication is that students should be explicitly taught strategies for constructing and verifying examples. One of the hardest parts of trial and error for the students was picking a construction to test. However by explaining how the examples in the course are constructed, it may be possible to guide the students beyond blinding picking parameters to test.

In this study, most of the students became convinced that a *prove or disprove* statement was true after constructing only one or two examples. However, when mathematicians obtain conviction from empirical evidence it is often from multiple examples or for unusual properties (Weber, 2013; Weber, Inglis, & Mejia-Ramos, 2014). Although it is unreasonable to assume that numerous examples should be constructed before trying to prove a statement, we need to teach students to consider the quality of the examples they construct and to view the examples as a collection. For example, a statement that is true for a prime number, a perfect square, and another composite number is far more believable than a statement evaluated only with a prime number. But students need to be taught to consider examples collectively rather than only individually.

Future research. Additional research needs to be completed on the instruction of example construction. How does instruction impact a provers ability to effectively use and construct examples? It is unclear whether or not such instruction will actually help the students learn how to construct examples effectively. Some studies suggest that instruction in problem solving frameworks alone does not help students become better problem solvers (Garofalo & Lester, 1985; Schoenfeld, 1980), so it is possible a similar phenomenon will occur here. This can only be established through additional study.

Finally, it is unclear whether effective example construction will positively impact proof writing. Iannone et al. (2011) found that generating examples provided no benefits to the students as compared to receiving a list of examples. One interpretation of this is that it does not matter where the examples come from, what matters is how the examples are used and what conclusions are drawn from the examples. As such, it is possible that knowledge in using examples effectively can improve a persons ability to successfully write proofs, but additional study is needed on this topic.

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An initial look at students' conveyed meanings for probability

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Probability is the central component that allows statistics to be a useful tool for many fields. Thus, the meanings that students develop for probability have the potential for lasting impacts. This report extends Thompson's (2015) theory of meanings through the notion of conveyed meaning: the constrained implications that a receiver attributes to the sender's statements. A student's conveyed meanings give insight into his/her initial and/or dominate meanings for a particular idea. This report shares the results of examining 114 undergraduate students' conveyed meanings for probability after they received instruction as well as their instructors' conveyed meanings. The worrisome presence of circular conveyed meanings carries implications for the teaching of probability.

Key words: probability, statistics, meanings, introductory statistics course

Probability is the engine that makes inferential statistics run. In particular, probability is the result of centuries of work towards one goal: the quantification of uncertainty. Since before the 1600s mathematicians, philosophers, logicians, and statisticians have attempted to resolve questions where uncertain outcomes dominate (Weisberg, 2014). Over the course of history, many scholars have engaged in what Thompson (2011) calls quantitative reasoning and quantification. In settling what a measure of uncertainty means (along with how to get a measure and what is meant by measuring uncertainty), scholars have taken different paths and arrived at their own meanings for the same notion, probability. Laplace considered the ratio of the number of outcomes of interest to the number of all possible outcomes under the assumption of "equally likely" outcomes (Weisberg, 2014). Von Mises (1981) considered repeating some process indefinitely to build a collective and that the limit of the relative frequency of an event of interest was the probability of that event. Kolmogorov's (2013) axiomatic, measure-theoretic approach has become the gold standard for probability theory. De Finetti (1974) and Savage (1972) regarded probability as dealing with measuring the amount of belief that an individual had for a particular outcome's occurrence that they called "subjective" or "personalistic" probability.

The opening statement of this paper is one that hardly any practitioner of statistics will disagree with. Regardless of which school of probability you ask (i.e., Frequentist, Bayesian, Conditional Frequentist, etc.) each acknowledges that the central ideas of probability allow us to move beyond merely describing a data set to using the data set as evidence for supporting or refuting claims. The members of these schools of thought have already carried out the quantification of uncertainty, something that students have yet to undertake. How practitioners think is often vastly different from how students think before, during, and after instruction. Kahneman and Tversky (1974, 1982) described how individuals will use different heuristics when making judgments under uncertainty. For example, how representative an event (sample) is to the parent process (population) to can influence a person's estimate of the probability of the event. Another heuristic that they found people use to measure uncertainty centers on the ease (or lack of) with which a person can imagine the event occurring; the more "available" an event is for the person to imagine, the larger the probability (the less uncertainty) there is for that event. Konold (1989) found that for some individuals, their way of thinking about probability did not match the use of heuristics nor was their thinking consistent with the schools of probability. Rather, these individuals appeared to view the goal of uncertainty to be the prediction of the next result; Konold referred to this way of thinking as the outcome approach to probability. Students also have a tendency to view events as equally probable when they do not perceive the many ways a compound event might occur (Lecoutre, Durand, & Cordier, 1990). Lecoutre et al. found that students and adults view the event of getting a five and a six as having the same probability as getting two sixes when rolling two dice. They hypothesized that not recognizing that event of (5, 6) is comprised of two smaller events. This way of thinking across multiple events is what they referred to as the equiprobability bias. Fielding-Wells (2014) found that when trying to pick the best card for playing addition-bingo, Year 3 students (7-8 years) operated as though all of the sums of the numbers 1 to 10 were equally probable. Saldanha and Liu (2014) reviewed much of the literature on students' understandings of probability and proposed that a key conceptual scheme for understanding the measurement of uncertainty is a stochastic conception. They define a stochastic conception as "a conception of probability that is built on the concepts of random process and distribution" (p. 393). They argue that in the quest to support students developing coherent probabilistic reasoning, instructors need to conceive of probability as ways of thinking rather than skills and design curriculum that supports this. While this review of the literature is brief, already apparent is the fact that the quantification of uncertainty is challenging. Individuals of all ages and backgrounds struggle just as mathematicians, statisticians have to construct a meaning for probability.

In a discussion with a university instructor about introductory statistics courses, I was surprised to hear this individual say "I skip by probability because my students don't really need it and we need the time to talk about doing hypothesis tests." This statement caught me off-guard for two reasons: 1) this instructor had a Ph. D. in Statistics, and 2) the instructor continued to talk about how she wanted her students to develop "rich and productive meanings for hypothesis tests and *p*-values". While I believe that students *can* and *will* develop meanings for hypothesis tests in the absence of a way of thinking about probability, I challenge the claim that students can develop "rich and productive" meanings. The sentiment that the instructor expressed about skipping probability is reminiscent of a position that Cobb and Moore (1997) took; "first courses in statistics should contain essentially no formal probability theory" (p. 820). Since they originally took this stance, the American Statistical Association through their Guidelines for Assessment and Instruction in Statistics Education (GAISE) have endorsed the notion of reducing probabilities role in introductory statistics classes (Aliaga et al., 2005). However, there is a critical distinction between Cobb and Moore's position and the aforementioned instructor's: Cobb and Moore's position (echoed in GAISE) is *not* that first courses should avoid discussing probability, bur rather there should be little to no emphasis on *formal rules of calculating* probabilities (e.g., what to for all types of cases for $P[A \cup B]$ or $P[A \cap B]$). They believe that discussing probability is still important for statistical inference. Liu and Thompson (2002) make an excellent argument that trying to debate the question of "What *is* probability?" is a fruitless endeavor in a first course. Rather, in a first course on statistics and probability, our focus should be on what we (us and our students) *mean* by the term "probability".

Students do not learn and instructors do not teach in a vacuum with only each other's company. Course materials such as textbooks play a role in student learning. A quick perusal of four introductory statistics texts indicates that these texts cover probability in the exact way that Cobb and Moore urged against; i.e., each focus almost entirely on how to calculate rather than how to think about (what do we mean by) probability. The introductory text *Statistics for the Life Sciences,* 4th edition (Samuels, Witmer, & Schaffner, 2012) devotes 15 pages to probability. However, there are only two sentences related to how to think about probability. Out of the 29 exercises provided for the students to use for homework, 26 ask for students to calculate a value of the probability of some event, 3 ask students to make a claim about whether or not two events are independent, and no questions ask students to interpret/make use of a way of thinking about probability. Likewise, *Introduction to the Practice of Statistics*, 7th edition (Moore, McCabe, & Craig, 2012) devotes 18 pages to probability and randomness. Of these pages, only 3 sentences (all variations of each other) focus on how to think about probability. There are only two questions of the 45 that focus on something other than a calculation of probabilities or judgment of independence; one asks whether or not a probability value is applicable to a larger set of colleges, and the other asks students to explain what a probability value means. In both of these cases, students' major takeaway is that probability is a calculation. In addition to these two traditional textbooks, I examined two open source texts; *Introductory Statistics* (Illowsky, Dean, & OpenStax College, 2013) and *OpenIntro Statistics*, 3rd edition (Diez, Barr, & Çetinkaya-Rundel, 2016). There are 51 pages devoted to the topic of probability in *Introductory Statistics*, while the *OpenIntro* gives 40 pages to the "special topic" of probability. There is only one sentence in each that focuses on how to think about probability. For the 128 homework questions in *Introductory Statistics*, only three ask for something other than a calculation of a probability value. Those three questions ask students to state what an expression such as $P[A \text{ OR } B]$ means in words. None of *OpenIntro*'s 44 questions ask students to interpret a probability value. I chose these four texts for different reasons. The Samuels text was the textbook used by the students I studied; the Moore text is one of the most popular introductory texts at the undergraduate level. We are seeing a surge in number of open access textbooks across all disciplines and I've had statistics educators recommend checking out both of the texts I did examine. My intent was not to conduct a full-scale text analysis, but rather just to get a sense of how textbooks treatment probability. Worth pointing out is that all four of these texts have publishing dates after the Cobb and Moore article and after the release/adoption of GAISE.

Given the recommendations in GAISE, what I found in the textbooks I examined, and the extant literature on people's understanding of probability, I began to question what meanings students could possibly have for probability *after* instruction. In particular, I wanted to answer the questions:

- What meanings do students convey for probability after they received instruction?
- Are there differences in the students' conveyed meanings based on which instructor they had?

Theoretical Background

To investigate my questions, I turned to the theory of meanings that Thompson and Harel have devised (see, Thompson, 2015; Thompson, Carlson, Byerley, & Hatfield, 2014). *Meaning* refers to the space of implications which includes actions, images, and other meanings that results from an individual assimilating some experience and thereby forming some understanding of that experience (Thompson, 2015). The space of implications, that is the meaning, is the inference that accompanies assimilation (Jonckheere, Mandelbrot, & Piaget, 1958). Central to the radical constructivist perspective is the belief that every individual builds his/her own knowledge through repeated experiences (von Glasersfeld, 1995). The meaning that an individual imbues an experience with is a product of their constructions. Thus, an individual's meanings are intensely personal and researchers do not have access to that individual's meanings. This problem is a familiar one to mathematics education researchers. Thompson (2013) tackled a similar question; if meanings are entirely within individuals, then how can people learn a meaning from someone else? Thompson's found an answer by turning to the notion of intersubjectivity and Pask's conversation theory. Intersubjectivity hinges on an individual having a mental image of another person that is free to think like and not like individual (von Glasersfeld, 1995). From Pask's theory, Thompson (2013) highlights that a conversation is more than just a verbal exchanges; conversation also includes all of the participants' "attempts to convey and discern meaning" (p. 63). He uses the following figure to highlight the blending of intersubjectivity and conversation theory:

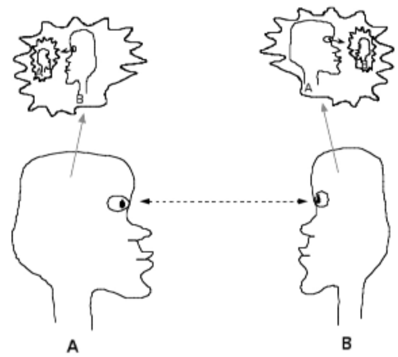

Figure 1. A "meaningful" conversation (Thompson, 2013, p. 64).

As 'A' and 'B' talk to each other, they must each keep in mind not only what they wish to communicate but also how the other person might interpret his/her words/actions. Both 'A' and 'B' build a model of the other person. Thompson's work provides an answer for how the conveyance of meaning from one individual to another might occur. Suppose that 'A' wants to communicate something specific to 'B'. When 'B' assimilates the experience to his schemes, he imbues that experience with a meaning that stems from two sources. The first source is his own meanings; the second is what he knows about 'A'. The meaning that 'B' gives to 'A's is what I refer to as the *conveyed meaning.* A conveyed meaning is the set of implications that a receiver attributes to the sender's message constrained by 1) the receiver's de-centering and 2) the receiver's belief that the sender made an honest effort to convey his/her thinking. These constraints lay the groundwork for intersubjectivity and keep both participants in the picture. While 'A's conveyed meaning might not be a perfect reflection of 'A's actual meaning, this is how 'B' understood 'A' and is the basis for which 'B' to now respond. The notion of conveyed meaning is useful in education in several ways. First, we can use this notion in research to attempt to discern what meanings our students have constructed for various topics. Second, we can use this notion in the planning of lessons. By trying to answer the question of "what have I conveyed to my students?" we can engage in de-centering and design meaningful conversations. This second use is easily extended to a third focused on the generation of curriculum materials such as activities and textbooks. With textbooks, we can imagine to types of conveyed meaning; the first being what the authors conveyed to us and the second being what the authors conveyed to our students.

By examining students' responses, we can characterize those responses by the meaning conveyed. However, to compare categories of conveyed meanings there must be an aspect of the theory that deals with productiveness of the meanings. Thompson (2015) proposes that productive meanings are those meanings that provide coherence to ideas that students have and those meanings which afford students a frame that supports the students in future learning (Thompson, 2015). *Productive meanings* are clear, widely applicable (within reason), and entail an awareness of and need to explicate any assumptions. Notice that the usefulness of a meaning is part of a productive meaning. I take a *useful meaning* to be any meaning that allows a student to meet some performance or learning goal. Consider the following meanings for the associative property: A) move parentheses around vs. B) the choice of which of two structures to impose ($[a+b]+c$ or $a+[b+c]$) does not change the result. Meaning A is useful; students can get correct answers, however, this meaning does not necessarily help students when there are more than three terms. However, meaning B is a productive meaning and useful.

Methodology

To answer my research questions I conducted an observational study at a large, public university located in the Southwestern region of the United States during the Spring 2014 semester. Students enrolled in an introductory statistics course aimed at life science majors took a written survey during a regular class period after they had received instruction about probability. I was only able to secure a five-minute window from the course instructors, so the survey consisted of two questions related to their meanings for probability. The course had four sections, taught by three instructors. Instructor A was a Ph.D. Math Education student who had not previously taught the course, Instructor B was a Senior Lecture (MS-Statistics) who had taught the course many times and served as the course coordinator, and Instructor C was as Ph.D. Statistics student who had taught the course the previous semester. Instructors B and C (B had two sections) followed the textbook (the aforementioned *Statistics for the Life Sciences* (Samuels et al., 2012)) closely. Instructor C made use of many of B's materials. Instructor A departed from the textbook, and instead focused on designing a course that sought to foster students' construction of productive meanings. Each of these instructors also responded to the written survey. The two questions students and instructors responded two were:

Question 1: How do you think about probability? That is, how would you explain probability to another person?

Question 2: Consider the following statement:

The probability of observing a value of 4 when looking at the product of two dice is 3/36.
How should someone think about (interpret) 3/36 given the above statement?

To analyze the responses I used a grounded method consistent with that described by Strauss and Corbin (1990). I initially used open coding for the responses and then I made use of an axial coding system. I used the axial codes in my analysis that follows. This approach partnered with the theoretical perspective is also consistent with the general methodology used in the development the Mathematical Meanings for Teaching Secondary Mathematics instrument described in Thompson (2015).

Results

I used the ten axial codes as shown in Table 1, five for each survey question along with the ordering of the meanings in terms of productivity. The categories of long-run relative frequency and percent of the time both point to the same underlying meaning: if you were to repeat some process an unlimited number of times, the probability of an event is the percentage of the time (i.e., the unlimited trials) or long-run relative frequency of that event. This meaning is the most productive out of the set presented. This meaning is applicable to a wide range of situations (discrete or continuous) and makes relatively few assumptions that are included in the statement (repeating a process indefinitely).

The categories of Frequency and Classical are similar and are the next two most productive meanings. The conveyed meanings in the frequency category appear to reflect a blending of Frequentist language and Classical notions. Classical conveyed meanings are consistent with Laplacian (set-theory based) notions of probability. This notion depends on the assumption of "equally likely" but none of the responses included this assumption.
The third category for Question 1 covers those students' responses that dealt with prediction. The responses that fall into this category are reminiscent of the outcome-approach of probabilistic thinking (Konold, 1989). Often these students only spoke about the very next time you carry out some process. This focus makes the meaning less productive than the proceeding ones. While the conveyed meaning in for the category "Fixed Number of Rolls" is similar to those in the "Classical" category in viewing 3/36 as two numbers; these

meanings convey a sense that we must repeat a process a fixed number of times and we will always see exactly the same number of the events of interest. For example, if we were to roll the dice 36 times, we would then see exactly 3 products of 4 ("fixed number of rolls"). This conveyed meaning points to a meaning that runs counter to all of the schools of thought about probability.

Table 1. Axial Codes for Conveyed Meanings from Survey Responses.

Q1: How do you think about probability?	**More Productive**	**Q2: Interpreting 3/36**
Long-run Relative Frequency (L.R.R.F): The student conveyed that the value emerges from carrying out some process a large number of times; the value is the relative frequency of how many times you see an event with respect to the number of trials.		**Percent of the Time:** the student conveyed that 3/36 was a single number that was the percent (fraction) of the time that you would see a product of four if you were to roll two dice "many, many times".
Frequency: The student conveyed that the value was the absolute or relative frequency of seeing an event but did not convey the existence of a repeating process.		**Classical:** the student conveyed that 3/36 was two numbers; the "first" ("upper") number tells you how many ways to get a four there are and the "second" ("lower") number tells you the number of ways you can roll two dice.
Prediction: The student conveyed that the value referred to the next trial (outcome approach) or the student conveyed that the value was a long-run prediction of something unspecified.		**Fixed Number of Rolls**: the student conveyed that 3/36 was two numbers; the 3 told you how many times you would see a product of 4 when you rolled two dice exactly 36 times (the second number).
Circular: The student conveyed that probability was just "chance" or "likelihood" of something happening.		**Chance:** the student conveyed that 3/36 was your "chance" of getting a product of four.
Other: The student's conveyed meaning did not fit any other category.		**Other:** The student's conveyed meaning did not fit any other category.
	Less Productive	

The fourth category of conveyed meanings for Question 1 is what I call "Circular". Typical responses that fall into this category are "Probability is the chance that something happens", or "the likelihood of some event". In trying to understand such student statements, I spoke with students who were willing to speak to me from the course as well as other students from the mathematics-tutoring center. During these discussions, the students who spoke of probability as being "chance" or the "likelihood" of some event, would often answer the follow up question of "What is chance/likelihood?" with the statements along the lines of "well, chance is, umm, just probability." Several times these students would also introduce the third term or even a fourth ("odds"). If I asked the student to explain probability but banned the usage of the terms "chance", "likelihood", and "odds", the students struggled to say anything. The way that these students thought about probability appeared to be a near unending cycle of labels with little meaning behind those labels. The seemingly only way these students broke out of this cycle was when they had to deal with a concrete situation and a specific value for them to speak about. However, this is did not always work. The conveyed meaning of Chance for Question 2 is a case in point. Responses here conveyed that 3/36 was merely the “chance” that you see a product of four with no further explanation. This circular conveyed meaning is the least productive meaning (other than the Other category). This meaning does not lay any groundwork for future learning and lends no coherence to the idea of probability. The final category of conveyed meanings for both questions is the “Other” category that captures any conveyed meaning not captured by the other categories.

What meanings do students convey for probability after they received instruction?

The following bar chart (Figure 2) shows the frequency of responses that fall into these categories for Question 1. Overwhelmingly, 89 students (78.1%) gave a response conveyed a circular meaning. Nineteen students (16%) appear to think about probability in terms of frequency/relative frequency. Of these students, 15 think about probability as the long-run relative frequency of some process.

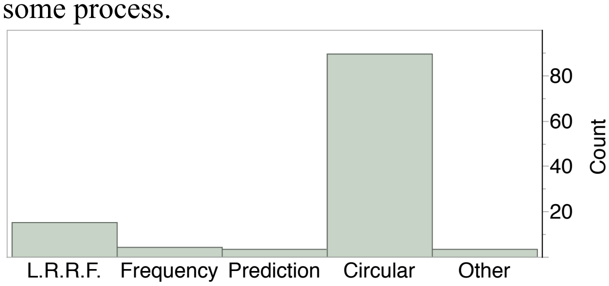

Figure 2. Students' conveyed meanings for Question 1.

As shown in the Figure 3, a majority of students interpreted the probability value (Question 2) as being about a fixed number of rolls of the dice and a fixed number of 4’s (40.5%). Only 17.1% (19) of the students thought about 3/36 as representing the percent of the time we would see a product of 4. Fifteen students appeared to use a “classical” way of thinking, while 19 just substituted “chance” for “probability”.

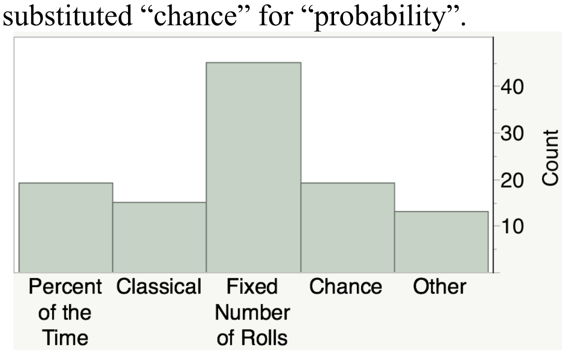

Figure 3. Students' conveyed meanings for Question 2.

A natural question that follows from the previous two questions, is how do the students’ responses to each question relate to one another? Table 2 shows the two-way contingency table for students’ responses to both questions. The vast majority of individuals who appeared conveyed that probability is the long-run relative frequency of some event also conveyed that a given probability value as the percent of the time we would see some event happen in the long run. The majority of students who conveyed that 3/36 as being two numbers separated by a bar (either Classical or Fixed Number) or as a “measure of chance”, also conveyed a circular meaning for probability. The wide range of interpretations given by students with a circular meaning is not surprising. Given that the students’ meaning for probability appears related to a word-exchange, the students would need to draw upon some other meanings to help make sense of the value 3/36. All but one student, who explained 3/36 as the "chance" of getting a product of 4, gave responses that indicated a circular meaning to Question 1.

Table 2. Students' responses to Question 1 by their responses to Question 2.

	Percent of the Time	Classical	Fixed Number of Rolls	Chance	Other	total
L.R.R.F.	12	1	0	1	0	14
Frequency	1	0	1	0	2	4
Prediction	2	1	0	0	0	3
Circular	4	12	42	18	10	87
Other	0	0	1	0	2	3
total	19	14	44	19	18	111

I will quickly mention that three students did not answer Question 2, thus the reduction in the grand total to 111. Taking these results together, we can see most students conveyed meanings that are less productive than what they could have. The overwhelming percentage of students who initial conveyed a circular meaning for probability for Question 1 is worrisome. The prevalence of such conveyed meanings and whether or not this conveyed meaning is truly indicative of the students' actual meanings require further research.

Are there differences in the students' conveyed meanings based on which instructor they had?

To explore this question, I used the ordering imposed on the conveyed meanings by the notion of productive meanings, to establish a multivariate ranking of the students across both questions (Wittkowski, 2004; Wittkowski, Song, Anderson, & Daniels, 2008). This non-parametric approach makes use of my theoretical perspective without adding any assumptions about the data that would be unreasonable (e.g., data are normally distributed). The Kruskal-Wallis test is an appropriate method to see if the there are differences between students' conveyed meanings by instructor. The test statistic has a value of 50.7783; thus, under a χ_2^2 distribution, the approximate probability of observing a value at least as extreme was we did is $p < 0.0001$. A post-hoc analysis using the Steel-Dwass method shows that Instructor A's students' conveyed meanings tend to be more productive than Instructor B's students' ($p < 0.0001$; large effect size of 0.6255) as well as more productive than Instructor C's students' ($p < 0.0001$; large effect size 0.5458). The conveyed meanings from Instructor B's students do not appear to be statistically different from Instructor C's students' in terms of productivity ($p = 0.9268$). To examine the site of the difference at the multivariate level, I also examined each question.

Table 3 provides a good visualization of how students described probability in Question 1. A striking aspect to notice is that all of the students who conveyed that probability is the long-run relative frequency all have Instructor A. Additionally, the vast majority of students for both Instructor B and Instructor C gave responses that conveyed a circular meaning for probability in general.

Table 3. Students' Responses to Question 1 by Students' Instructor

	L.R.R.F.	Frequency	Prediction	Circular	Other	total
Instructor A	15	2	2	8	0	27
Instructor B	0	2	1	58	1	62
Instructor C	0	0	0	23	2	25
total	15	4	3	84	8	114

To further explore this difference, I conducted a Kruskal-Wallis test with α = 0.05. The test statistic has a value of 58.0382. Thus under a χ_2^2 distribution, the approximate probability of observing a difference at least as extreme as we did is $p < 0.0001$. The post-hoc analysis using the Steel-Dwass method shows that Instructor A's students' responses are significantly different from the responses of Instructor B's students ($p < 0.0001$; large effect size of 0.6255) and significantly different from Instructor C's students ($p < 0.0001$; large effect size of 0.4807). However, the responses from Instructor B's and Instructor C's are not significantly different from each other ($p = 0.1752$).

Much like the prior question, a two-way contingency table provides insight into answering the question about the difference in how students interpret a given probability value in relation to the students' instructor. Notice in Table 4 that the vast majority of students conveyed that 3/36 as a percent of time have Instructor A and two-thirds of Instructor A's students gave this type of interpretation. None of Instructor C's students and only 1 of Instructor B's students gave a response that fell into this category. Given that the majority of Instructor B's and Instructor C's students conveyed a circular meaning for probability (see Table 3), the spread of their students' interpretations is not surprising.

Table 4. Students' Responses to Question 2 by Students' Instructor

	Percent of the Time	Classical	Fixed Number of Rolls	Chance	Other	total
Instructor A	18	3	3	1	1	26
Instructor B	1	10	27	12	11	61
Instructor C	0	2	15	6	1	24
total	19	15	45	19	13	111

I conducted another Kruskal-Wallis test (with α = 0.05) to test the difference between the students' responses in relation to instructor. The test statistic has a value of 32.2145. Under a χ_2^2 distribution, the approximate probability that we observe the differences we did or one greater is $p < 0.0001$. Post-hoc analysis using the Steel-Dwass method indicates that Instructor A's students' responses are significantly different from those of Instructor B's students ($p < 0.0001$; large effect size of 0.4995) and Instructor C's students ($p < 0.0001$; large effect size of 0.4262). The responses of Instructor B's students are not significantly different from Instructor C's students ($p = 0.9801$).

Thus, we can say that there does appear to be statistically significant difference between the students' conveyed meanings based upon which instructor they had. This difference exists at all levels of analysis and with consistently large effect sizes. One possible reason for this difference could be the meanings that the instructors conveyed during the course for probability. Table 5 shows how each instructor's responses to the same two questions the students answered fell under my axial coding system for conveyed meaning.

Table 5. Instructors' Conveyed Meanings for Questions 1 and 2.

	Response to Question 1 (probability in general)	Response to Question 2 (interpret 3/36)
Instructor A	L.R.R.F. (55.6%)	Percent of the Time (69.23%)
Instructor B	Circular (93.5%)	Classical (16.4%)
Instructor C	Circular (92%)	Classical (8.33%)

For Question 1, a majority of each instructor's students conveyed a meaning of the same category. However, for Question 2, this does not hold true for Instructors B and C. In both cases, the conveyed meaning that the majority of the students conveyed was the Fixed Number of Rolls (Instructor B: 44.3%; Instructor C: 62.5%). If the meanings conveyed by the instructors on the survey are consistent and representative of their conveyed meanings during instruction, then this data suggests that instructors' conveyed meanings serve as a limiting factor for students' construction of meaning.

Discussion

The vast majority (78.1%) of students conveyed a circular meaning for probability, with roughly 13% (of 114) conveyed probability as being about the long-run relative frequency of some event (given a stochastic process). Similarly, a majority of students (40.5%) conveyed that a probability value, 3/36, was about a fixed number of rolls (of dice) and observing exactly 3 outcomes that were the event of interest. There are differences in the conveyed meanings of each instructor's students. In the case of Instructor A, the most common conveyed meaning for probability (both in general and for interpreting) students gave is consistent with thinking about probability as the long-run relative frequency of an outcome of some repeatable process. For Instructor B's and Instructor C's students, the dominant conveyed meaning for probability focused on a circular word-exchange and a fixed number of trials. While further investigation into each teacher's actual meanings for probability as well as what meanings they convey during instruction is necessary, there is evidence to support the idea that a teacher's mathematical meanings serve as one of the key components of how that teacher teaches (Thompson, 2013).

Circular conveyed meanings

One finding of this study is that a vast majority (78.1%) of the students described probability in way that which I classified as conveying a circular meaning. As I mentioned previously, the origin of this category's label grew out of many interactions with students seeking help from this and other statistics courses in a tutoring center. Any time I worked with a statistics student, I would ask him/her to explain probability and much of the time the student would give a statement nearly identical to those the circular category. When I would press the student and ban any word in his/her cycle (i.e., "probability", "chance", "likelihood", "odds"), the student would often struggle. Only when I would move the student into a specific context did many (but not all) students break free of their circular statements. Since this original study, I've built a set of questions specifically to see test the circular nature of students' conveyed meanings. Preliminary results show that 31 of 35 (88.57%) students believe that at least two of the terms "probability", "likelihood", and "chance" are the same. Twenty of these students explicitly stated that they believe that all three terms as the same. Additionally, their responses to other items suggest that their circular conveyed meanings are reflective of their actual meanings and those meanings are context dependent.

Students are not alone in conveying a circular meaning for probability; textbooks, instructors, and everyday language equate these three terms. The aforementioned *Statistics for the Life Sciences* (Samuels, 2015) explicitly defines probability as "a numerical quantity that expresses the likelihood of an event" (p. 83). The authors go on to use "chance" interchangeably with "probability". The *OpenIntro Statistics* (Diez et al., 2016) uses "chance", "probability", and "likely" interchangeably as well. Even the Moore, McCabe, and Craig (2012) textbook includes "…what chance, or probability, each possible sample has" (p. 193). I will admit that in the past, I supported students in coming the think about "probability", "likelihood", and "chance" as all being the same. However, there do appear to be consequences for this. As instructors, we must be aware that the meanings conveyed to our students are not necessarily the meanings we have nor we intended to support students in constructing (Thompson, 2013, 2015). By intentionally making distinctions between the meanings we want students to construct for "probability", "likelihood", "chance", and "odds" (and "personal (un-) certainty") we can better help students develop more productive ways of thinking about situations involving uncertainty. To this end I propose that instructors, authors, and curriculum designers adopt the following distinctions:

- Probability refers to the long-run relative frequency of an event when you imagine repeating a stochastic process indefinitely. The description of the particular stochastic process includes any assumptions. Probability is a measure that focuses on the occurrence of events given our theory/assumptions.
- Likelihood refers to the long-run relative frequency of a parameter taking on a certain value (more generally, our assumptions being "true") given our observed data.
- Chance refers to the relative size of the subset of outcomes of interest with respect to the size of the sample space. While a stochastic process exists, there is an assumption that each simple outcome has the same chance of occurring as every other simple outcome. This assumption removes the need to carry out the stochastic process.
- Odds is a ratio formed by using one of the above measures and comparing an event's under that measure to the event's complement's under that same measure.
- Personal (Un-) Certainty refers to an individual's belief that a particular result will occur in a non-repeatable process; i.e. there is no stochastic process in the situation.

I must point out that the distinctions here are not new. The distinction I make between probability and likelihood is consistent with how statisticians already treat likelihood functions and cumulative/probability density functions. What I've called chance is also known as "Classical" or "Laplacian" probability (von Mises, 1981; Weisberg, 2014). Von Mises (1981) argued that his notion of probability built upon the notion of long-run relative frequency in the collective was not the same as Laplace's which hinges on the assumption of equal chances. Further he notes that "authors start with the 'equally likely cases', only to abandon this point of view at a suitable moment and turn aside to the notation of probability which is based on the frequency definition" (von Mises, 1981, p. 99). Savage's (1972) notion of "personalistic probability" serves as the basis for what I've called personal (un-) certainty; which he argued is fundamentally different from both probability and chance. What is new is my call for instructors (and researchers) to use different terms for each of these ideas rather than a single umbrella term ("probability") with a modifier ("classical", "frequentist", "subjective", etc.) that often gets dropped when writing. I hypothesize that by using different terms for the different ideas and discussing the differences, fewer students will develop meanings that lead them to convey a circular meaning for probability.

Limitations and Future Directions

A limitation to this study is that responses to two questions do not necessarily provide enough information to confidently describe an individual's meanings for a mathematical topic. However, when considering the role of conveyed meanings, this limitation is not as serious as initial thought. We cannot ignore the fact that the students chose to write what he/she did. I believe that the meaning that the students conveyed is strongly related to each students' initial and/or dominate meaning for probability. If the students' go-to explanation for probability is circular, then this suggests that their underlying meanings are not as productive as we would desire. Further research including a larger survey with a set of more items designed to get at students meanings for probability is in the works. As I mentioned in the previous section, preliminary results from piloting indicate that a majority of students convey a circular meaning for probability. Additionally, given that this was an observational study, we cannot definitively say that Instructor A is the cause for stark differences between the three sets of students' responses. However, given that Instructor A made the decision to follow a curriculum centered on assisting students in developing productive ways of thinking, there is evidence of a strong causal relationship. Further research could substantiate this claim.

The notion of productive meanings for probability joined with the notion of conveyed meaning is useful in the development of a scale to measure students' progress in developing coherent meanings. Such a scale serves as a progress variable which represents "(a) the developmental structures underlying a metric for measuring student achievement and growth, (b) a criterion-reference context for diagnosing student needs, and (c) a common basis for interpretation of student responses to assessment tasks" (Kennedy & Wilson, 2007, pp. 3–4). Establishing a progress variable for probability along with items that measure such a variable has the potential to change how we teach probability at all levels. Additionally, a progress variable for probability is vital for other areas of statistics education research including students' notions of *p*-values, hypothesis testing, and distributions of random variables.

This study serves as but a first step in examining how undergraduate students' conveyed meanings for probability after receiving instruction. I introduced my notion of conveyed meaning as the constrained implications that a receiver attributes to the sender's statements. Conveyed meanings can provide insight into the actual meanings that an individual has. The present study indicates that the dominant conveyed meanings for probability after instruction are circular and calculationally oriented. One section of the course, which used a curriculum aimed to support students in developing rich meanings, does have a number of students who conveyed a highly productive meaning for probability. Further work needs to be done in order to help more students develop a rich and deep meaning for probability that is coherent and does work for the students in statistics.

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Exploration of the Factors that Support Learning: Web-based Activity and Testing Systems in Community College Algebra

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A variety of computerized interactive learning platforms exist. Most include instructional supports in the form of problem sets. Feedback to users ranges from a single words like "Correct!" to offers of hints and partially to fully worked examples. Behind-the-scenes design of such systems varies as well – from static dictionaries of problems to "intelligent" and responsive programming that adapts assignments to users' demonstrated skills, timing, and an array of other learning theory-informed data collection within the computerized environment. This short paper presents background on digital learning contexts and describes the lively conversation with attendees at the conference poster session. The topics were the research design and early results of a cluster-randomized controlled trial study in community college elementary algebra classes where the intervention was a particular type of web-based activity and testing system.

Key words: Adaptive Tutoring System, College Algebra, Multi-site Cluster Randomized Controlled Trial

Research Questions

Funded by the U.S. Department of Education, we are conducting a large-scale mixed methods study in over 30 community colleges. The study is driven by two research questions:

Research Question 1: What student, instructor, or community college factors are associated with more effective learning from the implemented digital learning platform?

Research Question 2: What challenges to use-as-intended (by developers) are faculty encountering and how are they responding to the challenges as they implement the learning tool?

Background and Conceptual Framing

First, there are distinctions among cognitive, dynamic, and static learning environments (see Table 1). Learning environments can vary along at least two dimensions: (1) the extent to which they adaptively respond to student behavior and (2) the extent to which they are based on a careful cognitive model.

Table 1. Conceptual framework of the types of instruction based on adaptability and their basis in a theory of learning.

		Static	Dynamic
Is a particular model of learning explicit in design and implementation (structure and processes)?	No	Text and tasks with instructional adaptation external to the materials	Adaptive tutoring systems (Khan Academy, ALEKS, ActiveMath)
	Yes	Textbook design and use driven by fidelity to an explicit theory of learning	"Intelligent" tutoring systems (Cognitive Tutor)

Static learning environments are those that are non-adaptive without reliance on an underlying cognitive model – they deliver content in a fixed order and contain scaffolds or feedback that are identical for all users. The design may be based on intuition, convenience, or aesthetic appeal. An example of this type of environment might be online problem sets from a textbook that give immediate feedback to students (e.g., "Correct" or "Incorrect").

Dynamic learning environments keep track of student behavior (e.g., errors, error rates, or time-on-problem) and use this information in a programmed decision tree that selects problem sets and/or feedback based on students' estimated mastery of specific skills. An example of a dynamic environment might be a system such as ALEKS or the "mastery challenge" approach now used at the online Khan Academy. For example, at khanacadmy.org a behind-the-scenes data analyzer captures student performance on a "mastery challenge" set of items. Once a student gets six items in a row correct, the next level set of items in a programmed target learning trajectory is offered. Depending on the number and type of items the particular user answers incorrectly (e.g., on the path to six items in a row done correctly), the analyzer program identifies target content and assembles the next "mastery challenge" set of items. Above and beyond such responsive assignment generation, programming in a "cognitively-based" dynamic environment is informed by a theoretical model that asserts the cognitive processing necessary for acquiring skills (Anderson et al. 1995; Koedinger & Corbett, 2006). For example, instead of specifying only that graphing is important and should be practiced, a cognitively-based environment also will specify the student thinking and skills needed to comprehend graphing (e.g., connecting spatial and verbal information), and provide feedback and scaffolds that support these cognitive processes (e.g., visuo-spatial feedback and graphics that are integrated with text). In cognitively-based environments, scaffolds themselves can also be adaptive (e.g., more scaffolding through examples can be provided early in learning and scaffolding can be faded as a student acquires expertise; Ritter et al., 2007). Like other dynamic systems, cognitively-based systems can also provide summaries of student progress, which better enable teachers to support struggling students. Some studies have shown the promise of cognitively-based dynamic environments in post-secondary mathematics (Koedinger & Sueker, 1996).

Method

The study we report here is a multi-site cluster randomized trial (note: because the study is currently underway, we purposefully under-report some details). Half of instructors at each community college site are assigned to use a particular adaptive web-based system in their instruction (Treatment condition), the other half teach as they usually would (Control condition). The primary outcome measure for students' performance is an assessment from the Mathematics Diagnostic Testing Program (MDTP), which is a valid and reliable assessment of students' algebraic knowledge (Gerachis & Manaster, 1995).

Using a stratified sampling approach to recruitment, we first conducted a cluster analysis on all 112 community college sites eligible to participate in the study (i.e., in a state that was a study partner and offering semester-long courses in elementary algebra that met at least some of the time in a physical classroom or learning/computer lab). The cluster analysis was based on college-level characteristics that may be related to student learning (e.g., average age of students at the college, the proportion of adjunct faculty, etc.). This analysis led to five clusters of colleges. Our recruitment efforts then aimed to include a proportionate number of colleges within each cluster. The primary value of this approach is that it allows more appropriate generalization of study findings to the target population (Tipton, 2014). Recruitment for our first cohort of participants yielded a study sample of 38 colleges similar to the overall distribution across clusters that was the target for the sample (see Figure 1).

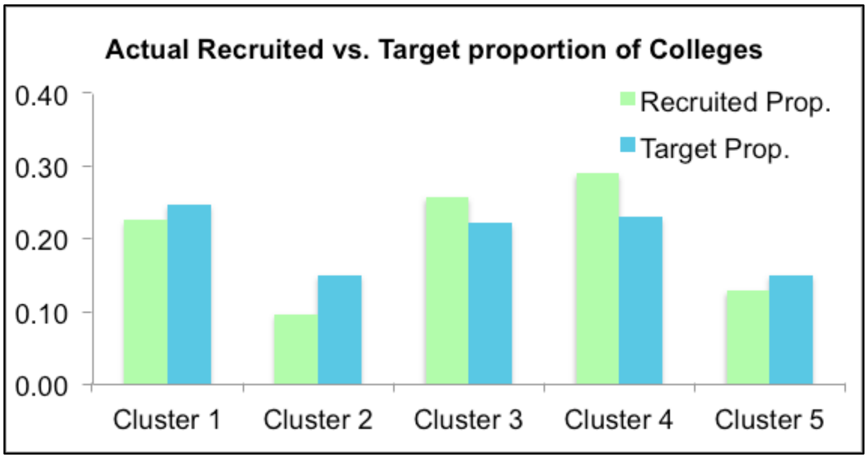

Figure 1. Recruited sample proportions and target sample proportions across clusters.

Sample for this Report

Initial enrollment in the study included 89 teachers across the 38 college sites. For this report on early results, we used the data from the participating students of 30 instructors across 19 colleges. Student and teacher numbers related to the data set reported on here are shown in Table 2.

Table 2. Counts of Teachers, Students, and Colleges in the Study.

Condition	*Teachers*	*Students*	*Colleges*
Control	19	147	15
Treatment	11	80	10
Total	*30*	*227*	*19*

Quantitative Analysis

The primary aim of the quantitative analysis was to address Research Question 1, how and for whom the particular adaptive computer environment might be effective. To this end, ultimately we will employ Hierarchical Linear Modeling (HLM) on the full data set. Models will include interaction terms between instructors' treatment assignment and covariates at different levels (e.g., students' history of course-taking, self-concept of ability), to explore the moderating impact of tool use on student learning. The primary post-test outcome measure is the MDTP elementary algebra assessment. A different but related MDTP pre-algebra diagnostic served as the measure of students' baseline knowledge. For this report, we have focused on the MDTP post-test as an indicator of algebraic knowledge.

Qualitative Analysis

To address Research Question 2, a great deal of textual, observational, and interview data are still being gathered. These data allow careful analysis of the intended and actual use of the learning environment and the classroom contexts in which it is enacted – an examination of implementation structures and processes. Indices of specific and generic fidelity derived from this work also will play a role in HLM generation and interpretation in the coming year.

Preliminary Results

Fall 2015 was the first full semester of data gathering for the project. It was our "practice" semester in that researchers were refining instruments and participant communication processes while treatment condition instructors were trying out the web-based learning tool with their classes for the first time. The "efficacy study" semester takes place in Spring 2016.

At the Conference: Poster Conversations

At the time of the conference, we had early results from the practice semester that suggested an aptitude by treatment interaction. Specifically, students in the Treatment group who started out with lower scores relative to the group mean on their algebra readiness pre-test, showed more benefit than Control group students (i.e., Treatment group students from the lower scores group had higher scores, relative to the group mean, on their post-test in elementary algebra). Some discussions in the poster session at the conference revolved around this interaction. For instance, one conference participant reported finding a similar result using a web-based technology: In his study, lower ability students exhibited higher grades when they were required to use the web-based tutor than when they were not. In another discussion, a conference participant hypothesized that instructors need to gain familiarity with technology before they can effectively use web-based learning tools for teaching. Indeed, after a semester of practice, Treatment but not Control instructors in our study reported an increase in their ability to use technology for teaching mathematics. Though not statistically significant (p = .12), the difference was consistent with the conference participants' hypothesis. Another key set of conversations at the poster were about the idea of an adaptive system that was based on a relatively stable "learning trajectory" or "genetic decomposition" as compared to a "cognitively-based" model approach that includes variability within a trajectory or decomposition, depending on the student, as the mechanism to guide selection algorithms when diagnosing and responding to student work in the computerized learning environment. We believe interactions such as those at the poster help to improve communication between the cognitive science research community and the RUME community.

Since the Conference: Updated Results

Since the conference, we have cleaned more data and have conducted analyses on this updated set. These analyses indicated that the aptitude by treatment interaction that was reported on the poster was no longer statistically significant: Estimate = -0.04, p = 0.71. Nevertheless, findings may continue to change as we continue to collect data in our efficacy semester.

Here we can add information about a new analysis of post-semester test scores that corrected for instructor clustering and students' scores on their algebra readiness pre-test. This analysis indicated that students in the treatment condition (adjusted M = 23.80, unadjusted SD = 6.67, N = 80) performed higher on their post-test than students in the control condition (adjusted M = 22.45, unadjusted SD = 8.27, N = 147), albeit these mean scores, at about 1 point difference, were not statistically different (Estimate = 0.93, p = 0.62). The effect size for this difference was Hedges' g = .12, which is considered small, but within expectation for efficacy trials of this type and is worth noting (Cheung & Slavin, 2015; Hill et al. 2008). As mentioned, this analysis included only a subset of students (data cleaning is ongoing) and results may continue to change as we collect, clean, and add more data to the analysis. Figure 2 shows box-plots of pre-test and adjusted post-test scores.

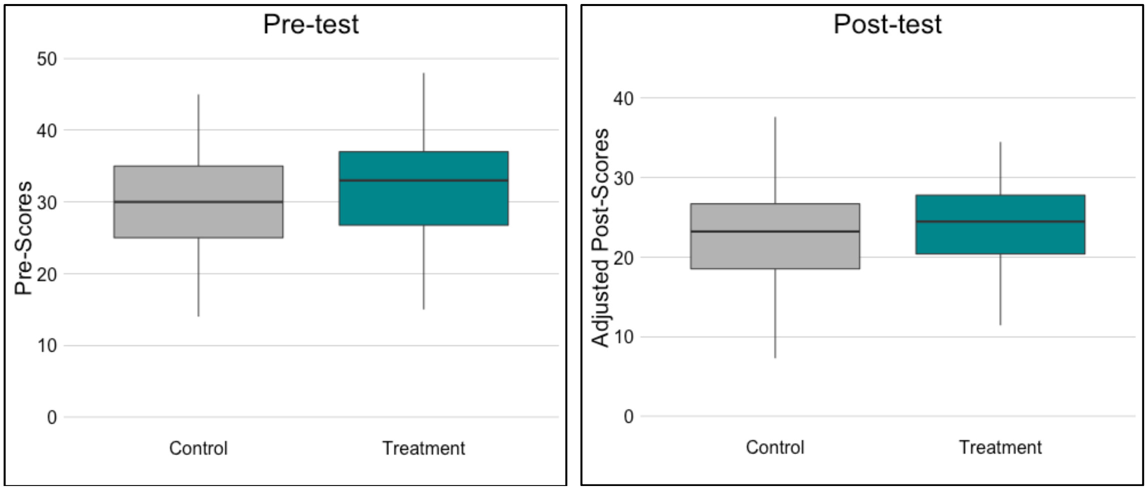

Figure 2. Left: Box-plot of pre-test scores. Right: Box-plot of adjusted post-test scores.

Next Steps

We will continue this study with a second cohort of new participants who will repeat the year-long study in the 2016-2017 academic year. Our specific objectives in the upcoming year are to (1) complete data collection from the first cohort for the primary efficacy study (i.e., data for hundreds of students for Spring 2016), (2) continue reporting findings from the Spring 2016 efficacy study of cohort 1, (3) recruit a second cohort of participants for another practice semester and efficacy study in 2016-17, and (3) begin the practice semester of the study with second cohort of participants.

Of particular interest is how the spread of information shown in Figure 3 might look for the efficacy (Spring 2016) data set. We look forward to having more to report and new questions to discuss at the 2017 conference.

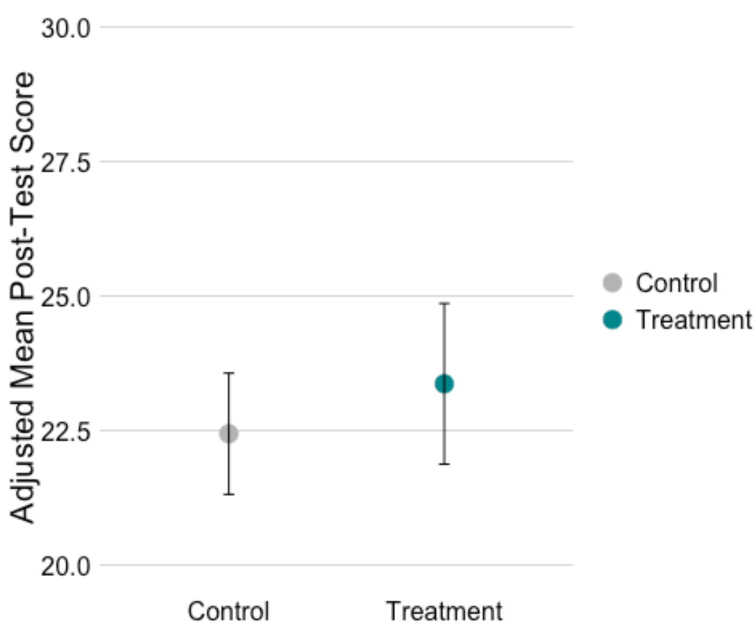

Figure 3. Adjusted mean post-test score by condition. Vertical bars represent standard errors of the means.

Acknowledgement

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How "Good" is "Good Enough"? Exploring Fidelity of Implementation for a Web-based Activity and Testing System in Developmental Algebra Instruction

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A web-based activity and testing system (WATS) has features such as adaptive problem sets, videos, and data-driven tools for instructors to monitor and scaffold student learning. Central to WATS adoption and use are questions about the implementation process: What constitutes "good" implementation and how far from "good" is "good enough"? Here we report on and illustrate our work to provide structure for such examination. The context is a study about implementation that is part of a state-wide randomized controlled trial examining student learning in community college algebra when a particular WATS suite of tools is used. Discussion questions for conference participants dug into the distinctions among intended, enacted, and achieved curriculum and the processes surrounding these as well as the challenges and opportunities in researching fidelity of implementation in the community college context, particularly the role of instructional practice as a contextual component of the research.

Key words: Fidelity of Implementation, College Algebra, Research Tools

Background

"How good is good enough?" has plagued humankind since the early cave dwellers wondered if killing three bison would get the family through the cold winter months. Even today, with our technological advances, we still ask questions such as "Do I have enough money for retirement?" "Have I practiced enough hours?" or "Is what I'm doing good enough?"

This ubiquitous question plagues social science researchers who are assessing the whats, whys, and hows of an intervention. Did the instructors have enough support to adequately implement the new curriculum? Were the materials adequate to provide enough practice hours for students? Was the instruction sufficient to prepare students to pass the final exam? Oh, if there were only an answer!

Study Context

We chose to attempt to answer this question of "good enough" in the implementation of a large project investigating relationships among student achievement and varying conditions of implementation for a web-based activity and testing system (WATS) used in community college algebra. We selected an implementation research approach that we had used previously and found to be helpful. In the new study we hope to replicate and to refine our earlier experience. Implementing the WATS is part of a statewide, randomized controlled trial examining student learning in community college algebra. WATS tools include adaptive problem sets, instructional videos, and data-driven tools for instructors to use to monitor and scaffold student learning. The WATS is accessed on the internet and is designed primarily for use as replacement for some in-class individual seatwork.

Research Questions

In what ways does a program-in-operation have to match the program-as-intended to be successful? Well, we have to identify what "success" means and also to identify alignment between intended and enacted implementation. Thus, two major research questions drive our attempt to answer the “good enough” question:

(1) What is the nature of alignment between how the program is implemented and how the developer/publisher envisioned it (i.e., what is the fidelity of implementation)?
(2) What are the relationships among varying conditions of implementation (differing degrees of fidelity) and the extent to which students are achieving the desired results?

Conceptual Framework

The theoretical basis for our approach lies in program theory, “the construction of a plausible and sensible model of how a program is supposed to work” (Bickman, 1987, p. 5). Having such a model in place allows researchers to conjecture and test causal connections between inputs and outputs, rather than relying on intuition or untested assumptions. As in many curricula projects, developers of the program in our study did pay attention to learning theory in determining the content in the web-based system, but the same was not true for determining implementation processes and structures. The pragmatic details of large-scale classroom use were under-specified. Developers articulated their assumptions about what students learned as they completed activities, but the roles of specific components, including the instructor role in the mediation of learning, were not clearly defined.

As Munter and colleagues (2014) have pointed out, there is no agreement on how to assess fidelity of implementation. However, there is a growing consensus on a component-based approach to measuring its structure and processes (Century & Cassata, 2014). *Fidelity of implementation* is the degree to which an intervention or program is delivered as intended (Dusenbury, Brannigan, Falco, & Hansen, 2003). Do implementers understand the trade-offs in the daily decisions they must make “in the wild” and the short and long-term consequences on student learning as a result of compromises in fidelity? Century and Cassata’s (2014) summary of the research offers five core components to consider in fidelity of implementation: Diagnostic, Procedural, Educative, Pedagogical, and Student Engagement (see Table 1).

Table 1. *Components and Focus in a Fidelity of Implementation Study*

Components	*Focus*
Diagnostic	These factors say what the “it” is that is being implemented (e.g., what makes this particular WATS distinct from other activities).
Structural-Procedural	These components tell the user (in this case, the instructor) what to do (e.g., assign intervention x times/week, y minutes/use). These are aspects of the *expected* curriculum.
Structural-Educative	These state the developers’ expectations for what the user needs to know relative to the intervention (e.g., types of technological, content, pedagogical knowledge are needed by an instructor).
Interaction-Pedagogical	These capture the actions, behaviors, and interactions users are expected to engage in when using the intervention (e.g., intervention is at least x % of assignments, counts for at least y % of student grade). These are aspects of the *intended* curriculum.
Interaction-Engagement	These components delineate the actions, behaviors, and interactions that students are expected to engage in for successful implementation. These are aspects of the *achieved* curriculum.

Method

The components in Table 1 are operationalized through a rubric, the guide for collecting and reporting data in our implementation study. A rubric is a "document that articulates the expectations for an assignment by listing the criteria, or what counts, and describing the levels of quality from excellent to poor" (Andrade, 2014). Each component has several factors that define the component. The project's research team has developed a rubric for fidelity of implementation, identifying measurable attributes for each component (for example, see Table 2 for some detail on the "educative" component).

Table 2. *Example of rubric descriptors for levels of fidelity, Structural-Educative component.*

Educative: These components state the developers' expectations for what the user needs to know relative to the intervention.			
	High Level of Fidelity	**Moderate Fidelity**	**Low Level of Fidelity**
Users' proficiency in math content	Instructor is proficient to highly proficient in the subject matter.	Instructor has some gaps in proficiency in the subject matter.	Instructor does not have basic knowledge and/or skills in the subject area.
Users' proficiency in TPCK	Instructor regularly integrates content, pedagogical, and technological knowledge in classroom instruction. Communicates with students through WATS.	Instructor struggles to integrate CK, PK, and TK in instruction. Occasionally sends digital messages to students using WATS tools.	Instructor CK, PK, and/or TK sparse or applied in a haphazard manner in classroom instruction. Rarely uses WATS tools to communicate with students.
Users' knowledge of requirements of the intervention	Instructor understands philosophy of WATS resources (practice items, "mastery mechanics," analytics, and coaching tools),	Instructor understanding of the philosophy of WATS tool has some gaps. NOTE: Disagreeing is okay, this is about instructor knowledge of it.	Instructor does not understand philosophy of WATS resources. NOTE: Disagreeing is okay, this is about instructor knowledge of it.
Users' knowledge of requirements of the intervention	Instructor understands the purpose, procedures, and/or the desired outcomes of the project (i.e., "mastery")	Instructor understanding of project has some gaps (e.g., may know purpose, but not all procedures, or desired outcomes).	Instructor does not understand the purpose, procedures, and/or desired outcomes. Problems are typical.

Results

Our focus here is two-fold. We first offer the preliminary results of rubric refinement from data collected through observation, interview, and teacher self-report in weekly surveys (also known as "teaching logs"). These results were shared on the poster (and handouts) at the conference. Then we summarize the highlights of the conversations about researching fidelity of implementation that emerged at the conference.

Defining and Refining Measures for the Fidelity of Implementation Rubric

The ultimate purpose of a fidelity of implementation rubric is to articulate how to determine what works, for whom, under what conditions. In addition to allowing identification of alignment between developer expectations and classroom enactment, it provides the opportunity to discover where productive adaptations may be made by instructors, adaptations that boost student achievement beyond that associated with an implementation faithful to the developers' view.

The example on the poster was for the procedural component from our WATS intervention (see Table 3, next page). The Structural-Procedural components tell the user what needs to be done (e.g., makes assignments for students to complete using the WATS tool). The table has four rows of expectations. Columns define high, moderate, and low fidelity followed by data sources and notes on the measures used.

We employ a mixed-method, feedback design to capture and communicate about fidelity of implementation. A feedback design for refining an intervention can be driven by qualitative research and supported by quantitative snapshots of student performance, teacher understandings, and systemic growth. Or vice versa. Our rubric (Table 3) lists primary, secondary and tertiary sources of data for gathering information about the four items on the procedural component of the fidelity rubric. These sources are WATS Application programming interface (API) – this provides data from the digital audit trail of WATS usage, occasional classroom observations for some instructors with an associated instructor interview, instructor self-report (through logs and surveys), and student survey. These measures were selected based on available sources and constraints on project time and funding.

We always dance between what we want to know about an intervention and what we are able to measure. Instructor self-report logs are highly useful as they can document what is happening with implementation. For example, logs can tell us how many times an instructor mentioned or used the intervention. And that accretion across weeks gives the area under the curve of what's going on across time, contributing to the big picture, of implementation.

In using the rubric, we assign a number to each level of fidelity. This can be as simple as a 3 for a high level of fidelity, 2 for a moderate level of fidelity, or a 1 for a low level; or the items can be weighted. Note on Table 3 under "amount of instruction – mindset lessons" we will know instructors' use of mindset lessons through logs and an interview question and can then assign a high, moderate, or low level of fidelity to the item (see Table 2, Notes on Metrics).

The score for the intervention will be the total number of points assigned in completing the rubric as a ratio of the total possible, across all instructors. It will also be possible to create a fidelity of implementation score on each row for each instructor – these data will be used in statistical modeling of the impact of the intervention as part of a "specific fidelity index" (Hulleman & Cordray, 2009). We first total points for the item, then the component, and finally all components for a single score as an index of implementation.

We anticipate having data that allow us to answer several questions related to "good enough." For example, for Research Question 1:

- To what extent did the instructors assign WATS activities?
- To what extent did the instructors encourage students to complete the WATS activities?
- To what extent were the mindset lessons implemented?
- How frequently was WATS assigned?

And, for Research Question 2: What is the relationship between level of mastery students achieved and number of WATS activities students completed or number of mindset activities students experienced?

Table 3. *Structural-Procedural.* These components tell the user what to do regarding instruction.

	High Level of Fidelity	*Moderate Level of Fidelity*	*Low Level of Fidelity*	*Primary Data Source*	*Second Data Source*	*Third Data Source*	*Notes on Metric(s)*
Assigns WATS	Instructor assigns all target WATS activities.	Instructor assigns 80% to 99% of target activities.	Instructor assigns fewer than 80% of target activities.	WATS API	Logs	Instruc. Interview	Source of boundary conditions: Developer, who said: "best guess at what a minimally effective dose is. Really, it has to be all."
Amount of instruction - Promotes Completion	Regularly encourages students to complete the mission.	Sometimes encourages students to complete the mission.	Rarely or never encourages students to complete the mission.	Logs	Instruc. interview	Obser-vation	*High & Moderate*: >2/3 of instructors say: Regularly: >75% of weeks (High) Sometimes: >35% of wks (Mod.) *Low*: <2/3 instructors OR Rarely: <35% of weeks
Amount of instruction - Mindset Lessons	Conducts 3 mindset lessons.	Conducts 2 mindset lessons.	Conducts at most 1 mindset lesson.	Logs	Instruc. Interview	N/A	Instructor Interview Q4: What plans do you have for mindset activities with students? Logs: What mindset lessons did your class do this week?
Frequency of instruction	WATS assigned at least 12 of 15 weeks in the term.	WATS assigned at least 7, up to 11 weeks in the term.	WATS assigned in 6 or fewer weeks in the term.	Logs	WATS API	Student survey	*High*: At least 2/3 of Instructors report WATS use in at least 75% of logs. *Moderate*: At least 2/3 of Instructors report use in 50 to 75% of logs. *Low*: not high or moderate.

At the Conference: Poster Conversations

The factors included in the poster were meant as a starting point for conversation. The poster shared the theory behind the protocol and was a touchstone for gathering ideas from RUME attendees on dissemination that might be productive as we move forward into the full study (2015 is a "practice" year for the study). Here we summarize the highlights of the conversations at the poster.

Participant comment: I never thought about this before, that somebody might pick up an activity that I designed and use it in a counter-productive way. Why would they even try it if they didn't think like I did about how to use it?

Response: There are myriad of reasons why someone would change the way they conduct an activity that another had designed. They can be from differences in content, pedagogical, or pedagogical content knowledge, or due to limitations of time or resources or even relate to someone wanting to "brand" the activity as their own. Just remember: what **you** call counter-productive may be seen as a helpful tweaking to someone else. The key lies in the impact of the change on the desired result (e.g., student learning gains).

Participant question: How do you do curriculum development when multiple people develop something for use by multiple people, including some who are not in the room?

Answer: Very skillfully! Your start with using data about your intended audience. As you design, you determine what you think will be your fidelity factors – the ones that drive your anticipated results. Next you implement your activity and then gather data to determine results and confirm the role of your fidelity factors. Further testing can show what happens when you vary a fidelity factor like contact time or dosage. These are all excellent opportunities to document "good enough." Remember you are dealing with human beings. Keep in mind the idea of "close approximation."

Participant question: How do you decide what the "it" is that is being implemented?

Answer. Excellent question. The "it" is the intervention, the project, the curriculum. Determining the "it" is answered in part by asking about a series of diagnostic factors that are part of our model. We start by interviewing developers, asking these diagnostic questions. One of the questions is how the intervention, project, or curriculum differs from others that are similar. Then we layer this information with observation of the training that developers give to faculty and the kinds of questions faculty ask about using the intervention during the training. You might think that determining the "it" is easy – sometimes yes, sometimes no. Unless you zero in on what the "it" is, you will never get to the level of specificity required to evaluate fidelity factors.

Participant question: I like to think of a three-way overlapping Venn diagram for an intervention: the intended curriculum, the implemented curriculum, and the achieved curriculum. Can your framework relate to this concept?

Answer: Our model is a fourth party that attempts to take in perspectives of all these aspects curricula. It can connect them as an important way of monitoring for efficacy (also, see the notes in the second column of Table 1, above).

Participant comment: I am surprised that a component is that instructors might need certain types of knowledge before they are ready to use a particular type of intervention.

Response: Usually something about an intervention is new. Maybe new content. Maybe new pedagogy. The instructor may not have learned whatever is required to carry out the intervention. One of the major reasons interventions fail is that participants are asked to do new things (such is the nature of interventions) for which they are given little or no training.

Participant question: I like the idea of descriptions of performance at the high, medium, and low levels. Can you develop materials that incorporate such descriptions on the front end?
Answer: Sure you can. Such descriptions can be used at each level from the beginning of an activity or program through the implementation and finally for the evaluation at the end.

Participant question: Does your framework help increase equity in any respect?
Answer: The specificity of what fidelity of implementation requires we include in the rubrics is an opportunity for us to address potential challenges to equity and inclusion in the implementation of an intervention. How to make college math accessible to all students is a theme of the work in in the WATS system we are studying. Investigating fidelity of implementation allows us to identify how curriculum and its implementation play a part in that accessibility process.

Participant question: Where is it explicit to a user what the developer's intentions are?
Answer: Sometimes the developers will tell you outright in the introductory material. Other times the intent is buried in the content, and you have to unearth it. Sometimes developers are very cognizant of their intentions; other times, oblivious. Regardless of level of transparency, intentions are always there.

Participant question: As a classroom instructor, where in the rubrics is my relationship with the WATS online resource? My perspective about its use in teaching and learning?
Answer: Yes, that's something we are wrestling with as we develop the details of the Educative rubric (Table 2). Right now, the rubric looks at the degree of knowledge instructors have about the intended relationship (e.g., about the philosophy behind the WATS tool), not at the alignment of the instructor's view with that perspective. We agree success of implementation may depend on how someone sees the resource, but is it necessarily an aspect of being faithful to the intentions of the tool? For an instructor, the resource can be a partner, or a distinctly separate support for teaching, or even an obstacle. The Concerns Based Adoption Model provides some ideas that we are pursing (Hall & Hord, 2014).

Participant comment: It's a new idea to me that implementation could be a major field of study.
Response: It has grown exponentially over the past 20 years, and we have learned much about the implementation process. You have probably heard the cliché, "We tried that once and it didn't work." What actually happens most often is a failure in implementation. Even the best ideas will collapse with insufficient or faulty implementation.

Implications for Practice

By definition, high fidelity implementation of an instructional tool is use that results in greater learning gains than non-use. Instructors and students are better equipped to implement with high fidelity when they have answers to questions like: What are the characteristics of good implementation? Among preferred actions in implementation, which are the highest priority? What are the trade-offs and consequences of making particular decisions about use of the tool?

Answers to these questions provide data for determining what is "good enough" and help users make the best decisions for program efficacy. As the field moves forward, we seek effective ways to communicate implications to college instructors, department chairs, as well as stakeholders in the larger public arena.

Acknowledgement

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A Framework for Mathematical Understanding for Secondary Teaching: The Mathematical Activity perspective

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Abstract: *A framework for mathematical understanding for secondary teaching was developed from analysis of the mathematics in classroom events. The Mathematical Activity perspective describes the mathematical actions that characterize the nature of the mathematical understanding that secondary teachers could productively use.*

Mathematics teaching at the collegiate level focuses on enabling students to develop solid understanding of mathematics. Although collegiate mathematics students often describe mathematics as learning specific topics and strategies and applying this knowledge to their work, their instructors may have additional but less explicit goals such as valuing the structure of mathematics, being able to create a deductive argument, or exploring and comparing systems of mathematics. These latter goals are especially important for prospective teachers of secondary mathematics, and college mathematics instructors are attending in new ways to the mathematical preparation of those who will teach mathematics.

Over the past three decades, mathematics education researchers and theorists have increased their focus on the mathematical knowledge of teachers that helps teachers reach their goals of promoting a more robust understanding of mathematics in their students. During that time, researchers have refined the focus from Shulman's (1986) construct of pedagogical content knowledge to constructs such as mathematical knowledge for teaching (MKT) (Ball, 2003; Ball & Bass, 2003; Ball & Sleep, 2007a; Ball & Sleep, 2007b; Ball, Thames, & Phelps, 2008) and knowledge of algebra for teaching (KAT) (Ferrini-Mundy, Floden, McCrory, Burrill, & Sandow, 2005; McCrory, Floden, Ferrini-Mundy, Reckase, & Senk, 2012). Work on MKT is, perhaps, the best known of the research programs focused on teachers' mathematical knowledge. MKT originated with a reflection on the mathematical knowledge involved in the mathematical work of teaching at the elementary level. MKT partitions the territory of mathematical thinking into categories such as specialized content knowledge, common mathematical knowledge, and mathematics at the horizon. While the MKT categories can partition mathematical knowledge at the secondary level as well as at the elementary level, those categories may not characterize the nature of mathematical thinking that seems to distinguish mathematics at the secondary school level.

In their work in secondary mathematics, students expand their mathematical knowledge to include new ideas such as irrational numbers, complex numbers, static and rotating objects, sample spaces, and a variety of ways to represent these ideas. But the differences between mathematics at the elementary and secondary levels are not solely extensions of the topics involved, but also a change in the nature of mathematical thinking involved. Whereas both elementary and secondary mathematics honor deductive reasoning, secondary mathematics places a much stronger emphasis on deductive thinking within a closed mathematical system. It is in the context of secondary mathematics that curricula focus on reasoning on the basis of a well-defined system of given properties and relationships. For example, the work of secondary students in the study of geometry is more likely to occur at the third or fourth van Hiele level (making deductive connections and constructing proofs) rather than the first or second levels (focused on visualizing or recognizing properties of geometric objects) that are more prominent at the elementary level. At the elementary level, students develop ways to represent mathematical relationships. As students progress through school mathematics, their repertoires of ways to represent mathematical relationships expands so that, as they engage in secondary mathematics, they can be expected to link representations of the same mathematical entities and to reason about a mathematical entity in one representation making conclusions about that entity in another representation.

Secondary teachers need to be able to reason flexibly enough to recognize and act on opportunities for their students to build capacities for reasoning in a closed system and for capitalizing appropriately on a range of representations. They need mathematical understanding that enables them to perform such activities as creating examples, nonexamples, and counterexamples of entities encountered in secondary mathematics, to identify special cases of broad categories of mathematical objects, and to explain when a general statement can or cannot be extended to a larger or different domain or set of mathematical objects. Secondary teachers need to make connections across mathematical systems. In order to facilitate learning secondary mathematics, the work or context of teaching requires a depth of specific mathematical understanding that incorporates the more subtle but important goals of mathematics teaching. Mathematics teachers must not only understand mathematics but they must enable others to understand mathematics in the fullest sense. They need to pose interesting questions and tasks that bring the structure of mathematical systems alive. They need to understand the mathematical thinking of students in order to correct or challenge their thinking. They need to be able to reflect on the curriculum and organization of mathematical ideas. The context of learning mathematics requires specific mathematical understanding beyond pedagogical knowledge.

The six faculty involved in our project on Mathematical Understanding for Secondary Teaching (Glen Blume, M. Kathleen Heid, Jeremy Kilpatrick, James Wilson, Patricia Wilson, and Rose Mary Zbiek), recognizing the need to understand better the nature of the mathematical understanding that could best serve secondary teachers, adopted the goal of developing a framework that could account for the mathematics a secondary teacher could productively use in the context of teaching secondary mathematics. We decided to began our inquiry by identifying mathematical opportunities secondary teachers actually encounter, and so we began in the classroom. As we embarked on our study of mathematical opportunities unfolding in the classroom, we recognized many of the ideas expressed by others who have attended to secondary mathematics (e.g., Adler & Davis, 2006; Cuoco, 2001; Cuoco, Goldenberg, & Mark, 1996; Even, 1990; McEwen & Bull, 1991; Peressini, Borko, Romagnano, Knuth, & Willis-Yorker, 2004; Tattoo et al., 2008). While the framework incorporates previous ideas, it attends directly to the secondary mathematics built on data from mathematics classes.

Our source of data was a set of what we came to call Situations. A Situation is a mathematical description, based on an actual event that occurred in the practice of teaching, of the mathematics that teachers could productively use in the work of teaching mathematics. Teams of mathematics education faculty at Penn State and at University of Georgia worked with dozens of doctoral students in mathematics education to develop more than 50 Situations.

Although any one Situation is too large to report in this paper, we provide a brief outline of one of the Situations (from Heid & Wilson, 2015) here. Each Situation includes a Prompt (a description of a mathematical opportunity–an event that one of the authors observed happening in the course of teachers planning or implementing a secondary mathematics lesson) and several Mathematical Foci (development of mathematics that a teacher could productively use in the context of that mathematical opportunity). A short statement about the nature of the mathematical understanding being targeted precedes each Mathematical Focus. Other parts of each Situation are Commentaries (a description of how the Mathematical Foci for the Situation fit together) and PostCommentaries. One of the Situations is outlined in Figure 1.

The Situations we (the cross-university teams) developed suggested a range of mathematical abilities, actions, and settings that could underlie potentially productive mathematical thinking on the part of the teacher. It was on the basis of these abilities, actions, and settings that we embarked on the challenging task of developing our Framework for Mathematical Understanding for Secondary Teaching. As we examined the Situations we had created, we recognized that we needed several different perspectives to explain the mathematics we had identified. Akin to Plato's allegory of the cave, the framework on which we settled consisted of three perspectives, each of which cast a different shadow representing a student's mathematical understanding (See Figure 2).

We adapted one perspective, Mathematical Proficiency, from *Adding it up: Helping children learn mathematics* (National Research Council, 2001). We found that we could identify the mathematical understandings in our Situations as examples of the strands of proficiency in this document: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition, supplemented by an additional strand focused on historical and cultural knowledge. This perspective accounted for mathematical knowledge and its use, but did not account for the mathematical actions that secondary teachers could productively take in the context of teaching mathematics. The second perspective addressed this focus on mathematical actions as Mathematical Activity. However, neither the first nor second perspective accounted for the settings in which teachers needed to call on their mathematical knowledge. The third perspective, Mathematical Context of Teaching, addressed the mathematical context in which teachers could productively call upon their mathematical knowledge.

The first perspective, Mathematical Proficiency, is likely to be familiar as a way to think about students' mathematical capability. The third perspective, Mathematical Context, provides a description of the mathematical understanding that is particularly relevant to teaching. This perspective was more implicit than explicit in our data, but we realized that the Mathematical Context of teaching indicates why it is critical to recognize and attend to the importance of Mathematical Activity. In this paper, we confine our discussion to the development of the second perspective, Mathematical Activity. The reader can find more detail on the Mathematical Proficiency and Mathematical Context perspectives in the Heid and Wilson book (Heid & Wilson, 2015).

CHAPTER 22. INVERSE TRIGONOMETRIC FUNCTIONS (originally identified and developed by Rose Mary Zbiek)

Prompt

Three prospective teachers planned a unit of trigonometry as part of their work in a methods course on the teaching and learning of secondary mathematics. They developed a plan in which high school students would first encounter what the prospective teachers called *the three basic trig functions*: sine, cosine, and tangent. The prospective teachers indicated in their plan that students next would work with "the inverse functions," which they identified as secant, cosecant, and cotangent.

Commentary

The Foci draw on the general concept of inverse and its multiple uses in school mathematics. Key ideas related to the inverse are the operation involved, the set of elements on which the operation is defined, and the identity element given this operation and set of elements. The crux of the issue raised by the Prompt lies in the use of the term *inverse* with both functions and operations.

Mathematical Focus 1

An inverse requires three entities: a set, a binary operation on that set, and an identity element given that operation and set of elements.

Secondary mathematics involves work with many different contexts for inverses. For example, opposites are additive inverses defined for real numbers and with additive identity of 0, and reciprocals are multiplicative inverses defined for nonzero real numbers and with multiplicative identity of 1. [Discussion follows about the nature of inverses, the role of an identity in inverses, and the importance of domain and range in consideration of inverses.]

…

Mathematical Focus 2

Although the inverse under multiplication is not the same as the inverse under function composition, the same notation, the superscript -1, is used for both.

[Discussion follows about notation used in different inverse relationships, and the specific use of that notation in consideration of trigonometric functions.]

…

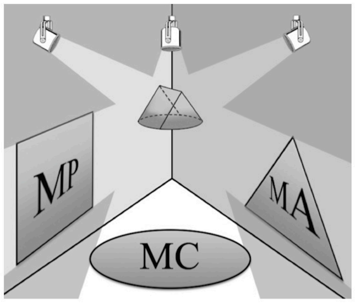
Mathematical Focus 3

When functions are graphed in an xy*-coordinate system with* y *as a function of* x*, these graphs are reflections) in the line* y = x *of their inverses' graphs (under composition).*

The graph of a function reflected in the line $y = x$ is the graph of its inverse, although without restricting to principal values, the inverse may not be a function. Justifying this claim requires establishing that the reflection of an arbitrary point (a, b) in the line $y = x$ is the point (b, a). [A geometric proof follows, using a coordinate plane representation of the reflection of a point (a,b) over the line $y = x$.]

…

Figure 1. Outline describing a Situation appearing in (Heid & Wilson, 2015).



MP Mathematical Proficiency
MA Mathematical Activity
MC Mathematical Context

Figure 2. Three perspectives of the Framework for Mathematical Understanding for Secondary Teaching (Heid & Wilson, 2015).

MATHEMATICAL ACTIVITY

We used the set of Mathematical Foci from the Situations as data from which to generate our Framework for Mathematical Understanding for Secondary Teaching. First we identified mathematical actions implicit or explicit in each of the Foci. We then developed categories that seems to capture those actions, including categories such as creating mathematical entities, interpreting mathematical representations, and orchestrating movement among them.

Creating

One set of mathematical actions that seemed to group into a single category included the following actions:

- Creating a counterexample for a given structure, constraint, or property
- Creating an example or non-example for a given structure, constraint, or property
- Creating equivalent equations to reveal information
- Creating problems to foreground a concept
- Creating a file (a computer application) whose creation requires mathematics beyond what the file is used to teach
- Constructing an object given a set of mathematical constraints
- Generating specific examples from an abstract idea
- Creating a representation for a mathematical object with known structure, constraints, or properties

Each of these actions involves the generation of a new mathematical entity. Some of the actions involve the generation of a new (to the creator) mathematical object such as an instance of a counterexample for a mathematical conjecture, some of the actions involve the creation of specific examples that illustrate mathematical concepts that have given sets of properties, and some of the actions involve the creation of mathematical tools such as computer files that display a representation of the motion of a bicycle. In every case, the mathematical entity that was created was one fitted to a given set of mathematical conditions.

Having grouped these actions into a single category, we developed a description of a mathematical action that encompassed these actions. In this case our description was “Creating a mathematical entity or setting from known (to the one creating) structures, constraints, or properties." An example of a specific mathematical action that might fit this category is the task of constructing a quadrilateral with specific characteristics. Other mathematical actions were developed in a similar fashion. A few of the final set of mathematical actions at this juncture, along with specific examples drawn from the Situations, are shown in Figure 3.

Extending

A similar process underpinned the development of a subcategory we called extending. Mathematical actions that fell into this category could be described as those involving extension of a domain, argument, or class of objects for which a mathematical statement is or remains valid. The following mathematical actions fell under the "extending" category:

- Structuring an argument so that it extends to a more general case;
- Determining mathematical extensions to a given problem or question;
- Recognizing mathematical relationships that allow one to extend a conclusion to a larger class;
- Considering a definition in an expanded sense or altering the "universe" being considered; and
- Extending domain while preserving structure.

Just as "extending" the domain of a mathematical relationship is an essential mathematical process for teachers, so is restricting or constraining the domain. Examples in the set of Situations that involved extending are: extending the absolute value function from the real to the complex domain; and extending the object "triangle" from Euclidean to spherical geometry

Category	Example
Create: Creating a new (to the one creating) mathematical entity or setting from known (to the one creating) structure, constraints, or properties	Sketch quadrilateral ABCD with $m\angle D = m\angle A = 90$ and $\overline{AB} \parallel \overline{DC}$ such that ABCD is not a parallelogram.
Recognize: Recognizing mathematical properties, constraints, or structure in a given mathematical entity or setting, or across instances of a mathematical entity	Recognizing that strategic choices for pairwise groupings of numbers are critical to one way of developing the general formula for summing the first *n* natural numbers
Choose: Considering and selecting from among known (to the one choosing) mathematical entities or settings based on known (to the one choosing) mathematical criteria	The mathematical meaning of a/b (with $b \neq 0$)) arises in different mathematical settings, including: slope of a line, direct proportion, Cartesian product, factor pairs, and area of rectangles. One might choose slope of a line as a setting to illustrate the need for $b \neq 0$.

Figure 3. A few of the set of mathematical actions that comprised the Mathematical Activity perspective of the Framework for Mathematical Understanding for Secondary Teaching, along with specific examples drawn from the Situations.

Use representations: For given representations, interpret them in the context of the signified, orchestrate movements between them, and craft analogies to describe the representations, objects, and relationships	Using tabular and graphical representations to estimate the value of $2^{2.5}$
Assess (interpret and adapt) the mathematics of the situation: Interpret and/or change certain mathematical conditions/ constraints relevant to a mathematical activity	Recognize the desirability of a modulus definition of absolute value in evaluating $f(x)=\left\|\sqrt{x-10}\right\|$
Extend: Extend the domain, argument, or class or objects for which a mathematical statement is/remains valid.	Extending: the absolute value function from the real to the complex domain; "triangle" from Euclidean to spherical geometry
Connect: By recognizing structural similarity, make connections between: representations of the same mathematical object; different methods for solving a problem; mathematical objects of different classes; objects in different systems; or properties of an object in a different system.	Identifying structural similarities of the Euclidean algorithm and the long division algorithm
Reason: Reason about a mathematical entity in more than one way, including, but not limited to: from mathematical definitions, from given conditionals, from and toward abstractions, by continuity, by analogy, and by using structurally equivalent statements.	Reasoning about the sum of the first n natural numbers by appealing to cases, by making strategic choices for pair-wise grouping of numbers, and by appealing to arithmetic sequences and properties of such sequences.

Figure 3, continued.

Creating the Mathematical Activity perspective

Finally, we organized the set of mathematical actions to account for the actions arising in the Situations as well as mathematical actions that we could readily imagine and that were not captured in the categories that were derived from the Situations. The final set of categories is displayed in Figure 4.

The final categories differed from existing frameworks in their mathematical nature. The mathematical actions we described derived from the mathematical decisions that teachers confront. Their work in mathematics classrooms would benefit from their ability to notice similar mathematical structures. Being comfortable enough with mathematical entities, properties, and structures to create and modify new representations would allow them the freedom to pursue their students thinking. They could productively use a flexible and robust repertoire of techniques for justifying their mathematical work.

The framework is intended to be a work in progress. It can serve as a research tool to study the mathematical understanding of secondary teachers. Researchers might investigate, for example, what collegiate mathematics courses contribute to the development of the capabilities suggested in each of the perspectives. They might also investigate how the aspects of secondary mathematics teachers' own mathematical understandings as described in the Framework influence the mathematics to which they expose their students.

I. Mathematical noticing: *Recognize and choose from among known mathematical entities or settings based on known mathematical criteria such as:*

- A. Structure of mathematical systems
- B. Symbolic form
- C. Form of an argument
- D. Connections within and outside mathematics

II. Mathematical reasoning**:** *Reason about a mathematical entity in one or more than one way, including, but not limited to: from mathematical definitions, from given conditionals, from and toward abstractions, by continuity, by analogy, and by using structurally equivalent statements.*

- A. Justifying/proving
- B. Reasoning when conjecturing and generalizing
    - 1. Investigate (*Take a mathematical action to find out more about structure, constraints, or properties of a mathematical situation or a mathematical object)*
- C. Constraining and extending

III. Mathematical creating. *Create (Creating a mathematical entity or setting from known (to the one creating) structure, constraints, or properties)*

- A. Representing
- B. Defining
- C. Modifying/transforming/manipulating

IV. Integrating strands of mathematical activity. *Coordinate (Coordinate mathematical knowledge, student mathematical thinking, school curricula, and knowledge development); Reflect (self-reflect) (Reflect on mathematical aspects of one's practice or on one's own doing math); and Apply (Employ algorithms, definitions, and technology in mathematical settings and/or real world quantitative settings when applicable.)*

Figure 4. Mathematical Activity Perspective of the Framework for Mathematical Understanding for Secondary Teaching (from Heid & Wilson, 2015)

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Secondary teachers' responses to embedded mathematical uncertainty: Cases from an assessment item on exponents.

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A growing body of research recognizes uncertainty as an inevitable part of teaching. Yet still little has attended to the potential of uncertainty as a space for mathematics teacher education. In this paper, we report on an analysis of interview data in which secondary teachers were asked to think aloud as they solved a task with embedded uncertainty. We present examples of teacher responses and discuss ways in which they assign valence to their experience of uncertainty. We found that teachers do not always engage with mathematical uncertainty, but also that their (dis)engagement did not necessarily signal their ability to reach the intended answer, and was not always predicted by the valence they assigned the uncertainty. These results suggest that, while tasks containing embedded uncertainty may be rich sources to draw on, facilitating the resulting discussions is likely to be complicated and needs to account for a number of factors.

Keywords: Secondary Mathematics Teachers, Uncertainty, Algebra, Teacher Education, Teacher Assessment

"Teaching is evidently and inevitably uncertain" (Floden & Buchmann, 1993, p. 373). In acknowledgement of this, education researchers have attended to what causes uncertainty and to how teachers might respond productively to uncertainty. We build on this work by attending to *mathematical* uncertainty, by which we mean uncertainty whose source is mathematical. In this study, we examine a set of interviews with secondary mathematics teachers who responded to a task involving mathematical uncertainty. In the sections that follow, we first locate this study in the literature on uncertainty in teaching and teacher education. We then discuss the task, called the *Williams item*, originally designed to assess mathematical knowledge for teaching, and discuss the mathematical uncertainty embedded in that task. We then describe our data and the context in which the teacher interviews were conducted, and outline our analysis process. In presenting the results of the analysis, we describe teachers' reactions to uncertainty as well as the valence they assign that experience. The term valence, in this paper, means one's opinion of uncertainty that can be positive, negative, both, or neutral. Our goal is to populate an example space within which teacher educators can think through what might make such reactions more or less productive. We finish by discussing how the types of reactions we observed suggest that tasks containing mathematical uncertainty may potentially serve as tools for mathematics teacher education.

Background

Uncertainty in thought, encountered while teaching, can be conceptualized as cognitive "perplexity, confusion, or doubt" (Dewey, 1933) that can originate in various sources and that may exist within or outside of the individual. Uncertainty can result from instructional content (e.g., Capobianco, 2010), pedagogy (e.g., Midthassel, 2006; Wheatley, 2005), and student traits or school culture (e.g., Labaree, 2003; Villaume, 2000). Education researchers have used the term uncertainty to describe everything from unpredictable classroom situations (e.g., Midthassel, 2006) to low self-efficacy (e.g., Wheatley, 2005). Some strands of literature describe it more as an external, contextual attribute (e.g., Floden & Buchmann, 1993) while others locate it as an emotion experienced by the teacher (e.g., Meister & Nolan, 2001). In other words, uncertainty is defined in the literature in various ways, and it covers a range of aspects of teaching.

Studies suggest that when teachers confront uncertainty and conceptualize teaching as open and fluid, they encounter greater opportunity to develop their practices and their subject matter knowledge (Floden & Buchmann, 1993; Labaree, 2003). And this may be as true for expert teachers as for novice teachers. Floden and Chang (2007) suggest the metaphor of a jazz score for teaching, where certain frames of reference can be nailed down and others are, of necessity, open to creativity and interpretation; a sign of expertise is the ability to make use of uncertainty rather than the ability to avoid uncertainty. Denying uncertainty may restrict teachers' opportunities to look for alternative teaching methods and in turn limit students' learning (D. Cohen, 1988; Helsing, 2007; Munthe, 2003).

In this paper we focus on uncertainty that is mathematical in nature. Mathematical knowledge can be open to revision and hence it does change (Ernest, 2004, 2015; Lakatos, 1976; Stinson & Bullock, 2012). In other words, uncertainty can be embedded in the subject matter, and this type of uncertainty will never be fully resolved no matter how well a teacher prepares, even if preparation can reduce some uncertainty. Mathematical uncertainty can be as irreducible as other uncertainties.

Although uncertainty is worth examining from the view of teacher education, little empirical research has been done in the context of *mathematics* teacher education. Some research has attended to unexpectedness in teaching mathematics (e.g., Cavanna, Herbel-Eisenmann, & Seah, 2015; Coles & Scott, 2015; Mason, 2015; Rowland, Hodgen, & Solomon, 2015; Rowland, Thwaites, & Jared, 2015), with more focus on classroom interaction and general pedagogical situations. Zaslavsky (2005) focused on providing teachers with tasks designed to provoke mathematical uncertainty by having teachers generate competing claims, work with unknown paths or questionable conclusions, or deal with non-readily verifiable outcomes. The teachers, in interviews after engagement with the tasks, showed appreciation and reported deep engagement due to the uncertainty. The mathematical uncertainty encountered allowed the participants to engage with the mathematics more deeply. As a result of this activity with teachers, Zaslavsky recommended that teacher educators reflect on task implementation and make modifications so as to invite teachers to confront mathematical uncertainty. Buchbinder and Zaslavsky (2008) also reported teachers being satisfied and excited after resolving mathematical uncertainty.

Conceptualizations of the work of teaching associated with mathematical knowledge for teaching suggest that teachers need to hold specialized content knowledge, a form of pure mathematical knowledge uniquely needed by teachers (Ball, Thames, & Phelps, 2008). This knowledge often manifests in the consideration of unconventional student solutions, which may need to be examined for mathematical coherence, validity, generality, efficiency, or appropriateness. This type of work is clearly a component of the daily work that teachers do, but is also *mathematical* work. The need to respond quickly to student-generated mathematics suggests that the more open a classroom is to student-generated ideas, the more likely it is that teachers will need to respond rapidly to unfamiliar mathematics. In other words, one can characterize responding to mathematical uncertainty as an inevitable part of the work of teaching. Jordan, Kleinsasser, and Roe (2014) suggested teacher educators might support teachers' professional growth by deliberately inducing uncertainty.

Previous work on uncertainty has conceptualized how a mathematical task can provoke uncertainty and has provided empirical evidence that confronting uncertainty can help teachers *as learners of mathematics* to acquire deeper mathematical understandings. They do not take on the question of how confronting mathematical uncertainty *in their roles as teachers* might be leveraged by teachers to, for example, provide a window into the student experience and model approaches to uncertainty for their students. This skill has taken new relevance in view of recent work on K-12 standards. The Common Core State Standards (CCSS) have set forth a list of key mathematical practices that students should be able to take in solving mathematical problems. We see a potential in mathematical uncertainty for providing opportunities for teachers to engage with multiple standards for mathematical practice, including "Make sense of problems and persevere in solving them" and "construct viable arguments and critique the reasoning of others" (Common Core State Standards Initiative, 2012). If students are to learn these practices, it is important for teachers to model the practices explicitly (Charalambous, Hill, & Ball, 2011; J. Cohen, 2015; Morrison, Robbins, & Rose, 2008). Though scholars have explored, more generally, productive approaches to uncertain mathematics from learners' perspectives (Cuoco, Goldenberg, & Mark, 1996; Komatsu, 2016), there has been less study of approaches to mathematical uncertainty as a part of the work of teaching and how teacher educators can support teachers' ability to leverage the emotional component that may inevitably accompany the experience of confronting uncertainty in front of a classroom full of students. To better support teacher educators utilizing uncertainty, it is important to understand the strategies teachers use and how they feel when confronted by uncertainty in the work of teaching.

For the purposes of this paper, we conceptualize mathematical uncertainty following Zaslavsky (1995): any mathematical situation in which competing claims, an unknown path or questionable conclusion, or non-readily verifiable outcome occurs. We take uncertainty to be both a condition of the situation (that something cannot be known) and the associated emotions. We use the term *valence* to describe a teacher's positive or negative opinion of the consequences of uncertainty. Valence may be neutral, as a person may not have a strong opinion, and it does not itself have a truth value. Valence is not often discussed in the literature, although it might be read into certain work that accepting uncertainty means assigning a positive valence to uncertainty (Jordan, Kleinsasser, & Roe, 2014; Meister & Nolan, 2001; Villaume, 2000; Zaslavsky, 2005). We question this assumption. Our research questions are:

- What did participants do to attempt to resolve uncertainty?
- What types of valence do participants assign to the uncertainty they experience? To what do they attribute the valence (the mathematics, the teaching situation, etc.)?

In this paper we examine a set of teacher responses to a teaching situation in which mathematical uncertainty was embedded. Our analysis is qualitative and interpretive. We do not attempt to generalize our claims to the general population of all teachers in forms such as "X% of teachers have valence type V". Rather, our goal is to generate a set of response patterns that teacher educators might consider in using such tasks and to consider the ways in which the participants' valence interacts with their attempts to resolve uncertainty. We begin by introducing the Williams item and its embedded mathematical uncertainty.

The Williams Item

This study focuses on teachers' responses to the Williams item (see Figure 1). This item was originally drafted as an assessment item to evaluate teachers' mathematical knowledge for teaching. Specifically, the item assesses whether the teacher is aware of limitations of the exponential identity $(x^a)^b = x^{ab} = x^{ba} = (x^b)^a$ in numerical expressions, in particular, for evaluating $\left((-9)^{\frac{1}{2}}\right)^2$. Although the identity does hold for nonnegative bases without issue, complications arise when the base is negative, or when $x < 0$. Applying the identity to the example expression yields a contradiction. The identity implies that $\left((-9)^{\frac{1}{2}}\right)^2 = (-9)^{\frac{1}{2}\cdot 2}$ $=(-9)^{2\cdot\frac{1}{2}}$ $=((-9)^2)^{\frac{1}{2}}$. However, $((-9)^2)^{\frac{1}{2}} = (81)^{\frac{1}{2}} = 9$, whereas $\left((-9)^{\frac{1}{2}}\right)^2$ is either undefined if square roots of negative numbers have not been defined yet, or $\left((-9)^{\frac{1}{2}}\right)^2 = (3i)^2 = -9$. In either case, the result is not equivalent to +9. Hence applying the exponential identity in this case constitutes an invalid application of exponential laws. The Williams item features the work of two students who have performed this invalid application of the exponential identity. One student, Craig, obtains +9, and the other, Katlynn, coincidentally obtains -9, the "correct" numerical answer, by an invalid process. The intended response is that neither student's application of the laws of exponents is valid. We use the term "correct" carefully here, as one complexity embedded in the item is that there is not an unambiguous correct evaluation of the expression the students have been given. It is true that over the complex number system the given expression evaluates to -9, but it is not clear whether the students are working in this system, and if they are assumed to be working over the real numbers, the "correct" evaluation of the expression is that it is undefined.

Ms. Williams is reviewing a set of homework problems on which students were asked to evaluate exponential expressions, including the expression $\left((-9)^{\frac{1}{2}}\right)^2$. Ms. Williams asks two students to share their work.

For each of the following student's work, indicate whether it demonstrates a valid application of the laws of exponents to solve the problem.

	Valid Application	**Not Valid Application**
Craig said: I used the exponent rule to change the order of the squared and the one half because that way you don't have to take the square root of a negative number. $(-9)^2 = 81$ Then the square root of 81 is 9.		
Katlynn said: I did it an easier way. 2 and $\frac{1}{2}$ cancel, so it's just -9.		

Figure 1. The Williams item (Copyright 2013 by Educational Testing Service).

There are a number of potential sources of uncertainty embedded in the item. A participant might be uncertain about what the students actually thought or did, or struggle to make sense of the student work as presented, although this was not a pattern we observed in the data and is not one we would classify as *mathematical* uncertainty. A teacher who is solving the item does not know in this situation whether Ms. Williams' students are working in the domain of real or complex numbers, an issue that makes the correct evaluation of the expression uncertain. This uncertainty does not affect the correct response to the question posed in the item, as laws of exponents have been applied incorrectly regardless, but *is* a source of mathematical uncertainty about the underlying mathematics in the problem presented to the students. This example resembles one presented by Tirosh and Even (1997) in which the expression $(-8)^{1/3}$ similarly can be argued to have different correct evaluations because the base is a negative number, and which Zaslavsky (1995) categories as a type of mathematical uncertainty called 'competing claims'. In this case, it is a claim that cannot be resolved because the uncertainty is embedded in the mathematics itself. Needless to say, a participant might simply not have encountered this mathematics before, leading to a different type of mathematical uncertainty, one that potentially could be resolved with greater familiarity.

Responses to this particular item, because of the embedded mathematical uncertainty, and because a large proportion of the teacher responses demonstrated uncertainty, afforded an ideal opportunity to examine patterns of reasoning in response to uncertainty. In particular, this item provides an opportunity to explore the strategies participants used to resolve their own mathematical uncertainty and the valences they attached to this process.

Methods

Context and Data

This study represents a secondary analysis of data originally collected as part of a larger project intended to provide validity evidence for a set of assessment items developed for measuring teachers' mathematical knowledge for teaching. The items used in this larger study were developed at the Educational Testing Service in 2013 following an assessment design theory that originated at the University of Michigan (e.g., Study of Instructional Improvement, 2008). The items require test takers to apply mathematical knowledge in the context of teaching. Teachers with secondary mathematics certification were selected to participate in think-aloud interviews in which they shared their reasoning for selecting each answer. Participants were selected for strong mathematical proficiency as measured by the Praxis secondary mathematics content test, and varied between 0 and 13 years of teaching experience. 14 participants were female and nine were male. The present study used transcript data collected for one item, the Williams item, because of the opportunity to observe engagement with mathematical uncertainty. The data set included 13 transcribed responses to the Williams item. Among these, four answered correctly that both students' work were invalid, eight responded that Craig's work was not valid but Katlynn's work was valid, and one responded both were valid.

Analysis

Drawing on thematic analysis, we conducted multiple iterations of theme searching from the transcripts. Thematic analysis is "a method for identifying, analyzing and reporting patterns (themes) within data" (Braun & Clarke, 2006, p. 79). Patterns, or themes, are not strictly defined in thematic analysis. A researcher may pick as a theme what was observed multiple times across the data. Alternatively, the researcher may pick impressive events that all together address the research questions. Because of such nature of thematic analysis, the themes are not easy to describe at the beginning of the analysis, but becomes clear towards the end of the analysis.

We began our analysis by each closely reviewing transcripts for the Williams item to produce an initial list of themes. We also noted the statements participants made about how they would deal with a situation such as that presented in their classroom teaching, which were occasionally prompted by the interviewer but more often spontaneous. We then narrowed the themes of interest to the following list, for which we systematically recoded the entire data set. Themes for which we coded data at this point were the presence (or absence) of explicit statements of uncertainty, the valence assigned to that uncertainty by the participant where applicable, the presence of incorrect mathematics in the participant's response, and the types of mathematical approaches each participant used to resolve uncertainty if approaches were taken.

Among these themes, the ones that stood out most strongly were those of type of mathematical approach and valence. Mathematical approaches were quite varied, in some cases sophisticated, and often showed signs of being potentially productive regardless of the end result. And statements reflecting valence were quite mixed, including some strong statements.

We grouped responses in which a mathematical approach was taken by type of approach, and we operationalized valence as a personal perception of uncertainty. Through multiple iterations of analysis with the data, we observed participants negatively framing their acknowledged uncertainty. Reading such cases from the transcripts, we came to see the gap between acknowledging uncertainty and welcoming uncertainty. We wanted to examine the relationship between valence, uncertainty, and teaching. From our observation of the data, we recognized that the participants can have positive, negative, or no valence to uncertainty. These observations informed us with refining the theme and selecting excerpts to share in this paper.

Results

We divide our results into two sections in which we describe teachers' responses to each of the research questions. First, we discuss the types of responses we saw from participants when confronted with uncertainty. Second, we discuss the valence they assigned it. Responding to each research question, we share major themes we encountered. These themes are major because they were worth attending to, not because they were most common. Thematic analysis allowed us to pick impressive events, instead of counting instances of each theme. We refer to participants by pseudonym numbers (e.g., "4016").

How Did the Participants Respond to Uncertainty?

Where participants were uncertain about the mathematics, responses fell into three categories. We first share the cases in which responses over-relied on the correctness of the numerical answer to determine validity. Then, we show cases in which responses demonstrated engagement in doing the mathematics in the moment to figure out the solution. These cases included a number of different and potentially productive mathematical approaches. Lastly, we present cases of seeking external resources to resolve the uncertainty.

Correctness of student answer

In responding to the item, some participants acknowledged their uncertainty but seemed to give the numeric answer the student produced disproportional weight as a signal of validity. For example, when determining the validity of Craig's work, a participant (4016) first determined the answer -9 using complex arithmetic and then compared her answer to Craig's answer, stating:

> So I disagree with Craig's answer as far as did he apply properties of exponents, yes he did, but he did not get the same answer that I deem valid, and therefore I think there is something wrong with his application.

The participant, however, did not explain what exactly is wrong with Craig's application of the law, and when prompted by the interviewer to guess at what might be wrong, stated, "I guess it's just some properties of exponents must not apply to imaginary numbers." This participant was aware of her uncertainty regarding the properties of exponents, and qualified her answer as a guess. Instead of exploring whether or not the properties of exponents applied, she relied on the correctness of the answers to determine validity. It is worth noting that this did not, for this participant, lead to an incorrect analysis of the second student's work. Participant 4016 concluded that Katlynn's work was not valid, even though it produced a correct answer, and engaged in exploration of why. However, the logic of discarding Craig's use of the exponent laws on the basis of it producing an incorrect answer is subtly flawed—in such a case Craig might have applied the laws correctly and have made an arithmetic error that produced the erroneous answer. Although it is mathematically valid to conclude that something about reasoning is incorrect when an incorrect answer is obtained, the item asks whether the error can be attributed to a particular source, not just whether the reasoning is on the whole invalid or valid.

It would be equally flawed to conclude that Katlynn's work was correct simply because it led to a correct answer. We did not see clear evidence of participants basing their answers only on Katlynn's answer, but we did see those in which responses seemed swayed by the correct answer. One participant (4023) simply stated that Katlynn's work was correct, and when pushed to explain why Katlynn could simplify the expression to -9 if Craig could not, said, "I mean she's got the right answer but actually I'll put a question mark in terms of like the way she did it," hinting that this participant must have considered the answer more heavily than the method in marking 'valid' as his answer for Katlynn.

In one striking case, interviewer error gave a window into how vulnerable a participant's reasoning can be to the correctness of answer. A participant (4020) engaged in strong initial reasoning about Craig's work, initially solving using complex arithmetic and correctly pointing to the sign of the radicand as a potential issue:

> If I do what this student [Craig] did I'm squaring –9 first, which gives me 81, and then I'm taking the square root of 81 and then that would be +9 so… So if, I guess if the number inside the radicand there was positive then it would be okay to assume that this would be good.

He goes on to conclude that Craig was making an "incorrect generalization," and when asked to explain the validity of Katlynn's work, he initially replied "1 and ½ I don't think that they cancel," indicating that he does not think Katlynn's work to be valid. The interviewer, who misheard the response, then made a mistake and asks how he would reconcile this with Katlynn's "correct answer", and the participant reported that he "didn't look at the final answer." He then changed his original answer, saying, "I'm squaring the square root, so actually that is valid, isn't it?" and went on to state that it would work for both negative and positive bases, which is not true. In other words, with only a slight prompt from the interviewer, a participant whose reasoning was quite strong to that point reversed course and presented contradictory reasoning to justify the student's correct answer, suggesting that the numeric answer influenced his thinking about the reasoning quite strongly. It is also worth noting that a number of participants, asked later in the interview how they approach figuring out if student work is valid in general, discussed the 'false positive' logic, when a student's answer is correct it could be simply coincidence and one needs in such case to dig more deeply into the reasoning. However, in several cases participants who made these statements still showed signs of being unduly influenced by the answer, indicating that perhaps even strong understanding of the pitfalls in general terms is vulnerable in the face of uncertainty around the underlying mathematics.

Mathematical investigation

The second type of responses we noted among our responses was that of engaging in mathematical investigations in an effort to resolve the uncertain mathematics. This is a response that the literature might describe as embracing uncertainty, and we view this type of engagement to be a positive response type in general as it has the potential to lead to deeper mathematical understanding. The examples we are sharing here, however, also show that *engaging* with uncertainty does not guarantee that the participant will respond correctly to the item. In some ways this may be a limitation of the data, which was not collected for the purpose of exploring uncertainty. Interviewers did not, in general, push participants to resolve uncertainty, as they were cautious to avoid leading the participants during the interview. Because of time constraints of the interview, they moved on from discussion once the desired evidence had been collected, and so we simply do not know whether participants might have reasoned through their uncertainty productively given sufficient time or support to do so. We describe the approaches below as *potentially productive* because from a mathematical perspective they are approaches from which one could reason to an understanding, not because these participants necessarily succeeded in doing so. Our purpose in noting this pattern of engagement is not to suggest a certain set of 'right' teacher approaches so much as to describe a set of reasonable approaches in ways that may support education researchers and teacher educators in anticipating possible teacher reactions to uncertainty in algebra. We discuss an algebraic approach, a graphical approach, and an approach involving examination of the boundary.

Algebraic approach

A number of participants engaged in an algebraic approach at some point during the course of responding to the item, in some cases in the course of evaluating the expression independently using complex arithmetic, and in some cases in unpacking the work of one or both students. In some cases the participant engaged in more extended mathematical investigation. We share the response of a participant (4028) who used an algebraic approach to understand Craig's work to solve the task in his own way, and then engaged in a more extended abstract investigation. When he first read Craig's work, it seemed plausible to him, and he unpacked line by line what he believed Craig to have done algebraically: "I said based on reading this sentence, yes, that makes sense. Powers of power, that's the rule, that how I remember them, powers of power, you multiply them, and when you multiply by two numbers is associative. So we can change the location, I just commute... So I said, oh that's going to work, so in that case 9^2 is going to give me 81, and then I do 81 to the 1/2 power that means square root, which is equal to +9." As he considered each line of Craig's work, he judged each step to be valid, and concluded initially then that Craig's work was valid, a reasonable conclusion. The participant goes on however, to do a deeper examination by solving the task in his own way:

> But after that, I said wait a minute, let me try this out on my own, and then I did my work down there if you see in that $(-9)^{1/2}$, and that give me the imaginary number I brought in the complex number system. And then $(3i)^2$ I got 9 times i^2 and in definition i^2 is –1. I got –9. So I said wait a minute, if I follow this the way to do it this outer from inner, from the inner computation go to the outer bigger picture, I got –9. But the first student got 9, okay that cannot be right.

The participant's algebraic examination, like other participants who solved the task in their own way, mainly involved computation with imaginary numbers. This algebraic examination led him to conclude that Craig's work is not valid because Craig's final answer was different from the answer he arrived at. At this point he potentially might have allowed the correctness of the numeric answer to overwhelming thinking, but instead he continued to explore, finding it necessary to figure out why algebraic steps that seemed reasonable to him would lead to an answer she believes is incorrect. He moved to a general case to explore further:

> I tried to write out the general case, that's……I'm thinking along the same rule, use letter A you see the numerical, algebraically, $(A^B)^C$ this is powers of power, this is the same as A^{BC} but since B times C we can commute we can rewrite as A^{CB} is equal to$(A^C)^B$. So that's really like what is the first student describe in words. And usually I take this view is correct, but based on the numerical calculation I did down there, it seems this answer going to change if A is negative in this case A is –9. So the correct answer should be –9 as I calculated. So that's how to me is a concept of my general experiment maybe not valid 100% depends what is A, B, C.

By moving to a more abstract representation of the rule Craig applied, the participant is able to articulate what it is he would generally consider correct and state that the contradictory evidence makes his question that conclusion. He continued to explore, stating:

> I'd written originally A, B, C belong to the real number, I do realize that, is this always true? That for B and C, that could be real? Whether it's a positive exponent or negative exponent I think that doesn't matter, but for the base... so that's why I erase A I put A belongs to R positive, or A is greater than or equal to zero I personally try to make a generalization here. But I'm not quite sure with this generalization I get is 100% right mathematically. And then I wrote a sentence, "If base A is a negative, then the rule does not always work."

The participant remained uncertain about whether the domain restriction he has identified is exactly correct as stated, but makes substantial progress through this exploration in figuring out what the key uncertain mathematics is and why it is uncertain.

Graphical approach

Another participant (4016) used a graphical approach to address her feeling of uncertainty about Katlynn's approach:

> I know that there is something weird about the way we define the square root, if you look at like the graph of $f(x)$ squared it's a parabola and it's in you know the first and the second quadrant but then if you take the inverse which would graphically be the equivalent of... reflecting over the line y equals x now you have a problem you would have like a sideways parabola in quadrant one and quadrant four but it won't pass the vertical line test, it won't be a function. So we just kind of say like oh no we're just going to take the positive part. And I think that kind of where some of the trouble occurs by the way that we define the square root. Because technically the inverse of squaring something is plus or minus the square root so it's not, the x^2 function doesn't have a true inverse, the inverse of the x^2 function is not a function so we have to have to kind of restrict the domain to make it a function, we say the square root has to be positive yeah. So then I was like well let me figure this out. So I decided to graph both of them like both ways like f of x equals the square root of x squared and when I did that I got the line $f(x)$ equals x but only when I took positive. Then I tried to graph it the other way and do $f(x)$ equals the square root of x^2 and I got the absolute value function.

The participant, who had already noted that Katlynn's answer was numerically correct, went on to attend to her method, and connected the cancellation to the relationship between a function y equals x^2 and its inverse, touched on the issue of restricting the domain, and used a graphic calculator to graph the functions. From this graphical investigation, the participant concluded that "squaring the square root doesn't give you back x all the times" and "the square root of x^2 doesn't always give you x all the time." The participant tried through this to come up with a viable argument on why Katlynn's method may not always work. That is, she actively explored mathematical uncertainty by taking a graphical approach. Although the participant did not articulate fully the set of domain restrictions under which Katlynn's method would be valid, her exploration was sufficiently productive to point her to the correct conclusion that Katlynn's method was not valid.

Examining the boundary

Some participants tried to determine the validity of student work by examining the boundary of the laws, specifically by selecting other test cases to see if the law would hold. For example, a participant (4020), once forced to confront the disconnection between Katlynn's 'correct' answer and her method, searched for example cases to justify why her method should be valid: "Maybe that is valid. Because if I would have done 100 times [inaudible] 1/2 to the square root I would get 10, 10^2 is 100. I'm squaring the square root, so actually that is valid, isn't it?" The participant applied Katlynn's strategy to 10, a positive number, and this selection may be intentional since the originally given value, −9 was negative. In general, selecting examples that are different from one another is a productive choice when testing the generality of a rule. In this case, however, it leads the participant to an erroneous conclusion because Katlynn's method does, in fact, work with positive bases regardless of the exponent. We note that we do not have strong evidence here to infer the classic error of using isolated examples to demonstrate the truth of a rule; this participant was not asked to provide a mathematical proof and was simply engaged in informal reasoning to try to make sense of an apparent contradiction. As mentioned above, he had been unintentionally prompted to conclude Katlynn's work was correct, so may have simply been looking for any explanation that would make this a plausible conclusion. His approach, however, had the potential to be productive, because testing a rule's generality by selecting well-chosen examples to test is typically a productive starting point.

Seeking external resources

A third pattern we observed was a willingness to draw on external resources. For example, when the interviewer asked a participant (4016) what she would do if Craig and Katlynn were in her classroom and were forced to respond, the participant said she would search on the web:

> I mean honestly I would just go try to Google and figure out what the problem is or I would tell them to think on it tonight and see what you can find online and I would definitely try to explore it further and come to like an answer. I don't have any problems with revealing my hesitancy in front of the students.

We point out this response type for a number of reasons. First, it is worth noting that it is also a potentially productive response to uncertainty. It might be argued, in fact, that in the moment of classroom teaching, pausing to engage in extended exploration might not always be appropriate, and finding a way to buy time by saving the question for another day is a strong pedagogical response. We often think of teachers as reasoning through difficult mathematical situations that come up in teaching in isolation and in the moment, but this participant reminds us that this need not be the case, that one can respond to students by simply saying 'I don't know but I'll find out" and by consulting with external sources—in this case via internet search, but perhaps also by consulting with colleagues or mentors.

Approaches were not mutually exclusive

We also note here that we have chosen to present responses from one participant (4016) as examples across three categories in order to illustrate that these response patterns were not mutually exclusive. Participant 4016 used one type of reasoning in considering Craig's work and another in considering Katlynn's work. She was willing and able to engage in mathematical investigation but still reported that if faced with such a situation in actual instruction she would likely turn to an external resource instead. Although she selected the intended answer to the item, that neither student's work was valid, she experienced uncertainty throughout her response, using phrases like "I'm not sure" and "my understanding is limited." But she also indicated a level of comfort with that uncertainty, seeming untroubled by revealing it to the interviewer, and stating explicitly that she would be comfortable revealing it to her students. This comfort with uncertainty hints at our second theme, the assignment of valence.

What Types of Valence Do Participants Assign to the Uncertainty They Experience?

In addition to how the participant reacted to the uncertainty, we examined statements in which they assigned valence to the uncertainty or to their experience of it. Not surprisingly, the valence varied. More surprisingly, we did not see many cases of negative valence even when respondents reported uncertainty or answered incorrectly, and the expressed valence did not show a strong connection with the approach taken to resolving the uncertainty; in other words, assigning a positive valence did not necessarily predict potentially productive investigation, nor did assigning a negative valence preclude it. In the next sections we describe what these statements of valence looked like when we saw them in the data.

Positive valence

One participant (4012) assigned a strong positive valence to uncertainty, demonstrating not just comfort but genuine excitement about the ways in which uncertain situations can be pedagogically useful:

> I have had times when I'm like whoa I didn't think about this, how did this happen? But the important thing is just that I just throw away back to my student. That's a very good observation, why did you think this happened? You know, it kind of makes

> them be more kind of curious about it, but then like I go home and I'm like dying, I'm like okay I got to figure this out why is this happening you know. But yeah... but it's very nice I love when situations like these come out because it forces to think of why, and also different ways to actually justify our thinking.

This particular participant did engage in mathematical exploration, and expressed explicit uncertainty throughout the interview, even saying "I had a hard time with this." In the end, despite her expression of need to figure out what is happening, she did not arrive at the intended answer to the item during the recorded interview.

Mixed valence

Participant (4014) expressed mixed valence. This participant, who self-identified to "have a pretty significant mathematical background," displayed a strong understanding of the inherent ambiguity in the expression and disliked the idea that Ms. Williams had presented such a task to students. As he commented:

> It gives you something, ambiguous is just like really the best word here it gives you something that's ambiguous. It doesn't define the meaning for you. Where mathematics is very precise in its meaning and its definition. And having such a question that creates such a gray area, kind of isn't that great, like you really shouldn't have that there but that's just my opinion.

This participant's negative valence originated from the task being "ambiguous," and unlike other responses seemed less about his own experience of uncertainty and more about his perception that the students would experience uncertainty in a pedagogically inappropriate way. He did, however, recognize the potential for other types of open tasks to elicit interesting classroom discussions:

> So there's this great problem where they're like "Can you raise an irrational number to an irrational number and get a rational number?" That's an amazing question and a great activity for a day or two for students to explore. And it's something I want to use with a different population of students who would be able to access that question more.

The two excerpts above shows that the participant appreciates uncertainty as what intellectually stimulates students to have mathematical discussions. We hypothesize that this participant has a positive valence to uncertainty that is originated from students' feeling, but has a negative valence to uncertainty that is from the mathematics in the task. In other words, it seems that the participant values students' mathematical exploration, but the exploration should only happen within a mathematically safe space. Teachers like this participant might not think that mathematical knowledge is open to revision (Ernest, 2004, 2015; Lakatos, 1976), hence inherently uncertain.

Discussion

In the first set of results we discussed a range of approaches that participants took toward the uncertain mathematics presented in the Williams item. We noted that the correctness or incorrectness of the given numeric answer seemed to be given disproportionate weight in some participants' thinking, at times overshadowing their thinking about the validity of the methods. Some participants explicitly mentioned being aware of the logical error in which one assumes that a correct answer means correct reasoning was used to reach it, but mentioning this did not seem predictive of resisting the influence of the answer. This may not be a shocking result—people do not always do in specific cases the things they describe as general approaches. But we wonder if it might also be a function of the uncertain mathematics. Confronted with uncertainty, there might be a tendency to fall back on methods for evaluating student work that are simply not valid methods even when the person doing so knows that they are not valid methods. Several participants fell into the accompanying error of assuming that because Craig's numeric answer was incorrect his method must also be without examining his work in detail. These examples suggest to us a number of points that might be taken up specifically in teacher preparation, including explicit discussion of the relationship (or non-relationship) between students' methods and their answers, but perhaps more importantly suggesting that prospective teachers need opportunities not just to discuss these things in general terms but also support in learning to recognize cases where these things might be happening in their own thinking.

Overall we observed many responses in which participants engaged in potentially productive mathematical investigations. We presented examples of an algebraic approach, a graphical approach, and an examination of the boundary, but recognize that this is only one way of categorizing these responses. Most of the algebraic approaches also might have been labeled as attempts to explore the generality of the rule, which has some overlap with exploration of boundaries, or we might have categorized those approaches in terms of use of examples or whether a general proof argument was attempted. We do not claim that our categorization is necessarily best or comprehensive, and certainly one would expect that different tasks would present opportunities to engage in different types of strategies. We called out these three approaches because we believe they are approaches that are likely to generalize to other task types and that therefore deserve attention in teacher preparation. The participants of this study demonstrated their *mathematical habits of mind* (Mark, Cuoco, Goldenberg, & Sword, 2010) by looking at extreme cases or using abstraction. That is, the embedded mathematical uncertainty provided an opportunity for the participants to practice their mathematical habits of mind. This seems particularly important to attend to in the context of the CCSS standards for mathematical practices, which ask teachers to help students learn how to engage in some of these very habits. Situations in which uncertainty is available might constitute the very opportunities teachers need to model this type of engagement to their students as they themselves think through the mathematics in a public way.

The second set of results in which we observed the valence assigned to uncertainty is harder to interpret. We expected to see case of strong emotional reaction to the item, as is often the case with tasks that participants find difficult. We also expected we might see a fairly simple relationship between valence and approach in that people who experienced uncertainty as positive might engage productively, but the story was more complex. We see from the data that we cannot predict the participants' correctness of their responses from their valence. We also see that with the valence, we cannot predict whether they would deeply engage with the uncertainty. We observed cases where valence was assigned to different aspects of the uncertainty at in different ways. All this are to say that the relationship between valence assigned uncertainty and meaningful mathematical investigation is not straightforward.

Being aware of this complexity between valence and resolution of uncertainty is important for teacher educators and researchers. Teacher educators should be careful not to assume that teachers with negative valence will not engage with uncertainty and those with positive valence will conduct an in-depth mathematical investigation or will know how to do so productively. This awareness is important for researchers, too, because teacher valence is often not explicitly addressed in the literature. While literatures encourage teachers to acknowledge uncertainty, we would argue that *acknowledging* uncertainty should not be interpreted as having a positive valence toward uncertainty. Our examination presents cases suggesting that the relationship between valence and engagement might be more complex than we thought, and therefore further examination on such matter is necessary to support better teacher learning.

We learn from literature that cultivating an orientation toward engaging with uncertainty is important for teacher development (e.g., Capobianco, 2010; Meister & Nolan, 2001; Munthe, 2001, 2003; Munthe & Thuen, 2009; Wheatley, 2005), but this is clearly only the first step. Teachers also have to have ways to work *productively* with that uncertainty, regardless of the valence they assign it, in order to figure the mathematics out for themselves. To teach students to engage in mathematics by exploration, teachers must think about modeling the practices as they engage in order to utilize uncertainties as opportunities to learn.

Conclusion

The examples we shared in the paper, rather than providing a catalogue of responses, present possible approaches teachers can take. The point of these examples is to illustrate that uncertainty is a slippery space in which teachers' mathematical and emotional response varies, and one that can easily arise even in the context of fairly simple mathematics. We do not advocate a "right" response to uncertainty, but think that these considerations might be useful to teacher educators in considering how to prepare teachers for dealing with such situations. We see a strong potential of uncertainty as a resource for teacher education. We draw on the notion of *specialized content knowledge* (Ball, Thames, & Phelps, 2008) to understand the potential of uncertainty. When responding to the Williams item, the participants used mathematics to understand student work. That is, uncertainty from the item invited the participants to use mathematics that are unique to the teaching situation.

We echo Zaslavsky's (2005) suggestion to include tasks with embedded mathematical uncertainty in teacher education to cultivate mathematical understanding. We take a step farther and claim that teachers should be supported in acknowledging uncertainty, confronting and engaging it, finding productive approaches, articulating those approaches, and thinking through how such moments could be transformed into "teachable moments." This is partly mathematical skill, partly an emotional response to uncertainty, and partly developing a disposition that values such opportunity. The field has acknowledged the importance of teachers' modeling the mathematical practices for their students (e.g., Charalambous, Hill, & Ball, 2011) and we posit that the best modeling may come in response to a genuine, unexpected mathematical problem in a case where the teacher truly does not know the answer. By providing teachers opportunities to engage in mathematical uncertainty, we expect them to become more skillful at productively responding to uncertainty, and in turn more able to model such practice in front of their students. Teacher educators needs to provide opportunities for novice teachers to engage in approximations of the work of teaching (Grossman, Compton, Igra, & Ronfeldt, 2009), where responding to the range of unexpectedness presented by classroom interactions (Coles & Scott, 2015; Mason, 2015) is part of that work of teaching, and one type of unexpectedness can be in the form of uncertain mathematics. Responding to mathematical uncertainty is not simple, and we see it comprises at least the following steps: (a) notice their uncertainty about what a student is proposing (b) understand that it might be worth pausing for or attending to (c) constructively direct the class in a way that leaves the question open and allows time to think (d) explore the uncertain mathematics and (e) make a judgment about how to engage the students productively depending on the result of that exploration. We believe that engagement with appropriate tasks can provide opportunities for prospective teachers to practice performing each of those steps, with appropriate support.

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Reasoning about changes: a frame of reference approach

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In a RUME 18 Theoretical Report, my co-authors and I presented our cognitive description of a conceptualized frame of reference, consisting of mental commitments to units, reference points, and directionality of comparison when thinking about measures. Here I present a pilot study on how a focus on conceptualizing a frame of reference impacts students' ability to reason quantitatively about changes. The two-part empirical study consisted of clinical interviews with several students followed by teaching interviews with three students, who were chosen because of their varying abilities to conceptualize a frame of reference. This initial evidence shows that the ability to conceptualize a frame of reference benefits students as they attempt to reason with changes.

Keywords: Frames of Reference, Quantitative Reasoning, Quantities, Changes, Additive Comparisons

In a RUME 18 Theoretical Report , my co-authors and I presented our cognitive description of a conceptualized frame of reference (Joshua, Musgrave, Hatfield, & Thompson, 2015). At the same time, my experiences working with reform curricula for Precalculus (Marilyn P. Carlson, Oehrtman, & Moore, 2013) and Calculus 1 (Patrick W. Thompson, Byerley, & Hatfield, 2013) at a large Southwestern public university led me to be surprised at how much students struggle with thinking and reasoning about changes. Rate of change is known to be a central idea in calculus (Marilyn P. Carlson, Jacobs, Coe, Larsen, & Hsu, 2002), and important to introduce as early as Algebra 1 with the idea of slope. But in order to reason about rate of change, a student must be able to conceptualize and reason about changes themselves.

My own anecdotal data and conversations with other Precalculus and Calculus 1 teachers showed that students frequently struggled with conflating the idea of a total quantity's measurement with a change in a quantity's measurement. These discussions led me to realize that a fundamental differentiation between measures of total quantity and change in quantity is coordination of reference points from which to measure. Hence, student struggles in reasoning with and about changes fit squarely within my current research topic of frames of reference.

Theoretical Perspective

A survey of both math education and physics education literature revealed few papers that focused on student thinking about frames of reference (Bowden et al., 1992; Dede, Salzman, & Loftin, 1996; Marshall & Carrejo, 2008; Monaghan & Clement, 2000; Panse, Ramadas, & Kumar, 1994; Shen & Confrey, 2010) and many of those did not explicitly state what they meant by a "frame of reference", taking that phrase to have a shared meaning between the authors and the readers; even a paper on alternative conceptions of frames of reference (Panse et al. 1994) only stated what a frame of reference is *not*. However, it is not clear that there is a shared definition of a frame of reference in these fields. The few concrete definitions we found in the literature or textbooks ranged from physical objects like "welded rods" (Carroll & Traschen, 2005), or "a set of observers" (de Hosson, Kermen, & Parizot, 2010) to a coordinate system (Young, Freedman, & Ford, 2011).

More significantly, none of these definitions focus on the cognitive actions of a student that is thinking about *measures within a frame of reference*. In tasks that involve thinking about a frame of reference (such as relative motion, special relativity, etc.), the focus of the task is never the frame of reference itself. Rather, it is that a student must organize the measures of quantities such as distance, velocity, and time within one or more frames of reference in order to keep track of what those measures mean. In a 2015 paper, my colleagues and I introduced our cognitive definition of what a conceptualized frame of reference entails:

> An individual can think of a measure as merely reflecting the size of an object relative to a unit or he can think of a measure within a system of potential measures and comparisons of measures. An individual conceives of measures as existing within a frame of reference if the act of measuring entails: 1) committing to a unit so that all measures are multiplicative comparisons to it, 2) committing to a reference point that gives meaning to a zero measure and all non-zero measures, and 3) committing to a directionality of measure comparison additively, multiplicatively, or both (Joshua et al. 2015).

One important implication of our definition is that it places frames of reference squarely within the larger construct of quantitative reasoning (Patrick W. Thompson, 1993a). Though people may frequently speak of a frame of reference as a noun, it is actually a set of decisions that a person makes about how to think about quantities and their measurements. Thompson writes "[a] person constitutes a quantity by conceiving of a quality of an object in such a way that he or she understands the possibility of measuring it" (Patrick W. Thompson, 1993b). In 2015 we expanded on his previous definition by defining a *framed quantity* "which refers to when a person thinks of a quantity with commitments to unit, reference point, and directionality of comparison".

Our definition expands the utility of frames of reference from its traditional application of relative motion to almost any situation in which quantitative reasoning is necessary. Of course, our definition provides a strong starting point to analyze student difficulties for working with relative motion. However, we have already identified additional contexts were these ideas can be applied: concavity of functions, rate of change, electric potential, and the difference between quantity and change in quantity.

Research Questions & Methodology

My initial hypothesis was that student struggles in reasoning with and about changes were due at least in part to the fact that measures of changes in quantities had reference points and directionality, but that they did not think about *framed quantities* – quantities whose total measures necessitated commitments to unit, reference point, and directionality of comparison in order to make sense of them. Therefore, they did not have parallel attributes with which to compare and contrast the ideas of quantities versus changes, and to distinguish the two in their minds. From this hypothesis I developed seven research questions for this pilot study, four of which are discussed in this paper.

Changes in Quantity vs. Values of Quantity:

- RQ#1: How do students conceptualize a change in a quantity versus the value of a quantity?

- RQ#2: Does a focus on frames of reference affect students' ability to reason about changes in quantity and values of quantity, by drawing explicit attention to reference points?

Changes in Changes:

- RQ#3: How do students think about changes in changes, in tasks such as being asked to identify whether a function is increasing/decreasing at an increasing/decreasing rate?
- RQ#4: Does a focus on frames of reference affect students' ability to reason about changes in changes, by drawing explicit attention to a directionality of comparison?

To investigate these research questions, I designed and carried out an empirical study on the connections between a student's ability to conceptualize a frame of reference and his or her ability to reason about changes that involved both clinical and teaching interviews. Clinical interviews are conducted for the purpose of gathering information about a student's current understanding; with such a goal it would be counterproductive to engage in any form of teaching, including questioning designed to help the student make connections. In contrast, teaching interviews developed as a data-collecting tool in the 1970s because "researchers explicitly acknowledged that mathematical activity in school occurs as a result of students' participation in teaching" (Steffe & Thompson, 2000). The goal of a teaching interview is to ascertain both a student's current understanding and to examine how and to what extent a student progresses in understanding in the context of teaching (such as probing or guiding questions, didactic objects (P. W. Thompson, 2002), conversations with the interviewer, direct instruction, etc.). Even with a teaching interview, a student's progress is still affected by the mental resources that she brings to the interview, and a teaching interview can help to illuminate those resources.

The first part of the study consisted of clinical interviews with seven students who had taken at least one algebra class and one physics class in college or high school. These interviews helped me form models of each student's current ability to conceptualize a frame of reference. I then picked three students that I found demonstrated varying abilities to conceptualize a frame of reference. Hayden, a recent college graduate in graphic design, consistently gave high-level responses that were situated within a frame of reference and referred to directionality and reference points when explaining his answers. Miranda, a biochemistry major in a reform Calculus 1 class, performed at a medium level, sometimes giving answers that indicated a conceptualized frame of reference. Luigi, a full-time worker with an Associate's degree, rarely gave answers situated within a frame of reference and often could not explain his answers. The second phase consisted of teaching interviews with these students to study to what extent their varying abilities to conceptualize a frame of reference would help them to move forward in their ability to reason about changes.

All interviews were videotaped and analyzed to form models of how the student thought about measures and measure comparisons before, during, and at the end of the teaching experiment. I also used the videos to build hypotheses about how these ways of thinking about measures (within a frame of reference or not) affected each student's ability to reason about changes. For Luigi's interview the video recording malfunctioned, so I wrote detailed notes with Luigi directly after his interview.

Results from Clinical Interviews

In this section I present results from the clinical interview stage of this study. I focus on tasks that elicited responses that helped me to build models of how Hayden, Miranda, and Luigi did or did not conceptualize a frame of reference.

Land of the Midnight Sun

The second part of our cognitive definition of a conceptualized frame of reference is that a person commits to a reference point that gives meaning to both a zero measure and all non-zero measures. To see to what extent my students could make such a commitment, I created the graph in Figure 2 of a flagpole's shadow length in an imaginary Land of the Midnight Sun, where there is sunshine in the middle of the night. The labels "length of shadow" and "o'clock" on the axes were left deliberately vague.

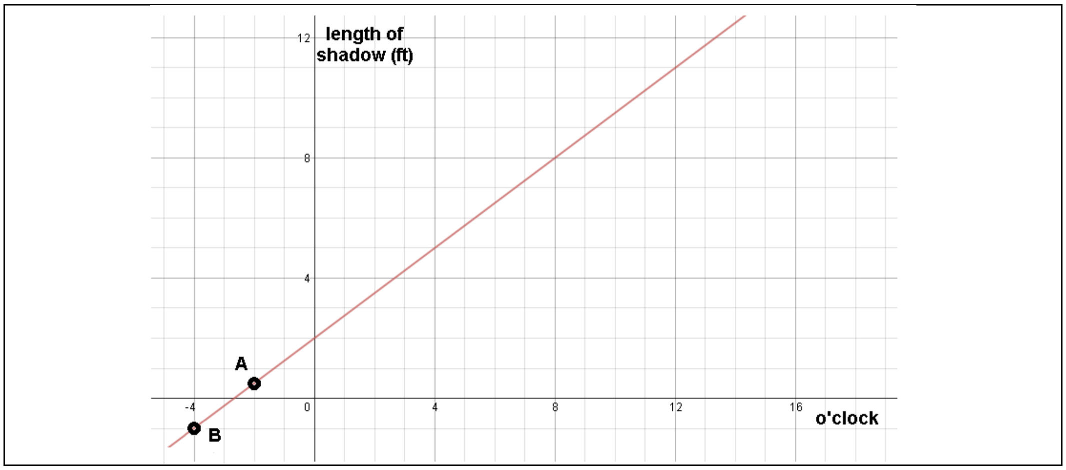


Figure 2. Land of Midnight Sun graph.

Part 1's purpose was to make sure that the students understood what the vague axes labels meant. They were shown the portion of the graph for x ≥ 0 and asked "What does this point mean?" while highlighting the points (4, 5) and (14, 12.5); all three easily said that at 4am the shadow was 5 feet long and at 2pm the shadow was 12.5 feet long.

For Part 2, I extended the visible graph to the domain x ≥ -3 so that point A was visible, and asked the students what the meaning of point A was. Hayden immediately said that at 10pm the night before the shadow's length was 0.5 feet. When asked how he decided that the time was 10pm, he said "It's like this 12 o'clock is the point of reference of when to use…". Miranda originally said that point A was the length of the shadow at 2 o'clock, but with some prompting was able to say 10 the night before. There was a clear difference in how Hayden and Miranda thought of measures of time; while Miranda seemed to have some idea of using zero/midnight as a reference point, her meaning for her own reference point did not help her to find meanings for negative measures. In contrast, Hayden's explicit mention of a reference point and immediate response showed that for him the reference point gave meaning to all non-zero measures both positive and negative, which is the second part of our 2015 definition of a conceptualized frame of reference introduced in the Theoretical Perspective above.

In Part 3 I showed the entire graph and asked the students to interpret the meaning of point B. Here Miranda started by assuming that the meaning of point B somehow related to the area under the curve. When asked to justify her answer she eventually said that the length had started increasing again as time went backwards. Luigi clearly stated that the shadow's length was 1 foot, and when questioned further he merely shrugged and smiled and said that he didn't know how to account for the negative. He did not seem upset by his lack of a systematic way to deal with the negative. Hayden was the most clear, saying that "the fact that it's going negative in feet now could mean one of two things. It could mean the shadow is just like going beyond existing, which is not realistically possible, so… The other explanation would be that the shadow is now going in the opposite direction. So, if the shadow was extending to the right, now it's extending to the left." Here Hayden was incorporating both a commitment to reference point and a commitment to directionality of comparison that allowed him to make these comments; these commitments are the second and third parts of our 2015 definition of a conceptualized frame of reference.

Hayden, Miranda, and Luigi revealed their varying abilities to commit to a reference point by their treatment of the vague axes labels. This task was designed to answer the question 'What meanings could each student give to the definition of zero on each axis so that negative measures made sense?' Miranda and Luigi were unable to give such productive meanings to 'zero o'clock' and 'zero shadow length' while Hayden could. This suggests that Hayden had ways of thinking – specifically, committing to a meaningful reference point and directionality of comparison - available to him that Miranda and Luigi did not have. Because of this he was able to work with a conceptualized frame of reference, as defined by our three-part definition above, to make sense of all the interview tasks in a coherent way.

Comparing Spousal Heights

The third part of our cognitive definition of a conceptualized frame of reference is that a person commits to a directionality of measure comparison additively, multiplicatively, or both. If a student always calculates changes by saying "3 less pounds" or "4 more pounds", then the student is constantly changing her directionality of comparison to keep the comparison measures positive. The measures themselves (3 and 4) do not carry all the information needed to make sense of them. While this is usually not a problem for a single comparison, it becomes extremely important when reasoning about multiple comparisons.

As with my unit task, I did not expect students to spontaneously commit to a directionality of measure comparison. Instead, I created a series of increasingly detailed questions and looked to see how much prompting (if any) could elicit such a commitment. Even though this was still part of the clinical interview, it was also the first time I had an opportunity to ask the students about changes (additive comparisons).

husband's height (ft)	wife's height (ft)	
5.6	6.2	a) Compare the heights of each couple.
3.2	4.9	b) How much taller or shorter?
6.1	5.9	c) Now only use the words "taller than"
4.8	3.2	d) Now only use the phrase:
5.9	7	"Wife is ___ taller than husband"

Figure 3. Comparing Spouses' Heights Task

All three students did not maintain a directionality of comparison for prompts a-c. When I asked them to maintain a comparison from one spouse to the other and also maintain the comparison "taller than" Luigi was unable to do, and Hayden and Miranda had similar answers but with very different levels of comfort, which demonstrated their varying abilities to conceptualize a frame of reference.

Hayden: So, the wife is point 6 feet taller than the husband in the first couple. Second couple; the wife is four…second couple, the wife is 1 point 7 feet taller. I guess I have to use the word taller?

Interviewer: Yeah. Than the husband.

Hayden: Than the husband. Okay. So, if I have to keep saying the wife first, then uh... use the word taller, then the wife is…The third couple, is negative point 2 feet taller than the husband. And for the fourth couple, the wife is negative point 6… negative 1 point 6 feet taller than the husband. And the last couple, the wife is 1 point 1 feet taller than the husband.

Interviewer: Okay. You hesitated before you did the third couple. May I ask why?

Hayden: Cause usually when talking about people's height, you don't really use a negative number. So I just had to think for a second… like, if that's the way that I wanted to describe it.

Interviewer: Okay. Can you tell me about why you decided that's the way you wanted to describe it?

Hayden: I mean, given the parameters, there's not really much else… other way you can describe it. If you have to use the word wife first, and if you have to say taller, there's not that many other ways to describe it.

Interviewer: Do you think that that is actually a mathematically legitimate way of describing the situation?

Hayden: I'd say it's a similar situation to the graph before [Midnight Sun task]. So, where in this case the negative number either means that there's such a thing as negative length. Like, theoretically, like measuring something that you would actually be able to measure negative amounts. Or it means that it's going in the opposite direction. In this case, it would technically mean shorter, even though you're using the word 'taller'.

Interviewer: All right.

Hayden: So, like if I'm …if someone's shorter, then basically negative taller would be shorter, or negative shorter would be taller.

Interviewer: Now, say that sentence, and be filling in the blank each time.

Miranda: Can I say… do I have to say taller?

Interviewer: You have to say 'the wife is blank taller than the husband.'

Miranda: Okay. The wife… For couple one the wife is point six feet taller than the husband. For couple two, the wife is one point seven feet taller than the husband. For couple three, the wife is not…hmm. For the wife…I don't think…

Interviewer: Just say whatever you feel should go in that blank. Honestly.

Miranda: There's two things that could go.

Interviewer: Okay. Go ahead.

Miranda: Okay, so the first thing is; the wife is not taller than the husband, but the husband is one point two taller than the…wait. Hold on. The wife is not taller than the husband, but the husband is taller…is point two taller than the wife in terms of the change between the wife and the husband. (Laughs)

Interviewer: Okay.

Miranda: Or you could just say the wife is negative point two feet taller than the husband.

Interviewer: Out of those three things, which one do you instinctively go towards, and also which one do you suspect is probably, like, the most mathematical way of saying it?

Miranda: Negative two is the most um…mathematical way of saying it.

Interviewer: And what was your instinct?

Miranda: I just want to switch it to make it positive. So, I just want to switch the…I want to switch the – the input and output …so it's the… it's supposed to be less…it's less confusing.

While maintaining a directionality of comparison in this task was not natural for either Hayden or Miranda, Hayden quickly adjusted to my request and made reference to earlier tasks in which he had made and maintained mental commitments so that all of his responses were coherent. His spontaneous explanations at the end of the excerpt and his analogy between the couple's height task and the midnight sun task showed that he thought about both kinds of measures within a consistent system – a conceptualized frame of reference. In contrast, Miranda had a vague understanding that a negative was a mathematical tool likely to be useful in this task, but was deeply uncomfortable with using it. Both her words and her body language indicated that she was giving her answers against her own better judgment, which indicates that if she was not repeatedly pressed to do so she simply would not commit to a directionality of comparison and therefore not conceptualize a frame of reference within which to think about such tasks.

The husband and wife height's task was followed by a red car vs. blue car speeds task that was formulated in the exact same way, except that one car's speed was always an integer multiple of the other's speed. In order to maintain a directionality of comparison a student would have to go from, for example, "Red car is 5 times as fast as blue car" to "Red car is 1/3 times as fast as blue car." I was surprised to see that all three students completed the car speeds task with much less discomfort! Their speed in coming up with statements could be explained by the fact that they were primed by the couple's heights task, but the difference in their comfort levels and body language was evident. This difference in comfort between maintaining a directionality of comparison additively and multiplicatively will be a point of focus in my further study, and I plan to have the cars going in different directions to see how students coordinate both additive and multiplicative comparisons.

The data from clinical interviews shows that Hayden, Miranda and Luigi had varying abilities to conceptualize a frame of reference. Luigi could not commit to either a reference point or a directionality of comparison, Miranda could partially commit to both in somewhat vague ways that were not always productive, and Hayden could commit to both a reference point and directionality of comparison in ways that allowed him to move forward on tasks.

Results from Teaching Interviews

Of the seven students who participated in the clinical interviews, I chose Hayden, Miranda, and Luigi to participate in teaching interviews because of their varied abilities to conceptualize a frame of reference and their willingness to verbalize their thinking. In this phase my primary goal was to examine these students' ability to reason about changes, and to look for any indications that their reasoning did or did not involve a conceptualized frame of reference.

Tom and Padma

The Tom and Padma task is designed to see whether students understand that all quantities are measured with respect to a reference point, and that any choice of reference point is arbitrary. A total quantity's measurement is therefore a change from a measurement of zero, and what we call a change in quantity is a change from any starting point. Yet even a measurement of zero depends on the definition of the quantity itself. In any given task "total water I drank" might be this day or week but is likely not all the water I have drunk in my entire life, "total wheat sold" is never defined as since man first crawled out of the oceans, and "velocity" by the principle of relativity (citations) has no absolute zero except with reference to another object taken to be at rest. Even what we commonly call "zero velocity" is only a change of zero in velocity with respect to the surface of the Earth.

To see whether the students could, with help, reason about velocity measurements as changes, I created the two-part Tom and Padma task. All three students got stuck and fell silent after 5-7 minutes, so I started asking them about what an observer from space would see to try and elicit a definition of velocity measurement that was specifically with respect to the surface of the Earth. My hypothesis was that once a student acknowledged that even our "everyday" velocity measurements were with respect to a reference point, they might conclude that all velocity measurements have a reference point as well.

> Tom watches Padma's car go by, makes some measurements, and announces that Padma's car is moving at a speed of 15 mph. What can you tell me about the car?
>
> What if I tell you that Padma's car continues at the same constant speed, and ends up making a 200 mile trip in only 4 hours? How can this be? Note: Tom did everything correctly.

Figure 4. The Tom and Padma task.

Miranda had a great deal of trouble with Tom and Padma, asking if Tom only measured the very beginning of her trip when she accelerated, Tom incorrectly measured the time, or that Tom made incorrect calculations. Then, nine minutes after she started working on the task, she made a sudden breakthrough.

Interviewer: How could I look at this [pencil on desk] and say, 'this is at rest?' It's at rest relative to what?
Miranda: My position in time?
Interviewer: Actually, your position in space, right?

Miranda: In space!

Interviewer: Not time. Right now, it's not moving with respect to you, right?

Miranda: No.

Interviewer: Anything else that's not -

Miranda: ***WAS HE RUNNING??!?!***

Interviewer: - moving with respect to you?

Miranda: Was... (laughs and smiles) If he's moving…or was he…. Does this mean…okay…if he's going…if this is slower…Okay…mmm… if this car….

Interviewer: Do you need this pencil?

Miranda: Yes! [Picks up both pens to represent Tom and Padma] (laughs) If this car is going this fast, and he's going really slow…[moves pens parallel to each other, one faster than the other] er, like, he's not at a constant place if this is going super fast, and he thinks it's going really slow…so…does that mean that…ah…is that…. (shakes pens)

Interviewer: Yes!

Miranda: Does….Yeah?

Interviewer: Yes! Yeah!

Miranda: Yeah? He's going really slow and the car's going really fast? That is how you think it is?

Interviewer: Right. Tell me…can you tell me-

Miranda: Like fifteen miles per hour?

Interviewer: Can you be really specific…exactly what's going on, so that he thinks… he honestly…like it is accurate to say, right? Isn't it accurate to say that this [pencil] is at rest with respect to Brittany right now?

Miranda: Mmm hmm.

Interviewer: It's also accurate to say that it's moving with respect to what? What is it moving with respect to?

Miranda: Space.

Interviewer: The problem is, what's space? (laughs) Give me something specific instead of space. Something else that's out there.

Miranda: The speed of light!

Interviewer: That is not an object….or a location.

Miranda: Hmmm.

Miranda was able to see that if Tom was running, as she put it, he could accurately measure Padma's speed as less than 50mph, but that qualitative description was all she could achieve. We spent a total of 45 minutes on the Tom and Padma task (at her request, she did not want to give up) and she could never tell me that Tom must be driving at 35mph on the road to measure Padma at 15mph. Luigi was able to say that Tom must be moving after only a few minutes of thinking, but also was never able to tell me how fast Tom must be going. Neither Miranda nor Luigi could think about the different velocity measurements within conceptualized frames of reference, and so could not coordinate them to finish the task.

In contrast, when Hayden realized that Tom could be moving, he conceptualized both measures of Padma's speed as framed quantities, and could then reason about a change between them. As for the other two students, after he was stuck for 6 minutes I engaged him in a discussion about how we measure velocity.

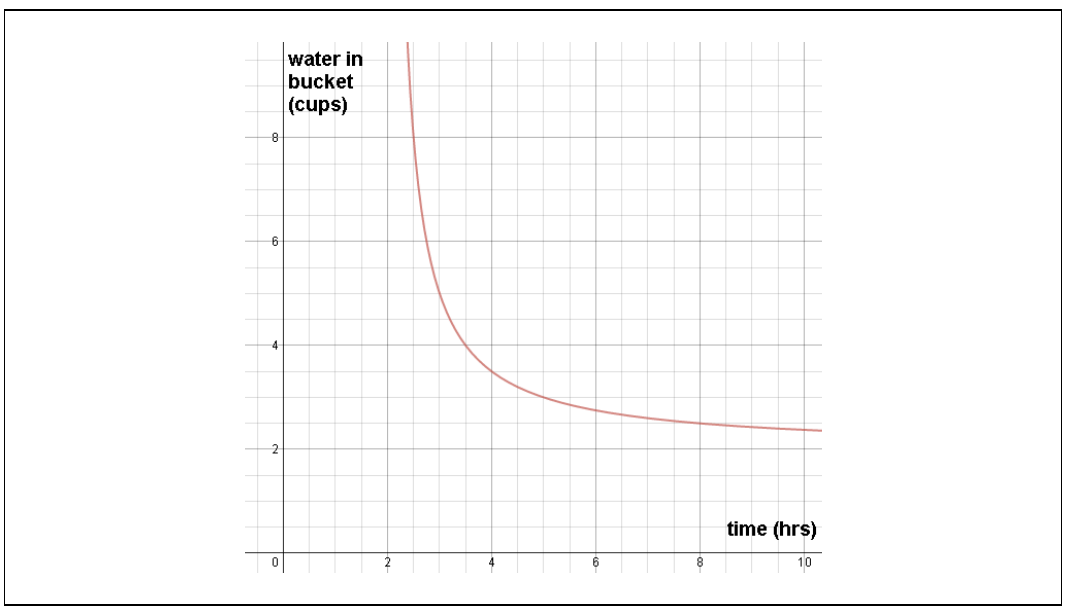
Interviewer: Now imagine the solar system. And the earth is whirling around the sun.
Hayden: Okay. Yes.
Interviewer: And the earth is spinning on its axis. And Padma's driving. Are you sure that her car's going at 50 miles an hour?
Hayden: Ohhhh. It's um, if you weren't factoring in the speed of the earth's rotation, she would be going 50 miles an hour, but if you factor that in, technically she would be going faster than that.
Interviewer: So…can that give you a hint as to what, maybe, is going on here?
Hayden: So, I was making the assumption that we are just considering like, just space… I was going to make the assumption that we were just talking about like, driving like from here to Las Vegas, or something…and just considering that distance on the earth's…
Interviewer: Okay.
Hayden: …just on the earth's surface itself.
Interviewer: Wonderful. On the earth's surface. Or with respect to the surface of the earth.
Hayden: Yeah
Interviewer: Good, okay. Now. You make that assumption, but that assumption is not here. So, how could Tom have taken this measurement and gotten a true 15 miles per hour?
Hayden: [after brief diversion where he tries to incorporate planetary speeds] So, if he's going… so if he's also moving, then technically he would get some reading that's less than what the speedometer is.
Interviewer: Could there ever be more?
Hayden: If he was going the opposite direction.
Interviewer: Wonderful. So, you're saying that he's moving in the same direction?
Hayden: Yes.
Interviewer: Okay. And do you know how fast he was going?
Hayden: [after first correctly saying that Tom was going 15/50 as fast as Padma] But, um… see, he would have to be going 35. Yes, 35.

Not only could Hayden reason that Tom was going in the same direction as Padma at 35mph with respect to the surface of the Earth, but also he spontaneously commented that Tom would measure Padma's velocity as more than 50mph if he was going in the opposite direction from Padma. He was able to use 15mph and 50 mph as framed quantities in different conceptualized frames of reference and coordinate the two frames to reason about changes between them.

There is a clear difference in the way Luigi, Hayden, and Miranda were able to move forward in reasoning about changes with my guiding questions about velocity that address Research Questions #1 and #2: 'How do students conceptualize a change in a quantity versus the value of a quantity?' and 'Does a focus on frames of reference affect students' ability to reason about changes in quantity and values of quantity, by drawing explicit attention to reference points?' Luigi was able to discern that Tom was moving after only a few minutes, and Miranda came to the same conclusion after nine minutes, which meant that they were eventually able to conceptualize velocity measurements as a change in velocity with respect to another object taken at rest (in this case Padma's measurement of her own speed held the surface of the Earth to be at rest, and Tom's measurement of Padma's speed took held *himself* to be at rest). However, neither were able to utilize these realizations to reason that when both measures are considered within a single conceptualized frame of reference, such as the one Padma used, then Padma's speed measurement is a quantity and Tom's speed measurement is a change in quantity. Without this ability to coordinate quantity versus change in quantity, neither were able to make a conclusion about how fast Tom must be going to make sense of the task. Hayden, whose clinical interview showed that he was able to conceptualize a frame of reference, could complete the task by coordinating changes, which required him to commit to reference points for both Tom and Padma's measurements of Padma's speed. Moreover, he was also able to move beyond the planned task to reason about how Tom might measure Padma's velocity as more than 50mph by committing to a directionality of comparison of [Padma's velocity with respect to Earth's surface] – [Tom's velocity with respect to Earth's surface] = [Tom's measurement of Padma's velocity].

Leaky Bucket Task

The leaky bucket task is designed to see if students can reason about changes in changes. The graph in Figure 5 shows the relationship between time and the number of cups of water left in a leaky bucket. Because Luigi was showing signs of discomfort and stress I skipped this task with him and only presented it to Hayden and Miranda.



a) What is happening to the water as time increases?
b) Starting with point (2.5, 8) can you illustrate how much the water changes as the time increases by 0.5 hours?
c) Can you illustrate changes in water along the entire graph for x-intervals of length 0.5 hours?
d) Look at the changes in the water as time increases. What can you tell me about the changes?
e) If necessary: Can you also tell me how the changes are changing?

Figure 5. Graph and Prompts for Leaky Bucket task.

Both students struggled with prompt d. To describe the set of changes drawn in prompt c, a person can say that the changes are negative, and that the changes are increasing as time increases (since the sequence -3, -1, -0.5 is increasing). In order to conclude that the changes are negative a student needs to choose a directionality of comparison $f(\text{time}_{\text{final}}) - f(\text{time}_{\text{initial}})$, and to conclude that the changes are increasing a student needs to continue to commit to that same directionality. While both students chose the above directionality and said that the changes were negative, both then broke that commitment and used a comparison [larger change] – [smaller change] to conclude that the changes were decreasing.

Interviewer: So, as time goes by, what is the water doing?
Hayden: So, the water is decreasing.
Interviewer: Decreasing. Okay…. First let's just look at the changes in time. So look at these changes. Are they increasing, decreasing, or staying the same?
Hayden: Staying the same. Each one's half an hour.
Interviewer: Now, what about these changes?
Hayden: The change is becoming less and less…each time.
Interviewer: Okay. So the change is…
Hayden: So the greater change is first, and it's getting to the less of a change.
Interviewer: Are the changes increasing, decreasing, or staying constant?
Hayden: Decreasing.

Miranda gave a similar response to Hayden. I then asked both students to estimate the value of each change. Hayden did so and initially tried to tell me whether the changes of the changes were increasing, decreasing, or staying constant, but when I redirected him to speak about the changes he said they were increasing.

Hayden: Oh, yeah. But like the size between, if I actually took like accurate measurements and put it on a timeline the size between these changes would start decreasing. So like, E [fifth labeled change] would be closer to D [fourth change] than B [second change] would be closer to A [first change].
Interviewer: So, as we're going from A to E are the changes increasing, decreasing, or staying the same?

Hayden: Okay. That's what you meant by that. I thought you meant something else.
Interviewer: Tell me what you meant by something else.
Hayden: I thought you meant like, was the distance between the two changes… like was it like a greater amount of change, or was it a less amount of change.
Interviewer: Right. Well, you're talking about comparing the changes of changes.
Hayden: Yeah.
Interviewer: But what about comparing the changes?
Hayden: The changes themselves, it's increasing each time.

When I gave Miranda the same prompt to estimate the changes, she found a negative increasing sequence but continued to say that the changes were decreasing. We spent 10 minutes talking about the meaning of the measures of changes but she was not able to independently say that the changes were increasing. In response to her increasing frustration I told her the sequence was increasing and explained why by placing the measures of changes on a number line and asking her to drag her finger from one change to another.

Interviewer: Why do you think that you saw negative three, negative one, and negative point five, and you said decreasing?
Miranda: The negative value.
Interviewer: What about it?
Miranda: That's why I said it's decreasing.
Interviewer: Oh, just because they were all negative?
Miranda: (Nods)

Hayden and Miranda's answers reveal their different abilities to reason about changes in changes which addresses Research Questions #3 and #4: 'How do students think about changes in changes, in tasks such as being asked to identify whether a function is increasing/decreasing at an increasing/decreasing rate?' and 'Does a focus on frames of reference affect students' ability to reason about changes in changes, by drawing explicit attention to a directionality of comparison?' Miranda repeatedly confounded comparisons of changes with changes themselves, (a parallel problem to confounding changes with quantities themselves) shown by her explanation that she thought negative changes meant that the changes were decreasing. Even when I directly instructed her to look only at the measures of the changes in changes she could not keep track of quantities, changes, and changes in changes in a way that allowed her to make sense of all the information she was given. In other words, since the only difference between the three aforementioned quantities are reference points and directionality of comparison, Miranda was not coordinating these ideas within a conceptualized frame of reference per our 2015 definition. Though Hayden struggled in the beginning by inadvertently looking at the changes in changes in changes, he recovered and then was able to reason about changes in changes by comparing the changes measures in ways that made sense to him. He considered the changes in changes by means of both measuring each with respect to its own reference point $f(\text{time}_{\text{initial}})$ for that x-interval, and also by committing to a directionality of comparison. His conceptualized reference frame is what allowed him to arrive at his final conclusion that the changes were both negative and increasing.

Conclusion

Through this pilot study, I found evidence that students need to commit to and think about a reference point in order to reason about changes versus total quantities (Research Question #1) and need to commit to and think about a directionality of comparison in order to reason about changes in changes (Research Question #3) as explained with the excerpts above.

I also found strong initial evidence that a student's ability to conceptualize a frame of reference – that is, commit to a unit, reference point, and directionality of comparison when thinking about measures - had a significant positive effect on their ability to reason about changes versus quantities (Research Question #2) as well as reason about changes in changes (Research Question #4). The evidence for this conclusion is two-fold. First, the students' abilities to reason through tasks about changes in the teaching interviews frequently reflected the initial positions of 'high' 'medium' and 'low' that I had placed them simply on their abilities to reason about a frame of reference in the clinical interviews. More significantly, the language that the students' used to explain their reasoning about tasks involving changes was often about a reference point or direction when the students – especially Hayden - were successful, and almost never about any of the three aspects of a frame of reference in our cognitive definition when the students gave up or were unsuccessful.

The results of this pilot study are being used to inform the creation of a larger study with more tasks, a longer time frame, and more students on how their capacity to conceptualize a frame of reference affects their ability to reason about changes. My preliminary results also suggest that a student's ability to reason about changes within a conceptualized frame of reference has implications for his ability to make sense of rates, first and second derivatives, concavity, addition of vectors, velocity, acceleration, and the principle of relativity, among other ideas. If so, this work can provide starting places in both math education and physics education for better instructional design.

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Connecting symbol sense and structure sense

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Novotná & Hoch (2008) propose that structure sense is an extension of symbol sense, which is an extension of number sense. Data collected initially to explore symbol sense were re-examined in terms of possible instances of structure sense. Preliminary thoughts on this re-examination are presented in this paper.

Key Words: symbol sense, algebraic notation, structure sense

One of the benefits of presenting early at a conference is the opportunity to have conversations with colleagues. In this case, I was able to have conversations with Stacy Musgrave, Duane Graysay, and others on possible connections between my notion of symbol sense and elements of structure sense in their work. These conversations prompted me to revisit my interview data with undergraduate mathematics students and mathematicians in search of possible evidence of structure sense. Task-based interviews were conducted with nine mathematicians and eleven undergraduate students enrolled in a proof-based mathematics course. Tasks for the interviews were selected to be accessible to a range of participants but to also provide enough complexity so that a participant's approach to notation becomes apparent and an explicit focus of the interview.

Characterizing Symbol Sense

In my short paper in these proceedings, I proposed a framework for characterizing algebraic symbol sense. Within work on a task, an individual may be *looking at, looking with,* or *looking through* the notation, with the assumption that for a fluent user of notation, these viewpoints are more interconnected, so that shifts between the viewpoints are fluid and reasoned. This supports my earlier definition of symbol sense as a coherent approach to algebraic notation that supports and extends mathematical reasoning (Kinzel, 2000). Brief examples from each viewpoint are presented here with the goal of connecting to elements of *structure sense.*

Looking at the notation involves noticing particular aspects of a notational form and considering potential actions. This can happen in both productive and non-productive ways. One mathematician participant (M8), when asked to solve the system of equations shown below, responded:

$$\begin{cases} xy = 100 \\ (x-5)(y+1) = 100 \end{cases}$$

> "What I see right off the bat is that I can subtract these. Left-hand side also has *xy* in it. Left with something in terms of *x+y*. Does that help me solve it though? If I subtract them, I know *x+y* is equal to something. Yeah. That will allow me to solve for *y* in terms of *x* plus a number and plug it back in and that will give me my answer."

This seems a productive instance of *looking at* the notation and considering the effectiveness of possible operations. In contrast, a student participant (S1) recognized the common term of 100, set the left sides equal to each other, and proceeded to solve for *y*, expecting to arrive at a numeric value. Surprised by the result of a linear equation in x and y, the student applied a familiar process and found the intercepts of this equation. In this case, *looking at* the (unexpected) linear equation prompted a known procedure that actually has limited relevance for the given task. Both instances seem to involve the participant considering structural aspects of the task and using their understanding of these aspects to guide choices.

Looking through the notation intends to capture those instances in which participants seemed to read through the symbols to the underlying relationships described by the symbolic forms. For example, a few participants (both students and mathematicians) recognized the system of equations shown above as representing products, and used this relationship to quickly find solutions to the system. One student participant (S7) found the solution (25,4) quickly without writing anything down. When asked if this were the only solution, he considered negative factors and proposed (-20,-5) as a second solution to the system. Another student participant (S10), surprised by finding two solutions to the system, was able to reconsider the original equations as products and reasoned:

> "Generally with a system of equations, you should be able to get the correct result using only one factor of x and y each. Considering that you have two multiplication problems, I suppose—instead of more addition and subtraction—I guess I can understand why it might be two factors. Because they can both be negative and then they can both be positive."

Although S10 did not seem to interpret the equations as products initially, his unexpected result prompted him to reconsider the equations and he was able to recognize them as products. Again, both participants seemed to draw on some aspect of the structure of the task within their work.

Looking with the notation is intended to capture instances in which an individual introduces or constructs notational forms to represent relationships within a context, specifically with the intent of understanding or manipulating the quantities involved. Once such forms are constructed, one's sense of algebraic structure could then be evidenced through articulations and choices made. It may also be that evidence of a sense of structure can be found within the construction process. Consider the Treasure Hunt task:

> A treasure is located at a point along a straight road with towns A, B, C, and D on it in that order. A map gives the following instructions for locating the treasure:
>
> a. Start at town A and go half of the way to C.
> b. Then go one-third of the way to D.
> c. Then go one-fourth of the way to B, and dig for the treasure.
>
> If AB = 6 miles, BC = 8 miles, and the treasure is buried midway between A and D, find the distance from C to D.

Constructing appropriate expressions for the distances traveled requires focusing on and coordinating various steps, and thus can be challenging even for fluent users of notation. Within their work on this task, several participants referred to expressions or terms within expressions as "this guy" or "this thing" and in some cases replaced a term or expression with a new variable. As a mathematician (M4) explained: "Sometimes introducing a simpler expression allows you to see things." She further stated that the simpler expression can allow you to see things from a different perspective, and perhaps indicate different potential operations. Such considerations may indicate strategies to isolate relevant structure within a representation.

Connections to Aspects of Structure Sense

Hoch (2003) suggested that structure sense includes the ability to recognize algebraic structure and to use appropriate features of that structure to determine useful operations. It seems this draws on aspects of both *looking at* and *looking through* the notation. M8's productive *looking at* the notation as described above could be interpreted as recognizing particular forms and using that recognition to guide his choices. Similarly, S7's recognition of the equations as products allowed him to apply a perhaps more efficient trial-and-error approach to this particular task. Some participants, when working on the Treasure Hunt task, wondered whether the location of the treasure is uniquely determined by the given information. Recognizing and drawing on the relationship between the number of variables and the number of constraints within a context may also be an aspect of structure sense

The tasks used in my interviews were chosen to elicit a range of interactions with algebraic notation and thus were not specifically aimed at exploring participants' ability to capitalize on algebraic structure. (See the list of tasks at the end of the paper.) All participants completed the Rational Equation, Treasure Hunt, Age Ratio, and System of Equations tasks; the Percent and Permutations tasks were used at the discretion of the interviewer. Of these, the Rational Equation and System of Equations tasks come closest to being aimed specifically at revealing an individual's use of algebraic structure.

Aspects of Structure Sense within the Rational Equation Task

Results. All participants were given the Rational Equation task as the first task in the interview: Solve for x: $\frac{x+16}{x^2-3x-12} = 0$. All nine of the mathematician participants and six of the eleven student participants (S1, S2, S5, S7, S10, and S11) focused primarily on the numerator and checked in some way that x=-16 would not also produce a value of 0 in the denominator. In fact, one mathematician participant (M5) commented that the identification of x=-16 was "so easy, it's got to be that -16 makes the denominator zero as well." This check was usually done by evaluating the denominator for x=-16, although M2, M4, and S1 initially attempted to factor the denominator. When this proved nontrivial, all three confirmed that x=-16 would not produce a zero value in the denominator and were satisfied with their solution. An additional student (S8) identified x=-16 as the solution, but with no explicit attention to possible implications for the denominator. It should be noted that one mathematician (M4) initially focused on the left side of the equation, thinking in terms of the shape of the graph, rather than noticing the value of zero on the right side of the equation. The remaining four students focused on the denominator, attempting to factor in order to cancel with the x+16 in the numerator. Most persisted with this approach even when it became apparent that (x+16) would not be a factor of the denominator.

Discussion. Hoch and Dreyfus (2004) include in a proposed definition of structure sense the ability to "recognize an algebraic expression or sentence as a previously met structure." All the participants discussed here seemed able to recognize the rational equation as a familiar structure, but they focused on different components of that structure. The nine mathematicians and six students seemed able to identify "expression = 0" as the most relevant structural aspect of the task and this allowed them to quickly propose x=-16 as a potential solution. They all went on to acknowledge that the left side being a rational expression required further verification of the solution. It is not clear whether the additional student (S8) recognized this aspect of the equation.

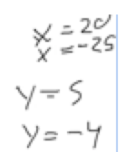
Further, M4 initially focused on the left side as a rational function and its related graph, only focusing on the "expression = 0" aspect later in her work. In contrast, the remaining four students seemed to attend more to the quadratic structure within the denominator, prompting them to apply known procedures for identifying factors of a quadratic expression. We can assume that all twenty participants had encountered rational expressions set equal to zero previously, and so this should exist for them as a "previously met structure." For the four student participants, however, the familiar quadratic structure of the denominator seemed to take precedence. This is perhaps an example of what Musgrave and colleagues (2015) refer to as responding to contextual cues rather than structural awareness; that is, a quadratic expression presented within a "solve" context may trigger factoring strategies and may limit or inhibit a participant's attention to the overall structure of the task.

Aspects of Structure Sense within the System of Equations Task

Results. The System of Equations task was presented above and some sample excerpts discussed. Nearly all of the participants successfully completed this task. One student (S1), as described above, found the intercepts for an intermediate linear equation and assumed these points would be solutions to the system. When this proved to not be the case, he was unable to make further progress on the task. Four additional students (S3, S6, S9, and S11) found appropriate values for one or both variables, but their articulations reveal potential problems in terms of algebraic structure. Although appropriate procedures were applied to the task, the interpretation of values as solutions to the system was problematic for these four participants.

When S6 encountered a quadratic equation within his solution process, he applied the quadratic formula to find two values for *x*; these values were written as (25, -20) and he stopped work on the task. When asked if he had solved the system, he replied "I think so. Is it right?" but did not check his solution in any way. A similar situation occurred in S11's work; he chose to solve for *y* and, finding that *y* could be -5 or 4, initially assumed that these two values constituted the solution to the system. He did eventually find -20 and 25 as values for *x*, and these were written under the appropriate values for *y*, but S11 does not clearly articulate that there are two ordered pair solutions to the system.

Both S6 and S11 initially assumed that the two possible values for one variable constituted a solution to the system. S11 did make further progress on the task, but did not clearly identify two ordered pairs as solutions. A similar result was seen in the work of S3; he also solved for *y* and used the first equation to find values for *x*: "*y* equals 5 or *y* equals -4. Easiest will be *xy*=100." These values are written but not explicitly connected as ordered pairs (see figure at right). Finally, S9 also found two values for *x* (-20 and 25) and two values for *y* (-5 and 4), but then created the four possible ordered pairs: (-20, -5), (-20, 4), (25, -5) and (25, 4) and checked each pair in the equations. In this way, he did identify the two ordered pairs that are solutions to the system but for him and the other students described here, there is clearly an issue with understanding the algebraic structure within this task.



Discussion. Participants who were successful on this task seemed to acknowledge the algebraic structure in some way, appropriately identifying the two ordered pair solutions and perhaps recognizing the equations as representations of products (as in the work of S7 and S10 described in the description of the framework). This recognition of structure seemed to allow S10 to make sense of his unexpected result of two solutions to the system. In contrast, the four less successful participants (S3, S6, S9, and S11) recognized the system as a familiar form and applied appropriate procedures but their interpretations of the nature of the solutions were limited and this limitation is likely related to not attending to the nonlinear nature of the given equations.

Aspects of Structure Sense within Contextual Tasks

The remaining tasks used in this particular set of interviews were presented as contextual situations and may or may not have introduced specific symbols. Returning to Hoch and Dreyfus' (2004) definition of structure sense as it applies to high school algebra, several abilities are included:

- see an algebraic expression or sentence as an entity,
- recognize an algebraic expression or sentence as a previously met structure,
- divide an entity into sub-structures,
- recognize mutual connections between structures,
- recognize which manipulations it is possible to perform, and
- recognize which manipulations it is useful to perform.

A full analysis of these interview data with respect to this list of abilities will not be attempted here. I will offer a few preliminary observations that seem to be related to the overall notion of structure sense.

Across interviews, when the task required the construction of a notational representation, the more fluent and/or successful participants often expressed the notion of being "done" once an appropriate representation was produced. Particularly on the Treasure Hunt task, once a (perceived) correct equation was constructed, participants felt the task had been completed and only carried out the algebraic manipulations for confirmation or as a reaction to the interview setting. That is, the "work" of algebra was accomplished in the construction of the equation. In the specific case of the Treasure Hunt task, this is an appropriate view of the structural aspect of the task: confidence in the appropriateness of the equation combined with confidence in one's manipulative skill should indicate successful completion of the task. Such a view was also seen in some participants' work on the Permutations task, particularly with respect to claim (a). For this task, a place-value based representation leads easily to noticing that the average of the permutations will be 37 times the sum of the three digits. Once this relationship was recognized, participants generally did not feel the need to produce a formal written proof for claim (a); the relevant structure had been identified so the "work" was done.

A second theme that emerged across interviews is the tendency to think in terms of a general case and how the notation is used in service of this. The Age Ratio, Permutations, and Percent tasks each are open to either a numeric or notational exploration. Both approaches were observed in the work of more than one participant on each task. The approaches did not align with mathematician or student status; that is, not all mathematicians took a notational approach and not all students took a numeric approach. Varying degrees of success were observed across participants. In some cases, a numeric approach was used in hopes of revealing the structure of the general case. In these cases, even when what was written involved specific values, the articulations indicated that these values were recognized as "stand-ins" for the general case. In contrast, other participants seemed to focus on the specific values themselves, and so were less likely to be able to recognize the values as stand-ins for generic quantities in the tasks. In both the Age Ratio and Percent tasks, the specific values of ages or salaries drop out when appropriate notational forms are constructed and manipulated, revealing the underlying relationship between the ratios or percentages. More fluent users of notation were able to see the notation as a tool capable of revealing underlying structure that is perhaps not evident in specific numeric instances.

Closing Thoughts

As I reexamined these interviews, I was struck by a few particular comments. These comments seem related to a sense of algebraic structure so I share them here as food for thought. Two students, S3 and S5, made comments that may indicate their awareness of structure but also potential limitations within that awareness. In contrast, comments from mathematician participants more frequently indicated not only an awareness of, but an intention to capitalize on aspects of structure.

As S3 began work on the Rational Equation task, he indicated a desire to cancel factors in the denominator with the numerator, and made this comment regarding the denominator: "I'm really hoping this isn't a quadratic." Further articulations indicate that he meant that he hoped that the (obviously quadratic) expression in the denominator would factor so that he would not need to use the quadratic formula. This may just be a careless use of "quadratic," but given that the goal was to cancel with the numerator, *not* seeing x-16 as a potential factor of the denominator should eliminate the quadratic formula as a useful manipulation in this case. Later in his work on the System of Equations task, S3 commented on his chosen process, "Hope I'm not backing myself into a corner." When asked how he might know, he responded, "For me, personally, probably wouldn't realize it until it was too late. I imagine people with more advanced algebraic skills would, like with chess, be thinking a few moves ahead." This comment seems to indicate that S3 may be aware that there is more structure inherent in tasks than he feels completely capable of recognizing and using to guide his work.

Related to the notion of where the "work" of algebra takes place, a comment from S5 on the System of Equations task is unsettling. She used substitution to approach the task and produced the quadratic equation $20 = y + y^2$. At this point, she paused and said, "I kinda already know what the answer is, but it's not, for me, solving." She was able to look at the equation and reason that y=4 satisfies the condition since "4 times 4 is 16 and plus 4 is 20." She did not see this as a legitimate "solving" process. I would argue that this is an indication of recognizing the represented relationship and a perfectly legitimate, perhaps even desirable, means of solving this particular equation.

Comments from mathematicians often referred to possible structural awareness, such as a general comment from M5 while working on the Treasure Hunt task: "Sometimes in mathematics, what you choose to let your variable be affects the complexity of the solution." A poignant comment from M2 reflects an *awareness of being aware*. He recounted his memory of his first "purely mathematical thought" at around age 12: "Rules for pushing symbols around were one thing, but the empty spaces being inhabited by virtual numbers was a different thing."

Other authors have indicated that it is difficult to capitalize on structures not "previously met"; ensuring that all students have the opportunity to meet all possible structures would seem a daunting task. It seems potentially more useful to consider how to develop the structural awareness indicated by the mathematician comments in the previous paragraph. The interview data examined here suggest that a possible approach would involve developing the ability *and tendency* to think in terms of a general case, coordinated with developing thoughtful fluency with manipulations of a variety of notational forms. Such an instructional approach should emphasize the role of notation as a tool capable of both representing and revealing algebraic structure.

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Tasks

Rational Equation Task

Solve for x:

$$\frac{x+16}{x^2-3x-12}=0$$

Treasure Hunt Task

A treasure is located at a point along a straight road with towns A, B, C, and D on it in that order. A map gives the following instructions for locating the treasure:

a. Start at town A and go half of the way to C.

b. Then go one-third of the way to D.

c. Then go one-fourth of the way to B, and dig for the treasure.

If AB = 6 miles, BC = 8 miles, and the treasure is buried midway between A and D, find the distance from C to D. (Charosh, 1965)

Age Ratio Task

The ratio of John's age to Mary's age is now r. If $1<r<2$, express in terms of r the ratio of John's age to Mary's age when John was as old as Mary is now. (Saul et al., 1986)

xy System Task

Solve the system for x and y:

$$xy = 100$$
$$(x - 5)(y + 1) = 100$$

Permutations Task

The average of the six permutations of 1, 2, 8, (that is, 128, 182, 218, 281, 812, and 821) is 2442 ÷ 6 or 407. It is observed that (a) the average is an integer; and (b) 1 + 2 + 8 = 4 + 0 + 7. Are these observations valid for any three digits? (Charosh, 1965)

Percent Task

A person's salary is reduced by *p* percent. By what percent would the salary then have to be raised to bring it back to the original amount? (Charosh, 1965)

Online Calculus Homework: The Student Experience

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Online homework is propagating rapidly across the nation, especially in large, introductory courses in STEM fields. The literature provides some evidence that the implementation of online homework is correlated with higher exam scores and course grades, but theory about how online homework supports learning is lacking, as is research on how students engage with and experience online homework. This study examines student experiences with online homework by identifying various homework environments, resource use, perceptions, and strategies that characterize diverse student experiences and learning opportunities. This portrayal demonstrates that aggregating student experiences through statistical analyses is sometimes an oversimplification that limits the inferences that can be made from achievement data and is evidence that qualitative data is vital to understand how online homework supports learning and inform improvements to its implementation.

Key words: online homework, calculus homework, screen recording, interviews, observation

Addressing the problem of persistence of students working toward a degree in the science, technology, engineering, and mathematics (STEM) fields is an enduring challenge of educators across the Unites States. It is estimated that between 40 and 60 percent of students who enter postsecondary education with the intention of pursuing a degree in a STEM field will switch their study to a non-STEM field (Bressoud, Mesa, Hsu, & Rasmussen, 2014). Poor instruction in mathematics and science courses, especially calculus, is cited as a reason that students decide to switch out of STEM fields (Seymour, 2006; Sonnert & Sadler, 2015). Even academically successful students sometimes leave STEM fields because of negative reactions to the pedagogy they experience in mathematics and science courses, meaning that factors other than student success should be considered to address the issue of STEM retention.

As the use of online calculus homework propagates across the country (for example, over 700 institutions use WeBWorK nationwide), online homework is taking on a growing role in students' experiences with calculus. My experiences as a student, tutor, and instructor led me to believe that online homework is large component of students' experience with calculus, and students have expressed both positive and negative perceptions about their online homework.

The goal of this study is to portray student experiences with online calculus homework to better understand how online homework facilitates learning and to inform instructional decisions about online homework. A deeper understanding about the strengths and weaknesses of online homework, especially about the ways in which online homework facilitates learning opportunities, can inform how online homework is implemented to maximize its effectiveness. For various reasons described later, online homework appears to be positioned as a permanent component of college calculus instruction, so improving its implementation is one way to improve students' experiences with calculus.

Why has Online Homework Propagated so Rapidly?

Two factors have contributed to the rapid adoption of online homework systems across the nation. First, there is evidence in the research literature (summarized in the next section) that students in calculus classes that use online homework perform slightly better, in terms of exam scores and course grades, than those without online homework. Second, several features of online homework provide instructional alternatives that indirectly contribute to improved student support. Online homework can free up department resources, allow for assigning students individualized homework sets, and can provide immediate feedback while automatically grading entire assignments. Although the evidence suggests that online homework may slightly improve student success in calculus courses, it is important to recognize that the following opportunities offered by online homework may have contributed to its propagation independent of a critical analysis about how online homework supports learning.

Freeing up department resources. In many university courses, especially large courses, assigning paper-and-pencil homework entails a large commitment to grading. Often, graders are hired to assist with grading, which incurs a significant financial burden on the departments. Online homework systems can automatically grade assignments, alleviating the financial burden of grading homework. Those resources can be redirected to other programs designed to support student learning, such as tutoring centers (Bonham, Beichner, & Deardorff, 2001; Carpenter & Camp, 2008; Richards-Babb, Drelick, Henry, & Robertson-Honecher, 2011; Zerr, 2007).

Immediate feedback and grading. Online homework systems offer instantaneous grading and feedback. This feature enables instructors to grant multiple solution attempts to students, which allows students to work through their mistakes without having to turn in an assignment and wait for grading and feedback (Bonham et al., 2001; Carpenter & Camp, 2008; Kortemeyer, 2014; Zerr, 2007). Furthermore, immediate feedback has been shown to improve learning opportunities (Epstein, Epstein, & Brosvic, 2001). Immediate grading also enables continual formative assessment, both because automatic grade reports can inform instructors about the progress of their students and by providing a means for students to self-assess (Demirci, 2007).

Individualized homework sets. Online homework systems can generate problems algorithmically, so students can be assigned unique problems. Problems can be authored using several random parameters, so that each student's homework sets have problems that are identical in structure but that have unique numbers. Many instructors see this capability as a means to deter cheating (Carpenter & Camp, 2008), but this capability can also be employed to allow students to generate an unlimited number of practice problems.

Does Online Homework Help Students Learn?

To determine the effectiveness of online homework in supporting student learning, assumptions must be made about the types of evidence that inform such an evaluation. The following vignette illustrates that one type of evidence that is valued is achievement data, gathered through an experimental comparison between control and experimental groups. This perspective undergirds much of the research about online homework, but the neglect of the student perspective has left a gap in the literature. This study portrays student experiences with online homework and demonstrates the importance of the student perspective in understanding how online homework supports learning.

Assumptions about Measuring Learning

During a course meeting while data for this study was being collected, a faculty member respectfully asked, "Do we even know if online homework helps students learn?" Although I attempted to provide some insights during the meeting, it is clear, in retrospect, that I should have been mindful of the differences in the assumptions about what evidence to answer that question might look like.

My colleague politely inquired for evidence from an experimental study, revealing his conception that the measurement of learning can be accomplished through comparison between a treatment group and a control group. That kind of evidence does exist; Hirsch and Weibel (2003), for example, found that students performed slightly, but significantly, better on a final exam in a general calculus course when enrolled in a section for which online homework was used, scoring 4% higher on average. In an introductory calculus-based physics class, Bonham, Beichner, and Deardoff (2001) found that students scored slightly better on tests in sections that employed online homework (78%) in comparison to sections using paper-and-pencil homework (75%). A similar effect has been reported in other introductory STEM courses (Cheng, Thacker, Cardenas, & Crouch, 2004; Cole & Todd, 2003; Richards-Babb et al., 2011).

This evidence, while providing some insights about the value of online homework, is incomplete. The gap in these results is clear; measuring the systematic improvement to the structure of a course is more complex than finding a 4% improvement on final exams. The results of that variety of research are unquestionably important, but student learning is more than complex than student outcomes on exams. We must include the student perspective to understand how learning opportunities are facilitated by online homework. Student success data is information about the results of students' experiences with online homework, but data from the student perspective is required to understand how students' experiences support learning.

A New Perspective

Each of the previously described studies have a commonality: they are all inquiries from the instructional perspective. The student perspective is missing from these results, and thus only half of the story has been described. Hirsch and Weibel (2003) found a substantial difference (of two letter-grades) between students who attempted every online homework problem and those "who did not attempt many" (p. 14). These findings describe, with more nuance, qualities of the student experience that we could try to facilitate to maximize the effectiveness of online homework as a learning tool. The evidence collected from the instructional perspective provides a limited portrayal of the mechanisms by which online homework might support learning, which is why we must look to the student perspective for more insight.

By examining the ways that students experience online homework, I was able to capture students' perceptions about how online homework supports their learning, the study behaviors facilitated by online homework, and the ways in which students use resources to complete their assignments. From this perspective, this study describes how online homework facilitates student behaviors and perspectives that lead to learning opportunities, and identifies homework environments and study strategies that shape students experiences with online homework.

Research Method

My research combined a quantitative survey with observation to determine general trends in tandem with providing a detailed portrayal of individual student experiences with online homework. My research is guided by the four following questions:

1. What homework environments do students experience during online homework?
2. What resources do students employ while completing online calculus homework?
3. What perceptions do students have about online homework?

4. What strategies do students employ to complete online homework?

Sample

The survey was administered to all students in the mainstream calculus course at a large public university via email and was completed with a 24% response rate.

Participants for the observation study were solicited via email from two sections taught by two different graduate teaching assistants. Both of the graduate teaching assistants had at least one full year of teaching experience and had previously participated in a teaching mentoring program, which is a typical of the graduate students assigned to teach calculus at the institution. Four students were selected to participate from a pool of 9 volunteers, with attention given to selecting participants with varying backgrounds in terms of experience with AP Calculus, with other calculus classes at the institution, and as repeat students for the same calculus course.

The Survey

The survey gathered data on students' demographic information, mathematical backgrounds, perceptions of online homework, study habits, and resource use. I analyzed the survey data by examining graphical representations and basic descriptive statistics to identify general trends.

Qualitative Study

I gathered two main forms of qualitative data: (a) video recordings of student homework sessions and (b) transcribed, audio-recorded interviews. As a secondary data source, I drew on notes from informal conversations with the participants, most of which occurred while meeting with students briefly to collect the video recording files.

Video recordings. The video recordings of homework sessions include two data streams. Screen-recording software (Screencast-O-Matic) was installed on the participants' computers to capture the details of their computer work. The screen recordings provide details about students' exact input into the online homework system, and also captured students online activity outside of the online homework system, such as browsing the internet for support and using online calculators such as WolframAlpha. A webcam was used to simultaneously to capture students' real-world activity, but did not capture the same level of detail that was captured by the screen recordings. In the webcam recordings, I can identify when students are working with paper-and-pencil during their online homework session, but it is not possible to determine exactly what students are writing.

Interviews. Each of the four students participated in an interview administered using a semi-structured protocol, with the goal of identifying individual study habits and perceptions about online homework. I also informally discussed the students' experiences with their online homework several times while I was transferring the videos from their computers.

Results

To analyze my data, I conceptualized students' experiences with online homework as being characterized across the following dimensions: (a) the homework environments that students create, (b) the resources that students utilize, (c) the perceptions that students have about online homework, and (d) the strategies that students employ to complete homework. The homework environments that I describe illustrate variation in the amount of time that students spend on online homework, the setting in which homework is completed and the people present in that setting, and the time when homework is completed. Then, I describe the resources that students use, along with variation in the ways that those resources are used. Next, I describe student perceptions about online homework, including the perceived usefulness of online homework as a learning tool. Finally, I will describe several strategies that students employ related to how and why they complete homework.

Homework Environments

The context that students create to complete their online homework determines learning opportunities afforded by that experience. I found variation in the homework environments across three dimensions: (a) the length of time that students study, (b) the physical setting, including the location and people, and (c) when students choose to study relative to the course schedule.

How much time do students spend on online homework? The survey, interviews, and observations all suggest that students spend about 5 hours per week working on online homework. Of course, the amount of time required to complete an online homework assignment varies from assignment to assignment and student to student, but it appears that students spend more time working on their online homework than on studying calculus through other means.

Survey data. Figure 1 is a histogram that displays the time that students reported spending on online homework, per week, as well as the amount of time that they spend studying calculus otherwise. This data was gathered through a free response question and responses were binned by 1-hour intervals.

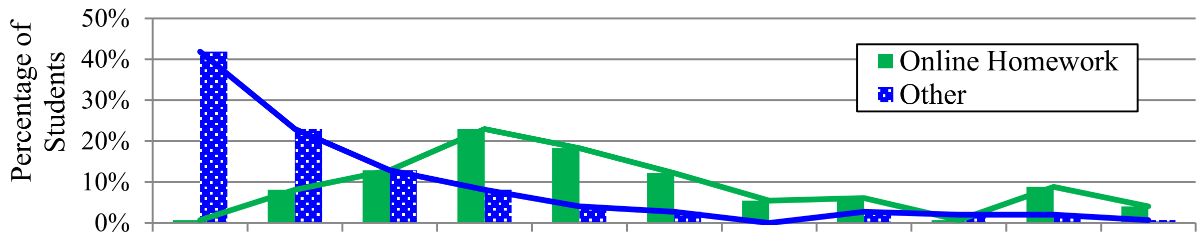

Figure 1: Amount of Time Studying Calculus (hours/week)

Interview data. Aesha and Brittany provided some insights into the amount of time that they spend working on online homework. Aesha reported that she "spends three hours on it to finish it, three for two assignments", so approximately 4.5 hours per week for the three assignments each week. Aesha also indicated that her study routine only consisted of working on online homework and doing practice exams, without studying otherwise, but Aesha was a high-achieving student who had taken a college calculus course previously. Brittany suggested that she worked on online homework for long hours, but was not explicit about the number of hours. Brittany did complain, however, that "Even though the [online homework] takes 90% of your time, it's worth like a fraction of what your grade is," indicating that a large portion of her study time is spent working on homework, which matches the trend in the survey data.

Observation data. The video recordings provided an exact record of the amount of time that students spent on each problem that that worked on during a recorded homework session. For the four observed students, the average time to solve one problem was 6.2 minutes (SD = 3.4) and the average time needed to complete one homework set was 35.0 minutes (SD = 15.5). All of the students completed multiple homework sets during one homework session on some occasions, and the average homework session time was 105.0 minutes (SD = 33.7). Being that three homework sets were assigned per week, the observed students spent about 105 minutes on online homework per week.

Where do students do online homework, and with whom? The setting in which students do online homework varies along two dimensions: (a) the physical space in which the homework is completed and (b) the people present in that space. I have grouped these components of students' homework context together because the place that students choose to study in determines, in part, the types of people that will be present.

Survey data. Students reported spending more time, per week, than in a collaborative setting, as shown in Figure 2. The tutoring center appears to facilitate collaboration.

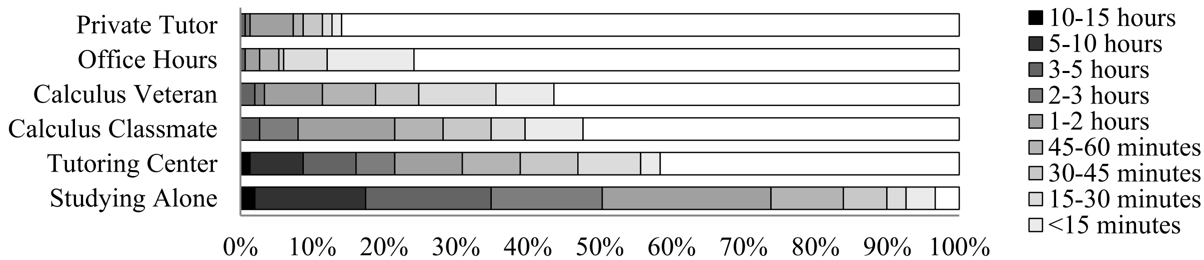

Figure 2: Homework Environments (time per week)

Interview data. I found variability in the physical homework setting described across the interviewed students. Aesha reported that studying alone in her dorm room to complete her online homework was the primary studying she did outside of class (A:65-78)[1]. Brittany started her online homework assignments alone in her dorm room, and then finished them in the tutoring center. While Brittany studied in the tutoring center, she usually worked alone, but was happy to work with other students, even though she not seek out study partners (B:433-443). Chris usually studied with his roommate, who was enrolled in a different section of the same calculus course; they worked together on entire problem sets and compared work to progress through the assignment (C:286-300). Dan studied with his friend occasionally, but studied alone is his dorm more often than he had in a previous semester, when he was taking the same course (D:127-140).

Observation data. All of the students that were observed recorded homework sessions in their dorms; Brittany was the only one who recorded another setting when she recorded one homework session in the tutoring center. The variety of students' study settings is not well-captured by my research, due to limitations of the technology. While the screen recording software runs seamlessly in the background of the computer, the recording of the student requires an external webcam that is awkward to set up in a public setting.

When do students do online homework? The day of week (and time of night) that students complete online homework is influenced by the due dates placed on that homework. Homework deadlines tend to determine when students start assignments, and coinciding due dates for multiple assignments tend to determine how assignments are grouped into homework sessions. Note that during this study, homework assignments opened 3-7 days before the content was scheduled in class (it was a uniform course) and was due the following Thursday. Quizzes over the material were given in class on the Friday *before* the homework was due.

Survey data. Although the survey response options shown in Figure 3 were both vague and overlapping, a trend is still apparent: large numbers of students complete the majority of their online homework assignments close to the due date.

[1] References to interview transcripts are formatted as ([Appendix]:[Line Numbers]). Appendices are available electronically: A: https://goo.gl/bajW1x; B: https://goo.gl/Uq88SN; C: https://goo.gl/CPKREb; D: https://goo.gl/jWO8kz

"**When do you typically do the majority of online assignments?**"

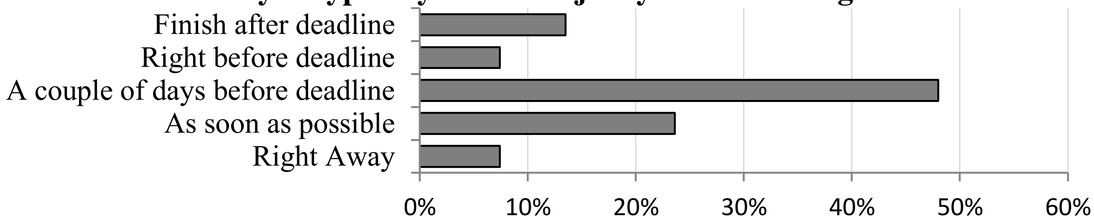

Figure 3: Student-reported Homework Timing

Interview data. The students who were interviewed all described ways in which their studying was structured, at least partially, around online homework. Aesha explained, "I'm like a lazy person...If I don't have the homework which is due like in 1 or 2 days, I just let it go and not study until the exam is coming." Similarly, Chris talked about online homework helping him to structure his studying, "It is like an annoying mother that actually ends up helping," but he also explained that he attributed his success in precalculus, in part, to his habit of working on his online homework as soon as possible following the lesson on that material (C:237-268). Chris explained that he had recently been waiting until just before the due date to complete his online homework, which was set for the week following the quiz, and had felt unprepared for quizzes as a result (C162-180). Dan also explained that the misalignment of the homework due dates and the quizzes had negatively impacted his quiz preparation (D39-63).

During this study, all three of the homework assignments from any given week were due on Thursday the following week, at the same time. As a result, Aesha, Brittany, and Chris all explained that they completed multiple online homework assignments during the same homework session (A:79-89; B:99-106; C:241-252). The interviews clarify the survey data displayed in Figure 3, which shows that many students complete online homework close to the deadline, meaning that they are likely grouping the content in ways that are similar to those described by Aesha, Brittany, and Chris.

Observation results. Video recordings confirmed that students often work on multiple online homework assignments as a group during a single homework session. The recordings captured the students working on multiple assignments with the same due date during the same session.

Resources

Students use a variety of resources when working on online homework. Popular resources include class notes, viewing a similar example within the online homework system (which will be referred to as See Similar Example hereafter), online calculators, YouTube videos, and informational websites. Traditional resources such as the textbook, on campus tutoring centers, office hours, study groups, and private tutors were used less frequently. See Similar Example is a feature of the online homework system in use for the calculus course that was studying; there is a button on the interface, which is available on each problem, that is a link to a solution of a problem that is similar to the assigned problem, in PDF format. An instructional website is a website, such as *Paul's Notes*, that provides explanations of math content and reads much like a textbook. Online calculators include WolframAlpha and Symbolab, which have advanced capability to solve essentially any calculus problem, both symbolically and questions posed in common language, such as "What is the derivative of x^2?" Online help forums allow students to post questions that other users can answer (i.e. physicsforums.com, Yahoo Answers, and mathforum.org), and preserves the communication so that students can find answers to questions by Google searching for homework questions using the text of the problem verbatim; often full solutions and even exact answers can be located using this strategy.

Survey data. Figure 4 shows student response data about the frequency of resource use during online homework. Students reported using the "See Similar Example" feature the most, followed by class notes and online calculators. Traditional resources, such as the textbook and office hours, were used sparingly.

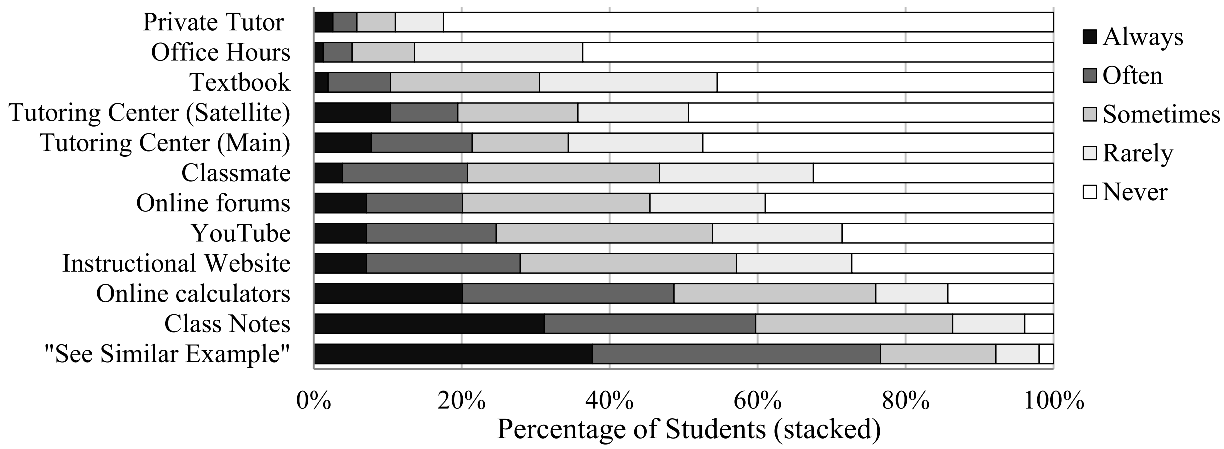

Figure 4: Resource Use for Online Homework

Interview data. Interview data pertaining to the frequency that various resources are used is described below, but descriptions of the strategies related to different resources are presented in the following section of the paper. Interviews revealed variability in the types of resources students use when completing online homework, but the use of online symbolic calculators (i.e. WolframAlpha) was mentioned by all four interviewed students. Aesha shared that she never uses the textbook, but often used WolframAlpha, the "See Similar Example" feature, and her class notes (A:96-117). Similarly, Brittany questioned her decision to buy the textbook because she primarily used online resources including online calculators and informational websites as resources; Brittany also indicated that she worked in the tutoring center for 30-40% of her online homework (B:225-248;427-430). Chris estimated that his use of online calculators accounted for half of his total resource use, with the other half split between looking at similar examples and going to the tutoring center (C:308-335). Dan also indicated that he used online calculators and the "See Similar Example" feature, while also using the textbook (D:134-142).

Observation data. Observations of the students revealed complexities in the ways that students use online resources, particularly with four electronic resources: (a) the "See Similar Example" feature, (b) online calculators, and (c) instructional websites and online forums.

"See Similar Example". Observation data aligned with the survey and interview data in that I observed students using this feature on many problems; this is often the first source that students turn to when they are stuck. Sometimes students look at the similar example immediately after they read the problem, other times they look at the example after they have attempted the problem and have submitted an incorrect answer, but students almost always turn to this resource first when they are not able to complete the problem on their own. It is difficult to infer exactly how students are interacting with the PDF, but informal conversations with the students match the observed behavior in that it appears as though the primary goal is to determine similarities in the structure of the problem to try to figure out how the numbers in the example can be manipulated to yield the answer to the problem at hand. In this sense, the similar example serves more as a recipe to complete the necessary calculation, rather than a support for students to make sense of the problem that they are trying to complete.

Online calculators. Observations of the students revealed two distinct uses of online calculators, which I will call troubleshooting and circumventing. First, when students are troubleshooting using online calculators, they have already worked through the problem on their own, submitted an incorrect answer, and are trying to find their error. Students talked about this behavior informally, explaining that they used online calculators to double check complex arithmetic and algebra because it is difficult to maintain exact precision throughout the calculations that involve unruly numerical values. Second, students sometimes use online calculators to circumvent a problem, meaning that they use the technology to calculate the answer without first attempting the problem on their own. Brittany and Chris spoke informally about sometimes completing their online homework by circumventing problems using online calculators, but both recognized that they tried to avoid the strategy because they thought it was hurting their learning.

Instructional websites and online forums. In the observations, students accessed instructional websites (e.g. Paul's Notes) and online forums (e.g. YahooAnswers) exclusively through the use of a search engine, such as Google, which is why I have grouped these two resources together. Consider the two following searches queried by the students during recorded homework sessions: (a) Brittany: "How do you find a second derivative?" and (b) Aesha: "Use interval notation to indicate where on the interval [-2, 4] the function $f(x)$ is differentiable." The results of Brittany's search were links to several instructional websites that explained the process of finding second derivatives, one of which Brittany read before going back to her notebook to complete her problem. The results of Aesha's search were links to several online help forums, one of which Aesha visited and found the full solution to the exact problem that she was working on. Aesha read through the explanation and several comments, and then copied the posted answer and submitted it for credit. The slightly differing strategies for internet searching led to quite different kinds of resource use and learning opportunties.

Student Perceptions of Online Homework

The survey and interview data were used to determine student perceptions about online homework. Many students reported that the online homework was a useful tool for learning calculus, but several aspects were identified that could improve students' experiences.

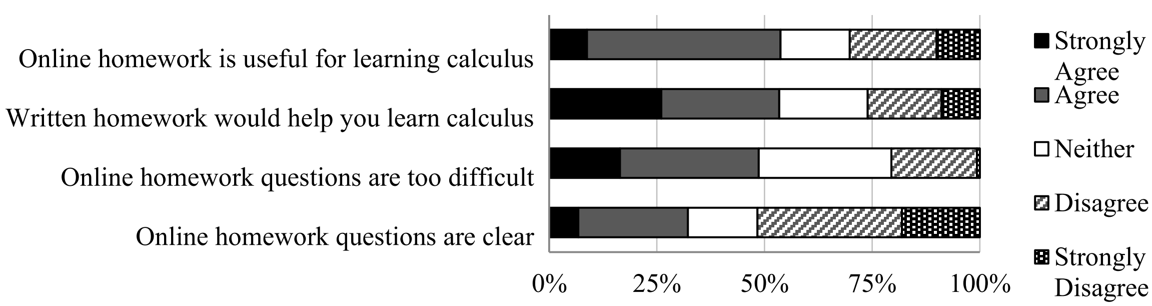

Figure 5: Student Perceptions of Online Homework

Survey results. Figure 5 shows student responses to Likert-type survey questions about perceptions of online homework. Students generally perceive online homework as a useful tool for learning calculus, but also expressed a need for written homework as a compliment.

Interview results. Each of the four students who were interviewed stated that online homework helps them learn and provided details about the specific aspects that were helpful, including: (a) using online homework to structure and schedule studying, (b) receiving immediate feedback, and (c) having an opportunity to practice problems. Several issues were identified including: (a) being overburdened by the size of assignments, (b) misalignment of due dates, (c) the numerical complexity of problems, and (d) the need for written homework to supplement online homework.

Structuring studying. Aesha described how online homework helps her to keep up with her homework and hypothesized that she would do less homework without online homework assignments (A:20-33). Chris agreed that a more structured schedule, with more frequent due dates, would help his learning, joking that "it is like an annoying mother that actually ends up helping." Dan explained that the online homework was easy to study from (D:29-30).

Immediate feedback. Both Brittany and Chris found the capability of online homework to provide immediate feedback useful for their learning (B:291-296; C:127-136). Brittany noted this capability as advantage over written homework, explaining that "you don't really get feedback about if it's right or wrong until like after you turn it in and you're sort of like detached from it 'cause it might be like a week before you get it back." Chris described, "I like the online part because then you can know what the answer is after a while like eventually, but if you are doing homework and you've got the answer wrong you might keep going in your homework and continuously do that wrong, like, yeah, this is how to do it and there is no way to check unless you go to the learning centers...[online homework] gives you that instant…What's right and what's wrong."

Practice. Brittany and Chris both explained that the practice provided by doing online homework problems was helpful (B:30-33). Chris suggested that adding a feature to regenerates new problems would be helpful (C:217-230), "I would make the problems able to be solved infinite amount of times. Like in [chemistry]... And I would like if the problems would change when you try to do them." This feature has been added since the time of this study and both students and tutors in the tutoring center have found the feature useful, anecdotally.

Overburdening . Brittany and Dan both explained that the learning opportunity presented by online homework is diminished when they feel overburdened by the size of the assignments (B:6-37; D:164-168). Brittany explained, "it's not helpful when there's like you know, 15 questions and trying to finish them in an hour and you're just putting it all into WolframAlpha and getting the answers...I end up doing a lot of that because it's like, crunch time, got to get this done." Note that Brittany talked about online homework in a positive light overall.

Due date misalignment. Brittany, Chris, and Dan each explained that their study habits, in terms of when they do their homework, are at least partially determined by the due dates on the homework assignments. It was problematic for students that the assigned due dates were after the weekly quizzes. Chris explained that he completes his online homework after he takes the corresponding quiz, "Most of the time, I don't really study that much and I'll take the quiz and then do my [online] homework... most of the time I feel kind of unprepared [for the quiz]." Dan echoed Chris's sentiment, "This semester I have noticed it was like a week off, a couple of days off...what I've noticed now is that with the things are off, I'm not doing too hot on the quizzes."

Numerical complexity. Aesha, Chris, and Dan each talked about the numerical complexity of some of the online homework problems as being a barrier to their learning (A:43-56; C:19-32; D:19-33). Each student is assigned the same problem set, but individual problems are generated using random parameters, leading to answers that are not easily simplified. Students are not allowed to use calculators on quizzes and exams, so complicated answers are intimidating to students and can be difficult to input into the system with all of the parentheses.

Written homework. Aesha thought one advantage of written homework is that it allows the teacher to provide feedback about the entire solution process, but she admitted that she would not do written homework that was not graded (A:169-187). Brittany found the optional written homework provided by her instructor useful (B:217-221), and estimated that her work on those written assignments accounted for approximately 70% of her learning (B:314-344), but she thought that most students would not do homework that was not graded (B:291-299).

Strategies

Each of the four students interacted with their online homework in unique ways, so I will provide a short description of each students overall strategy. I will provide some references to the formal interviews, but much of the description is informed by informal conversations and hours of video data that are more difficult to reference specifically.

Aesha. Aesha generally attempted each problem on her own before turning to resources; she her class notes, WolframAlpha, the See Similar Example feature, and occasionally Googled the question when she ran into problems. According to Aesha, "see similar example in online homework is more useful than the notebook. Sometimes the notebook doesn't cover the knowledge they want us to do." If she could not solve a problem using those resources, which was rare, she would, "…sometimes Google it…really, really seldom to the Google." She would find an exact solution on an answer forum, would read the accompanying explanation, and would submit the answer provided in the forum.

Aesha recognized that working on problems was integral to her learning process and appreciated that online homework provided a structure for her to do that. She identified one limitation of online homework, "usually on the [online homework] you only show the answer of the result, but on the written paper you need to show your processing… and if there any wrong in your processing…maybe the instructor can like talk about these question." Aesha's recognition of this shortcoming demonstrates the value that she places on homework extends beyond its value in the grade book; she truly values homework as a learning opportunity. She often spoke about the importance of completing her online homework to her understanding; on several occasions Aesha dismissed her temporary struggles with confidence that she would understand the content after she completed her online homework.

Even though Aesha claimed, "so I'm like a lazy person. If I don't have homework, I usually don't do much effort on math," she completed nearly all of her online homework independently (A: 65-68). Her learning routine was consistent throughout her experience: (1) attend class and take notes, (2) complete the online homework individually using notes, textbook, and online resources, (3) succeed on weekly quizzes, (4) study for the exam, and (5) succeed on the exam.

Brittany. Brittany's experience with calculus was more like a mission to complete her online homework, which aligns with her perception of the role on online homework in her learning. Brittany appreciated that she was able to persevere to complete those assignments, but her frustration that the grade weighting was so low (10%) demonstrates that she valued online homework more for the grades rather than the learning experience. She explained, "…even though the [online homework] takes like…. 90% of your time, it's worth like a fraction of what your grade is which I think is kind of dumb…I've really tried to work really hard on the [online homework] just because… rather than like an exam if you don't know the answer you just get it wrong, but at least on the homework you can get help to get the answers right." Brittany explained that, "in some ways [online homework] can be helpful…because it is practice…we do some example problems in class and the [online homework] works…to practice applying those skills like if we were in class," but she never specifically identifies how online homework supports her learning or talks about online homework as a way to master the material.

Brittany often procrastinated her online homework, which led to study habits that she recognized as being less productive than was possible:

> It's not helpful when there's like you know, 15 questions and trying to finish them in an hour…and you're just putting it all into WolframAlpha and getting the answers…I end up doing a lot of that because it's like, crunch time got to get this done… and I've no idea what I just did but, it's green so….that's when I think it's not helpful…it's when it gets to the point where you're just getting it done, to get done and not really focusing on how to do it.

Brittany recognized that doing her homework herself was valuable, and explained that she would always attempt problems before she turned to other resources, namely the MLC and online calculators (B:30-47; 409-430). Brittany explained, "I try not to just get the answers on the computer and then put it right into my work and not really even process what that means. 'Cause then, that's not gonna help me on a quiz or an exam when I don't have the computer to use," which further demonstrates the value that she placed on online homework as a means to practice problems similar to those on the quizzes and exams. When she talked about online homework as a way to practice, language about the importance of doing those problems as a part of learning the materials was always missing, which indicates that she sees her online homework more as a component of her grade than as a learning tool that would help her master the content. In other words, Brittany was concerned about completing the online homework problems rather than learning *how* to complete the online homework problems.

Chris. Chris valued online homework based on his previous experience and success in a precalculus class that utilized online homework, but his study habits did not align with those values, and he recognized the discrepancy. He often worked on his precalculus online homework immediately following the classes in which the content was covered, but abandoned that strategy for calculus and instead completed his calculus homework based on the due date, which was after his quizzes. He recognized that his study strategy was less beneficial than more regular studying, and suggested that pushing students to work more regularly through more structured due dates would be, "like an annoying mother that actually ends up helping." Chris clearly recognized that online homework was most helpful when he worked on it more regularly, but there were a number of factors that contributed to his habit to utilize the online homework in a less effective way.

Like Brittany, Chris valued online homework as a way to practice problems, and suggested, "I would make the problems able to be solved infinite amount of times." Chris compared his experience with online calculus homework to his experience with online chemistry homework and explained that the online homework helps him learn by providing him with example problems, and that having an unlimited number of problems helped him use online homework more effectively in his chemistry class (unlimited problems, solutions provided) than in his calculus class (specific homework set, no solutions provided). This is indicative of the value that Chris ascribed to online homework as being a way to practice problems and learn the material by looking through solutions to try to find patterns, which implies that he values the online homework as a way to perfect his solution methods rather than a way to improve his conceptual understanding of the content.

Dan. Dan valued online homework primarily as an organizational tool. He explained, "It's like easy to study from, so you know what to study from. You don't have to like trace through notes and all that." The misalignment of the homework with the rest of the course was his main critique of his experience with online homework, "this semester it doesn't usually match with what we are doing in class and when I do it, it is just like wait, we're not doing this, it is distracting and sometimes I didn't get it done."

Dan recognized that working on problems was important for his learning, but he was critical of the types of problems presented within the online homework, saying, "some of the web work problems are a lot like more challenging than you ever get… sometimes it is just too broad. It always asks about the trig function ones, I rarely see those [on quizzes or exams]." He compared online homework problems to some optional problems assigned by his instructor, "he has a lot of conceptual ones which I think that helps a lot where a lot more whereas these are, just like, you got to learn how to plug it in and all that," which demonstrates that, after having worked through the online homework during his previous semester with calculus, he is aware of the different kinds of learning that the different types of homework problems support.

Discussion and Implications

The range of student experiences portrayed demonstrates the complexity of student interactions with online homework. Given the same homework assignment in the same class, one student might productively struggle through a 15-problem homework set prior to a quiz while another student may not even start that same assignment until after the quiz. If the average achievement (e.g., course grade) of these two students is employed as a way to understand the effect of online homework on student learning, the nuances of the students' experiences are lost and that data might tell a misleading story. To determine the effect of online homework on student learning, we must examine the mechanisms by which online homework supports learning, and we can only do that by understanding how students interact with their online homework to create learning opportunities. Once we have a better understanding of various student experiences, instructors can identify learning associated with specific experiences and can leverage teaching practices to facilitate the interactions that are most productive.

Homework Environments

The context in which online homework is completed shapes the learning opportunities afforded by that task; working on an online homework set alone in a dorm room is different than working on that same homework during office hours with a group of peers. Instructional design should draw on the strengths of online homework and should account for the shortcomings. I found that students tend to work on online homework alone and tend to complete assignments based on the deadline, regardless of how it aligns with the course schedule. Addressing those issues is one way that we might seek to improve how online homework supports learning.

One way that instructors might counteract the tendency of students to complete online homework alone is to supplement online assignments with collaborative written assignments. Students perceived optional, written homework as useful supplements to online homework both because written homework provides a space for them to express their thinking process as well as a common experience upon which to base classroom discussion and demonstration. Although interviews revealed that motivation may be an issue when the written homework is not graded, the value that students ascribed to written homework that is conceptually-based suggests that finding creative ways to assign and incentivize written homework is worthwhile. Shifting some student tasks away from online homework can also help to shift students' focus away from online homework as the only medium for studying, which is currently the case for many students as over 40% if students report spending less than an hour per week studying calculus outside of their online homework (Figure 1).

The majority of surveyed students (68.9%) reported that they complete their online homework either the day of or a couple of days before the due date. Chris and Dan both explained that they were unprepared for calculus quizzes because they completed homework assignments on the due dates that were set after the quizzes. It is no surprise that online homework that is completed following an assessment is perceived as less useful than homework completed before, so instructors should consider structuring online homework due dates in ways that facilitate the study behavior that is desired, even if the structure seems too prescriptive.

Perceptions, Strategies, and Resource Use

Student perceptions about the usefulness and purpose of online homework influenced the strategies and resources that students employed to complete their assignments, and thus affected the learning opportunities experienced. Some students viewed the completion of the online homework assignments as being important, in its own right, because of the graded component of the homework. This contrasts sharply with the perceptions of students who valued online homework as learning experience to practice the procedures and master the material. The learning opportunities shaped by the homework environments, strategies, and resource use associated with these contrasting perspectives differ substantially, and illustrate the importance of understanding online homework from the student perspective.

Brittany, Chris, and Dan all espoused perceptions about their online homework that suggested that they leaned towards valuing online homework because it was a graded component of the course. They employed strategies and resources to expedite their studying, often at the expense of deteriorating their learning opportunities. These students often turned to online forums, online calculators, the “See Similar Example” feature, and sometimes the tutoring center for assistance in performing the required procedure to calculate the correct answer to specific problems. In doing so, they essentially circumvented the experience of solving those problems on their own, diminishing their learning opportunities.

Although Brittany, Chris, and Dan mentioned the value of online homework as a learning experience, Aesha clearly explained that she valued her online homework as an important component of her learning and demonstrated a significant commitment to strategies and resources that supported her learning. When she struggled with a problem, she turned to the textbook, class notes, and instructional websites to try to understand the problem, and to make sense of the solution method as a whole, rather than seeking the solution to that specific problem. Aesha often turned to an online calculator to find an error in a solution that had already been worked out, as opposed to the strategy of employing an online calculator to simply calculate the answer to a problem, but she would turn to online forums when all else failed.

Student perceptions about the usefulness and the role of online homework are a determining factor in how students interact with online homework, and thus determine the learning opportunities that are experienced. The students that participated in my research demonstrated that a number of productive strategies exist, along with numerous strategies that degrade the quality of the learning experience. In particular, when students prioritize the credit they earn towards their grade for completing the online homework over the learning opportunity inherent in the struggle to solve a problem, they often employ strategies that degrade their learning opportunities. Nonetheless, these students perceived online homework as a useful learning tool, and likely benefited from the alignment of the online homework with the course exams. Because of this, an inquiry focused on student experiences can reveal the strengths and issues with online homework that would be overlooked by an evaluation focused on student success as measured by exam scores and course grades. Future work about online calculus homework should be directed towards the student perspective to more fully understand the ways that it can support student learning and inform instruction that facilitates productive study strategies.

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Lacking confidence and resources despite having value: A potential explanation for learning goals and instructional tasks used in undergraduate mathematics courses for prospective secondary teachers

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In this paper, I report on an interview-based study of 9 mathematicians to investigate the process of choosing tasks for undergraduate mathematics courses for prospective secondary teachers. Participants were asked to prioritize complementary learning goals and tasks for an undergraduate mathematics course for prospective secondary teachers and to rate their confidence in their ability to teach with those tasks and goals. While the mathematicians largely valued task types and goals that mathematics education researchers have proposed to be beneficial for such courses, the mathematicians also largely expressed lack of confidence in their ability to teach with these task types and goals. Expectancy-value theory, in combination with these findings, is proposed as one account of why, despite consensus about broad aims of mathematical preparation for secondary teaching, these aims may be inconsistent with learning opportunities afforded by actual tasks and goals used.

Key words: secondary teacher education, mathematicians' instructional dispositions

Each year, many prospective secondary teachers are enrolled in undergraduate programs intended to prepare them to apply mathematical knowledge to their future teaching practice. The field has called for improving these programs, including teachers' mathematical preparation. Although effects of teacher programs have proven difficult to assess in general, there are multiple studies on secondary mathematics teacher education suggest that secondary teacher do not connect their mathematical preparation and teaching practice. Even if connections could made in theory, it seems unlikely that teachers would leverage them if they believe there is no connection.

In many institutions, mathematics departments offer courses required for these programs (Conference Board of the Mathematical Sciences, 2010). However, there are few studies of how mathematics faculty teach (Speer, Smith, & Horvath, 2010), including why mathematics faculty make the instructional priorities that they do, and what resources are available to support these priorities. In this paper, I begin with an overview of policy and textbooks that inform mathematics courses for prospective secondary teachers. Then I report on a small interview study in which 9 mathematicians were asked about their priorities for goals for these courses. The responses provide a potential explanation for why current priorities may be difficult to change, even if there is some agreement on how to change.

Background

Potential to connect mathematical preparation to teaching

Approaches to mathematical preparation for teaching. In the US, committees of the Mathematical Association of America set policy for undergraduate mathematics courses taught through mathematics departments. Policy documents for mathematics courses for prospective mathematics teachers include reports written by the Committee on the Undergraduate Program in Mathematics Panel on Teacher Training in the 1960's to 1980's and Committee on the Mathematical Education of Teachers in the 1990's. Contemporary policy documents include the *Mathematical Education of Teachers* published by the Conference Board of the Mathematical Sciences (2001, 2012).

As a set, these policies describe different approaches to designing mathematical experiences for prospective teachers: mathematics from an advanced standpoint; trajectories to disciplinary content and practices; and mathematics as it is applied to recognizing, responding, understanding mathematical issues that arise in the context of teaching.

Each of these approaches connect mathematics to teaching in different ways. Mathematics from an advanced standpoint (Klein, 1908) seeks to provide an underlying mathematical structure for school mathematics, lending mathematical coherence to K-12 mathematics. In teaching, knowledge of school mathematics from an advanced standpoint provide explanations of foundational mathematical ideas that position learners to connect the mathematics across the curriculum. Mathematics that leads to disciplinary content and practices can be used to give intellectual purpose to school mathematics because they provide potential trajectories from school mathematics to compelling problems and practices of the discipline. An example of this approach is developing the mathematics necessary to prove the impossibility of classical straight-edge and compass constructions. In teaching, knowing such trajectories may provide perspective in ways similar to the role of horizon content knowledge (Ball, Thames, & Phelps, 2008), and can inform how to portray ideas so they are consistent with generalized mathematical systems learned later.

The third approach is a “practice-based” approach because it applies mathematics to teaching practice. The phrase references Ball and Bass’s (2003) introduction to mathematical knowledge for teaching as a “practice-based theory”. It is the mathematics that teachers draw upon to carry out the work of teaching, for instance, to select mathematical tasks that are amenable to whole group discussion, recognize what students are thinking, and strategize how to respond to students’ thinking toward particular mathematical points.

There is a fourth approach represented by policy for teacher preparation, which I call mastery of conventional mathematics. By conventional mathematics, I refer to the mathematics of a conventional course of study. A common notion in the US spanning back more than a century and codified by policies such as No Child Left Behind (2002) is that the teacher should generally know mathematics that is the equivalent of one degree level higher than the level of instruction. Teachers must at least know what they will teach at the level that the students are intended to learn. In teaching, teachers at least need to solve the mathematics problems that students are assigned.

Empirical results on the effectiveness of mathematical preparation for secondary teachers. Although experiences in mathematics from an advanced standpoint and that lead to disciplinary content have been received well as professional development (e.g., Watson, 2008; Eisenbud, 2015; Educational Development Center, 2015), there are also studies suggesting that prospective teachers are dissatisfied with their mathematical preparation, do not see their preparation as connected to their teaching. In Zaskis and Leikin’s (2010) survey of 52 practicing secondary teachers, and Goulding, Hatch, and Rodd’s (2003) survey of 173 prospective teachers, many participants found their mathematical preparation disconnected to teaching. Ticknor (2012) interviewed five prospective secondary teachers and a mathematics faculty instructor, and found that instructor saw connections between abstract algebra and teaching school mathematics, the teachers were engaged with the course, and yet the teachers perceived the course as disconnected from teaching. Wasserman, Villanueva, Mejia-Ramos, and Weber (2015) interviewed 14 secondary teachers and found similar results for real analysis. Thus, though there are theoretical arguments for why experiences in mathematics from an advanced standpoint and that lead to disciplinary content are beneficial for teachers, but it is harder to establish these benefits in practice. Wasserman et al. (2015) and Ticknor

(2012) concluded that these benefits may be more likely to occur if the content taught were more explicitly and intentionally situated in teaching situations – in other words, to adopt a more practice-based approach to the content. Scholars have made the argument for a practice-based approach to teacher preparation more generally (e.g., Ball & Cohen, 1999; Stylianides & Stylianides, 2014).

Practice-based knowledge and its links to teaching quality and student outcomes. Practice-based knowledge is linked to teaching quality and student outcomes at elementary and secondary level (Hill et al., 2008; Hill, Rowan, & Ball, 2005; Baumert et al., 2010), and at least two studies have documented that practice-based knowledge is more so linked than mastery of school mathematics or mathematics from an advanced standpoint. Baumert et al. (2010) found that performance on their measure of content knowledge had less to do with teaching quality than performance on their measure of pedagogical content knowledge, and Rockoff, Jacob, Kane, & Staiger (2011) found that teachers' performance on the LMT MKT assessment (Hill, Schilling, & Ball, 2004) of practice-based mathematical knowledge related to their student achievement outcomes, whereas the relationship between teachers' performance on SAT math student achievement outcomes was positive but statistically insignificant. These results give credence to the potential value of a practice-based approach to teacher education.

Prevalence of practice-based resources for the mathematical preparation of teachers

Given the evidence for the potential of practice-based approaches to mathematical preparation of teachers, it is natural to ask to what extent existing resources, such as policy documents and textbooks, support practice-based approaches. In short, policy and texts support practice-based approaches for elementary teacher education, but not secondary teacher education, and this difference began in the 1990's and 2000's.

Surveying historical policy documents for mathematics departments, the dominant approach to secondary mathematics education is to require prospective teachers to either pursue mathematics majors or to take many courses that overlap with the requirements of mathematics majors (Ferrini-Mundy & Findell, 2004). It is striking that current policy, as codified in the *Mathematical Education of Teachers, 2* (CBMS, 2012), does not propose practice-based knowledge as an organizing principle of mathematics courses specific to secondary teacher education, despite highlighting the importance of mathematical knowledge for teaching in the introductory chapters of the document. The principles advocated are "treat high school mathematics from an advanced standpoint", "take up a particular mathematical terrain related to high school mathematics and develop it in depth", and "develop mathematics that is useful in teachers' professional lives" (p. 63). The last principle may seem practice-based, but the examples given to define it are "classical ideas that are not normally included in a mathematics major but are of special use for teachers, such as the classical theory of equations or three-dimensional Euclidean geometry" (p. 63).

Historically, policy for elementary teacher education resembles secondary teacher education in its emphasis. As the 1971 Report of the Panel on Teacher Training explained of its proposed sequence for elementary teachers, "the arithmetic of the rationals finds justification and natural applications in elementary probability theory. … Also, the algorithms of elementary arithmetic lead naturally to flowchart and to a study of the role of computers" (p. 166). Indeed, the objectives of teacher training were "understanding the concepts, structure, and style of mathematics", "facility with its applications", "ability to solve mathematical problems", and "development of computational skills" (p. 161).

In contrast, the *Mathematical Education of Teachers, 2* states for elementary teacher education, "A major advance in teacher education is the realization that teachers should study the mathematics they teach in depth, and from the perspective of a teacher. … It is also not enough for teachers just to study mathematics that is more advanced than the mathematics they will teach" (p. 23). Illustrative activities for domains to be studied in coursework include examining and critiquing student reasoning, examining common student errors, and using common classroom manipulatives to explain elementary mathematics ideas. The domains include key mathematical ideas, their general principles, and how their structures connect to the structure of ideas later in the curriculum. Contemporary policy for elementary teacher education emphasizes practice-based knowledge that draws on structure of elementary mathematics. In this sense, the approach to elementary teacher education can be interpreted as practice-based drawing upon elementary mathematics from an advanced standpoint. The change toward practice-based knowledge in policy began in the 1990s.

Textbooks for teacher education reflect the policy. Beckmann's (2003) textbook for prospective elementary mathematics teachers, a highly rated and common text (National Council on Teaching Quality, 2008), contains many examples and tasks that use representations that are typically used at the elementary level and not beyond the elementary level (e.g., Cuisenaire rods, fraction strips), ask for a student-accessible explanation, feature embedded student work or a teaching goal, or that ask for explanation or interpretation based on different ways of thinking about the same ideas. In contrast, common textbooks for prospective secondary mathematics teachers, such as Usiskin et al. (2003) and Bremigan, Bremigan, and Lorch (2011) contain some but comparatively fewer practice-based tasks, and with the majority of problems focusing on mathematics from an advanced standpoint.

Elementary and secondary teacher education paint different pictures of mathematical preparation, as evidenced by MET2 and historical changes in national policy documents for mathematics departments. This backdrop raises a question of why practice-based approaches have not taken hold of secondary teacher education, as well as whether such change would be favorable. I focus here on the first issue. To do so, I looked at expectancy-value theory, which provides one explanation for how people make decisions. This paper reports on a study addressing the questions: *What goals and tasks do mathematicians value for mathematics courses for prospective secondary teachers? What expectancy do mathematicians hold for these goals and tasks? What factors influence value and expectancy?*

Theoretical framework

Expectancy-value theory

Expectancy-value theory is a frame for understanding choices that people make. Broadly speaking, many studies have shown that a person's success in attaining a goal is strongly shaped by how much the person values the goal intrinsically, the person's confidence that they could attain the goal, and the quality of the person's ability to conceive of implementation intentions (statements of the form "If X happens, then I will do goal-attaining behavior Y"). (See Eccles and Wigfield (2002)'s review of research on the effects of motivational beliefs and values on goal attainment, and Gollwitzer and Sheeran (2006)'s review on the effect of implementation intention on goal attainment). To represent confidence, I use expectancy, that is, a person's belief about how well they will do at a task (Atkinson, 1964), as used in Eccles and colleagues' extensively validated expectancy-value theory that relative value and perceived probability of success influence achievement-related choices (e.g., Eccles, 1983; Eccles, Wigfield, Harold, & Blumenfeld, 1993). The phrasing of this study's interview questions on expectancy and value were adapted from those described in Eccles, Wigfield, Harold, and Blumenfeld (1993). Expectancy-value theory holds that a person is most likely to choose tasks for which they hold high value and expectancy. If value is high, but expectancy is low, then success may be sacrificed because a person chooses to engage in tasks that are counter to their values. A classic example of this phenomenon is quitting an addiction. Addicts may know the value of quitting, but because they do not expect to succeed at quitting, they may act in ways that enforce the addiction. Knowing the value of changing habits is not enough to support actual change.

Practice-based design

Following the conceptualizations of researchers in mathematical knowledge for teaching (Ball & Bass, 2003, Hill, Schilling, & Ball, 2004; Gitomer, et al., 2014), a task or goal is considered to follow a "practice-based" design if it intentionally and explicitly connects mathematics and teaching. That is, it applies mathematical knowledge to teaching situations, situates teachers in work of teaching, the kind of work to which the mathematics is applied is important and recurrent to teaching, and the mathematical knowledge addresses K-12 mathematics. Ball and Bass (2003) introduced mathematical knowledge for teaching as a practice-based theory, and the phrase has been subsequently codified in work that attempts to systematically catalogue the work of designing tasks that assess mathematical knowledge for teaching (e.g. Gitomer, et al., 2014).

The propositional mathematical knowledge drawn in the work of a practice-based task can include mathematics of secondary school and beyond. The Green Task gives an example of former and the Blue Task the latter. What makes them practice-based is that they both require mathematical and teaching knowledge to interact for the purpose of teaching secondary mathematics. In the Green Task, a teacher must apply knowledge of factoring to infer that discussion of different solution paths is best supported by the radicand with the most possible perfect square factors, in this case, 72. In the Blue Task, a teacher must apply knowledge of functions to a student accessible explanation.

Approaches studied

In a previous study on proofs for pedagogical purposes, some mathematicians evaluated the value of a proof differently depending on the mathematical ideas emphasized (Lai & Weber, 2014). Because the mathematical content of a practice-based task can range from secondary content to beyond, and the argument for mathematics from an advanced standpoint is that it underlies explanation of foundational ideas, I decomposed practice-based tasks into two types: practice-based drawing on strictly secondary content and from mathematics from an advanced standpoint. Thus the approaches to tasks studied were: (1) mathematics from an advanced standpoint, (2) mathematics that leads to disciplinary content, (3)(b) practice-based drawing on strictly secondary content, (3)(a) practice-based drawing on mathematics from an advanced standpoint, (4) ensuring mastery of school mathematics. Because the mathematics involved in a particular goal can vary more than it might for a specific task, I did not do a similar parsing of practice-based goals.

Study

Rationale

To examine factors influencing value and expectancy of the different approaches, mathematicians were asked to perform three card sorts using a think-aloud protocol. The card sets were statements of 4 content-specific goals, statements of 4 content-generic goals, and 6 tasks representing the different approaches to mathematics courses for prospective secondary teachers described above. Mathematicians were asked to place the cards horizontally to represent the value of the card to the mathematical preparation of secondary teachers, and then vertically to represent their expectancy in teaching the task or toward the goal.

Sorting on value was designed to simulate prioritizing. A common choice encountered in teaching is selecting among tasks and goals to accomplish in a limited amount of time. Even if many goals or tasks are valuable, it is often not possible to address all goals and tasks deemed valuable. Sorting on expectancy was meant to elicit relative confidence in ability to carry out the goals and tasks. Relative value and expectancy informs the goals and tasks that may be chosen when others must be sacrificed for the sake of time.

Card sorts with content-specific goals, content-generic goals, and tasks were used both to elicit relative value and expectancy and also to examine construct consistency among the participants. Common wisdom provided to instructors across K-16 education suggests identifying goals for the course as a whole, identifying goals for each lesson, and then selecting or constructing tasks that serve the goal. Thus one would expect consistency in how instructors perceive course-level goals, lesson-level goals, and tasks. At the same time, if inconsistencies arose, their reasons should be considered in explaining instructional choices.

Method

Participants. Mathematicians were recruited for an interview study as follows. Email invitations were sent to a list of mathematicians who had participated in workshops on teacher education discussing practice-based approaches to conceptualizing the knowledge needed for teaching. The invitation specified that the study sought mathematics faculty and mathematics graduate students who had or would, if given the opportunity, teach a course for prospective high school teachers. A total of 9 mathematicians agreed to participate. Prior to the interview, participants were asked to complete a survey about their background and experiences as mathematics instructors. The mathematicians had between 0 and 10 years experience teaching courses for prospective secondary teachers, 0 to 5 years experience teaching courses for prospective elementary teachers, 0 to 3 years experience teaching courses for practicing secondary teachers, and 0 to 3 years experience teaching courses for practicing elementary teachers. All mathematicians had at least 1 year experience teaching prospective teachers of some level. In the pre-interview survey, all mathematicians responded that, on a scale of 1 (not at all) to 7 (very much), being good at teaching courses for teachers was 5 to 7, and being good at teaching in general was either 6 or 7. The mathematicians were also asked, in comparison to other courses, how well did they think they taught courses for teachers; responses were between 3 and 7, with mostly 3's and 4's. The mathematicians taught at a variety of institutions from small liberal arts schools to research-intensive schools and ranged in appointment, including assistant professors, associate professors, full professors, and lecturers.

Materials and procedures. This interview took place in the context of a longer interview with six parts: (a) describing a favorite task used when teaching prospective teachers, its goals, and its enactment, (b) considering the appropriateness of content-specific goals for a particular task shown to participants, (c) (card sort 1) content-specific goal sort as goals independent of the task, (d) (card sort 2) task sort, (e) (card sort 3) content-generic goal sort, (f) wish list. This study focuses on the results of the three card sorts, with the responses to the wish list questions used to provide additional context for card sort comments. The card sort contents were based on existing resources for teacher education when possible. Prior to the interviews, two mathematicians and two mathematics educators independently reviewed the contents favorably for consistency with intended approach.

I now briefly describe the protocols for the three card sorts and the wish list. For each, participants were first asked to sort the cards horizontally to indicate value and then vertically to indicate expectancy. Participants were prompted to explain the relative positioning of cards for each sort.

Card Sort 1: Content-specific goals. Four content-specific goals were shown to participants representing the approaches to goals. Table 1 shows the goals.

Card Sort 2: Tasks. Six cards containing tasks were shown to participants representing the approaches. The tasks were named by colors (Pink, Orange, Yellow, Green, Blue, Purple). The cards contained tasks representing approaches to tasks, with one task (Orange) that situated the Pink Task in a teaching situation. Table 2 shows the tasks.

Card Sort 3: Content-generic goals. Four content-generic goals were shown to participants representing approaches to goals. Table 1 shows the goals.

Wish List. At the end of the interview, participants were asked: If you could make a wish list for resources for getting better at teaching math courses for teachers, what are some things that wish list would contain? What is inadequate about existing resources, such as textbooks that are already out there, or talking with other people who have taught this course before in your institution or beyond?

Analysis. For each card for each participant, cards were assigned horizontal and vertical coordinates with values between 1 and 5 based on the approximate location of the center of the card as placed by the participant where horizontal coordinates represented value and vertical represented expectancy.

Interview transcripts were chunked into statements discussing beliefs, goals, knowledge, or action plans, following Schoenfeld's (1998) theory of instructional decision making. The 9 participants made 144 statements in total during the card sorts and wish lists. Each statement was then coded for whether it discussed value or expectancy. Among these, 17 statements were coded as "neutral", meaning they did not give a specific reason (e.g., a belief statement such as "I think the orange card should be placed higher than the pink card" or "I feel confident about this"). Among the remaining, 61 statements addressed expectancy and 66 statements addressed value. The collection of statements with reasons were analyzed for themes (Strauss & Corbin, 1994). Patterns noted in card placements were triangulated with interview statements.

Results

Table 3 shows scatterplots of card sort placements. It is worth noting that generally, these participants valued practice-based tasks and goals more than they were confident in being able to carry out practice-based tasks and goals. The practice-based data points tend to fall beneath the diagonal.

We now discuss how participants' card sorts and interview responses addressed themes addressing value and expectancy, and how this might play into these participants' disposition toward practice-based tasks and goals. For brevity, we focus on only the dominant themes. We present tables summarizing all themes in the types of comments that participants provided, but give representative illustrations or only the most dominant themes for each value and expectancy. See Tables 4 and 5. The themes discussed are bolded.

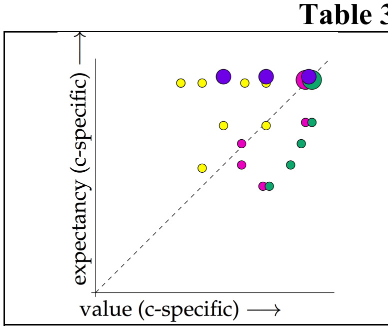
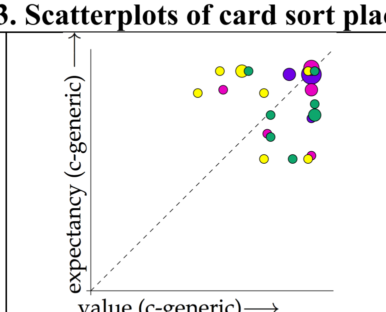
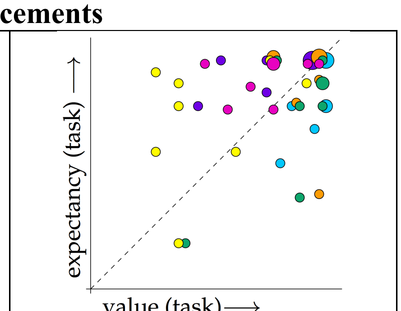
Table 1. Content-specific goals, content-generic goals, and their correspondence

Content-specific goals	*Content-generic goals*
Understanding the relationship between the definition of an equation, the definition of graph, and the definition of relation.	Connecting ideas from higher mathematics to secondary mathematics
Seeing how "circles" can look very different depending on the metric used.	Experiencing secondary mathematics as a rigorous, challenging, coherent body of mathematics.
Analyzing incorrect solutions for foundational ideas that may be misunderstood.	Analyzing mathematical teaching situations
Mastery in graphing relations of two variables, especially involving absolute values.	Ensuring that teachers would be able to do the problems they are responsible for teaching K-12 students to do.

Table 2. Tasks and their correspondence to approaches

Green Task: Practice-based drawing on secondary content[1] Ms. Madison wants to pick one example from the previous day's homework on simplifying radicals to review at the beginning of today's class. Which of the following radicals is best for setting up a discussion about different solution paths for simplifying radical expressions? 1. $\sqrt{54}$ 2. $\sqrt{72}$ 3. $\sqrt{120}$ 4. $\sqrt{124}$ 5. Each of them would work equally well. Explain your reasoning.	*Yellow Task: Secondary mathematics leading to disciplinary content*[2] During a lesson on exponentiation, Ms. Waller's students came across the expression $\left((-4)^{\frac{1}{2}}\right)^2$. Two students obtained different answers when they tried to evaluate this expression. **Anna**: *I got* -4. *I started with* $(-4)^{\frac{1}{2}} = \sqrt{-4}$. *And* $\sqrt{-4} = 2i$. *So* $(2i)^2 = 4i^2 = -4$, *and so* $\left((-4)^{\frac{1}{2}}\right)^2 = -4$. **Brenda**: *My answer was* 4. *I did* $\left((-4)^{\frac{1}{2}}\right)^2 = (-4)^{\frac{1}{2}\cdot 2} = (-4)^{2\cdot\frac{1}{2}} = ((-4)^2)^{\frac{1}{2}} = 16^{\frac{1}{2}} = \sqrt{16} = 4$. Explain the apparent contradiction between Anna's and Brenda's answers in terms of a multi-valued exponential function.
Pink Task: Secondary mathematics from an advanced standpoint Suppose $x \neq 0$. Prove that $x^0 = 1$. You may use the additive law of exponents ($a^{b+c} = a^b a^c$ for all $a \in \mathbb{R}$, $b, c \geq 0$, and $b, c \in \mathbb{Z}$) and the definition that $a^1 = a$ for all $a \in \mathbb{R}$	*Purple Task: Mastery* Find three different pairs of functions g and h such that $g \circ h = (x+3)^2$.
Orange Task: Practice-based drawing on mathematics from an advanced standpoint In Ms. Swain's Algebra I class, a student says the following. *I don't know why* $x^0 = 1$. *Is it just a convention? To me, it seems like it should be 0 because anything times 0 is 0.* Write two different explanations that Ms. Swain can give to address what the student said. The explanations should address the underlying mathematics and be accessible to Algebra I students.	*Blue Task: Practice-based drawing on mathematics from an advanced standpoint* During a lesson on functions and their graphs, a student asks Mr. Loman: *The vertical line test talks about graphs and lines and points. But I thought we said that a function was something that has to do with rules about inputs and outputs. What do lines going through points have to do with rules about inputs and outputs?* Write an explanation that Mr. Loman can give to address the student's question. The explanation should address the student's concerns about the connection between the underlying mathematics.

Table 3. Scatterplots of card sort placements

		
Content-Specific Sort Value vs. Expectancy	Content-Generic Sort Value vs. Expectancy	Task Sort Value vs. Expectancy

Key: Green, Orange, Blue = Practice Based, Yellow = Disciplinary Content, Pink = Advanced Standpoint, Purple = Mastery. Larger circles represent more participants placing the card in that approximate location.

[1] Source: Thames (2006), p. 6.
[2] Adapted from an instrument developed by the Educational Testing Service © 2013, with permission

Factors influencing value. The most dominant themes for determining value were importance to teaching practice, how the knowledge represented by the task or goal might enhance teaching or how well the nature of mathematics is conveyed, and proximity to secondary mathematics.

Importance to teaching practice. There were 24 statements about the value of a task or goal in terms of its importance to work that teachers do, the knowledge needed for teaching, or whether a task applied mathematics to teaching. All 9 mathematicians made statements to this effect. These statements were used to justify moving practice-based tasks and goals more to the right in the card sort, designating higher value. Typical of this reasoning was Joe's and Cantor's thinking on the content-specific goal sort as they moved the practice-based goal more to the right. As Joe commented, (italics are my emphasis):

> Because their students are going to misunderstand. There are going to be foundational ideas when they're teaching that they they'll need to convey and these ideas are often confusing. So *having them identify you know, what's wrong with this, this solution gives them experience. I guess it gives them experience teaching*.

Joe is pointing out that the work of teaching entails conveying ideas and parsing student work, and the value of the goal is that it seems close to practice. The work of teaching highlighted by participants generally concerned identifying and remedying student misconceptions and explaining mathematics. Calling out work of teaching as a reason for valuing a task or goal occurred in all three card sorts.

The potential for knowledge represented to be useful in teaching was used most typically used to justify value for the practice-based and advanced standpoint tasks and goals. As Heidi commented (italics mine), "And I did put the analyze the incorrect solutions slightly lower and then the understanding relationship and that's partially because I think *you need to understand the relationship first to really be able to analyze the incorrect pieces*." Heidi later commented that making connections between different concepts in secondary math allows a person to teach in a "flexible creative way". One participant did use application to teaching to justify the connections to disciplinary content task. However, this participant still placed the practice-based and advanced standpoint tasks higher.

Finally, the potential for a task to prompt teachers to apply mathematics to teaching was used to justify the value of practice-based tasks and goals. As Cantor reasoned (italics mine), "I guess if teachers can answer the pink one, they probably have a good answer to the orange one, but the orange one felt like more the situation a teacher would get…interact with, *that's more contextualized. So it got moved farther to the right."* All participants valued the Orange Task more than the Pink Task, although not all participants provided explicit reasoning for why. I suggest that one reason may be that, as Cantor pointed out, the Orange Task situates the mathematics of the Pink Task in a teaching situation.

Proximity to secondary mathematics. The closeness of the knowledge to secondary mathematics was generally used to justify moving cards representing advanced standpoint and practice-based with advanced standpoint cards more to the right and connections to disciplinary content more to the left, although two participants also used this to justify moving mastery cards more to the right. As Heidi commented about the connections to disciplinary content task:

> The detail needed for this particular questions is probably beyond the necessary knowledge of somebody whose teaching algebra 1 or middle school math or geometry. … so that's kind of why I put them in the middle.

Participants generally valued advanced standpoint tasks more than the connections to disciplinary content task.

Potential to enhance understanding the nature and foundation of mathematics. Whether a task or goal captured a "foundational" or "fundamental" aspect was used to justify valuing advanced standpoint and practice-based drawing on advanced standpoint, as well as to justify less value in the practice-based drawing on strictly secondary mathematics. Typical of this reasoning was Dan:

> And I feel like…the things that seem most important to me are understanding why things work the way they do and I think it might be important to have a discussion of how there are multiple ways you can come to a correct answer. And some of these problems might illustrate these better than others, but I think the other questions get at deeper foundational issues.

Dan is pointing out both that the more that a task gets at foundational mathematics, the more important it is, and the less that a task foundational mathematics, the less important it is.

Four participants pointed out the importance of connections to disciplinary content as exposure to the nature of mathematics. However, these participants did not place these cards rightmost, hedging that the content might be important, but not as important. As Jim said in the content-specific goal sort:

> I guess well you've made your horizon knowledge to understand these connections and have a deeper appreciation for what we're doing in geometry. We have to push them a little and expose them to a broader viewpoint. But what is the most important thing? Well they better know how to graph something and explain that to someone. Right?"

Two participants pointed out a "neat factor" in connections to disciplinary content, but that this might not be as important as mathematics that is more foundational.

Factors influencing expectancy. The two most dominant themes influencing expectancy were the demands of taking a practice-based approach and related past experiences.

Difficulty of taking a practice-based approach. Participants cited the demands of a practice-based approach for why they felt less confident in teaching practice-based tasks or towards practice-based goals. In particular, participants commented on the difficulty of designing good quality practice-based tasks as well as assessing solutions when the language used could not rely on the typical formal language used in the discipline.

As Cantor observed of practice-based questions, "I tend to be good at recognizing, 'This is really good, these are interesting questions.' But coming up with them, creating them myself I think I tend to struggle." Eight of the participants commented in the wish list section that they wanted "more problems like the colored tasks". That the participants had encountered difficulty in writing practice-based tasks is unsurprising. Although there is increasing knowledge in the field of how to write practice-based tasks well (Gitomer et al., 2014; Hill, Schilling, & Ball, 2004) and of how practice-based tasks function (Howell, Lai, & Phelps, 2016; Howell & Phelps, 2016; Lai, Jacobson, & Thames, 2013), the current state of the art still generally requires multiple reviews and revisions involving persons with different expertise (Gitomer et al., 2014). Combined with the lack of available examples of practice-based tasks for secondary mathematics teaching in commonly used resources, it should be expected that an individual express difficulty expressed in writing practice-based tasks.

Assessing and understanding solutions to practice-based tasks also challenged participants. As Heidi commented on the placement of a practice-based task card, "It's hard seeing inside a student's head to be absolutely sure that you're communicating correctly no matter how much assessment you do, we use words differently. So that's why it's just a little bit lower." Other participants commented that "proofs are something I can do", going on to explain that they have language for proof. I suggest that in contrast, explanations that are student accessible or that are about connections between representations may require using language that is nonstandard, and therefore should seem more difficult than language that is familiar to mathematicians because they are trained in formal language of proofs.

One participant, Margaret, also commented on the difficulty of teaching prospective teachers to see mathematics from their future students' point of view: "You have to teach them how to look … They're viewing it from the right, the student's viewing it from the left, they're seeing mirror images, they're seeing different things, and they haven't even started to think about how this student is seeing it." Margaret saw this perspective as necessary for engaging with practice-based tasks, but expressed reservation in knowing how to cultivate this perspective in prospective teachers.

Related past experiences. Participants described their experience with teaching practice-based tasks to justify expectancy for success in teaching the practice-based tasks or toward the practice-based goals shown. Participants both used positive experiences with practice-based tasks to raise their expectancy of success with these practice-based tasks, and used lack of experience as a reason for lower expectancy of success. For instance, Jim, on the content-generic goal of analyzing mathematical teaching scenarios, commented, "I think it would still be a challenge for me because of lack of experience. … I never took a course on classroom management or, or how to think about how to organize a course." On the other hand, Dan and Heidi, who had both had positive experiences teaching from Beckmann's textbooks for prospective elementary teachers featuring many practice-based tasks, expressed confidence in carrying out further practice-based tasks, even if the tasks were for a different course. As Dan reasoned, "I'm confident that I could, create situations where they could do well or learn to do well, based on having done that in other courses." Previous experience was also used to justify high expectancy on mastery tasks and goals. Although these participants generally did not value mastery tasks and goals as highly as practice-based and advanced standpoint tasks and goals, they at the same time expressed higher confidence in ability to teach for mastery because they had taught more courses with mastery goals. These comments are consistent with findings in the expectancy-value literature; previous achievement-related experience, positive and negative, have been shown to factor into a person's expectancy of future success (e.g., Eccles & Wigfield, 2002).

Table 4. Themes for how participants determined the value of a task or goal

Ways mathematicians determine value	# statements (# distinct participants)
Importance to teaching practice	**24 (9)**
• It arises in the work of teaching	8 (6)
• The work of teaching requires more math than what is being taught to students.	6 (5)
• It is important for teachers to apply content to teaching. If it does not apply as much to teaching, it is not as important.	10 (6)
Proximity to secondary mathematics	**16 (8)**
Potential to enhance teachers' understanding of the nature or foundation of mathematics	**10 (6)**
How mathematical the task or goal is	4 (4)
Not knowing it would be bad for teaching	5 (4)
Needs of particular teachers being taught	4 (3)
Priority for beginning teaching	3 (1)
Total statements	66

Table 5. Themes for factors influencing participants' expectancy

Factors influencing mathematicians' expectancy	# statements (# distinct participants)
Difficulty of taking a practice-based approach	**17 (9)**
• Coming up with practice-based problems is hard	6 (5)
• It is hard to teach prospective teachers to take a teaching viewpoint	2 (1)
• Getting away from formal language is hard	9 (4)
Lack of experience / ability to draw on experience	**27 (7)**
Difficulty of explaining complex mathematics	12 (5)
Perceptions of prospective teachers' beliefs and attitudes	5 (3)
Total statements	61

Discussion

These participants' interviews generally revealed value in practice-based approaches to mathematical preparation of teacher, as well as mixed expectancy for carrying out such approaches. Interview analysis revealed that reasons for valuing practice-based approaches included importance to teaching, proximity to secondary mathematics, and potential to enhance teachers' understanding of the nature and foundation of mathematics. Reasons for higher or lower expectancy included the demands of taking a practice-based approach as well as presence or absence of experience with a practice-based approach.

There are two limitations of the study due to the aim and scope of the study. The first is that only nine mathematicians were interviewed. This is a small sample, and although commonalities emerged from the participants, there is no warrant for sample-to-population generalizations. Nonetheless, this study does document a particular phenomenon—that practice-based tasks could be held in high value in combination with mixed expectancy, and that expectancy can be informed by previous experience. The mathematicians here all participated in workshops that emphasized the need for mathematical knowledge for teaching, so it is unsurprising that they espoused practice-based values. What is more striking in this context is the lack of expectancy. Future studies can look at the degree to which mathematics faculty hold contrasting values and expectancies for practice-based approaches and other approaches, more generally.

The second limitation of this analysis is the scope. The participants examined only a limited number of tasks designed with a particular frame of approaches. It is possible that a different set of tasks or goals would have elicited different value and expectancy ratings. The consistency of the ratings across the three card sorts is promising, although the task and content-specific card sorts could have primed the content-generic card sort that followed. It is also possible that there are approaches to the mathematical preparation of teaching that have not been documented in policy documents for mathematics departments. Yet in the current time, these approaches are the ones that are available and therefore likely the ones that instructors will select among, so understanding why they are theoretically valued as well as valued in practice, are important to understand. The current study identifies some values that may be used in prioritizing among approaches.

The literature on expectancy-value theory suggests that successful goal-oriented decision making is more likely when a person values the goal, has high expectancy at achieving that goal, and has the resources to support goals and build expectancy. When expectancy and resources are not in place, even if a person's values resonate with a goal, the person may not choose to make choices that support that goal. Mathematicians may actually share math educators' values but do not take on practice-based tasks because they lack the confidence and resources to do them.

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Classifying combinations:
Do students distinguish between different categories of combination problems?

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In this paper we report on a survey study to determine whether or not students differentiated between two different categories of problems involving combinations – problems in which combinations are used to count unordered sets of distinct objects (a natural, common way to use combinations), and problems in which combinations are used to count ordered sequences of two (or more) indistinguishable objects (a less obvious application of combinations). We hypothesized that novice students may recognize combinations as appropriate for the first category but not for the second category, and our results support this hypothesis. We briefly discuss the mathematics, share the results, and offer implications and directions for future research.

Key words: Combinatorics, Discrete mathematics, Counting

Introduction and Motivation

Discrete mathematics, with its relevance to modern day applications, is an increasingly important part of students' mathematical education, and national organizations have called for increased teaching of discrete mathematics topics in K-16 mathematics education (e.g., NCTM, 2000). Combinatorics, and the solving of counting problems, is one component of discrete mathematics that fosters deep mathematical thinking but that is the source of much difficulty for students at a variety of levels (e.g., Batanero, Navarro-Pelayo, & Godino, 1997; Eizenberg & Zaslavsky, 2004). The fact that counting problems can be easy to state but difficult to solve underscores the need need for more research about students' thinking about combinatorics.

One fundamental building block for understanding and solving combinatorial problems are combinations (i.e., C(*n,k*), also called binomial coefficients due to their role in the binomial theorem). Combinations are prominent in much of the counting and combinatorial activity with which students engage, and yet little has been explicitly studied with regard to student reasoning about combinations. This study contributes to our understanding of students' reasoning about combinations, and, in particular, beginning students' inclination to differentiate between typical combinatorics problems. This study addresses the following research question: *Do early undergraduate students use binomial coefficients to express the solution to two different categories of combination problems?*

Theoretical Perspective

Lockwood (2013; 2014) has argued for the importance of focusing on *sets of outcomes* in solving counting problems. In an initial model of students' combinatorial thinking, Lockwood (2013) proposed that there are three inter-related components that students may draw upon as they solve counting problems: formulas/expressions, counting processes, and sets of outcomes. Formulas/expressions are terms involving numbers or variables that reflect the answer to a counting problem. Counting processes are the series of procedures, either mental or physical, in which one engages as they solve a counting problem. Sets of outcomes are the complete set of objects that are being counted in the problem, and the cardinality of the set of outcomes gives the answer to the counting problem (Lockwood, 2013). Lockwood also pointed out that a given expression might reflect a counting process, and that various counting processes may impose different respective structures on the set of outcomes. For example, consider solving a problem like *How many arrangements are there of the letters A, B, C, and D?* A 4-stage counting process to solve this problem is to first place one of the four letters in the first position (there are 4 choices), then to place one of the three remaining letters in the second position (3 choices), then to place one of the two remaining letters in the third position (2 choices), and finally to place the last remaining letter in the last position (1 choice). Because at each of these stages the number of choices is the same and is independent of the result of the previous stages, we can multiply the number of options at each stage.[1] This counting process thus yields an expression of 4•3•2•1, which is 24. Note that this counting process also imposes a structure on the set of outcomes, namely organizing them according to first, then second, then third, then fourth letter. The lexicographic listing of the set of outcomes in Figure 1 demonstrates the set of outcomes that might be associated with the specific counting process.

ABCD	BACD	CABD	DABC
ABDC	BADC	CADB	DACB
ACBD	BCAD	CBAD	DBAC
ACDB	BCDA	CBDA	DBCA
ADBC	BDAC	CDAB	DCAB
ADCB	BDCA	CDBA	DCBA

Figure 1 – A lexicographic listing of arrangements of A, B, C, and D

The point of this example is that a given formula or expression (4•3•2•1) may have a combinatorial process that underlies it (the 4-stage process described above), which in turn structures the set of outcomes in some specific way. Lockwood (2013) argued that it would behoove students to have flexibility in their counting, especially in understanding that different counting processes may organize the sets of outcomes in different ways, and vice versa.

Lockwood (2014) emphasized the importance of sets of outcomes, and argues for a set-oriented perspective toward counting. In this perspective, counting "involves attending to sets of outcomes as an intrinsic component of solving counting problems" (p. 31). In this paper, we frame our work from this perspective, namely that *sets of outcomes* are the key factor in determining what counting situation one is in, and thus what counting process and formula may be appropriate. That is, as we discuss "combination problems," we consider a problem to be a combination problem if the set of outcomes can be appropriately modeled in a particular way (specifically, as sets of distinct objects), as opposed to being attuned to problem features like particular key words or contexts.

The nature of our data is such that we are, at times, only able to examine a student's written expression and must make inferences about their counting process based on that expression. In this way, we draw on Lockwood's (2013) model and its position that counting processes can and do underlie formulas and expressions. Those underlying processes can suggest how students might be conceiving of the set of outcomes, if at all.

[1] This is due to the multiplication principle, which is a fundamental yet subtle combinatorial idea. See Tucker (2002) for a statement of the principle and Lockwood, Reed, & Caughman (2015) for a more in-depth discussion of its mathematical subtleties.

Literature

Combinations as a Unifying Combinatorial Topic

Combinations, or binomial coefficients, are one of those unifying topics that connect a variety of combinatorial contexts and ideas. They are commonly used to count particular kinds of objects, namely k-element subsets of n-element sets, or, equivalently, the number of ways of selecting k elements from an n-element set. But this fact is applied meaningfully in a variety of important combinatorial settings, establishing combinations as a foundational "big idea" in combinatorics. Specifically, in addition to solving counting problems, these binomial coefficients also show up as entries in the rows of Pascal's Triangle as a model for solving multichoose problems, and as the coefficients in the binomial theorem, $(x+y)^n = \sum_{k=0}^{n} C(n,k)x^n y^{n-k}$, because to find the coefficient of a given x^k term we simply choose which k of the n binomials will contribute an x term. Because of this important relationship, combinations are known as binomial coefficients and arise in a number of mathematical domains. Finally, combinations appear in many combinatorial identities, and they necessitate combinatorial proof, which is another key topic in combinatorics. The variety of contexts in which combinations naturally arise points to the invaluable role that they play in the domain of combinatorics. Given how pervasive and fundamental they are, we were motivated to better understand how students might reason about and learn this foundational combinatorial idea.

Combinations in Mathematics Education Literature

There is much documented evidence for the fact that students struggle with solving counting problems correctly. Some reasons for such difficulty are that counting problems are difficult to verify (Eizenberg & Zaslavsky, 2004) and that it can be difficult to effectively encode outcomes in terms of objects one knows how to count (e.g., Lockwood, Swinyard, & Caughman, 2015b). We seek to address potential difficulties by focusing on better understanding students' application and use of combinations – which are fundamental in enumeration. Piaget & Inhelder (1957) studied students' mental processes as they solved arrangement and selection problems, and they took a special interest in determining whether permutations, arrangements, or combinations would be most difficult for students. Dubois (1984), Fischbein and Gazit (1988), and Batanero, et al. (1997) have also investigated the effects of both implicit combinatorial models and particular combinatorial operations on students' counting, again considering differences in reasoning about particular problem types such as permutations and combinations. We extend existing work that focuses on students' mental processes of foundational combinatorial ideas, seeking specifically to explore the extent to which undergraduate students appear to distinguish between two categories of combination problems (we define these two categories in the following section).

Other researchers have looked at students as they explore the variety of settings in which binomial coefficients arise. For example, Maher, Powell, and Uptegrove (2011) documented several episodes in which students of their longitudinal study make meaningful connections between binomial coefficients, certain counting problems, and Pascal's Triangle. More specifically, Speiser (2011) documented one 8th grade student's reasoning about problems involving block towers, which can be solved as combination problems, and connections she made to the formula C(n,r). In a similar vein, Tarlow (2011) reported on eight 11th grade students who could make sense of a well-known binomial identity using both pizza and towers contexts. These studies highlight the many connections that binomial coefficients afford in combinatorial settings, suggesting that it may be beneficial for students to have a sophisticated understanding of binomial coefficients that could facilitate these kinds of valuable connections. We build upon this work by Maher et al., (2011) by focusing on whether or not students see two distinct settings as both involving combinations.

Our work also builds on a recent study by Lockwood, Swinyard, & Caughman (2015a) in which two undergraduate students reinvented basic counting formulas, including the formula for combinations. Based on the students' work on combination problems, Lockwood, et al., (2015b) suggested the importance of being able to correctly encode outcomes combinatorially (by which they mean the act of articulating the nature of what is being counted by associating each outcome with a mathematical entity such as a set or a sequence). More specifically, the students could solve combination problems but not others, which was curious given their overwhelming success overall. In our current study, we were able to mathematically characterize a fundamental difference between some common combination problems based on characterizing a difference in their sets of outcomes. We wanted to explore further whether, for novice students, this distinction in the set of outcomes would pose a significant hurdle to their ability to recognize binomial coefficients as equally useful in the solutions to both situations. This resulted in us differentiating two different, different "categories" of combination problems and speculating about whether or not, for students, this difference matched their reality. We introduce and explore this distinction in the next section.

Mathematical Discussion – Classifying Two Categories of Combination Problems

In this section we outline mathematical details of combinations, and we highlight a distinction between two "categories" of combination problems based on their sets of outcomes that is the focus of our investigation. A combination is a set of distinct objects (as opposed to a permutation, which is an arrangement of distinct objects). Combinations can also be described as the solution to counting problems that count "distinguishable objects" (i.e., without repetition), where "order *does not* matter." The total number of combinations of size k from a set of n distinct objects is denoted C(n,k) and is verbalized as "n choose k."[2] So, the binomial coefficient C(n,k) represents the set of all combinations of k objects from n distinct objects. As an example, combinations can be used to select, from eight (distinguishable) books, three books to take on a trip (order does not matter) – the solution is C(8,3), or 56 possible combinations. In contrast, other combinatorial problems and solution methods, such as permutations, are frequently organized in relation to the different possible constraints – see Table 1.

Table 1: Selecting *k* objects from *n* distinct objects

	Ordered	**Unordered**
Distinguishable Objects (without repetition)	Permutations $\frac{n!}{(n-k)!} = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-k+1)$	Combinations $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
Indistinguishable Objects (with repetition)	Sequences $n^k = \underbrace{n\cdot n\cdot n\cdot\ldots\cdot n}_{k}$	Multicombinations $\left(\binom{n}{k}\right) = \binom{k+n-1}{n-1}$

In this paper we refer to *combination problems* as problems that can be solved using binomial coefficients, in the sense that parts of their outcomes can be appropriately encoded as *sets* of distinct objects. Sometimes this encoding is fairly straightforward, as the outcomes are very apparently sets of distinct objects. For instance, in the above problem of selecting three books from eight books to take on a trip, the books could be encoded as the numbers 1 through 8

[2] The derivation of the formula for $C(n,k)$ as $n!/((n-k)!k!)$ is not pertinent to the study; Tucker (2002) provides a thorough explanation.

(because they are different books), and the outcomes are fairly naturally modeled as 3-element sets taken from the set of 8 distinct books. Any such set is in direct correspondence with a desired outcome; there are C(8,3) of these sets. We call such problems Category I problems.

In other situations, fairly typical combination problems may still appropriately be solved using a binomial coefficient, but recognizing how to encode the outcomes as sets of distinct objects is less clear. For example, consider the Coin Flips problem (stated in Table 2). A natural way to model an outcome in this problem is as an ordered sequence of length 5 consisting of 3 (identical) Hs and 2 (identical) Ts, such as HHTHT. Solving the problem then becomes a matter of counting such sequences. One way to count the number of such sequences is to arrange all of the five letters (in 5! ways), and then divide out the repetitive outcomes based on the fact that there are three identical Hs (3!) and two identical Ts (2!) – a solution of 5!/(3!2!). One could think of first treating all five of the letters as distinct and then "un-labeling" identical Hs and Ts. However, we note that the problem also can be (and frequently is) solved using a binomial coefficient – but, in order to do this, the outcomes must be encoded appropriately as a set of distinct positions in which the Hs are placed. Given the five possible distinct positions (i.e., the set: {1, 2, 3, 4, 5}), the outcome HHTHT would be encoded as the set {1, 2, 4}. This sufficiently establishes a bijection between outcomes (sequences of Hs and Ts) and sets of the numbers 1 through 5 because every outcome has a unique placement for the Hs (the Ts must go in the remaining positions). In this way, the answer to the Coin Flips problem is simply the number of 3-element subsets from 5 distinct objects (i.e., positions 1 to 5), which is C(5,3). This gives an identical formula of 5!/(3!2!) and it is another way of solving the problem.[3] We call these problems, which can naturally be modeled as *sequences* of *identical* objects, but which can be encoded so as to be solved via a binomial coefficient, Category II problems (See Table 2).

Table 2: Characterizing two different categories of fairly standard combination problems

	Description	Example problem	Natural Model for Outcomes
Category I	*An unordered selection of distinguishable objects*	*Basketball Problem.* There are 12 athletes who try out for the basketball team – which can take exactly 7 players. How many different basketball team rosters could there be?	{(1,2,3,4,5,6,7), (1,3,5,7,9,11,12), …}
Category II	*An ordered sequence of two (or more) indistinguishable objects*	*Coin Flips Problem.* Fred flipped a coin 5 times, recording the result (Head or Tail) each time. In how many different ways could Fred get a sequence of 5 flips with exactly 3 Heads?	{(HHTTH), (HTHHT), (TTHHH), …}

In light of various ways of encoding outcomes that facilitates the use of combinations, we point out that it may *seem* that combinations are actually being used to solve two very different kinds of problems. The outcomes in the Books problem are clearly unordered sets of distinct objects, but the outcomes in the Coin Flips problem are actually *ordered* sequences (not unordered) of two kinds of *indistinguishable* (not distinct) objects (Hs and Ts). Combinations are

[3] We can use Lockwood's (2013) language to describe this phenomenon, in which two different counting processes might yield the same expression. Indeed, the point is that just by looking at a particular formula or expression that a student writes for a problem, it may require some interpretation as to what their underlying counting process to solve the problem might have been. We can hypothesize and look for supportive evidence, but it is possible for multiple counting processes to yield identical expressions.

applicable in both situations, but we argue that there could be a difference for students in identifying both problems as counting combinations. Indeed, although both categories can be, and frequently are, thought of in terms of counting (choosing) sets, using combinations to solve Category II problems involves an additional step of properly encoding the outcomes with a corresponding set of distinct objects. We thus posit that Category I problems may be more natural for novice students, more clearly representative of combination problems than Category II problems. In spite of the widespread applicability of combinations, we posit that students may not recognize both categories of combination problems as problems involving combinations. This may be due in part to the fact that students tend not to reason carefully about outcomes (e.g., Lockwood, et al., 2015b), and because "distinguishable" and "unordered" are not always natural or clear descriptions of the situation or outcomes.

However, we note that it is important and useful for students to be able to solve these Category II problems (and use combinations to do so) because combinations often arise as a *stage* in the counting process. Furthermore, using combinations in this way can facilitate productive and efficient solutions. For example, consider a problem such as *Passwords consist of 8 upper-case letters. How many such passwords contain exactly 3 Es?*. This problem can be solved by using combinations as a stage in the counting process – first we select 3 of the 8 positions in which to place Es (there are C(8,3) ways to do this), and then we fill in the remaining position with any of the 25 non-E letters (there are 25^5 ways to do this). Thus, the two-stage process yields an answer of $C(8,3)*25^5$. Here, we recognize that a Category II combination problem can help us complete the first stage of the counting problem. Importantly, realizing that a combination is useful in the first step makes the problem much easier than if one were to try to only to use permutations to do so. If we tried to answer the Passwords problem without first selecting places for the Es, and instead took a permutation approach, it would be possible, but it would require a large number of complex case breakdowns.[4] Thus, we argue that being able to solve Category II problems by using combinations demonstrates the utility of combinations as a powerful tool, and yet it also represents a sophisticated understanding of what combinations count, because it involves encoding the set of outcomes in a particular way. Given that the ability to solve Category II combination problems may allow students to solve a wide range of problems, and that it can reinforce a more complete understanding of what binomial coefficients can do, we are motivated us to investigate whether or not students actually respond differently to the two different problem categories.

Methodology

We designed two versions of a survey, and although the surveys contained a number of elements, we focus in particular on features of the survey that serve to answer the research question stated above. Each survey consisted of 11 combinatorics problems, and each problem was designed with categories in mind that included problem category (I or II) and complexity (Simple, Multistep, or Dummy).[5] *Simple* combination problems refer to those that can be solved using a single binomial coefficient, in the sense that their outcomes can be appropriately encoded as sets of distinct objects; *multistep* combination problems would require multiple binomial

[4] For example, we would have to consider how many of the non-Es are distinct and then arrange them. So we would have to account for cases in which a) all of the non-E letters are identical, b) exactly one of the non-E letters is identical (and the rest are distinct), c) exactly two of the non-E letters are identical (and the rest distinct), there are exactly two pairs of identical non-E letters, and so on. This process can generate the right answer, but it is complex.

[5] We also coded the tasks according to other criteria that we do not report on in this paper, such as: sense of choosing (Active or Passive), and whether an object or process is to be counted (Structural or Operational).

coefficients in the solution (see Table 3 for problems in Survey 1 and the Appendix for Survey 2). The authors coded the problems independently before finalizing the coding for each problem. Each version of the survey contained the same number of problems of each category and complexity, as well as the same two "Dummy" problems to discourage students from assuming that every problem could be solved with a combination. Each problem was selected for one version of the survey with a companion problem in mind for the other version in order to compare responses with respect to the various coding categories. In this report, however, we primarily use the two surveys as additional support that the phenomenon observed is not limited to particular problems on one survey, but is consistent across a larger variety of these different categories of problems.

Table 3: Survey 1

Description		Problem
Category I	**Simple**	*There are 12 athletes who try out for the basketball team – which can take exactly 7 players. How many different basketball team rosters could there be?*
Category I	**Simple**	*There are 8 children, and there are 3 identical lollipops to give to the children. How many ways could we distribute the lollipops if no child can have more than one lollipop?*
Category II	**Simple**	*There are 3 green cubes and 4 red cubes. Sam is making "towers" using all of the 7 blocks by stacking the cubes on top of each other. How many different "towers" could Sam make?*
Category II	**Simple**	*Computers store data using binary notation - an ordered sequence of 0s and 1s. A particular piece of computer data is 95 digits in length, and it has exactly 12 1s. How many possible sequences fit this constraint?*
Category II	**Multistep**	*Stella is stacking ice cream scoops onto a cone. She has 3 scoops of chocolate, 5 of vanilla, 2 of pistachio, and 6 of strawberry. How many different ways can she stack all of the ice cream scoops onto the cone?*
Dummy		*In Montana, a license plate consists of a sequence of 3 letters (A-Z), followed by 3 numbers (0-9). How many different possible license plates are there in Montana?*
Category I	**Simple**	*There are 12 points, all different colors, drawn on a sheet of paper (and no three points are on a line). How many different possible triangles can be made from these 12 points?*
Category I	**Simple**	*Bob got a new job and is at a store looking for new ties. The tie rack has 196 different ties to choose from. In how many ways can Bob select 10 ties to buy?*
Category II	**Simple**	*Fred flipped a coin 25 times, recording the result (Head or Tail) each time. In how many different ways could Fred get a sequence of 25 flips with exactly 11 Heads?*
Category I	**Multistep**	*There are 8 females and 10 males who would like to be on a committee. How many different committees of 6 people could there be if there need to be exactly 2 females on the committee?*
Dummy		*From an Olympic field of 15 athletes competing in the 100-meter race, how many different possible results could there be for gold, silver, and bronze medals?*

We targeted Calculus students because they were believed to have been likely to have encountered combinations at some point in their mathematical careers (perhaps in middle or high school) without having studied them in detail, making them informed but novice counters. Although we collected some additional demographic information about their previous mathematics courses and experiences with combinatorics, we have yet to incorporate this into our analysis. Overall, 281 people started the survey; however, many of these did not answer a single question on the survey, leaving n=126 people (65 for Survey 1, 61 for Survey 2), who responded to at least one of the combination questions. We included a reminder in the beginning of the survey that offered a brief standard overview of combinations and permutations. The goal was for students to be reminded of the formulas and notation for combinations and permutations that they likely had encountered previously at some point in their education.

The prompt for the combination problems asked participants to use notation that would suggest their approaches, rather than just numerical values. We gave this prompt because we wanted to be able to know what the student's counting process was, and so we would not have to make inferences based only on numerical responses. Specifically, the prompt was:

> Read each problem and input your solution in the text box. **Please write a solution to the problem that indicates your approach.** If you're not sure, input your best guess. NOTE: Appropriate notation includes: 9+20, C(5,2), 5C2, 21*9*3, 5*5*5*5=5^4, 8!, 8!/5! = 8*7*6 = P(8,3), C(10,2)*3, Sum(i,i,1,10), 12!/(5!*7!), etc. *Only* if you individually count *all* of the outcomes should you input a numerical answer, such as 35.

In addition, for three of the problems, participants were prompted to expand particularly on how they understood the set of outcomes they were counting (see Appendix for this prompt.) The intent was to provide further evidence from the students about their counting process and the set of outcomes. Notably, however, many students had difficulties with these instructions, frequently providing numerical answers in their initial responses. We discuss further our analysis approach.

In general, to investigate the research question we wanted to compare their solutions to the different categories of combination problems. Thus, we coded each response in three different ways. First, we coded whether the response was "correct" or "incorrect." In this case, *any* problem the participants answered was determined to be correct or incorrect (with two exceptions: if the notation they used was not standard, e.g., "C(3,5,2,6)," – which happened four times – or their answer was unclear, e.g., "a small fraction of 95^{95}," to where we could not judge correctness – which happened five times). Importantly, *many* of the participants only included numerical answers, for which we could just confirm correctness, but not process. Therefore, secondly, for those participants that wrote a solution indicating their approach (i.e., followed the instructions) we coded the "definite" method that characterized their solution. If they used a combination, we coded whether they used the combination correctly (CC) or incorrectly (CI); if their answer involved a permutation, we coded it a P; if their answer was essentially multiplying numbers, we coded it M; if it only involved factorials that were not in a permutation or combination formula, we coded it F; if it involved exponents, we coded it E, if they just summed numbers, we coded it S; if they just used a single number from the problem, we coded it N; and if it was did not fall into any of the previous categories, we coded it O.

Notably, for three of the problems, participants had an opportunity to explain more in relation to the sets of outcomes. For some participants who wrote numerical expressions, it was here that they explained their process. Therefore, when the numerical answer and their process matched up in these cases, we included a "definite" method based on their explanation. As mentioned previously, from some solution responses the method would not be completely clear. Particularly considering combinations, we had to determine the meaning of a response such as: "8!/(5!3!)." Thankfully, there were not many such instances. For participants who appeared to be using the verbatim combination formula, such as "8!/(5!(8-5)!)," we coded this as a combination; however, for participants who wrote "8!/(5!3!)," if there was a clear indication from the set of outcomes responses or if they had used combination notation on other problems, e.g., C(20,10), we presumed their response to be indicative of a permutation approach.[6] Thirdly, if numerical answers seemed to have a clear process – such as the problem including the numbers 8 and 5, and participants answering 40 – we coded the "probable" method that characterized their response.

We consider this last coding involving "probable" codes as the best set of codes for our analysis for two reasons. First, some students gave correct numerical responses, such as 4,457,400, which we viewed as most likely indicative of using C(25,11), because coming up with this answer in some other way would be very difficult. This allowed us to include C(25,11) as their probable method – to the extent possible, we wanted to give students the benefit of the doubt. Second, because so many responses were only numerical, this also allowed us to include some more responses in the analysis for which we could be fairly sure of their method.

For the purposes of this paper, we limit our analysis only to the most basic combination questions – the four *simple* Category I problems and the three *simple* Category II problems on each survey. These seven problems provide us with the most basic comparison between their responses to these two categories of problems. So, using the third coding ("probable" method), first we computed the proportion of instances on which participants used a combination approach at all for Category I problems, and the proportion of instances on which participants used a combination approach at all for Category II problems, and compared these two proportions. We also computed the proportion of these responses that were correct uses of a combination approach. Notably, however, each individual participant answered up to seven such questions. Thus, for each individual, we secondly computed whether or not that individual had used a combination approach at all on at least one Category I problem, and whether they had used a combination approach at all on at least one Category II problem. We also computed the proportion of participants that had used the combination approach correctly on at least one of each category of problem. This allowed us the ability to compare the proportion of distinct participants (not distinct problems) who had used a combination approach correctly on the differing problem categories. Thirdly, given that many participants' methods were significantly off track – i.e., many simply multiplied numbers in the problem without ever using a different method – we limited our participants to only those that had used a combination approach on at least one problem. This allowed us to probe further whether participants that at least found a combination approach useful on one problem were differentiating between the two problem categories. For each proportion, we used a proportion *t*-test to determine whether the proportions were significantly different, and then used Cohen's *h* to determine the relative effect size of the differences. (We used standard cutoffs from Cohen (1988) of: 0.2, small effect; 0.5, medium; and 0.8, large.) Notably, when separating the analysis by survey, the results in every analysis were similar, and so we present the combined analysis across both surveys.

Findings

In this section we present the different analyses of our data, which all support the singular finding that students do indeed use a combination approach more regularly on Category I problems than they do on Category II problems. We regard this result as indicating that from a

[6] Indeed, one participant (1009), had correctly responded "20C4" to one problem, explaining it as, "You choose any 4 out of 20. Order is irrelevant," and had responded "8!/((5!)*(3!))" on another problem, explaining it as, "It's like having five dots and placing 3 dots in between the five. This is similar to arranging them in any way but not caring how they are arranged among themselves."

learners' perspective there is a meaningful difference between these two categories of basic combination problems.

Overall Participant Responses. Of the 126 participants, there were 117 of which we were able to give a "probable" approach code on at least one Category I problem that they answered, yielding 380 total responses to Category I problems for which we probably knew the participants' approach. There were 116 participants for which we were able to give a "probable" approach code on at least one Category II problem, yielding 261 total responses to Category II problems for us to consider. Table 4 indicates the results. The data show that about 24% of the responses to Category I problems used a combination approach, whereas only 16% of responses to Category II problems did – a significant difference, but with a small effect size. Interestingly, most of the combination approaches to Category I problems were correct (only 2/90 were incorrect), whereas about one-quarter of the combination approaches to Category II problems were incorrect (11/43 were incorrect). We see this as another indication that even if a student attempts to use a combination approach on Category II problems, they are doing so in ways that are, in fact, incorrect. Similarly, when isolating unique participants, we also find a statistically significant difference, with relatively small effect size, between the proportion of participants that were using a combination approach on Category I compared to Category II problems.

Table 4. Comparison of Overall Participant Responses

	Category I Simple		**Category II Simple**		***p*-value**	**Cohen's *h***
On what proportion of **responses** is a combination approach **used at all**?	90/380	~24%	40/261	~15%	p<0.01	0.212 (Small)
On what proportion of **responses** is a combination approach used **correctly**?	88/380	~23%	29/261	~11%	p<0.001	0.324 (Small)
What proportion of **participants** used a combination approach **at all** on at least one problem?	38/117	~32%	24/116	~21%	p<0.05	0.268 (Small)
What proportion of **participants** used a combination approach **correctly** on at least one problem?	37/117	~32%	18/116	~16%	p<0.01	0.385 (Small)

Reduced Participant Responses. Since many participants never used a combination approach on any problem in the survey, as is evident from the overwhelmingly small proportions in Table 4, we reduced our participants to only those 42 who had used a combination approach on at least one problem – either Category I or Category II. These 42 participants yielded 153 responses to Category I problems, and 114 responses to Category II problems. Table 2 indicates the results. Notably, for even these seemingly more knowledgeable participants there is a significant difference in the use of a combination approach to these two categories of problems. In fact, by reducing our population to only those who at least appear to have some idea that a combination might be a useful approach to solve a counting problem, we see larger effects in the differences. Given that these participants most closely match our desired population of participants – students who had likely been introduced to combinations but not studied them extensively, and who can use combinations appropriately in some settings – we see these results as the most telling.

Table 5. Comparison of Reduced Participant Responses

	Category I Simple		Category II Simple		*p*-value	Cohen's *h*
On what proportion of **responses** is a combination approach used **correctly**?	88/153	~58%	29/114	~25%	$p<0.001$	0.664 (Medium)
What proportion of **participants** used a combination approach **correctly** on at least one problem?	37/42	~88%	18/42	~43%	$p<0.001$	1.009 (Large)

Discussion

Despite the fact that both Category I and Category II problems could be naturally encoded as combination problems, our findings suggest that the participants do not view the problems in this way. We found statistically significant differences in students' use of combinations to solve Category I versus Category II problems. In this way, our study offers quantitative evidence of what had been an anecdotally observed phenomenon.

In terms of the model of students' combinatorial thinking (Lockwood, 2013) and the set-oriented perspective toward counting (Lockwood, 2014), this study suggests that students are not recognizing that outcomes of Category II problems can be appropriately encoded as *sets* of objects. That is, even though natural bijections exists that would allow students to leverage binomial coefficients in a variety of contexts, our research suggests that students are either not aware of this fact or are not able to use that bijection to encode outcomes effectively. Indeed, a large majority of students were simply not able to answer Category II questions in a correct manner, regardless of the method – there were no other correct non-combination responses to the Category II questions, except one participant who correctly solved three Category II questions in a permutation manner. However, we want to acknowledge that it is not necessarily surprising that students would struggle to see this distinction. Indeed, familiar descriptions of "unordered" and "distinct" do not seem to apply – at least in the most natural way to model the outcomes. Students can tend to associate counting with key words, specific contexts, and mantras like "order doesn't matter," and they tend not to think about counting in terms of the outcomes they are trying to count (Lockwood, 2014). If this is a student's perspective on counting, it would follow that they would not be attuned to the importance of encoding outcomes and might not realize that they have the flexibility to encode outcomes in creative ways. Our study thus offers further evidence that students would benefit from focusing on the nature of the outcomes as the determining factor in what counting processes (and, ultimately, formulas) are most appropriate in a given situation.

Conclusions, and Implications

In sum, students may need additional exposure to combinations and may benefit from explicit instruction about how Category II problems can be encoded in a way that is consistent with Category I problems. Generally, this point underscores a need for students to become more adept at combinatorial encoding (Lockwood, et al., 2015b). Encoding outcomes as sets is an inherent part of the field of combinatorics, but students may need particular help in making this connection explicit. Also, these findings provide evidence for the fact that it may not be productive for students to be exposed to formulas initially if they are not pushed to understand those formulas.

In terms of implications for instruction, then, we feel that teachers should explicitly direct students toward focusing on what they are trying to count. This means thinking of combination problems not exclusively as those problems whose outcomes can be encoded as sets of objects. Given how difficult (and seemingly unnatural) it is for students to encode outcomes of Category II problems, instructors may need to give examples of ways to encode outcomes of Category II problems and to clearly establish relevant bijections. Discussing the relationship between how one models the set of outcomes and the pertinent solution approach may also be particularly meaningful in this context. For example, listing outcomes as 5 Hs and 3 Ts might lead to a permutation approach of 8!/(5!3!), whereas further encoding the outcomes in terms of the 5 distinct positions, for which three will be heads, might lead to C(5,3) as the natural solution approach. This is not to claim one approach as preferential over another, but we regard having both the flexibility to see different solution approaches as viable in this situation, as well as connecting sets of outcomes with particular solution methods, as highly important in developing an understanding of combinatorics. This might mean that instructors should first familiarize themselves with this distinction and to be able to understand and articulate what the distinction is between these two categories and why students might perceive them as different. Being able to use binomial coefficients flexibly and in a variety of settings can be a powerful tool for solving enumeration problems, but without a robust understanding – including how and why Category II problems can be solved using them – students may possess a tool they do not really understand how to use. As instructors, we should invest time and energy in helping students to understand this tool and the various ways in which it can be effectively used.

There are natural next steps and avenues for further research. We plan to investigate more questions and hypotheses with the data we have, such as analyzing effects of demographic data and investigating other relationships and potentially contributing factors in students' responses. We also want to explore the multistep problems more, and we wonder if perhaps the multistep problems might be similar in some way to the Category II problems, because both of these use a binomial coefficient as part of a process as opposed to the complete set of solutions. We could also see investigating similar kids of questions with counters with more experience than the novice calculus students, such as discrete mathematics or probability students. Our findings also indicate that further investigating students' reasoning about encoding with combinations through in-depth interviews may give insight into the development of more robust understandings.

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Appendix

Prompt 2: "On the previous page, you entered your solution to different combinatorics problems. **On this page, we would like you to expand upon how your solution is related to the "set of outcomes" for a few of the problems.** That is, we want you to list **some (but not necessarily all)** of the outcomes that you are counting. Explain your thinking for your solution and its relation to what is being counted.

For example, for the problem, "If we have four distinct toy cars (Red (R), Blue (B), Green (G), and Yellow (Y)), how many different subsets of 2 of them are there?", your solution to the problem might have been: C(4,2). On this page, the intent is to expand on how that solution, C(4,2), relates to the set of outcomes. You might write something like, "The set of outcomes includes the following pairs of cars: BR, RG, GY, BG. I used the combination C(4,2) because the outcomes were "pairs" (2) from the 4 different colored toy cars. I did not include RB because this would be the same as BR in this case."

Table: Survey 2

Description		Problem
Category I	**Simple**	*You are packing for a trip. Of the 20 different books you consider packing, you are going to select 4 of them to take with you. How many different possible combinations of books could you pack?*
Category I	**Simple**	*There are 9 justices on the Supreme Court. In theory, how many different ways could the nine justices come to a 7:2 vote in favor of the defendant?*
Category II	**Simple**	*Stella is ordering an ice cream cone that is 8 scoops tall. She orders 5 chocolate scoops and the rest vanilla. How many different ways can the employee stack the ice cream scoops?*
Category II	**Simple**	*There are 55 elementary students standing in a line. The teacher has 35 identical red balloons and 20 identical blue balloons, and gives each student either a red or a blue balloon. How many different outcomes are possible in this process?*
Category II	**Multistep**	*Sam is making "towers" from 3 green, 4 red, 2 yellow, and 8 orange blocks. Using all 17 blocks, how many different "towers" could Sam make?*
Dummy		*In Montana, a license plate consists of a sequence of 3 letters (A-Z), followed by 3 numbers (0-9). How many different possible license plates are there in Montana?*
Category I	**Simple**	*There are 15 people in a room. Everyone shakes hands with everyone else. How many different handshakes take place?*
Category I	**Simple**	*There are 250 kittens at a shelter. Sally is adopting 6 of them. In how many ways could she adopt 6 kittens?*
Category II	**Simple**	*A professor writes a 40-question True/False test. If 17 of the questions are true and 23 are false, how many possible T/F answer keys are possible?*
Category I	**Multistep**	*There are 19 students in your class. How many ways are there to split the class into 3 different groups - one group of size 5, another of size 6, and another of size 8?*
Dummy		*From an Olympic field of 15 athletes competing in the 100-meter race, how many different possible results could there be for gold, silver, and bronze medals?*

When we grade students' proofs, do they understand our feedback?

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Instructors often write feedback on students' proofs even if there is no expectation for the students to revise and resubmit the work. It is not known, however, what students do with that feedback or if they understand the professor's intentions. To this end, we asked eight advanced mathematics undergraduates to respond to professor comments on four written proofs by interpreting and implementing the comments. We analyzed the student's responses using the categories of corrective feedback for language acquisition, viewing the language of mathematical proof as a register of academic English.

Keywords: Proof Writing, Proof Grading, Proof Instruction, Proof Revision, Student Thinking

Introduction and Research Questions

Rav (1999) claimed that proofs "are the heart of mathematics" and they play an "intricate role […] in generating mathematical knowledge and understanding" (p. 6), and multiple authors have claimed that proof is how the discipline grows (cf., Lakatos, 1976; de Villiers, 2003). Consequently, proof is perhaps the dominant feature of the advanced undergraduate mathematics curriculum. For example, Mills (2011) estimated that approximately half of all class time consists of instructors presenting proofs at the board while the presentation of definitions, examples, algorithms and theorems (as well as explanations of the same) together constitute less than half of class time. Similarly, student homework and exams in advanced mathematics classes generally include a number of prompts asking them to produce written proofs of given claims (cf., Fukawa-Connelly, 2015). That is, as Weber stated, developing proof proficiency "is often the primary goal of advanced mathematics courses and typically the only means of assessing students' performance" (2001, p. 101).

While proof proficiency may be the primary goal of advanced undergraduate mathematics courses, a significant body of research has demonstrated that students have great difficulty writing proofs, and a body of writing about mathematicians' personal teaching experiences supports this claim as well. For example, Epp (2003) claimed:

> Often their efforts consisted of little more than a few disconnected calculations and imprecisely or incorrectly used words and phrases that did not even advance the substance of their cases. My students seemed to live in a different logical and linguistic world from the one I inhabited, a world that made it very difficult for them to engage in the kind of abstract mathematical thinking I was trying to help them learn. (886)

Due to space constraints, we do not report all of this research here, rather we note that there are multiple handbook chapters (e.g., Harel and Sowder, 2007) describing the literature, and a comprehensive summary of undergraduate mathematics majors' difficulties with proof is given in Selden and Selden (2008). This difficulty is further illustrated by the emphasis on reforming proof instruction as evidenced by the large number of "how-to" textbooks on proofs (e.g., Chartrand, Polimeni, & Zhang, 2012; Franklin & Daoud, 2011; Smith, Eggen & St. Andre, 2014), and practitioner articles about improving student proof writing (e.g., Strickland & Rand, in press; Zerr & Zerr, 2011). Of particular importance to the present study is Moore's (1994) claim that students are unfamiliar with the language of mathematical proof. Yet, the language of proof writing is largely unstudied, and thus, there is little information about how mathematicians understand this language; instead, we rely on individual reflections and theoretical analyses.

We argue that the language of proof is a particular academic register of English. A register is a variety of language used for a particular purpose or within a particular social context. Specifically, it "is the set of meanings, the configuration of semantic patterns, that are typically drawn upon under the specific conditions, along with the words and structures that are used in the realization of these meanings" (Halliday, 1978, p. 23).To support our claim that the language of mathematical proof constitutes a register, we note that Scarcella (2003) claimed that academic English was a particular register "used in professional books and characterized by the specific linguistic features associated with academic disciplines" (p. 9) and that each academic discipline had its own particular subregister. Halliday (1978), Pimm (1987) and Moschkovich (1999) have similarly argued that the language of mathematics constitutes a particular register. Thus, "we can refer to a 'mathematics register,' in the sense of the meanings that belong to the language of mathematics [...], and that a language must express if it is being used for mathematical purpose" (Halliday, 1978, p.195).

When considering how students might learn and use this register, exposure and practice is certainly important (Pinker, 2009). Yet, without feedback on this practice, students are unlikely to improve their proof writing. Moreover, mathematicians act on the belief that giving students feedback is critical to their learning by making marks and notes on student proof productions (e.g., Hattie & Timperley, 2007; Moore, 2016). Yet, this feedback improves student learning only if students read, make sense of, and incorporate it into their future work. Few, if any, studies have explored students' understanding of this process of giving feedback as a means to improve their proficiency. Thus, in this study we investigate the following questions:

1. What do students claim to do with the professor's feedback on their proofs?
2. How do students interpret and explain the rationale for the professor's marks and comments on student-written proofs?
3. How do students' responses to the professor's comments align with the way that is normative in the discipline, as described by mathematically enculturated individuals?

Literature and Theory

Theoretical orientation

The theoretical orientation for this study is acquisition of language through which we consider the register of mathematical proof. In the case of proof in an advanced undergraduate mathematics classroom, there is an interesting duality: there are the proofs that the professors present and those that the students produce. Professors report that they often present proofs with small gaps in them; giving multiple reasons for doing so, such as not allowing technical details to obscure the big ideas, conserving limited class time, and providing the gaps as learning opportunities for students to help them better understand the content and support their proof writing (Lai & Weber, 2014; Lai, Weber, & Mejia-Ramos, 2012).

Language acquisition

While there is little research on how students learn formal mathematical language, there is a significant body of research on language acquisition (both first and second language acquisition) which has relevance to the acquisition of a register. Pinker (2009) described four criteria for a language, or in this case, a register, to be learnable:

1. From among the class of all possible languages, the target needs to be identified.
2. Learners need an environment in which to learn it.
3. Learners need a learning strategy, meaning a learner-created algorithm that uses information in the environment to create "hypotheses" about the target language, and then to determine whether they are consistent with the input information from the environment.
4. Learners determine a success criterion, meaning the hypotheses are related in some systematic way to the target language. A learner may waver among a set of hypotheses, one of which is correct. They may arrive at a hypothesis identical to the target language, or they may arrive at an approximation to it.

All of these are fulfilled by the advanced mathematics classroom: there is a particular register to be acquired, an environment in which to learn it, means for trying out usages of the language (proof writing), and a success criterion (sufficient fluency to earn passing marks).

Pinker continued by noting that language learning was a case of induction—developing uncertain generalizations from the observed instances of the language used in the environment. In the case of the formal mathematical register, students often get specific instruction describing the use of some phrases as well as the syntax, and they observe their professors using the register, although mixed with colloquial mathematics (Lew, et a., 2016). Moreover, professors frequently write only the algebraic steps of a proof on the board while saying aloud many of the connecting phrases and logical underpinnings (Fukawa-Connelly, 2012). In contrast, when reading proofs the professors focus on the phrases and logical descriptions in the written language, acting as if the algebraic manipulations are of secondary importance (Hodds, Alcock, & Inglis, 2014). Thus, the sample of the register on which the students are inducting is problematic and suggests an over-importance of the symbolic argument, thus increasing the number of generalizations likely to be present in the superset of the language that the students might develop. It is from these uses that students must develop hypotheses about the symbols, phrases and syntax, and yet as noted above, unlike in colloquial language, formal mathematical language must be unambiguous and correct, thus increasing the challenge for students.

Most important, language acquisition research suggests vital roles for learners' language production, including that learners "try out new language forms and structures as they stretch their interlanguage to meet communicative needs; they may use output to see what works and what does not" (Swain, 1998, p. 68). That is, students learning the mathematical register will write proofs in which they will commonly use correct grammar and syntax but also incorrect grammar and syntax, and they do this, partially, in order to try to learn the language and solicit feedback. This testing of language is not typically a conscious goal, rather we appear to be biologically disposed to do so. While to a mathematical expert, novice proof productions are filled with errors of logic, grammar, syntax and more, under the lens of language learning we instead view them as attempts to communicate in the style of the community where the rules are, at best, partially mastered and often tacit. On the basis of this research, we assert that professors' feedback on student proof productions has perhaps an unparalleled role in students' learning to produce proofs.

The role of feedback in language acquisition

We adopt Leeman's (2007) definition of feedback, meaning a mechanism that provides a learner with information regarding the success or failure of a given production, such as a written proof. Much research has focused on negative feedback, which is feedback that in some way indicates an error or issue with the production (Gass, 2008; Herschensohn & Young-Scholten, 2013). In particular, Lyster and Ranta (1997) described six types of corrective feedback, each of which can also be seen as indicating issues with the learner's production. Their types of feedback included:

1. Explicit correction: the teacher indicates that the utterance is incorrect and provides the correct form.
2. Recast: the teacher reformulates the student's production and remediates the error or provides the correction, such as writing out the correct sentence, without specifically indicating the error in the student's work.
3. Clarification request: the teacher asks the student to reformulate their production without specifically indicating what the error is.
4. Metalinguistic clues: the teacher poses questions, typically with a yes/no answer, or gives comments related to the student's production but does not provide the correct form or specifically indicate the error (e.g., Do we use this symbol here?").
5. Elicitation: the teacher directly elicits the correct form from the student by asking questions or starting the phrase. These questions typically expect more than a yes/no answer, such as, "Where do we use this hypothesis?"
6. Repetition: the teacher repeats the student production but highlights the error in the delivery (e.g., highlighting).

Lyster and Ranta found that recasts and explicit corrections did not result in subsequent improvement in student productions and hypothesized that it was because students are then limited to repeating the correct form the teacher provided (Tedick & de Gortari, 1998). Gass (2008) argued that explicit negative feedback can only teach about surface-level phenomena and cannot teach abstraction. Consequently, students may use the correct grammar and syntax in the future but without developing reasons to support their choices. The other five types of responses do not provide learners with the correct form and require that the learners provide it. Lyster and Ranta claim that when learners must restate, they are having to actively engage with the feedback, which is a critical feature in learning from feedback.

Student and mathematician misunderstandings

While we argue that professor comments on student proof productions have a significant role in students' learning to produce proofs, we also have reason to believe that students are likely to misinterpret them. Research on student understanding of lectures suggests that students sometimes develop significantly different understandings of the presented material and meaning for professor actions than the professor intends (cf. Lew, et al., 2016; Weinberg, Weisner, & Fukawa-Connelly, 2014). The work of Ko and Knuth (2013) and Selden and Selden (2003) has shown that students often fixate on the form, such as the presence of mathematical symbols, rather than the content of the proof, when reading mathematical arguments.

We use this prior research on misunderstandings and “misses of understanding” to form hypotheses about how students are likely to interpret a professor’s comments on proof productions. (To learn about the kinds of comments professors leave on proofs, see Moore, 2016). In particular, we hypothesize that they are likely to:

- not apprehend some feedback,
- develop only a surface-level understanding of some feedback, and
- interpret feedback in ways that differ from what mathematical experts would do.

Moreover, we argue that the latter two of these hypotheses are supported by the language acquisition literature described above.

Methods

Participant selection

The participants were 8 students, 4 men and 4 women, with advanced undergraduate standing at two institutions, four from each institution. Each participant had taken at least two proof-based undergraduate mathematics classes, including a transition-to-proof course. We purposely selected participants who had experience with writing proofs and receiving feedback from their professors so as to give the best possible chances for their success in understanding the proofs and interpreting the professor’s comments in this study.

Data collection

We engaged each participant in a 90-minute task-based interview where the primary task was to describe and interpret a professor’s comments about proofs. The interviews were audio-recorded and pencast with Livescribe pens. Each interview began with basic demographic questions and reflective questions about the participant’s typical use of their professors’ feedback, as follows:

1. What is your major(s) and which college-level math courses have you taken? Which math courses have emphasized proof writing?
2. When a professor returned homework papers, how often were you asked to revise and resubmit your work?
3. When you were not asked to revise and resubmit, what did you do with the feedback your professors gave you? Can you provide a specific example?
4. If you were asked to revise and resubmit, what did you do with the feedback?
5. Do you think it is important to read the comments on your proofs? Why or why not?

Subsequently, the main part of the interview consisted of a sequence of three or four proofs, as time allowed. We ensured that each proof had a mix of professor marks and comments related to notation and presentation, and logical issues. The proofs and professor comments were taken from a previous research project exploring four mathematics professors’ proof grading practices (Moore, 2016). From among the various marks and comments that the professors in Moore’s study wrote on the proofs, we selected the ones that appeared on the proofs in the present study. An example proof, Proof A, with comments, is shown in Figure 1. (Note that the participants did not see the numbers beside the comments; we inserted them later for our convenience in referring to the comments).

First, we handed a proof to the participant, told her it had been written by a student, and asked her to read and understand it as best she could. Next, we presented a marked proof to the participant with a professor’s feedback written in red ink. To determine whether the participant’s interpretation of the mark or comment matched our own, we asked the participant to explain why the professor had written each individual mark or comment and what changes she thought the professor wanted. Finally, we asked the participant to rewrite the proof in order to allow us to further explore her interpretation of the comments and see how she implemented the professor's recommendations.

Theorem A. Transitive Relation

Define a relation R on the set of real numbers by $x\ R\ y$ if and only if $x - y$ is an integer, that is, two real numbers are related if and only if they differ by an integer. Prove that R is transitive.

Proof A.

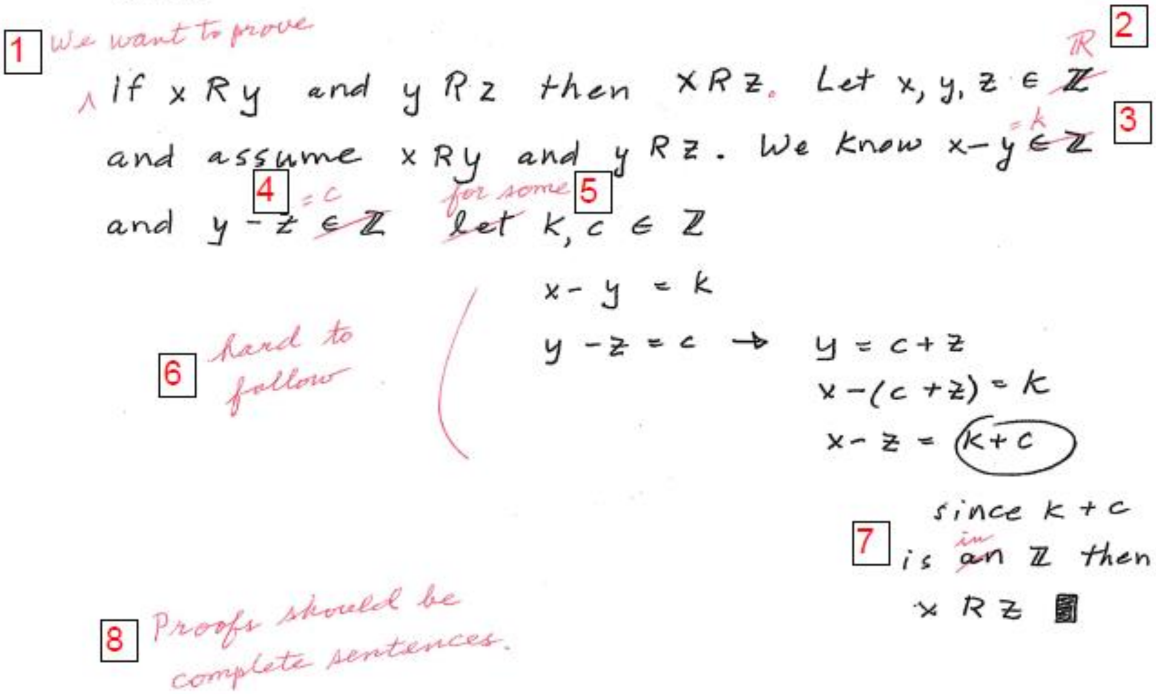


Figure 1. Proof A with the professor's comments.

Data analysis

For each comment in each proof, each of the researchers wrote a description of what change he or she believed the professor wanted and a rationale for the change. Based on these individual notes, we created a consensus description of what each mark and comment was asking the participant to change and the reason for the change. We also classified each comment using Lyster and Ranta's (1997) types of corrective feedback. Thus, each comment was classified along three dimensions: what change was requested, the reason for the change, and the type of corrective feedback.

We transcribed each interview and then chunked it at a number of levels. We parsed the demographic and reflective questions in one piece and the participant's discussion of each proof in additional pieces. We partitioned the discussion of each proof by identifying the participant's initial reading of the proof as a whole, and then her conversations about the individual comments. In cases where participants discussed multiple comments in the same utterance, we looked across interviews, and when it was common, we treated the comments as a single unit to parse all interviews similarly. We made a final block of the talk-aloud proof-writing process, for which we chunked the participant's utterances around the comments and linked those to what she wrote when she revised the proof.

To code the participant's utterances about each comment we first wrote a brief holistic summary, and then we developed a more detailed coding sheet that recorded:

- what the participant identified as the part of the proof the comment addressed,
- what the participant's response suggested should change in the proof,
- any reason the participant gave to explain the intended change and the underlying logic,
- a summary of what the participant changed in her revised proof,
- a comparison of each of the above points to our consensus expert interpretation, and
- an explanation of how an unanticipated change exhibited during the proof writing could be understood as a logical interpretation of the professor's comments.

We then created summaries, first by looking at individual proof comments across participants, and then by looking across the different categorizations of comments. For example, in the case of explicit corrections we described how the participants interpreted the requested changes, the kinds of reasons the participants provided for them, and the changes the participants included in their revised proofs. Similarly, we aggregated all of the changes that, according to our expert consensus, were recommended by the professor for a particular reason, such as "cultural convention," described the kinds of reasons the participants provided for them, and noted the changes the participants included in their revised proofs.

Results

Overall, the participants were very successful at interpreting what a professor wanted them to do in response to the comments. For example, for each of the eight comments on Proof A, 100% of participants correctly identified an acceptable part of the proof to be changed, and they all executed a change in a manner logically consistent with our understanding of the comment. However, their explanations of the rationale for the comments were not always consistent with our expert understanding. In the sections that follow, we explore the participants' work and thinking about the professor's comments on Proofs A and B.

Explicit corrections and recasts: feedback that specified the change

When the professor's comment specified a change to make to the proof, such as in an explicit correction or recast, the participants were consistently able to identify and state the change the professor wanted, and in the case of explicit corrections, what the professor considered incorrect. In their revised proofs, all of the participants consistently adopted the professor's suggested changes. For example, six of the professor's eight comments on Proof A (see Figure 1) were explicit corrections or recasts. Five of them indicated that something in the proof should be crossed out and replaced. In these instances, the participants always identified what they believed the professor wanted them to revise and implemented the revisions in a way that conformed with expert understanding.

Comment 1, the only recast comment on Proof A, was also specific but suggested the addition of new text, namely, the phrase "We want to prove," rather than the replacement of existing text. Seven participants added the recommended phrase, and one participant, Adam, showed some individuality by assuming that *xRy* and *yRz* and then writing, “We will prove that *xRz*,” which we judged to be consistent with our understanding of the professor’s comment.

In summary, when the professor’s comment was an explicit change of or an addition to the proof-text, the participants’ identification and implementations of the recommended changes were largely consistent with the expert consensus, or made a related change that addressed the intent of the comment. Thus, it appears that the participants demonstrated that they could appropriately identify and use explicit corrections and recasts on proofs.

Elicitations and clarification requests: feedback that did not specify the change

The professor’s comments on the four proofs included eleven elicitations and clarification requests. None of these comments gave specific directions about how to revise the proof. For example, comment 6 on Proof A, which says “hard to follow,” was a clarification request that did not indicate how to revise the proof. The participants generally interpreted this comment to mean that the algebraic steps lacked readability. Adam said about this comment:

> When I was reading the original proof, I guess it took me a little bit of time to follow how they laid it out, and maybe the layout could have been a little bit easier to follow. So maybe we should change some of this so it’s a little bit easier to get from certain points to certain points.

Our interpretation of Adam’s statement is that the comment asked for a change in layout in order to help a reader understand how the proof flows from point to point.

Bella and Don interpreted comment 6 differently and focused on the idea that the professor was asking for more detail. Bella focused on adding algebraic steps, whereas Don suggested clarifying and justifying the steps: “I would take that to mean use more detail, I suppose, describing what I’m actually doing at each step, even it were just to label, you know, 1, 2, 3, and then preferably say why 1, why 2, why 3.”

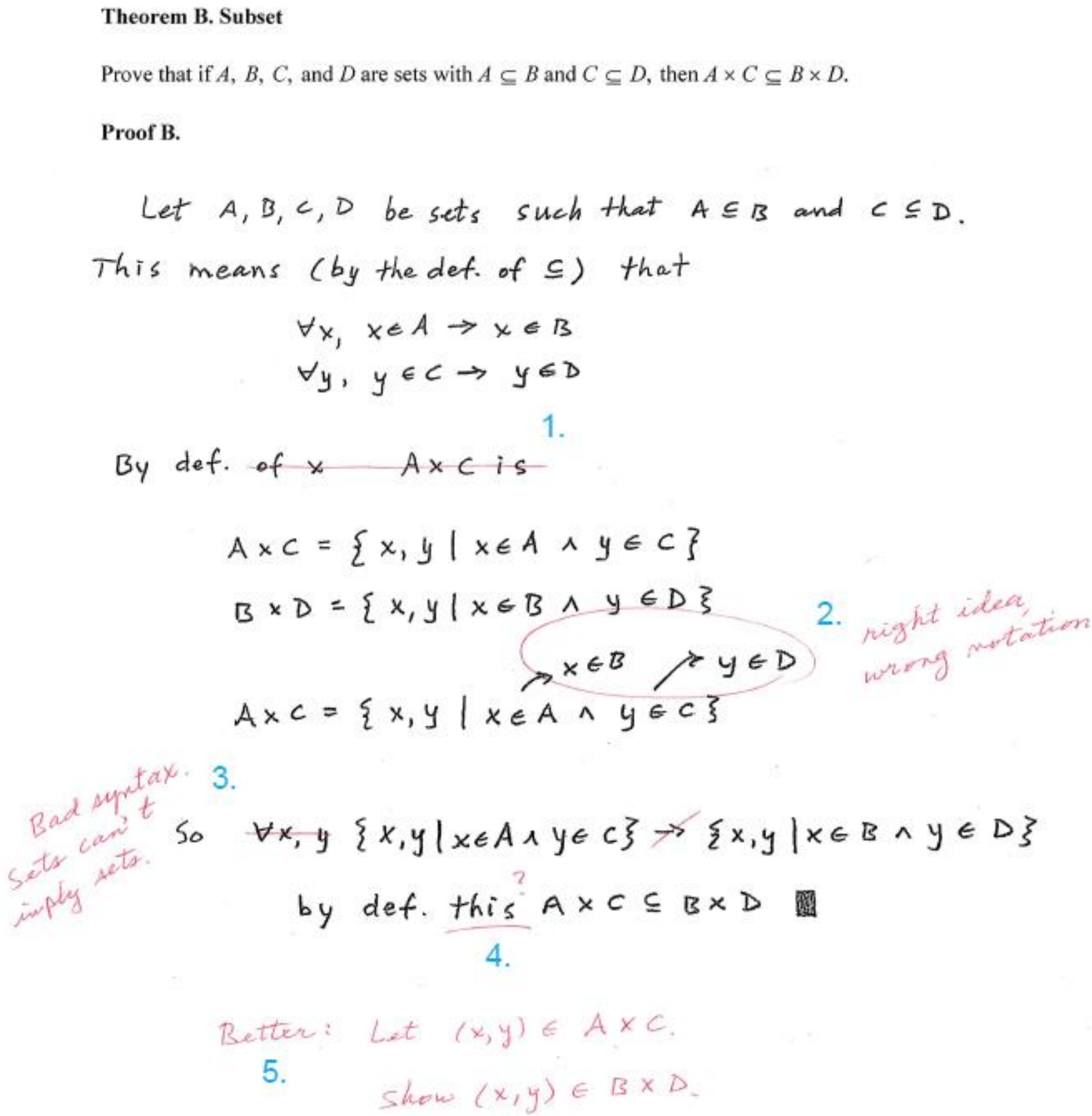
Two participants, Ruby and Nancy, noted specific changes that they would make to the proof. Ruby suggested that, “If there were sentences there, it wouldn’t be as hard to follow.” Nancy explained how she would add transition phrases and justifications, when she said:

> Normally you say more instead of just writing a bunch of random equations. ... You could say like “using substitution, we know this part” ... instead of just writing down random arrows and saying “Ha, ha! We got it!” … You can figure out what it’s trying to say, it’s not hard to follow in that sense, but it’s hard to follow in the sense of formal proof writing. It doesn’t have the normal, yeah, words, English.

Nancy has claimed that text is a “normal” part of proofs and a “bunch of random equations” is not appropriate, and her interpretation of the comment appears to be influenced by this belief.

On Proof B, shown in Figure 2, we classified comments 2 and 3 as clarification requests. For comment 2, the professor circled a part of the student’s work and wrote, “right idea, bad notation.” For comment 3, the professor crossed out two parts of the student’s proof and wrote, “Bad syntax. Sets can’t imply sets.” Note that in comment 2, the professor did not specify the error in what was written, and for neither of these two comments did the professor indicate how to revise the proof, which is consistent with the definition of a clarification request.

When the participants read comment 2, “right idea, bad notation,” all but one were able to identify that the professor was indicating that the student’s arrow notation was incorrect and needed to be changed, but most of them had difficulty articulating exactly what was incorrect and how to correct it. For example, Charles said, “I suppose this tells me they did it wrong. I would have to look up to see what the proper notation was, if I didn’t know what it was off hand.”



Theorem B. Subset

Prove that if A, B, C, and D are sets with $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$.

Proof B.

Let A, B, C, D be sets such that $A \subseteq B$ and $C \subseteq D$.

This means (by the def. of $\subseteq$) that

$\forall x,\ x \in A \rightarrow x \in B$

$\forall y,\ y \in C \rightarrow y \in D$

1.

By def. ~~of $\times$ $A \times C$ is~~

$A \times C = \{x, y \mid x \in A \wedge y \in C\}$

$B \times D = \{x, y \mid x \in B \wedge y \in D\}$

2. right idea, wrong notation

$\rightarrow x \in B \quad \rightarrow y \in D$

$A \times C = \{x, y \mid x \in A \wedge y \in C\}$

Bad syntax. Sets can't imply sets.

3.

So $\cancel{\forall x, y}$ $\{x, y \mid x \in A \wedge y \in C\} \cancel{\rightarrow} \{x, y \mid x \in B \wedge y \in D\}$

by def. this? $A \times C \subseteq B \times D$ ■

4.

Better: Let $(x, y) \in A \times C$.

5.

Show $(x, y) \in B \times D$.

Figure 2. Proof B with the professor’s comments.

Two participants suggested a means to deal with the comment by changing everything into written language without using symbols. Genevieve claimed, “that should be sentences instead of arrows.” Nancy noted that the text is “not written in a mathematically formalized way. It’s more written in a shorthand of reminding yourself” and suggested “actually writing it out in a way that says ... it in a mathematical proof kind of way.” Nancy’s response provided insufficient evidence to determine what she thought would make the argument more formal, but when asked, suggested writing out the inclusions in words: "Here I might write, 'by definition of subset,' and then write, 'x is in B and y is in D,' or something like that."

The participants' responses to comment 3 were similar to those for comment 2. They recognized the arrow between the two sets as problematic but did not give much more information. Bella stated, "It's not something you can do, it's not like, it's just maybe a rule." Multiple participants commented that the arrow indicated implication, such as Adam, "Yeah, the arrow means implies. Syntax. … you have to make sure you're using the right ones to mean what you want it to mean. And implies doesn't work for sets. You just don't do that." Genevieve said, "It's not grammatical when you substitute implies in place of arrow." Thus, although the participants generally recognized the problematic usage of the arrow, their verbal explanations of their interpretations of this comment offered little insight into what was wrong about the arrow and what they would change. Moreover, all of the participants avoided the problematic arrow usage in their revised proofs, typically by writing the inclusion in sentence form, often significantly restructuring the last few lines of text.

Participants' descriptions of the logic of specified changes

The participants were also asked to describe why they thought the professor had specified each of the changes to the proof text, and their answers revealed a variety of ways of thinking about the language of formal mathematics. For most of the comments, some participants were able to express the logic supporting the comment and some were not, and this was consistent across comment types.

With regard to the "We want to prove" recast comment at the beginning of Proof A, the expert consensus was that a proof should not begin with the conclusion that is to be proved, and the correction indicates that the first sentence expresses the goal of the proof. Two participants clearly articulated the logical issues with first line of the proof. Here is Adam's response:

> I think what the professor meant by this is you want to make a statement saying this is what we are going to prove or trying to prove.... They didn't say we're trying to prove this or this is the conclusion we're going to come to. And you want to make that clear.

In contrast, Bella explained that she understood the reason for the comment as "that's just one of the proper ways to start a proof, that from what I've learned, yeah, it's just the way to start a proof." That is, her thinking appeared to focus on the form of proofs, rather than on the logical function of the statement. The other five participants described the added phrase as clarifying the presentation, but they did not clearly articulate the logical issue. For example, Don said, "This is just to me good syntax. It's a way of setting it up to be understand better and to be read more easily." Thus, we suggest that the logic the professor intended to motivate by this comment was not successfully communicated to six of the participants.

The participants also initially showed mixed understanding of the reasons for the professor's explicit corrections on Proof A (Figure 1). For comment 2, which specified changing $\boldsymbol{Z}$ to $\boldsymbol{R}$, the experts agreed that the change was logically necessary because the relation R is defined on the set of real number. Six of the participants gave an explanation that approached that of the experts, including Genevieve who said, "there is no reason to believe that x, y, z are in the integers. The theorem never states that they are in the integers. [The theorem states] on the set of real numbers." Don gave a mixed explanation, initially saying "they [$\boldsymbol{Z}$ and $\boldsymbol{R}$] are both correct but the real numbers are more applicable, in most cases," but later in the interview noted that the theorem specifies that x, y, z are real numbers. Don's explanation for the requested change did not reject the original statement as inappropriate for the proof.

In reference to comment 5, only two participants, Don and Nancy, articulated the distinction between *let* and *for some,* whereas the other participants said little about this comment or gave explanations that did not fully align with the expert consensus. For instance, Charles said: "... when you make the mark *for some k, c,* that's traditional writing. But it also helps clarify how k and c are related to the two previous corrections. Um, whereas the student seems to be setting it up as its own separate idea, it's important to clarify that k and c are related to x, y, and z." Charles's comment about "traditional writing" suggests that he viewed the explicit correction as stylistic, i.e., to bring the writing into alignment with writing norms, although he also noted the logical issue.

The participants' interpretations of the clarification requests were similarly met with mixed success. Consider comment 2 on Proof B (Figure 2). The participants generally did not know the correct notation, and as a result, gave only an explanation about the importance of correct notation. For example, Adam said, "Wrong notation. Let me think here. (pause) I don't necessarily know if I know the right notation to put this in now (laughs), but I knew that wasn't the right notation." Genevieve, on the other hand, was able to articulate the notational issues with this part of the proof by saying "The arrows are sort of sometimes ambiguous with people because it generally means implies, but sometimes people use arrows in different ways, and so I think to actually put it in a sentence and make it clearer would be a lot better."

Similarly, for comment 3 on Proof B the participants agreed that "sets can't imply sets" but struggled to articulate the reason. Nancy and Ruby attempted to give a rationale in terms of *if-then* statements but had difficulty expressing themselves clearly and succinctly.

The participants' revisions of the proofs

When the professor gave an explicit correction or recast comment, the participants always incorporated it, or revised the proof in such a way as to avoid the problem. When the professor used an elicitation or a clarification request, the participants had to develop their own change. One such comment was "proofs should be complete sentences," which is somewhat directive in that it suggested a course of action. Four of the participants rewrote the entire proof in paragraph form, eschewing the string of equations, while the other four displayed the string of equations and wrote the concluding part of the proof in complete sentences. Both are reasonable interpretations of the professor's note, and either is stylistically acceptable.

In sum, the participants were generally quite capable of writing revised proofs that remediated the issues indicated by the professor's marks and comments, even when they could not fully explain the rationale for the comments.

What the participants claimed to do with professor feedback

To begin the interview, we asked the participants to describe what they typically did with the professors' comments on their graded proofs. Generally, they claimed to make relatively little use of them. Bella noted that she did not consistently read the comments, noting that "sometimes it would just go on the pile of homework, to be honest." Five more claimed to consistently read them. Nancy claimed to do more than simply read the comments:

> I would read through it, um, and I would try to kind of make mental notes. I would often not try to go back and rewrite it, but if I then did another proof that was similar, I would ... try to

make a mental note of it and use that. I did try to use it in other proofs. I wouldn't always redo those, but I definitely would use that information the next time that I wrote a proof. And I would actually get it out and look at it the next time I was writing a proof and try to see what did I do, what could I do differently to make this better.

Adam, too, talked about making additional use of the comments, and in his case it was in the context of his abstract algebra course. During this course, the teacher would sometimes return proof papers in class and go over them, and in that case Adam would write notes on his proofs for later reference. Adam claimed that he would use the written and oral comments to help him prepare to rewrite the same proofs during exams. But it seemed that without that motivation, he would normally do no more than read the comments.

Six of the participants claimed to make use of their professors' comments when they were asked to revise and resubmit proofs, but they all noted that they were seldom asked for revisions. This evidence suggests that students generally do not meaningfully engage with the comments on their graded proofs unless required to do so. The language acquisition literature (e.g., Lyster & Ranta, 1997) and composition research (e.g., Bean, 2011) argue that such engagement is critical for learning from feedback.

Conclusion

This study contributes to undergraduate mathematics education in that it is the first study to describe and analyze how students interpret and respond to their graded proof papers. Given that mathematicians consider proof grading to be an important means of teaching students to write proofs (Moore, 2016), the study opens a new line of research on the teaching and learning of mathematical proof at the undergraduate level.

We report three principal findings in response to the research questions. The first important finding is that when the professor wrote an explicit correction or a recast comment on a proof, the participants correctly identified the changes recommended and generally, but not always, could provide some rationale for the changes, including what was incorrect about the original proof. We suggest that the participants were able to invent reasons by comparing the original text and the corrected text; that is, they were able to develop a hypothesis about usage via induction (Swain, 1998). In contrast, when the professor's comment was a clarification request or an elicitation and did not provide new text, the participants struggled to provide a rationale for the change and fell back to more general claims that did not explicitly identify what was incorrect about the original proof. We note that the literature on second language acquisition suggests that cognitive engagement with the feedback is important for the students to incorporate the new information, and that clarification requests and elicitations promote cognitive engagement better than explicit corrections and recasts (Lyster & Ranta, 1997). Yet in this study, when the professor wrote clarification requests and elicitations, our participants were not reliably successful in responding to the comments because they did not appear to fully understand the comment and the mathematical language to correct the problem. Instead, they relied on general claims that are insufficient to support future proof-writing attempts. On the other hand, in the context of an actual course, if the professor requires students to revise their work, clarification requests and elicitations may promote cognitive engagement by directing students to use available resources to learn how to address the issues raised by the comments on their proofs.

The second important finding is that, regardless of the type of feedback the professor offered, the participants could not reliably describe normatively correct logic for the changes and why the original proof production was incorrect or problematic. While explicit corrections and recasts gave the participants a means to develop hypotheses by contrasting the two proofs, their inferences were sometimes incorrect or incomplete. It appeared that, generally, the participants could only explain the logic for requested changes when they already could identify the issue prior to reading the professor's comments. Moreover, it appears that they would often fall back to claiming that a requested change was "cultural" or "how you do it in formal mathematics." This observation suggests that the way professors currently write comments is not an effective way to communicate the reasons underlying the comments, which they often claim is the most important thing they are attempting to convey in their instruction (Lew, et al., 2016). Our data suggests that professors should write more about the logic that they are attempting to convey to the students, as well as distinguish between logical errors and readability concerns. What form such feedback might take to be most effective and efficient, whether direct statements, clarification requests, or elicitations, is a productive direction for future research.

A third finding is that students' written proofs are insufficient to distinguish between those who have some level of conceptual understanding and those who work only procedurally. When they revised the proofs, the participants successfully implemented the suggested changes in nearly all instances, regardless of the type of comment, even when they did not fully understand the rationale for the changes. This ability to write a correct proof without an understanding of the underlying logic is problematic for the teacher who may conclude that students understand more than they actually do. This potential mismatch suggests that perhaps professors have good reason to focus on teaching students the important mathematical logic and let the language, symbols, and grammar sort themselves out over time.

We recognize that this is a single, exploratory study with a small number of elementary proofs, a small number of participants, and only analytical generalizations. Moreover, we note three significant limitations of this study that suggest the need for further work. First, the participants were reading and writing proofs on mathematical topics that most of them had not worked with in some time, possibly since their introduction to proof class. Second, we asked the participants to interpret comments on proofs they had not written, thus imposing a need to make sense of another student's proof attempt prior to interpreting the comments. More research is needed to explore students' ability to interpret feedback in the context of their own proof writing. This first exploratory study provides a body of empirical evidence for future directions and more theoretical work. We note a third limitation: initially the four experts did not always agree on the reasons for the changes. While we could come to a consensus interpretation, there were significant differences in our initial interpretations, which means that different researchers, or a different mix of researchers, might have arrived at a different consensus interpretation of the professor's comments. This limitation of the study suggests an avenue for future research, motivated by Weber's (2014) argument that proof is a cluster concept. We hypothesize that while professors might share instructional goals about proof and use similar notes and language to communicate with students, in reality they may be attempting to convey very different content via the same notes, which has significant implications for students.

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Students' symmetric ability in relation to their use and preference for symmetric heuristics in problem solving

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Advanced mathematical problem solving is marked by efficient and fluid use of multiple solution strategies. Symmetric arguments are apt heuristics and eminently useful in mathematics and science fields. Research suggests that mathematics proficiency is correlated with spatial reasoning. We define symmetric ability as fluency with mentally visualizing, manipulating, and making comparisons among 2D objects under rotation and reflection. We hypothesize that symmetric ability is a distinct sub-ability of spatial reasoning which is more accessible to students due to inherent cultural biases for symmetric balance. Do students with varying levels of symmetric ability use or prefer symmetric arguments in problem solving? How does symmetric ability relate to insight in problem solving? Results from initial analysis indicate that, among mathematics undergraduates, there is variation in symmetric ability. Methods, future research, and implications are discussed.

Key words: [Symmetry, Problem Solving, Heuristics, Cognitive Spatial Reasoning]

Context

Geometry is the branch of mathematics most connected to the world as humans experience it; geometry is the math we see in the natural and manufactured structures around us. Cultures around the world have used geometry to build, navigate, make art, and predict motion. It is how we measure the earth and space. Shapes and angles provide the vocabulary we use to describe our surroundings and communicate about them with others. The utility of geometry to students is as plain as a line they draw and the plane in which they draw it. A traditional geometry curriculum in the United States is capped around age 16 with proof-based construction geometry and the introduction of trigonometry. Broadly speaking, visualizing in mathematics is one of the most important tools we have to offer students to help them better understand the application of mathematics to the physical world around them. Wai, Lubinski, and Benbow (2009) found that spatial ability was a key factor in determining success of students in advanced STEM education and career paths. Others have found that spatial skills can be taught and retained, which may increase participation in STEM fields (Uttal et al., 2013). Unfortunately, robust visual geometric understanding is often not the outcome of a standard mathematics education. Undergraduate students have trouble with geometric transformations, including symmetrical relationships (Rizzo, 2013). The 2011 TIMSS survey found that only half of United States 4th graders could complete a shape to have line symmetry. Further, in both 4th and 8th grades in the content area of geometry and measurement, American students underperformed in comparison to their overall mathematics achievement (Mullis, Martin, Foy, & Arora, 2011).

Symmetry. It has been frequently suggested, to the point of inclusion in both NCTM and CCSS standards (Common Core State Standards Initiative, 2012; National Council of Teachers of Mathematics, 2000), that studying symmetry is an important part of a well rounded mathematics education. Mathematicians have long noted symmetry as a major convergence point of mathematics and beauty (Drefus, T., Eisenberg, 1990). Symmetry is a persistent part of visual language worldwide (Dreyfus & Eisenburg, 2000; Hargittai & Hargittai, 1994; Shaw, 1990). Molecules, gems, butterflies, flowers, human faces, sculpture, architecture, musical notation, and even galaxies exhibit symmetry in both generation and propagation. Many cultures use and have used symmetry in their creative expressions. Classifying and decoding the meaning of symmetry in historical artifacts is an important endeavor to sociologists (Washburn & Crowe, 1987). Further, symmetry has great sway in terms of value judgments of mathematical, musical, literary, scientific, and aesthetic objects (Weyl, 1952). Symmetry is relevant in all branches of school mathematics. While most apparent in geometry and its study of transformations on Euclidian spaces, topics in algebra like symmetric functions, symmetric systems of equations, and symmetric graphs also rely on this construct (Hilton & Pedersen, 1986). In calculus, there is symmetry in integral construction. In statistics and probability, there is symmetry in distributions. And in trigonometry, there is symmetry in the identities (Dreyfus, T., Eisenberg, 1990). Symmetry has great importance to life and earth sciences, and it has served as a strong basis for investigation since the early geometers. The relatively new field of group theory in mathematics is based in large part on notions of symmetry and has been applied to the study of crystals, particle physics, and phyllotaxis (the study of branching in plants) (Dreyfus, T., Eisenberg, 1990). For these reasons, fluency in symmetry not only taps into an innate part of the human experience, but also can provide rich opportunities for engagement across academic disciplines.

Spatial Ability. There are many forms of spatial ability and reasoning, as well as tests and probing research on this aptitude. Spatial ability, the precision and robustness of visual perception, correlates positively with mathematical achievement (Battista, 1990). Spatial ability includes skills like differentiating foreground and background, performing mental shifts of orientation, seeing the effects of single and multiple reflections, identifying rotationally symmetric objects in two- and three-dimensions, correlating elevational and cross-sectional plans, mentally rearranging objects, and recreating visual patterns. Clements and Battista (1992) define spatial reasoning as “the set of cognitive processes by which mental representations for spatial objects, relationships, and transformations are constructed and manipulated.” Spatial reasoning is further split by some psychometricians into several factors like spatial orientation: “The ability to perceive spatial patterns or to maintain orientation with respect to objects in space,” and spatial visualization: “The ability to manipulate or transform the image of spatial patterns into other visual arrangements” (French, Ekstrom, & Price, 1963). The primary difference between these two constructs being the active or passive nature of the observer. These definitional distinctions are hotly debated and other such terms have been proposed and defended. Many facets of spatial ability have been studied in contexts such as gender differences and mathematical achievement (Battista, 1990; E. H. Fennema & Sherman, 1978; E. Fennema & Sherman, 1977; Harris, 1981), problem solving (Tartre, 1990), implications for instruction (Bruce & Hawes, 2014; Ferrini-Mundy, 1987) with the aid of engineering drawings (Olkun, 2003), learning disabilities (Garderen, 2006), and relationship to fields such as: chemistry (Bodner & Guay, 1997), geology (Orion, Nir, Ben-Chaim, David, Kali, 1997), kinematics (Kozhevnikov, Motes, & Hegarty, 2007), and music and sport (Pietsch & Jansen, 2012). Evidence suggests that spatial ability peaks in humans during preadolescence, further justifying inclusion and serious treatment of symmetry in middle and high school curricula (Ben-Chaim, Lappan, & Houang, 1988). Overall, spatial ability is an important cognitive function that strongly correlates (.30-.60 (Battista, 1990)) with mathematical achievement. This correlation becomes more pronounced as task difficulty increases and is of approximately equal weight to that of verbal reasoning (E. Fennema & Sherman, 1977). The perception of symmetry is an unknown portion of spatial ability.

Problem Solving. Problem-solving and critical thinking are main pillars of reformed K-12 curricula (Common Core State Standards Initiative, 2012; National Council of Teachers of

Mathematics, 2000) and are pervasive in PCAST reports (Holdren & Lander, 2012). Schoenfeld (1987) demonstrates that metacognitively aware problem solvers read, analyze, explore, plan, implement, and verify during their problem solving process. The "analyze" phase involves creating hypotheses of how a problem can be solved – i.e. the generation and recall of solution strategies. Polya (1957) offered up some of the first solution strategies for devising a plan of mathematical action: use a related problem or alter the problem to make it easier. Since then, more and more solution strategies have been named and studied as they relate to specific mathematical problem areas, many of them algorithmic. Multiple solution tasks are those that can have many solution paths. Almost all mathematics problems can be solved more than one way. A simple example might be the different algorithms one can use to add three-digit numbers. A more complex task might be to prove that a parallelogram with congruent diagonals is a rectangle (Levav-Waynberg & Leikin, 2012). Some research on problem solving focuses on flexibility which is, "knowledge of multiple solutions as well as the ability and tendency to selectively choose the most appropriate ones for a given problem and a particular problem-solving goal" (Star & Newton, 2009). This research shows that flexibility can be trained for and that contrasting cases are particularly useful in this endeavor (Rittle-Johnson & Star, 2007). The skilled mathematician is fluid in their use of solution strategies.

Symmetric Heuristics. Symmetry is often an easy path in problem solving and may have broad appeal to students and mathematicians alike. Goldin & McClintock (1980) call the application of symmetry in problem solving insight and enumerate several kinds of symmetry in problem solving: overt, construction, representational alteration, and hidden. They make recommendations for teaching to exploit symmetry. Drefus & Eisenberg (1990; 2000) emphasize that for symmetry to cut through the restrictions of a problem, like a hot knife through butter, it must first be seen and imposed, a challenging task. The utility of symmetry as heuristic has applications in multivariate calculus, organic chemistry, applied engineering and design, and physics (Hilton & Pedersen, 1986). Previous research with in-service teachers shows that they generally do not use symmetric solution strategies, and are skeptical of the mathematical validity or sufficiency of such solution strategies when working on multiple solution tasks (Leikin, Berman, & Zaslavsky, 2000; Leikin, 2003). Similarly they believe that conventional solution strategies (relying on calculus, algebra, or geometric definitions) are more trustworthy and they have more confidence in teaching them. The insightful recognition of when to use symmetric arguments, and preference for/against symmetric arguments has not been investigated with students before. The terms Goldin & McClintock offered for categorizing symmetric heuristics have not been observed in practice and cataloged as of yet.

Mathematics attitude. Research indicates that affective/attitudinal factors greatly influence mathematical achievement (E. Fennema & Sherman, 1977) as well as selective processes like choosing a career (Betz & Hackett, 1983). Put differently, one's attitudes about mathematics influence one's mathematics performance as well as one's decisions having to do with mathematics. We propose to look at mathematics attitude in relation to how one ranks solution strategies in order of preference. Having some measure of self-conception is important when trying to form a well-rounded picture of ability level. Bandura (1993) posits that a students' self-efficacy is composed of several factors, one of them being selective processes. Choice as mediated by ability is a new avenue in mathematics research.

Research Questions and Goals

In this research we propose to investigate the relationship between students' symmetric spatial abilities, their attitudes toward mathematics, and their use and preference for symmetric heuristics. We define symmetric ability as: a student's ability to mentally visualize, manipulate, and make comparisons among 2D geometric objects and as applied to cultural material in terms of reflectional and rotational symmetry. A research instrument was developed for the purpose of measuring this sub-ability of spatial reasoning. In response to the research context and instructional significance presented here, this research project is designed to answer the following research questions:

1. How does students' symmetric ability relate to their use and preference for symmetric heuristics in problem solving?
2. How does students' symmetric ability track with confidence and self-efficacy in mathematics learning?

The conclusions of this research could serve as a basis for the continued curricular expansion of treatments of symmetry within geometry and other mathematics curricula, and provide insight into how students currently think about symmetry as a heuristic. Future research might seek to find out how one's symmetric ability can be built upon in challenging problem solving situations as we know that spatial ability can be refined through training (Uttal et al., 2013). This research was undertaken from a cognitivist theoretical framework, meaning that the perspectives and views of subjects were assumed to be accessible through verbal and visual communication with them. This framework is appropriate because subjects' conception of validity and justification of symmetry as heuristic is the focus of this study.

Research Methods

Two research instruments were developed for this investigation: a survey and an interview protocol.

Symmetric Ability Survey. A survey was developed to provide a quantitative measure of symmetric ability. Our definition of symmetric ability mimics those that Olkun (2003) summarizes: spatial ability in reasoning, relations, and visualizations with an added cultural component influenced by research on ethnomathematics (Abas, 2004; D'Ambrosio, 2001; Eglash, Bennett, O'Donnell, Jennings, & Cintorino, 2006). Therefore this survey has three sections (based on our definition of symmetric ability and our research goals dealing with mathematics attitude): cultural, mathematical, and attitudinal. Sample items can be seen in Figures 1-4. For both the cultural and mathematical symmetry sections, detailed written instructions and worked examples were part of the survey.

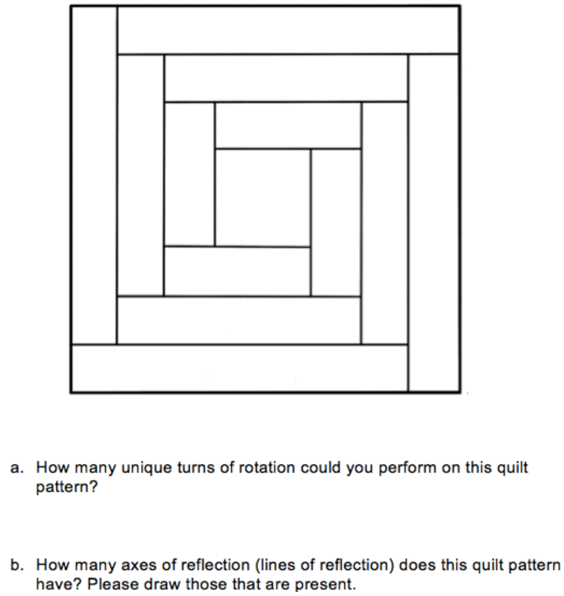

Figure 1. Cultural item from symmetric ability survey, one of five. All items used the same questions (a.,b., and implied c. - drawing)

The cultural symmetry section of this survey was developed through both informal (research workshop) and formal (pilot sample) validity measures. Cultural objects were drawn from traditional northeastern art and design while question vocabulary was drawn from school-aged curricula.

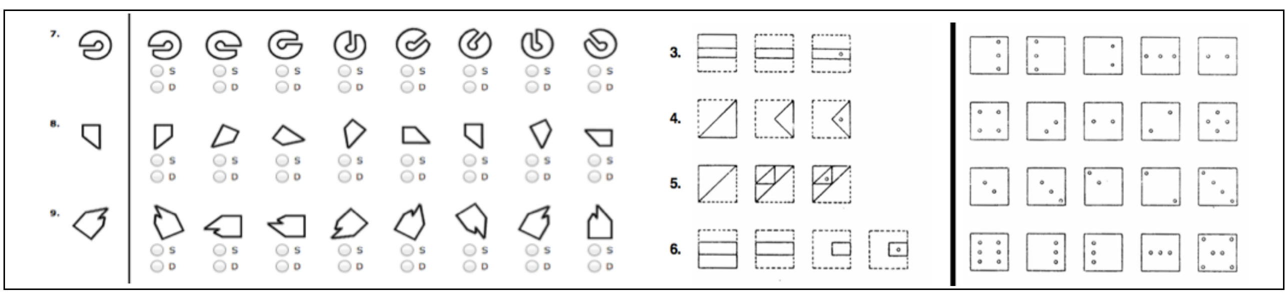

Figure 2. (left) Geometric 2D items from symmetric ability survey, tests rotational ability. Students mark whether the eight images are rotationally symmetric (S) or different (D) from the leftmost image. Three of ten.

Figure 3. (right) Geometric 2D items from symmetric ability survey, tests reflectional ability. Images to the left of the bold line show a series of folds and hole punches done on a square piece of paper. Students choose the image to the right that corresponds to the correct pattern of holes when the paper is unfolded in place. Four of ten.

The mathematical symmetry section was composed of two factor analysis tests taken from the literature. An exhaustive search was performed to catalog all the research-validated tests of spatial ability. The Card Rotations Test and Paper Folding Test (Ekstrom, French, Harman, & Derman, 1976) were chosen as they highly correspond to the tasks of mental rotation and mental reflection in two dimensions. This portion of the survey was timed in accordance with the Educational Testing Service's administration protocol (Ekstrom et al., 1976).

I am challenged by math problems I can't understand immediately.

Disagree				Agree
1	2	3	4	5

I study mathematics because I know how useful it is.

Disagree				Agree
1	2	3	4	5

I don't think I could do advanced mathematics.

Disagree				Agree
1	2	3	4	5

Generally I feel secure about attempting to learn mathematics.

Disagree				Agree
1	2	3	4	5

Figure 4. Attitudinal items from the symmetric ability survey. Four of twenty-five.

The attitudinal section of the survey was composed of the confidence and self-efficacy sub-scales of the Fennema-Sherman attitudinal scales (Fennema & Sherman, 1976), Figure 4.

Symmetric Heuristics Interview. An interview protocol to qualitatively describe use and preference for symmetric arguments was developed. This interview protocol centered on four problems. Sample problems can be seen in Figures 5-6.

You are programming a robot to transport items across a warehouse. Your robot starts at point A, moves to a moveable point P on line segment $\overline{CD}$ to pick up a box of cereal. It then delivers this box to point B. What is the shortest distance you can have this robot travel?

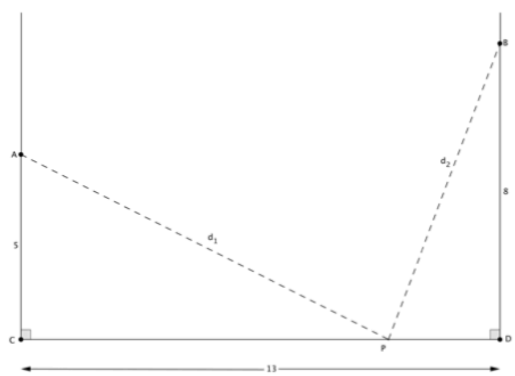

Figure 5. Students may respond to this open-ended question with several solution strategies. Expected solutions include parameterization and minimization, guess and check, reflection of B through CD to form the straight path AB' and application of the Pythagorean theorem, and using proportional reasoning from the related case of AC=DB. Problem statement modified from Goldin & McClintock (1980).

Which solution to the following system of equations do you prefer?

$$\begin{cases} 3x + 2y + z = 30 \\ x + 3y + 2z = 30 \\ 2x + y + 3z = 30 \end{cases}$$

1. To solve a system of three equations with three variables we can solve by pairs of equations. Taking the first and second equations together and the second and third equations together we can remove z from both by multiplying out the z coefficient and subtracting equations as shown.

$$\begin{array}{cc} 3x + 2y + z = 30 & x + 3y + 2z = 30 \\ x + 3y + 2z = 30 & 2x + y + 3z = 30 \\ & \\ 2(3x + 2y + z) = (30)2 & 3(x + 3y + 2z) = (30)3 \\ 1(x + 3y + 2z) = (30)1 & 2(2x + y + 3z) = (30)2 \\ & \\ 6x + 4y + 2z = 60 & 4x + 2y + 6z = 60 \\ -1x + 3y + 2z = 30 & -3x + 9y + 6z = 90 \\ \hline 5x + y = 60 & x - 7y = -30 \end{array}$$

We now have a system of two equations in two variables. Solving for x or y in either and substituting will yield an answer of $x = 5, y = 5$, and $z = 5$.

2. Notice that the coefficients of x, y, z cycle through the equations: $\{3, 2, 1\}, \{1, 3, 2\}, \{2, 1, 3\}$. For all of these equations to equal 30 at the same time, $x = y = z$. The first equation can be rewritten:

$$3x + 2x + x = 30$$
$$6x = 30$$

Therefore, $x = 5, y = 5$, and $z = 5$.

3. Use a software program like Mathematica, MATLAB, or WolframAlpha to solve. Finding that $x = 5, y = 5$, and $z = 5$.

Figure 6. Students respond by ranking their preferred solution strategy. Problem statement from Leikin et al. (2000).

These problems were taken from various sources: the literature, an undergraduate physics text, and the mathematics cannon. Two of the problems were open-ended and could be solved in multiple ways, for example: algebraically, with calculus, by guess and check, or with symmetry. These problems were intended to gauge native use of symmetry as heuristic (Fig. 5). The remaining two problems presented students with three worked solutions that they were asked to rank. Criteria for this ranking were not specified. Instead, students were asked to reflect on the reasoning behind their selections, and further to hypothesize on how expert mathematicians would rank the solution strategies. These problems were intended to gauge students' preference for symmetric heuristics and identify reasoning for these preferences (Fig. 6). One of each type of problem, open-ended or ranked solution, had a symmetric solution strategy relying on reflection and one of each type of problem had a symmetric solution based on rotation. Sample cognitive interview prompts can be seen in Figure 7.

Use:
1. Can you explain why you decided to try solving the problem this way?
2. Can you think of any other ways to solve this problem?
3. How might a mathematician approach this problem?
4. What are you studying in math class right now?

Preference:
1. Can you rephrase this solution strategy in your own words?
2. Why do you prefer this strategy to others?
3. Can you think of any other ways to solve this problem?
4. Which would a mathematician prefer?

Figure 7. Sample prompts from the problem solving interview protocol.

Primary data collection occurred during the 2015-16 academic year at a large northeastern university. The symmetric ability survey (Fig. 1-4) was administered to n=95 students concurrently enrolled in integral (39%) or multivariable (61%) calculus courses. The majority of students (78%) were engineering majors (mechanical, chemical, electrical & computer), 4% were math majors, and 14% were women. A subsample, currently n=5, was selected to take part in the symmetric heuristics interview (which was designed as a think-aloud, problem solving format). Calculus students were chosen as an audience to ensure that calculus solution strategies would be available to students.

Data Analysis

Symmetric Ability Survey. The three sections of the symmetric ability survey were scored separately. A rubric was developed to establish the cultural section as a quantitative measure. Cultural symmetry questions (Fig. 1) were scored in a binary fashion for each of the three problem parts: the number of possible rotations, the number of reflectional axes, and the correct drawing of axes on the image. To establish inter-rater reliability of this rubric, three raters reviewed overlapping portions of the data set. We found agreement > 95% of the time. Card rotation items (Fig. 2) and Paper folding items (Fig. 3) were scored following the guidelines provided by the distributors (Ekstrom et al., 1976). Likert scale attitudinal data (Fig. 4) were scored using the reverse coding method (Field, 2009). This method calls for Likert data to be coded against the response pattern of an ideal subject, thus giving a measure of confidence and self-efficacy as compared to a responder with perfect self-image of confidence and efficacy. All measures were scaled for comparison purposes. In future data analysis we hope to search for trends within the card rotations and paper folding test having to do with angular difference (Cooper, 1975) and fold complexity. Further, we intend to catalog the persistence and variety of misconceptions students have with using school language symmetry with the cultural items.

Symmetric Heuristics Interview. Survey participants were invited to participate in the think-aloud symmetric heuristics interview. We used high and low symmetric ability and high and low attitude as selection criteria for the interview. Students whose scores were $\pm$ 1 standard deviation from the mean in any of two symmetric ability scales and on either of the attitudinal scales were contacted for participation. The matrix in Figure 8 describes the distribution of interview participants.

Figure 8. Distribution of symmetric ability survey participants based on $\pm$ 1 standard deviation performance on multiple measures of symmetric ability and at least one measure of mathematic attitude. Darkened circles indicate current interview sample, effort was made to interview high and low performance students.

In addition to these subjects, we conducted a pilot interview with a small group of mathematics graduate students in order to solicit broader idea generation and to practice fruitful questioning tactics. Inclusion of one or more expert (professorial) interviews is expected. Transcription of these interviews is in process. We will undertake a grounded theory reading of the interviews searching for trends in student reasoning (Corbin & Strauss, 2008).

Preliminary Results

Symmetric Ability Survey. In the survey sample population we see some variation in symmetric ability (Fig. 9), indicating that this instrument can parse differences in student performance. However, we noted that on all measures members of the inter and upper quartile performed very well, getting more than half of all questions correct. This contrasts with previous findings from a pilot study (n=11) of undergraduate 'exploratory mathematics' students (Fig. 10). Further, there seems to be a positive correlation between symmetric ability and mathematics attitude (Fig. 11), though this correlation is weak and not statistically significant (r^2 = .11).

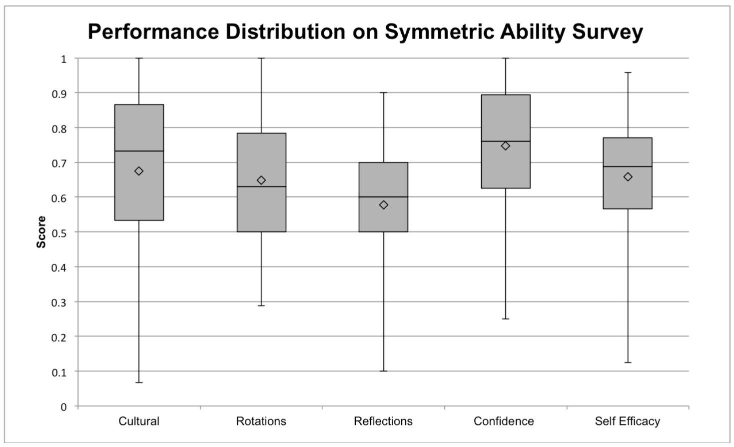

Figure 9. Summary statistics for each section of the symmetric ability survey.

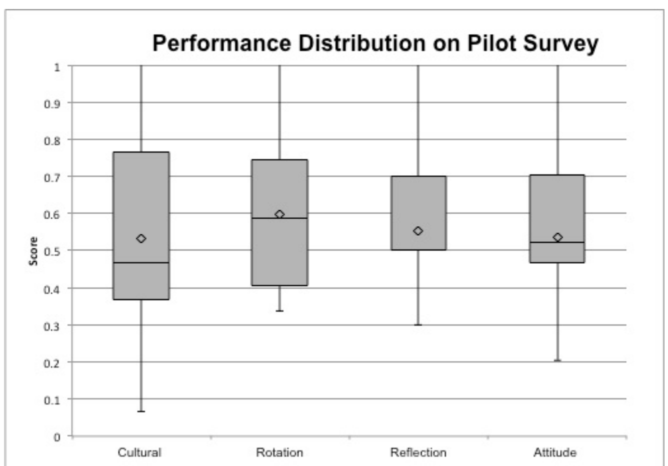

Figure 10. Summary statistics for each section of the symmetric ability survey, pilot population. Note: In a previous version of the survey the attitude item scales were incomplete. Cultural, rotational, and reflectional sections did not undergo revision and are cross-comparable.

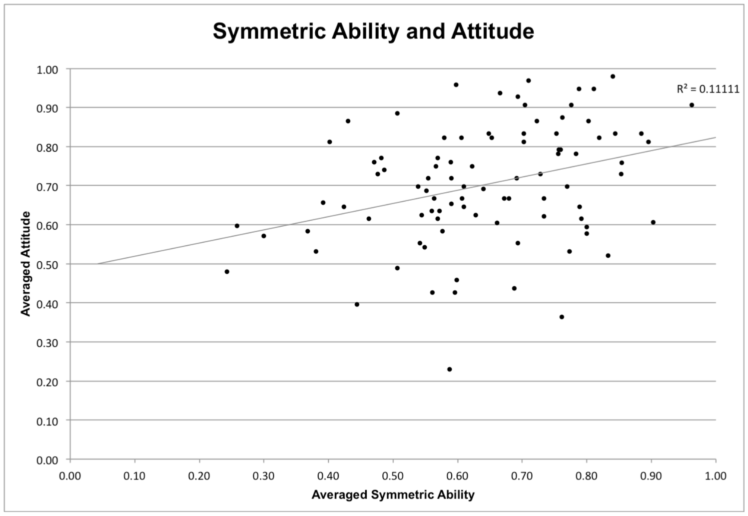

Figure 11. All three measures of symmetric ability weighted equally in comparison to an equal weighting of attitudinal scales.

Symmetric Heuristics Interview. Initial investigation revealed a few avenues for further inquiry. Students displayed expected conventional solution strategies as seen in Figures 12-13 but there was more of a mix of preference when ranking traditional and symmetric solution strategies both personally and when projecting on to experts.

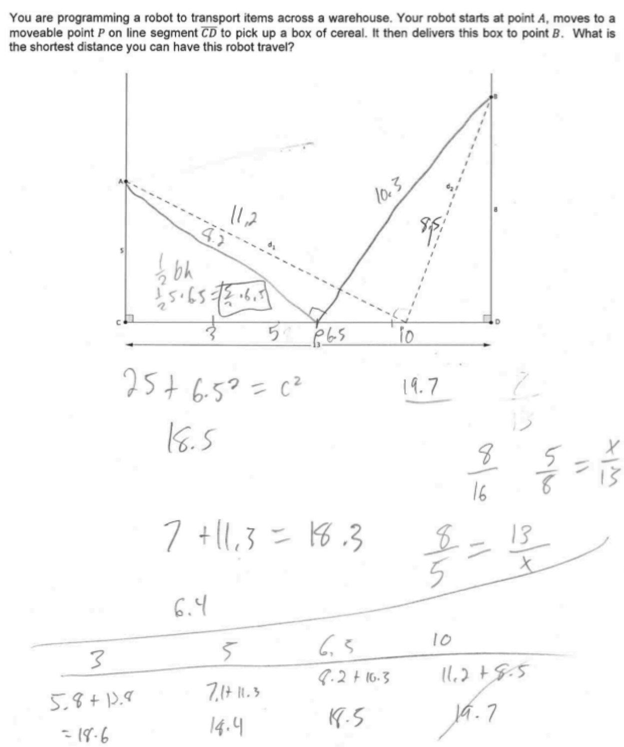

Figure 12. Example of expected 'guess and check' solution strategy.

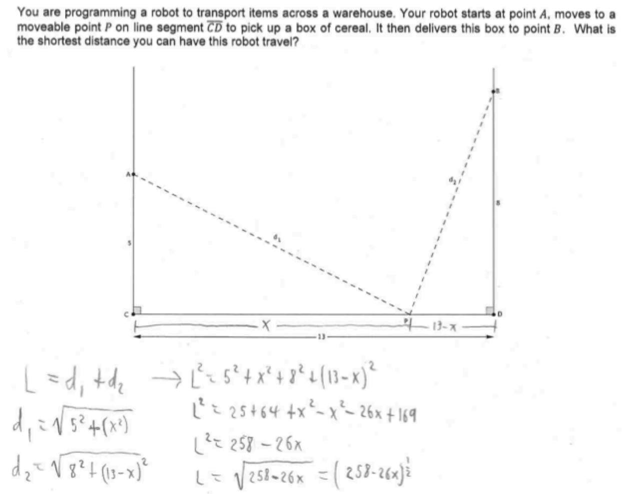

Figure 13. Example of expected calculus minimization solution strategy.

The student who successfully applied calculus (Fig. 13) later expressed preference for a traditional solution strategy while working on the system of equations problem (Fig. 6).

> *"I'd like to think that my professor would choose the first solution, but I do know some professors that would be just like, oh yea, that (the second solution strategy) is how equations work. My professor definitely sees patterns in the numbers a lot better than I do sometimes. I'd like to think they would use the first one just because it is a good rote way to do it, showing us how to do it. But if they were to do it off the top of their head, I think they might have been able to do the second one."*

This preference for the traditional solution strategy extends beyond personal to include a preference for how this student thinks the content should be taught, recognizing that the symmetric strategy requires a certain level of intuition which causes some self doubt. This might support the findings of Leikin (2003) that there is appreciation for symmetric heuristics but qualities like validity and perceived ease detract from overall preference and use.

Implications and Future Inquiry

Based on preliminary findings it seems that this population does not exhibit, with this measure, a meaningfully varied range of symmetric abilities. While there is a statistically insignificant positive correlation between symmetric ability and mathematics attitude, students performed very highly on both. It could be that a sample of quite advanced mathematics students would have previously self-selected to continue in mathematics based on good symmetric ability and/or mathematics attitude. This hypothesis suggests that differences may exist between students with high or low symmetric ability, but our research population was not suited to find this. Future research plans include expanding symmetric ability survey sample size to less proficient mathematics students. Possible further interview findings include: high symmetric ability students prefer but do not natively use symmetric heuristics, low symmetric ability students do not prefer and do not natively use symmetric heuristics, or any combination therein. Further, this line of inquiry will provide a characterization of how students think about symmetry as a heuristic.

Research has shown that it is possible to train and develop varieties of spatial ability (Uttal et al., 2013) and that using symmetric heuristics can enhance one's problem-solving endeavors (Ho, Ho, & Jaguthsing, 2012). We share the perspective of others that symmetry is a powerful and yet lightweight tool that could be used as a vehicle for inspiring deeper mathematical learning. We sought to find a connection between symmetric ability and symmetric heuristic use and preference which would be an important finding for the continuing efforts of establishing curricular recommendations that align with this perspective.

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Understanding and advancing graduate teaching assistants' mathematical knowledge for teaching

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Graduate student teaching assistants (GTAs) usually teach introductory level courses at the undergraduate level. Since GTAs constitute the majority of future mathematics faculty, their image of effective teaching and preparedness to lead instructional improvements will impact future directions in undergraduate mathematics curriculum and instruction. Yet GTA training, when offered, tends to focus on the practice of teaching rather than the mathematics being taught. In this paper, we argue for the need to support GTAs' in improving their mathematical meanings of foundational ideas and their ability to support productive student thinking. By investigating GTAs' meanings for average rate of change, a key content area in precalculus and calculus, we found evidence that even mathematically sophisticated GTAs possess impoverished meanings of this key idea. We argue for the need, and highlight one approach, for supporting GTAs' to improve their understanding of foundational mathematical ideas and how these ideas are learned.

Key words: graduate student teaching assistant, mathematical meanings, average rate of change, precalculus

Many mathematicians believe that undergraduate curriculum and teaching is unproblematic. After all, it worked for them! However, evidence exists to suggest otherwise (Bressoud, et al., 2012; Seymour, 2006). As universities continue to explore how to improve the mathematical success of their students, we propose that consideration for the preparation of future faculty be a central component of shifting introductory university level instruction. Graduate student teaching assistants (GTAs), namely, those individuals who constitute future faculty, teach many of the courses that undergraduate students first encounter in college (i.e., courses up to and including calculus). As such, GTAs who offer high quality instructional experiences for their students can impact a large number of students early in their studies. Yet GTA instructional support and professional development, when offered, frequently focuses on general discussion of best practices in pedagogy and specific logistical and student issues for a given institution. As novice instructors, this information is important in supporting GTAs' development of facility with the day-to-day practices of teaching, but it does little to support them in engaging deeply with the mathematical content in a way that will translate to meaningful mathematical discussions in their classrooms.

As other researchers have noted, having completed many mathematics courses, as most GTAs have, does not necessarily improve a teacher's understandings and teaching practices (Speer, 2008; Speer, Gutmann, & Murphy, 2005). Speer and Wagner (2009) further argue that the work done by mathematics instructors in providing analytic scaffolding, which entails recognizing and figuring out the ideas expressed by students to build upon and push the mathematical discussion forward, involves *mathematical* work. Instructors with impoverished or strictly procedural meanings for the mathematical content of a lesson will struggle to engage their students in conceptually oriented discussions around this lesson. Thompson, Carlson & Silverman (2007) claim that:

> If a teacher's conceptual structures comprise disconnected facts and procedures, their instruction is likely to focus on disconnected facts and procedures. In contrast, if a teacher's conceptual structures comprise a web of

> mathematical ideas and compatible ways of thinking, it will at least be possible that she attempts to develop these same conceptual structures in her students. We believe that it is mathematical understandings of the latter type that serve as a necessary condition for teachers to teach for students' high-quality understanding (pp. 416-417).

The possible attempts to support student learning are revealed in the nature of the instructor's questions, her questioning patterns and the quality of the discussion she leads. All of these activities are built from how the teacher makes sense of her students' thinking, which itself is done through the lens of the teacher's own mathematical meanings (Teuscher, Moore & Carlson, 2015).

In this paper, we argue the need for attention to the GTAs' mathematical meanings, fostering attentiveness to student thinking, and supporting GTAs to *speak with meaning* during their training (Clark, Moore, & Carlson, 2008). In the context of an intervention to support mathematics GTAs to act in productive pedagogical ways, we investigated GTAs' meanings for average rate of change (AROC), a key content area in precalculus and calculus. We found evidence that even mathematically sophisticated GTAs possess meanings that vary widely in productivity for teaching prior to our intervention, and that shifts in meanings required prolonged engagement in professional development focused on making meaning, building coherence and communicating with precision.

We provide an expanded description of our goals for high quality instruction, how the intervention supports GTAs in making progress towards those goals, and describe what research has reported to be a productive meaning for the idea of AROC (Thompson, 1994). We then report on results that reveal the varied fluency among participants in speaking with meaning about AROC when probed before, during and after the intervention. We conclude with a discussion about the impact of the intervention and implications for other GTA training and professional development for undergraduate mathematics instructors.

Theoretical Framework

We provide a brief explanation of our theoretical perspective on mathematical meanings and say what we mean by high quality instruction. We also include a detailed description of the intervention used to support GTAs in developing more robust meanings of the concepts of AROC.

Researchers have proposed *mathematical meanings* as the organization of an individual's experiences with an idea, also referred to as a scheme (Thompson, 1994). It is through repeated reasoning and reconstruction that an individual constructs schemes to organize experiences in an internally consistent way (Piaget & Garcia, 1991; Thompson, 2013; Thompson, Carlson, Byerley, & Hatfield, 2013). For example, an individual's meaning for the idea of *average rate of change* might consist of the calculation for the slope of a secant line, or simply $\Delta y/\Delta x$. An individual who has committed to memory that the average rate of change is the slope of a secant line does not possess the same meaning as someone who sees the slope of a secant line as the constant rate of change that yields the same change in the dependent quantity (as some original non-linear relationship) over the interval of the independent quantity that is of interest. These two individuals hold different meanings for the same idea, and the consequences of such differences can be profound. For instance, the former individual may or may not have an understanding of AROC as a *rate of change*, but instead conceptualizes AROC as the numerical output of a computation. This individual will struggle to generate problem solving strategies relying on AROC, for instance, in situations where linear approximations could help generate characterizations for how two quantities covary over a given interval.

In studying individuals' meanings for various mathematical constructs, researchers are at a disadvantage – we cannot *see* schemes. Rather, we only have access to what an individual says, writes and gestures when engaging in mathematical activity. In our research and in the intervention, we focus on an individual's *expressed meaning of an idea*, or the spontaneous utterances that an individual conveys about an idea. From these utterances we can make inferences about how an individual has organized her experiences with the idea. We find the expressed meanings of particular interest in studying instructors' mathematical meanings because it is these spontaneous verbalizations that emerge during class instruction when students ask questions or pose solutions that deviate from what the instructor had prepared. What the instructors say *in the moment*, in turn, affects the ways of thinking their students develop about the mathematical idea(s) central to the discussion. We further propose that these expressed meanings provide insight into an instructor's meanings, which influence nearly all instructional activities, including: how the instructor interprets and responds to students' utterances and written products, the nature of the questions the instructor poses to students, and the selection and implementation of curricular tasks during class meetings.

We further note that an individual's meanings can be more or less productive for teaching. By more productive for teaching, we intend to describe those meanings that would support the individual in constructing new mathematical ideas and connections to other ideas, and correspondingly create the possibility for the individual to foster those connections within her students. An individual's meaning for specific ideas can be further developed through reflection, which occurs when the individual is faced with perturbations to her current meanings for those ideas (Dewey, 1910). With this perspective in mind, the intervention for GTAs is designed, in part, to perturb the GTAs' thinking about mathematical content areas specific to the courses they teach, namely precalculus and beginning calculus.

High quality instruction

For the purposes of this paper and the intervention being discussed, we delineate some of the characteristics we take as essential components of high quality instruction. A teacher is engaging in high quality instructional practices when she

- Supports students in constructing deep understandings and rich connections among central ideas of a course,
- Supports students in developing flexible problem solving abilities that enable them to solve problems that are novel to them and require that they apply their understandings
- Interprets and acts on student thinking when teaching, and
- Reflects on student thinking and learning to improve teaching.

One of the behaviors that might indicate high quality instruction to an observer is that the instructor is speaking precisely and meaningfully about mathematical ideas (Clark, Moore, & Carlson, 2008). Another indicator is that the instructor asks questions that elicit student thinking and then interprets, analyzes, clarifies and, whenever possible uses students' contributions to push forward the mathematical activity in the classroom (Johnson, 2013; Steffe & Thompson, 2000). A teacher's ability to engage in these practices depends heavily on the teacher's mathematical meanings and her mathematical knowledge for teaching (Ball & Bass, 2003; Ball, Hill, & Bass, 2004; Silverman & Thompson, 2008).

The intervention

The graduate students involved in the intervention volunteered to participate in the program and were compensated for their participation. All but one had met their respective university requirements to teach and agreed to further engage in a yearlong professional development program, which we call *the intervention*. The GTAs participated in a 2-3 day workshop before the start of their first semester of teaching with research-based, *Pathways Precalculus* curriculum materials (Carlson, Oehrtman, & Moore, 2015). During this intensive workshop, the GTAs completed mathematical tasks that were designed and sequenced to support graduate students in constructing a productive meaning for the key ideas in the precalculus curriculum, including the ideas of constant rate of change, AROC, exponential growth, angle measure, and the sine and cosine functions[1]. The graduate students confronted problems and questions designed to perturb their meanings for these topics. Sometimes these questions were as simple as "Explain the meaning of _____." Other times, the workshop tasks were more advanced mathematical problems that could be solved with precalculus tools, provided the individual had a deep understanding of the mathematical ideas central to the lesson. Still other intervention tasks were situated in the act of teaching, requiring participants to respond to hypothetical student questions and responses. The intent of these questions and tasks was to prompt reflection and subsequent shifts in the GTAs' meanings for the mathematical ideas that were the focus of their instruction.

The intervention leaders conducted workshops in a manner to encourage the participants to speak with meaning about the ideas. This meant GTAs were encouraged to avoid using vague language (e.g., use the quantity descriptions instead of pronouns, as in "The distance increases" instead of "It goes up.") and to explain basic vocabulary (e.g., *proportional*) instead of taking terms as understood. As this type of communication is new to most of the participants, this focus on speaking with meaning about the mathematics continued during the intervention. During the fall and spring semesters, the GTAs attended weekly 90-minute seminars concurrent with teaching a course using *Pathways Precalculus* materials, instructional resources designed using research on student thinking (e.g., Carlson, 1998; Carlson et al., 2002; Moore, 2012; Strom, 2008; Smith, 2008; Engelke, 2007) and scaffolded to support students' construction of key ideas of precalculus that are foundational for calculus.

The primary goals of the weekly seminars were to support the graduate students in developing more productive meanings of the key ideas to be taught during the upcoming week, and to support them in clearly explaining their meanings for those ideas to others. As part of the intervention to support them in achieving these goals, each of the graduate students reviewed the course materials prior to the weekly seminar and came prepared to discuss what is involved in understanding and learning the ideas central to the lesson for the subsequent week. They also came prepared to give presentations to their peers about how they intended to support their students' learning of these ideas. During the seminars, participants frequently worked in small groups to develop mini-presentations focused on implementing the most conceptually challenging tasks to be used in the lessons for the upcoming week. At some universities, GTAs were further asked to videotape their instruction and write reflections on their instruction with respect to the ways of thinking they supported during that class, the types of questions they posed to students, and how they might have responded differently taking student thinking into account.

The intervention had a primary focus on the mathematical content and what was entailed in understanding and learning key ideas of each lesson. Student thinking was discussed and described using constructs from relevant research literature and the GTAs' classroom experiences. Student thinking was analyzed for the purpose of describing student reasoning and understanding, and identifying productive ways for advancing students' learning. To

[1] The idea of *average rate of change* is the culminating idea of the first instructional unit of the

achieve the goal of improving the GTA's ability to make sense of and advance student thinking, the intervention leaders and accompanying instructor materials had a primary focus on supporting the GTAs in developing rich and well-connected meanings for the mathematical ideas that were the focus of each lesson.

The content for the course began with a focus on developing students' ability to conceptualize quantities in a problem context and consider how pairs of varying quantities change together—these reasoning abilities have been identified to be necessary for constructing meaningful formulas and graphs (Moore & Carlson, 2012) to define functional relationships in applied contexts. As students explore the patterns of change for non-linear functions, such as exponential, quadratic, polynomial, rational, and trigonometric functions, they use the idea of *average rate of change* to characterize and compare function behavior over intervals of a function's domain. Since the idea of *average rate of change* is an important cross cutting idea that is challenging to teach with a conceptual focus, we selected this idea for studying the GTAs' mathematical meanings.

A productive meaning for the idea of average rate of change

Constructing a rich meaning of *average rate of change* entails conceptualizing a hypothetical relationship between two varying quantities in a dynamic situation. Given a relationship between the independent quantity A and the dependent quantity B, and a fixed interval of measure of quantity A, the *average rate of change* of quantity B with respect to quantity A is the *constant rate of change* that yields the same change in quantity B as the original relationship over the given interval. In order to understand this complex idea meaningfully, an individual must first conceptualize the idea of *quantity* as a measurable attribute of an object (e.g., the distance a car travels from home, number of minutes elapsed since noon). Next, provided a situation in which two quantities vary in tandem, an individual must develop an understanding for what it means to describe the *rate of chang*e of one quantity relative to the other. Namely, the individual must conceptualize the multiplicative comparison of changes in the two quantities (the change in the output quantity is always some amount times as large as the change in the input quantity). In the special case that the relative size of changes in one quantity relative to the other remains constant, we say the quantities vary with a constant rate of change (CROC) (see Figure 1 for the mental actions involved in constructing the idea of CROC). Individuals with a robust meaning will draw connections between AROC and CROC and view those two connected ideas as a means for approximating values of varying quantities in dynamic scenarios.

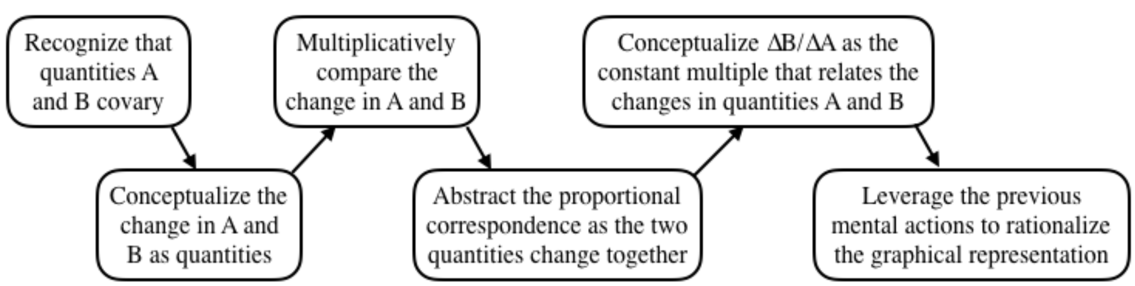

Figure 1. Mental actions involved in understanding constant rate of change

Methods

We collected data from mathematics graduate students and instructors at three large, public, PhD-granting universities in the United States. Participants' teaching experience varied between zero and 11 years, at both the K-12 and tertiary level. We conducted semi-structured clinical interviews with 19 graduate teaching assistants, all of whom had at least one semester experience teaching a small section or acting as the recitation leader for a Precalculus course that used the *Pathways Precalculus* materials (Clement, 2000). The lead author conducted interviews to probe shifts in their beliefs about the roles of students and teachers in teaching and learning mathematics, and gain insights about their understandings of mathematical ideas, teaching practices and goals for student learning. Interviews were recorded using both a video camera and Livescribe technology to capture audio-matched written responses to sample teaching scenarios provided during the interviews. Interviews lasted 1-2 hours, and were transcribed and coded by three members of the research team. Members of our team analyzed videos in pairs at first, identifying themes of interest relative to our conception of a productive expressed meaning for AROC before working individually to continue coding and reconvening as a group to discuss our findings (Strauss & Corbin, 1990).

Results

We first share data that reveals the expressed meanings that the graduate students conveyed for the idea of AROC when entering the program. We then illustrate their varied fluency in describing their meaning for AROC by sharing excerpts from clinical interviews with experienced participants. This is followed by our contrasting this data with their written descriptions of participants' meaning for AROC provided the week after they had completed teaching the investigations on AROC in a precalculus course using the Pathways materials for the first time. Collectively, our data reveals that the meanings for AROC conveyed by the novice and experienced GTAs differ in terms of the level to which they are able to spontaneously provide a meaningful description of what is involved in understanding AROC; however, it is noteworthy that even after completing the intervention, some GTAs in our study did not shift to speak fluently about the idea of AROC.

Pre-Intervention Meanings for AROC

As a warm-up activity for the start of a Summer 2015 teaching assistant workshop, we asked seven math graduate students to describe the meaning of "average rate of change." Each participant's response is recorded in Figure 2, in the order in which they verbalized their meaning to the group. Their responses align with the authors' prior experiences with both students and teachers at the secondary and tertiary levels; most of the participants provided computational or geometric interpretations based on imagining a secant line between two points on the graph of a function. In particular, we see that Alan[2] described AROC both computationally (i.e., $\Delta y/\Delta x$) and geometrically as a line, instead of highlighting the key attribute of the line—its slope. Another student, Frank, provided two equivalent descriptions of how to *compute* the AROC over a given interval, but did not convey what the result of those computations represent. When prompted by the workshop leader to explain the meaning of the result of the described computations, Frank struggled to communicate his thinking beyond referencing speed, saying, "[The result] represents how fast you're moving in effect. I always think of it as distance and time. It's difficult when you don't know what the quantities are." This issues of describing a meaning for AROC devoid of context arises repeatedly, even among the most experienced GTAs.

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[2] Psuedonyms are used throughout the reporting to protect the identity of participants.

Responses to the question: What does "average rate of change" mean to you?	
Alan:	Delta *y* over delta *x*. A straight line between two points on a graph
Edgar:	Rate of change over an interval and talk about interval as a whole. Describe the rate of change.
Frank:	The amount the dependent variable changes divided by the amount the independent variable changes. Delta *y* divided by delta *x*.
Cassie:	Steepness of a graph, like how steep or how flat it is.
Diane:	Steepness of a graph. […] Uh, I don't have actual words.[…] Slope or derivative.
Brian:	As one variable changes for every one unit, how much is the other variable changing. Slope. Speed.
Greg:	I lost all the words…It's the predictive effect of changing one variable and the amount and how it's going to affect the other variable. One quantity affecting change in another quantity.

Figure 2. Pre-intervention participant descriptions of AROC

Cassie and Diane spoke explicitly about a graph's steepness, a visual aspect of a graph that is simultaneously restricted to the Cartesian coordinate system and, in that setting, is potentially misleading when the coordinate axes do not have the same scale (see Figure 3). Another GTA, Brian, also mentioned slope, though he did so while conveying the idea that slope is an amount of change in the dependent quantity for each unit change in the independent quantity, a restrictive meaning for slope as it fails to support reasoning about variation when changes in the independent quantity have magnitude other than 1. His mention of speed suggests he might have been imagining more than he communicated, but his inability, or perceived lack of need, to coherently communicate with precision about his thinking is exactly one of the characteristic behaviors of the novice GTAs the intervention aims to transform.

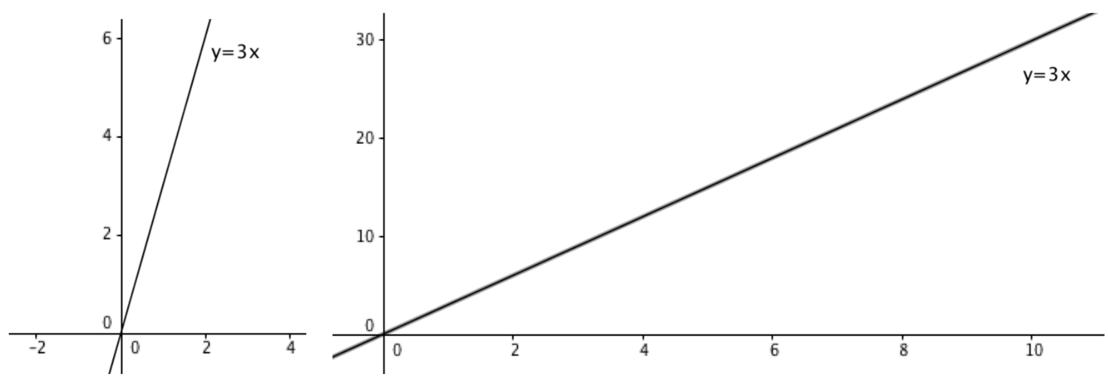

Figure 3. Same relationship but different visual "steepness"

Edgar's description of average rate of change as the rate of change over a whole interval lacks specificity and fails to communicate new information to the label of "average rate of change" beyond involving an interval. Taken literally, saying the AROC describes the rate of change loses all nuances about the possibility of a situation involving a varying rate of change and how AROC allows one to capture information about a dynamic situation even in the absence of information about the actual rate of change. Greg commented on the "predictive" quality of AROC, making him the only participant to explicitly highlight the idea that AROC provides an alternate means for characterizing how two quantities change together. This thinking, however, is missing many elements of what we have previously characterized as a productive meaning for AROC.

The pre-intervention participants' expressed meanings were predominantly geometric, computational, or restricted to particular representation formats (i.e., a graphical representation of a function in the Cartesian coordinate system); moreover, only one of the seven participants spontaneously hinted at the idea that the AROC serves as a tool for characterizing a function's change over some interval of its domain.

Post-Intervention Meanings for AROC

We analyzed 19 clinical interviews with participants who experienced at least one summer workshop and one semester of our intervention. In contrast to the predominantly geometric and computational descriptions of AROC from our pre-intervention participants, 15 of the 19 participants attempted to describe a meaning for AROC that conveyed some significance of the concept beyond a computation to perform (i.e., $\Delta y/\Delta x$) or an image to consider of a particular representation (i.e., secant line connecting two points). These descriptions can be classified as: the productive, general meaning described in our theoretical framework; a special case of that meaning for average speed; or, in one instance, a distinct interpretation the participant called "linearization." The other four participants offered explanations that fall strictly into the last four categories described in Table 1.

Table 1. Experienced participant descriptions of AROC

Expressed Meaning Category	*Sample Excerpts from Clinical Interviews*	*Number of Instances**
Productive – General	[Students] have to understand constant rate of change because the average rate of change is the constant rate of change someone else would have to go, and I'm talking about average speed now, to achieve the same change in output for a given change in input. So, if you don't have meaning for constant rate of change, well, then average rate of change is just this number.	9
Average Speed	[AROC] is a constant rate of change for that specific time and distance, or uh, you know how I mean…	8
Conceptual Other	I would like to say linearization. Right, this idea of approximating something that isn't linear in a linear fashion.	1
Computational	… this final minus initial over the outputs and this final minus initial over the inputs and that's a rate.	4
Geometric	Average rate of change is the constant rate of change to go between two points.	2
Incorrect	I want my students to understand that constant rate of change is a special case, I guess of average rate of change. It's this special case that exists when the corresponding changes in our two quantities are proportional.	3
None		1

* Total exceeds 19 because some interviewees conveyed more than one expressed meaning.

The excerpts in Table 1 highlight the fact that the impact of the intervention on participants is far from uniform. One participant failed to provide a clear statement of a meaning for AROC as he talked around the issue for 14 minutes during his interview. We found this surprising in light of the fact that this participant had four semesters of experience teaching the idea of AROC using curriculum materials designed to support the productive meaning described above. Of the 9 people who conveyed the productive meaning for AROC, four started with the more specific version of average speed and then generalized to a description in terms of general quantities. The other five produced explanations with various degrees of fluency, taking one participant almost 4 minutes to construct his explanation. Four of the eight participants who conveyed a meaning tied to average speed did not convey a meaning for AROC beyond the context of comparing distance and time. In fact, two of those four GTAs only discussed average speed when prompted from the interviewer following their initial response of describing a computation.

The sample excerpt for an incorrect meaning suggests that the participant developed a meaning for AROC linked to CROC in a non-standard way; conventional treatment of the two ideas typically describes AROC as a CROC approximation instead of viewing CROC as a special case of AROC. Another GTA stated that an average rate of change can be found by adding up and dividing, conveying a meaning for AROC that is conventionally identified as an arithmetic mean. Yet another participant proclaimed, "I will forever think of average rate of change as the slope of the secant line." The fact that some of these GTAs did not immediately produce the meaning for AROC supported both by the intervention and the curriculum materials points to the complexity of the idea of AROC and the difficulty that even graduate students had in modifying their strongly held geometric and computational images of the idea of AROC to a more robust scheme with connections that are rooted in a quantitative meaning that can be expressed in multiple representational contexts.

Nonetheless, many participants' expressed meanings did align with our productive meaning for AROC as a way to describe a characteristic of a relationship between varying quantities, even if only in the special case of average speed. Recall that during the intervention, leaders encouraged participants to *speak with meaning* as a tool to support their students in reasoning about quantities. Participants were asked to use appropriate language, describe the underlying meanings of specialized vocabulary (e.g., reference quantities instead of using pronouns like "it", explain "proportional" instead of just using that word), and offer multiple ways of explaining a concept. We see evidence of this practice in the "Productive Meaning" excerpt from Table 1 that was conveyed by a participant with 3 years of experience with the intervention, first as a participant and more recently as a leader. Not only did she express a productive meaning for AROC, using appropriate descriptions that highlighted *changes* in quantities as opposed to *values* of quantities, she further made explicit the connection between CROC and AROC and described the mental imagery she hopes her students develop. She later elaborated the importance of students imagining a second object or scenario that displays a CROC relationship that would yield the same change in output over the given interval of the input quantity.

Similarly, Hannah stumbled slightly, but ultimately described AROC in terms of changes in quantities, as seen in the following interview excerpt:

> *I think one needs to understand that average rate of change means that […] two quantities are varying but not necessarily at a constant rate of change—like the output quantity can, umm, not have a constant factor with respect to the input quantity. But the average rate of change of that relationship would be like if the…if there was a constant rate of change, the same output would be covered for a given amount of input. I think the easiest one for students to*

understand with that is the example of like distance and speed. So if you're driving your car at a constant speed and I am stopping and going and slowing down and speeding up, we will cover the same amount of distance in the same amount of time. And your—the constant rate that you go—is the same with my average rate. But I find that with that example it's really […] hard for students to talk about things not in terms of time. I also find that using the word "average" is confusing to students.

She continued to reflect on a driving context as a familiar example to support students' reasoning about AROC, but demonstrated an awareness of student thinking by highlighting that particular example as potentially problematic for students to generalize beyond contexts dependent on time. She also expressed an awareness of student difficulties with the multiple meanings of the word "average" appearing in the phrase *average rate of change*.

Interestingly, though Hannah demonstrated a relatively high level of fluency in speaking with meaning about AROC, she pointed out that this particular idea is usually difficult for her to discuss with her students, saying:

I was struggling with it, and […] it's just hard to word it in terms of input and output and varying quantities without having a concrete example. And so, to me I'm not even sure that [students are] not getting it so much as that they're not able to articulate it.

Other GTAs expressed similar difficulties in discussing the idea, both during the interview and while teaching. For those GTAs who were not actively involved in the intervention and were also not the lead course instructor (they led the break-out recitation sections while a professor or lecturer provided the lecture and assigned student grades), their discomfort in expressing their meaning for AROC appeared to be more severe; not only were they unable to verbalize a coherent meaning for AROC, but they appeared more uncomfortable in being asked to do so. Examination of the video data revealed that they were more likely to squirm in their chairs, cross their arms, or move away from the interview desk. After the interviewer asked one GTA to describe his meaning for CROC, the GTA first explained how many months it had been since he taught that idea. As the interview progressed and he was asked to explain his meaning for AROC, he stumbled through saying:

The average rate of change is…um…rather than having a, a, um, ok. So the average rate of change is the, um, is again, the relation of two quantities…[explanation omitted]…ahhh…blah. You understand what I'm getting at, I hope. I'm just putting it into words poorly.

Another GTA paused for several seconds as she debated how specific to be in her response and if she would "mess up writing it or verbally saying it." Both of these participants eventually conveyed the productive meaning for AROC, but not without discomfort, lengthy periods of reflection and scratch paper.

Mid-Intervention Meanings for AROC

Because of the extreme difficulty some of the GTAs had in describing a meaning for AROC, we gathered written data from six GTAs at one university while they were actively engaged in the intervention. Participants wrote responses to several tasks asking about their meanings for CROC and AROC during the week after they taught the idea in their respective classes, either as lead instructor or as recitation leader. We attribute some of the variation in productivity of responses to whether or not the GTA was lead instructor or recitation leader (see Table 2). All but one of the GTAs wrote a description of AROC as a CROC satisfying some condition. The first GTA produced a written description of AROC that conveys the productive meaning for AROC *and* uses the same level of precision in language supported in the intervention and curriculum materials. This GTA had extensive experience studying student thinking around the idea of AROC outside of her participation in the intervention and had been selected to help lead the weekly meetings of the intervention during the academic year.

In sharp contrast to the pre-intervention computational and geometric expressed meanings, GTAs 2-6 all refer to the CROC needed to achieve the "same" or "given" change in one quantity relative to the "same" change in the other quantity. At this stage of the intervention, the sharpest criticism of these responses is that they neglect to clarify what these changes are the same *as* (i.e., a mention of an existing relationship between two quantities and the corresponding changes under inspection). In the same set of tasks, the GTAs were further prompted to construct new problem scenarios that would require a student to use the idea of AROC to understand. All five of these GTAs produced coherent problems that did, in fact, rely on the productive meaning of AROC in the context of distance and time. Interestingly, GTA 1 was the only GTA to construct a problem involving two *non*-time quantities, with her example asking a student to determine a vehicle's average rate of change of miles traveled with respect to gallons used based on an odometer reading and information about number of gallons purchased at a gas pump.

Table 2. Written responses to the question "What is *your* meaning for 'average rate of change'?"

<table>
<tr><th>GTA #</th><th>Type of GTA</th><th>Written response</th><th>Meaning Category</th></tr>
<tr><td>1</td><td>Lead instructor</td><td>I imagine 2 quantities x and y changing together, not necessarily at a constant rate of change. As x changes from x_1 to x_2, I imagine that y changes from y_1 to y_2. x_1 and y_1 stand for an arbitrary ordered pair that is fixed. The average rate of change of y with respect to x over the interval (x_1, x_2) is the constant rate of change needed to have the same change in x (that is $x_2 - x_1$) and the same change in y (that is $y_2 - y_1$) as the original function.</td><td rowspan="5">Productive - General</td></tr>
<tr><td>2</td><td>Lead instructor</td><td>The constant rate of change needed to cover the change in the output given the change in input.</td></tr>
<tr><td>3</td><td>Recitation Leader</td><td>The constant rate of change needed to produce the same change in output given the same change in input.</td></tr>
<tr><td>4</td><td>Recitation leader</td><td>AROC is the constant rate of change required to cover the same change in the dependent variable over the same change in the independent variable.</td></tr>
<tr><td>5</td><td>Recitation leader</td><td>It is the constant rate of change that you have to apply to your variable to get a fixed value of your answer. For example, if it is speed: it is the value of speed that you should pick to get a given distance in a given amount of time.</td></tr>
<tr><td>6</td><td>Recitation Leader</td><td>It is the constant rate of change to have traveled the same amount distance in the same amount of time.</td><td>Average Speed</td></tr>
<tr><td>7</td><td>Not teaching</td><td>Sum all the rate of change, the divide by the total number.</td><td>Incorrect</td></tr>
</table>

Surprisingly, GTA 7, who participated in the intervention by attending the summer workshop and the weekly seminar, in hopes of getting a teaching assignment during the subsequent semester, again gave a response highlighting the stability of a meaning for AROC as the arithmetic mean by describing AROC as the arithmetic mean of rates of change. Even more than the interview setting, getting this type of response from a GTA in writing points to the difficulties of getting individuals, even those with sophisticated mathematical backgrounds, to shift their thinking and communication about the idea of AROC.

Discussion

The vast majority of participants held impoverished meanings for the idea of AROC at the beginning of the study, primarily focused on computation or a geometric interpretation restricted to graphical representations of function relationships in the Cartesian coordinate system. Once the GTAs participated in a summer workshop and taught using Pathways research-based curriculum designed to support the productive meaning for AROC, the meanings GTAs conveyed in written responses more closely matched the meaning supported by the intervention. Our interviews with participants who were no longer attending the weekly seminars revealed that these GTAs had retained aspects of the productive meanings for the idea of AROC that we desired. In particular, the majority of post-intervention interviewees attempted to give a meaning for AROC that went beyond a computational or geometric meaning for the idea. What remained variable, however, was the fluency with which they conveyed these somewhat more productive, albeit unstable, meanings.

The initial impoverished meanings expressed by graduate students were widespread across all three institutions, suggesting that that the relatively impoverished meanings expressed by mathematics graduate students' prior to the intervention is likely a widespread phenomena and is in need of further investigation. Moreover, it bears noting that some of these graduate students had prior teaching experience, typically as recitation leader for a range of calculus classes. Yet their experiences in teaching these classes did not support them in building strong, connected meanings for ideas of slope, rate of change, constant rate of change and average rate of change. These findings challenge the assumptions that graduate students in mathematics have strong meanings of fundamental ideas of mathematics, and further, that having taught a course guarantees such meanings will be constructed. In fact, when asked what it means to talk about "rate of change," one GTA responded by saying, "Actually, I've taught calculus before, like Calc 1. Usually in that class, I don't talk about rate of change to them. So I haven't thought about that before." Such a statement suggests that interventions such as what we have described are necessary in fostering reflective teaching practices among future faculty that might lead to more productive and coherent mathematics instruction. Failure to take action to debunk these faulty assumptions by supporting GTAs' development of conceptually coherent and meaningful conceptions of key mathematical ideas may have severe consequences for improving the predominantly procedural focus that exists in many introductory undergraduate courses in colleges and universities across the United States of America (e.g., Tallman & Carlson, 2012).

Graduate mathematics students who hold a meaning for AROC that is strictly geometric (i.e., slope of secant line) will be unable to support their students in developing a quantitative meaning for AROC. It is the quantitative meaning for AROC that an individual leverages to build connections between accumulation functions and rate of change functions, an activity foundational to applying the tools of calculus in contextualized problem scenarios. In particular, those bound by a geometric meaning for AROC are similarly limited to thinking about derivatives in a calculus class as the result of some limiting process for slopes of secant lines. Individuals with this meaning will be hard pressed to make connections between this slope-of-tangent-line meaning and, say, using the derivative to predict population values given a model of population relative to time.

Though not the focus of this research, there was mention during some of the interviews of how the Pathways materials exposed the participants to new ways of thinking about the mathematical ideas. As one GTA said, her participation in the intervention gave her the experience of "living conceptual learning…all of a sudden [she] was being asked questions about linear functions [she] couldn't answer and that really transformed [her] image of what precalculus was and the math that [she] was teaching." Other GTAs emphasized that their new ways of thinking further had positive effects on their own performance as mathematics PhD students. Interestingly, however, these new ways of thinking did not necessarily translate to what the participants had as goals for their students' learning. There was evidence of tension between what the GTAs had experienced as undergraduate students and what they were being asked to do as instructors with regards to discussing mathematics meaningfully.

On a similar note, experiences in working with the graduate students during the interventions produced encouraging anecdotal evidence that the opportunity to reconceptualize fundamental ideas may have a lasting impact on their image of what effective mathematics teaching entails. GTAs with prolonged exposure to the intervention (more than one year, and sometimes in conjunction with a leadership role within the intervention) more frequently commented on both wanting students to develop a more coherent view of the mathematics and the role of understanding their students' thinking in helping them become more effective instructors. One GTA even suggested that, depending on the level of influence his future position will offer, he could envision leading a similar type of program for instructors of a common course to provide a space for discussing the mathematics, student thinking and ways of supporting student learning. This leaves us optimistic that ours and other similar efforts might motivate mathematicians to engage in work to make undergraduate mathematics instruction more meaningful for students.

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Changes in Assessment Practices of Calculus Instructors while Piloting Research-Based Curricular Activities

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We report our analysis of changes in assessment practices of introductory calculus instructors piloting weekly labs designed to enhance the coherence, rigor, and accessibility of central concepts in their classroom activity. Our analysis compared all items on midterm and final exams created by seven instructors prior to their participation in the program (386 items) with those they created during their participation (495 items). Prior exams of the seven instructors were similar to the national profile, but during the pilot program increased from 12.4% of items requiring demonstration of understanding to 33.5%. Their questions involving representations other than symbolic expressions changed from 37.0% to 59.8% of the items. The frequency of exam questions requiring explanations grew from 5.2% to 18.0%, and they shifted from 1.0% to 5.1% of items requiring an open-ended response. We examine qualitative data to explore instructors' attributions for these changes.

Key words: [Calculus, Assessment, Cognitive Level, Representations, Problem-Solving]

One component of the recent national study of calculus programs in the United States (Bressoud, Mesa, & Rasmussen, 2015) examined the assessment practices of instructors of these courses. Tallman et al. (2016) analyzed the content of 150 Calculus 1 final exams sampled from a variety of post-secondary institutions in the larger study. Using their Exam Characterization Framework (ECF), Tallman et al. described the cognitive orientation, mathematical representations, and answer format of each item in their sample. The study demonstrated that few final exam items required a demonstration or application of understanding of the material, primarily involved only symbolic representations, and rarely required explanation or involved open-ended responses. One explanation of these results may be that faculty assessment practices simply reflect the expectations of institutionally adopted curricula. Lithner (2004), for example, found that a majority of exercises in calculus textbooks could be solved by choosing examples or theorems elsewhere in the text based on surface-level features and mimicking the demonstrated procedures.

We examined the assessment practices of pilot instructors implementing activities in their calculus courses designed to simultaneously enhance the coherence, rigor, and accessibility of student learning throughout the course. Project CLEAR Calculus provided weekly labs in which students participated in group problem-solving activities to scaffold the development of central concepts in the course along with instructor training and support to implement the labs. While the project did not address student assessment through exams, we hypothesized the conceptual focus in the labs and requirements of student write-ups would impact the instructors' assessment practices. Our study was guided by the following research questions:

1. How do the pilot instructors' exam questions compare to their previous exams along the three ECF dimensions (cognitive orientation, mathematical representation, and answer format)?
2. What factors do the pilot instructors attribute for any shifts in their assessment practices?

Background

Limit concepts are at the core of mathematics curriculum for STEM majors, but decades of research have revealed numerous misconceptions and barriers to students' understanding. Building off of work by Williams (1991, 2001), Oehrtman (2009) identified several cognitive models employed by students that met criteria for emphasis across limit concepts and for sufficient depth to influence students' reasoning. Williams noted that students frequently attempt to reason about limits using intuitive ideas associated with boundaries, motion, and approximation. Oehrtman found that, unlike most other cognitive models employed by students, the structure of students' spontaneous reasoning about approximations shares significant parallels with the logic of formal limit definitions while being simultaneously conceptually accessible and supporting students' productive exploration of concepts in calculus defined in terms of limits. These findings suggest the commonly presumed dichotomy between a formally sound, structurally robust treatment of calculus on the one hand and a conceptually accessible and applicable approach on the other (Tucker, 1986) is false. By adopting an instructional framework utilizing approximation and error analyses, we designed labs based on criteria listed in Figure 1 intended for weekly use in an introductory calculus sequence.

Design Criteria 1.	Language, notation, and constructs used in the labs should be conceptually accessible to introductory calculus students.
Design Criteria 2.	The structure of students' activity should reflect rigorous limit definitions and arguments without the language and symbolism of formal ε-δ and ε-N notation that is a barrier to most calculus students' understanding.
Design Criteria 3.	The labs should present a coherent approach across all concepts defined in terms of limits and effectively support students' exploration into these concepts.
Design Criteria 4.	The central quantities and relationships developed in all labs should be coherent across representational systems (especially contextual, graphical, algebraic, and numerical representations)
Design Criteria 5.	All labs should foster quantitative reasoning and modeling skills required for STEM fields.
Design Criteria 6.	The sequence of labs should establish a strong conceptual foundation for subsequent rigorous development of real analysis.
Design Criteria 7.	All labs should be implemented following instructional techniques based on a constructivist theory of concept development.

Figure 1. Design criteria for the CLEAR Calculus labs.

When left unguided, students' applications of intuitive ideas about approximations are highly idiosyncratic (Martin & Oehrtman, 2010a, 2010b; Oehrtman, 2009). To systematize students' reasoning concerning approximation ideas and support an accessible yet rigorous approach to calculus instruction, students engage in labs that contain contextualized versions of the questions in Figure 2. These questions develop coherence between structural components, reveal operations performed on these components, and highlight relationships among the operations, all of which is foundational for the generation of new understandings (e.g., Piaget, 1970; von Glasersfeld, 1995; Ernest, 1998).

Question 1.	Explain why the unknown quantity cannot be computed directly.
Question 2.	Approximate the unknown quantity and determine, if possible, whether your approximation is an underestimate or overestimate
Question 3.	Represent the error in your approximation and determine if there is a way to make the error smaller.
Question 4.	Given an approximation, find a useful bound on the error.
Question 5.	Given an error bound, find a sufficiently accurate approximation.
Question 6.	Explain how to find an approximation within any predetermined bound.

Figure 2. Approximation questions consistent across most labs.

Exam Characterization Framework

Tallman et al. (2016) developed a three-dimensional framework to analyze a sample of post-secondary calculus I final exams to get a snapshot of the skills and understandings that are currently being emphasized in college calculus. Their *Exam Characterization Framework* (ECF) characterizes exam items according to three distinct item attributes: (a) *item orientation*, (b) *item representation*, and (c) *item format*.

Item Orientation

Tallman et al. adapted the six intellectual behaviors in the conceptual knowledge dimension of a modification of Bloom's taxonomy (Anderson & Krathwohl, 2001) to characterize the cognitive demand of exam items. The six categories of item orientation are hierarchical with the lowest level requiring students to remember information and the highest level requiring students to make connections (see Table 1).

Table 1
Item orientation codes (Tallman et al., 2016, p. 113)

Cognitive Behavior	**Description**
Remember	Students are prompted to retrieve knowledge from long-term memory.
Recall and apply procedure	Students must recognize what procedures to recall and apply when directly prompted to do so.
Understand	Students are prompted to make interpretations, provide explanations, make comparisons or make inferences that require an understanding of a mathematics concept.
Apply understanding	Students must recognize the need to use a concept and apply it in a way that requires an understanding of the concept.
Analyze	Students are prompted to break material into constituent parts and determine how parts relate to one another and to an overall structure or purpose.
Evaluate	Students are prompted to make judgments based on criteria and standards.

Create	Students are prompted to put elements together to form a coherent or functional whole; reorganize elements into a new pattern or structure.

Item Representation

The item representation domain of the ECF involves classification of both the representation of mathematical information in the task as well as the representation the task solicits in a solution (see Table 2). A task statement or solution may involve multiple representations. Since many tasks can be solved in a variety of ways and with consideration of multiple representations, we observed Tallman et al.'s recommendation of considering only the representation the task requires.

Table 2
Item representation codes (Tallman et al., 2016, p. 116)

Representation	Task statement	Solicited solution
Applied/ modeling	The task presents a physical or contextual situation.	The task requires students to define relationships between quantities or use a mathematical model to describe a physical or contextual situation.
Symbolic	The task conveys information in the form of symbols.	The task requires the manipulation, interpretation, or representation of symbols.
Tabular	The task provides information in the form of a table.	The task requires students to organize data in a table.
Graphical	The task presents a graph.	The task requires students to generate a graph or illustrate a concept graphically.
Definition/ theorem	The task asks the student to state or interpret a definition or theorem.	The task requires a statement or interpretation of a definition or theorem.
Proof	The task presents a conjecture or proposition.	The task requires students to demonstrate the truth of a conjecture or proposition by reasoning deductively.
Example/ counterexample	The task presents a proposition or statement.	The task requires students to produce an example or counterexample.
Explanation	Not applicable.	The task requires students to explain the meaning of a statement.

Item Format

The third and final dimension of the ECF is item format. The most general distinction of an item's format is whether it is multiple-choice or open-ended. However, there is variation in how open-ended tasks are posed. For this reason, Tallman et al. define three subcategories of open-ended tasks: short answer, broad open-ended, and word problem. A short answer item is similar in form to a multiple-choice item, but without the choices. A student can anticipate the form of the solution of a short answer item upon reading the item. In contrast, the form of the solution of a broad open-ended item is not recognizable upon immediate inspection of the item. Broad open-ended items therefore elicit various responses, with each response typically supported by some explanation. Word problems can be of a short answer or broad open-ended format, but prompt students to create an algebraic, tabular and/or graphical model to relate specified quantities in the problem, and may also prompt students to make inferences about the quantities in the context using the model. Also, tasks that require students to explain their reasoning or justify their solution can be supplements of short answer or broad open-ended items.

Exam Characterization Results of the National Sample

Tallman et al. only coded 14.83% of items in their randomly-selected sample of 150 post-secondary calculus I final exams, collectively containing 3,735 items, at the "Understand" level of the item orientation taxonomy or higher. Their coding also revealed that 34.55% of items in their sample involved representations other than symbolic in either the problem statement or solution. Additionally, Tallman et al. found that only 2.4% of the items required explanations from students and just 1.34% of items in their sample were broad open-ended questions.

Methods

Twelve instructors piloted up to 30 labs in 24 different first and second semester calculus classrooms at eight different institutions from Fall 2013 to Spring 2015. Training began with in-person and online meetings with pilot instructors before the start of the fall semesters, and most of the instructors attended a three-day workshop outlining the goals, strategies, and activities of the project. We supported their implementation of the labs throughout the fall and spring semesters with online meetings with project personnel. The project website (http://clearcalculus.okstate.edu) provided instructors with student materials, instructor notes for each lab, solutions, grading rubrics, supporting handouts, and virtual manipulatives. Support meetings frequently included discussions of assessing lab write-ups but did not include discussions of creating or grading exams.

To document changes in the pilot instructors' assessment practices, we collected mid-term and final exams from the calculus classes the instructors taught prior to implementing CLEAR Calculus labs and from the classes in which they were implementing the labs. Five of the instructors either had not previously taught calculus or were required to give exams that were created by other faculty, so exams from these instructors were removed from the comparative sample.

A lead researcher in the development of the ECF and its application in the national study trained two members of our team to code with the framework resulting in 89% agreement between coding the training sample. Subsequent training focused on discrepancies. One member of our team has coded 386 items from 24 exams given by seven instructors prior to using CLEAR Calculus labs and 495 items from 27 exams given by the same instructors while implementing the labs. A random sample of 13% of the items was coded by the second member resulting in 92% agreement.

We also collected self-reported characterizations on the impact of pilot instructors' teaching and exams through their implementation of CLEAR Calculus labs.

Results

Our analysis of exams given by our pilot instructors prior to participating in the project revealed a pattern very similar to the national profile found by Tallman et al. (2016) as shown in Table 3. In contrast, while implementing the labs the instructors nearly tripled the frequency at which they asked questions requiring a demonstration or application of understanding (from 12.4% to 33.5%) and included representations other than symbolic expressions at over 1.5 times the previous frequency (37.0% to 59.8%). They asked for explanations nearly 4 times as often (5.2% to 18.0%) and included broad open-ended items over 5 times as often (1.0% to 5.1%).

Table 3
Shifts in CLEAR Calculus pilot instructor's assessment practices

	Tallman et al. National Sample (3735 items)	Pilot instructors prior to CLEAR Calculus (355 items)	Pilot instructors with CLEAR Calculus (417 items)
Items requiring understanding or higher level reasoning	14.8%	12.4%	33.5%
Items involving representations other than symbolic	34.6%	37.0%	59.8%
Items requiring explanation	2.4%	5.2%	18.0%
Broad open-ended items	1.3%	1.0%	5.1%

Items Requiring Understanding or Higher Level Reasoning

When looking at the individual instructors, five out of seven instructors were below the national average prior to CLEAR Calculus. However, with CLEAR Calculus all seven were above the nation average, which is shown in Figure 5. Moreover, the degree of the shifts in particular is noteworthy.

As an example of the types of shifts we observed, we compare an exam question from one pilot instructor's prior exam in Figure 3, to an exam question from the same pilot instructor while using the labs in Figure 4.

1. Use implicit differentiation to find $\frac{dy}{dx}$ for $3xy - 4 = y^2 + 2x$.
2. Use your answer to question 1 above to find the equation of line tangent to the curve $3xy - 4 = y^2 + 2x$ at the point (2, 4).

Figure 3. Question from a pilot instructor's exam prior to CLEAR Calculus.

Consider the function $f(x) = \arctan(x)$ a) Find the tangent line approximation to $f(x)$ near $x = 1$. (Hint: $\arctan(1) = \frac{\pi}{4}$.) b) Use the equation you found above to approximate $\arctan(1.1)$. c) Is you approximation of $\arctan(1.1)$ an overestimate or an underestimate? Explain how you know.

Figure 4. Question from the same pilot instructor's exam with CLEAR Calculus.

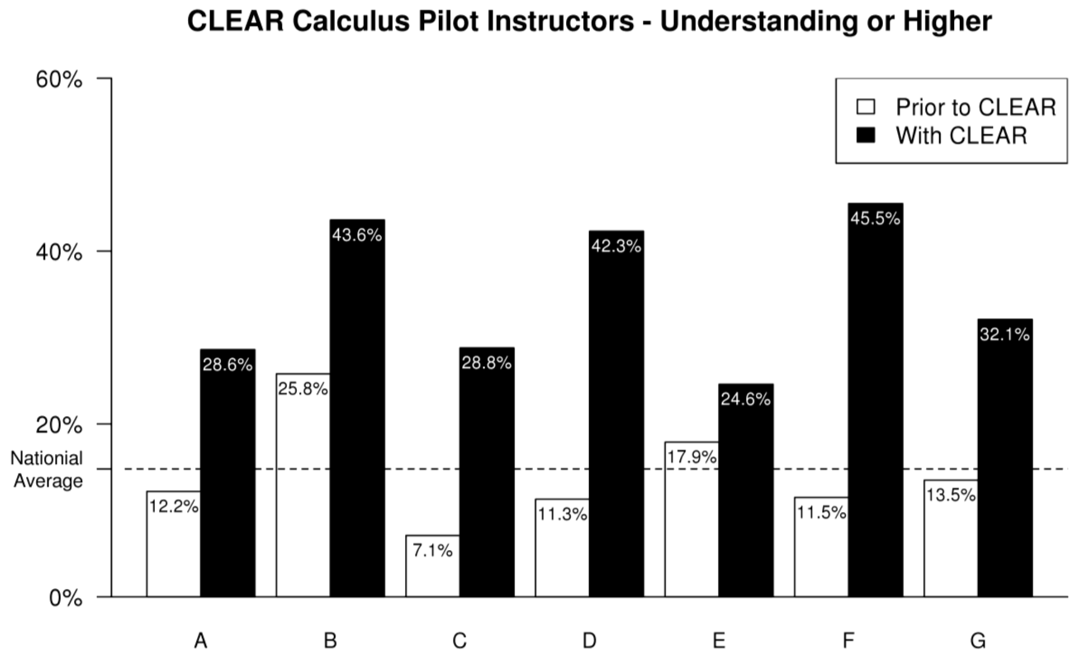

Figure 5. Shits in items requiring understanding or higher level reasoning.

The questions in Figure 3 are standard procedural questions, which demand cognitive behavior at the recall and apply procedure level. However, to perform the task in Figure 4, a student would have to solve a procedural task equivalent to the task in Figure 3 and then apply understanding of the tangent line.

The conceptual emphasis of the labs was a common theme attributed to the shift to more conceptual exam problems in the instructor interviews, highlighted by the following quotation from one pilot instructor.

> The labs had an effect on my learning goals for the class as well as my assessment of them. Because of the deeper nature of lab problems, I was able to focus more on the problem solving process as opposed to answers. When many textbook problems have only one step, it is often difficult to distinguish between these.

Additionally, the following quotation demonstrates one instructor's desire to test more conceptually demanding problems on an exam, despite using conceptually demanding coursework.

> I don't look at the labs as being an assessment. I would categorize them more as an activity. The main value for the students is doing it, and I want them to learn something by doing it, and I'm not so focused on using it as a measure of what they know. I think that in addition to being helpful ways of understanding the calculus

concepts, the notion of approximations and knowing how good they are worthwhile things for students to know themselves, even if they're not kind of inherently part of Calculus. So for that reason, they showed up a lot in class and they showed up on the exam.

Items Involving Representations Other Than Symbolic

In the national study, only 34.6% of the items involved representations other than symbolic. Thus for almost two thirds of the items from the national sample both the question statement and the solution elicited from the students were represented symbolically. Four of the seven pilot instructors in our study were below the national average prior to using CLEAR Calculus. However, while using the labs, six out of seven instructors were significantly above the nation average.

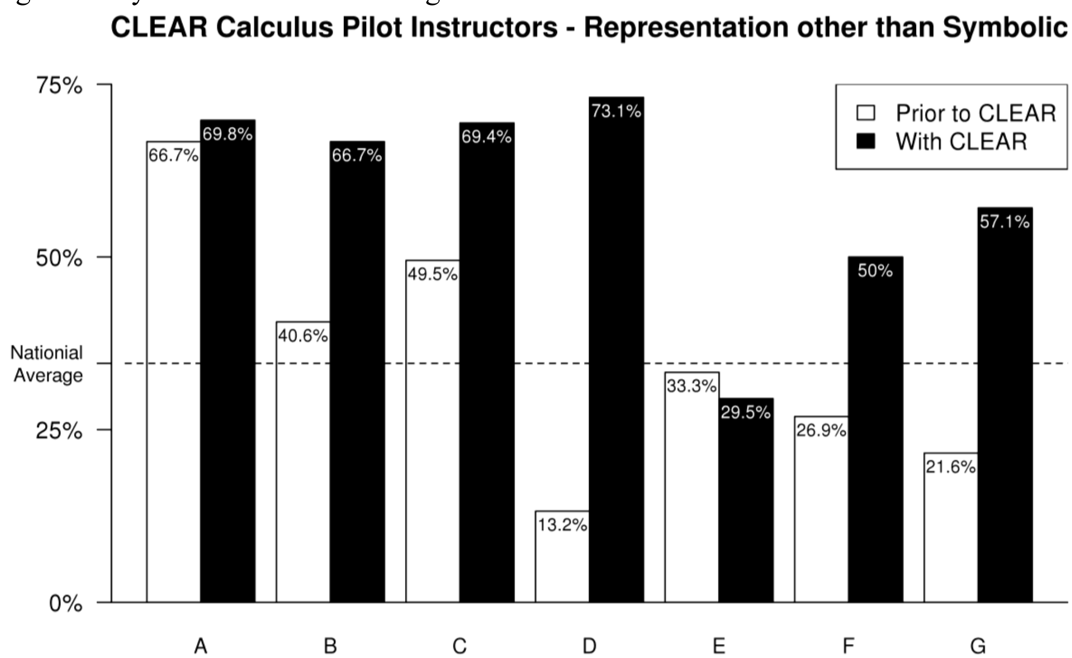

Figure 6. Shifts in items involving representations other than symbolic.

An example demonstrating the shift toward representation other than symbolic is given in Figures 7 and 8.

Determine the critical points of the function $f(x) = ax^2 + \frac{b}{x}$.

Figure 7. Question from a pilot instructor's exam prior to CLEAR Calculus.

Artisanal rocking chair construction has been going particularly well, and the work-space needs to be expanded. The original building has a square footprint of 75 ft by 75 ft. You buy 100 feet of drywall to construct the walls of the new extension. You are going to build a rectangular extension (see figure) to the building using one wall of the existing building and the other three walls from the new drywall. What dimensions should the new extension be to maximize the floor area of the new room?

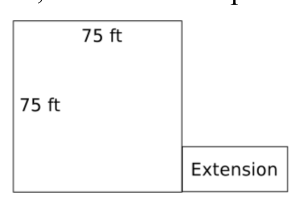

Figure 8. Question from the same pilot instructor's exam with CLEAR Calculus.

In Figure 7, both the presented task and the required solution are represented symbolically. However, Figure 8 presents and applied task, in which students will need to write a symbolic formula similar to the one given in Figure 7 in the process of solving the problem.

In the instructor interviews, the shift to representations other than symbolic was attributed to increased treatment of real-world problems through the labs. Tallman et al. found in the national sample problems were rarely stated in real-world contexts (13.2% of the tasks and 7% of the solutions). In our study, the percentage of applied/modeling items in either the statement or the solution increased from 17.1% to 32.7%. One pilot instructor explained,

> I think that they get at a little more numerical understanding of these ideas that is often not really part of the traditional calculus class. The traditional calculus class, you're world is elementary functions. If you're given an elementary function for something, then you have all these computational tools to deal with it. But in traditional calculus classes, if you have a function that's not an elementary function, then you are stuck. But that's not a good reflection of reality, and in real life you don't get elementary functions handed to you for any particular data that you're trying to understand, even if calculus tools might be useful for that. I think it's even more true in this day and age when anything dealing with elementary functions is very easy for a computer to do. So I think that this was sort of a way to help them have tools that apply more broadly, and if they understand at a conceptual level what a derivative is, and these more numerically based ideas about how to work with them in more than an algebraic context, then that will be more valuable to them.

Items Requiring Explanation

The numbers of items on the majority of the pilot instructors' exams prior to CLEAR Calculus was above the above the national average of 2.4%. While using the labs, all the pilot instructors were significantly above the national average.

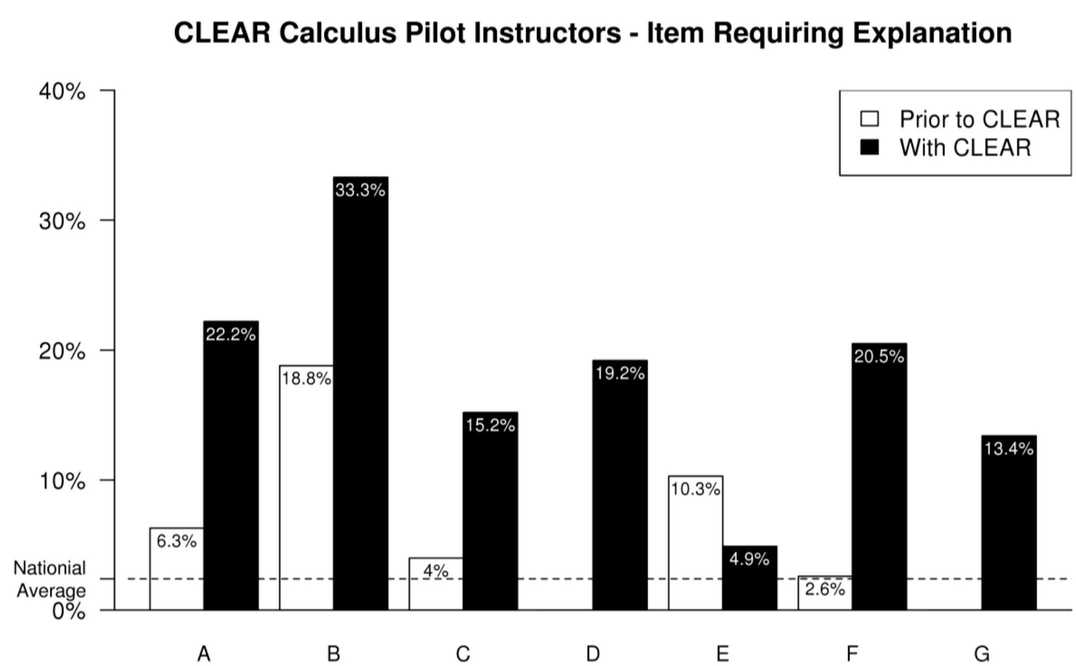

Figure 9 *Shifts in items requiring explanation*

The following Figure is an example from a pilot instructor's exam with CLEAR Calculus.

The following table gives the number of cars, $C = f(t)$, in millions, in the US in the year t.

t	1940	1950	1960	1970	1980	1990	2000
C	27.5	40.3	61.7	89.2	121.6	133.7	133.6

a) Is $f'(t)$ positive or negative during the period from 1940 to 1980? What does this mean about car sales during this period?
b) Is $f''(t)$ positive or negative during the period from 1940 to 1980? What does this mean about car sales during this period?
c) Estimate $f'(1975)$, and include the units for this number. Interpret this answer in terms of passenger cars.

Figure 10. Question from a pilot instructor's exam with CLEAR Calculus.

All three items in Figure 10 require an explanation. This examples also demenstrate a broader shift in assessement practices, which were not neccessarly related to the approximation framework.

One of the goals of the CLEAR Calculus project was mathematical rigor. The following quotation from one pilot instructor highlights how the approximation framework together with explanations is analogous to the rigours $\varepsilon - \delta$ and $\varepsilon - N$ arguemnts in analysis.

> I wanted them to show an understanding of that process, the understanding that this process does guarantee a sufficient accuracy. And what I was hoping to lead them away from was, "Oh I'll just choose a difference quotient between 1 and 1.000001," where they choose an arbitrary number of zeros, "and I'm sure that will probably be good enough." The point here is that this method gives you a guaranteed error bound, right? You know for sure that you are within this interval of the correct answer. And I was hoping for them to be able to articulate that.

Broad Open-Ended Items

Due to the nature of broad open-ended items we would not expect a typical calculus I exam to contain a large number of broad open-ended items. Tallman et al. found that less than 1% of the items in there study were broad open-ended. Three of the pilot instructors increased their inclusion of broad open-ended items on their exams.

The following example taken from a pilot instructor's exam with CLEAR Calculus demonstrates a broad open-ended task.

If you are trying to approximate $g'(2)$, is it possible that $\frac{g(2.1)-g(2)}{.1}$ and $\frac{g(2)-g(1.9)}{.1}$ are **both** underestimates? If so, give an example of such a function g and explain why this is the case. If not, explain why it is impossible.

Figure 11 *Question from a pilot instructor's exam with CLEAR Calculus*

In the task presented, students first have to decide if the situation presented is possible or not.

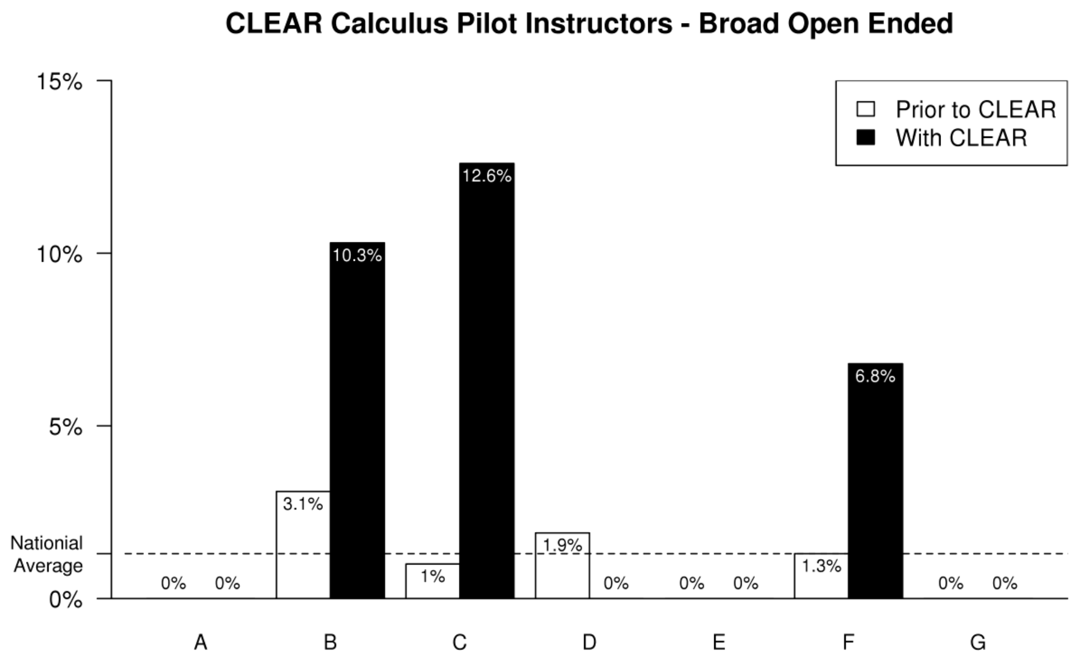

Figure 12 *Shifts in broad-open ended items*

A common theme in the instructor interviews was the goal of conceptual understanding. The following instructor attributed the shift to more broad open-ended items to this goal of conceptual understanding.

> I want to know what they're doing. If they say "overestimate" and they get it right, I don't know if that's because they actually know something about derivatives. This is based on the Approximation Framework... So what I was hoping to find out, do they understand and are they able to connect it with the concavity. It seemed like the most natural way to do that was to do it more open-ended, rather than to try to come up with a question that is designed in such a way that if they don't understand that, they can't get the right answer. It seemed just a little more natural if I'm looking for them to understand a specific thing is to ask them to tell me about that specific thing.

Discussion

All but one of the seven pilot instructors participating in this study demonstrated notable shifts in their assessment practices. Prior to implementing CLEAR Calculus labs in their classes, their exams resembled the national averages in requiring conceptual understanding, using multiple representations, and requiring explanations. While piloting the CLEAR Calculus materials, the seven instructors averaged a 270% increase in the proportion of exam questions requiring conceptual understanding, 162% increase in use of representations other than symbolic, and 346% increase in items requiring explanations. Three of these instructors also asked more questions involving open-ended responses. Due to starting near the low national averages, large shifts are certainly possible with even moderate changes in practices. Nevertheless these shifts were large enough that many of the pilot instructor's exams while using the labs would have constituted outliers in each category among the random national sample analyzed by Tallman et al. (2016).

Pilot instructors viewed their shifts in assessment practices as positive and attributed them to multiple aspects of the CLEAR Calculus labs aligned with the design goals as well as to other broader factors. The important quantities and their interrelationships emphasized in the Approximation Framework guiding the labs are salient, useable, and consistent across all of the primary mathematical representations used in calculus (Figure 1, Design Criteria 4). Instructors cited this coherent use of multiple representations throughout the lab materials for establishing meaningful ways to assess their students using representations other than only symbolic. The labs also engage students in conceptually accessible approximation concepts in a consistent way across all of the calculus concepts defined in terms of limits (Figure 1, Design Criteria 1 & 3). Instructors cited these features for providing important global ideas and a foundation of common reasoning to assess (for example, choosing appropriate tools to find an underestimate or overestimate as required). While remaining conceptually accessible, the labs are also designed to engage students in activities that reflect the structure of standard ε-δ and ε-N definitions and proofs (Figure 1, Design Criteria 2). The pilot instructors cited the connection to rigorous mathematical argumentation as providing an opportunity to require additional explanation or justification from students (for example, justifying that their approximation is sufficiently accurate).

Our continued analysis of this data will explore these patterns in greater depth and provide a foundation for further interviews with the pilot instructors. For example, the most experienced of the pilot instructors was the sole exception to the shifts displayed by the others. Thus, we plan to explore the role of background and experience with each of the instructors to identify their potential connections to changes in assessment practices. Similarly, we seek to better understand the rationale behind shifts in use of questions that are not directly derived from the Approximation Framework questions (Figure 2). Finally, by pairing items on exams and our common Calculus Concepts Assessment administered to most students across the pilot sites, we will identify and characterize any correlation of assessment focus and differences in student learning.

Acknowledgment

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Covariational and parametric reasoning

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Researchers have argued that students can develop foundational meanings for a variety of mathematics topics via quantitative and covariational reasoning. We extend this research by examining two students' reasoning that created an intellectual need for parametric functions. We first describe our theoretical background including different conceptions of covariation researchers have found useful when analyzing students' activities constructing and representing relationships between covarying quantities. We then present two students' activities during a teaching experiment in which they constructed and reasoned about covarying quantities. We highlight aspects of the students' reasoning that we conjecture created an intellectual need for parametric functions, a need we capitalized on in a later session. We conclude with implications the students' reasoning has for future research and curriculum design.

Key words: Covariational reasoning; Quantitative reasoning; Parametric Functions; Cognition

An increasing number of researchers have made contributions to the literature base on students' quantitative and covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Carlson, Larsen, & Jacobs, 2001; Castillo-Garsow, 2012; Confrey & Smith, 1995; Ellis, 2007; Ellis, Ozgur, Kulow, Williams, & Amidon, 2015; Johnson, 2012a; Thompson, 1994a, 1994b). These contributions have been with respect to students' understandings of various content areas (e.g., function classes, rate of change, and the fundamental theorem of calculus) and to their enactment of important mental processes (e.g., generalizing, modeling, and problem solving). Although maintaining the common intention of understanding students' covariational reasoning, researchers' treatments of covariation are varied. For instance, Confrey and Smith (1994, 1995) approached covariation in terms of reasoning about discrete *numerical values*, finding patterns in these values, and interpolating patterns between them. In contrast, Thompson and Saldanha (Saldanha & Thompson, 1998; Thompson, 2011) approached covariation in terms of coordinating changes in two continuous *magnitudes*, thus not constraining covariation to the availability of specified numerical values.

In this work, we illustrate students conceiving of and representing covarying quantities in ways compatible with Thompson's and Saldanha's descriptions of covariation. We identified this reasoning while exploring the research question, "What ways of reasoning do students engage in during activities intended to emphasize reasoning about relationships quantitatively and covariationally?" Before characterizing students' reasoning during a study framed around this research question, we provide our theoretical background including the perspectives on covariation that informed our work. We also describe the methods—teaching experiments and clinical interviews—we used to investigate students' covariational reasoning. We then focus on the students' actions during the closing sessions of the teaching experiment to discuss how the students used graphs in the Cartesian coordinate system to represent relationships they perceived to constitute some situation or phenomena. We highlight the parametric nature of the students' reasoning that we recognized during on-going analysis and describe our efforts at affording the students opportunities to become explicitly aware of the parametric nature of their reasoning. We then provide a discussion of our findings and relate our results to research examining students' understandings of parameters and parametric functions. We conclude by describing research and curricula implications of this study.

Theoretical Background

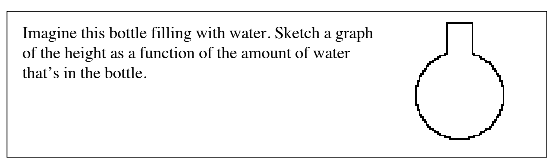
Researchers have drawn from interpretations of Piagetian and radical constructivist theories of knowing and learning to develop definitions and frameworks that describe the mental processes and conceptual structures entailed in reasoning about relationships between quantities (Carlson et al., 2002; Johnson, 2012a, 2012b; Moore & Thompson, 2015, in preparation; Steffe, 1991; Thompson, 1994a, 2011). Of importance to this report, Carlson et al. (2002) presented a framework that allows for a fine-grained analysis of students' covariational reasoning. The authors identified mental actions students engage in when coordinating covarying quantities including coordinating *direction of change* (quantity A *increases* as quantity B *increases*; MA2), *amounts of change* (the *change* in quantity A *decreases* as quantity B *increases in equal successive amounts*; MA3), and *rates of change* (quantity A *increases at a decreasing rate* with respect to quantity B; MA4-5) (Figure 1a).

Carlson et al. (2002) described the aforementioned mental actions in relation to the Bottle Problem (Figure 1b). As they described, when coordinating the relationship between volume of water and height of water in the bottle as water is poured into the bottle, a student first conceives that the two quantities are changing (MA1). The student can then coordinate that as volume of water increases, height also increases (MA2). Next, the student can coordinate that for equal changes in volume, represented by each colored cross sectional area in Figure 1c, successive increases in height decrease until the widest part of the bottle, at which point increases in height increase until the neck of the bottle (MA3). As the student re-constructs this relationship, she may coordinate the average (MA4) and instantaneous (MA5) rate of change of liquid volume with respect to liquid height as water is poured into the bottle. In this report, we focus on students' enactment of MA1-3 as part of their conceiving and representing relationships between covarying quantities.

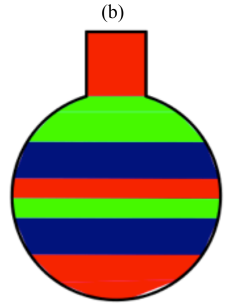
Mental Actions of the Covariation Framework

Mental action	Description of mental action
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable
Mental Action 4 (MA4)	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function

(a)



(b)



(c)

Figure 1: Carlson et al.'s (2002) framework and the Bottle Problem.

We also leverage the work of Thompson and Saldanha to characterize student thinking in this study. Saldanha and Thompson (1998) stated, "Our notion of covariation is of someone holding in mind a sustained image of two quantities' values (magnitudes) simultaneously" (p. 298). The researchers argued students' images of covariation are developmental and described an operative image of covariation in which a student is capable of imagining two quantities being tracked for some amount of time with the correspondence of the two quantities an emergent property of this image. Such an understanding entails a student coupling the two quantities to form a multiplicative object. Saldanha and Thompson (1998) described, "As a multiplicative object, one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value" (p. 298).

Extending this description, Thompson (2011) provided a first-order model of such an understanding in which an individual conceives of a quantity's value, x, varying over (conceptual) time, t. The individual could then conceive of covering the domain of t-values using intervals of size ε, and consider the variation of x within these intervals (i.e. considering x_ε as the set of x-values $(x(t), x(t+\varepsilon)) = x(t_\varepsilon)$). Thompson (2011) concluded his description, "I can now represent a conception of two quantities' values covarying as $(x_\varepsilon, y_\varepsilon) = (x(t_\varepsilon), y(t_\varepsilon))$. I intend the pair $(x_\varepsilon, y_\varepsilon)$ to represent conceiving of a multiplicative object—an object that is produced by uniting in mind two or more quantities simultaneously" (p. 47). Apparent in this description of covariational reasoning is the parametric nature of covariational reasoning; a student imagines two quantities varying with respect to (conceptual or experienced) time, eventually coordinating these two quantities in a way that variation in either quantity necessarily entails variation in the other quantity to form a multiplicative object.

Drawing on the aforementioned perspectives of covariation, Moore and Thompson (2015, in preparation) defined *emergent shape thinking* as a student conceiving graphs in terms of an emergent, progressive trace constituted by covarying magnitudes. We use Figure 2 to represent instantiations of an emergent image of a trace representing liquid height and liquid volume in a bottle covarying as liquid is poured into the bottle. Adopting Thompson's (2011) notation, the student understands $h_\varepsilon = h(t_\varepsilon)$ and $v_\varepsilon = v(t_\varepsilon)$ both increase as time, t_ε, increases. A student with such an image of a graph understands that the magnitude of the blue segment represents the height of liquid in the bottle and the magnitude of the red segment represents the volume of liquid in the bottle at a certain moment of (experiential or conceptual) time, and that the resulting trace is a product of tracking how these two quantities covary with respect to (experiential or conceptual) time. That is, the student understands the graph as being formed by a trace of the multiplicative object $(h_\varepsilon, v_\varepsilon) = (h(t_\varepsilon), v(t_\varepsilon))$ for all values t_ε.

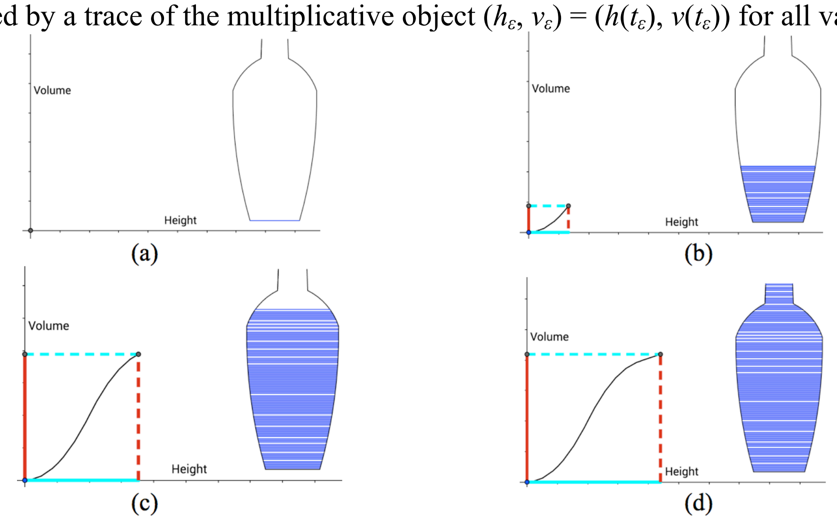


Figure 2: Four instantiations of an emergent conception of height and volume of liquid in a bottle covarying as liquid is poured into a bottle.

Subjects, Methodology, and Methods

We conducted a semester long teaching experiment with two undergraduate students, Arya and Katlyn (pseudonyms). The students were enrolled in a secondary education mathematics program at a large state institution in the southern U.S. Both students were juniors (in credit hours taken) who had successfully completed a calculus sequence and at least two additional courses beyond calculus. The teaching experiment consisted of three individual semi-structured task-based clinical interviews (per student) (Clement, 2000) and 15 paired teaching episodes (Steffe & Thompson, 2000). Each clinical interview and teaching episode lasted approximately 1.25 hours. We video and audio recorded the sessions and we captured and digitized records of the students' written work at the end of each episode.

We used semi-structured task-based clinical interviews (Clement, 2000) as individual pre- and post-interviews to determine the meanings and reasoning the students brought to the teaching experiment and to examine if any shifts had occurred throughout the teaching experiment. Clinical interviews provided us insight into students' meanings without intending to create shifts in their meanings. We used minimal heuristics to probe student thinking as we did not intend to promote shifts in student thinking (Hunting, 1997); we note that although we did not intend to promote shifts, such shifts might have occurred as addressing tasks or questions caused the students to reflect on their activity in such a way that raised perturbations for the students.

We used the teaching experiment as an exploratory tool, giving us firsthand experiences with the students' mathematics and allowing us to explore the mathematical progress the students made over the semester (Steffe & Thompson, 2000). Consistent with the teaching experiment methodology, we began the teaching experiment with a hypothesis we intended to examine, but we also created and tested many hypotheses during the teaching experiment. This sometimes led to our abandoning our initial research question to explore new conjectures created in the moment of interacting with the students (Steffe & Thompson, 2000). Specific to this report, we describe students' activities resulting from one such instance in which we made a conjecture regarding the students' understandings during our on-going analysis that we explored in a later session.

When analyzing the data we conducted a conceptual analysis—"building models of what students actually know at some specific time and what they comprehend in specific situations" (Thompson, 2008, p. 60)—to develop and refine models of the students' mathematics. By *students' mathematics*, we mean the students' body of understandings that are fundamentally unknowable to us as researchers (Steffe & Thompson, 2000). With the goal of building viable models of the students' mathematics in mind, we analyzed the records from the teaching episodes using open (generative) and axial (convergent) approaches (Clement, 2000; Strauss & Corbin, 1998). Initially, we identified instances of Arya's and Katlyn's behaviors and actions that provided insights into each student's mathematics. We used these instances to generate tentative models of the students' mathematics that we tested by searching for supporting or contradicting instances in their other activities. When evidence contradicted our constructed models, we made new hypotheses to explain the students' ways of operating and returned to prior data with these new hypotheses in mind for the purpose of modifying previous hypotheses or characterizing shifts in students' ways of operating.

Task Design

Throughout the teaching experiment, we provided Arya and Katlyn tasks prompting them to represent relationships between covarying quantities. We followed certain principles when designing these tasks. (a) We designed tasks to include situations that would be familiar and accessible to the students, with most tasks including videos, applets, or images of phenomena (e.g., circular motion). (b) We avoided providing specific values for quantities as we were interested in the students' capacity and propensity to engage in reasoning that was magnitude based (i.e., emergent shape thinking). (c) We often asked students to construct multiple graphs related to a situation to explore if, and if so how, the students leveraged their images of the quantities and covariation between quantities when creating multiple graphs that may or may not differ in appearance. This principle was based on our findings (Moore, Silverman, Paoletti, & LaForest, 2014; Moore, Stevens, Paoletti, & Hobson, 2016) that students' meanings for functions and graphs can become problematic when attempting to differentiate and represent numerous relationships (possibly under numerous axes orientations).

To illustrate these design principles, we used a variation of the aforementioned *Bottle Problem*, which was designed by the Shell Centre (Swan & Shell Centre Team, 1985) that several researchers have used productively to investigate students' covariational reasoning (e.g., Carlson et al. (2002), Carlson et al. (2001), Johnson (2012, 2015), Stalvey & Vidakovic (2015)). Reflecting (a) and (b), we provided the students with a pictured bottle that provided no numerical information and asked them to imagine the experience of filling the bottle with liquid. We then asked them to graph the relationship between volume and height of liquid in the bottle as it filled with liquid. Reflecting (c), and after they constructed a graph for a given bottle and a bottle for a given graph, we altered the prompt to ask the students to imagine liquid evaporating from the bottle. We then asked the students to represent the relationship between liquid height and liquid volume in the bottle for this new scenario.

Results

We first summarize the students' activities when creating graphs to represent how the liquid height and liquid volume covaried as a bottle filled. We then present their activities addressing liquid evaporating from the bottle in order to illustrate the students representing an additional aspect of the situation in their graph: the direction in which they imagined the graph tracing. We conclude by highlighting the students' activities on a task that we implemented during a later clinical interview in which we asked the students questions that we conjectured would explicitly raise the notion of a parametrically defined function.

Overview of students' activities addressing the (filling) Bottle Problem

As the teaching experiment progressed, the students exhibited activities indicative of reasoning about graphs as emergent traces representing two covarying quantities they conceived as constituting some situation. For instance, during the first part of the Bottle Problem each student conceived that the two quantities increase in tandem and then determined how the volume of liquid changes for equal successive increases in liquid height; each student coordinated how the volume and height of liquid in a bottle covaried in terms of direction of change (MA2) and amounts of change (MA3). Each student then created a graph while maintaining an explicit focus on how all drawn points and traces represented the relationship she conceived between the height and volume of liquid.

As an example, consider Katlyn's activity as she created her graph (see Figure 3c). Katlyn had already marked equal changes of height in her bottle (Figure 3a) and shaded the first three areas representing the volume in these height intervals (Figure 3b) (Excerpt 1).

Excerpt 1. Katlyn coordinates the relationship between liquid volume and liquid height using the bottle.

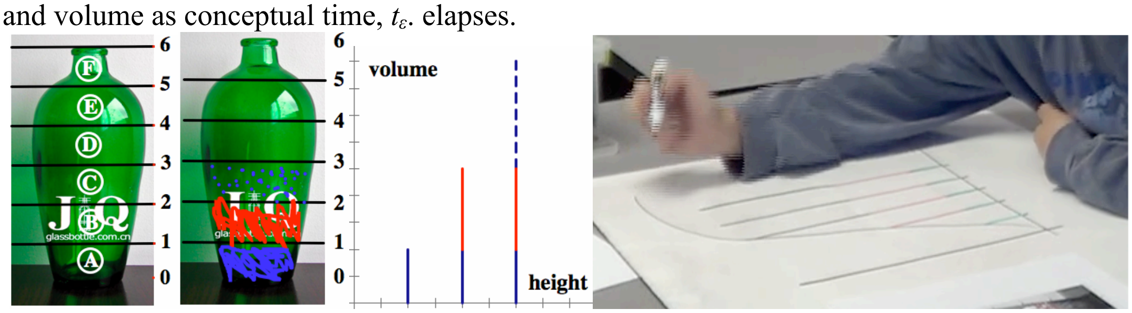
Katlyn: This volume [*the volume in* (A)] is obviously smaller then this one [*the volume in* (B)]. But they're, like when you add them together like that's the height, or that's the volume at that height. And then when we move from here to here [*pointing to* (B) *and* (C)] um this volume [*the volume in* (C)] is also bigger then the one before it [*the volume in* (B)] so when we add it on again it's like continuing to grow… the volume is increasing [*pause*] more with equal changes in height

Katlyn's image of the situation entailed MA1-3 relative to the relationship between liquid height and liquid volume as the bottle filled. Katlyn then drew coordinate axes, marked equal changes along the horizontal height axis, and constructed her graph (Excerpt 2),

Excerpt 2. Katlyn represents the relationship between volume and height in a graph.

Katlyn: This first equal change in height [*pointing to* (A) *in* Figure 3*a*] gives you like this much volume every single time [*draws segments representing the volume in* (A) *at* *each height increment, recreated as solid blue segments in* Figure 3*c*]. So you're always going to have that 'cause we're adding them up. Um and then the next one is going, like when we move up in height again [*motioning from* (A) *to* (B) *in* Figure 3*a*] the next volume that you're going to add on is bigger. But since we're adding them we can add that one to the rest of them [*draws segments extending from the original segment at the second and successive equal changes of height representing the volume in* (B) *recreated as solid red segments in* Figure 3*c*].

Katlyn continued in this way as she constructed her graph representing the relationship she inferred from the situation. She reasoned about the magnitudes of the color-coordinated segments she drew as representing amounts of volume within specific height intervals, understanding that each added segment corresponded to an amount of volume added to the total volume. Katlyn's careful attention to the quantities and use of magnitudes indicates Katlyn maintained an understanding of the trace of her curve as representing two magnitudes as represented in Figure 2. Using Thompson's (2011) notation, she conceived her graph as composed of coordinate points $(h_\varepsilon, v_\varepsilon) = (h(t_\varepsilon), v(t_\varepsilon))$ with $h(t_\varepsilon)$ and $v(t_\varepsilon)$ representing height and volume as conceptual time, t_ε. elapses.



(a) (b) (c) (d)

Figure 3: (a) Katlyn's bottle (numbers and letters added for referencing), (b-c) Katlyn representing total volume with respect to height in the situation and graph, and (d) Katlyn's resultant graph.

Addressing water evaporating from the bottle

After the students had constructed and discussed graphs for multiple bottles, we asked them to graph the relationship between height and volume of liquid in the bottle in Figure 3a as the liquid evaporated. We requested they complete this graph on the same board as a graph representing the relationship between height and volume of liquid in the bottle as the bottle filled. Indicating they did not anticipate that their previous graph could represent the posed relationship, the pair first drew a new set of axes then spent two minutes considering this new relationship and deciding how to label their axes. After they decided to label the horizontal axis height, as in Katlyn's original graph (Figure 3d), Arya noted they should start at "full volume, full height." Katlyn then pointed to her original graph (Excerpt 3).

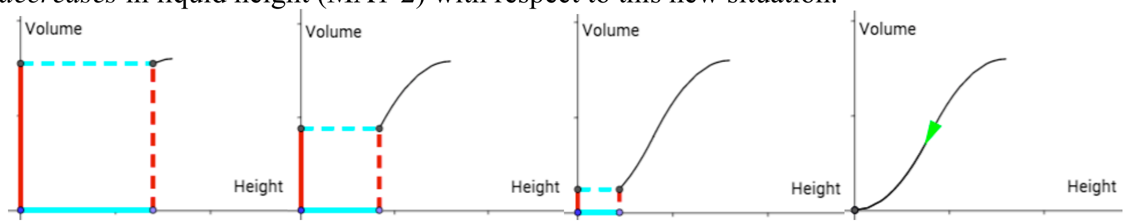
Excerpt 3. Katlyn describes how to represent the relationship between height and volume as water evaporates from the bottle.

Katlyn: It's going to look backwards… We can literally just travel this way instead [*motioning over the completed prior graph from the top-right most point back to the origin*]. [*To the researchers*] We're done, we're just going to travel this way [*again motioning over the original curve from the top-right most*].

[*Shortly after this, Katlyn elaborated as to how she conceived the graph as representing water evaporating from the bottle*]

Katlyn: If we're looking at it like equal changes of height… if we start at this, this height [*pointing to the maximum height value on the horizontal axis*] and this volume [*points to the maximum volume value on the vertical axis then to the top-left most point on the graph*], and then, oh some water evaporated, now we're at this new height [*pointing to tick 4 in* Figure 3b], now we're going to be at this volume [*placing her marker on her curve corresponding to the height and volume at tick 4*]. Okay more water evaporated [*pointing to tick 3 in* Figure 3*b*] now were at this height this volume [*placing her marker on her curve corresponding to the height and volume at tick 3*]… We don't need a new graph, it's the same."

As when addressing water entering the bottle, Katlyn's careful attention to the magnitudes of quantities represented on the axes are indicative of emergent shape thinking. Katlyn now conceived the graph as $(h_{2\varepsilon}, v_{2\varepsilon}) = (h(t_{2\varepsilon}), v(t_{2\varepsilon}))$ with $h(t_{2\varepsilon})$ and $v(t_{2\varepsilon})$ decreasing as conceptual time in this second situation, $t_{2\varepsilon}$, elapses (recreated in Figure 4a-c). Katlyn described this relationship by identifying how liquid volume in the bottle *decreased* for equal *decreases* in liquid height (MA1-2) with respect to this new situation.



(a) (b) (c) (d)

Figure 4: (a)-(c) A recreation of the students' graph as an emergent trace and (d) a recreation of their graph with the added arrow representing the direction of the trace.

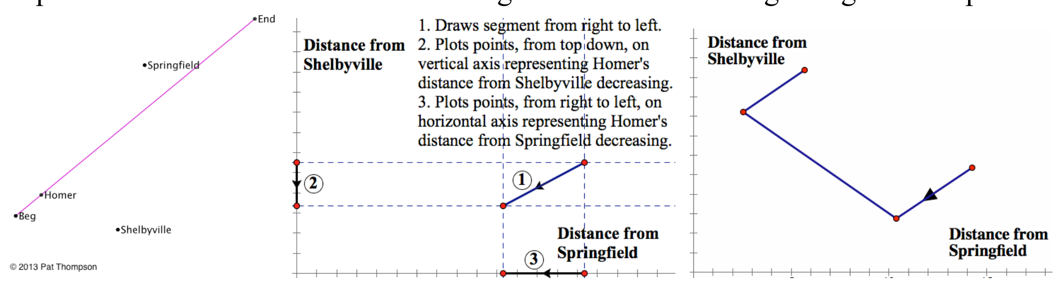
To investigate if using the same curve for a new context created a perturbation for the students, a researcher asked, "Is the situation the same? You're ending up with the same graph." Katlyn responded, "No, I just want to draw little arrows... we're going this way now [*draws an arrow on the curve pointing towards the origin, recreated in* Figure 4d]." As she addressed the displayed graph representing two different situations, Katlyn represented this difference by adding an arrow to indicate the direction in which the graph is traced out with respect to the second situation; Katlyn parameterized her graph (from our perspective) with respect to conceptual time to differentiate how the displayed graph is traced out depending on the situation. Adopting Thompson's (2011) notation, Katlyn understood the displayed graph as composed of points (h, v) representing the appropriate magnitudes of height and volume of liquid in the bottle, regardless if liquid is entering or leaving the bottle. In the first scenario, she understood $(h, v) = (h_{1\varepsilon}, v_{1\varepsilon}) = (h(t_{1\varepsilon}), v(t_{1\varepsilon}))$ with $t_{1\varepsilon}$ representing conceptual time as liquid enters the bottle. In the second scenario she understood $(h, v) = (h_{2\varepsilon}, v_{2\varepsilon}) = (h(t_{2\varepsilon}), v(t_{2\varepsilon}))$ with $t_{2\varepsilon}$ representing conceptual time as liquid evaporates from the bottle.

Addressing the Car Problem

During our on-going analysis, we conjectured that the students' activities addressing how the same graph could represent the relationship between volume and height of liquid either as it enters or evaporates from a bottle may have led to their experiencing an intellectual need (Harel, 2007) for parametrically defined functions. Harel (2007) described, "The term *intellectual need* refers to a behavior that manifests itself internally with learners when they encounter an intrinsic problem—a problem they understand and appreciate" (emphasis in original, p. 13). Specifically, we conjectured the students experienced an intrinsic problem as they attempted to represent a new situation in a graph created to represent a different situation but, ultimately, appearing the same as another completed graph. The students resolved this by adding an arrow to their graph to indicate the direction of the trace producing the graph. Further, we hypothesized that this reasoning had the potential to support them in becoming explicitly aware of the parametric nature of their reasoning as well as possibly bringing to the surface parametric functions.

We intended to explore the extent that we could support the students in bringing the parametric nature of their reasoning to the forefront as they addressed the *Car Problem* that Saldanha and Thompson (1998) designed and used to investigate students' covariational reasoning. We asked the students to represent the relationship between an individual's (Homer) distances from two cities (Shelbyville and Springfield) as he travels back-and-forth on a road (Figure 5a). Because the relationship is such that neither distance is a function of the other distance, we conjectured raising the idea of function after each student constructed a graph might support her in reasoning about an explicitly defined parametric function.

Both students initially described the directional variation (MA2) of each distance (e.g., as Homer moves from the beginning of his trip, the distance from each city decreases). As Arya attempted to represent this relationship in her graph, she drew a segment from right to left getting closer to the horizontal and vertical axis (indicated by (1) in Figure 5b). After Arya re-described the directional relationship she conceived (MA1-2), she moved to her graph and marked points on each axis to confirm her graphed segment represented that Homer's distance from each city was decreasing (MA2) (indicated by (2) and (3) in Figure 5b). Arya continued this process to draw her completed graph (Figure 5c). As in previous situations, Arya conceived her graph as an emergent trace representing two projected covarying magnitudes, indicated by her explicit attention to the magnitudes of the quantities represented on the axes when drawing each segment. Further, and similar to the students' activities addressing the Bottle Problem, Arya added an arrow to her completed graph (Figure 5c) to represent an additional aspect of the situation: the direction the graph was traced in correspondence to how Homer traveled along the road from the beginning of his trip.



(a) (b) (c)

Figure 5: (a) The Car Problem applet, (b) a recreation of Arya's work, and (c) a recreation of Arya's final graph.

Arya subsequently described that her graph did not represent either distance as a function of the other. Hoping to raise the idea of a parametrically defined function, a researcher asked, "What if your input was total distance traveled and your output was two-dimensional?" He then described the output as being composed of both distance from Springfield and distance from Shelbyville. Arya stated that this relationship represented a function as each total distance input corresponded to exactly one pair of distances.

Similarly, addressing whether the relationship with the same two-dimensional output but with 'distance on the path' as the input represented a function, Katlyn identified, "Well that's what [my graph] shows, right?" Katlyn described that for any of Homer's distances on the path there was only one corresponding coordinate point on her graph, concluding that this relationship represented a function. Katlyn added, "I understand, like, what I've been drawing this whole time is like, how I'm traveling on like this purple path. But I don't, I never thought of that as my input, but it really is." Both students were able to assimilate a question concerning a one-dimensional input and two-dimensional output to consider a parametrically defined function after they had engaged in constructing the relationship and the graph as an emergent trace representing their conceived relationship.

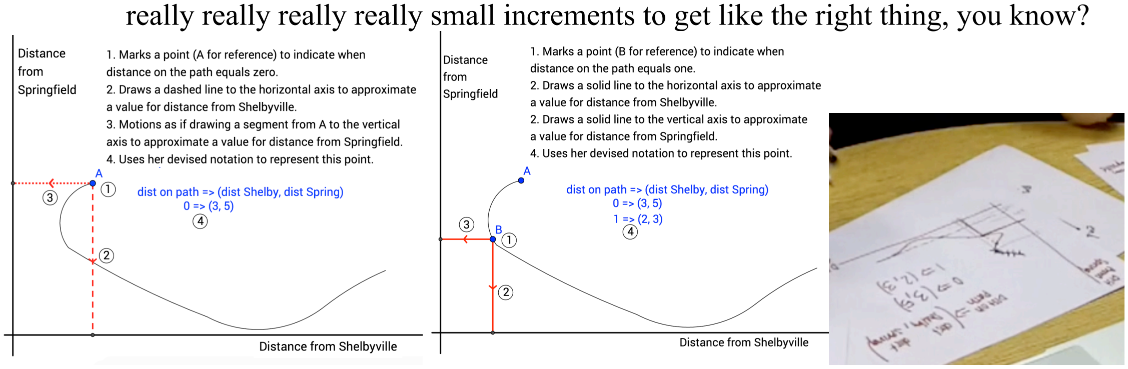
Katlyn spontaneously continued to consider the parametrically defined function. Katlyn wrote "Dist. on Path => (dist. Shelby, dist. Spring)" as a notational system she used to consider specific instances of Homer's distance on the path and the corresponding instances of the two-dimensional output. After labeling the point representing Homer's position on the path at Beg (labeled A for reference in Figure 6a), Katlyn continued (Excerpt 4).

Excerpts 4. Katlyn re-constructs her graph as defined by a parametric function.

Katlyn: So my distance zero on my path [*writes 0 =>, see* Figure 6*a*], and my output, distance from Shelbyville is [*marking a dashed line from A to the horizontal axis, indicated by (2)*] like, um, three. [*laughing*] Whatever, it doesn't matter, and then the distance from Springfield [*motions as if drawing a line from A to the vertical axis, indicated by (3)*], five [*finishes writing 0 => (3, 5)*]. So now I'd say I would start [*pointing to A*] at three five. Then I would do like one [*writes 1 => as seen in* Figure 6*b*]. Pretend that this is one [*tracing along her graph from A to B*].

KM: So by, what do you mean by pretending this is one?

Katlyn: [*pointing to B in* Figure 6*b*] I guess like what, this is like [*pause, points to a position on the road on the computer screen*] one. I don't know, the first portion that I've decided to call one on this purple curve [*using her fingers to indicate an interval starting at Beg along the road in* Figure 5*a*]. This is [*pointing to B in* Figure 6*b*] distance on my path, I've traveled a distance of one on my path then I'm this far away from Shelbyville [*drawing a solid segment from B to the horizontal axis, indicated by (2) in* Figure 6*b*] which is probably like two and then I'm [*drawing solid line from B to the vertical axis, indicated by (3) in* Figure 6*b*] this far away from Springfield which is like [*finishes writing 1 => (2, 3), indicated by (4)*]. And so it makes sense but I'm just like, you'd have to go at like really really really really small increments to get like the right thing, you know?



(a) (b) (c)

Figure 6: (a) and (b) recreations of Katlyn's work and (c) an image of her final work.

As Katlyn conceived of and represented the relationship with a one-dimensional input and two-dimensional output, she created her own notational system for such a relationship. Further, she considered that creating an accurate graph using this definition would require "really really really really small" incremental changes in her input. We note that although she choose specific values for each quantity (e.g., 0 => (3, 5)), to Katlyn the actual values were not critical to her thinking (e.g., "distance from Shelbyville is like, um, three. [*laughing*] Whatever, it doesn't matter"); her reasoning was focused on comparing the magnitudes of the distances from each city with respect to changes in distance on the path and she used fictitious numbers to support her in doing this. Throughout, Katlyn's understanding of the graph as an emergent trace of points representing a coordination of the distance from each city (i.e. a multiplicative object) supported her reasoning parametrically and becoming conscious of this reasoning.

Discussion

In this section we highlight important findings relative to the students' covariational and parametric reasoning. We also relate our findings to existing research on students developing understandings of parameters and parametric functions.

Students' emergent shape thinking

The students developed and maintained images of covariation we interpreted to be compatible with the descriptions of Thompson, Saldanha, and Moore (Moore & Thompson, 2015, in preparation; Saldanha & Thompson, 1998; Thompson, 2011). Specifically, the students' words and actions (e.g., Katlyn saying "we're going this way now" as she traced along her curve recreated in Figure 4), along with their careful attention to the quantities' magnitudes represented along the axes, indicate they imagined their constructed graphs as traces of a point, with the point representing a multiplicative object and the trace involving coordinating two covarying magnitudes.

We highlight the interplay between the mental actions described by Carlson et al. (2002) and the students thinking of graphs emergently. Specifically, the students first constituted the situations in terms of covarying quantities (i.e., at MA1-3 in the Bottle Problem and at MA1-2 in the Car Problem). After conceiving of covarying quantities in the situation, the students represented this relationship using a graph by carefully attending to the quantity on each axis and representing the directions (MA2) and/or amounts of change (MA3) they inferred from their activity constructing the situation. Throughout this process, the students' mental actions (MA1-3) provided a root for the students accurately representing the relationships they conceived. That is, while the students anticipated that their graphs represented emergent traces coordinating two covarying quantities as described by Moore & Thompson (2015), in order to construct an accurate graphical representation they enacted MA1-3 as they constructed relationships with respect to the situation and then enacted MA1-3 in a graphical context in order to construct a graph representing an equivalent relationship.

An intellectual need and explicitly coordinating three quantities

Recall Harel (2007) described, "*intellectual need* refers to a behavior that manifests itself internally with learners when they encounter an intrinsic problem—a problem they understand and appreciate" (emphasis in original, p. 13). We conjecture the students' activities addressing the Bottle Problem raised an intellectual need for parametric functions, a need that we capitalized on with the Car Problem. When addressing water evaporating in the Bottle Problem, the students' activities resulted in their encountering an intrinsic problem: the students came to understand one curve as corresponding to two different experiential situations. This resulted in them seeking to determine how to differentiate between the two situations while using the same completed curve. We conjecture that this intellectual need, which was supported by their emergent shape thinking, was critical to the students considering the parametric nature of the relationships they represented. By understanding one curve as representing two different emergent traces, the students became explicitly aware of their thinking about the curve in terms of two related quantities *and* conceptual time.

When addressing the Car Problem, we interpreted each student's initial activities to indicate her reasoning parametrically about the relationship between Homer's distance from the two cities covarying as Homer's total distance or 'distance on the path' varied. However, the students did not explicitly conceive their graph parametrically until we described a relationship or function with a one-dimensional input and two-dimensional output.

Addressing this question, the students brought to the surface a particular conception of the graph, a graph as an emergent trace, in relation to "function" (i.e. a uniqueness mapping). Both students described such a parametrically defined relationship as representing a function with Katlyn explicitly addressing the novelty of this reasoning (e.g., "I never thought of that as my input, but it really is").

In one study examining students' understandings of parametric functions and parameters, Keene defined dynamic reasoning as "developing and using conceptualizations about time as a dynamic parameter that implicitly or explicitly coordinates with other quantities to understand and solve problems" (2007, p. 231). Arya and Katlyn's reasoning was compatible with Keene's (2007) definition of dynamic reasoning with their initial activities in each problem being compatible with Keene's description of *implicitly* coordinating time with other quantities. Although the students engaged in reasoning that was parametric or dynamic in nature when responding to both tasks, the students did not exhibit activities to indicate they were *explicitly* aware of the parametric nature of their reasoning until they addressed later questions that we designed to focus in this area.

Developing parametric function understandings and the "same graph"

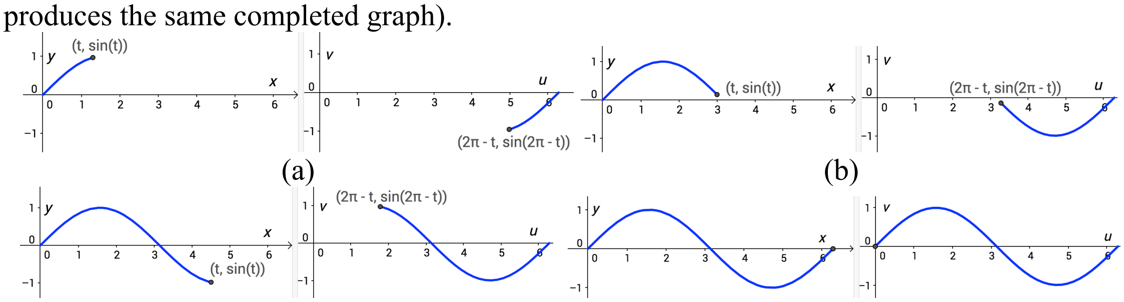
Our data provides evidence that conceiving of and representing relationships between covarying quantities can lead to more formal parametric function understandings. For example, Katlyn's activity addressing the Car Problem indicate how a student who engages in emergent shape thinking and then considers the parametric nature of this reasoning can spontaneously conceive of and use a notation for parametric functions and consider the importance of using small incremental changes of the input to draw an accurate representation of a parametrically defined relationship. Katlyn's activity is compatible with other researchers' findings that indicate students can leverage their quantitative and covariational reasoning to develop foundations for more formal mathematical ideas (Ellis et al., 2015; Johnson, 2012a; Thompson & Carlson, in press).

Related to these findings, Stalvey and Vidakovic (2015) investigated how students conceive an invariant relationship between two simultaneously varying quantities with respect to a third quantity and how this reasoning relates to parametric functions. They were interested in how students might conceive of the rate of change of two quantities as independent from the third quantity (i.e., the parameter). Similar to our prompt involving water evaporating from a bottle, the researchers asked the students to consider water emptying from two identical coolers at different rates to investigate how students might conceive that the rate of change of height with respect to volume is the same regardless of the rate at which water emptied from the cooler.

Stalvey and Vidakovic (2015) presented a genetic decomposition for parametric functions (a model for how an individual might construct an understanding of a specific mathematical concept), which included some features we observed in Arya and Katlyn's reasoning. For instance, Stalvey and Vidakovic described a student imagining a point tracing out a curve in the coordinate system. Further, the researchers' main focus (i.e., the last step of their genetic decomposition) involved students conceiving "a relationship between x and y as invariant" (ibid, p. 193) across two different experiential situations, which is compatible with Arya's and Katlyn's understanding of the graph in Figure 4d as $(h, v) = (h(t_{1\varepsilon}), v(t_{1\varepsilon}))$ and $(h, v) = (h(t_{2\varepsilon}), v(t_{2\varepsilon}))$.

Although our students' activities are compatible with some of Stalvey and Vidakovic's genetic decomposition, our main interest was not on students conceiving of an invariant relationship between the rate of change of height and volume. Instead, our interest was students understanding one graph as being able to represent more than one possible trace. As such, our results provide a contrast to Stalvey and Vidakovic's (2015) genetic decomposition in regards to students' developing parametric function understandings. Specifically, by engaging in emergent shape thinking the students in our study were attentive to the direction they conceived the graph being traced out; these students did not conceive of their graph for water entering and evaporating from the bottle as being "the same" or invariant in that the students understood that these situations resulted in different *emergent*, *progressive* traces.

We note Stalvey & Vidakovic's (2015) interest was in students developing an understanding that the rate of change of two parametrically defined quantities *with respect to each other* does not depend on the parameter and our focus was on students coordinating two covarying quantities without an explicit focus on the rate of change of the two quantities. We conjecture these two foci complement each other. If students first construct and represent different relationships via emergent shape thinking they likely would be better suited to coordinate relative changes of one graphed quantity with respect to the other graphed quantity with the explicit understanding that the rate of change of the two quantities *with respect to each other* does not depend on the parameter. For example, consider two parametric functions, $t \to (x,y)$ and $t \to (u,v)$, $0 \le t \le 2\pi$, such that $(x, y) = (t, \sin(t))$ and $(u, v) = (2\pi - t, \sin(2\pi - t))$. A student who engages in reasoning compatible with Thompson's and Salandha's descriptions of covariation (Moore & Thompson, 2015, in preparation; Saldanha & Thompson, 1998; Thompson, 2011) imagines (x, y) and (u, v) as producing different emergent traces (Figure 7a-c) that result in the same completed graph (Figure 7d). Compatible with Stalvey and Vidakovic's (2015) focus, as the student re-constructs her images of the phenomena and relationships between the quantities, she can choose to consider the relationships between the paired quantities without the parameter t, possibly leading to the student identifying invariances between the relationships defined by $y = f(x)$ and $v = g(u)$ including their respective rates of change. Further, the interplay of these two approaches may lead to students engaging in even more sophisticated and nuanced covariational reasoning (e.g., conceiving in this case as x *increases* from 0 to $\pi/2$, y increases at a decreasing rate and as u *decreases* from $\pi/2$ to 0, v decreases at an increasing rate and produces the same completed graph).



(a) (b) (c) (d)

Figure 7: Four instantiations of the trace of (x, y) and (u, v).

Implications and Future Research

Unlike other researchers who have set out to examine students' understandings of parameters and parametric function in differential equations or calculus settings (Keene, 2007; Stalvey & Vidakovic, 2015; Trigueros, 2004), we intended to examine students' understandings of pre-calculus concepts through their quantitative and covariational reasoning; although this reasoning is parametric in nature (see Thompson, 2011), we did not expect to examine the students developing explicit *parametric* understandings. The fact that the students spontaneously engaged in reasoning that we interpreted as creating an intellectual need for parametric functions has both curricular and research implications.

First, these findings support researchers' conjectures (Carlson et al., 2002; Kline, 1970, 1974; Oehrtman, Carlson, & Thompson, 2008; Smith III & Thompson, 2008; Thompson, 2011; Thompson & Carlson, in press) that curricular approaches providing students opportunities to reason about relationships between quantities can be productive for students' function understandings. Although this report highlighted the students' construction of parametric functions, Arya and Katlyn's activities considering a one-dimensional input and two-dimensional output gives more general insights into the students' function understandings. Each student understood a question about the 'function-ness' of a relationship required explicitly defining input and output quantities then considering if there was a unique output value for each input value. Hence, researchers and curriculum designers might examine how providing students opportunities to construct and represent relationships between covarying quantities can support their developing formal function understandings.

Second, future researchers and curriculum designers might examine how providing students with experiences in constructing graphs as emergent traces can provide foundations for more explicit and formal introductions to parametric functions. For instance, and stemming from the current study ending before we could more extensively pursue the students' reasoning about parametric functions, researchers and educators should further explore how using different situations that result in students constructing and reasoning about the same completed graph via different emergent traces has the potential to create an intellectual need for parametric functions.

Concluding Remarks

Before the onset of the teaching experiment we conjectured that students reasoning about relationships quantitatively and covariationally could be productive for their mathematical learning and set out to further explore students' reasoning in situations we intended to provide them to opportunities to engage in such reasoning. Although we *did not* set out to explore students developing understandings of a particular mathematical concept, the students engaging in reasoning about relationships quantitatively and covariationally led to their constructing, re-constructing, or creating intellectual needs for several content areas typically presented in K-16 school mathematics (e.g., parametric functions, inverse functions, and function more generally).

We find this notable for several reasons. First, this leads us to reiterate researchers' calls (Carlson et al., 2002; Oehrtman et al., 2008; Smith III & Thompson, 2008; Thompson & Carlson, in press) for increased attention to students' quantitative and covariational reasoning in K-16 school mathematics. Second, we highlight that if mathematics education researchers are too strictly focused on students constructing specific content in K-16 school mathematics (e.g., certain function classes), then we as researchers limit our scope to these specific ideas. In such cases, as with Stalvey & Vidakovic (2015), we can construct descriptions that viably explain how someone with a sophisticated understanding of a concept may have developed such an understanding but overlook how students who are novices with these concepts can develop foundational understandings that later support them in developing these more sophisticated understandings (Thompson, 2002). For instance, by focusing on different ways students might conceptualize graphs as emergent traces and the products of such reasoning, we identified students engaging in reasoning that supported them in developing function and parametric function understandings. We conjecture engaging in quantitative and covariational reasoning can support students in developing foundational ideas for other content areas in K-16 mathematics (e.g., coordinate systems, constant and varying rates of change, differentiation) in ways that can be overlooked if we only focus on specific content topics.

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Re-claiming: One way in which conceptual understanding informs proving activity

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Abstract: In this research, I set out to elucidate the construct of Re-Claiming - a way in which students' conceptual understanding informs their proof activity. This construct emerged during a broader research project in which I analyzed data from individual interviews with three students from a junior-level Modern Algebra course in order to model the students' understanding of inverse and identity, model their proof activity, and explore connections between the two models. Each stage of analysis consisted of iterative coding, drawing on grounded theory methodology (Charmaz, 2006; Glaser & Strauss, 1967). In order to model conceptual understanding, I draw on the form/function framework (Saxe, et al., 1998). I analyze proof activity using Aberdein's (2006a, 2006b) extension of Toulmin's (1969) model of argumentation. Reflection across these two analyses contributed to the development of the construct of Re-Claiming, which I describe and explore in this article.

Key words: Mathematical Proof, Conceptual Understanding, Abstract Algebra

Mathematical proof is an important area of mathematics education research that has gained emphasis over recent decades. Inherent in the process of proving is the notion that one must validate (or refute) some mathematical relationship that one might not necessarily know before he or she engages in the activity of proving. Each proof involves the statement of a mathematical relationship, which is either intuitively driven or presented to the individual, and the validity of which is either in question or taken as unknown. The individual then sets out to draw on his or her specific notions about the concepts involved in the relationship in order to show that the relationship is valid relative to his or her own mathematical reality, logic, reasoning, and perception of expectations within a mathematical community in which he or she might intend to communicate such proof activity. Once the relationship is validated (or refuted), there is new potential for the prover to begin to incorporate this new relationship into his or her understanding of the concepts involved (perhaps slowly and over time, perhaps quickly and with immediate consequences). In this brief description of the proving process, one might identify two general interactions: the ways in which a prover's current conceptual understanding informs proving activity in the moment and the potential the individual has to alter his or her understanding of the very concepts about which he or she is proving.

The majority of empirical research in proof focuses on individuals' proof production (e.g., Alcock & Inglis, 2008), individuals' understanding of or beliefs about proof (e.g., Harel & Sowder, 1998), and how students develop notions of proof as they progress through higher-level mathematics courses (e.g., Tall & Mejia-Ramos, 2012). Researchers have also generated philosophical discussions that explore the purposes of proof (e.g., Bell 1976; de Villiers, 1990). Much of this latter discussion centers on the explanatory power of proof (e.g., Weber, 2010), with the primary focus being on the techniques and methods involved in a given proof (e.g., Thurston, 1996), rather than the development of concepts or definitions (Lakatos, 1976). Few studies, however, use grounded empirical data to explicitly discuss the relationships between an individual's conceptual understanding and his or her engagement in proof (e.g., Weber, 2005). Rather, research tends to isolate proof as a discipline in-and-of itself – relatively decontextualized from the specific mathematical conceptions the prover brings to bear in a given situation. In this research, I focus on individual students' engagement in Abstract Algebra proofs that involve inverse and identity. Specifically, I seek to investigate the question: "*How might students' conceptual understanding of identity and* *inverse relate to their proof activity?*" In this article, I introduce a construct, re-claiming, that addresses one way in which conceptual understanding informs proving activity. I then provide examples from one participant's interview responses that illustrate the underlying interactions between conceptual understanding and proof activity involved in re-claiming.

Theoretical Frameworks

In this research I operationalize participants' conceptual understanding using Saxe et al.'s constructs of form and function (Saxe, Dawson, Fall, & Howard, 1996; Saxe & Esmonde 2005; Saxe et al, 2009). Throughout the literature, forms are defined as cultural representations, gestures, and symbols that are adopted by an individual in order to serve a specific function in goal-directed activity (Saxe & Esmonde, 2005). Three facets constitute a form: a representational vehicle, a representational object, and a correspondence between the representational vehicle and representational object (Saxe & Esmonde, 2005). Saxe focuses on the use of forms to serve specific functions in goal-directed activity as well as shifts in form/function relations and their dynamic connections to goal formation. Through this framework, learning is associated with individuals' adoption of new forms to serve functions in goal-directed activity as well as the development of new goals in social interaction. Saxe, Dawson, Fall, and Howard (1996) describe how one might think of learning using *form/function* relations, saying, "Mathematical development in the form/function framework can be understood as a process of appropriating forms that have been specialized to serve developmentally prior cognitive functions and respecializing them such that they take on new properties" (p. 126). Accordingly, the *form/function* framework provides an appropriate theoretical framing for investigating the ways that individuals' understanding of identity and inverse relates to their engagement in the goal-directed activity of proving.

Saxe (1999) discusses how a *form* can be schematized as a vehicle for mathematical meaning. This schematization involves a representational vehicle, a representational object, and a semantic mapping (correspondence) between vehicle and object (p. 23). Saxe (1999) goes on to state that, "inherent in the microgenetic act is a schematization of a correspondence between the latent qualities of the vehicle and object such that one can come to stand for the other" (p. 24). He continues, "individuals structure cultural forms … into means for accomplishing representational and strategic goals. This dynamic process allows for the flexibility of forms to serve different functions in activity, in that the same forms may be structured into means for accomplishing different ends" (Saxe, 1999, p. 26). These quotes draw focus toward the ways in which *forms* are able to shift during goal-oriented activity. This aspect of the *form/function* framework informs the focus of the current study by drawing attention to the ways in which participants structure the *forms* and *functions* upon which they draw during proof activity, specifically with regards to the ways in which specific forms might support varied reasoning within different proving contexts.

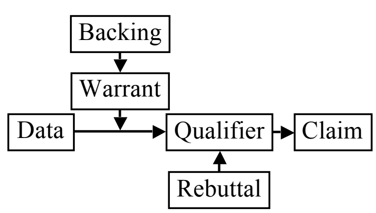

Figure 1. Visual representation of Toulmin models

In order to model participants' proof activity, I use Aberdein's (2006a) adaptation of Toulmin's (1969) model of argumentation. Several researchers have adopted Toulmin's model of argumentation to document proof (e.g., Fukawa-Connelly, 2013). This analytical tool organizes arguments based on the general structure of claim, warrant, and backing. In this structure, the claim is the general statement about which the individual argues. Data are general information, facts, rules or principles that support the claim and a warrant justifies the use of the data to support the claim. More complicated arguments may use backings, which support the warrant; rebuttals, which account for exceptions to the claim; and qualifiers, which state the resulting force of the argument (Aberdein, 2006a). This structure is typically organized into a diagram, with each part of the argument constituting a node and directed edges emanating from the node to the part of the argument that it supports (Figure 1).

Aberdein (2006a) provides a thorough discussion of how Toulmin models might be extended to organize mathematical proofs, including several examples relating the logical structure of an argument to a Toulmin model organizing it. Using "layout" to refer to the graphic organization of a Toulmin model, Aberdein includes a set of rules he to coordinate more complicated mathematical arguments in a process he calls combining layouts: "(1) treat data and claim as the nodes in a graph or network, (2) allow nodes to contain multiple propositions, (3) any node may function as the data or claim of a new layout, (4) the whole network may be treated as data in a new layout" (p. 213). The first two rules are relatively straightforward – the first focuses on the treatment of the graphical layout, as for the second, one can imagine including multiple data sources in the same data node. The third and fourth rules provide a structure for combining different layouts and rely on organizational principles that Aberdein uses. He provides examples of combined layouts (Figure 2).

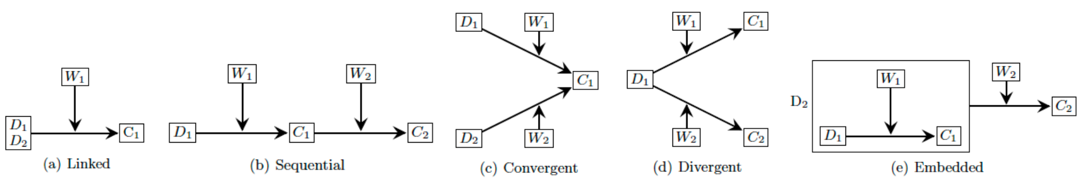

Figure 2. Five Ways of Combining Layouts (Aberdein, 2006a, p.214)

Together, these two theoretical frameworks support the development deeper models of the participants' conceptual understanding and proof activity. Further, juxtaposition of these models, affords more holistic insight into the motivations of participants' proof activity, specifically contextualizing the argumentation that a participant develops in the moment relative to documented consistencies in his or her ways of thinking as well as affording insight into subtle, in-the-moment shifts in the ways that participants draw on and use the very concepts about which they are proving. This directly addresses a student's proof activity as situated relative to his or her conceptual understanding, each of which is treated as an emergent, dynamic facet of the student's mathematical reality. In short, the combined use of these frameworks allows researchers hold both concept and activity as integral and integrated, each continually informing the other throughout the student's proving process.

Methods

Data were collected with nine students in a Junior-level introductory Abstract Algebra course, entitled *Modern Algebra.* The course met twice a week, for seventy-five minutes per meeting, over fifteen weeks. The curriculum used in the course was *Teaching Abstract Algebra for Understanding* (TAAFU; Larsen, 2013), an inquiry-oriented, RME-based curriculum that relies on Local Instructional Theories that anticipate students' development of conceptual understanding of ideas in group theory. Three individual interviews (forty-five to ninety minutes each) took place at the beginning, middle, and end of the semester, respectively. These interviews were semi-structured (Bernard, 1988) and used a common interview protocol so that each participant was asked the same questions as the others. Unplanned follow-up questions were asked during the interview to probe students' descriptions and assertions. The goal for each interview was to evoke the participants' discussion of inverse and identity and engage them in proof activity that involved inverse and identity. I developed initial protocols for these interviews, which were then discussed and refined with fellow mathematics education researchers.

Each interview began by prompting the student to both generally describe what "inverse" and "identity" meant to them and also to formally define the two mathematical concepts. Additional follow-up questions elicited specific details about what the participant meant by his/her given statements, figures, etc. The interview protocol then engaged each participant in specific mathematical tasks aimed to elicit engagement in proof or proof related activity. Each participant was asked to prove given statements, conjecture about mathematical relationships, and describe how he or she might prove a given statement. As with the questions about defining, each of these tasks had planned and unplanned follow-up questions so that all participants were asked at least the same base questions, but their reasoning was thoroughly explored. Throughout the interviews I kept field notes documenting participants' responses to each interview task. I also audio and video recorded each of the interviews, and all participant work and the field notes were retained and scanned into a PDF format. As I shifted to analysis of the data, I narrowed my focus to three of the participants (John, Tucker, and Violet) and transcribed each interview, including thick descriptions of participants' gestures and the timing of pauses in participants' speech.

The retrospective analysis of the three participants' interview responses consisted of three stages, which I ordered so that each stage built upon the previous stages toward a resolution of the general research question. This consisted of an iterative coding process to generate thorough models of the participants' conceptual understanding and engagement in proof and proof-related activity. I analyzed each participant's data separately, coordinating each analysis chronologically so that the model of each participant's conceptual understanding corresponded with his or her proof activity over the semester. I then investigated relationships between each participant's conceptual understanding and proof activity, exploring instances in which meaningful interactions between understanding and activity occurred.

Modeling individual students' conceptual understanding

The form/function analysis for participants' understanding consisted of iterative analysis similar to Grounded Theory methodology (Charmaz, 2006; Glaser & Strauss, 1967). This analysis is differentiated from Grounded Theory most basically by the fact that the purpose of this specific analysis was not to develop a causal mechanism for changes in the students' conceptual understanding, but rather that it was used to develop a detailed model of students' conceptual understanding at given moments in time. For each interview transcript, I carried out an iteration of open coding targeted towards incidents in which the concepts of inverse and identity were mentioned or used. In this iteration, I focused on the representational vehicles used for the representational objects of identity and inverse and pulled excerpts that afforded insight into the correspondence that the participant was drawing between the representational vehicle and object in the moment. Along with the open codes, I developed rich descriptions of the participants' responses that served as running analytical memos. After the open coding, I carried out a second iteration of axial coding using the constant comparative method, in which open codes were compared with each other and generalized into broader descriptive categories. These categories emerged from the constant comparison of the open codes and were used to organize subsequent focused codes until saturation was reached. Throughout this process, I wrote analytical memos documenting the decisions that I made in forming the focused codes and, in turn, providing an audit trail for the decisions made in the development of the emerging categories. This supports the methodology's reliability (Charmaz, 2006).

Modeling proof activity

In the second stage of analysis, I first separated statements that conveyed a complete thought, initially focusing on complete sentences and clauses. I then reflected on the intention of each statement, focusing on prepositions and conjunctions that might serve to distinguish the intentions of utterances that comprise the sentence or clause. Following this, I compared these utterances to the model's constructs, focusing on which node an utterance might comprise. I constantly and iteratively compared each utterance relative to the overarching argument in order to parse out how the utterance served the argument in relation to other statements within the proof. For each proof, I then generated a working graphic organizer (i.e., a figure with the various nodes and how they are connected), including corresponding transcription highlighting the structure of the participant's argument. I then iteratively refined the graphical scheme to more closely reflect the structure of the argument as the participant communicated it. After this process, I completed a final iteration in which I compared the scheme to the participant's communication of the proof in its entirety to ensure that the model most accurately reflected the participant's communication of the proof. An expert in the field then compared and checked the developed Toulmin schemes against transcript of the interview in order to challenge my reasoning for the construction of the scheme, supporting the reliability of the constructions of the Toulmin schemes.

Relating conceptual understanding and proof

During the third and final stage of analysis, I focused on the participants' use of forms and functions within nodes of the Toulmin scheme, comparing the roles that specific forms and functions served in various nodes within the argument. I also focused on the shifts in which the participants' generated new, related arguments, specifically attending to concurrent shifts in forms and functions. I compared across arguments, looking for similarities and differences between the forms upon which the participant drew and the functions that the forms serve within the respective arguments. As in the previous stages, the analysis across conceptual understanding and proof centered on an iterative comparison of the patterns emerging across the analyses of the three participants' argumentation. In this comparison, I noted differences and similarities in the overall structures of Toulmin models for arguments. Further, I attended to the aspects of form/function relations that served consistent roles across similar types of extended Toulmin models. I continuously built and refined hypothesized emerging relationships through constant comparative analysis and memos. Through this process, I characterized constructs that unify the patterns found between the roles forms and functions of identity and inverse served across Toulmin schemes for the three participants.

Results

In this section, I discuss data from Tucker's second (midsemester) interview in order to demonstrate the construct of re-claiming that emerged during the third stage of analysis. I first discuss specific aspects of the form/function model of Tucker's understanding of inverse and identity. These codes of Tucker's conceptual understanding are relevant for discussing selected parts of his response to Question 7 of the protocol, which asked the participants to prove or disprove whether a defined subset H of a group G was subgroup of G (Figure 3). The reader might recognize the set H in question as the normalizer of the element h. This is can equally be thought of as the subgroup of elements that commute with h or the set of elements that fix h under conjugation, as the definition in the problem statement is structured.

Importantly, at the time of the interview, the notions of conjugation and normalizer had not been address in class. The students were familiar with proving whether subsets were subgroups, although, to this point in the semester, this often occurred with specific instantiations of subsets that the students could enumerate, rather than set notational definitions, although the participants were familiar with notation used to define subsets.

"Prove or disprove the following: for a group G under operation * and a fixed element $h \in G$, the set $H = \{g \in G : g*h*g^{-1} = h\}$ is a subgroup of G."

Figure 3. Asking participants to prove about the normalizer of *h*

For the sake of space, I have chosen to share three sub-arguments of Tucker's proof in response to question 7, specifically because they help demonstrate the construct of re-claiming without shifting between different participants' conceptual understanding and proof activity or shifting the focus of mathematical content too drastically. Further, I focus on Tucker's understanding of inverse so that the reader might gain a better sense of how Tucker drew on forms of inverse to serve specific functions in his goal-oriented activity. Tucker's response to Question 7 lasted about 40 minutes and involved several shifts between the three subgroup rules. Because of this, Tucker's proving activity in response to the prompt was modeled with Toulmin schemes for 11 sub-arguments, three of which are discussed in this article. These three sub-arguments provide a glimpse into Tucker's proving process in his effort to show that *H* satisfies the inverse and closure subgroup rules (that *H* contains each of its element's inverse and that the product under the group operation, *, of any two elements in *H* is also an element of *H*).

Form/function codes for Tucker's understanding of inverse

Tucker's discussion throughout the interviews supported the development of several codes for functions of inverse, three of which he used during the parts of his proof activity discussed herein: an "end-operating" function of inverse in which Tucker operates on the same end of both sides of an equation with a form of inverse, a "vanishing" function of inverse in which an element and its inverse are described as being operated together and are removed from an algebraic statement, an "inverse-inverse " function of inverse characterized by an element serving a function of inverse in relation to its inverse, and an "inverse of a product" *function* of inverse in which the inverse of a concatenation (or product) of elements is the reverse order of the inverses of those elements (e.g., $(g*h)^{-1} = h^{-1}*g^{-1}$) . Throughout his proof activity in these excerpts, Tucker draws on the "letter" form of inverse to serve these functions. Although these functions of inverse are consistent with ways inverses are typically used in formal mathematics, Tucker drew on these functions with varying fidelity to their formal treatment depending on the problem contexts in which he was engaged. Tucker continually demonstrated that he was able to use the "letter" form of inverse to serve these functions of inverse in order to appropriately manipulate equations and maintain logically consistent equations. When considering these functions of inverse relative to the Toulmin models of Tucker's proof activity, the functions tended to serve as warrants within Sequential layouts, connecting equations (serving as data) to logically equivalent equations (serving initially as a claim, then as data for the next claim in the sequence). This is evident in two of the three sub-arguments discussed in this article.

Proving that *H* satisfies the inverse subgroup rule

As mentioned, I will discuss three sub-arguments of Tucker's proof that help to explicate the construct of re-claiming. During his response to Question 7, Tucker read over his work saying, "I- you know what I might do actually?" (line 1078). He then explained

So, right now, we have *g* star *h* star *g* inverse is equal to *h*. We want to get to somewhere that looks like- … Want to show. *g* inverse star *h* star *g* is equal to *h*. In order for the inverse of *g* to satisfy this (points to definition of *H*) right here. Cause that's what you do when you put in the *g* inverse. (lines 1084-1086).

With this excerpt, Tucker began a subargument (Figure 4) of his broader, overarching proof for Question 7 in which he attempted to show that the set *H* contains the inverse of each of its elements. He began with the equation used to define *H*, saying, "right now, we have *g* star *h* star *g* inverse is equal to *h*" (line 1085), which serves as initial data (Data1.1) for the argument. He then described wanting to show that $g^{-1}*h*g = h$, which serves as the claim in the subargument (Claim1). He supported this claim by explaining that this goal means that g^{-1} satisfies the given equation, saying, "Cause that's what you do when you put in the *g* inverse" (line 1087). This warrants the claim by reflecting Tucker's previous activity in which he replaced *g* in the equation used to define *H* with its inverse and drew on the "inverse-inverse" function of inverse to rewrite the equation ($g^{-1}*h*g = h$). Although he had discussed the need to show that the set *H* satisfies the inverse subgroup rule earlier in his response, this the first time during his response that Tucker outlined a plan for demonstrating that the set contained inverses. This constitutes a shift in Tucker's description of what it would mean for the set *H* to contain inverse elements. Specifically, he anticipates manipulating the definition of *H* to result in the same equation he had obtained by substituting g^{-1} into the definition of *H*.

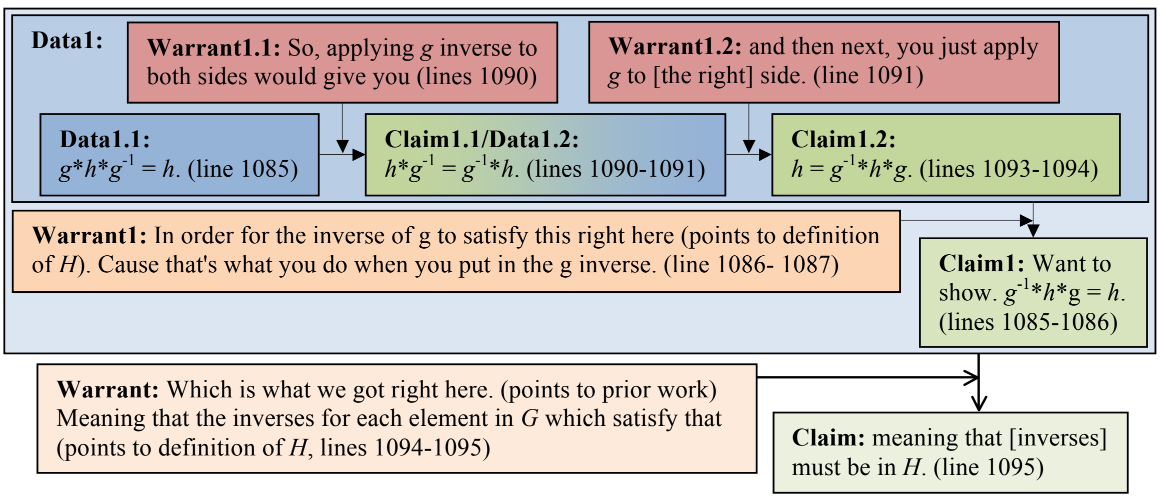

Figure 4. Tucker's inverse subproof in response to Interview 2, Q7

Tucker then continued, explaining how he might manipulate the first equation so that it looks like the second equation. Tucker began by left-operating with g^{-1}, saying, "let's apply the *g* inverse to that. So, applying g inverse to both sides would give you *h* star *g* inverse is equal to *g* inverse star *h*" (Warrant1.1, lines 1089-1091). This process comprises a warrant that draws on the "end-operating" and the "vanishing" functions of inverse to support the claim that a new equation (Claim1.1/Data1.2) can be produced. This equation then serves as data as Tucker describes right-operating with *g* to produce the equation $h = g^{-1}*h*g$ (Claim1.2). Similar to the left-operation with g^{-1}, this draws on the "end-operating" and "vanishing" functions of inverse to warrant the new claim. However, this action also subtly draws on the "inverse-inverse" function of inverse in that Tucker is using the element g as the inverse of its own inverse in order to cancel the g^{-1} on the right end of the left-hand side of the equation. Tucker then interpreted the result of this activity, saying, "Which is what we got right here. Meaning that the inverses for each element in *G* which satisfy that (points to definition of H), mean that must be in H" (lines 1093-1095), which comprises a warrant and claim for the overarching argument that H contains the inverses of its elements.

Tucker's work in this instance exemplifies the construct of re-claiming (Figure 5), which I define as the process of reframing an existing claim in a way that affords an individual the ability to draw on a specific form and the functions that it serves in meaningful (to the student) ways to support the new claim, which the student is then able to connect back to the original claim. In this study, it was often the case that re-claiming occurred when a participant was asked to prove or disprove a general statement and, in response, interpreted the general statement using a specific form to produce a new claim in terms of this form. For instance, in this example, Tucker substituted the "letter" form of inverse into the definition of H to produce the equation $g^{-1}*h*g = h$. An important part of successfully re-claiming is the consistency between the original claim and new claim. In this case, Tucker's new claim successfully reflects the assumption that g^{-1} satisfies the definition of the set H. The individual must also be able to interpret any possible hypotheses or assumptions of the original claim with respect to the new form upon which he or she draws. Again, Tucker successfully uses the assumption that g satisfies the definition of H and uses this as initial data in his construction of an argument. Once the individual generates appropriate initial data from the given hypotheses and assumptions, he or she is then able to draw on the new form to serve specific functions, which affords the development of meaningful argumentation toward the new claim. The various functions of inverse that Tucker was able to draw on to manipulate the equation in the initial data allowed Tucker to produce meaningful connections between the initial data he generated and his re-claimed claim. Finally, after supporting the new claim, the individual should be able to provide a warrant for how or why this claim supports the original claim.

Proving that *H* satisfies the closure subgroup rule

Among the other sub-arguments that Tucker generated during his response to Question 7, he produced two sub-arguments to show that H satisfies the closure subgroup rule which also help demonstrate re-claiming. Tucker describes this subgroup rule, saying, "[we] have to prove that all of those star themselves will yield you back another one" (line 762), which constitutes Claim of this sub-argument (Figure 5). He follows this by describing his approach, saying,

> *a* be an element of *H* and *b* be an element of *H*, then by definition of the set, *a* star *h* star *a* inverse is equal to *h* and *b* star *h* star *b* inverse is equal to *h*. And it looks like we can just, like, set these equal to each other. So, *a* star *h* star *a* inverse is equal to *b* star *h* star *b* inverse. See, what we're trying to do here is prove that *a* star *b* is also in *H* to prove that there's closure. That's what you've got to do to prove closure. So- In order to prove- I'm just gonna write this out. Trying to show that *a* star *b* is an element *H*, which means *a* star *b* (mumbles, writing) - I'm just gonna need to write star between that - star *h* star *a* star *b* inverse is equal to *h*. So that's what we gotta, um, we need to prove. Um- Okay. I think it would be easiest to rearrange (points to $a*h*a^{-1} = b*h*b^{-1}$) this so that we get that (points to $(a*b)*h*(a*b)^{-1}$ = h). Hm. That's a good question. I don't know how we're supposed to do that, though. (lines 764-775)

This excerpt provides transcript of the entirety of this sub-argument, except for Tucker's initial claim. Notice that Tucker begins by fixing two elements of H, a and b. These serve as initial data (Data1) in a sequence in which Tucker draws on the definition of the set to warrant (Warrant1) the generation of two equations: $a*h*a^{-1} = h$ and $b*h*b^{-1} = h$

(Claim1/Data2). These equations then serve as data in order for Tucker to generate a third equation ($a*h*a^{-1} = b*h*b^{-1}$) comprised of the left-hand sides of the first two equations set equal to each other, serving as a second claim that is warranted by the phrase “it looks like we can just set these two equal to each other” (lines 766-767). Together, these data, warrants, and claims follow the structure of a sequential argument that serves as data that Tucker indicates should eventually support the claim that the element $a*b$ must satisfy the equation used to define H.

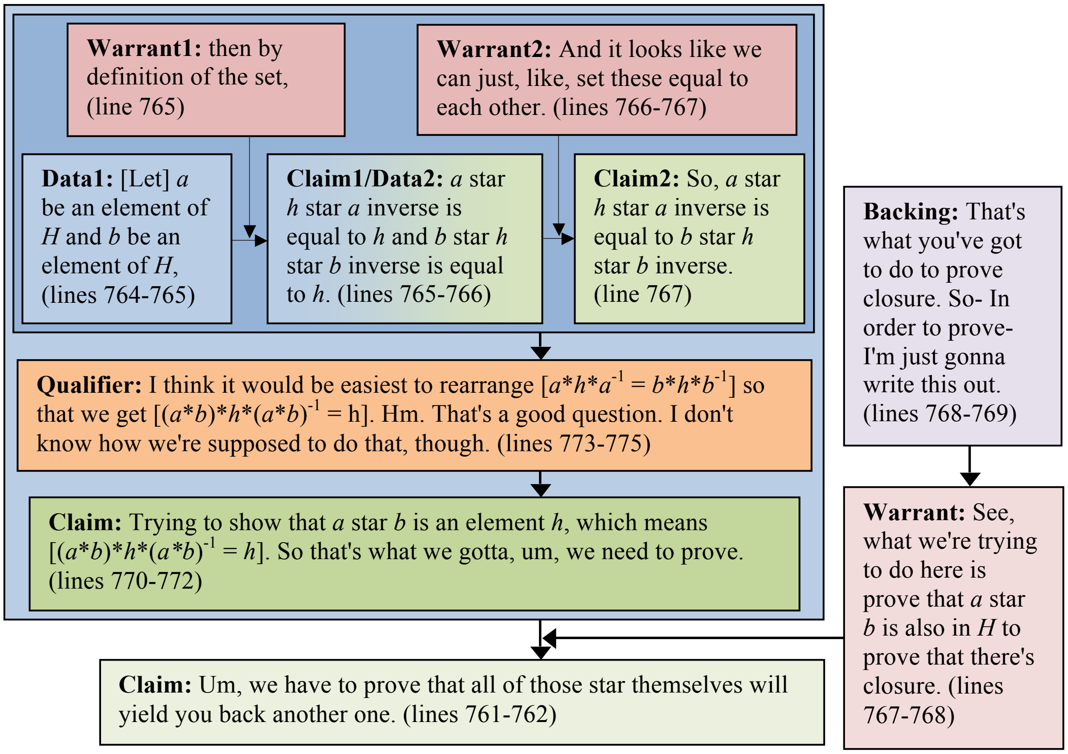

Figure 5. Tucker’s first closure sub-argument in response to Interview 2, Q7

Tucker follows this initial sequential argument by rephrasing the original claim to reflect the definition of the set H. Specifically, similarly to Claim1/Data2, Tucker replaces g and g^{-1} with the algebraic statement “($a*b$),” yielding the equation “$(a*b)*h*(a*b)^{-1} = h$.” This shifts the goal of the proof from a more general description of closure to an algebraic framing that reflects the data that Tucker has produced. He then provides a series of statements that qualify, warrant, and back the embedded argument to support this new claim. First, he alludes to rearranging the equation “$a*h*a^{-1} = b*h*b^{-1}$” to “get that” while pointing to the written “$(a*b)*h*(a*b)^{-1} = h$,” but admits that he is unsure how he might do this, which qualifies his argument. Tucker then provides a warrant for how his data might serve the claim that elements in H satisfy closure, saying, “See, what we're trying to do here is prove that a star b is also in H to prove that there's closure” (lines 767-768). He adds, “That's what you've got to do to prove closure. So- In order to prove- I'm just gonna write this out” (lines 768-769), which serves as backing for this warrant. Tucker’s argument then stalls as he again expresses his uncertainty in the qualifier, saying, “This is a little trickier. I'm open to suggestions here” (line 784). At this point, Tucker has reframed the original claim that he has set out to prove and generated an equation that might serve as initial data in this argument, but reaches an impasse as he is unsure how he might be able to connect the two equations.

Later in his response to Question 7, Tucker returned to the closure subgroup rule, making a new sub-argument (Figure 6) to show that H satisfies the closure rule. Tucker initially indicated that he is still unsure of how to prove that the set satisfies closure. When asked what tools he could use, Tucker replies, "existence of inverses and identity. I don't know" (line 1148). The interviewer reminds Tucker that associativity holds and suggests, "Or you could just un-group stuff" (line 1154). This likely provided Tucker with some insight into a possible approach to the proof, as he said, "And, like, kinda like modify this right side as well, or- (4 seconds) Okay, so, if we bring that over-" and began writing. He then began this sub-argument by saying, "So, we know that a works and b works" (line 1163), which serves as the first data (Data1) in support of the claim, "We wanna show that $a*b$ works" (line 1164, Claim). It is likely that Tucker is using the word "works" here to indicate that they satisfy the definition of H. This sense is supported by Tucker generating the equations $a*h*a^{-1} = h$ and $b*h*b^{-1} = h$. Tucker then provides an argument that reflects his anticipated goal in the qualifier of the prior sub-argument, which he had described, saying, "I think it would be easiest to rearrange [$a*h*a^{-1} = b*h*b^{-1}$] so that we get [$(a*b)*h*(a*b)^{-1} = h$]" (lines 773-774).

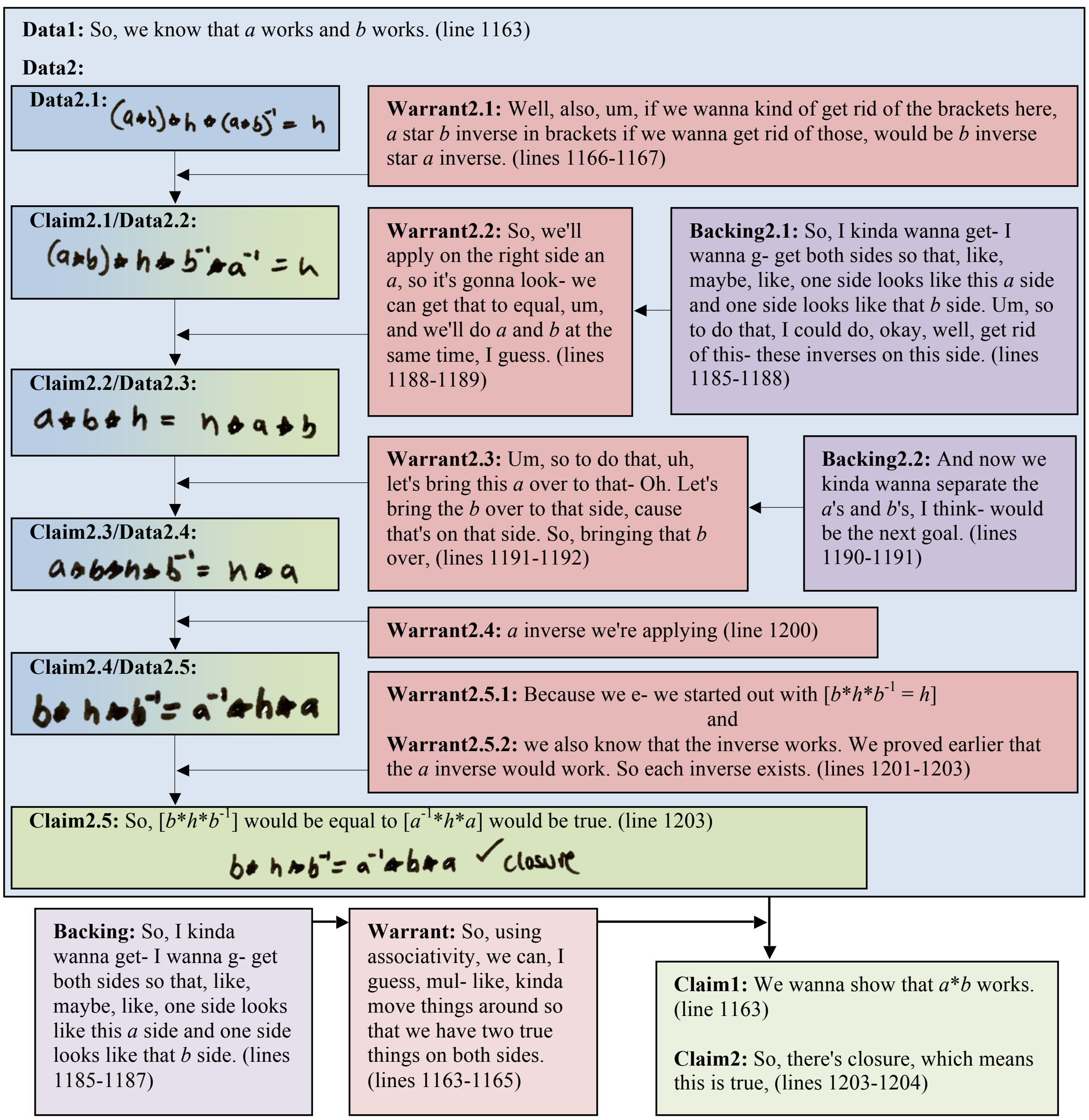

Figure 6. Tucker's second closure sub-argument in response to Interview 2, Q7

Tucker describes how he expects to approach the proof, saying, “So, using associativity, we can, I guess, mul- like, kinda move things around so that we have two true things on both sides” (lines 1163-1165), which serves to warrant Tucker’s re-claiming. He then draws on the “inverse of a product” function of inverse (Warrant2.2) to support rewriting the equation $(a*b)*h*(a*b)^{-1} = h$ (Data2.1) as $(a*b)*h *b^{-1}*a^{-1} = h$ (Claim2.1/Data2.2). Tucker draws on this new equation to serve as data for developing yet another equation, first backing his activity by describing his reasoning for changing the equation, saying,

> So, I kinda wanna get- I wanna g- get both sides so that, like, maybe, like, one side looks like this *a* side and one side looks like that *b* side. Um, so to do that, I could do, okay, well, get rid of this- these inverses on this side. (lines 1185-1188)

This supports a strategy that it seems Tucker anticipates helping to eventually generate the desired equation. Tucker generates the new equation by drawing on the “end-operating” and “vanishing” *functions* of inverse to simultaneously remove the a^{-1} and b^{-1} from the right end of the left-hand side of the equation and concatenate a and b on the right end of the right-hand side of the equation, resulting in the new equation $a*b*h = h*a*b$ (Claim2.2/Data2.3). In doing so, he warrants his activity by saying, “So, we'll apply on the right side an a, so it's gonna look- we can get that to equal, um, and we'll do a and b at the same time, I guess” (lines 1188-1189).

Tucker continues by drawing on the same *functions* of inverse to remove the b that he had just concatenated on the right-hand side of the equation and concatenate b^{-1} on the left-hand side of the equation, essentially undoing part of his activity in which he generated the equation in (Claim2.2/Data2.3). He explains,

> And now we kinda wanna separate the *a*'s and *b*'s, I think- would be the next goal. Um, so to do that, uh, let's bring this *a* over to that- Oh. Let's bring the *b* over to that side, cause that's on that side. So, bringing that *b* over… (lines 1191-1192)

This serves as warrant and backing for Tucker to generate the equation $a*b*h*b^{-1} = h*a$ (Claim2.3/Data2.4). Tucker immediately follows Claim2.3 by saying, “a inverse we’re applying” (line 1200), which warrants the equation $b*h*b^{-1} = a^{-1}*h*a$. Throughout this entire data-warrant-claim sequence, new equations are generated by Tucker’s manipulation of the previous equation, drawing primarily on the “end-operating” and “vanishing” *functions* of inverse, the “inverse of a product” *function* of inverse, and also, implicitly, the “inverse-inverse” *function* of inverse. Having generated the equation $b*h*b^{-1} = a^{-1}*h*a$, Tucker interprets his work, saying, “…we started out with this and we also know that the inverse works. We proved earlier that the a inverse would work. So each inverse exists. So, that would be equal to that would be true. So, there's closure, which means this is true” (lines 1201-1204). In the first part of this excerpt, Tucker provides two warrants (Warrant2.5.1, Warrant2.5.2) that support the claim that the equation $b*h*b^{-1} = a^{-1}*h*a$ is true, seemingly drawing on the equation $b*h*b^{-1} = h$ in Data1 and his prior sub-argument showing that H satisfies the inverse rule for subgroups, which supports the equation $h = a^{-1}*h*a$. However, Tucker does not explicitly draw on these two equations or any sense of transitivity, instead saying “we started out with this” (lines 1201-1202) while pointing to the statement $b*h*b^{-1}$ and “a inverse would work. So each inverse exists” (lines 1202-1203), while pointing to the statement $a^{-1}*h*a$. Tucker finally concludes, “So, there’s closure, which means this is true” (lines 1203-1204, Claim2).

In this sub-argument, Tucker’s re-claiming activity is slightly different from the prior sub-argument. As before, Tucker begins by drawing on a specific form of inverse to reframe the original claim that H is closed under the operation (*). He also generated initial data from the hidden hypotheses of the original claim (considering two elements that are elements of H; Data1). However, the data that Tucker chooses to manipulate in order to reach some conclusion in the Embedded, Sequential Toulmin scheme is the assumption that $a*b$ satisfies the definition of H. He then develops a chain of reasoning that leads back to an equation that Tucker recognizes as verifiably valid based on his assumption in Claim1 that a and b each satisfy the definition of H. This reflects a not-uncommon proof approach in which the prover assumes the result that he or she intends to show and deductively shows that the result is equivalent to the assumption(s) he or she is able to make about the conjecture. Still, Tucker's activity reflects a type of re-claiming in which he was able to draw on a specific form to generate a new claim and initial data. From this claim and data, Tucker leveraged specific functions of inverse to demonstrate that the initial data and new claim were logically equivalent. Tucker's work is warranted by his initial claim that he would try and generate and equation that was true from his assumption that $a*b$ satisfied the definition of H.

Conclusions

This discussion of Tucker's proof activity should afford a sense of the various facets involved in re-claiming. Specifically, in re-claiming, it is not sufficient, to only reframe a claim. Rather, one must likely also reframe its related (often hidden) hypotheses. These aspects of reclaiming reflect the frequently taught proof mantras of "what do I know?" and "what do I want to show?" In this case, Tucker describes needing to show that $g^{-1}*h*g = h$ and begins with the equation $g*h*g^{-1} = h$, which reflects the assumption that g satisfies the definition of H. In the context of the form/function framework, these restated hypotheses serve as initial data (drawing on a specific form of identity or inverse) in a new argument in which the participant is able to draw on the form of identity or inverse with which the data is reframed to serve appropriate functions of identity and inverse in support of the new claim. The individual should then be able to reason that this new argument supports the original claim. In this sense, Re-Claiming provides a type of proof activity in which an individual's conceptual understanding (forms upon which an individual draws and the functions that these forms are able to serve) informs his or her proof approach. Specifically, the access to a form that is able to serve specific functions affords the individual an opportunity to generate a meaningful argument that he or she would likely not have been able to produce without Re-Claiming the initial statement. This activity is not necessarily an inherent necessity of a given conjecture, but rather depends on the individual's understanding in the moment.

The current research was constrained by several factors. First, my focus on three students' responses to individual interview protocols limits analysis of the relationships between conceptual understanding and proof activity, warranting further analysis of different participants' conceptual understanding and proof activity. Also, although this analysis was informed by the broader contexts of the classroom environment, the focus on the individual interview setting affords insight into a specific community of proof in which argumentation develops differently than in other communities. For instance, the structure of the interview setting necessitated that participants developed their arguments solely relying on their own understanding in the moment and for the audience of a single interviewer. My early observations of and reflections on the development of argumentation in the classroom and homework groups included the mutual development of argumentation in which participants' argumentation was informed by their interactions. Accordingly, analysis of the other collected data is warranted.

This research contributes to the field by drawing on the form/function framework to characterize students' conceptual understanding of inverse and identity in Abstract Algebra.

This affords insight into the forms upon which students participating in the TAAFU curriculum might draw as well as the various functions that these forms are able serve. The broader research also contributes to the field by providing several examples of how Aberdein's (2006a) extension of Toulmin's (1969) model of argumentation might be used to analyze proofs in an Abstract Algebra context. Further, this research draws attention to an aspect of the relationships between individuals' conceptual understanding and proof activity. These results situate well among the work of contemporary mathematics education researchers. For instance, Zazkis, Weber, and Mejia-Ramos (2014) have developed three constructs that also draw on Toulmin schemes to model students proofs in which the researchers focus on students development of formal arguments from informal arguments. These constructs provide interesting parallels with the three aspects of relationships between conceptual understanding and proof activity developed in the current research. Zazkis, Weber, and Mejia-Ramos (2014) describe the process of rewarranting, in which an individual relies on the warrant of an informal argument to generate a warrant in a more formal argument. However, the current research focuses more on the aspects of conceptual understanding that might inform such activity.

Moving forward from this research, I intend to analyze the data from other participants' individual interviews in order to develop more form and function codes for identity and inverse, affording deeper insight into the various form/function relations students in this class developed. Such analysis should also explore the proof activity of the other participants in the study, which would provide a larger sample of proof activity, in turn affording new and different insights into the relationships between mathematical proof and conceptual understanding. I also intend to analyze the sociomathematical norms and classroom math practices within the classroom. This will afford insight into the sociogenesis and ontogenesis of forms and functions at the classroom and small group levels in order to support and extend the individual analyses – which are focused on microgenesis – in the current research.

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Physics students' construction and use of differential elements in an unconventional spherical coordinate system

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As part of an effort to examine students' understanding of non-Cartesian coordinate systems when using vector calculus in the physics topics of electricity and magnetism, we interviewed four pairs of students. In one task, developed to compel students to be explicit about the components of specific coordinate systems, students constructed differential length and volume elements for an unconventional spherical coordinate system. While all pairs eventually arrived at the correct elements, some unsuccessfully attempted to reason through spherical or Cartesian coordinates, but recognized the error when checking their work. Analysis of student work identified actions and aspects of students' concept images when constructing differential vector elements. Findings suggest that attention to multiple components and dimensionality were useful aspects of students' concept image, while students had difficulty appropriately applying the idea of projection.

Key words: Coordinate Systems, Differential Elements, Physics, Vector Calculus

It may seem obvious that the farther a physics student progresses within their degree program the more difficult the mathematical computation becomes, but physics does not employ mathematics as merely a calculational tool. It is equally important for students in upper-division physics courses to reason using mathematics in the development of their physical models when problem solving. Many problems in upper-division courses require students to walk a fine line between physical context and mathematical understanding, only to turn around and carry this mathematical reasoning back across the metaphorical high-wire to make sense of the physical context in which they started. For students in junior-level electricity and magnetism (E&M), this is no easy task. In fact, students in upper-division E&M courses have difficulty setting up appropriate calculations, interpreting mathematical results physically, accounting for underlying spatial situations, and accessing appropriate equations or methods of solutions (Pepper et al. 2012). Thus we see that not only is mathematical understanding important, it is an area of difficulty for many students.

E&M courses expect students to employ knowledge of single-variable calculus, multivariable calculus, and vector calculus. Notably, physics students' exposure to vector calculus and non-Cartesian coordinate systems often occurs in physics courses before these topics are taught in a mathematics course sequence. Physics students typically learn relevant concepts and theorems of vector calculus in their E&M course.[1] The reason for the emphasis on vector calculus instruction is largely due to the need for physics students to use vector calculus as a conceptual reasoning tool, as well as the particular application of vector calculus in non-Cartesian coordinate systems. While infinite sheets or plates of charge can easily be solved with use of a

[1] A greater asynchrony occurs in some introductory mechanics courses, in which mechanical work is defined as a line integral. Many departments have a "mathematical methods for physics" course that surveys a wide range of mathematical concepts and techniques. Still, in many cases vector calculus is taught far later in the course than is needed in a parallel E&M course.

Cartesian coordinate system, symmetries exist in common physical systems that favor other coordinates: point charges, spherical volumes, and spherical shells are more easily thought about in spherical coordinates, just as line charges and problems involving straight wires can be made quick work of with cylindrical coordinates. The choice of a coordinate system is one of the many tasks left to the student when problem solving.

Hand in hand with the choice of a coordinate system, students are expected to use differential line, area, and volume elements, as well as position vectors that describe the locations of charges distributed over volumes, surfaces, and lines, in order to set up appropriate integrals. As many E&M equations involve vector calculus (line and area integrals), differential line and area elements are used as vector quantities and thus have a specific direction, while the volume elements are scalar. The ubiquitous nature of the differential element, coupled with coordinate system confusion, present an area of student difficulty at the boundary of mathematics and physics that is ripe for study.

Related Literature

Various physics education researchers have explored student difficulties with the mathematics applied in Electricity and Magnetism (E&M). Early work addressed student use and understanding of integration (Doughty *et al.,* 2014; Meredith & Marrongelle, 2008; Nguyen & Rebello, 2011). As a part of this work, student understanding of the differential element began to come into focus. Nguyen and Rebello identified student difficulty determining the appropriate differential element when setting up integration, as well as particular difficulty with discerning the meaning of a differential area element (Nguyen & Rebello, 2011). Expanding upon this, later work identified what the differential represented in students' conceptual understanding of integration (Hu and Rebello, 2013). In many cases, students interpreted the differential as an identification of the variable of integration; in these cases the differential possessed no further physical meaning. This work highlights the fact that many students often disregard the true meaning of differential when performing calculations and is a common finding in the literature (Artigue *et al.,* 1990; Jones, 2013; Nguyen & Rebello 2011; Sealey & Thompson, 2016).

Little other work has looked at student understanding of differentials in a multivariable or vector calculus setting, where differentials have particular meaning in regards to a specific path of motion or area over which a flux is taken. As part of a larger paper detailing the many student difficulties in E&M, Pepper and colleagues found that students had difficulty articulating complex symmetry arguments and noted student mistakes with differential elements during a homework help session (Pepper *et al*., 2012). One group incorrectly wrote a spherical differential area as $da = d\theta d\phi$, without the necessary $r^2 \sin\theta$ scaling factor needed to make it an appropriate area. Another group used $dxdydz$ as a length element when calculating a line integral and became confused when they recognized that this possessed features of a volume. These types of errors are consistent with both a limited understanding of the differential and difficulties reasoning mathematically when accounting for the underlying spatial situations./

Physics education researchers have identified difficulties in student understanding when applying Gauss's and Ampère's Laws, two key equations of E&M courses that involve a surface integral and line integral, respectively (Guisasola et al., 2008; Manogue et al., 2006; Pepper et al., 2012; Wallace & Chasteen, 2010). Much of this work has looked at how students recognize and apply specific symmetries, due to the constraint of these problems requiring highly symmetric cases. The need for symmetry comes from the "inverse nature" of both laws (Manogue *et al*., 2006); rather than attempting to solve the line or area integral to obtain a result, students must make arguments about the direction and magnitude of the electric and magnetic fields, in order to extricate the field terms from the dot product and integral. Other work in vector calculus has looked at how students have addressed calculation, understanding, and application of gradient, divergence, and curl in both mathematics and physics settings (Astolfi and Baily 2014; Bollen *et al*., 2015). In particular it was found that while students were adept at mathematical calculation, they had little ability to reason physically about gradient, divergence, and curl when applied to vector fields.

When it comes to the teaching and mathematical application of vector calculus, Dray and Manogue highlight numerous disciplinary conventions that can hinder students understanding (Dray and Manogue 1999; 2003; 2004). In particular, the lack of standardization of cylindrical, and spherical coordinates is of significant concern (2003). In mathematics courses, beginning with the introduction of two-dimensional polar coordinates, θ is typically used as the azimuthal angle (rotating about the z-axis). When moving to a three-dimensional coordinate system, it is common practice for mathematicians to use ϕ as the polar angle (measured from the z-axis). In physics the roles of the angles are swapped, with θ as the polar angle and ϕ the azimuthal angle. While the authors do not highlight any student work in particular, results from work published in 2010 looking at students abilities to write $\vec{r}$ in spherical coordinates for six points, each located on a Cartesian axis, touched upon this as a student difficulty (Hinrichs, 2010). Of 28 volunteers, no student was able to correctly answer the original question by writing $\vec{r} = r\hat{r}$, and only slightly fewer than half of the students were able to write the correct r, θ, and ϕ for each point. The most common mistakes were with the writing of the angles, with 20% of the total switching the values for θ and ϕ. While disciplinary conventions can be an obstruction to student understanding early in the course, even when these are addressed, students can still have difficulty constructing these differential elements.

Research Design and Methodology

The overarching goal of this study is to address student understanding and use of vector calculus in upper-division physics. By the end of the first semester of a year-long, junior-level E&M sequence, students are expected to have a working knowledge of spherical coordinates. As part of instruction in the course, students are taught how to build each term of a differential length element, a procedure replicated in the course textbook (Griffiths, 2013). The majority of the following semester is then spent using this coordinate system (as well as cylindrical and Cartesian coordinate systems) in particular problems. This involves making determinations of differential elements to use in integration and construction of position vectors.

Given the importance of these differential elements and different coordinate systems to the calculations, the current focus of the research seeks to address the following questions:

- How do students make sense of and work with coordinate systems, specifically cylindrical and spherical coordinates?
- How do students construct differential vector elements in a given coordinate system?

In order to address these questions further clinical think-aloud interviews were conducted with pairs of students (N=8) at the end of the first semester of a year-long, junior-level E&M sequence. Pair interviews allowed for a more authentic interaction and sharing of ideas between students with minimal influence from the interviewer. This report focuses on a task based on an unconventional spherical coordinate system (see Figure 1). Students were asked to conclude whether the system was feasible, and to build and verify the differential line and volume elements. As students work through these tasks, we are able to see how they reason about the differential elements in a specific coordinate system, thus giving insight into the choice and use of these elements in their problem solving.

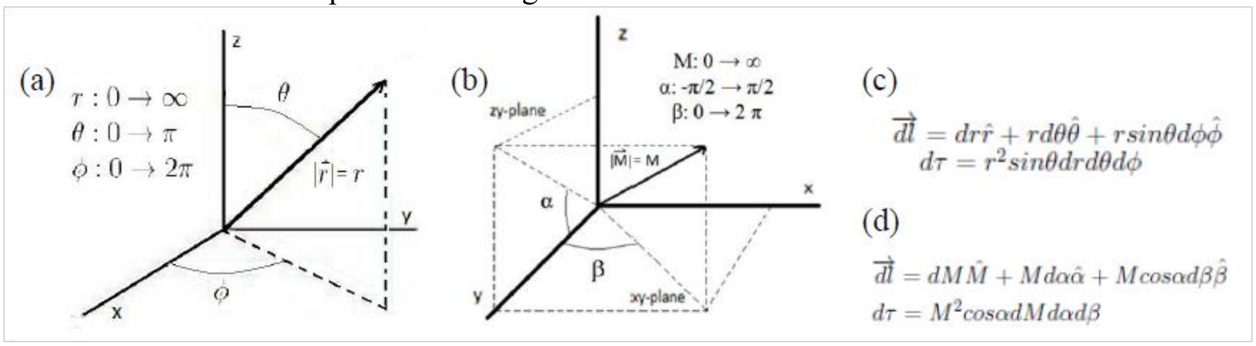

Figure 1. (a) Conventional (physics) spherical coordinates; (b) an unconventional spherical coordinate system given to students, for which they were to construct differential length and volume elements. The correct elements for each system are in (c) and (d), respectively.

The unconventional system, schmerical coordinates, is left-handed and depicts both angles as measured from the y-axis. The left-handed nature allows us to determine if any Cartesian elements presented by students are the result of recall or accurate (but unnecessary) projections. This also means β is placed differently than the analogous φ in spherical coordinates while not changing the length element. Likewise, the placement of α is different than θ. It should be noted that by symmetry and arguments based on the range of the angles, α can also be placed between the vector **M** and the xy-plane. The variation in the placement of the angles from spherical coordinates requires students to recall and use various techniques of building differential elements.

The theoretical framework we adopted for this preliminary stage of the study is that of *concept image* and *concept definition* (Tall and Vinner 1981). A student's concept image represents that student's entire cognitive structure about a particular idea, including any properties, processes, and mental pictures a student may recall. Unlike the actual concept definition, the concept image is a dynamic construct dependent upon specific contexts and may contain elements that are contradictory or false. The idea of concept image has recently been adopted by physics education researchers studying student use of mathematics, particularly in the context of vector differential operators and integration in electromagnetism courses (Bollen et al., 2015; Doughty et al., 2014).

A relevant framework from the physics education research literature is that of *resources* (Hammer, 2000), which is an extension of diSessa's knowledge-in-pieces framework (diSessa, 1993). These frameworks consider the elements of knowledge used in student problem solving in physics. More recent work has extended these to include reasoning elements and epistemological perspectives in the context of using mathematics in introductory physics (Tuminaro & Redish, 2007) as well as mathematical procedures in more advanced physics (Wittmann & Black, 2015).

We are limiting our analytical framework here to that of concept image; it is the more prevalent in the mathematics domain, which is where this work is situated. Future efforts will reconcile this framework with the resources model and possibly align the two frameworks for interdisciplinary work.

With the concept image framework in mind, each interview was videotaped and transcribed for analysis using modified grounded theory (Strauss & Corbin, 1998; Bryant & Charmaz, 2007). The tasks administered here provided us with views of students' concept images of the differential length and volume elements. We also used the activity of element construction to observe the aspects of the concept images presented in student discussion. Analysis of student responses to the unconventional coordinate system allows us to develop a clearer picture of student understanding, as well as specific student difficulties (Heron, 2003) and successes when working with coordinate systems that they apply to particular problems throughout the semester.

Results

In addressing the feasibility of the unconventional coordinate system, all students correctly determined that schmerical coordinates would allow one to map any point in space. Students also made the determination that the unconventional system would be useful for solving the same types of problems as spherical coordinates. The similarity to spherical coordinates was challenging for some students, but upon recognizing that α and θ covered the same range of π radians, students recognized the similarity of the systems.

Following a decision on the feasibility of the system, students were asked to construct a differential length element for the system. Analysis of students' concept images has allowed us to identify four particular aspects that students associate with and use when constructing a differential element as part of our interviews. Table 1 defines each aspect and provides an example of how students attended to and drew upon these aspects during construction.

Aspects	Explanation	Example 1	Example 2
Component & Direction	Writing sum of terms Each component displaced independently	(67) F: Yeah, so like there, dl, there are **three different dl's**. There is dl with respect to M, dl with respect to a, α, and dl with respect to β…	(67) F: …you can change M and you can change α and you can change β, but **if you change α the effects are different from if you change β or you change M.**
Dimensionality	Units of length in each term	(28-30) A:… This **doesn't have any units of length**…so, it needs to have **some M term**.	(89) C:…but **sine** of something **isn't a length,** so we're going to have to also have **something else in there.**
Differential	Small changes (of displacements)	(74) C: Right. So you have **a change in your $\hat{M}$** is going to be your **dM, it's your change in your M**.	(147) E: …ok, **dl_α, so that means we're moving a little bit in α** so we need the r, that's that, there and then times a little α, $d\alpha$.
Projection	Use of cosine/sine explicitly (not rote recall)	(90-92) D: I mean, it's like $M\cos\alpha$ **would put us where we're… down** in the B hat range.	(273) E: …but if we're pointed way up here, then **we need to take the cosine** so that we're, we have the **component of r that is actually in the β plane**, and then we can move a $d\beta$ length amount...

Table 1. Examples of the four aspects identified from student work when building differential length elements in the interviews.

The *component and direction* aspect involved students' attention to the summation of three different components as well as the idea that each component of the vector equation is displaced independently. Many of the students placed emphasis on the aspect of *dimensionality,* specifically attending to the need of each component to have units of length. Students used the aspect of *differential* to talk about needing small displacements or changes in specific directions. Due to the nature of the coordinate system, the aspect of *projection* (obtaining a component of a vector in a particular plane) is relevant to appropriately explain the need for a $\cos\alpha$ in the β-component. However many students did not apply this last aspect to their construction.

In addition to identifying necessary concepts for building there were several actions that students took during the interviews: *rote recall* of length elements from other systems*; mapping* of the variables to spherical or Cartesian; and *grouping* of elements, typically based on variable. *Grouping* as we identify it here is distinguishable from the grouping resource identified by Wittmann and Black (2015), where terms in a differential equation are combined into a single new term. We highlight instances where students engaged in these actions in Table 2.

Actions	Explanation	Example
Grouping	Combining elements by like variables or terms	(47) H: You've got $rdr\,\hat{r}$ plus, **is it sin $\boldsymbol{\theta\ d\theta}$ or is there an $\boldsymbol{r}$ in there?**
Mapping	Direct matching of variables from existing coordinate system	(35-37) B:...so now we have just to compare so we have $\boldsymbol{r}$ **it is $\boldsymbol{M}$, $\boldsymbol{\theta}$ is $\boldsymbol{\alpha}$=...=$\boldsymbol{\varphi}$ is $\boldsymbol{\beta}$.**
Rote Recall	Writing elements from Cartesian or spherical	(112-116) G: **dτ in spherical** is r^2 $sin\theta$=... =$d\theta$ dr=...=$d\varphi$

Table 2. Examples of identified actions taken by students when constructing differential length elements in the interviews.

Students application of recall and mapping versus building of length terms

Each group of students appeared to approach the problem in different ways. Some attempted to reason about the length elements through direct mapping from spherical or Cartesian coordinates. Whether a student chose to build the differential length element from the necessary concepts and ideas or recalled ideas from memory provided insight into how students approach multivariable differential elements in integration in E&M.

When asked to construct a differential length element, the graduate students (A and B) each initially took a different approach.

> A: Alright, let's try, dl, well let's do the easy one first, dM, and I know you don't like this but=
> B: Yes, [laughs]
> A: =it's easy for me, um [draws $\widehat{M}$] So these angles are a bit more difficult, say you do this $d\alpha$. This doesn't have any units of length=
> B: [independently writes differential length element from spherical coordinates]
> A: =so, it needs to have some M term. I think it is just like that, isn't it [writes $Md\alpha$]. For α? [sweeps arm down as if covering the space of the angle] Yeah.
> B: You can, you can check from this, um…
> A: For α it doesn't have any dependence on this other angle over here, but when you're talking about β, um [looking at the spherical $\overrightarrow{dl}$ that B wrote]
> B: So this is dl [gestures to spherical differential he wrote], okay? dr r [hat], $rd\theta$ θ [hat],=
> A: No, I have this backwards (erases α terms)
> B:= $r\sin\theta\ d\varphi\ \varphi$ [hat], so now we have just to compare so we have r it is M, θ is α=
> A: (writes β's in place of α terms)
> B: =φ is β.

We see from this exchange that student A is attempting to reason using the aspects of *component and direction* and *dimensionality*, while student B is attempting to make use of the existing spherical coordinates using *recall* and *mapping*. Once student B makes the direct mapping, the two students work together and finish the construction of the differential element so that it mirrors the spherical length element and includes $\sin \alpha$.

It is notable to mention that the actions made by student A in the last few lines of the transcript were later illuminated as due to a use of the mathematical convention for spherical coordinates. This would be acceptable as long the angles were also changed in a student's description of the differential element, which was not the case for student A. Using limits for the angles from the mathematical convention coupled with a physics interpretation of the differential volume element results in a value of zero for integration (of $\sin \alpha \, d\alpha$ from $-\pi/2$ to $\pi/2$) along with potential for several conceptual inconsistencies, as seen here. The two students drew a spherical coordinate system and student B instituted the physics convention, allowing A to fix his mistake.

The second pair of students, C and D, progressed in a similar fashion but spent more time discussing the choices and reasons for their actions. The pair began building using all four aspects but did not attend appropriately to the differential when constructing the β length component, as depicted in the following exchange.

> D: I mean, it's like $M \cos \alpha$ would put us where we're =
> C: I like, no
> D: = down in the b[β]-hat range. And so judging by what you're saying is we just need that there [writes a "d" in front of $M \cos \alpha$ to make a dM],

After further difficulties in building, C and D recall the differential volume element from spherical coordinates to reason about the components of the differential length element for schmerical coordinates. While they had previously recognized the appropriate term for projection, the direct mapping resulted in the incorrect use of $\sin \alpha$ in the β length component, as it had for the graduate students.

The third pair of students, E and F, provided a contrast to the previous two groups. The two students resolved to build the integral from scratch and made a deliberate choice to not "fog their minds with preconceived notions of how things should work." As a result they spent the interview weaving together aspects of *component and direction, differential,* and *dimensionality,* building each component of the length vector independently; later they added each component together to represent the entire differential length element. The aspect of *projection* was entirely absent from their reasoning. At one point they made a comparison to spherical but agreed that they should not include a $\sin \alpha$ term, given that they could not justify the need. As a result, their differential element lacked any trigonometric function.

The final pair, G and H, focused entirely on *rote recall* and *mapping*. Neither student, however, could appropriately construct a spherical differential length element, due to lack of consideration of *dimensionality* coupled with difficulty with *grouping* terms by variable (as is done in integration) rather than by appropriate length component. This difficulty pushed them toward building an element in Cartesian coordinates using the form $xdx\hat{x} + ydy\hat{y} + zdz\hat{z}$. They then decomposed $\vec{M}$ into x, y, and z components for a right-handed system, rather than the given left-handed coordinates. Recognizing the determined differential element was in Cartesian coordinates and not in schmerical coordinates, the students returned to the idea of building the differential length element later in the interview by recalling the method of construction they had learned in class at the beginning of the semester.

Themes in students building of differential elements

Identification of these four building aspects and three actions afforded us the ability to determine the order and grouping of these aspects as students progressed through the interviews. We were then able to generalize across the interviews and observe recurring patterns in students' construction. We were able to identify which aspects or combination of aspects were productive and when the absence, or misapplication, of an idea in student thinking hampered further construction. Analysis across all of the interviews identified specific difficulties faced by individual groups or that were commonly held by several students, which we were able to connect to the scope and sequence of aspects that emerged in the interviews.

Productive combination and sequence: Component and direction *and* dimensionality

Analysis across groups identified that the use of *component and direction* coupled with *dimensionality* was very productive for students in the first three pairs when considering the differential length element as a whole. For the third pair of students, the combination of these two aspects was additionally beneficial when constructing each individual components of the differential.

F: So then if you have β—
E: $d\beta$.
F: Oh, yeah.
E: So you're going to have a length component in the β-hat direction.

For each term, the pair would isolate a specific direction of movement and then discuss what a length element in that direction was comprised of.

The role of dimensionality

In general, d*imensionality* alone was a strong factor for the students that used the aspect. Students C and D were particularly adamant about accounting for units of length.

C: ...it's going to be like, so if it's going to be some trig thing but sine of something isn't a length so we're going to have to also have something else in there.

C and D used the aspect of *dimension* to reason about the variables of each term, to an extent that later in the interview they could not recall whether or not differential angles or unit vectors gave units of length to their vector components. While the pair made a comparison to the spherical volume element, the concern persisted as they continued to construct terms. Other students often did not provide additional reasoning for including an M in their construction, as was seen in early transcripts.

A: … This doesn't have any units of length…so, it needs to have some M term.

However, student E specifically addressed the idea of arc length, combining aspects of *direction, dimension,* and *differential,* which made using the radius of length M apparent.

E: =so it's M times some Δ, I think it's M times $\Delta\beta$, a small β, because it's like if you take r times its small θ then that is the arc length=
F: Yeah.
E: =around a circle.
F: Yeah, okay.
E: Right, so like $rd\theta$ would be like the length component around a circle, so this would be $Md\beta$.

The final pair of students did not attend to dimensionality and subsequently had difficulty with early recall from spherical and Cartesian coordinate systems.

The role of the differential

Particular concept images of differentials were important to students' reasoning abilities. The treatment of differentials in terms of small amounts of motion (Artigue et al., 1990; Roundy et al., 2015; Von Korff & Rebello, 2012) was helpful to the building of terms. This idea is trivial for students here, but other views may be coming into play. C and D had particular trouble constructing the α and β components due to difficulties with reasoning about the differential, thinking only in terms of changes rather than small motions applied to the $\vec{M}$, and more specifically not attending to the need to have a $d\beta$ with the β-hat term.

Grouping terms within components

Groups CD and GH assembled the differential length component by sorting terms in the recalled spherical differential volume element. Both pairs grouped angular terms based on variables, rather than length components (Figure 2). Some attention to dimensionality was present in both cases, however. Initially both pairs of students wrote the $\sin\theta$ term with $d\theta$ in a similar fashion to how it would appear when looking at the integration of each term for a differential volume element. In typical E&M problem solving, the writing of differential volume elements (e.g. $r^2\ sin\theta dr d\theta d\varphi$ for spherical coordinates) involves a grouping of terms in a way that dissociates the variables from their particular length component. This was something student C paid attention to when writing her differential volume element.

> C: ... I was trying to figure out which I guess um I don't know vector direction each come from, um, because I feel like, right? This is right, right? We just write it r^2 for convenience right? It comes from separated out [terms].

C and D were later able correct the incorrect grouping of terms when reasoning through how the differential element was constructed in spherical coordinates. For student H, this dissociation of variable from the length components went further:

> H: You've got $r\ dr\ \hat{r}$ plus, is it $\sin\theta\ d\theta$ or is there an r in there?
> G: I think there is an r there, it's an r because you want, you want at that radius uh, plus a small angle.

Student H seems to have a concept image where the grouping of terms based on like variable is dominating the correct ideas for each length component. If this were the case, all the r terms in the differential length component would have been grouped with $\hat{r}$. This goes further to show how a lack of reasoning about *dimensionality* can hamper problem solving in E&M.

$$d\tau = \underbrace{r\sin\theta d\theta}, \underbrace{r d\phi}, \underbrace{dr}$$

$$r^2 \sin\theta\ d\theta\, d\phi\ dr$$
$$r dr\ \hat{r} + r\sin\theta\ d\theta\ \hat{\theta} +$$

Figure 2. (left) Incorrectly recalled differential spherical volume element written by student C and (right) unsuccessful attempt to reconstruct differential spherical length element by students G and H.

Construction and checking of differential volume elements

When asked to construct a volume element for schmerical coordinates, pairs AB and CD immediately knew to multiply the length components together. While this may seem trivial for the pair CD, having constructed the length elements through recall of the spherical differential volume, the pair acknowledged that they understood the volume element originated from the multiplication of length components. Both pairs of students also knew they could check their volume element by integrating to a constant radius to attain the equation for the volume of a sphere, but due to their incorrect trigonometric function, both checks of the differential volume resulted in an integration to zero over the domain of α. At this point, students recognized an error in their differential length element. Student A immediately recognized the mistaken projection due to the direct mapping and articulated that the change in placement of the angle makes $\cos\alpha$ the appropriate projection. Both students C and D were stricken by the result of the integration and only postulated that the error may have been with the $\sin\alpha$, recognizing this caused the integration to zero. Shortly following this exchange, they were also able to recognize the mistake in the projection.

When asked to check their volume element, F attempted to reason dimensionally, saying the integration of the M terms would given units of length cubed, so it didn't matter what the remaining integrals yielded. Unconvinced, E suggested the integration of the full differential volume element. The resulting π^2 in their answer convinced them that a sine or cosine was needed, but the justification around the aspect of *projection* was not immediately seen. Eventually E and F arrived at the correct differential length element.

When it came to the differential volume element, G and H mapped from the spherical differential, as they were both comfortable with that element. Early on in the construction they accounted for the different placement of α but the decided a direct mapping would be okay and included a $\sin\alpha$ term. After reconstructing a differential length element and obtaining correct terms, G and H were asked if they still satisfied with their differential volume element, which included a different trigonometric function from their length element.

H: I still like our volume element=
G: Yeah, I think so.
H: = I don't know about you, this one over here, I still think that
G: They're the same, yeah.
Interviewer: Okay, and can you check that that volume element is correct?
G: Isn't that kind of the same question?
H: Oh, you want us to actually do this integral out.
G: Oh. No, but see in down here we've gone with the $\cos\alpha$.
H: Oh, we've gone cosine, oh yeah.
G: And so we might want cosine. Yeah, I think we do, oh wait, let's see, oh not that's, alright, yeah we do want these, we want these to agree so they need to be, this needs to be a cosine [in the volume element].
...[After the exchange the pair start working out the integration]
G: Why did we change it to cosine?
H: I'm sorry?
G: Actually wait no, because the negative sign, the negative $\sin\frac{\pi}{2}$ is one=

As G and H are checking their differential volume element, they were hesitant to change their volume element and even after deciding that the length and volume element should agree, student G questioned the switch in trigonometric functions as the pair began to check their volume element, but was then comfortable with the choice, seeing that the mathematics arrived at the expected answer. The overreliance on spherical coordinates as well as the overly direct mapping from one coordinate system to another by the first two pairs of students is reminiscent of *x,y syndrome* (White & Mitchelmore, 1996), where a particular process in remembered in terms of symbols rather than how it comes about. Likewise, the symbols of the differential are remembered in the form that they are first taught and lose any particular meaning.

Conclusion

Our results suggest students do not have a robust understanding of how to build differential elements, but are able to check the validity of these elements and adjust terms appropriately. When working in an unconventional spherical coordinate system, students used a mixture of approaches to construct differential length and volume elements. Some attempted to reason about the length elements through direct mapping from spherical or Cartesian coordinates. We found students could implement successful strategies using necessary concepts. Particular attention to *component and direction* as well as *dimensionality,* both a coupling of the aspects and as individual aspects, allowed students to think productively about terms. Using the aspect of *differential* to think in terms of small changes was also useful to students. Interviews also highlighted a number of difficulties students face when working with differential length elements. An overreliance on rote recall and mapping led to difficulties for a number of students. It was also noted that students had particular difficulty grouping terms within recalled spherical length and volume elements. Students' inattention to *dimensionality* and *projection* hampered construction of terms. The successes and difficulties surrounding dimensionality speak to the importance of reasoning about units and dimensions when it comes to work in physics.

However, the determination and subsequent checking of the volume element seemed to help elicit ideas or connections between aspects that were not used when working on the length components. This suggests that while some students have an incomplete understanding of the coordinate systems due to misapplication of particular aspects, for others the requisite aspects are present but not accessed.

Student interviews have been particularly useful in exploring student understanding in this area, since for written problems seen earlier in the course sequence, rote memorization may have taken the place of conceptual understanding without hampering students' ability to arrive at the correct answer. We see similar results in the findings of Bollen and colleagues (2015) where students are able to correctly perform the calculations for differential vector operators but are unable to explain what the results mean. These cases are reminiscent of Tall and Vinner (1981), who stated that a restricted concept image can develop when students work for long periods with the same application of a given formula, even if the student is first presented with the formal definition. Then when students are met with a broader context, the student may be unable to cope. This implies that in order to improve student instruction in differential vector elements in E&M, more focus should be given to how length, area, and volume elements are constructed and determined when problem solving. Results of this study will be used to develop instructional resources to be used in E&M or the methods of theoretical physics course.

Further work seeks to explore how students reason about differential area elements, given the prevalence of these in flux integrals in E&M. We also seek to explore the role differential vector elements play in students' understanding when these elements represent abstract quantities such as electric fields in physics problems, as well as make comparison about students' thinking and use of differential vector elements between the contexts of math and physics. Additionally, we are analyzing student responses in terms of symbolic forms (Sherin, 2001), identifying a range of forms that students invoke when constructing and interpreting vector expressions as well as building from partially remembered equations.

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Students' interpretation and justification of "backward" definite integrals

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The definite integral is an important concept in calculus, with applications throughout mathematics and science. Studies of student understanding of definite integrals reveal several student difficulties, some related to determining the sign of an integral. Clinical interviews of five students gleaned their understanding of "backward" definite integrals, i.e., integrals for which the lower limit is greater than the upper limit and the differential is negative. Students initially invoked the Fundamental Theorem of Calculus to justify the negative sign of the integral. Some students eventually accessed the Riemann sum appropriately but could not determine how to obtain a negative quantity this way. In our research, we analyze various concept images for the differential, and see the primary obstacle here as interpreting the differential as a width, and thus an unsigned quantity, rather than a difference between two values.

Key words: Definite integrals, Calculus, Differential

In this preliminary report, we examine the role of the differential in the "backward" definite integral, $\int_b^a f(x)dx$ where $a < b$. The definite integral is a fundamental concept in calculus, with applications throughout mathematics and science. Studies of student understanding of definite integrals reveal several difficulties (Bajracharya, Wemyss, & Thompson, 2012; Bezuidenhout & Oliver, 2000; Jones, 2013; Lobato, 2006; Orton, 1983; Sealey, 2006, 2014; Sealey & Oehrtman, 2005). The existing literature on definite integrals tends to support a specific approach to developing an understanding of the definite integral, specifically by recognizing it as the sum of infinitely small products, which are formed via Riemann sums (Jones, 2013; Meredith & Marrongelle, 2008; Sealey, 2014). Additionally, Sealey (2006) and Jones (2013) point out that recognizing the Riemann sum as a sum of products of the function value $f(x)$ and the increment on the *x*-axis (Δx) is necessary for students to understand the meaning of the *area under the curve,* which is, arguably, the most prominent metaphor/interpretation of the definite integral. On the other hand, reasoning about a definite integral as area under the curve may limit students' ability to apply the integral concept (Norman & Prichard, 1994; Sealey, 2006; Thompson & Silverman, 2008).

Another aspect of the definite integral that leads to student difficulties is the meaning of the differential itself. Students treat the differential as an indicator of the variable of integration rather than a fundamental element of the product in integration of both single- and multivariable functions (Artigue et al., 1990; Hu & Rebello, 2013; Jones, 2013; Nguyen & Rebello, 2011). This could stem from, or lead to, a failure to understand the product layer of the integral (Sealey, 2014; Von Korff & Rebello, 2012). In other recent work, students treat *dx* as a graphical width (Bajracharya et al., 2012; Wemyss, Bajracharya, Thompson, & Wagner, 2011) or a small amount or quantity of whatever the *x*-axis represents (Artigue et al., 1990; Hu & Rebello, 2013; Roundy et al., 2015), rather than an infinitesimal difference or change in *x* (Von Korff & Rebello, 2012), both for positive and negative integrals.

Interpreting the sign of the integral has been shown to be difficult for students. Definite integrals that have a negative result are of particular difficulty geometrically. Students often do not treat the area as a negative quantity, effectively associating it with spatial area rather than the quantity represented by the product of $f(x)\,dx$. This is true for integrals for which $f(x)$ is negative, i.e., below the x-axis (Bezuidenhout & Oliver, 2000; Lobato, 2006; Orton, 1983; Rasslan & Tall, 2002), as well as those for which dx is negative, i.e., the direction of integration is in the negative direction (Bajracharya et al., 2012). The former type of negative integral is more common, but the latter also has relevance to applications in physical situations (e.g., finding thermodynamic work during the compression of a gas).

The notion of dx as a signed quantity is somewhat controversial, depending on the way one defines the differential. The perspective here, which is consistent with applications in physics and other fields, is that dx is defined as an infinitesimal *change* in the quantity *x,* akin to the limit of the change in x for the products in a Riemann sum: $\Delta x = \frac{b-a}{n}$; $dx = \lim_{n\to\infty} \frac{b-a}{n}$. This is consistent with Von Korff & Rebello (2012), who argue that infinitesimal quantities and infinitesimal products are important for an understanding of the meaning of definite integrals. Generally the sign of these quantities is not of interest, since $b>a$ in most cases. However, if $b<a$, then Δx, and thus dx, are negative. In Stewart's (2016) most recent text, he explains that the backward integral is negative because Δx is negative, but does not explicitly refer to dx as a signed quantity.

Theoretical Perspective

Given the prior work in this area, we wanted to explore the facets of students' concept image (Tall & Vinner, 1981) of the definite integral that apply to the sign of the integral. As described earlier, it was noted in the literature that the differential can be viewed as a variable of integration, as a width of a rectangle, as a small amount of a given quantity, or as a difference or change in a quantity. In particular, the role of the differential in a backward integral, $\int_b^a f(x)dx$, is crucial in interpreting the sign of the integral. We suspected that students would not recognize the fact that the differential would be negative for backward integrals, and we were interested in exploring whether certain concept images for the differential were more productive for viewing the differential as a negative quantity. Thus the backward integral had the potential to illuminate students' understanding of the meaning of differentials, definite integrals, and to some extent, the Riemann sum, beyond what has been seen in the literature to date.

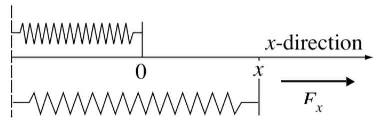
Methods

During clinical interviews, students were asked a series of questions about the relationship between forward and backward integrals. As this was a pilot study, we chose to interview five students at various levels: two second-semester freshmen (both double majors in math and physics and concurrently enrolled in a second-semester calculus course), one junior math major, one senior math major, and one first-semester Ph.D. level graduate student in math/math education. The interview subjects were volunteers who were either former students or teaching assistants of one of the authors. Interviewees received a $10 gift card at the conclusion of the interview.

Prior to the interviews, we developed an interview protocol and agreed upon the order in which the questions would be asked of the students, starting with the open ended general expressions shown below and concluding with a physical example.

1. General expressions: $\int_a^b f(x)dx$ and $\int_b^a f(x)dx$

2. Specific expressions: $\int_1^3 2x\,dx$ and $\int_3^1 2x\,dx$
3. Physical scenario: Work required to stretch a spring, $\int_{x_1}^{x_2} F\,dx$, where $F = kx$ (see Fig. 1)



Force needed to hold spring (with "stiffness" k) in place at position x away from equilibrium

$$F = kx$$

Work – Energy transferred into spring – to move end of spring from x_1 to x_2

$$W = \int_{x_1}^{x_2} F\,dx$$

Figure 1. Information given to interview subjects for spring question.

In each case we gave the forward integral first, then asked about the backward integral of the same expression. Interviews were semi-structured, following the same order of topics for each of the five interviews, but allowing for off-script questions in order to clarify our understanding of the students' responses. For example, with the general expressions, we first gave the students a paper containing the integral expression $\int_a^b f(x)dx$ and asked the students how they would read that expression. Next, we asked them to, "Explain what you know and understand about that expression." In the third section of the interview, students were asked about the work done on a spring. Bajracharya et al. (2008) found that students could justify the sign of a negative integral represented graphically by imagining a physical context that could be represented by the given graph. This led us to include a physical scenario involving a backward integral in order to explore its effect on evoked student concept images. Our initial task provided the integral expression as well as the expression for the force on a spring and a figure to demonstrate the scenario. Students were asked to interpret this integral for an extension of the spring (i.e., where $x_1 < x_2$), and then to interpret the integral with the limits reversed, rather than explicitly asking about a compression of the spring. During the students' explanations, we asked clarifying questions in order to refine our understanding of the students' responses and their meanings. Since this was a pilot study, we also allowed for time at the end of each interview to go off script and ask additional questions.

Interviews were videotaped and transcribed for analysis. During the first round of data analysis, both authors viewed the videos multiple times and made notes of the ways in which the students explained the sign of the integral in both the forward and backward integral. In subsequent analysis of the videos and the transcripts, we paid particular attention to how (or if) the students could describe the definite integral as a sum of products, and how (or if) they described the definite integral as area under a curve. Finally, we noted which of the concept images of the differential each student held, which were evoked during each interview question, and also attempted to identify other concept images that emerged from the data.

Data and Results

All five students were able to describe the definite integral in terms of area under a curve, and most students sufficiently described it as the sum of very small rectangles, whose heights were values of $f(x)$. Some students described the width of the rectangles as dx and some as Δx, and often used the two quantities interchangeably. Overall, we were satisfied with the students' ability to describe a definite integral as the sum of products.

All students initially used the Fundamental Theorem of Calculus (FTC) to justify why $\int_a^b f(x)dx = -\int_b^a f(x)dx$. Specifically, they were able to state that $\int_a^b f(x)dx = F(b) - F(a)$, where $F(x)$ is the antiderivative of $f(x)$, and then that $\int_b^a f(x)dx = F(a) - F(b)$, which would have the opposite sign. Graphically, the students had much more difficulty explaining why one of the quantities should be negative. In the following analysis, we discuss students' use of area under a curve and its connection to the FTC, their concept images of the differential, and the role of the application problem in their understanding of the backward integral.

Using area under the curve and the Fundamental Theorem of Calculus

All of the students seemed comfortable discussing the integral as the area under the curve. While they were able to consider the total area as the sum of small rectangles (or trapezoids), all students had difficulty explaining why the backward integral was negative in terms of area under a curve, and thus this ended up being an interesting part of our analysis.

Sara, a sophomore mathematics and physics double major, evaluated $\int_1^3 2x\,dx$ by finding the area of the large triangle (Fig. 1a) and subtracting the area of the small triangle (Fig. 1b) to obtain the desired area (Fig. 1c). She noticed that these calculations corresponded to the values she obtained when applying the FTC to the same problem: the area of the large triangle corresponded to $F(3)$, and the area of the small triangle to $F(1)$. Then, when computing $\int_3^1 2x\,dx$, she reversed the order of her subtraction, subtracting the area of the large triangle (Fig. 1a) from the area of the small triangle (Fig. 1b), and said, "But I'm not sure why that order is. I mean I know why for the integral [symbolically] because it's written that way, but if you were to solve this geometrically, I don't know why you would change the order of the subtraction."

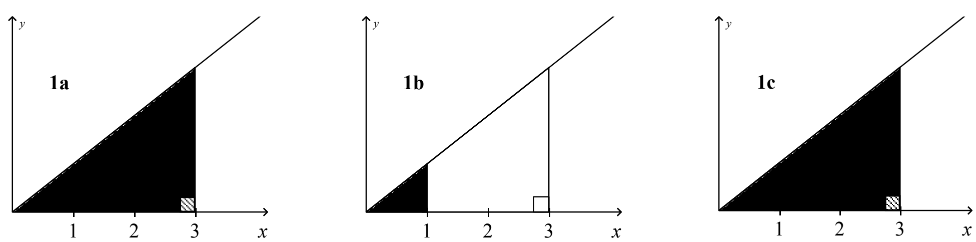


Figure 2: Sara's method of computing the area

Matt, a junior math major, also was able to justify the relationship between the forward and backward integral symbolically using the FTC, but also struggled to justify the result graphically. When computing the area under the function $2x$ between $x = 1$ and $x = 3$, he recognized it as a trapezoid. Instead of using Sara's method of subtracting the smaller triangle from the larger triangle (Fig. 1), Matt added the area of the lower rectangle (Fig. 2a) to the area of the upper triangle (Fig. 2b) to obtain the total area (Fig. 2c).

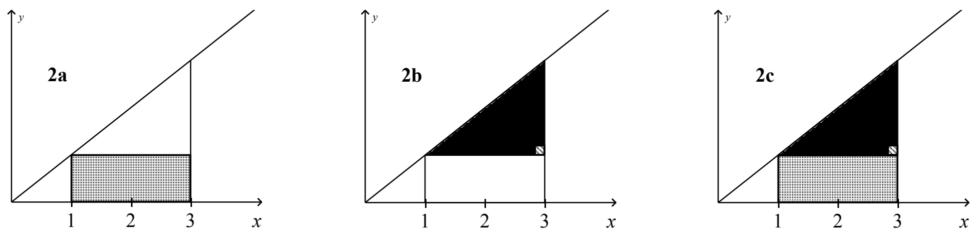


Figure 3: Matt's method of computing the area

Matt's solution is perfectly valid, but did not mimic the calculations from the FTC, as did Sara's method. Matt tried several different ways to graphically justify the negation of the backward integral but was never completely content with his justification. He noted that the backward integral represented the same area as the forward integral, but the backward integral would have to be negative since the limits were reversed "because I already know that, like as a fact, that it's a negative if you want to flip the bounds." He did state that he believed there *should* be a graphical justification, but he did not know what one would be. We do not mean to imply that Sara's solution was in some way better than Matt's, but simply note the connection to the FTC in Sara's solution. In fact, both Sara and Matt used solutions that sidestep the need for thinking about the Riemann sum and the *dx* specifically.

Student thinking about the differential

Most of the students were able to think about *dx* in at least two ways. Many of the students mentioned that the *dx* refers to the variable of integration, and most also were able to describe Δx, but not *dx* as the width of individual rectangles under a curve. Subsequent data analysis will note which concept images for *dx* and Δx were evoked in different circumstances, which concept image was evoked first, and if/when the students changed the way in which they thought about the quantities Δx and *dx*. Of particular interest to us is whether or not the students can conceive of *dx* as a signed quantity, as either a negative width, or as a negative value obtained from $x_2 - x_1$. According to our preliminary analysis, none of the students thought about *dx* as a signed quantity on their own accord, but with prompting from the interviewers, some were able to do so.

Near the end of Sara's interview, we pushed her to consider each rectangle under the curve, which she had described at the beginning of her interview. Sara was comfortable with $f(x)$ being negative or positive, depending on if it was above or below the *x*-axis, but when she was directly asked if *dx* could have a sign, said, "Well no, I don't think *dx* would ever be negative because it's just a distance, it's not like an actual value."

Matt eventually was able to think about Δx as a negative quantity and described *dx* as the limit as Δx approached zero. After many attempts from Matt, the interviewer asked him if *dx* could be negative. His response indicated that he was not confident in his answer, saying,

> "That's probably the hidden spot that I couldn't figure out before. Yeah I would say that this *dx* would be negative [from *a* to *b*] and this one would be positive [from *b* to *a*] because it's approaching 0 so this [from *a* to *b*] would still stay positive […] and this one [from *b* to *a*] would stay under, yeah I'm going to say this *dx* here [from *b* to *a*] is negative and this *dx* is a positive *dx* [from *a* to *b*], and I guess that's where it's hidden and that's what their difference is? I don't know."

Anna, a senior math major, spoke often of Δx, but rarely of *dx*. She had no trouble thinking about Δx as a negative width, but did not seem comfortable thinking about *dx* as being positive or negative. Her explanation of why the backward integral was negative was because the width was negative, and she used an explanation referring to the sum of the areas of a finite number of rectangles to explain, "You're going to have that negative width times a positive value, which is going to give you a negative number, so you're going to get the addition of a bunch of negative numbers." Much later in the interview, one of the interviewers asked Anna if it was possible for dx to be positive or negative, and Anna responded, "I've actually never thought of that. So I'm not sure. I mean I guess it could, but I just always viewed the dx as the indication of what term to integrate to. So I'm not actually sure, I guess."

Nick, a mathematics graduate student, focused his explanation as to why the backward integral was negative on direction. He said that the dx represents a change, and that change implies motion. He seemed to be thinking about the variable x representing time, and mentioned more than once that the backward integral would be like playing a movie in reverse. On another note, Nick spent a great deal of time during the interview talking about the two terms that made up the product in the definite integral, namely the $2x$ and the dx in $\int_3^1 2x\,dx$. He knew that when multiplying two quantities to obtain a negative result, exactly one of the terms multiplied must be negative. He debated if the x turned negative or the dx turned negative. He "voted" for the dx to be negative, again tying this to the direction. But this answer did not sit well with him: "I've just, in my head for so long, always assumed dx to be a positive, you know, change in x and for so long never thought this deep into the pieces." He said to be sure, he would have to go back to the definition of Δx in the textbook to see if he was right.

The role of the physics context in student thinking

Our data suggest that the spring task seems to evoke the concept images of dx as a small but finite amount, change, or difference in the position of the end of the spring in students who already were engaging with the idea of the differential.

Matt had indicated a concept image of dx as a variable indicator, but during the spring extension discussion, he said that it would be "the change in the distance from each, like, individual from like your x_1 to x_2 and it's going to be every, like, small small distance." Upon prompting from an interviewer about the change in his view of dx, Matt responded, "when you think about just, like, the pure math problems, that's all you really think about — just the fact that dx is just telling you […] what variable to use […] But […] here, it represents, it represents something…" Given physical meaning, Matt was able to interpret dx as a small displacement of the spring.

Similarly, Anna pointed out that before the interview, she thought of dx as "the term you're supposed to integrate to." She had a clear sense of the product layer and of the adding up pieces conceptualization of the definite integral, multiplying $f(x)$ by Δx "to get the area limits" for a Riemann rectangle. She immediately realized that Δx is negative for the backward integral in the first task, and how that leads to the negative sign on the result: "you're going to have that negative width times a positive value, which is going to give you a negative number, so you're going to get the addition of a bunch of negative numbers." During the discussion of the spring compression integral, she stated that dx is "kind of representative of that infinitely small piece which would still be able to have a negative change as we are heading smaller, because it's directional so because it's getting smaller it's negative."

However, we also saw a case where this context did not promote this particular concept image of dx as an infinitesimal signed quantity. In the math contexts, Alex displayed reasonable area under the curve and antiderivative images of the definite integral, although he did show some confusion graphically with the integrand representing the area. He never mentioned dx in his discussion (except when reading the definite integrals aloud); instead focusing on more macroscopic differences, either Δx or the whole interval $(b - a)$. His image of Δx was primarily as a width (he says "length") on the x-axis; this could be a signed quantity, since reversing the limits of the integral "would make your width negative." But Alex had a strong aversion to the idea of negative area; for him the negative sign was added when limits were reversed "to negate the negative distance" because otherwise you would have an negative area, "which isn't really a thing." Similarly, he did not connect the sign of the backward integral (for a positive function) to the sign of Δx.

With the spring task, Alex considered that the forward work integral represented the energy put into the spring and that the backward integral was energy lost. He discussed the meaning of this: "It is totally okay for there to be negative work, but that's usually, that's defined upon reference frames. Whether work was done on a system or by a system." As Alex continued, his discomfort with a negative integral emerged:

> "Whether you start at x_1 or you start at x_2, it's still the same amount of force over the same amount of distance so there's no reason why the flipped bound, this second, x_2 to x_1, there's no reason why that shouldn't be able to have a positive answer because it is still the same amount of force over the same amount of distance."

It seems as if Alex used the idea of reference frame to justify the arbitrary positive sign of the integral and avoided the need to consider the sign of the differential quantity. Our interpretation of Alex's response is that his concept image was limited by his failure to acknowledge dx in the discussion and thus he failed to recognize the role of dx in determining the sign of the integral.

The inclusion of the physical context of the spring, as well as Nick's evoking of motion or time contexts for the integrals, seemed to help subjects make sense of the differential itself and also of the sign thereof. This supports our reasoning for including the spring task in the interviews, and is consistent with earlier reports of students invoking physical contexts on their own in order to interpret a negative integral, either a backward integral of a positive function or a forward integral of a negative function (Bajracharya et al., 2012; Wemyss et al., 2011). On the other hand, it is possible that any reasoning on the spring task about the differential in backward integrals may have arisen due to priming of this issue in the previous two tasks of the interview. While Alex could be considered a counterexample of this, to be certain, additional data would need to be collected with a rearranged or otherwise varied protocol.

Discussion and Conclusion

Students recognized the negative value of the backward integral based on the FTC/antiderivative difference formula, but when asked for a geometric interpretation, most said they hadn't thought about it before and had difficulty making a reasonable interpretation on their own. Most students' graphical explanations of why the backward integral yields a negative result seemed to invoke the direction of the integration, treating the area as a macroscopic negative quantity, but failed to recognize the role of the differential in generating that sign. We know from the literature and our own prior research (Bajracharya et al., 2012; Sealey, 2006; Thompson & Silverman, 2008) that students often lack an understanding of *why* or *how* area under a curve is a representation of a definite integral. Our subjects, who we acknowledge may be more advanced than the average calculus student, did not seem to have this difficulty: they were able to describe the definite integral as the sum of the areas of very small rectangles, and adequately described the product layer that makes up these small rectangles. They could all explain that $f(x)$ represented the height of the rectangles and that Δx (and sometimes dx) represented the width of the rectangle.

However, thinking about the backward integral adds another level of difficulty to describing the definite integral in terms of area. The students did not always recognize that Δx and dx could be negative values. Instead of thinking about Δx as a difference, (e.g., as $(x_{i+1} - x_i)$ or as $\frac{b-a}{n}$), they initially thought of Δx as the width of a rectangle, and usually assumed it was always a positive value, leading to the assumption of a positive backward integral. Alex's use of this reasoning above in the spring task is consistent with previous work in this area (Bajracharya et al., 2012; Wemyss et al., 2011), and suggests the consequence of a limited concept image of the differential.

We certainly do not mean to imply that Δx and dx should never be thought of as a width. In fact, research by Hu and Rebello (2007) suggested that dx-as-width is an important perspective for problem solving in physics. Instead, we emphasize the necessity for being able to think about dx as positive or negative widths *and* the change between two quantities. With moderate prompting, most of our research subjects were able to do this; future research will examine what type of instruction or intervention enables students to make this connection.

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Supporting preservice teachers' use of connections and technology in algebra teaching and learning

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The Conference Board of the Mathematical Sciences recently advocated for making connections and incorporating technology in secondary mathematics teacher education programs, but programs across the United States incorporate such experiences to varying degrees. This study explores opportunities provided by secondary mathematics teacher preparation programs for preservice teachers (PSTs) to (1) expand their knowledge of algebra by making connections and using technology and to (2) learn how to incorporate their own use of connections and technology in algebra teaching. We explore the research question: What opportunities do secondary mathematics teacher preparation programs provide for PSTs to learn about connections and encounter technologies in learning algebra and learning to teach algebra? We examine data collected from five teacher education programs chosen from across the U.S. Our data suggest not all secondary mathematics teacher preparation programs integrate experiences with making connections of different types and using technology to enhance learning across mathematics and mathematics education courses. We present overall findings with examples.

Key words: Algebra, Technology, Connections, Secondary Teacher Training

Algebra plays a prominent role in mathematics education reform efforts because it is valued both as a foundational subject in mathematics and as a gatekeeper for college entrance and careers (Kilpatrick & Izsák, 2008). Particularly in the United States, preparing future secondary mathematics teachers to teach algebra has gained importance because of changing populations and changing perspectives on effective algebra teaching. First, in response to algebra-for-all initiatives, more states include algebra as a high school graduation requirement (Teuscher, Dingman, Nevels, & Reys, 2008). Due to these new requirements, not only are more secondary mathematics teachers teaching algebra in their first professional position, but these new teachers are also expected to teach algebra to a more diverse population of students than ever before (Stein, Kaufman, Sherman, & Hillen, 2011). Hence it is critical to study how teaching programs prepare preservice teachers (PSTs) for teaching algebra. Second, reform initiatives (e.g., NCTM, 1989, 2000; CCSSO, 2010) have changed expectations of teachers. In *The Mathematical Education of Teachers II (METII)*, the Conference Board of the Mathematical Sciences argued that "a careful look at the mathematics [e.g., algebra] that is taught in high school reveals that it is often developed as a collection of unrelated facts that are not always justified or precisely formulated" (p. 56, CBMS, 2012). CBMS recommended increased attention to how future mathematics teachers are supported in developing a deep understanding of mathematics, with algebra as one large area of focus. Although may experiences may occur in mathematics education courses, CBMS recommended that experiences supporting the development of connected, coherent understanding of mathematics should also occur in mathematics courses.

The current study is situated within a larger project that explored opportunities that teacher preparation programs provide PSTs to learn algebra and learn to teach algebra. In this smaller study, we have chosen to focus particularly on two approaches to learning algebra and learning to teach algebra: making algebraic connections and using technologies strategically.

Standards for both secondary mathematics content and teacher preparation have emphasized the importance of developing PSTs' abilities to make connections and to use appropriate educational technologies in their own mathematical learning and in their future mathematics teaching. Particularly with respect to PSTs' mathematics courses, *Mathematics Education of Teachers II (METII)* recommended that instructors of mathematics courses support PSTs in "forming connections" (p. 56) and that experience with technology "should be integrated across the entire spectrum of undergraduate mathematics" (CBMS, 2012, pp. 56-57).

Standards developed for teacher preparation program accreditation agencies have emphasized the importance of developing PSTs' abilities to see mathematics as a complex, connected system woven through other non-mathematical disciplines as well as a way to make sense of the real world (Council of Chief State School Officers [CCSSO], 1995; National Council of Teachers of Mathematics [NCTM], 2012). Although different research and policy documents may use the term *making mathematical connections* in slightly different ways, here we mean, broadly, identifying relationships between mathematical ideas, objects, structures, etc. For example, identifying the many ways functions may be represented or used within a particular domain, across multiple mathematical domains, in real-world relationships, or in the mathematics of other disciplines.

PSTs must think about mathematics as a "whole fabric" as they make connections among mathematical topics and in relation to others (NBPTS, 2010). To support this view of mathematics, PSTs need to make connections within algebra, and between algebra and other mathematical fields, while linking algebra with real-world situations. PSTs should prepare to teach using "rich mathematical learning experiences" and provide their future students with opportunities to "make connections among mathematics, other content areas, everyday life, and the workplace (NCTM, 2012). Further, PSTs should also be able to prepare to support their future learners in reflecting "on prior content knowledge, link[ing] new concepts to familiar concepts, and mak[ing] connections to learners' experiences" (CCSSO, 1995).

Teacher preparation standards have emphasized the importance of PSTs' encountering technologies in mathematics and mathematics education courses (CBMS, 2001, 2012; CCSSO, 1995; NCTM, 2012). In this study, we define technology broadly as electronic software, such as calculators, mathematical software, and online applets. *METII* recommended that PSTs have experiences with technology "integrated across the entire spectrum of undergraduate mathematics" (p. 56). *METII* warned that technology should not be used "in a superficial way, without connection to mathematical reasoning" (p. 57) but that PSTs should have opportunities to encounter a variety of technologies and to learn to use technology in a variety of ways; for example, to offload routine computations, operations, representations; to apply previous knowledge to more complex problems; and to explore and experiment to understand subtle mathematical concepts. Teachers also need support in critically evaluating and strategically using technology in mathematics teaching and learning (CBMS, 2012; CCSSO, 1995; NCTM, 2012). That is, PSTs should have opportunities in mathematics and mathematics education courses to learn to use a variety of technologies in support of three big ideas: using technology as practical expedient, using technology to enhance learning, and thinking critically about the choice and use of technologies in mathematics teaching and learning.

This study explores opportunities provided by secondary mathematics teacher preparation programs for PSTs to (1) expand their knowledge of algebra by making connections and using technology and to (2) learn how to incorporate their own use of connections and technology in algebra teaching. We explore the following research question: "What opportunities do secondary mathematics teacher preparation programs provide for PSTs to learn about connections and encounter technologies in learning algebra and learning to teach algebra?" Making connections in the service of algebra teaching and learning might include making connections within algebra, between algebra and other mathematical fields, between algebra and non-mathematical fields, and between ideas in advanced algebra and school algebra. Encounters with technology in the service of algebra teaching and learning might include using or learning about a variety of algebra-appropriate technologies, as well as thinking critically about technology use.

Method

This study is part of *Preparing to Teach Algebra (PTA),* a mixed-methods study that explores opportunities provided by secondary mathematics teacher preparation programs to learn algebra and to learn to teach algebra. The *PTA* project consists of a national survey of secondary mathematics teacher preparation programs and case studies of five universities. The current study is a qualitative analysis of the case studies focusing on the opportunities provided to PSTs to encounter technology and to make connections in learning algebra and learning to teach algebra.

The *PTA* project purposefully chose to explore secondary mathematics teacher preparation programs at five universities. We refer to these universities as Great Lakes University (GLU), Midwestern Research, Midwestern Urban, Southeastern Research, and West Coast Urban Universities. We chose the first three before beginning the project, and the latter two after analyzing results of a national survey. We initially chose to focus on three universities (Great Lakes, Midwestern Research, Midwestern Urban) to deliberately vary the type of institution, location, demographics, and type of program. Some details are shown in Table 1 below. Great Lakes, Midwestern Research, and Midwestern Urban Universities varied in their context, Carnegie Classification, and the average number of graduates. For example, Midwestern Research University was a doctorate-granting institution with very high research activity, while Great Lakes and Midwestern Urban Universities were large master's-level universities. In addition, Great Lakes University was unique in that the mathematics educators and mathematicians both were in the Department of Mathematics (mathematics educators at Midwestern Research and Midwestern Urban Universities were in the College of Education). Midwestern Urban University was unique in its ethnic diversity: only 40% of its students were Caucasian with 32% Hispanic and 10% African American. Great Lakes and Midwestern Research Universities had populations of 91% and 83% Caucasian, respectively.

We chose two additional sites (Southeastern Research and West Coast Urban) based on responses from a national survey that we developed and administered early on in the project. While Great Lakes, Midwestern Research, and Midwestern Urban Universities are all located in the Great Lakes region of the United States, Southeastern Research is in the Southeast and West Coast Urban is in the Far West. West Coast Urban University was chosen because its program is post-baccalaureate, so students do not take any mathematics course requirements in the program, but only mathematics education and general education courses.

Table 1.
Demographic Characteristics of the PTA Case Study Programs and Universities.

	Great Lakes	Midwestern Research	Midwestern Urban	Southeastern Research	West Coast Urban
Context	Small City	Mid-size City	Large City	Mid-size City	Large City
Carnegie Classification[1]	Master's L[2]	Doctorate RU/VH[3]	Master's L[2]	Doctorate RU/VH[3]	Master's L[2]
Degree-Seeking Undergraduate Race/Ethnicity Enrollment Percentages[4]	2% Asian 5% Black 4% Latin@ 84% White 3% Multiracial	4% Asian 4% Black 5% Latin@ 71% White 4% Multiracial	9% Asian 10% Black 35% Latin@ 37% White 2% Multiracial	9% Asian 7% Black 5% Latin@ 73% White 3% Multiracial	16% Asian 5% Black 59% Latin@ 8% White 2% Multiracial
Avg. Number of Graduates[5]	34	22	12	39	30
Degree upon Completion	4-year Bach	4-year Bach	4-year Bach	4-year Bach	None (Post Bacc)
Secondary Math Program Academic Home	Mathematics Department	College of Education	College of Education	College of Education	College of Education

[1] The Carnegie Foundation for the Advancement of Teaching (2010).
[2] Master's L: Master's Colleges and Universities (larger programs)
[3] RU/VH: Research Universities (very high research activity)
[4] Race-Ethnicity enrollment statistics were collected from the Common Data Set for each university (2013-14). Only categories with at least 2% in at least one of the universities were included in the table. Nonresident aliens and race/ethnicity unknown categories were not included.
[5] Average annual number of graduates across three academic years (2009-10, 2010-11, 2011-12).

We compiled data by conducting two focus groups of 3-4 PSTs and 10-13 instructor interviews at each site and collected corresponding instructional materials from each instructor we interviewed. In the instructor interviews at each site, we included required mathematics, mathematics for teachers, mathematics education, and general education courses selected for potential algebra content. West Coast Urban University, as a post-baccalaureate program, required no mathematics courses and so we conducted only 3 interviews with instructors of mathematics education and general education courses.

Among other interview questions, we asked instructors which types of technologies they used in a particular course and how they supported PSTs in making connections in algebra. Similarly, we asked PSTs in focus groups to identify required courses that incorporated opportunities to make connections or to use technology in learning algebra or learning to teach algebra. We asked PSTs explicitly about their required or shared experiences with connections and with technology.

Prior to considering the data for mentions of connections or technology, the *PTA* project team had coded data for algebraic content. In analyzing data, four researchers worked in pairs, reading the interview and focus group transcripts and discussing potential opportunities reported by instructors or PSTs. For this study, we focus more narrowly on Abstract Algebra, Linear Algebra, and Secondary Mathematics Methods courses at the five universities. Different universities required different sequences of Secondary Mathematics Methods: Great Lakes required two courses, Midwestern Research required two, Southeastern Research required three, and Midwestern Urban and West Coast Urban required one each. Also, we asked mathematics instructors approximately how many PSTs were in each course. Table 2 shows the percentages of PSTs in the Abstract Algebra and Linear Algebra courses at each university. Note that West Coast Urban University did not require these courses, so they are not listed. The Midwestern Research Abstract Algebra course was a mathematics for teachers course, so it only included PSTs.

Table 2.
Number of Students and Proportion of Preservice Teachers in Linear Algebra and Abstract Algebra courses at Great Lakes, Midwestern Research, Midwestern Urban, and Southeastern Research Universities.

University	Linear Algebra	Abstract Algebra
Great Lakes	21 (50%)	20 (70%)
Midwestern Research	50 (unknown)	*Algebra for Teachers:* 19 (100%)
Midwestern Urban	25 (50%)	15 (about 50%)
Southeastern Research	33 (33%)	24 (60%)

For connections, the two researchers individually coded data sources based on the major four types of connections (e.g., connections within algebra, connections between algebra and mathematics) and met to make consensus on the coding. We then developed summary documents of each university, including tables of the number of opportunities and quotations in each course. We will analyze the quotations to document different types of opportunities that were reported (e.g., algebraic topics that PSTs were exposed, specific activities that PSTs engaged with, or/and opportunities to help PSTs learn to teach connections).

For technology, the two researchers have considered instructors' interviews and instructional materials. We captured types of technologies mentioned by course instructors, as well as details of the experiences the rationale (if any) given by the instructor detailing why (or why not) technology were used (e.g., "dulls the mind" or "representations help students understand quantitative situations"). Based on previous research, we will analyze instructors' reports of technology use to understand why opportunities are or are not provided in particular mathematics or mathematics education courses, and to understand the types of experiences provided, whether the experiences are as a "practical expedient," or to "advance learning," or to provide opportunities for PSTs to think critically about choice and use of technology by engaging with potential affordances and limitations (CBMS, 2001).

Results

For the purposes of this paper, we focus on finding exemplars of *types* of experiences provided to PSTs across the four different programs and focus on experiences in Abstract Algebra, Linear Algebra, and Secondary Mathematics Methods courses. That is, we are not evaluating the programs; rather, exploring strengths and challenges of each program to understand what rich experiences across a program's offerings could look like, and to understand the challenges that arise. First, we consider *connections*; we describe the types of experiences making algebraic connections that instructors told us they provided PSTs. Then, we consider *technologies* and the types of encounters with technologies and algebra that instructors reported providing.

Connections

In terms of connections, we describe the types of connections that the instructors reported PSTs had opportunities to learn and learn to teach about, the purposes of making these connections, and the ways of providing such connections to PSTs in Linear Algebra, Abstract Algebra, and Secondary mathematics methods courses.

Types of connections. We focused on four types of connections provided in these courses: connections (a) within algebra, (b) between algebra and other mathematics, (c) between algebra and non-mathematics, and (d) between school and college algebra. We provide examples of each type of connections in this section.

Of the connections within algebra, instructors of Linear Algebra and Abstract Algebra discussed connections around common structures or big ideas. For example, an Abstract Algebra instructor at Great Lakes said, "We certainly emphasize connections within algebra. In looking at common structures, and themes behind different number systems and the way algebra works in those number systems." Similarly, a Linear Algebra instructor at Midwestern Urban remarked, "Once they have studied how to solve systems of equations, they see that everything that is being done in linear algebra is modeled using that. So that is the big idea. So we spent quite a number of time, quite a number of classes at the beginning studying how to solve systems of equations." In the former example, the instructor described students' connecting common algebraic structures across number systems; in the latter, the instructor described students making connections by recognizing a big idea in Linear Algebra: that systems of equations are used to model "everythign that is being done" in Linear Algebra.

Instructors of Secondary Mathematics Methods emphasized how they could help PSTs make connections within algebra. An instructor at Great Lakes, for example, reported that "my focus a lot of times is on thinking about how they can make connections within the algebra themselves, how they can consolidate those ideas." He continued, "I'm the one who's making the connection. I feel much more comfortable that you make the connection." That is, his emphasis was on supporting PSTs to make their own connections rather than simply being told about connections. Similarly, an instructor at Midwestern Urban described that students made representational connections within algebra: "having students make connections among graph, different representations. So graphs, tables, words and symbols." Other instructors also mentioned different ways of engaging PSTs by using tasks and discussions that address algebraic connections.

Of the connections between algebra and other mathematics, examples vary depending on topics that can be connected to algebra. A Linear Algebra instructor at Midwestern Research, for example, mentioned concepts around a tangent: "When you're doing calculus you're essentially approximating a graph by its tangent line, finding its tangent line, and then you use that to approximate and that generalizes to a tangent plane, of a surface, or tangent space of a manifold." Hence, the students would connect a basic algebraic idea of linear functions to ideas in Calculus. An Abstract Algebra instructor at Southeastern Research discussed algebraic connections to geometric topics: "One of the things in this course, really the culminating section of the course talks about the non-constructability of certain geometric objects…You can't square the circle, you can't double the cube, you can't trisect angles and that's something that is a connection with high school geometry." A Secondary Mathematics Methods instructor at Great Lakes provided a broader role of algebra: "We connect to geometry, probability, measurement, statistics. One of the conceptions of algebra that we talk about is algebra as kind of -- especially symbolic algebra -- as kind of the written language of mathematics and how being able to make a symbolic representation of an idea or a relationship gives us access to so many other tools -- that's a powerful problem-solving tool."

Connections between algebra and non-mathematics were also discussed, making connections to subjects other than mathematics or to real-life situations. A Linear Algebra instructor at Great Lakes discussed building relationships between algebra and computer science when PSTs studied population dynamics. An Abstract Algebra instructor at Midwestern Research mentioned algebraic connections to cryptography: "Algebra and this cryptography is probably a connection between algebra and other fields because cryptography was done mostly by electrical engineers who are actually building machines." A Secondary Mathematics Methods instructor at Great Lakes discussed connecting algebra to real-life situations, such as the price for a car wash: "Someone was looking at systems of linear equations and they were really struggling. I said, 'I saw there was a gas station that was offering, it was this price for regular without a car wash and it was this price for regular if it was a car wash. Would something like that be helpful?'" Secondary Mathematics Methods instructors at other universities also discussed different examples of real-world connections, including the border problem at Midwestern Urban (generalizing the context using variables) and connections to other subject areas (e.g., geography) at Midwestern Research.

Lastly, connections between school and college algebra were discussed by only mathematics instructors. A Linear Algebra instructor at Southeastern Research provided a specific example of making connections between eigenvalues of matrices and polynomials: "When we went to find the eigenvalues of this matrix, so that came finding the characteristic polynomial which in this case was this [writing on board]. And then finding its roots so using the quadrant so there's a bunch of high school algebra, finding this polynomial, figuring out its roots, so those are x equal 1 plus or minus root 5 over 2." Abstract Algebra instructors discussed such connections around topics of polynomials and set theory. An instructor at Great Lakes said, "We talk about polynomials, for instance, we talk about the relationships between problems that we might ask at this level and the corresponding problems at the high school level." Another instructor at Midwestern Urban remarked, "I mainly see connections between is the college-level algebra and school algebra and this abstract notion of algebra in terms of let's start with number systems you're familiar with and then expand it to these more abstract settings." Such connections were not reported by Secondary Mathematics Methods instructors, but shared by Mathematics for Teachers instructors (These results are discussed in other publications.)

Purposes or ways of providing opportunities related to connections. Several purposes of making connections were described by the instructors: (a) to emphasize the application aspect of algebra, (b) to address the usefulness of algebraic connections, (c) to help PSTs teach connections to their students, and (d) to develop PSTs' conceptual understanding of algebraic concepts. The instructors also discussed diverse ways of providing such opportunities, including the use of previous chapters or other subject areas connected to algebra, the use of concepts or experiences that are familiar to PSTs, and discussions around tasks and textbooks.

Of the Linear Algebra instructors, two instructors (at Midwestern Urban and Southeastern Research Universities) reported using connections for showing its applications in other mathematics. Instructor at Southeastern Research, for example, said, "Certainly algebra with other mathematical field, the reason we do Linear Algebra at this point in their career is that it has applications in so many other parts of math. And I bring those up whenever I can." Instructors also described different ways of providing opportunities related to connections. An instructor at Midwestern Research described the use of related chapters to make connections. He reported, "The beginning chapter is building up the theory, understanding what a linear transformation is, its kernel, its image, being able to do row operations to answer various questions. The later chapter always use that stuff, it's always talking in terms of that language that we've learned about." An instructor at Great Lakes mentioned making connections by using relationships between algebra and computer science. He remarked, "we study population dynamics. We use this as kind of a model for dynamical systems." Lastly, an instructor at Southeastern Research described a way to connect trigonometry with the eigenvalues of matrix. These examples are unique in that each Linear Algebra instructor reported different ways of providing opportunities related to connections.

The Abstract Algebra instructors at Midwestern Urban and Midwestern Research universities reported the usefulness of algebraic concepts by making connections to other mathematical fields or real-life situations. For example, the instructor at Midwestern Urban said, "In terms of between algebra and other mathematical fields, I think they see the usefulness of number theory in the course. They see the usefulness of set theory." An instructor at Midwestern Research reported, "my goal is to introduce how the modern algebra is used in real life." A notable way to teach connections was to teach concepts that PSTs are familiar with. An instructor at Midwestern Urban said, "I have them think about what it means to solve for x in an equation they're familiar with, which is from something they're familiar with (college algebra, previous math classes) and then going to the abstract notion of what we mean by a group." Another way of teaching connections was to promote discussions around relationships between school and college algebra. An instructor at Great Lakes reported, "We talk about the relationships between problems that we might ask at this level and the corresponding problems at the high school level. And the same thing with the integers." The instructor also described how to connect the machinery in the division algorithm with the way that kids learn about division.

In Secondary Mathematics Methods, examples of making connections for PSTs' future students were shown in Great Lakes and Southeastern Research universities. Instructors described why making connections in college level is important in helping their students. A Secondary Mathematics Methods II instructor at Southeastern Research, for example, said, "We tried to help our students to be a little bit creative and try to find the connections and make it coherent for their students." As a way to achieve this goal, the instructor described that "We talked about making connections between algebra and geometry to help students understand what is going on within the curriculum here in this State." Another purpose of making connections reported by Secondary Mathematics Methods instructors was to develop PSTs conceptual understanding of concepts. A Secondary Mathematics Methods instructor at

Southeastern Research said, “We’ve focused so much on the cognitive demand and the conceptual understanding we try to show the connections to our students and bring it in and have them the investigations I try to have the kids go through is investigating ways that they can generalize so they’re not just memorizing rules.” An instructor at Great Lakes explained the ways that he helped PSTs make connections. He reported, “The connections that I’m hoping that they’re going to make are based on some of the workshops that they’ll do, the reading workshops as they’re doing that and seeing the connection to their own experience what’s going on in the classroom.” Other instructors also mentioned several ways of providing opportunities, such as discussions around mathematical tasks and textbooks (Midwestern Research) and discussions after watching a video tape that involves population growth (Midwestern Urban).

Technologies

In analyzing PSTs’ encounters with technology in learning algebra or to learning to teach algebra, we considered the types of technologies that instructors of the three courses reported using, their descriptions of their purposes in using particular technologies, and their descriptions of the ways that they or the course students used technologies. In considering how technologies were used, we thought about whether they were used for pragmatic use, to enhance learning, or to support critical thinking about the use of technologies.

Types of Technologies. For four universities, the widest variety of technologies used occurred in Secondary Mathematics Methods courses. Of the Linear Algebra courses, only Great Lakes and Midwestern Urban instructors reported using technologies for class. At Great Lakes University, the Linear Algebra course had a computer lab component, where the class met in a computer lab one day each week; they used graphing calculators, Java applets created by the instructor, and Maple software. At Midwestern Urban University, the Linear Algebra instructor reported use of graphing calculators; at Midwestern Research, the instructor said he encouraged students to do some work using their favorite technology outside of class and also gave them access to *Mathematica* if they chose to use it, but said, “I’m interested in how this works, the brain, not how that [technology] works, so I don’t need to see the results of their computations.” At Southeastern Research University, the instructor said, “You’re not even allowed a calculator in the exams. Because I think it dulls the mind. Makes them rely on it instead of doing, learning mental arithmetic.”

Of the Abstract Algebra instructors, three instructors (at Great Lakes, Midwestern Urban, and Southeastern Research Universities) reported using a classroom management system for organization and communication of the course. Beyond that, the types of technologies varied: the Great Lakes instructor described use of GoogleDocs, Midwestern Research – any calculators, Midwestern Urban – Wolfram Demonstrations Project, and Southeastern Research – an iPad App called “Show Me.”

In comparison, Secondary Mathematics Methods instructors reported a wide range of technologies including mathematics websites (i.e., Wolfram|Alpha, GeoGebra, Khan Academy), virtual manipulatives or applets (i.e., National Library of Virtual Manipulatives, National Council of Teachers of Mathematics: Illuminations), mathematics software (i.e., Geometer’s SketchPad, TinkerPlots, Fathom, GeoGebra, Excel), teaching websites or resources (i.e., TIMSS videos, classroom videos, Moodle), and hardware (i.e., SmartBoards, tablets).

Ways of Using. We saw examples of each of our three categories of using technology: as a practical expedient, to enhance learning, and to support thinking critically about technology use.

In Linear Algebra, an example of using technology as a practical expedient was allowing students to perform row reduction operations on matrices using a calculator instead of by hand. Great Lakes and Midwestern Urban University instructors reported encouraging students to use graphing calculators (Great Lakes and Midwestern Urban) or Maple (Great Lakes). The Southeastern Research University instructor also encouraged students to row-reduce matrices "using their favorite technology," but instead of asking students to report only their final answer, he said he asked students to explain or justify their answers. In Abstract Algebra courses, Great Lakes, Midwestern Urban, and Southeastern Research University instructors reported using course management systems for organization.

As examples to enhance learning, Abstract Algebra instructors at two universities described their use of technology as supporting students' exploration of and experimentation with mathematical ideas. At Midwestern Urban University, the instructor explained that he chose particular representations from Wolfram Demonstrations Project to support students in exploring symmetries and to "clearly understand what the different operations do." At Midwestern Research University, the instructors asked students to use any calculator to perform experiments and said the use of calculators gave students a "better chance of seeing the pattern and coming up with better conjectures." At Great Lakes University, a Secondary Mathematics Methods instructor also described using Wolfram|Alpha as a tool for student experimentation: "you're making conjectures, you're gathering data, and now you're trying to go about proving it." The Midwestern Research Secondary Mathematics Methods instructor described a big idea of his course as using "technology as a tool for learning."

The Secondary Mathematics Methods instructors at Great Lakes, Midwestern Research, Midwestern Urban, and Southeastern Research described opportunities PSTs had to think critically about using technology in algebra through use of different tasks and activities. At Great Lakes, Midwestern Research, and Southeastern Research Universities, instructors described discussions with PSTs focused on consideration of how different schools provide access to different technologies, so considering what is or is not available and how to consider why and how to use technology to support learning. The Great Lakes instructor described discussions with students asking, "what technology is available [at a school] and what can you do with it? And why use it?" Similarly, the Midwestern Research instructor had discussions asking "What does GSP [Geometers SketchPad] afford? What does – what can you do if you don't have something like this? In terms of developing ideas about co-variation of quantity as a basis for functions." The Southeastern Research instructor explained, "I feel like we try to emphasize not using the technology and the resources for the sake of using them but making sure that there is a purpose and a reason behind why are we using this." The Midwestern Urban instructor described one activity that PSTs engaged in as choosing a particular technology and writing a lesson plan that would use the technology.

Discussion

Our preliminary results show different types of opportunities that PSTs were provided related to the learning of algebraic connections and the use of technology to learn and learn to teach algebra. There was a wide range of opportunities that instructors provided related to algebraic connections: some instructors provided lists of topics and ways that they made connections (e.g., Linear Algebra at Great Lakes); others reported specific activities that engaged PSTs to make connections (e.g., Secondary Mathematics Methods at Midwestern Urban). At

Great Lakes University, mathematics instructors described how they emphasized different types of connections, while the mathematics education instructor focused on how PSTs made connections in his class. At Midwestern Urban University, instructors described connections among not only algebraic topics (e.g., systems of equations, variables), but also practices that can be used in different courses and grade levels (e.g., proofs, generalization), along with how technology can be used to make such connections (e.g., Linear Algebra instructor).

We heard concerns from both mathematics and mathematics education instructors that technology could impede PSTs' learning. Some mathematics education instructors argued, to the contrary, that use of technology enabled PSTs to increase their understanding of algebra topics in ways that were not possible otherwise. At each university there was at least one opportunity for PSTs to think critically about their future educational use of technology, but experiences varied.

Endnote

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Examining the role of a secondary teacher's image of instructional constraints on his enacted subject matter knowledge

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I present the results of a study designed to determine if there were incongruities between a secondary teacher's mathematical knowledge and the mathematical knowledge he leveraged in the context of teaching, and if so, to ascertain how the teacher's enacted subject matter knowledge was conditioned by his conscious responses to the circumstances he appraised as constraints on his practice. To address this focus, I conducted three semi-structured clinical interviews that elicited the teacher's rationale for instructional occasions in which the mathematical ways of understanding he conveyed in his teaching differed from the ways of understanding he demonstrated during a series of task-based clinical interviews. My analysis revealed that that the occasions in which the teacher conveyed/demonstrated inconsistent ways of understanding were not occasioned by his reacting to instructional constraints, but were instead a consequence of his unawareness of the mental activity involved in constructing particular ways of understanding mathematical ideas.

Key words: Mathematical Knowledge for Teaching; Enacted Knowledge; Instructional Constraints; Trigonometry.

Introduction

> [S]tudents' mathematical learning is the reason our profession exists. Everything we do as mathematics educators is, directly or indirectly, to improve the learning attained by anyone who studies mathematics. Our efforts to improve curricula and instruction, our efforts to improve teacher education, our efforts to improve in-service professional development are all done with the aim that students learn a mathematics worth knowing, learn it well, and experience value in what they learn. So, in the final analysis, the value of our contributions derives from how they feed into a system for improving and sustaining students' high quality mathematical learning (Thompson, 2008, p. 31).

Research in the area of teacher knowledge in mathematics education has progressed significantly since Shulman's conception of *pedagogical content knowledge* (Shulman, 1986). However, this domain of mathematics education scholarship still has much to contribute to the development of instructional, curricular, and pedagogical innovations that seek "to improve the learning attained by anyone who studies mathematics" (ibid.). The overwhelming majority of research in this area has attended to one, or more, of the following foci: (1) characterizing the nature of mathematical and pedagogical knowledge teachers must possess to support students' conceptual mathematical learning (e.g., Ball, Thames, & Phelps, 2008; Fennema & Franke, 1992; Rowland, Huckstep, & Thwaites, 2005; Shulman, 1986, 1887); (2) understanding the experiences by which teachers might construct such knowledge (e.g., Harel, 2008; Harel & Lim, 2004; Silverman & Thompson, 2008); (3) developing assessments to measure teachers' knowledge (e.g., Hill, Ball, & Schilling, 2008; Herbst & Kosko, 2014, Thompson, 2015), and (4) demonstrating the causal link between teacher knowledge and student achievement (e.g., Baumert et al., 2010; Campbell et al., 2014; Hill, Rowan, & Ball, 2005) or instructional quality (e.g., Charalambous & Hill, 2012; Copur-Gencturk, 2015; Even & Tirosh, 1995; Hill et al., 2008). Stated succinctly, research on teacher knowledge in mathematics education has largely focused on what teachers need to know, how they might come to know it, how one might measure it, and the effect of this knowledge on instructional quality and student performance. While these seemingly comprehensive foci are essential to the enterprise of improving students' mathematical learning, they neither identify nor characterize the effect of the factors that mediate the enactment of teachers' knowledge in instructional contexts, which is the only knowledge that has the potential to affect our field's raison d'être: *the mathematics students have the opportunity to learn*. A focus on characterizing, developing, assessing, and discerning the effect of teachers' professional knowledge without attending to the factors that compromise its enactment, while necessary, is not sufficient for ensuring "that students learn a mathematics worth knowing, learn it well, and experience value in what they learn" (ibid.).

The knowledge teachers leverage in the context of practice is regulated by a host of cognitive and affective processes that have thus far not received sufficient attention in the literature on teacher knowledge in mathematics education. Identifying the influences that condition the knowledge teachers utilize in the context of practice, and ascertaining the effect of such influences on the nature and quality of teachers' enacted knowledge, is imperative for satisfying Thompson's (2008) exacting proposal that our research contributions should "feed into a system for improving and sustaining students' high quality mathematical learning" (p. 31). For current scholarship on mathematics teacher knowledge to realize its intended effect of ensuring teachers are equipped to engage students in experiences that support their construction of rich mathematical ways of understanding and their development of productive ways of thinking, it is crucial to apprehend the effect of those factors that condition the enactment of the knowledge teachers *do* possess in addition to characterizing the knowledge teachers *should* possess. Ascertaining the factors that mediate the knowledge that resides in teachers' minds and the knowledge they bring to bear to support students' mathematical learning is indispensible for fashioning well-informed teacher preparation programs and professional development initiatives that take seriously the effect of teacher knowledge and those influences that compromise it. The present study seeks to contribute to this end.

In this paper, I present the results of a study in which I examined the effect of an experienced secondary mathematics teacher's image[1] of instructional constraints on the nature and quality of his enacted subject matter knowledge of trigonometric functions. While there are many factors that potentially mediate the enactment of teachers' professional knowledge, my decision to focus specifically on an in-service teacher's image of instructional constraints was motivated by the well-documented pervasiveness of teachers' resistance to contemporary educational reform initiatives, particularly within British and American educational systems. Such initiatives are often championed by those who conclude that deficits in teacher aptitude and motivation lie at the heart of modest student achievement. Opponents of this perspective contend that teachers operate under progressively crippling circumstances—many of which they consider occasioned by current reforms—that severely obstruct teachers' ability to engage students in high quality learning experiences. For instance, one often hears it said, at least in the United States, that engrained cultural practices, an overreliance on high-stakes testing and accountability, policymakers' aspirations to privatize education, decaying teacher-student relationships, the de-professionalization of teaching, unsupportive administrators and colleagues, insufficient emphasis on early childhood education, inadequate instructional and curricular resources, increasing proportions of students living in poverty, and the escalation of unsupportive parents all impose obstacles that severely limit what many teachers believe they can achieve in the classroom (Hargreaves

[1] The "image of" qualifier suggests my constructivist approach to defining instructional constraints, which I discuss in the section entitled, "Theoretical Framing."

& Shirley, 2009; Nichols & Berliner, 2007; Ravitch, 2010, 2013; Sahlberg, 2004, 2006, 2011; Stigler & Hiebert, 1999). However, the specific ways in which mathematics teachers' appraisal of, and accommodations for, such instructional constraints inhibits them from enacting the full extent of their professional knowledge to support students' learning has not yet been characterized. To address this limitation, the present study explored the following research questions:

RQ 1: Are there incongruities between an in-service secondary mathematics teacher's subject matter knowledge and the subject matter knowledge he enacted while teaching?[2]

RQ 2: If so, in what ways did the teacher's image of instructional constraints condition the nature and quality of his enacted subject matter knowledge?

It is essential in the contemporary climate of mathematics education to apprehend whether teachers do indeed teach what they know. When mathematics teacher educators assume teachers leverage the full extent of their professional knowledge in an uncompromised way, they design preparation programs and professional development experiences that focus on supporting teachers in constructing more advanced knowledge structures—knowledge that ultimately might not inform teachers' instructional practices and thus the mathematics students have the opportunity to learn. A focus on the factors that condition the enactment of teachers' knowledge has the potential to inform instructional and curricular designs that seek not only to advance teachers' knowledge but also to equip them with skills and strategies to minimize the unfavorable effects of such factors.

Theoretical Framing

The "image of" qualifier in the title of this paper suggests my radical constructivist approach to defining instructional constraints. I take the position that environmental circumstances per se in the absence of a teacher's construal of them cannot constrain his or her practice, but the teacher's construction and appraisal of environmental circumstances can and often does. For this reason, I contend that particular circumstances cannot maintain an ontological designation as instructional constraints, however consensual are teachers' construction and appraisal of such circumstances. Therefore, in consonance with radical constructivism's skeptical position on reality, I define instructional constraints as an individual teacher's subjective construction of the circumstances that impede the teacher's capacity to achieve his or her instructional goals and objectives. Such subjective constructions are the only "constraints" that maintain the potential to influence teachers' instructional actions. Accordingly, I locate instructional constraints in the mind of individuals, not the environment. This conceptualization stands in stark contrast to the common perception of instructional constraints as external pressures that exert influence on the quality of teachers' instruction. According to this view, the pressure comes from without instead of from within. My interest in understanding how a secondary teacher's image of instructional constraints conditioned the mathematical ways of understanding and ways of thinking he utilized in the context of teaching necessitated my constructing a model of the teacher's construction of those circumstances he appraised as constraints on his practice.

As a result of my view that instructional constraints are subjective constructions that reside in the minds of teachers, I consider anything that a teacher appraises as an imposition

[2] I note that the identification of incongruities between the teacher's subject matter knowledge and the subject matter knowledge he enacts while teaching is from my perspective. Similarly, characterizing the effect of a teacher's image of instructional constraints on his enacted mathematical knowledge is also a characterization from my perspective.

to achieving his or her instructional goals and objectives to be an instructional constraint. The appraisal need not even be of an external circumstance. A teacher may appraise internal characteristics such as his or her mathematical self-efficacy, social endowments, creativity, tolerance, attitude, perseverance, temperament, empathy, confidence, etc., as imposing limits on the quality of his or her instruction. Since a teacher's appraisal of such intrinsic characteristics is a subjective construction in the same way that a teacher's appraisal of external circumstances is, both types of appraisals have the capacity to influence teachers' practice in the same way.

Methods

My experimental methods proceeded in three phases. In the first phase, I conducted a series of nine task-based clinical interviews (TBCIs) (Clement, 2000; Goldin, 1997; Hunting, 1997) that allowed me to construct a model of the participating teacher's (David's) mathematical knowledge of various topics associated with trigonometric functions. In the second data collection phase, I used video data from 37 classroom observations to construct a model of the mathematical knowledge David utilized in the context of classroom practice. Finally, I employed a phase of three semi-structured clinical interviews to construct a model of David's perception of instructional constraints and to discern the role of this image on the quality of his enacted mathematical knowledge.

The goal of the series of task-based clinical interviews was to facilitate my construction of a model of David's *ways of understanding* and *ways of thinking* (Harel, 2008) relative to angle measure, the outputs and graphical representation of sine and cosine, and the period of sine and cosine. Constructing a model of an individual's cognition by projecting or imputing one's cognitive schemes to the individual constitutes developing a *first-order model* (Steffe & Thompson, 2000). This is in contrast to developing a *second-order model*, in which the researcher attempts to make sense of the individual's actions by interpreting them through the lens of his or her model of the individual, not through his or her own cognitive schemes (ibid.). It is important to note that the goal of the series of task-based clinical interviews I conducted was to construct a second-order model of David's mathematical knowledge. Although I constructed a second-order model of David's mathematics, this model does not constitute a direct representation of David's knowledge, but rather a viable characterization of plausible mental activity from which his language and observable actions may have derived. Constructing such a model involved my generating prior to, within, and among task-based clinical interviews tentative hypotheses of David's ways of understanding that explained my interpretation of the observable products of his reasoning. I developed these provisional hypotheses by attending to David's language and actions and abductively postulating the meanings that may lie behind them. I designed and modified tasks for subsequent interviews to test, extend, articulate, and refine my tentative hypotheses of David's mathematical knowledge.

All task-based clinical interviews took place in David's classroom after school on the days that best suited his schedule. I attempted to schedule the interviews so that there was at least one day between each to accommodate for ongoing analysis, and accomplished this with the exception of the last two task-based clinical interviews. In each interview, I obtained video recordings that captured David's writing, expressions, and gestures. I also created videos of my computer screen via *QuickTime Player* to capture the *didactic objects* (Thompson, 2002) David and I discussed as well as any work David completed on the computer. Additionally, I collected and scanned all written work that David produced during the interviews.

I collected daily video recordings of two of David's Honors Algebra II class sessions over a seven-and-a-half-week period, which resulted in 37 videos of classroom teaching. The only days I did not intend to collect videos of David's teaching were those days students were testing or the days David was teaching content unrelated to the angle measure, sine, or cosine. While the classroom observations did not demand the type of ongoing analysis that was part and parcel of the series of task-based clinical interviews, I documented, in the form of memos, the mathematical understandings and ways of thinking David afforded his students the opportunity to construct. I must emphasize that I characterized the ways of understanding and ways of thinking David *allowed* his students to construct, and not the understandings and ways of thinking his students *actually* constructed. In essence, I documented the understandings that I would be able to construct, and the ways of thinking that I would be able to develop, were I an engaged student in the class with sufficient background knowledge, uninhibited by unproductive understandings or disadvantageous ways of thinking.

The objective of the third phase of my experimental methodology was to obtain data that allowed me to construct a model of David's image of those aspects of his environmental context that he appraised as constraints on the quality of his instruction, and to determine the way in which this image conditioned the mathematical knowledge he employed in the context of teaching. Constructing such a model and determining the effect that David's image of instructional constraints had on his enacted subject matter knowledge involved my conducting a series of three semi-structured clinical interviews after David completed his instruction of trigonometric functions.

The content of these semi-structured clinical interviews was heavily informed by my analysis of the data I obtained from the series of task-based clinical interviews as well as from David's teaching. Based on my analysis of this data, I selected video clips to discuss with David during the clinical interview sessions to discern the role of David's image of instructional constraints on the quality of his enacted mathematical knowledge. I devoted particular attention to ascertaining David's rationale for those instructional actions in which the mathematics he allowed students to construct differed from the mathematical ways of understanding he demonstrated during the series of task-based clinical interviews. It is essential to point out that I did not assume David recognized the discrepancies I noticed in the videos excerpts I selected to discuss. Therefore, after having presented pairs of videos to David that I believed demonstrated him conveying/supporting discrepant meanings, I asked him to compare the ways of understanding he communicated in both videos. My rationale for doing so was to determine if David recognized the same inconsistencies that I noticed in the ways of understanding he demonstrated/conveyed.

Analytical Framework

I leveraged explicit formalizations of *quantitative reasoning* (Smith & Thompson, 2007; Thompson, 1990, 2011) in the design of the present study and my analysis of its data. A growing body of research (e.g., Castillo-Garsow, 2010; Ellis, 2007; Moore, 2012, 2014; Moore & Carlson, 2012; Oehrtman, Carlson, & Thompson, 2008; Thompson 1994, 2011) has identified quantitative reasoning as a particularly advantageous way of thinking for supporting students' learning of a wide variety of pre- and post-secondary mathematics concepts. Additionally, this body of research has demonstrated the diagnostic and explanatory utility of quantitative reasoning as a theory for how one may conceptualize quantitative situations.

Quantitative reasoning is a characterization of the mental actions involved in conceptualizing situations in terms of *quantities* and *quantitative relationships*. A *quantity* is an attribute, or quality, of an object that admits a measurement process (Thompson, 1990). One has conceptualized a quantity when she has identified a particular quality of an object and has in mind a process by which she may assign a numerical value to this quality in an appropriate unit (Thompson, 1994). It is important to note that quantities do not reside in objects or situations, but are instead constructed in the mind of an individual perceiving and interpreting an object or situation. Quantities are therefore conceptual entities (Thompson, 2011).

Conceptualizing a quantity does not require that one assign a numerical value to a particular attribute of an object. Instead, it is sufficient to simply have a measurement process in mind and to have conceived, either implicitly or explicitly, an appropriate unit. *Quantification* is the process by which one assigns numerical values to some quality of an object (Thompson, 1990). Note that one need not engage in a quantification process in order to have conceived a quantity, but must have in mind a quantification process whereby she may assign numerical values to the quantity (Thompson, 1994). Defining a process by which one may assign numerical values to a quantity often involves an operation on two other quantities. In such cases we say that the new quantity results from a *quantitative operation*—its conception involved an operation on two other quantities. Quantitative operations result in a conception of a single quantity while also defining the relationship among the quantity produced and the quantities operated upon to produce it (Thompson, 1990, p. 12). It is for this reason that quantitative operations assist in one's comprehension of a situation (Thompson, 1994). It is important to note the distinction between a quantitative operation and a numerical or *arithmetic operation*. Arithmetic operations are used to calculate a quantity's value whereas quantitative operations define the relationship between a new quantity and the quantities operated upon to conceive it (Thompson, 1990).

Results

On several occasions David demonstrated ways of understanding during the series of task-based clinical interviews (TBCIs) that were inconsistent or incompatible with the ways of understanding his instruction supported. I selected three such occasions to discuss with David during a phase of clinical interviews I conducted after David completed his instruction of trigonometric functions. Specifically, I presented David with three pairs of videos, each containing an excerpt from the series of task-based clinical interviews and an excerpt from his classroom teaching. From my perspective, these pairs of videos exemplified David communicating discrepant mathematical meanings. My purpose in presenting David with these pairs of videos was to determine if he willingly compromised the quality of his enacted mathematical knowledge in response to the circumstances and events he appraised as instructional constraints. The following is a presentation of my analysis of our conversation around two of these three pairs of video excerpts. I do not discuss my analysis of David's and my conversation around the third pair of video excerpts since the conclusions drawn therefrom are consistent with those I present below.

I presented David with a video excerpt from Lesson 1 in which he explained that the measure of an angle is unit-less since the standard linear units one uses to measure the length of the subtended arc and the circumference of the circle containing the subtended arc "cancel" when one computes the ratio of these lengths (see Excerpt 1). During this episode David was discussing the image displayed in Figure 1.

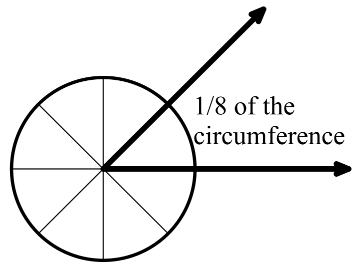


Figure 1. Angle measure as a fraction of the circle's circumference.

Excerpt 1

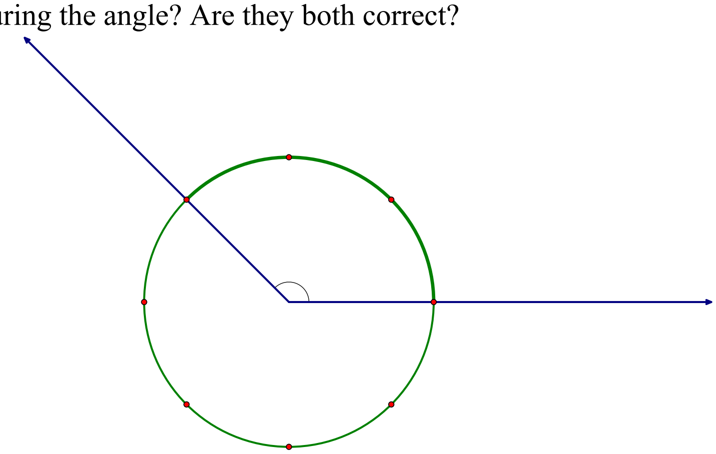
David:	So the angle subtends $1/8^{th}$ of the circumference of the circle. (*Pause*) Now do units matter here? … Why do units not matter here?
Student:	'Cause you're using a proportion.
David:	Why does that matter? …
Student:	Because even though you're making the radius larger you're also making the whole circle larger.
David:	So what happens when you do your proportion? Think in science class. (*Long pause*) 'Cause we're comparing it to our circumference, right? We're comparing arc length to circumference? What would happen to the units then? (*Long pause*) So lets just say for the sake of argument $1/8^{th}$ could be a circumference of, uh, a circumference of 16, that would mean that the arc length would be two, if it's an eighth. So two inches divided by 16 inches is?
Student:	One-eighth.
David:	One-eighth. What are the units now? (*Long pause*) What happens when you put—and again think in terms of science class—what happens when you put two inches divided by 16 inches (*writes* "2in/16in"), your science teacher would say that's $1/8^{th}$. What are the units?
Student:	It doesn't matter.
David:	It does matter. What are the units?
Student:	Inches.
David:	Inches divided by inches give you inches?
Student:	No.
David:	What does it give you?
Student:	One-eighth.
David:	What are the units?
Student:	It doesn't have units.
David:	It doesn't have units? Why not?
Student:	Because the inches cancel.
David:	'Cause inches cancel inches! … 'Cause I'm not just measuring arc length. What am I measuring? I'm measuring arc length and comparing it to what?
Student:	Circumference.
David:	Circumference! How am I comparing them?
Student:	By length.
David:	By length? What operation is going on here? Am I subtracting the circumference? (*Pause*) It's division! We're creating a ratio! Then do the units matter?
Student:	No. …

David:	What happens when we do the ratio? The units stop mattering, right? Because the units end up canceling. We're interested in the ratio. We're not interested in the units from the ratio because the units are going to reduce.

Excerpt 1 demonstrates that David did not provide students with an opportunity to interpret the division of subtended arc length and circumference as the numerator measured in units of the denominator but rather as the ratio of two values. David therefore supported students' understanding of the ratio of subtended arc length to circumference as an arithmetic operation as opposed to a quantitative operation. After showing David the video, I asked him to describe the meaning he conveyed. David's response, "Because we're comparing proportions of arc length to circumference, then we no longer care about the units of measure" led me to believe that he interpreted the video in the way I expected him to. I then showed David a video excerpt from TBCI 3 in which he responded to the task in Table 1. Excerpt 2 contains David's response to the task.

Table 1
Task 5 from TBCI 3

Nick claims that the measure of the angle shown is $3/8^{\text{ths}}$ and Meghan claims that the measure of this angle is three. How is Nick thinking about measuring this angle? How is Meghan thinking about measuring the angle? Are they both correct?



Excerpt 2

Michael:	Taking a look at this picture, what do you see in this picture here?
David:	Uh, I see an, a central angle inscribed inside a circle. Uh, we have an arc, um, that seems to be in a bolder line that, um, subtends the circle and that would be our angle measure.
Michael:	Okay. What about the red dots?
David:	Uh, the red dots seem to be evenly spaced around the, um, around the circle. So it appears that we have basically divided the circle into, um, eight pieces. …
Michael:	Okay. So let's assume that that's the case. Let's assume that these red dots are evenly spaced around the circumference of the circle so that we're splitting up the circumference into eight equal pieces. And let's suppose that Nick claims that the measure of this angle is $3/8^{\text{ths}}$ and let's say that Meghan claims that the measure of this angle is three. Um, how might Nick, who claimed that the measure of the angle is $3/8^{\text{ths}}$, how might he be thinking about measuring this angle?

David:	(*Pause*) Well if he's thinking of it as 3/8^{ths} then he is thinking that the arc is 3/8^{ths} of the whole. So his unit of measure is not the red dots, his unit of measure is the full circumference. Uh, so, um, where Meghan is saying it's three so her unit of measure is the, um, individual, um, arc lengths, uh, between the dots. And so she's saying that the arc is three units of measure where eight units of measure, where eight make up the whole. And he is saying it's 3/8^{ths} of the whole. And so his unit of measure is the whole not the parts.
Michael:	So are they both correct in their, in saying that the measure of this angle is, in Nick's case 3/8^{ths} and in Meghan's case three?
David:	Yes. It's just that they're using different units.

David's remarks in Excerpt 2 suggest that he recognized that Nick measured the subtended arc length in units of circumference and Meghan measured the subtended arc length in units of 1/8^{ths} of the circumference. David therefore assimilated Nick and Meghan's claims as two different instantiations of the same process—measuring the subtended arc length in a particular unit—and in doing so demonstrated a quantitative way of understanding angle measure. After having watched the video clip in which he responded to the task in Table 1, David reluctantly made the remark in Excerpt 3.

Excerpt 3

David:	Now that I'm thinking about it and listening to what I said, I'm not even sure if I would call the full circle a unit. I think he's thinking more in terms of a proportion of the whole. I'm not really even sure if I want to call it a unit. … Meghan is using a unit; she's using three quips or whatever. … He's thinking of it as a proportion. Like I said I'm not sure if I would even want to call it a unit.

After having reflected for a few minutes, David recognized, "The only units that 3/8^{ths} could go with is circumference. It's 3/8^{ths} of circumference. That's the only unit, if we were to assign it a unit that's the only unit that I can think of that would be appropriate." David's hesitation and reluctance suggests that his interpretation of the way of understanding he conveyed in his response to the task in Table 1 was not entirely consistent with my interpretation. David nonetheless acknowledged that the circumference is only unit to which Nick's measure of 3/8^{ths} could refer.

Once I assessed the extent to which David's interpretation of the video excerpts was consistent with mine, I asked him determine whether the ways of understanding he conveyed in the video clip from Lesson 1 (wherein he explained that angle measures are unit-less because units cancel when one computes the ratio of subtended arc length to circumference) was consistent with the way of understanding he conveyed in the video clip from TBCI 3 (in which he recognized that both Nick and Meghan measured the length of the subtended arc, albeit in different *units*).

Excerpt 4

Michael:	Is the way of understanding that you discuss in this clip (*TBCI 3*) consistent with the way of understanding you conveyed in this one here (*Lesson 1*)?
David:	Um, yeah because we're trying to get to like Nick's way of thinking. So like Nick was just talking about the proportion and how he like didn't really kind of have units. … So we were kind of leading to Nick's way of thinking about it so it does kind of match along with Nick's way of thinking.

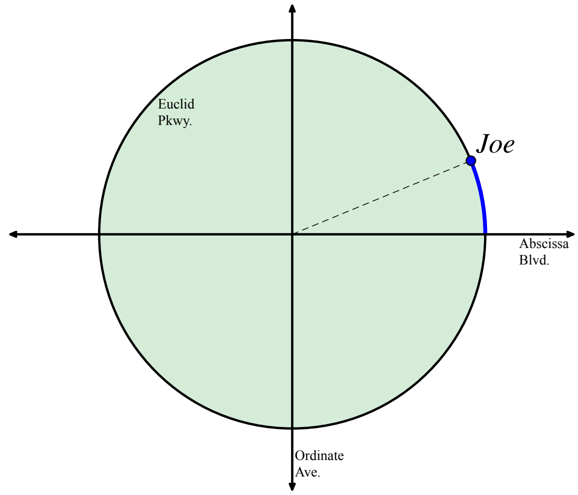
David's response in Excerpt 4 reveals that he did not recognize the discrepancy I noticed in the way of understanding angle measure he conveyed in both video excerpts. David interpreted 3/8^{ths}, the measure of the angle Nick proposed, as being a unit-less measure, which is consistent with the way of understanding he emphasized in the video clip from Lesson 1. I subsequently asked David to describe what he would change about the instructional episode depicted in the video excerpt from Lesson 1. David responded, "I don't know if I would actually change any of it because I do like the fact that when we get to radians that radians are kind of without units. So I do like the fact that we're talking about the units kind of canceling." David's failure to recognize that he conveyed what I consider vastly different meanings of angle measure in the two video excerpts, as well as his assertion that he would not change anything about his instruction from the Lesson 1 video, even after having seen a video in which he demonstrated a productive way of understanding, suggests that David was not consciously aware of the mental actions that comprise the meanings he intended to promote in his instruction. Such conscious awareness would likely have equipped David with the cognitive schemes to recognize the discrepant and incompatible ways of understanding he conveyed in both video excerpts.

I subsequently presented David with a video from the fourth task-based clinical interview in which he used an applet (see Table 2) to successfully approximate the values of sin(0.5) and cos(¾). During this interview David interpreted the task of approximating the value of sin(0.5) as, "Estimate how many radius lengths is Joe north of Abscissa Boulevard when the angle traced out by his path is 0.5 radians." In particular, David interpreted the 0.5 as representing the number of radius lengths that Joe had traveled along Euclid Parkway and sin(0.5) as representing Joe's distance north of Abscissa Boulevard in units of radius lengths. David similarly interpreted the task of approximating the value of cos(¾) in the following way: "Estimate how many radius lengths Joe is to the east of Ordinate Avenue when his path has traversed an arc that is 3/4^{ths} times as long as the radius of Flatville." David's response to the task of using the applet in Table 1 to approximate the values of sin(0.5) and cos(¾) suggests that he had constructed the outputs of sine and cosine as quantities; that is, as measurable attributes of a geometric object. After David watched the video except from the fourth task-based clinical interview, he described the way of understanding he demonstrated in a way that was consistent with my interpretation.

Table 2

Applet Designed to Support a Quantitative Understanding of Sine and Cosine Values

Suppose Joe is riding his bike on Euclid Parkway, a perfectly circular road that defines the city limits of Flatville. Ordinate Avenue is a road running vertically (north and south) through the center of Flatville and Abscissa Boulevard is a road running horizontally (east and west) through the center of Flatville. Assume Joe begins riding his bike at the east intersection of Euclid Parkway and Abscissa Boulevard in the counterclockwise direction.



After David watched the video excerpt from the fourth task-based clinical interview, I presented him with a video excerpt from Lesson 7 (which occurred four days after the fourth task-based clinical interview) in which he defined the outputs of sine and cosine relative to the following two cases: (1) when the radius of the circle centered at the vertex of an angle has a measure of one unit and (2) when this radius does not have a measure of one unit. Specifically, in the video excerpt David claimed that if the radius of the circle has a measure of one unit, then the sine and cosine values of the angle's measure are respectively equal to the y- and x-coordinates of the terminus of the subtended arc. David then explained that if the radius of the circle centered at the angle's vertex does not have a measure of one unit, then the values of sine and cosine are given by the respective ratios of the y- and x-coordinate of the terminal point to the length of the radius. It is noteworthy that David's explanation did *not* support students in conceptualizing sine and cosine values as the measure of a quantity in a particular unit. In other words, David's explanation in Lesson 7 did not support students in being able to answer the question, "What are the attributes to which sine and cosine values may respectively be applied as measures and in what unit are these attributes being measured?" In contrast to the quantitative way of understanding the outputs of sine and cosine David demonstrated in the fourth task-based clinical interview, during Lesson 7 David conveyed sine and cosine values as respectively representing y- and x-coordinates of the terminal point, or as arithmetic operations (i.e., $\sin(\theta) = y/r$ and $\cos(\theta) = x/r$).

After David viewed the two video excerpts, I asked him to determine if the way of understanding he supported in the excerpt from Lesson 7 differed from the understanding he employed to approximate the value of sin(0.5) and cos(¾) in the excerpt from the fourth task-based clinical interview (Excerpt 5).

Excerpt 5

Michael:	Is there any way that the understanding of sine and cosine you convey in this clip (*Lesson 7*) is different from what you did here (*Task-Based Clinical Interview 4*)?
David:	Only in the units of measure that we started with to obtain the ratio, but in the end we end up with an output that is a proportion of the entire radius. So in the end, no [they aren't different]. In the end they end up giving me the same thing.

David did not appear to recognize the way of understanding he demonstrated in the first video excerpt as being fundamentally different from the way of understanding he conveyed in the second. David's remark in Excerpt focused primarily on the outcome of his application of two discrepant (from my perspective) ways of understanding instead of attending to the ways of understanding themselves. Like several occasions in other interviews in which David demonstrated an incapacity to attend to ways of understanding—either his own or his students'—his remarks in Excerpt demonstrate that he had not achieved clarity relative to the mental activity involved in his own ways of understanding, nor of those he intended to support in his teaching. Had David done so, he would likely have been positioned to notice the discrepant meanings he conveyed in the videos I presented. David similarly failed to identify the inconsistent meanings he communicated in the other two pairs of video excerpts I presented to him.

Discussion

To investigate the role of David's image of instructional constraints on his enacted subject matter knowledge, I provided opportunities for him to rationalize occasions in which the ways of understanding he supported in his teaching differed from the ways of understanding he demonstrated during a series of task-based clinical interviews. My analysis of our conversation around all three pairs of video excerpts revealed that David failed to notice the discrepancy in the ways of understanding he conveyed/demonstrated in these excerpts. David's inability to recognize such discrepancies suggests that he was not consciously aware of the mental actions that comprise the meanings he intended to promote in his teaching, as such awareness would likely have equipped David with the cognitive schemes necessary to recognize the inconsistent and often incompatible ways of understanding he conveyed in the excerpts we discussed. My analysis further revealed that the occasions in which David conveyed/demonstrated discrepant, inconsistent, or incompatible ways of understanding were not occasioned by his responding to or accommodating for the circumstances and events he appraised as constraints on his practice, but were rather a consequence of his unawareness of the mental activity involved in constructing particular ways of understanding mathematical ideas.

The results of this study suggest that inconsistencies between mathematics teachers' subject matter knowledge and their enacted subject matter knowledge do not necessarily result from teachers' making conscious concessions to the quality of their enacted knowledge in the process of accommodating for the circumstances and events they appraise as constraints on their practice. Such inconsistencies might be a byproduct of teachers' unawareness of the mental activity that constitute their ways of understanding mathematical ideas. Accordingly, the main findings of this study suggest that the mathematical knowledge required for effective teaching involves more than powerful understandings of mathematical ideas; it involves an awareness of the mental actions and operations that constitute these understandings. Therefore, what might be called *mathematical content knowledge for teaching* entails both strong subject matter knowledge as well as an awareness of the mental processes that characterize such knowledge. Pre-service mathematics teacher educators and in-service professional development specialists should thus make an effort to provide opportunities for teachers to have explicit answers to questions like, "When my students read the symbols 'sin(θ)' what do I want them to imagine?" and "When my students look at an angle and think about measuring it in radians, what do I want them to visualize?" While it is beyond the scope of this paper to substantiate this claim, I believe that Piaget's (2001) notion of *reflected abstraction* can be leveraged as a particularly powerful instructional design principle for supporting teachers in becoming consciously aware of the mental processes that comprise their own ways of understanding mathematical ideas. Providing opportunities for teachers to achieve such conscious awareness of the mental activity involved in particular ways of understanding might minimize the potential that teachers will not leverage the full extent of their subject matter knowledge to support students' mathematics learning.

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Mathematicians' ideas when formulating proof in real analysis

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This report presents some findings from a study that investigated the ideas professional mathematicians find useful in developing mathematical proofs in real analysis. This research sought to describe the ideas the mathematicians developed that they deemed useful in moving their arguments toward a final proof, the context surrounding the development of these ideas in terms of Dewey's theory of inquiry, and the evolving structure of the personal argument utilizing Toulmin's argumentation scheme. Three research mathematicians completed tasks in real analysis thinking aloud in interview and at-home settings and their work was captured via video and Livescribe technology. The results of open iterative coding as well as the application of Dewey's and Toulmin's frameworks were three categories of ideas that emerged through the mathematicians' purposeful recognition of problems to be solved and their reflective and evaluative actions to solve them.

Key words: proof construction, Toulmin argumentation scheme, inquiry, real analysis, mathematicians

Writings of mathematicians and mathematics education researchers note that the mathematical proving process involves a formulation of ideas; specifically, for mathematicians, there is a reflection, reorganization of ideas and reasoning that "fill in the gaps" so a proof will emerge (Twomey Fosnot & Jacob, 2009). Byers (2007) described an idea as the answer to the question "what's really going on here?", and Raman, Sandefur, Birky, Campbell, and Somers (2009) observed three critical moments in the proving process in which there were opportunities for a proof to move forward. Tall and colleagues (2012) gave a description of proof for professional mathematics that "involves thinking about new situations, focusing on significant aspects, using previous knowledge to put new ideas together in new ways, consider relationships, make conjectures formulate definitions as necessary and to build a valid argument" (p. 15). Rav (1999) stated that the term "proof" can describe the written product used to "*display the mathematical* machinery for solving problems and to justify that a proposed solution to a problem is indeed a solution" (p. 13, italics in original); however the process of constructing proof involves informal and formal arguments to find methods to attack the problem as well as incomplete proof sketches (Aberdein, 2009). Despite these writings, little research describes the context around the formulation of ideas that a professional mathematician finds useful and how these ideas influence the development of the mathematical argument. This study focused on describing mathematicians' development of these ideas when constructing proofs in real analysis made evident in changes in the structure of the argument (Toulmin, 1958/2003) utilizing Dewey's (1938) theory of inquiry to describe the problem situation.

Research Questions

Part of a larger project, this report focuses on the findings for the research questions: What ideas move the argument forward as a professional mathematician's personal argument evolves? What problem situation is the mathematician currently entered into solving when s/he articulates and attains an idea that moves the personal argument forward?

Theoretical Perspective

This research conceived of the mathematical proving process as an evolving personal argument. The *personal argument* is a subset of one's total cognitive structure associated with the proof situation (described as a *statement image* by Selden and Selden (1995)) that the individual deems relevant to making progress in proving the statement. The personal argument is graded in that some aspects of the statement image may be central and others may lie on the periphery. The personal argument evolves or moves forward when an individual develops an idea that s/he sees as useful in making progress in proving the statement. The focus of this study was to describe the ideas incorporated and the inquirential context surrounding that development.

Toulmin's (1958/2003) argumentation model provided a means of describing structurally the evolution of the personal argument as the individual incorporated new ideas. The framework notes the content of the statements given in the argument (either explicitly or not) as well as the purposes that those statements serve. The framework classifies statements of an argument in six different categories. The claim (C) is the statement or conclusion to be asserted. The data (D) are the foundations on which the argument is based. The warrant (W) is the justification of the link between the grounds and the claim. Backing (B) presents further evidence that the warrant appropriately justifies that the data supports the claim. The modal qualifiers (Q) are statements that indicate the degree of certainty that the arguer believes that the warrant justifies the claims. The rebuttals (R) are statements that present the circumstances under which the claim would not hold.

New ideas result from periods of ambiguity or when engaged in non-routine problem solving (Byers, 2007; Lithner, 2008). John Dewey (1938) posited in his theory of inquiry that persons engage in two types of experiences: non-cognitive experiences (everyday life) and inquirential experiences (engagement in the intentional process to resolve doubtful situations). Moreover, the theory of inquiry insists that new knowledge or ideas are developed when one is engaged in active, productive inquiry into a problem. The inquirential process involves the cyclical process reflecting on problem situations to select or create tools to apply to the situation; acting to apply those tools (actively or hypothetically); and evaluating the effectiveness of the tools (Hickman, 1990). The term tool encompasses theories, proposals, actions, or knowledge chosen to be applied to a situation. Persons engaged in everyday, non-inquirential thought may also act and deploy tools; however, choosing and deploying tools is not an experiment requiring reflection and constant evaluation. Dewey's theory provided a framework for understanding the context surrounding the emergence of new ideas from the participant's point of view.

Related Literature

This research followed the lead of other researchers who have conceived of the proof construction process as a particular type of problem solving (i.e. Savic, 2012; 2013; Weber, 2005). Selden and Selden (1995; 2013) maintained that there is a close relationship between problem solving and proof, and that two kinds of problem solving could occur in proof construction: solving the mathematical problems and converting an informal solution into a formal mathematical product. Building upon extensive work in understanding the problem solving process and investigating the problem solving processes of twelve mathematicians, Carlson and Bloom (2005) developed a Multidimensional Problem Solving framework providing a description of the cyclical progression through the phases of problem solving (orientation, planning, executing, and checking), cycling, and problem-solving attributes.

Savic (2013) found that the four phases of Carlson and Bloom's framework could be used to code and describe most portions of the proving process. However, he found some differences including the mathematician cycling back to orienting after a period of incubation and one participant not completing the full cycle; Savic hypothesized additional problem solving phases could be added.

Some research has been conducted and documented the existence of and provided initial descriptions of the types of ideas that this study sought to describe. Raman (2003) characterized three types of ideas involved in the production of a proof: heuristic ideas (ideas based on informal understandings linked to private aspects of proof), procedural ideas (ideas based on logic and formal manipulations), and key ideas (heuristic ideas that can be mapped to formal proofs). In later work Raman and colleagues (Raman, Sandefur, Birky, Campbell, & Somers, 2009) identified the potential for three critical moments when constructing proof (1) attaining a *key idea* (later termed *conceptual insight*; Sandefur, Mason, Stylianides, & Watson, 2012) that gives a sense of why the statement is true; (2) gaining a *technical handle* for communicating a key idea, and (3) the culmination of the argument into a standard form. The potential for a key idea to exist apart from a technical handle exists when a prover is engaged in some informal mathematical reasoning. Sandefur et al. demonstrated how students could achieve alignment amongst the conceptual insight and technical handle when using examples in their proving process. This work as documented critical moments that may occur in the proving process characterized by the generation of certain types of ideas. Sandefur and others later showed how example-use can aid students in developing and utilizing these ideas.

Although they did not describe them as ideas, Inglis, Mejia-Ramos, & Simpson (2007) found mathematics graduate students used warrants based on both formal mathematical deductions (*deductive warrants*) and non-deductive reasoning including inductive reasoning (*inductive warrants)*and intuitive observations or experiments with some kind of mental structure (*structural-intuitive warrants*). Noting these ideas' existence is interesting but calls for further research into descriptions of how these ideas are developed and what kinds of ideas are deemed important when formal or informal reasoning is utilized. It is unclear whether these ideas identified are the only useful ideas. Additionally, there is a need for more description of the thinking and perception of the problem situation surrounding the emergence and evaluation of these ideas, how these ideas are tested and utilized in the development of the argument.

Methods

Three professional mathematicians with faculty appointments at four-year universities who specialized in researching or in teaching courses in real analysis served as the participants for this study. Each participant worked on a task or tasks in a "think-aloud" interview setting, continued to work on the tasks on their own, turned in their at-home work captured via Livescribe technology, participated in a follow-up interview replaying the video and Livescribe capture of their previous work, and repeated this process with new tasks in the next interview. Each participant worked on three to four tasks in total.

Data analysis proceeded in two phases. In the preliminary analysis of the participants' work on the tasks, I noted moments where participants articulated insights, observations, or hypotheses, and these acted as markers in the transcripts. I hypothesized Toulmin models of the participants' personal argument as well as the inquiriential context while these ideas were formulated (perceived problem, contributing actions and tools, and anticipated outcomes of applying the tools) prior to and following these markers. These hypotheses informed the questions asked at the follow-up interview. In the primary analysis, the follow-up interviews provided information to complete and modify the initial analyses. For each task, I wrote stories of the participant's complete work on the task sectioned by the ideas in order to capture the evolution of the argument. I conducted open iterative coding of each idea, the problem situation encountered, the tools that influenced the generation or articulation of the idea, and the anticipated outcome of said tools. Most analysis was inductive; however, I borrowed language from the literature when elements fit the descriptions given by other authors. I analyzed across the ideas of each participant and across participants along the common tasks to look for emerging themes and patterns. This paper reports findings regarding the types of ideas formulated and the problems encountered when ideas were articulated.

Results

In presenting these results, I first give an overview of the characteristics of the ideas that moved the argument forward and then brief descriptions of each idea category and idea type. I illustrate how these ideas developed through one participant's work on a task. Finally, I describe the problems that participants were entered into solving when they developed these ideas.

Ideas that moved the argument forward

The ideas that moved the argument forward either were accompanied by a structural shift in the personal argument captured by a Toulmin diagram, provided a means for the participant to communicate their personal argument in a logical manner, gave a participant a sense that his way of thinking was fitting, or were explicitly referred to by the participant as a useful insight. While pictures, examples, or individual actions were not included as ideas, the insights extracted from performing and reflecting upon these tools or a collection of tools were included. Ideas were coded in terms of the work they did for the participant. In total, I identified fifteen sub-type ideas grouped into three categories: ideas that focus and configure, ideas that connect and justify, and monitoring ideas (see Table 1). Note that three of the idea sub-types that connect and justify are meant to keep in the spirit of the descriptions given by Inglis et al. (2007), *inductive warrants, structural-intuitive warrants, and deductive warrants*.

An action or evaluation of that action from one particular moment could solve multiple problems or give rise to multiple feelings. Therefore, multiple idea-types at times characterized a single moment. For example, an insight that provided a *deductive warrant* could also give the prover a sense of *I can write a proof*.

Dr. C's work on the additive implies continuous task

To illustrate some examples of these idea-types and the mechanism for their development and incorporation into the personal argument, consider Dr. C's work on the task: *Let f be a function on the real numbers where for every x and y in the real numbers, f(x + y)= f(x)+ f(y). Prove or disprove that f is continuous on the real numbers if and only if it is continuous at 0.*

Upon his initial reading of the problem, Dr. C declared that he believed the statement was true for the rational numbers but not generally true for the real numbers.

> Dr. C: I was thinking about the well-known fact that the only continuous linear functions in the reals to the reals are those of the form $y = mx$ for some fixed m. And one shows that those are continuous on the rationals fairly easy, linear functions are continuous on the rationals pretty easily by doing some induction.

Table 1
Descriptions of ideas that moved the argument forward sub-types

Idea sub-type	Description
Ideas that focus and configure	Ideas that gave a sense of what was relevant, what claims to connect to the statement, fitting strategies to achieve connections, and how to structure and articulate the argument
Informing statement image	Ideas that broadened or narrowed the conception of the situation.
Task type	Assessments about what tools or ways of approaching developing connections between the conditions and the claim would be fitting
Truth proposal	Participant-generated conjectures about the validity of a given claim based on a warrant of any type
Identifying necessary conditions	A sense that "The statement can't possibly be true unless this condition is fulfilled"
Envisioned proof path	A proposal of a series of arguments that will lead to a solution that may be missing connections
Logical structure & representation system of proof	Decisions regarding structuring and communicating the formal argument
Ideas that connect and justify	Warrants and backing, the means of connecting data with claims
Deductive warrant*[1]	Reasoning based on generalizable logical statements
Inductive warrant*	Reasoning based on specific examples
Structural-intuitive warrant*	Reasoning based on a feeling that is informed by structure or experience
Syntactic connection	Symbolic manipulations deemed useful to connect given evidence to a claim that may not be supportable by deductive reasoning or attend to the mathematical objects that the symbols represent
Proposed backing	Proposed support for previously identified non-deductive warrants or vague senses of what would underlie a possible warrant
Ideas that monitor the argument evolution	Ideas or feelings about the mathematicians' progress
Truth conviction	Personal belief as to why a statement must be true
"I can write a proof"	A feeling of formulating the connections necessary to communicate the argument in a final proof
Unfruitful line of inquiry	An idea that persuaded the participant that the tools or actions pursued or considered were not optimal for achieving the set goal
Support for line of inquiry	A sense that one's actions were fitting

[1] The asterisks indicate that the titles of idea sub-types of deductive warrant, inductive warrant, and structural-intuitive warrant borrow from the descriptions of reasoning given by Inglis et al. (2007).

Dr. C was reflecting on the problem of determining the truth of the statement. Dr. C based his initial claim about the truth of the statement to be proven on a connection between the additive property of the function and linearity of functions as well as his past experience with linear functions. This was coded a *truth proposal* idea-type based on a *structural-intuitive warrant*. The Toulmin structure is after this utterance is given in Figure.

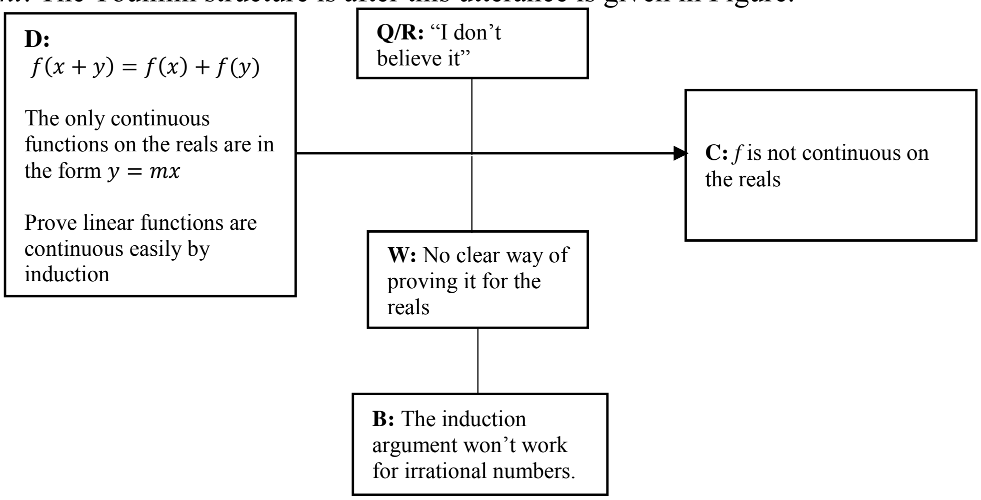

Figure 1. Toulmin model of Dr. C's personal argument with initial truth proposal.

Dr. C then set about the task of determining a means of supporting his initial inclination by looking for a counterexample to the statement. He proposed a counterexample function that was continuous at zero and the rational numbers but discontinuous on the reals, namely, the piecewise defined function that has an output of zero when the input is rational and the value of the input otherwise.

Dr. C: Well, I knew that it had to work for the rationals. So I thought I would try something that had one definition in the rationals and something else in the irrationals. And it seemed to me that *x* in one case and zero in the other case would be the easiest thing to try as a first effort

He then moved to verify the function could serve as a counterexample to the statement by checking to see if it satisfied the additive property. He chose to input two irrational numbers whose sum was rational and found the function not possess the additive property. He paused for a moment while he was working before concluding that the given statement might be true.

Dr. C: It turned out that didn't work. And if the easier ones didn't work, then the harder ones probably wouldn't either. Matter of fact, if the easier one didn't work, then it seemed likely that none of the harder ones would work.

I: Okay. So I was going to ask about that. So after you found that it didn't work, it didn't satisfy it. You paused for a while. Was it because you were trying to think of different examples, or were you convincing yourself that it-

Dr. C: Yeah. I was trying to convince myself that if this didn't work, then nothing would.

Dr. C recognized an *unfruitful line of inquiry*, moved back to the problem of determining the truth of the statement. The deliberation was inaudible but Dr. C reflected on his previous ideas about the types of functions that would not be continuous in light of the results of exploring the example function. He gave a new *truth proposal* which was based on the generated example coupled with his knowledge of functions (an *inductive warrant)*. These moment characterized ideas that moved the argument forward as illustrated in the Toulmin model in Figure.

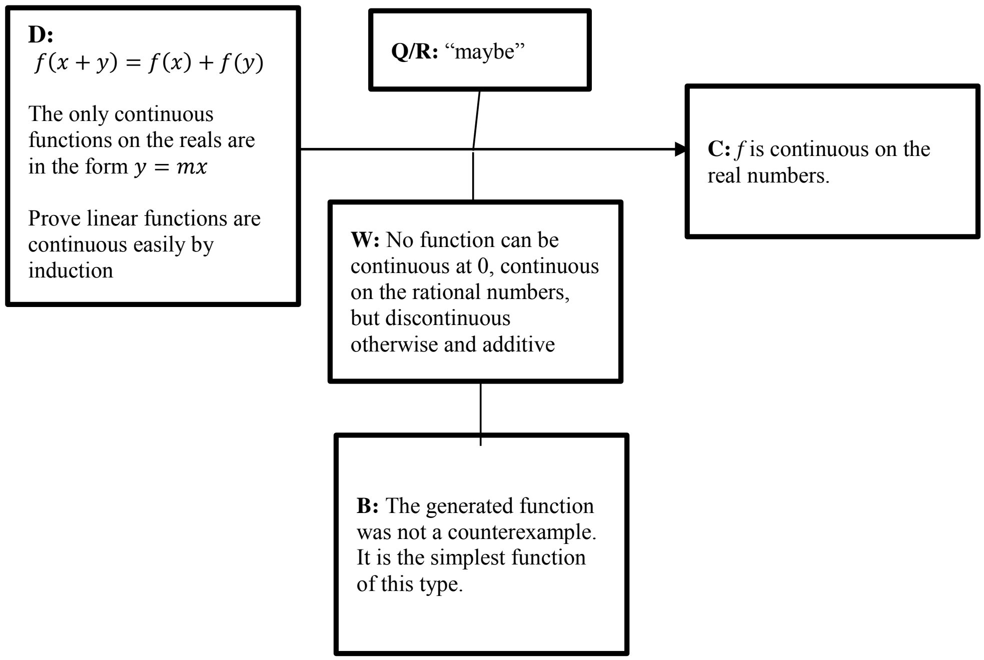

Figure 2. Toulmin model of Dr. C's personal argument after reevaluating his truth proposal.

He then moved to try to prove the statement was true (look for a deductive warrant). In exploring, he developed a string of inequalities based on instantiations of the definition of continuity and logical mathematical deductions, and he identified the *necessary condition* that $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$. He recalled a proof that $f(0) = 0$ and that the function was given to be continuous at zero to fulfill the condition. Dr. C symbolically evaluated that his written assertions were correct and declared a sense that he could now write the proof based on his deductive warrants. Because his work in proving the task was based on deductive warrants within the representation system of proof, the writing of the proof did not require the formulation of any new ideas.

Process of developing ideas: Problems encountered

No distinct pattern involving the types of problems and tools that contributed to the generation of certain ideas. However, the ideas that moved the argument forward were developed as a result of the pattern of a participant proposing or articulating an idea or tool, testing the usefulness of the proposed idea or tool or the prior ideas against the consequences of the new idea, and then articulating a new idea or evaluation. This process involved the passing through, perhaps multiple times, the inquirential cycle of reflecting, acting, and evaluating against the ideas' abilities to solve a perceived problem. The problems posed played a role in the decisions the mathematicians made about what tools or propositions would be useful. The focus of this paper is to elaborate on the problems encountered and tackled when ideas emerged. The participants transitioned through the following four phases of problems to tackle or tasks to complete in order to finish the construction of the proof.

1. Understanding the statement and/or determining truth
2. Determining a warrant of some kind
3. Validating, generalizing, or articulating those warrants
4. Writing the argument formally

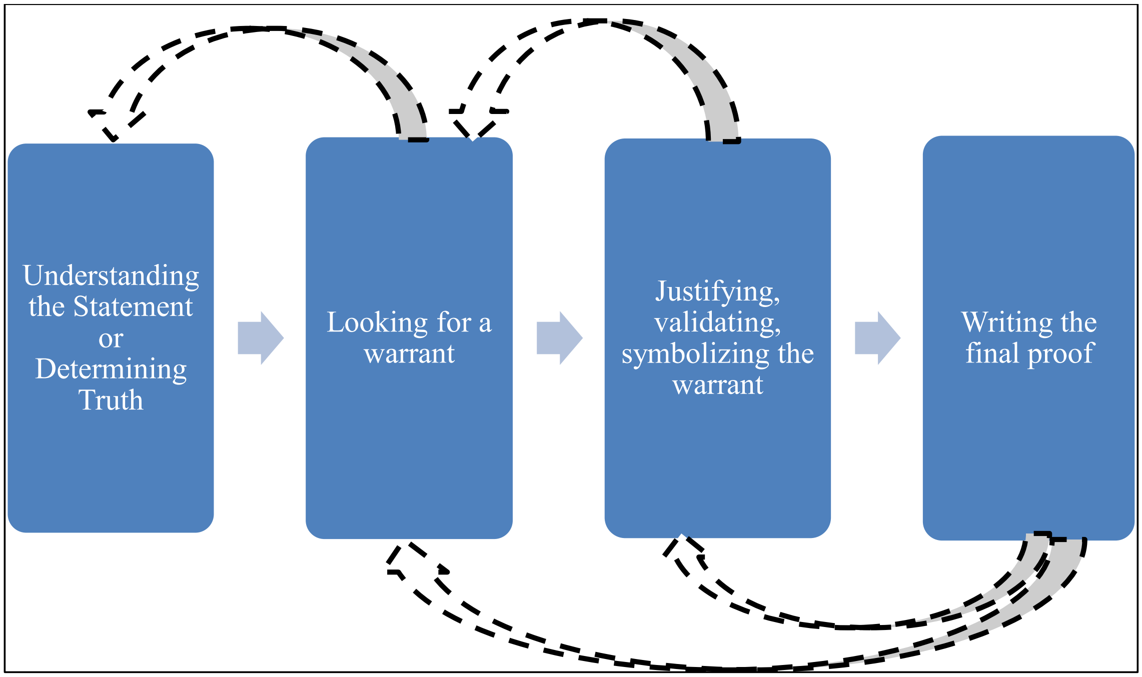

Figure 3. Illustration of the problem phases observed and the potential to cycle back.

The first phase involved the mathematicians engaging in efforts to get a sense of what the statement meant, definitions of objects described in the statement, and how objects in the statement related. On the one "prove or disprove" task, the mathematicians also engaged in determining the truth value of the statement.

The second phase of determining a warrant encompassed work to find a reason that a statement or participant-generated conjecture is true that they could eventually render into a final written proof. In the example above, Dr. C did look for a counterexample which, if found, could have been used in a final, formal proof. However, this phase also encompassed the work to find inductive and structural-intuitive warrants that provided personal feelings about understanding why the statement was true.

Once the mathematicians proposed a warrant, they worked to test the warrant, looked for ways to generalize the warrant, or articulate it in a symbolic or written way. If the warrant found previously was based on logical deductions, then this phase sometimes coincided with writing a rough draft of the proof formally. In the example above, Dr. C tested the function he generated to determine if it could serve as a warrant. Once the mathematicians articulated or felt they could articulate their reasoning for why the statement would be true in general, they indicated they were ready to write the final proof, the fourth phase listed above.

The first three phases listed could coincide with genuine problems for the mathematicians in the inquiriential sense. If a problem was encountered, the participants passed through the cycle of reflecting, acting, and evaluating until an idea that moved the argument forward was developed. Sometimes, however, the mathematicians passed through without problem as they could enact a previous insight or follow symbol manipulation through. For example, once Dr. C had formulated his truth proposal and proposed a possible counterexample, it was not problematic for him to find an efficient way to test the function.

In addition to the four above major problems to solve, the participating mathematicians also tackled problems parallel to or embedded within these problems (phases) such as dealing with a found problem with a tool. Writing the argument formally typically was not problematic for the professional mathematician once they had developed a deductive warrant.

At times the participants proceeded linearly through the four phases; however, there were instances where participants needed to cycle back to a previous phase when a proposed idea or tool was not fitting or if no tool (or proposition) could be found to solve the current problem (see Figure 1). Dr. C's work above provides an example of cycling back from the third phase of validating the warrant to the second phase of determining truth, he began (1) determining the truth of the statement, (2) worked to develop a counterexample (a deductive warrant), (3) worked to validate the warrant and found that he could not. In light of this new information, he cycled back to the first phase as he (1) reevaluated his truth determination. This prompted (2) searching for another warrant for why the statement would be true that he could render into a proof. This study also observed mathematicians cycling back from phase (3) to phase (2) and back from phase (4) to phases (2) or (3).

Discussion and Conclusions

The purpose of this larger research project was to identify ideas that mathematicians generated that pushed their arguments forward and to provide context to the situation when these ideas emerged. Dewey's theory of inquiry was

The mathematicians in this study developed ideas that moved their arguments forward in that their personal arguments structurally evolved upon the development of these ideas. The multiple idea-types are grouped into three categories. Every participant on each task identified ideas from each of the three idea categories. As was described above with Dr. C, the evolution of the personal argument was not linear in identifying focusing and configuring ideas, identifying connections and justifications, and then making monitoring decisions. Instead, multiple idea-types from any or all of these categories could characterize a single moment. The process of articulating ideas, testing the new idea or previous ideas against these new ideas, and then proposing new ideas was apparent. The process of testing ideas varied, but the process involved active, productive inquiry in that ideas were tested against their abilities to do work in solving a perceived problem. Four major types of problems or phases of the process of constructing proof were identified to coincide with the emergence of ideas. The mathematicians progressed through these four phases but needed to cycle back to a previous phase when the ideas that the mathematicians had previously incorporated into the personal argument were insufficient in resolving a situation in a later phase.

The four identified phases of understanding the statement or determining truth, looking for a warrant, working to validate, generalize, justify or articulate their warrant; and writing the formal proof are reminiscent of findings of other researchers. The following aspects have been identified as part of the proof construction process: understanding the statement or described objects (Alcock, 2008; Alcock & Weber, 2010; Carlson & Bloom, 2005; Savic, 2013); determining the truth of the statement (Sandefur et al., 2012); determining why the statement is true (Raman et al., 2009; Sandefur et al., 2012); translating ideas into analytic language (Alcock & Inglis, 2008; Alcock & Weber, 2010; Weber & Alcock, 2004); and justifying a previous idea (Alcock, 2008; Alcock & Weber, 2010).

This research is unique in its specific efforts to identify the problems encountered as participants developed new ideas and in its use of Dewey's theory of inquiry to explain how ideas were developed and tested against these problems. Using Dewey's theory of inquiry as a framework to describe the context surrounding the generation of these ideas focused the research on the question of identifying the problems encountered instead of the actions performed in isolation. We know that students and novice provers can apply heuristic strategies in non-purposeful ways (e.g. Alcock & Weber, 2010), and they may not recognize generated statements as relevant and useful ideas (e.g. Raman et al, 2009). The mathematicians in this study appear to have developed (perhaps unconsciously) a routine for approaching proof problems. There may be value in viewing a proof task as consisting of sub-tasks to be explored and resolved.

The findings presented in this paper are linked to a larger study, and there are limitations. The participants in this study selected some of the tasks. This resulted in imprecise formulations of two instances (out of ten) of tasks used. Additionally, the study was limited to three male mathematicians working in isolation on tasks within the realm of real analysis. The context of the interview situation was not representative of a research mathematician's typical practice. While the tasks presented genuine problem solving situations, they were still "school tasks". Therefore, the participants, informed by their training in school mathematics, brought with them conceptions about "hints" in the statement formulations, what were reasonable expectations for a solution, and what theorems they were allowed to assume.

The choice to conceive of the proof construction process as involving an evolving personal argument was made due to a desire to talk about all the ideas, relationships, concepts, pictures, and so on that an individual personally judges as important to providing a final proof and the relationships amongst these elements at various points in time. This conception allowed for attending to moments when ideas were generated that the prover saw as useful which broke the construction process into significant events to illustrate the story of the argument's evolution. As researching the proving process in this manner is relatively unexplored, many avenues of research are open to explore how these ideas develop, how they are tested, and the consequences their development provides for the evolution of the argument. The findings of this study were descriptive and exploratory and the fifteen idea sub-types found may not be salient in other studies. It is probable that varying the mathematical content area or narrowing the research questions would provide new and clarifying findings to refine the categorizations or provide insight as to how the proof construction process compares across mathematical content.

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Exploring pre-service teachers' mental models of "doing math"

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This study explored the mental models pre-service teachers hold of doing math. Mental models are cognitive structures people use while reasoning about the world. The mental models related to mathematics would influence a teacher's pedagogical decisions and thus influence the mental model of mathematics that their students would construct. In this study, pre-service elementary teachers drew images of mathematicians doing math. Using comparative judgements, they selected an image that best represented a mathematician doing math. The drawings and participants' responses to prompts were analyzed for common themes. The pre-service teachers generally believed that mathematicians do math through teaching and mathematicians really enjoy doing math. The appearances of the mathematicians generally adhered to stereotypes found in the literature.

Key words: Mental Models, Drawing Research, Pre-service Teacher Mathematics Beliefs, Comparative Judgement

In a recent article of the *MAA FOCUS* magazine, Francis Su, newly installed president of the MAA was asked the following question, "What is your earliest memory of doing mathematics?" Dr. Su spoke of solving arithmetic problems on worksheets, prior to being of kindergarten age, given to him by his father. He further clarified that, at that time, this was what he believed mathematics to be (Peterson, 2015). What does it mean to do math? What does it mean to do math? Due to their early interactions with students and mathematics, better understanding of teachers' perceptions regarding this question is important. This current study aims to explore mental models held by pre-service elementary teachers to better understand their perceptions of what it means to *do* math.

Mental Model Theory

Mental model theory is a theory of how people reason about the world. A mental model is a cognitive structure constructed by an individual as a representation of a possibly real, imaginary, or hypothetical external reality (Gentner, 2002; Jacob & Shaw, 1999; Johnson-Laird, Girotto, & Legrenzi, 1998; Jones, Ross, Lynam, Perez, & Leitch, 2011). Due to cognitive limitations of an individual, models cannot contain every detail of the reality and thus are not complete or technically accurate representations (Gentner, 2002; Jones et al., 2011; Norman, 1983/2014). However, structural relations present in the reality will have analogous representations in the individual's mental model (Johnson-Laird, 1998). Thus, a model will have structural features in common with the represented domain and be as iconic as possible (Johnson-Laird, 2004).

An individual constructs a mental model through experience, by perceiving or imagining the reality, or by understanding discourse and gaining formal knowledge (Jacob & Shaw, 1999; Johnson-Laird et al., 1998; Jones et al., 2011). An individual uses mental models as conceptual frameworks through which to interpret, understand, and reason about the world (Gentner 2002; Jacob & Shaw, 1999). New information filters through the model (Jones et al., 2011), and the individual reasons about situations, leading to predictions and decisions through mental manipulations of the models (Johnson-Laird, 2005). Because of how models are constructed, a mental model is contextually bound, constrained by an individual's experiences with the represented domain (Norman, 1983/2014). In addition to experience, an individual's goals and motives for construction of the model also influence the structural aspects of the reality that end up being represented in the model (Jones et al., 2011).

In addition to representing physical aspects of a particular domain, mental models also incorporate an individual's beliefs related to the domain; thus, mental models are reflective of belief systems (Libarkin, Beilfuss, & Kurdziel, 2003; Norman, 1983/2014). This connection allows an exploration of belief systems through an individual's mental model. Yet, being internal constructs, mental models are difficult to explore. While one method of exploration is the direct questioning of an individual's beliefs, people generally have difficulty clearly articulating their beliefs (Gentner, 2002). As a result, novel methods can be useful in constructing external representations of internal mental models (Jones et al., 2011).

Efforts continue in order to improve methods for constructing such representations. Mental models are more general instances of a mental image. Hence, underlying any mental image is a mental model, with the image being the projection of the mental model's visualizable aspects (Johnson-Laird, 1998; Johnson-Laird, Girotto, & Legrenzi, 1998). Some recent studies have explored mental models via participant-made drawings, which would be physical manifestations of mental images. For example, drawings were analyzed to explored elementary and middle school students' mental models of circuits (Jabot & Henry, 2007), pre-service teachers' mental models of themselves as teachers of science (Thomas, Pederson, & Finson, 2001), pre-service agriculture teachers' mental models of effective teaching (Robinson, Kelsey, & Terry, 2013), and pre-service teachers' mental models of the environment (Moseley, Desjean-Perrotta, & Utley, 2010). While not explicitly using mental model theory, other studies have used a drawing methodology to explore pre-service elementary teachers' visual images of themselves as mathematics teachers (Utley & Showalter, 2007) and middle and secondary students images of mathematicians at work (Aguilar, Rosas, Zavaleta, & Robo-Vazquez, 2014; Picker & Berry, 2000; Rock & Shaw, 2000).

In their work, Picker and Berry (2000) theorized how a stereotypical cultural image of mathematicians and their work is formed. A young learner, someone unfamiliar with the stereotypical cultural view of mathematics, begins school. Through exposure to cultural stereotypes via media, adults, and peers, through interactions with teachers lacking rich images of mathematics, through a pedagogy that reinforces stereotypes, and through the lack of clear intervention by the mathematics community, the student begins forming a deficient image of mathematics. Stereotypes fill the void left vacant by desirable alternatives, and the student's forming mental model is validated through experience. Teachers play a key early role in inculcating students into the stereotypes of mathematics. However, the teachers would need to hold a healthy model of mathematics themselves to have any positive effect, as a teacher's beliefs influence the mathematical experiences they have with their students and so can influence the model that the students form (Mewborn & Cross, 2007). If students do not have healthy images of mathematics, they may choose to pursue other vocations, potentially robbing society of valuable mathematical innovation. Thus, exploring pre-service teachers' mental models related to mathematics is of importance.

Doing **Math**

From a survey of twenty-five post-secondary mathematics professors, Latterell and Wilson (2012) formulated a working definition of doing math, stating that in order to be considered doing math, mathematicians must be *creating* new mathematics. Schoenfeld (1994) stated, "research – what most mathematicians would call *doing* mathematics – consists of making contributions to the mathematical community's knowledge store" (p. 66). As a result of their definition, Latterell and Wilson excluded teachers of mathematics from being considered as mathematicians and only included mathematics professors if they were engaged in research mathematics. However, the general populace does not necessarily hold to this same understanding.

Through a survey of children in grades K-8, Rock and Shaw (2000) determined that the students believed mathematicians did the same kind of math the students did in the classroom, only with larger numbers. Students also tended to believe mathematicians solved the hard problems no one else wanted to do. Many images drawn by the participants showed a mathematician in a classroom setting. Picker and Berry (2000) found similar results when they explored the images that 12-13 year olds had of mathematicians at work. About one-fifth of the drawings were of a teacher. The images of mathematicians adhered to some stereotypes found in the research of images of scientists; most of the images were of men, and some of the drawings resembled Einstein. In a follow-up prompt, the plurality of students mentioned that mathematicians were hired to teach math, suggesting that students actually did not have a clear idea of what mathematicians did. As a result, Picker and Berry suggested that mathematicians and their work were basically invisible to the students.

From a study of images of mathematicians at work created by high-achieving high school students attending a mathematics and science school, Aguilar, Rosas, Zavaleta, and Romo-Vázquez (2014) discovered that while the images were mostly male figures and contained many images of teachers, the students had a richer conception of what mathematicians did. They suggested this richer view developed from more exposure to advanced mathematics. Also, since many of the images contained items found in school settings, the students' limited interactions with math, mainly in the schools, heavily influenced their image of what it means to do math.

Due to the important role that teachers and the school setting play in the formation of a student's mental model of mathematics, this study explored the following question:

What shared mental model of *doing mathematics* is held by pre-service elementary teachers in a mathematics content course?

Theoretical Framework

This study used participant-made drawings in order to explore the mental models preservice teachers have of doing math. The use of drawings to explore concepts has its origins in Goodenough's Draw-a-Man psychological test developed in 1926. The test was adapted through the years, notably as the Draw-a-Scientist test in 1983 by Chambers (Finson, 2002). Participant drawings were analyzed through the Farland-Smith framework (2012) as adapted by Bachman, Berezay, and Tripp (2016). To analyze scientists at work, Farland-Smith (2012) suggested analysis along the dimensions of the appearance of the scientist, the location in which the scientific activity is taking place, and the scientific activity being conducted. In analyzing participant drawings of scientists at work, Farland-Smith used the categories of appearance, location, and activity. Using these categories, Bachman, Berezay, and Tripp coded participant drawings of themselves doing math to analyze students' mathematical affect related to doing math. Including issues of affect makes sense in light of other research into drawings of mathematicians; for example, research into images of mathematicians and mathematics has included extreme images, including images suggesting violence (e.g., Picker & Berry, 2000; Lee & Zeppelin; 2014). Images such as these may have little mathematical content but would contain valuable insight into peoples' beliefs about math and doing math.

In the current study, as the original drawing prompt requested students to draw a picture of a mathematician doing math, analysis of the drawings focused on the mathematician, what the mathematician was doing, and what elements in the drawings could be considered mathematical in nature. Thus, the categories of analysis were *Action*, *Mathematics*, *Appearance*, *Location*, and *Affect*. The action and mathematics categories were a splitting of the activity category from the Farland-Smith framework in order to better capture elements that the participants consider to be mathematical.

The drawings created by the participants were assumed to be external representations of their own mental images, which were in turn the projections of the visualizable aspects of their corresponding internal mental model. An individual's mental model was influenced by the culture (classroom) to which he or she belonged, forming shared mental model, which is an overlapping of mental representations of members of the culture (Van den Bossche, Gijselaers, Segers, Woltjer, & Kirschner, 2011). That is, the formation of the shared mental model occurred in a fashion as described by Picker and Berry (2000).

Methodology

The study was conducted at a regional university in the southeastern United States. Participants, or pre-service teachers (PSTs), in the study were undergraduate students in a teacher preparation program. The PSTs were enrolled in one of three sections of a mathematics content course for pre-service teachers. The course was the third in a sequence of four mathematics content courses required by the program. Forty-six PSTs were enrolled in the sections. The PSTs were divided between two disciplines, early childhood education (31, 67.4%) and special education (15, 32.6%). Of these students, 4 (8.7%) were male and 42 (91.3%) were female. Additionally, 2 were Hispanic (4.3%), 10 were African-American (21.7%), and 34 (73.9%) were Caucasian.

During the sixth week of classes, PSTs responded in an at-home activity consisting of several drawing activities. Germane to this current study was the prompt: Draw a picture of a mathematician doing math. PSTs had approximately one week to create the drawings. The drawings were subsequently collected and scanned to create electronic files. The *Mathematician doing Math* drawings were uploaded to the No More Marking website (nomoremarking.com), a website that facilitates and calculates comparative judgements to explore preferences. Comparative judgement is a method to measure qualities that are subjective in nature, such as individual's beliefs, and is based on the idea that a person assigns a value to a phenomenon; when asked to choose between two phenomena, the person will base the decision on a comparison of the phenomena's values; the values are based upon a shared consensus of those making the judgements (Pollitt, 2012). In other words, with many judges participating, the preference of a phenomena is based upon the shared cultural preferences of the judges. As an example, Jones and Alcock (2014) used comparative judgement to explore whether or not calculus students performed well as peer assessors in the absence of assessment criteria. Through a website, students were presented 20 pairs of student work and asked to judge which one showed better conceptual understanding. The work receiving the highest overall score, determined through appropriate formulae, was the work that showed the most understanding, as judged by the students.

During the ninth week, for an at-home activity, PSTs were invited to perform comparative judgments on the two sets of drawings with the following question: Which best represents a mathematician doing math? Furthermore, PSTs were instructed to compare each drawing and choose the one they believed best answered the questions, to give honest responses, and to not judge the pictures on artistic merit. Each PST made 40 comparisons per data set. Figure 1 shows what the PST would see on his or her screen while judging. The image receiving the highest overall score, estimated by using the Bradley-Terry model and calculated internally on the website, was taken to be the image that best represented a mathematician doing math in the opinion of the participants. That is, the image was taken to

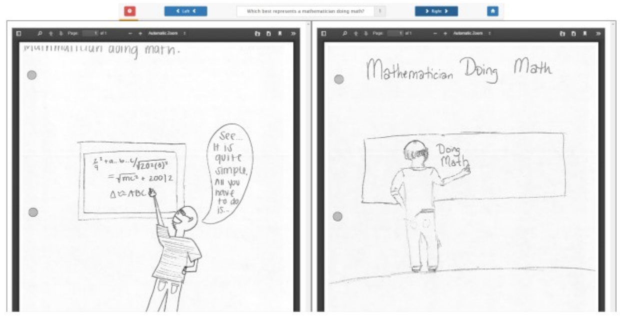


Figure 1. Screen judges while judging at No More Marking website.

represent the ideal image based upon shared standards of the students. Finally, during the twelfth week, for an at-home bonus activity, PSTs were shown the image selected through comparative judgment as the best representative of a *Mathematician doing Math* and answered the following prompts: 1.) Why do you believe this picture was selected as the best representation of a mathematician doing math? 2.) To what extent does this picture align with your beliefs of what it means for a mathematician to do math? 3.) To you, what does it mean to be a mathematician?

The participant drawings of mathematicians doing math were analyzed using the categories of action, mathematics, appearance, location, and affect as modified from the Farland-Smith framework. The drawings were analyzed one category at a time with the analysis focused on commonalities across the drawings. Colored pencils were used to circle common elements across the drawings and then a name was chosen for the subcategories to represent the common themes. The drawings were reanalyzed until elements in the drawings were exhausted. The cyclical process continued for the other categories in the framework.

In order to analyze the responses to the prompts in the bonus activity given during week 12, the participant responses were analyzed using an open coding approach, reading through the responses, then rereading each response and underlining common themes across each response to create categories. Finally, each response was reread and coded according to the themes present, with frequencies in each category tallied. As needed, categories were adjusted until the essence of each response could be categorized.

Results

This section contains the results of the analyses of the drawings and the participant responses to the first prompt on the bonus assignment.

Participant Drawings

Action

The focus of analysis for action found in the participants' drawings was on identifying what a person was doing in the drawing. Thus, a person needed to be visible in the drawing in order to identify an action. Table 1 includes the analysis of the drawings along the action category. Only one drawing did not have a person visible; this drawing showed a construction site with a crane and a building. Thus, this image was not coded for an action. Six categories of action evolved from the analysis. The first four categories were mutually exclusive while the remaining two categories could be action found in tandem with another category action.

Table 1

Actions Performed by Mathematicians

Action Category	Description	Frequency
Writing	The person has finished writing, is in the act of writing, or will be writing in the future. The person is holding a writing implement such as a pencil or a piece of chalk. The writing occurs on a vertical surface such as a chalk board or on a horizontal surface such as a piece of paper.	Vertical: 24/43 (55.8%) Horizontal: 2/43 (4.7%) Total: 26/43 (60.5%)
Pointing	The person uses a pointer or hand to direct the audience's attention to another element in the drawing.	9/43 (20.9%)
Presenting	The person is standing, facing outward to the audience with arms outstretched, in a "ta-da" pose. The person stands in front of some mathematical writing.	4/43 (9.3%)
Manipulating	The person uses physical manipulatives pieces such as those found in elementary mathematics classrooms.	1/43 (2.3%)
Talking	A speech bubble is apparent or a mouth is drawn in a way to suggest speech.	10/43 (23.3%)
Pondering	The person has a thought bubble filled with utterances related to a problem, is in a position identified as related to thought, or other elements related to thinking are present.	9/43 (20.9%)

Mathematics

This category focused on those elements in the picture that could be construed as intending to be mathematical. Two equations well-known in everyday culture, the Pythagorean theorem equation and the energy-mass equation, were present in nearly half of the drawings. The Pythagorean theorem appeared in some form, either being named or as an equation, in 13 out of 43 drawings (30.2%). Ten drawings (23.2%) contained either the equation $E = mc^2$ or the expression mc^2. As 3 drawings contained both equations, about 46.5% of the drawings contain one, the other, or both of the equations. In 12 drawings (27.9%), the person had written an algebraic expression containing alphanumeric symbols, not including the Pythagorean theorem equation, although some drawings had both the Pythagorean theorem equation and other algebraic expressions. Moreover, 28 drawings explicitly included the equal sign, while another 5 drawings contained notions of equivalence, such as geometric congruency or arithmetic problems written in a stacked algorithm, meaning that 76.7% of all drawings conveyed a notion of equivalence. In eight drawings (18.6%), a geometric drawing such as a triangle or a geometric concept such as angle was included. A doubling sum, such as $1 + 1$ occurred in 8 drawings (18.6%), while another 6 drawings (14.0%) contained basic operations involving single-digit whole numbers such as $1 + 2 = 3$. Symbols such as π or ∞ were found in 8 drawings (18.6%). Four drawings (9.3%) contained overly complex mathematics or overly complicated expressions. For example, one drawing contained the formula for a Taylor polynomial, including the remainder term formula. This same image also included the number 1729, written out in its two taxicab decompositions. A calculator could be found in four drawings (9.3%). Finally, 2 drawings (4.7%) had elements of a proof and 2 drawings (4.7%) showed physical manipulatives.

Appearance

The physical appearances of the people present in the drawings were analyzed. One drawing did not contain any people; thus, this image was not analyzed for appearance. One drawing contained seven people, one teacher standing at the front of the room at a board with six students in two rows of desks. The other drawings contain one individual person. In many of the drawings, people were standing (33/43, 76.7%). To be counted as standing, the feet of the person needed to be visible in the drawing. People were sitting in 4 images (9.3%). When standing, the person was nearly always standing near a vertical surface containing writing. The person in the drawing was either drawn in a facing (24/43, 55.8%), profile (10/43, 23.3%), or back position (8/43, 18.6%). To be considered facing, both eyes needed to be visible; in profile, only one eye was visible. Of the 34 drawings in the facing or profile position, in 15 of the drawings (44.1%; 34.9% overall), the person was wearing glasses. The Einstein effect (Picker & Berry, 2000) appeared in 13 drawings (30.2%); 4 drawings were a facsimile of Albert Einstein, and in 9 drawings, the mathematician had a wild hair style, usually sticking upward. To be considered a wild hair style, the upstanding hair needed to have an exaggerated appearance and not merely be due the participants' crude art skills. That is, the exaggerated hair style needed to appear purposeful.

While it appeared that most of the people in the images were male, this facet of analysis was not pursued in more detail as gender in many of the images was ambiguous in nature; additionally, many of the drawings were of stick figures. Furthermore, due to the stick figure nature of many drawings, an analysis of their clothing was not attempted.

Location

The location category contains descriptions of the environment surrounding the mathematician. Physical objects depicted in the drawings were noted and counted. A vertically-positioned rectangle containing some form of symbols was present in 37 drawings (86.0 %). Many of these rectangles represented chalk boards or white boards as they were either named as such in the drawing, a board was present on the rectangle similar to those found on chalk or white boards, or a tray along the bottom of the rectangle containing erasers, chalk, and/or markers was drawn. Seven drawings contain either a table or a desk (16.3%). As previously mentioned, one drawing was of a classroom setting with both a teacher and students present; this drawing contained both a vertical board and desks. Paper with writing was usually present on the tables/desks (5/43, 11.6%). Books were not present in any of the drawings.

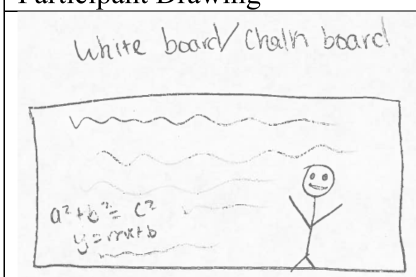
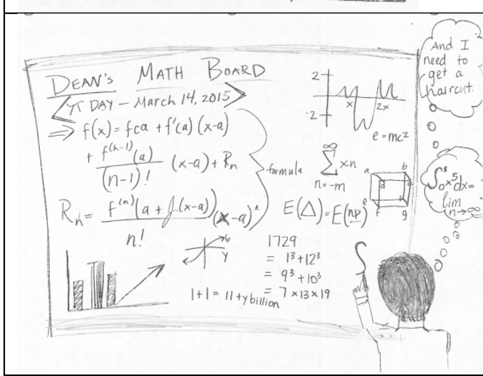
Affect

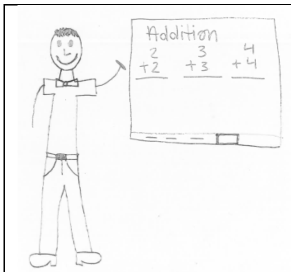
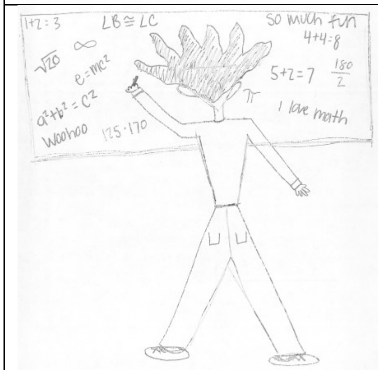
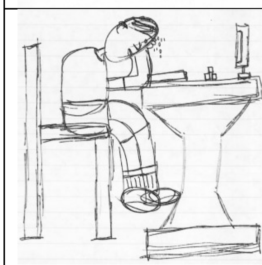
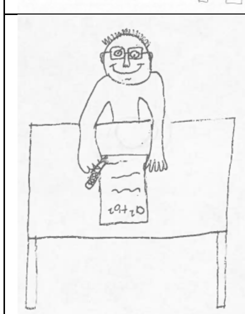
Each drawing was analyzed for affective factors related to the doing of mathematics. One aspect analyzed was the mouth on each person that was facing or in profile, totaling 34 drawings. Of these 34 drawings, 24 contained a person with a smiling mouth (64.7%);

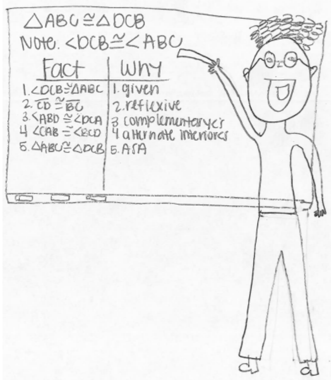
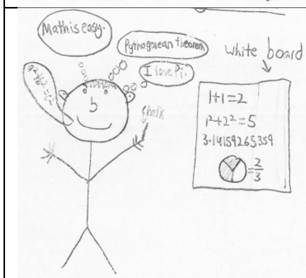
omitted from these possibilities were those drawings in which the person was determined to be talking. Additionally, 4 drawings contained the people saying or thinking positive words about math or their abilities. “Math is easy” & “I love Pi” and “Doubles, doubles, I can do doubles!” were written in thought bubbles with smiling people, while “See…it is quite simple” and “I’ve almost got it” were in a speech bubble and not already counted. Furthermore, 2 drawings contained positive writing on boards; “I love Math!” was written on the board with a smiling person, and the phrases “Woohoo,” “so much fun,” and “I love math” are written on the board with a person in a back position. Potentially negative elements in the drawings were scant. One board in a drawing of a smiling person contained the writing “Math…Blah Blah Blah.” The classroom setting drawing showed a sleeping student, with “zzz” emanating from the head of the student. Some of the pictures showing a pondering person were neither positive nor negative in nature; instead, these drawings depicted the struggle involved while solving problems. In two drawings, the person had question marks in thought bubbles, while another drawing showed the person saying, “Hmm,” with a tilde for a mouth. A more extreme pondering drawing showed perspiration dripping from the head of the person as he slumped his head forward into his hands. Overall, 26 of the 43 (55.8%) drawings contained positive elements, 4 of the 43 (9.3%) depicted the uncertainty and struggle in problem solving, 1 drawing had both positive and negative elements, 1 drawing had only negative elements, and 12 drawings were not coded for affect elements due to either a back position, talking, or no discernible affect elements (including the 1 drawing without a person).

Table 2

Example Drawings and Their Analysis

<table>
<tr><th>Participant Drawing</th><th>Themes</th></tr>
<tr><td>
</td><td>Action: Presenting
Mathematics: Pythagorean, algebra, equivalence
Appearance: Facing, Standing
Location: Board
Affect: Smile/Positive</td></tr>
<tr><td>
</td><td>Action: Writing, pondering
Mathematics: Energy-mass, geometric, infinity, complex, equivalence
Appearance: Back
Location: Board
Affect: None</td></tr>
</table>

<table>
<tr><td>
</td><td>Action: Writing
Mathematics: Doubles, equivalence
Appearance: Facing, Standing
Location: Board
Affect: Smile/Positive</td></tr>
<tr><td>
</td><td>Action: Writing
Mathematics: Pythagorean, energy-mass, geometric doubles, basic, infinity, equivalence
Appearance: Back, Standing, Einstein effect
Location: Board
Affect: Writing/Positive</td></tr>
<tr><td></td><td>Action: Pondering
Mathematics: Manipulatives/cubes
Appearance: Profile, Sitting
Location: Table/desk
Affect: Problem solving</td></tr>
<tr><td></td><td>Action: Manipulating
Mathematics: Manipulative/attribute blocks
Appearance: Facing, Standing
Location: Table
Affect: Smile/Positive</td></tr>
<tr><td></td><td>Action: Writing
Mathematics: Pythagorean
Appearance: Facing, Sitting, glasses
Location: Table
Affect: Smile/Positive</td></tr>
</table>

	**Action: Pointing, Talking** **Mathematics: Proof, geometric, equivalence** **Appearance: Facing, Standing, glasses** **Location: Board** **Affect: None**
	**Action: Writing, Talking, Pondering** **Mathematics: Pythagorean, Doubles, Basic, pi** **Appearance: Facing, Standing** **Location: Board** **Affect: Smile, words/Positive**

Extra Credit Prompt

The image in figure 2 was selected as the drawing that best represented a mathematician doing math. A reliability of 0.89 was achieved, suggesting a stability of the results. Furthermore, the interrater reliability was 0.77. In the first extra credit response, participants explained why they believed the particular drawing was chosen as the best drawing through comparative judgement. Through these responses to the prompt, students commented on five aspects of the drawing: *Artistic merit*, *Stereotypes*, *Teaching activity*, *Affect*, and *Mathematical content*.

Artistic merit

Despite being asked to not judge the drawings on artistic quality, 20 out of 37 participant responses commented about the artistic quantity. Comments were similar to the following examples: “For one, I have to say that the drawing is really good” and “I believe the artist is really talented at drawing and the depiction is detailed, realistic and aesthetically pleasing.”

Figure 2. Image selected as best representing a mathematician doing math.

Stereotypes

Participants expressed a belief that stereotypes about mathematicians were found in the drawing. Specifically, they commented that the mathematician was male, he was wearing glasses, and that he was dressed in a specific manner. Of the 37 responses, 18 (48.6%) mentioned the fact the mathematician wore glasses. Additionally, some participants added that the glasses made the mathematician appear smart, implying intelligence was a quality needed in a mathematician. For example, one participant stated, "It includes the standard picture of a man with glasses, which is a stereotype for smart people." Another said, "He also is wearing glasses, and we tend to think that people who wear glasses are smarter."

To be counted as a male stereotype, the participants had to specifically mention the choice due to being male and not just refer to the mathematician as a man. Thirteen participants (35.1%) mentioned the male stereotype. Example comments included, "I think of mathematicians as a male," "This mathematician is a man, which we sometimes automatically think of when we think of a mathematician," and "The person is a guy and generally when you think of a mathematician, it is a guy."

Furthermore, eleven participants (29.7%) commented that the person was dressed in the way a mathematician would dress, explaining that the mathematician was dressed in a fashion that was stereotypical. For example, one participant claimed, "The mathematician looks how a lot of people think a mathematician does look, with the glasses, buttoned up shirt, and sweater vest." "The character appears to be conservatively dressed which indicates an organized approach to problem solving," another participant stated.

Teaching activity

When responding to why they believe the drawing was chosen as the best representation of a mathematician doing math, 21 of the 37 participants specifically mentioned that the mathematician was teaching (56.8%). Here are some example comments:

> To me, the picture looks like that of a teacher and we also perceive our math teachers as mathematicians.
>
> I think the first example of a mathematician that comes to our minds is a math teacher so that would be one of the first things we would draw.
>
> I also see him at a board and it looks like he is teaching math. I see a mathematician as someone who not only works out math problems but teaches as well.
>
> I believe that this picture was chosen because, not only do mathematicians sit there and solve math problems all day in an office, but they also share their solutions and findings with the world. … Most math teachers that I have experienced get very excited about specific topics because that is what makes them happy and they cannot wait to share it with the world.

In these statements, participants professed the belief that teachers of math were mathematicians and thus doing math, and vice versa.

Affect

Of the 37 participant responses, 20 (54.1%) of them commented on the positive elements found in the drawing, suggesting that overall, mathematicians enjoy doing mathematics. Common words appearing in the responses were *enjoy*, *smile*, *happy*, and *excited*. Comments displaying the joy of math were similar to the following:

> Math makes mathematicians happy.
>
> The man is also smiling, and it looks like he is enjoying teaching the math.
>
> Mathematicians obviously love doing math, and this person [the illustrator] wanted to show that.

> I think that in order to be a mathematician, the mathematician must enjoy doing the math. Then with the smile, it makes me believe that this individual is sharing his knowledge and teaching it to others.
> There is a smile on his face meaning that he is excited to teach the class about math and enjoy the subject overall. This mathematician seems happy to be able to teach his children the things that he has planned on the board.

Mathematical content

Participants commented on the fact that the mathematician was working with formulas or mathematical content, leading to the conclusion that the person was a mathematician doing math. More specifically, several participants commented on the familiarity of the material on the board. For example, one participant commented, "Our class was learning about the things on the board." Another comment read, "I also believe it was picked because of the familiar formulas he is depicting." Similarly, a participant commented on choosing drawings with familiar mathematics, stating, "Many of us may have felt a connection to the mathematical concept being presented on the chalkboard since we recently discussed Euler's formula. It appears we picked what we were the most familiar to."

Discussion and Implications

This study explored the shared or cultural mental models pre-service teachers have of doing mathematics by analyzing participant-made drawings and analyzing student commentary of a prototypical mathematician doing math drawing selected through comparative judgement. The results of the analyses of the participant drawings and prompt responses were used to draw conclusions about the PSTs mental models.

One of the themes of the analysis was that familiarity breeds comfort. Martin and Gourley-Delaney (2014) suggested that students will consider activities more mathematical in nature if they have actual experience with the activity. Thus, many students comments on being familiar with the math in the prototypical drawing. Furthermore, much of the mathematics in the drawings was basic in nature or very common equations in everyday culture such as the Pythagorean theorem equation or the energy-mass equation. Moreover, pre-service teachers would generally have had limited experience with mathematicians. As their experiences in what they consider math up to this point in their lives has occurred in school settings, the only people they could possibly consider to be mathematical would be teachers.

The other possible exposure to mathematicians would be in popular culture such as television or movies in which mathematicians generally adhere to the stereotypes named by the participants. This exposure to mathematics through culture could also explain the phenomena of the Einstein effect, including Einstein as a mathematician. With a limited knowledge of mathematicians and what they do, Einstein could be a ready placeholder. Regardless, PSTs' mental models appear to contain stereotypes of mathematicians and limitations on the power of mathematics.

Another explanation for choosing teachers of mathematics as mathematicians was mentioned by several participants and best encapsulated by the following comment: "I believe that this picture was chosen because, not only do mathematicians sit there and solve math problems all day in an office, but they also share their solutions and findings with the world." Thus, according to the PSTs, a mathematician is a teacher; he or she solves difficult problems and then must effectively communicate this knowledge to others. Hence, part of doing math is teaching math, passing along knowledge. That PSTs emphasized the teaching aspect recalls the communication process standard of the NCTM. That is, mathematics is not merely solving problems, reasoning, proving, connecting, and representing. Communication is important, and communicating appeared to be a necessary facet of the PSTs' mental model of doing math.

The PSTs also believed that mathematicians loved interacting with mathematics and reveled in the joy of mathematics. As one participant wrote, "Math makes mathematicians happy." This facet of the mental model could be a bit troubling. If people do not feel joy when doing math, then perhaps they would believe they could not be a mathematician, or perhaps they are not doing math at all. While these last statements are a bit speculative, that strong, positive feelings for doing mathematics is an important feature of the mental model cannot be denied.

While more data would need to be collected from different populations, there does appear to be some misalignment between the mathematics community and the general population regarding doing math. Perhaps discussion within the mathematics and education communities would be warranted in order to help PSTs develop a mental model of math that would encourage robust models within students. If students view doing math as just teaching math, then they may become discouraged from entering the mathematics field. Or perhaps a new definition of doing mathematics should be promoted. For example, Chick and Stacey (2013) explained that mathematics teachers act as applied mathematicians in order to solve teaching problems. Such a definition would align with the results of this study.

Overall, the pre-service teachers were sensitive to the importance of mathematics in the world and their future place in facilitating a positive of mathematics in there students. As one participant so eloquently put it:

> A mathematician is someone that does not only deal with numbers, equations, or solutions. A mathematician is a contributor to the world; whether they are a teacher, an engineer, or a scientist. They share their ideas with the entire world, all the way from a 6th grade student to an elderly man reading a research article. They never stop questioning the world around us and are always looking for new ways to solve problems. When a mathematician is a teacher, all they are doing is trying to instill the passion they have for their career into a student, so they will hopefully go on and question ideas, and maybe one day discover something nobody knew existed, or maybe inspire a student to be a mathematician.

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Assessing student-learning gains from video lessons in a flipped calculus course

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Abstract

The flipped classroom has garnered attention in post-secondary mathematics in the past few years, but much of the research on this model has been on student perceptions rather than its effect on the attainment of learning goals. Instead of comparing to a "traditional" model, in this study we investigated student-learning gains in two flipped sections of Calculus I. In this paper, we focus on the question of determining learning gains from delivering content via video outside of the classroom. In particular, we compare student-learning gains after watching more conceptual videos versus more procedural ones. We share qualitative and quantitative data gathered from surveys and quizzes, as well as results from in-class assessments. We conclude by sharing some implications for future research.

Keywords: Flipped Classroom, Video Lessons, Learner-Centered Teaching, Calculus

Background

Learner-centered or active classrooms are those which change the role of the instructor from "sage on the stage" to "guide on the side" and encourage students to construct their own meaning while engaging in authentic problem-solving. Recent research has consistently showed that active classrooms improve student learning in a variety of fields. For example, in 2014 the National Academy of Sciences published a meta-study of 225 studies on student performance and failure rates in undergraduate science, technology, engineering, and mathematics [STEM] classrooms employing active learning components. Their analysis suggests that students in traditional lecture classrooms are 1.5 times more likely to fail than students in classrooms including any type of active learning technique, and failure rates in lecture classrooms are 55% higher than in active classrooms (Freeman et al., 2014). In addition, a 2013 report from the President's Council of Advisors on Science and Technology [PCAST] called for 1 million more college graduates in STEM over the next decade (PCAST STEM Undergraduate Working Group, 2013). Given the current retention rate in STEM majors during the first two years of college and the decreased failure rate in active learning classrooms reported in Freeman et al. (2014), a majority of this goal could be met by employing more active learning and less lecturing during classroom time.

The *flipped* (or *inverted*) classroom structure is one example of an active learning method that has become increasingly popular. This classroom structure takes on many forms, but the common trait is that most of the initial content delivery happens outside of the classroom while in-class time is spent solving problems, often in small groups, to assimilate the new knowledge and to deepen understanding. Some instructors deliver content through assigned readings from a text or other source, while others use videos curated from those available online or create their own videos. The core idea is to use classroom time for challenging problem-solving where students can draw support from their peers and instructor; this design more effectively uses the experience and knowledge of the instructor to guide students through the topic at hand.

Literature Review

Much of the initial literature on flipped classrooms only described the varying structures of such classrooms or the particular technologies employed by teachers using a flipped classroom. The controlled studies published on this classroom model have often focused on student perceptions of and attitudes towards the structure rather than its effect on the attainment of learning goals. For example, Foertsch, Moses, Strikwerda, and Litzkow (2002) described the use of a specific video streaming software in an engineering classroom, and reported student opinions of the videos and software, and Ford (2015) described the activity structure in a math content course for pre-service elementary teachers. Strayer (2007) gathered data on a traditional and flipped introductory statistics classroom to evaluate the learning environment of each structure, and found that students enjoyed the innovation and cooperation in the flipped class, but had a low "comfortability" with the learning activities in this environment. Roach (2014) found that 76% of students in an economics class believed that video lectures helped them learn, and the same percentage would take another class using the flipped format. Bishop and Verlager (2013) did a meta-analysis of the literature on flipped classrooms in all areas of STEM, as well as economics and sociology, and found that there were few studies examining student achievement and advocated for more controlled research.

While lecturing has been a staple of academia for close to a millennium, the flipped classroom structure might be seen as a return to an even older system of teaching where classroom time was centered around academic debate and discussion rather than the transmission of information. Modern flipped classrooms are now returning to this classroom structure and also taking advantage of newer technologies like video and the Internet. This recent resurgence dates to at least the mid-1990s when Eric Mazur, a physics professor at Harvard, started using team learning and in-class activities as ways to stop lecturing (Mazur, 1996). Jonathan Bergmann and Aaron Sams (2012) started using video lectures in the mid-2000s and are often credited with pioneering the flipped classroom and its current popularity. Since then, many educators in a variety of fields and at a wide range of institutions have started using this structure. For example, Gaughn (2014) wrote about their experiences running a flipped history classroom, and Findlay-Thompson and Mombourquette (2014) published research from their flipped business classroom. Additionally, research has been done on flipped classrooms at levels ranging from from high school (Johnson, 2013; Moore, Gillett, & Steele, 2014) to upper division medical courses (Sharma, Lau, Doherty, & Harbutt, 2015). Education-focused video repositories like Khan Academy are available on the web, and many have spoken about their experiences with various forms of the flipped classroom at local and national professional meetings (e.g., in 2014 the Joint Mathematics Meetings included a session titled *Flipping the Classroom* with 37 different talks).

As the flipped classroom has gained popularity among undergraduate STEM educators, more research studies are using classroom data to evaluate the success of flipped classrooms. Lape et al. (2014) and Mason, Shuman, and Cook (2013) compare grades on individual assessment questions in engineering between flipped and traditional sections of the same course and found few cases of statistically significantly higher scores in the flipped classroom, but no cases where students in a lecture section outperformed students in a flipped section. Similarly, Day and Foley (2006) compared grades on several course components in a senior level computer science elective and found that the flipped section earned higher average scores on every component of the grade, with statistically significant differences in the case of homework based on lectures/video lectures. Moravec et al. (2010) found statistically significant score increases over previous years in matched exam questions related to topics delivered in an inverted fashion in a large introductory biology course.

In mathematics in particular, McGivney-Burelle and Xue (2013) flipped a unit in a Calculus II course and showed that student grades on exams and homework were higher for the flipped section than the traditional section. Wilson (2013) found that students in a flipped section of statistics outperformed their lecture counterparts on exams and the course post-test. Love, Hodge, Grandgenett, and Swift (2014) found that students in a flipped linear algebra course had greater improvement in exam scores than those in a traditional section, and had higher averages on the final exam. Additionally, Problems, Resources, and Issues in Mathematics Undergraduate Studies (PRIMUS) has a forthcoming special issue on research in flipped classrooms that will increase the literature within mathematics education.

Some researchers have also considered the format, use, and effectiveness of video lectures, both in flipped classrooms and in general. For example Zappe, Leicht, Messner, Litzinger, and Lee (2009) investigated how students used online lecture videos to learn in an undergraduate engineering course, including the percentage of videos watched, students reviewing unclear segments, and time spent per video. Mayer and colleagues have published a number of papers considering specific attributes of videos, like the use of graphics and animations, or the style and tone of the voice in the video, and how they help or hinder student learning (e.g., Mayer, Hegarty, Mayer, & Campbell, 2005; Mayer, Sobko, & Mautone, 2003).

Research Question

Since students in the flipped classroom model do introductory learning of topics outside of the classroom, it is prudent to investigate the effectiveness of the content delivery method. The classroom in our study most often introduced new content outside of class through the use of the instructor's own video-recorded lessons. In this study we investigate the effectiveness of these videos on the learning gains made by students enrolled in two sections of a standard first semester undergraduate calculus course. In particular, we explore student-learning gains from watching videos outside the classroom to determine students' development of conceptual understanding and procedural skills in calculus.

Methods

Participants

The participants were undergraduate students in a first semester calculus course at a large comprehensive public university in the Mid-Atlantic United States. Of the 59 students in the study, 51 (86%) were freshmen, 5 (8%) were sophomores, 2 were juniors, and 1 was a senior. The majority of the students were male (64% male, 36% female). Four students withdrew from the course before the end of the semester. More than 80% of the students had previously had a course in calculus, generally in high school. The majority of the students were majoring in STEM fields. The students were divided into two sections (34 students in one section, 25 in the other) and generally covered the same material on the same days.

Classroom

The data was collected during the instructor's third semester running a flipped Calculus I classroom. Before each class, students had a pre-class assignment, such as watching a video or completing a reading. Nearly all class sessions started off with a short open-note quiz related to their pre-class assignment. The majority of class time was spent on group-work activities.

These activities were often sets of questions designed to reinforce, clarify, deepen, and extend the content in the pre-class assignment, address misconceptions, and provide practice. At times throughout the semester, activity were discovery or guided-inquiry, meant to allow students to develop a key concept or idea on their own through the use of a carefully chosen set of leading questions and problems. The students worked in groups of two to four students and the instructor would interact with the groups one-on-one. Students were also given homework and practice problems to be completed outside of class.

Data Sources

Over the course of the semester, we gathered qualitative data from the students, including student feedback about specific video lectures (for example, questions like "What did you find confusing?" or "What helped clear up confusion?"), student answers to post-video or post-activity questions or problems (calculus content questions to evaluate learning gains), and student surveys about their perceptions of the class structure and their learning gains. Aggregate quantitative data, such as assessment scores and exam grades, were also recorded. We used video recording on certain class days to help the instructor objectively evaluate and improve student-teacher interactions in the classroom. Collected data was used to make changes to course structure and activities in order to increase potential learning gains.

Analysis

We created rubrics to analyze the students' responses to assessment questions. For example, the rubric shown in Table 1 was used to analyze responses to a conceptual question asking students to describe L'Hôpital's Rule. We then used two-tailed pairwise comparisons ($\alpha = 0.05$) to compare groups of students (e.g., students who had previously viewed a more conceptual video about the mathematical content versus students who had viewed a more procedural video) or to compare pre- and post-assessment results. Written responses were also categorized so that we could view trends in the data.

Table 1.
Rubric Used for Scoring Responses to Conceptual L'Hôpital's Rule Question

Score	Explanation
0	Answer was blank or made no mention of tangent lines.
1	Answers either lack "functions act like their tangent lines", or say something about tangent lines but neither "slope" nor "compare".
2	Answer states that functions act like their tangent lines near a point, and that one can find limits of $f(x)/g(x)$ (or compare $f(x)$ and $g(x)$) which have indeterminate forms by comparing the slopes of their tangent lines.

Students were also given in-class surveys consisting of Likert-scale and multiple-choice questions. The surveys generally asked students about their perceptions of the class structure and their learning gains. Their answers were categorized to look for trends in the responses. Aggregate quantitative data, such as scores on specific questions from class assessments, were also used as a measure of student learning gains.

Results

In this section we share a subset of results from our larger study. In particular, we share students' overall opinions about video use and data on content learning for three individual topics, including one class period specifically designed to help us see differences in the ways students learn conceptual and procedural content via video.

First, we share data on the students' beliefs about video usage. Several times throughout the semester, students were given surveys where they could voice their opinions about the structure of the class. When asked to compare learning a new topic outside of class via reading assignment versus watching a video, students overwhelming preferred videos (86%). However, when asked what part of their class structure had the greatest positive impact on their learning, 56% of students said the pre-class videos and readings, whereas 46% said the in-class activities and interactions.[1] We also asked the students to state their beliefs on how the videos increased both their conceptual understanding and computational skills in the class (see Table 2). For both questions, the majority of the class believed the videos greatly or significantly helped their mathematical understanding and skills, although more of the students found video helpful for their conceptual understanding than their computational skills.

Table 2.
Students' Beliefs About Video Usage

	Greatly	**Significantly**	**Moderately**	**Slightly**
Conceptual understanding	38%	38%	24%	0%
Procedural skills	20%	40%	30%	10%

The data suggests that the students believed the videos contributed to their content learning, but what objective evidence for learning gains can be seen in the students' work in the classroom? Prior to an in-class activity about L'Hôpital's Rule, we had the students watch an introductory video about the topic. However, we split the classes into two groups: one group watched a more conceptual video, and the other watched a more procedural video (n = 23 for each group). At the beginning of class, the students were given a content-driven assessment about L'Hôpital's Rule, with one question asking for a more conceptual explanation ("Describe how L'Hôpital's Rule works geometrically.") and the other asking for a more procedural explanation ("How does one calculate a limit using L'Hôpital's Rule?"). We then assigned the students to groups of two to three so that each group contained at least one student who had watched each video. We videotaped the class session to capture the students' interactions with and explanations to each other. At the end of class, students were given the same assessment as before to measure what changes in their understanding occurred due to their group discussions.

We scored their responses to the pre/post assessment using rubrics similar to the one described above (0–2 scale). The students' average results can be found in Table 3. The results indicate that students who watched the more conceptual video were able to answer the more conceptual question on the pre-class assessment, but were not able to answer the more procedural question. The opposite was true for the students who had watched the procedural video.

Table 3.
Average Scores on L'Hôpital's Rule Assessment

	Conceptual Question		**Procedural Question**	
Group	Pre	Post	Pre	Post

[1] Percentages add up to more than 100% because students could choose more than one answer.

Watched conceptual video	1.39	1.48	0.09	1.26
Watched procedural video	0.04	1.35	1.74	1.57
Significantly different? (p-value)	Yes $p < 0.001$	No $p = 0.210$	Yes $p < 0.001$	No (barely) $p = 0.057$
Effect Size (r)	0.839	0.119	0.909	0.235

After working with their peers, both groups of students were generally able to answer the conceptual and procedural questions. No statistically significant differences were found in the two groups' post-class assessment average scores. However, the difference in their post-assessment scores for the procedural question was just barely insignificant. (We will discuss this finding further in the next section.) These are preliminary analyses, but they seem to indicate that students gained mathematical knowledge from watching the videos and were able to share that knowledge with other students.

We also found that the learning gains from the L'Hôpital's Rule videos were similar to the gains from other videos in the class. For example, two of the videos the students watched covered the formal definition of the limit and the intermediate value theorem. After each, the students had an in-class activity to explore the topic in more depth. This was similar to how the students were introduced to L'Hôpital's Rule

After watching each of these two videos, students took post-video surveys. For the definition of the limit, they were asked to write the definition in their own words. For the intermediate value theorem, students were asked to explain the importance of assuming continuity in the statement of the theorem. We created rubrics (0–2 scale) and scored their responses on these items (see Table 4). The average score on the limit definition survey was 1.07, and the average score on the intermediate value theorem survey was 1.26. However, these scores may hide the range of solutions given by the students. For example, on the limit definition survey more than 75% of the students had at least some understanding of the limit definition. In both cases, students exhibited at least moderate content learning gains after only watching the videos.

Table 4.
Score Distribution for Two Surveys

Score	**0**	**1**	**2**
Limit Definition survey	18%	57%	25%
IVT survey	10%	43%	36%

For the limit definition and for L'Hôpital's rule, we gave the students an additional assessment question after they had watched the video and discussed the topic in class. (For the limit definition, this was an exam question, and for L'Hôpital's rule this was the post-activity assessment.) We found the students had similar results on these assessments, with an average score of 71% on the L'Hôpital's Rule question and 75% on the definition of limit question.

Last, we compared the students' solutions to specific questions on their final exam (see Table 5). We looked at the results from the final exam about three different topics from the course: two of which were introduced via video and one of which was introduced via a guided-inquiry activity, the second most common form of content delivery in this classroom. Topics in this table are listed in the order they were covered in the course. The students' final exam scores on these three topics were virtually identical, which seems to indicate learning gains for these two methods of content delivery are nearly equivalent for these students.

Table 5.
Final Exam Results

Topic	Delivery Method	Percent
Definition of Limits	Video	76%
Definition of Derivatives	Guided-Inquiry Activity	78%
L'Hôpital's Rule	Video	75%

In summary, the results from our data imply that students are gaining at least some conceptual and procedural understanding of the mathematical content from video lessons. In the next section we discuss some possible implications of these results, along with areas of future research.

Discussion

Our study contains many different types of research data that may at first seem disconnected. This was our first attempt to quantify student learning gains in a flipped calculus classroom, specifically via video lectures. We collected data from a wide range of sources to help narrow down the research questions we wanted to explore in more depth in later studies. As such, this data may give us an initial overall picture of student learning gains through the semester but it does not allow us to go into depth for any one topic. However, we will use this data to help us design future studies to investigate the role that video lectures play in student learning in a flipped classroom.

In reviewing the results of the pre-activity assessment for the videos on L'Hôpital's Rule, we were not surprised by how well the students did on the question that related to the video they had viewed. However, more than 80% of the students in the class had taken at least one calculus class before, so we predicted that some students would initially be able to answer both questions successfully, which was not the case. Also, we were surprised by the students' improvement in both conceptual and procedural understanding after working in groups. Our results seem to indicate that students learned conceptual and procedural content from the videos and were able to share that knowledge effectively with their peers.

There are still some open questions from the data. The L'Hôpital's Rule post-activity assessment scores between the two groups of students on the procedural question were just barely insignificantly different and students felt the videos helped them more with conceptual knowledge than with learning procedures. This could mean we need to take into consideration what content educators deliver via video. However, because of the small number of students in this study, more research needs to be done to determine if there is a statistically significant difference in learning gains from more procedural videos than more conceptual ones.

One result that stands out is the low average on the post-video assessment on the limit definition. While in most other cases, the average scores were typically 65–75% on post-video assessments, in this case the average was just over 50%. As this topic is one of the most difficult and most conceptual of the entire course regardless of delivery method, this may not be as surprising as it initially seemed. It is also significant to note that this topic appeared early in the course, and it is possible that students had not yet developed productive ways of interacting with the video.

There are two important caveats to our findings. First, we had no control group against which to compare our results from the flipped course. We can anecdotally compare to our prior experiences teaching Calculus I at other institutions and with our colleagues' experiences in other sections of this course, but the goal of this study was not to make comparisons. Instead, we wanted to investigate learning gains in the flipped classroom, although we do advocate for research comparing learning gains from different pedagogical techniques. Second, while we attempted to specifically investigate learning gains from videos watched outside the classroom, we must remember the videos were not used in isolation. In some of our data, such as exam data, the effect of the videos on the students' understanding is difficult to separate from the effects of the other learning activities that happen afterwards (e.g., in-class discussion and activities, homework, office hours, studying for exams). As we continue our research, we hope to be able to isolate the learning gains from videos and also investigate how the structure of the video-watching experience affects student learning, which we will discuss in more detail below.

Last, teachers thinking about using videos in their classes should know that students will at least get a basic understanding from videos, whether the videos are more conceptual or procedural. Moreover, some may be disappointed that our results indicate that introducing material via video does not necessarily improve learning gains as compared to learning via other methods. However, our data does seem to indicate that student content learning gains from video are at least equivalent to those from the other content delivery methods used in this course.

Implications for Future Research

While the results of this preliminary study seem to indicate that students can and do learn mathematical content from video lessons, our data have also opened other lines of future inquiry. For example, one might investigate what balance of conceptual and procedural videos should be used to have the greatest impact on student-learning gains, or the effect of video-recorded lessons on specific student demographics. Another possible avenue of investigation would be to determine the effect of this classroom structure on student communities of learning. Anecdotal evidence suggests that students may form cohorts within a flipped classroom that persist in future courses.

It is important to note that flipped classrooms do not consist solely of video lessons. Moving primary instruction out of the classroom creates time in class for students to clarify and reinforce content through discussion with peers and to actively participate in authentic problem-solving. This inversion allows instructors to be present while students engage with deep mathematical ideas, which is a more effect use of their knowledge and instructional abilities.

In future research we plan to focus specifically on the ways in which students interact with videos. We want to determine if they are actively engaging with the video lessons or passively listening as though they are in a lecture. We are also curious about their video-watching habits and what they do when they are confused during a video. One of our primary goals is to find ways to structure the video-watching experience to improve student learning, including helping them build mathematical integrity (knowing what you do and do not know about mathematics). If students can accurately assess what they do and do not understand from the videos *before* each class session, the instructor can more productively run the class session. The results of this research could provide instructors with ways to make videos more effective and help students interact with videos more productively.

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Impact of Abstract Algebra Knowledge on the Teaching and Learning of Secondary Mathematics

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This study explores how in- and pre-service teachers make connections between advanced mathematics, specifically abstract algebra, and secondary mathematics. To better understand these connections, we draw on three areas of research: mathematical knowledge for teaching (e.g., Ball, Thames, & Phelps, 2008), mathematical practices (e.g., Council of Chief State School Officers [CCSSO], 2010; RAND, 2003) and habits of mind (e.g., Cuoco, Goldenberg, & Mark, 1996). In order to investigate how exposure to and instruction in abstract algebra impact the way teachers understand secondary mathematics and approach instruction, this study utilizes two main frameworks. To characterize mathematics understanding, we used the action, process, object, and schema (APOS) framework (Asiala, 1997). To unpack approaches to discussion, we use the construct of knowledge of content and teaching, part of pedagogical content knowledge in mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008). Our analysis uncovered two broad themes related to understanding and approach to instruction—understanding and interpreting inverse, and use of mathematical language.

Keywords: Algebra and Algebraic Thinking, Teacher Knowledge, Advanced Mathematical Thinking.

There has been a longstanding debate in the mathematics and mathematics education communities concerning the knowledge secondary mathematics teachers need to provide effective instruction. Central to this debate is what content knowledge secondary teachers should have in order to communicate mathematics to their students, assess student thinking, and make curricular and instructional decisions. Many researchers believe that mathematics teachers should have a strong mathematical foundation along with the knowledge of how advanced mathematics is connected to secondary mathematics (Papick, 2011). But according to others, more mathematics preparation does not necessarily improve instruction (Darling-Hammond, 2000; Monk, 1994). Therefore, it is important that, as a field, we investigate the nature of the present mathematics content courses offered to (and required of) prospective secondary mathematics teachers to gain a better understanding of which concepts positively impact teachers' instructional practice.

The Mathematics Education of Teachers II (MET-II) (Conference Board of the Mathematical Sciences [CBMS], 2012) calls for prospective mathematics teachers to have opportunities to examine the connections between the mathematics taken at the university level and the mathematics taught in high school. The importance of understanding such connections is highlighted by the fact that in order for secondary school teachers to be able to develop mathematical reasoning skills in their students, they must themselves have a coherent view of the structure of mathematics and the way in which new knowledge can be connected to and develop from prior knowledge (CBMS, 2012). However, while many have defended this idea that the mathematics education for future secondary teachers must involve "seeing the discipline as a coherent body of connected results derived from a parsimonious collection of assumptions and definitions" (CBMS, 2012, p. 56), less is known about how this might happen in traditional mathematics courses, such as abstract algebra.

This exploratory study aims to advance our understanding of how prospective mathematics teachers make sense of and understand mathematical connections between secondary mathematics and abstract algebra. This work furthers the field's understanding of what connections could be leveraged to positively impact teachers' knowledge of the structure of mathematics as well as their ability to use such structure to develop their own students' mathematical reasoning skills. The research questions for this study are as follows: (1) How does exposure to and instruction in abstract algebra impact the way teachers understand secondary mathematics? (2) How does exposure to and instruction in abstract algebra impact the way teachers approach secondary classroom instruction?

Background

We consider connections between advanced and secondary mathematics to be ones that encompass both mathematical content and ways of thinking about and engaging with that content. To better understand these connections, we draw on three areas of research: mathematical knowledge for teaching (e.g., Ball, Thames, & Phelps, 2008), mathematical practices (e.g., Council of Chief State School Officers [CCSSO], 2010) and habits of mind (e.g., Cuoco, Goldenberg, & Mark, 1996).

Since Shulman's seminal work on content and pedagogical content knowledge (Shulman, 1986), researchers and policy-makers have been investigating how teachers' understand the mathematics they teach, and how this understanding can help them clearly present it to students. Because of the growing interest in teacher knowledge and its relation to student learning (e.g., Bolyard & Moyer-Packenham, 2008; Hill, Rowan, & Ball, 2005; Piccolo, 2008), teacher education programs in the U.S. have begun to include a variety of courses designed specifically to improve teachers' content and pedagogical content knowledge.

Secondary mathematics teachers are generally required to complete the equivalent of an undergraduate degree in mathematics for certification. However, research has shown that more mathematics preparation does not necessarily improve instruction (Darling-Hammond, 2000; Monk, 1994). In fact, some research has shown that more mathematics preparation may hinder a person's ability to predict student difficulties with mathematics (Nathan & Koedinger, 2000; Nathan & Petrosino, 2003). Even so, many still believe that mathematics teachers should have a strong content foundation that includes understanding connections within and between mathematical topics (Papick, 2011). But questions remain about what secondary content stems from advanced connections, which connections are important, and how knowledge of such connections may impact classroom practice.

Mathematical knowledge for teaching (MKT) (Ball et al., 2008) incorporates both subject-matter knowledge and pedagogical content knowledge. One component in the larger domain of subject-matter knowledge we focus on is *horizon content knowledge* (HCK). We believe this specific aspect of MKT is particularly useful for thinking about what advanced content knowledge prospective mathematics teachers at the secondary level need for teaching. HCK focuses on the relation of a sequence of mathematical concepts and considers how understanding is intended to progress across a curriculum.

One component in the larger domain of pedagogical content knowledge we focus on is *knowledge of content and teaching* (KCT) (Ball et al., 2008). This particular component highlights the interaction between specific mathematical knowledge and a teacher's understanding of pedagogical issues that could impact a student's learning. In particular, teachers need to draw on their mathematical knowledge to make decisions about sequencing content for instruction, choosing examples, and evaluating "the instructional advantages and disadvantages of representations used to teach a specific idea and identify what different methods and procedures afford instructionally" (Ball et al., p. 401). This knowledge is also important as teachers facilitate classroom discussions. That is, teachers draw on their KCT to make decisions regarding when they need to ask for clarification of a student's understanding or offer alternative explanations. Each of these requires an "interaction between specific mathematical understanding and an understanding of pedagogical issues that affect student learning" (Ball et al., 2008, p. 9).

To expand the notion of MKT, it is also useful for us to consider what secondary teachers need to know beyond content and concepts and to encompass mathematical habits of mind (e.g., Cuoco et al., 1996) and engagement in mathematical practices (e.g., CCSSO, 2010). These include looking for patterns, making conjectures, attending to precision, utilizing visualizations, and connecting representations. Such habits and practices in mathematical thinking and learning extend across content areas and levels of mathematical study. Therefore, as we consider how advanced mathematical content impacts teachers' knowledge and understanding of the teaching and learning of secondary mathematics, it is important for us to consider habits and practices that may also influence how advanced ideas are learned and interpreted for teaching.

We drew on these tasks to engage practicing middle school mathematics teachers in mathematics tasks highlighting a particular connection between abstract algebra and secondary mathematics. This study seeks to show how connections can be not only mathematical in nature and relate directly to subject-matter knowledge, but to also illustrate how connections can go beyond knowledge *of* mathematics and encompass engagement *in* mathematics through the lens of mathematical knowledge for teaching, mathematical habits of mind, and mathematical practices.

Analytical Framework

Teacher Understanding of Specific Content

When considering a course such as abstract algebra, a course typically required in a traditional mathematics major, Cuoco and Rotman (2013) have argued that topics such as groups, rings, and fields are not well connected to high school mathematics in university courses. One possible explanation is that university professors are unaware of the connections between the content they teach and secondary mathematics (Hodge, Gerberry, Moss, & Staples, 2010). Therefore, in order to build prospective teachers' content knowledge in a way that could support their teaching, courses such as abstract algebra should provide occasion to encounter links between advanced and secondary mathematics (Blömeke & Delaney, 2012).

In order to investigate how exposure to and instruction in abstract algebra impact the way teachers understand secondary mathematics, this study utilizes the action, process, object, and schema (APOS) framework (Asiala, Brown, DeVries, Dubinsky, Mathews, & Thomas, 1997). APOS characterizes the learning and understanding of mathematics according to four levels of mathematical understanding. The most fundamental level of understanding of a mathematical concept is referred to as an *action*—a transformation of mathematical objects which are perceived to be, at least in part, external to the individual. Actions are composed of previously constructed mental or physical objects. For the purpose of our study, these objects are the set of numbers and mathematical symbols that comprise mathematical language. Any operation carried out on these mathematical objects is done in response to external cues; an individual simply manipulates the given objects to form another object. When an individual can perform these operations mentally, not necessarily according to a prescribed algorithm, these actions have been interiorized, marking the attainment of the *process* level of understanding (Breidenbach, Dubinsky, Hawks, & Nichols, 1992).

At the process level, a learner is carrying out operations while simultaneously considering the object produced by these operations. Another characteristic of process-level understanding is the ability to reverse an action by decomposing an object produced by a binary operation back into the original objects and operation. When an individual is able to transform a process by using actions, they are said to have *encapsulated* the process, thereby transforming it into an *object* (Clark et al., 1997). At object-level understanding, a learner is able to view processes in terms of both its operational and elemental components.

The highest level of understanding for a mathematical concept is the *schema* level, or the "individual's collection of actions, processes, objects, and other schemas which are linked by some general principles to form a framework in the individual's' mind that may be brought to bear upon a problem situation involving that concept" (Dubinsky & McDonald, 2001, p. 3). This approach towards inverse, in form of function, is also aligned to the perception of 'undoing' to reach a certain outcome. Thus, the different perspective of inverse laid confusion regarding the different methods of determining inverse. In elementary and middle school, when students are primarily familiar with four basic operations, then inverse is determined an operation nullifying the effect of another, but with advanced grade level, the operational approach changes to functional approach. According to Wasserman (2016), it is the responsibility of the teachers to develop the overarching theme of inverse despite the different ways inverse is referred to during different course level of study. Below we discuss how the APOS framework can be used to unpack one's understanding of inverse, identity, and binary operation, the building blocks of the important structures at the core of abstract algebra.

Understanding inverse, identity, and binary operation

The conceptual understanding of inverse changes as students progress through grade levels. In elementary school, children learn the fundamental arithmetic operations of addition, subtraction, multiplication and division and the relationship between them. In considering the relationships between addition and subtraction, and multiplication and division, this is perhaps the first time that children are introduced to the concept of inverse operation. According to Wasserman (2016), the primary focus of elementary school education is to develop and refine students' procedural skills as well as their ability to justify mathematical results. Gilmore and Bryant (2008) stated, "Improvements in conceptual understanding can lead to advances in procedural skill and vice versa" (p. 302). This bidirectional relationship between procedural skill and conceptual understanding constructs the foreground of students' conception of inverse.

In middle school, students' perception of number systems faces a challenge with the introduction of negative and rational numbers and mathematical operations on these numbers (Wasserman, 2016). During this time, the inverse operation of subtraction, taking one positive number from itself, (e.g., $8 - 8$), tends to be reconsidered as the equivalent operation of adding one positive number with its opposite, (e.g., $8 + {}^{-}8$). Under such circumstances, though the operations are different, the ultimate outcome is same. At this time, students often struggle with the transition of the concept of inverse operation to inverse element. Eventually, students become familiar with one of the most ubiquitous components of secondary mathematics, the concept of function. A function and its inverse are connected to each other in several ways. For example, the domain of the function is the range of the inverse function and vice versa. Also, if we consider composition as the binary operation associated with the set of functions, inverse elements become inverse functions.

Advanced content knowledge helps teachers understand the different applications and conceptions of inverses. Advanced knowledge can also help teachers and students further their conceptual understanding of inverse beyond action and process levels towards understanding inverse as "objects in a set based on connecting inverse function to abstract algebra"

(Wasserman, 2015, p. 22).

Language in the Mathematics Classroom

As presented above, a teacher's knowledge of content and teaching plays a role in the decision-making a teacher needs to sequence and choose examples and tasks and facilitate classroom discussions. Therefore, as we consider how exposure to and instruction in abstract algebra impact the way teachers may approach secondary classroom instruction; we use the construct of KCT and its relation to the use of language in the classroom.

A widely held belief about mathematics knowledge is that it is precise and unambiguous. This view extends to the language in the mathematics classroom. According to Barwell (2005), when it comes to mathematical language, "Any ambiguity, that is, any possibility of more than one interpretation for a mathematical expression arises from sloppy use of language rather than any uncertainty of mathematical ideas" (p. 118). However, Barwell argues that as students explore mathematical ideas, they are better able to participate in mathematical practices and further develop their mathematical thinking. In this way, ambiguity "acts as an important resource for students and teachers, serving as a means of articulating between thinking and discourse" (p. 125).

Hauk, Toney, Jackson, Nair, and Tsay (2014) use the idea of discourse in the mathematics classroom as a central tenet of an expanded model of PCK for secondary and postsecondary mathematics focused on the use of language. This expanded model aims to bring attention to "mathematical appropriateness, clarity, and precision that are integral to thinking, learning, and communicating, especially in advanced mathematics" (p. A24). In their framework, Hauk et al. connect the elements of PCK to *knowledge of discourse*, to help address the fundamental question: "What is the interplay among advanced mathematical understandings, teaching, and culturally mediated communication in defining and growing pedagogical content knowledge" (p. A21). This knowledge of discourse helps relate PCK components to ways of thinking. In particular, KCT is connected to knowledge of discourse through *implementation thinking*. Implementation thinking draws on knowledge of discourse and mathematics-specific instructional practices, such as questioning, as a teacher adapts teaching based on content and classroom context.

For example, Hauk et al. (2014) present vignettes of a novice undergraduate instructor, Pat, to illustrate his development in implementation thinking. In his first semester, Pat's implementation thinking focuses on students getting the correct answer. He evaluates students' contributions and student-to-student interaction as less important. In Pat's fifth semester of teaching, he begins to attend to student thinking and works to make sense of and reason about the mathematics content with his students, which indicates growth in his implementation thinking. The goal of presenting these vignettes is to illustrate how effective teaching needs to extend "beyond precise and accurate transmission of facts or uptake by students of information and includes taking into account the background and experiences (mathematical and otherwise) of the people in the room" (p. A34).

As we consider the relationship between approaches to instruction, KCT, and implementation thinking, we work to uncover how language in the classroom can impact the teaching and learning of mathematics. In particular, we consider how exposure to and instruction in abstract algebra may change one's perspective on their use of language, and thus alter how one draws on discourse to inform the sequencing and choice of examples and tasks and the facilitation of classroom discussions.

Methods

To explore how exposure to and instruction in abstract algebra impact teachers' understanding of secondary mathematics and approach to instruction, we conducted a pilot study with pre- and in-service teachers enrolled in a mathematics master degree program with a concentration in mathematics education. In this pilot, we developed an instructional unit focused on solving equations. The goals of the unit were to have participants consider mathematical properties used when solving equations and how these properties were related to algebraic structures. One of the researchers, Murray, used this unit to teach one 2.5-hour class session. The class was videotaped and all written artifacts were collected. A more detailed description of the participants, data collection, and data analysis follows.

Participants

Participants were 12 students in a mathematics education course, titled *Selected Topics in Mathematics Education*, which focuses on topics often taught during high school or the early years of college. Students in this course are exposed to subject matter viewpoints from advanced college-level mathematics courses such as abstract algebra, geometry, number theory, and analysis. The class met once a week for 2.5 hours over a 15-week semester. Of the 12 participants, four were male and eight were female with zero to fifteen years teaching experience. One out of the twelve participants was a special education teacher, two were pre-service teachers, and nine were in-service high school and middle school mathematics teachers.

Data Collection

We relied upon three data sources: audio and video classroom data, written artifacts, and participant interviews. The instructional unit began by engaging participants in an investigation of the mathematical properties used to solve three different linear equations. Participants were asked to provide rationale for every step used to solve the equations. The purpose of this section was to challenge teachers' understanding of mathematical properties used for solving equations and to consider how attention to the algebraic structures and their properties may inform procedures and solutions.

After conversation of the initial problems, Murray facilitated a discussion on abstract algebra, specifically groups, rings, fields and their properties. Throughout the class, participants were encouraged to engage in small group discussions and explore different aspects of each algebraic structure with the help of various examples. Every small-group discussion was followed by whole-class discussion where participants shared their own solutions and rationale supporting their answers. The session was filmed to capture participant responses for future analysis. At the end of the session, participants reflected upon their classroom experience by answering a series of four questions asking about the impact the class might have on their perception of solving equations.

Following the classroom lesson, participants were emailed a request to participate in a follow-up interview; the purpose of which was to clarify ideas discussed in class and to probe participants' thinking on the impact of tertiary knowledge on the understanding of secondary mathematics and instruction. Only one participant volunteered for the interview session and his replies were recorded for future analysis.

Data Analysis

Due to the exploratory nature of this study, we used qualitative methods to analyze the data. APOS theory is well-suited to characterize conceptual understanding of mathematics; for our purposes, we used this framework to describe teachers' knowledge of secondary mathematics, particularly with regard to the concept of inverse. Having the ability to assess teachers' understanding helped us to answer our first research question, as it provided a starting point from which to discern changes in participants' understanding of the use of inverses before and after exposure to and instruction in abstract algebra. We additionally employed the MKT framework, and specifically KCT, to address our second research question. In particular, we used this framework to investigate how participants discussed secondary mathematics instruction after exposure to and instruction in abstract algebra.

We conducted preliminary analysis of the audio and video data with the intention of isolating episodes that contained connections between abstract algebra and secondary mathematics. For the initial analysis, we independently reviewed the video data and isolated episodes where connections were believed to have occurred. The researchers then came together to discuss the isolated portions of data. Once the significance of these episodes was mutually ratified, the segments were transcribed. The segments of data were then re-analyzed by reading through the transcripts. Initial or "open" coding methods were used, searching for words or phrases that showed evidence of participants' understanding of secondary mathematics or connections between secondary mathematics and abstract algebra. Although the analysis of our transcribed data was guided by our initial research questions, open coding was the preferred method of initial analysis as it allowed for the development of tentative codes that led to further inquiry, thereby allowing the study to take direction organically (Saldana, 2009).

Once initial codes had been elicited from the transcribed data, the codes were refined by employing inductive, constant-comparative methods (Merriam, 2009). This structural coding was particularly well-suited for our study as the study was exploratory in nature, incorporated multiple participants, and elicited data via semi-structured data-gathering protocols. We were able to assign more focused codes that served as a labeling and indexing device for our preliminary analysis. Finally, we used axial coding to categorize and organize the codes that had been generated during the initial and secondary coding phases. This coding resulted in two broad themes related to abstract algebra and teachers' understanding of secondary mathematics—*understanding and interpreting inverse*, and *use of mathematical language*.

Results

We report results from classroom data, e.g., transcriptions from the video-taped lesson and written artifacts. Although we also interviewed one participant after the instructional unit, this data continues to be analyzed, and will not be discussed here.

Impact on Understanding Secondary Mathematics

In considering how exposure to and instruction in abstract algebra impact the way secondary teachers understand secondary mathematics, we found some participants moving from an action or process-level of understanding towards an object-level of understanding. To illustrate this change in understanding, or lack thereof, we present evidence from the data focused on the concept of inverse. To explain this characterization of participant knowledge, we adapted the definitions from the APOS framework for the concept of inverse (see Table 1).

	Definition (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997, p. 400).	Our Characterization
Action	[A]n *action* is a transformation of mathematical objects that is performed by an individual according to some explicit algorithm and hence is seen by the subject as externally driven.	An *action* conception of inverse involves being able to use inverses to perform particular mathematical tasks. In the case of this study, we began by discussing the use of inverses to solve equations. In this instance, when the students saw addition or multiplication as an external cue, they thought subtraction or division in response. That is, they were using the inverse to isolate the variable and determine the solution to a problem.

Process	When the individual reflects on the action and constructs an internal operation that performs the same transformation, then we say that the action has been interiorized as a process.	A *process* conception of inverse involves more generally thinking about inverses, so that an inverse is an operation, as well as a mathematical property that one can use in a variety of situations. For example, inverses were being thought of as operations and the students discuss properties of equality as a way to think about solving equations.
Object	When it becomes necessary to perform actions on a process, the subject must encapsulate it to become a total entity, or an *object*. In many mathematical operations, it is necessary to de- encapsulate an object and work with the process from which it came.	An *object* conception of inverse involves seeing the inverse as an element of a set, consisting of elements and a binary operation. Here, students think about the inverses as being elements in the set, but then are able to use the process to determine what that element is for each specific member of the group.
Schema	A *schema* is a coherent collection of processes, objects and previously constructed schemas that are invoked to deal with a mathematical problem situation. As with encapsulated processes, an object is created when a schema is *thematized* to become another kind of object, which can also be *de-thematized* to obtain the original contents of the schema.	The *schema* conception of inverse includes acceptance of the operational/elemental duality of inverses. The utility of inverses as not only an element with an associated binary operation, but also its relation to other mathematical properties and applications to problem solving are understood.

Table 1: *Characterization of Student Understanding of Inverse According to the APOS Framework*.

In the sections to follow, we illustrate how before exposure to and instruction in abstract algebra, participants relied on external cues within problem structure to determine and describe solution strategies. Some participants possessed a more generalized view of inverses and were able to establish a connection between the use of operational inverses and existing properties of equality. After the unit on abstract algebra, some participants were able to demonstrate characteristics of object-level understanding, yet still tended to rely on process to participate in the tasks. In all cases, conversation on inverses predominantly centered around the operational aspect; those students who attempted to reconcile the operational-elemental duality of inverses were viewed to demonstrate essential characteristics of object-level understanding and reasoning.

Understanding Prior to Exposure to Abstract Algebra

To begin the lesson, participants were provided three linear equations, $x+5=12$, $3x=12$, and $5x+7=3+2x$, with worked out solutions. They were asked to describe the steps and properties used to solve the equations. For the first example, participants cited the mathematical properties of "inverse of addition" and the "subtraction property of equality" to justify the step of subtracting 5 from both sides of the equation. [Author] followed up by asking, "Are those two things different?" Participants replied both yes and no, which led to the follow-up question of why they might be different.

P1: I think they are different because, um…the, calling it an additive inverse is referring to why they chose subtraction. Subtraction property of equality refers to the fact that its subtraction on both sides.

P2: That's, that's true.

Murray: So how are they the same then?

P1: They're both subtracting. I mean, if you're calling it the inverse of addition, that's subtraction, and the subtraction property of equality is obviously subtraction.

This brief episode at the beginning of the lesson sheds light on the participants' prior knowledge of secondary mathematics, particularly with respect to the use of inverses as an aspect of performing algebraic operations. In stating, "additive inverse is referring to why they chose subtraction," participants used the presence of addition as an indicator within the structure of the equation to employ subtraction as a solution step. Reliance on external cues embedded in the problem demonstrated an action-level understanding of inverse. The participants also viewed the inverse of addition as the subtraction property of equality, an established mathematical property, thereby signifying an understanding of the utility of inverses beyond that of an external process for 'undoing'. This interpretation indicates a more general perception of inverse and shows that some viewed inverses as serving an algebraic application other than reversing an operation, indicating a process-level of understanding.

The steps used to solve the second equation, $3x=12$, were viewed similarly. When asked to describe the step of dividing by 3 students replied,

P1: Division property of inequality, equality?
Murray: Division property of equality. Anyone calling it anything different than that?
P2: Inverse multiplication?
Murray: Inverse multiplication. So, is that the same thing what just happened with the…
P2: Mmhm.
Murray: So, when you talk about the inverse property, so you said inverse property of, what?
P2: Multiplication.
Murray: Of multiplication. So, what's the inverse?
P2: Division?
Murray: Ok. So you are saying, you are saying that because multiplication division are inverse operations? All right. Is that how other people are interpreting that?
P3: Multiplied by the reciprocal.
Murray: We are multiplying by the reciprocal. What's the reciprocal?
P3: Of three it's one-third.
Murray: One-third. So will that look different if I was thinking about it as multiplying by the reciprocal?
P3: Same thing. It's dividing by three.
P4: It will look, it would look different written in the equational form, obviously because instead of having the division sign it is having the multiplication sign.

As in the first problem, participants demonstrated a reliance on cues within the equation's structure to describe operations. In this equation, students observed that 3 and x were multiplied together to yield 12; the use of multiplication prompted the participants to employ division to reverse multiplication. When considering the use of the multiplicative inverse as an alternative to division, one participants stated, "it would look different written in the equational form, obviously because instead of having the division sign it is having the multiplication sign," demonstrating that some were still concerned with differences in notation in equivalent solution steps. According to our framework, the reliance on external cues to select or justify an operation indicates that the participants demonstrate an action-level of understanding of inverse.

Continuing the discussion, the students posit that the equation can be solved in two ways, by dividing both sides by three or multiplying by the multiplicative inverse. Although there was still some confusion over exact language use, all participants agreed that the outcome, or the object produced by either action would be the same. By arriving at the same result through the use of different actions, participants demonstrate a process-level understanding of inverse.

For the equation, $5x+7=3+2x$ participants were asked to justify the operation of subtracting

$2x$. As they discussed this step, rather than discussing additive inverse as before, a new idea came to light.

Murray: Ok then for the last one, what happens…what are we doing there? So for the first side we are subtracting the $2x$, what's that?

P1: Combining like terms.

Murray: Combining the like terms? What would other people say about that?

P2: It's still subtraction property of equality.

Murray: Ok. So it's still subtraction. So, what makes it different than the last problem? Like why it…why would combined like terms maybe pop-up there?

P3: Because it is a variable?

Murray: Still they look different right? So, we are kind of combining like terms but in order to do that we have to use the property.

P2: I only use combining like terms when things are on the same side of the equal sign, maybe it's my preference in my class, but like I, I try to distinguish them because when I say combining like terms I mean there are things on the same side of the equal sign that can be combined. And then when we are moving things from side to side, I use subtraction property or addition property just so that they know kind of distinction I am trying to make. But I also teach pre-algebra kids, so I try to create the distinction.

Unlike the discussion of the first two problems, in this segment we did not find any discussion of the use of inverses. Instead one participant explained this step as "combining like terms." There was no mention of using subtraction to reverse the operation of adding $2x$, nor was there mention of the additive inverse, $-2x$. Students viewed this problem as structurally different than the first two because they performed actions on a variable rather than a constant term. By performing intermediate steps to isolate the variable or combine like terms, students did not recognize they were utilizing the same properties as the first problem, namely inverse of addition. By slightly altering the structure of the problem and requiring students to reconcile operations on a variable term, at least one participant thought differently about essentially the same operations. The participants' perception that a different idea must be used in this situation showed reliance on external cues within the structure of the problem, evidencing action-level reasoning. In summary, prior to the introduction of abstract algebra content there is evidence that most participants possess an action and/or process-level understanding of inverses.

Understanding After Exposure to Abstract Algebra

After the introduction to abstract algebra, participants were prompted to explain the nature of inverse operations within a group structure. During this conversation, some participants displayed an object-level understanding of inverse, while others continued to show understanding of inverse at an action-level. Below is an exchange that illustrates the object level of understanding.

P1: We said when you were talking about a group that each element needs an inverse element and that in order to find an inverse element, you are thinking about an inverse operation.

Murray: So it means something different, when I say inverse operation I mean something different than when I say additive inverse.

P1: Sort of. When we are talking about the set of integers not that a specific element like 2, and I want to find its inverse, I am thinking about subtraction in a way to come up with additive inverse, I am thinking about the inverse operation.

The requirement that each element in the group has a corresponding inverse element is a property unique to the group structure, and knowledge of this requirement changed the direction of discourse on inverses. For the first time, participants discussed whether an inverse may or may not exist. Participants attended to the relationship between an element, its inverse, and the associated binary operation. By contemplating the existence of an inverse element for a given element with respect to a single operation, participants began to consider the elemental-operational duality of inverses. Within the APOS framework, by accommodating the notion that inverses consist of both an elemental and operational component, some participants were able to encapsulate the concept of inverse and demonstrate object-level understanding. These participants continued to display object-level understanding in a later activity in the unit.

One of the final activities included solving linear and quadratic equations in Z_5 and Z_6. For solving the linear equations, most participants used a guess-and-check method by either referring to the Cayley tables they generated or solving the equations over the set of real number and determining the solutions' equivalence class in Z_5 or Z_6. During small group discussion, however, Murray encouraged participants to consider using properties to solve the equation $x+5=1$ in Z_6:

P1: Like I can use the additive inverse of 5 on both sides or something. Is that what you are saying?
Murray: Maybe. Yeah. So if you did that...
P2: You want something that's one. Ah, well, but that's…
P1: Because that would give me a zero, which is what I was looking at.
Murray: So you're just saying, so she was like, if I added the additive inverse to both sides.
P2: Of five.
Murray: Of five. Yeah.
P1: Which, I have to think about.
Murray: Yeah.
P2: Because the additive inverse of five in six will be
P1: Is one?
P2: One.
Murray: One. So, if I did $x+5$,
P2: Add one to both sides.
P1: So add one to both sides.
Murray: Which would give me this.
P1: And get me x plus zero equals two.

In this discussion, participants considered the implications of adding the additive inverse of 5 to 1 in Z_6, which would result in x plus the additive identity on the left side of the equation, and a new object - the solution on the right. Furthermore, the participants considered what the solution would look like within the domain of Z_6 rather than on the set of real numbers. This indicated an object-level understanding of inverse as participants were thinking about the inverse as an element of Z_6, but were using a process to determine the solution.

However, not all participants were able to develop this level of understanding. In another activity, participants were asked to determine if Z_5 or Z_6 satisfied the field axioms. One participant, Steven (pseudonym), observed that multiplicative commutativity held for all elements within Z_5 and Z_6. But confusion arose around a discussion of the inverses. Steven claimed "The inverse is zero. Both the additive and multiplicative inverse is the same. 0 times a is 0." A few moments later Steven continued, "here they are saying, the inverse of a times a is the additive identity.” In testing Z_5 and Z_6 against the field axioms, Steven seemed to be emphasizing the importance of the operations, but ignoring the significance of the objects created by these operations. It appeared that Steven misread the multiplicative inverse axiom, taking *e* to be the additive identity. Steven remained focused on the operation to be performed, in this instance—multiplication, but ignored the fact that the axiom was dedicated to producing the *multiplicative* identity by multiplying an element with its *multiplicative* inverse.

When determining if Z_5 and Z_6 satisfy the multiplicative commutativity and identity axioms, Steven relied heavily on the notation within the axioms to analyze the structure of these residue classes. In all field axioms, the binary operation is multiplication, however; Steven failed to realize that multiplying *a'* and *a* should yield the multiplicative identity. Steven attended to the operations prescribed by the axioms without regard for the resultant object, thereby overlooking the fact that *e* represented different elements depending on the operation. The application of operations without consideration of the object produced provided evidence of action-level understanding.

Impact on Approaches to the Use of Language in Secondary Classroom Instruction

In considering how exposure to and instruction in abstract algebra impact the way teachers may approach secondary classroom instruction, we found participants focusing on language use in the classroom. To illustrate this focus, we present evidence from the data primarily after the introduction of a particular classroom scenario about solving quadratic equations.

It is interesting to note that throughout the instructional unit, Murray tried to engage participants in discussion about the influence of advanced mathematical concepts on secondary school mathematics. For example, after the extensive discussion about inverse operation and inverse elements associated with different groups, Murray stated:

> “So, when I talk about inverse operation, I wonder how much that maybe helps or hinders students’ ideas about what an inverse is. [W]e have all these inverses, right? I have additive inverse. I have multiplicative inverse. I have functional inverse…So this word inverse keeps coming up over and over and over again, but it means different things in different situations.”

Even with prompting comments such as this, participants did not seem to consider the impact of abstract algebra on their instruction until the final activities in the unit.

Recall that as participants solved linear and quadratic equations in Z_5 and Z_6, we saw some moving towards an object understanding of inverse. While solving quadratics, participants realized was there were four solutions in Z_6; namely the zero divisors in the ring. Immediately following discussion about the nature of the quadratic solutions for Z_5 and Z_6, [Author] introduced a vignette about a high school algebra teacher, Clark Freeman. The scenario below appeared in the instructional unit:

> *Clark Freeman is teaching in an Algebra classroom. He has mostly first year high school students, but there are some second and third year students repeating the course. The class has been discussing quadratic functions for a week, and has discussed finding the x-intercepts, y-intercept, and axis of symmetry. During one class, Mr. Clark is presenting solving simple quadratic equations. He first shows the students how to solve $x^2 - 4=0$. He describes adding four to both sides to obtain $x^2=4$, then takes the square root of each side to obtain $x = 2, {}^{-}2$. He then shows the students $x^2+4=0$. He describes subtracting four from both sides to obtain $x^2={}^{-}4$. He says, “When we try to take the square root of both sides, we have to take the square root of a negative number, which we can’t do, so there are no solutions.”*

Participants then considered the questions: (1) What do you think of Clark’s presentation?

(2) What does Clark mean by "no solution?", and (3) Is there any situation where the equation, $x^2+4=0$ would have a solution?

By connecting their experience of finding four solutions for the quadratic equation $x^2+2x+3=0$ in Z_6 to Mr. Freeman's instruction, participants considered the struggle students go through when first introduced to quadratic equations with complex solutions. Specifically, participants agreed that it is the responsibility of the teachers to choose, make, and use mathematical representations effectively (Ball et al., 2008), and use mathematical language in the classrooms carefully to avoid conflicts in students' understanding of mathematical concepts. One participant stated, "There's gotta be a distinction between 'no solution' and 'no real solution'. The wording has to be a little more precise."

Additionally, within the written responses at the conclusion of the unit, participants reacted to the question, "In what ways, if any, have these discussions impacted your thinking about instruction of solving equations for students?" Several responses indicated a focus on language use in the classroom and its impact on student learning. Comments included, "to be careful with wording of answers—consider all possibilities and their perspectives," "I care about words a lot and will think about that more," and "[there is a] difference of having absolutely no solution and having a complex number as a solution."

Conclusions and Implications

Prior to the instructional unit on abstract algebra, participants demonstrated action and/or process-level understanding of inverses. In most cases, participants viewed the inverse as an operation that could be used to undo another operation performed on the unknown variable. This indicated that use of inverse was prompted by cues within the structure of the equation, evidencing action-level understanding. Some participants saw the link between inverses and established properties of equality, demonstrating the algebraic utility of inverses as a problem-solving tool, which showed process-level understanding.

Post-instructional unit, some participants exhibited object-level understanding, while some remained at action or process-level. None of the participants displayed schema-level understanding. Specifically, when solving $x+5=1$ on Z_6, participants attended to inverses with respect to a binary operation, solution elements in the domain of Z_6, and the additive identity. The incorporation of the operational and elemental aspects of inverse along with contemplation of the object created as a result of the operation showed object-level understanding. Participants' knowledge of group structure, an essential component of abstract algebra, caused them to consider inverse elements with respect to a given binary operation, facilitating the transition from action/process-level understanding to object-level understanding. According to Wasserman (2016), this movement from an action/process-level of understanding of inverses towards viewing inverses as objects in a set reflects the APOS framework. This illustration of object-level understanding was indicative of the participants expanding their sense of number and operation. In fact, we claim the participants who were able to transition from the action/process-level of understanding toward an object-level understanding were able to do so because they began to view inverses as elements rather than operations.

We also found that participants were not concerned about the existence of solutions until they themselves were baffled by four solutions of the quadratic equation, $x^2+2x+3=0$, in Z_6. This result is opposed to the Fundamental Theorem of Algebra, according to which a polynomial of degree n should have at most n roots. The contradiction of this mathematical theorem, that participants themselves have used since middle school, challenged them. Later, when Murray put forward the story of Mr. Clark, participants realized the conflict faced by students of secondary mathematics as they accommodated a new, unfamiliar number system. By the end of the study, we found our participants became aware of the importance of maintaining a proper progression of related mathematical concepts for effective teaching and learning practice. This realization about sequencing mathematical concepts is aligned with KCT (Ball et al., 2008). According to KCT, effective teaching is the outcome of the perfect combination of knowledge of teaching and content. Teachers should be able to "choose which examples to start with and which examples to use to take students deeper into the content" (Ball et al., 2008, p. 401).

Even so, there are benefits to ambiguous language in the mathematics classroom for the continued development of concepts (Barwell, 2005). In fact, it may not necessarily be a problem that students are exposed to ambiguities, such as "no solution" versus "no real solution". Rather, it is up to the teacher to confront ambiguities in productive ways that allows students to develop a deeper understanding of a concept. In the context of solutions, this could mean that students themselves can question the idea of no solution and come to understanding that solutions to equations are closely related to the domain over which the equation is being solved. Therefore, while it is not surprising that our participants were questioning the use of language as they themselves experienced disequilibrium around the number of solutions, we claim that their students should have the opportunity to expand upon and alter their understandings of particular words, like solution, as their mathematical knowledge increases.

In this exploratory study, we investigated the impact of exposure to and instruction in abstract algebra on mathematics teachers understanding of secondary mathematics and approach to instruction. Many researchers of mathematics and mathematics education may intuitively understand how secondary mathematics teachers' deep knowledge of mathematics is related to the ability to be an effective mathematics instructor in secondary schools. However, the field still lacks deep understanding of how secondary teachers use their knowledge of tertiary mathematics during instruction. This understanding could lead to a better sense of the kinds of mathematics courses that can provide teachers with the content and pedagogical content knowledge they need to make best use of these connections to improve their own understanding and during classroom instruction.

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Service-learning in a precalculus class: Tutoring improves the course performance of the tutor

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We have introduced an experiment: as part of a Precalculus class, university students have been tutoring algebra prerequisites to students from the community via an academic service-learning program. The goal of the experiment was to improve university students' mastery of basic algebra and to quantitatively describe benefits of service-learning to students' performance in mathematics. At the end of the experiment, we observed 59% decrease of basic algebraic errors between experimental and control sections. The setup and analysis of the study have been informed by existing research on tutor learning and cross-age tutoring, as well as mathematical pedagogy for social justice.

Key words: Precalculus, design experiment, service-learning

Introduction and Research Questions

Academic service-learning consists of two integral components: a useful service to the community, and a meaningful learning opportunity to the students, which is relevant to the material covered in the course (Hadlock, 2013). The type of service is largely determined by the needs of the target community, and can range from a one-time day-long event to a project that spans several months. Service experience is linked directly to an academic course, and is guided by the course instructor, as well as by the supervisors from the community. While providing service, students practice and directly apply skills and knowledge acquired in the course, and bring to the classroom a real-life perspective on the material which they may have until now viewed as purely theoretical. Astin, Vogelsang, Lori, Ikeda, and Yee (2000) found that service-learning showed positive effects on academic performance (GPA, writing and critical thinking skills) and values of participating students.

Academic service-learning has traditionally been associated with social sciences, psychology, counseling, and social work, while service-learning in mathematics courses has recently been gaining prominence (Hadlock, 2005). By engaging in mathematical service-learning, students are meeting the needs of the community by providing mathematical and statistical modeling to local organizations, by offering tutoring in the fields of mathematics and STEM disciplines, or by organizing Math Fests for schoolchildren. Our present study was motivated in part by the need for a quantitative analysis of the benefits of service-learning to students' mathematical performance.

Our second motivation was the ever-present need to improve student success and retention in Calculus in effort to increase attrition in STEM disciplines (Bressoud, Mesa & Rasmussen, 2015). Results from a recent national study show that "a total of 48 percent of bachelor's degree students and 69 percent of associate's degree students who entered STEM fields between 2003 and 2009 had left these fields by spring 2009. Roughly one-half of these leavers switched their major to a non-STEM field, and the rest of them left STEM fields by exiting college before earning a degree or certificate" (U.S. Department of Education, 2013). Edge and Friedberg (1984) show that solid algebra skills are one of the main factors determining success in Calculus. Success of the service-learning project raises students' fluency in algebra and leads to a stronger chance of their mastering Calculus, and staying within their chosen technical field. At the same time, students in the community receive additional mathematical one-on-one instruction, raising their chances of high school graduation. University students, especially athletes, often serve as role models for those who may consider going to university.

Our study explored the following research questions:

Question 1: Will students who engage in tutoring algebra pre-requisites to middle-school and returning students demonstrate fewer 'fundamental' mistakes than students from the control section without the tutoring experience? By 'fundamental' mistakes we mean the following:

1. *Mistakes that result from misunderstanding the addition/subtraction algorithm of*
    a. *numerical fractions*
    b. *rational expressions*
2. *Cancellation mistakes in*
    a. *numerical fraction arithmetic*
    b. *rational expressions*
3. *Mistakes in operations on radicals*
4. *Mistakes in operations with exponents.*
5. *Mistakes in basic factoring using formulas.*
6. *Other – may be added after consultations with other mathematics faculty, or after marking the final exam.*

Question 2: What will be the reaction of students to the service-learning experience introduced in a scientific course that has not traditionally been associated with community work at this and other institutions?

Theoretical Perspective

Our framework for the present study follows a standard pseudo-experimental setup as described by McKnight, Magid, Murphy, and McKnight (2000): baseline performance for experimental and control sections is determined via a diagnostic test; the two sections receive equivalent instruction for the duration of the course, except for the difference in the tutoring service-learning component. The two sections are given identical final exam, and their performance is analyzed via a rubric. Qualitative data is also compared. To our knowledge ours is the first study that quantitatively analyses benefit to mathematical performance of service-learning students engaged in tutoring.

In designing the course and analyzing the results of the experiment, we consider sources in literature on understanding the phenomenon of tutor learning. Our idea to use tutoring as a means to help student-tutors learn mathematics starts with a well-known saying 'I hear and I forget. I see and I remember. I do and I understand'. We rely on work of Allen and Feldman (1976), as well as Gartner, Kohler, and Riessman (1971), who show that cognitively demanding tasks of organizing subject matter knowledge, as well as explaining and questioning in the process of working with a tutee, contribute to tutor knowledge. In designing reflective mathematical journal activities to reinforce tutoring experience, we rely on the work of Roscoe and Chi (2007), who name reflective knowledge-building, "which includes self-monitoring of comprehension", as fundamental to tutor learning.

Inspired by the stated mission of Seattle University to promote justice, our overall study is situated in the context of teaching and learning mathematics for social justice, as described by Gutstein (2006).

When designing and implementing the service-learning structure, we closely follow the suggestions and project design outlined in the Special Issue on Service-Learning in Mathematics, PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate

Studies (2013), particularly Schulteis' (2013) experience of building a course with university students' satisfying the tutoring needs of local institutions and non-profit organizations.

Methodology

Our work took place at Seattle University: a medium-sized urban Catholic university in the heart of Seattle, WA, with a long tradition of incorporating service-learning and community work into students' coursework and extra-curricular education. University's Center for Community Engagement runs Seattle University Youth Initiative which has provided tutoring to schools in close proximity to the University. In years 2014-2015, 112 members of Seattle University faculty implemented academic service-learning in their courses. Such high participation rate is largely due to the Center for Community Engagement that carries most of the administrative work in organizing student placements and in establishing effective communication with community partners.

Mathematics Department at Seattle University has traditionally been involved in outreach activities. Annually, several sections of a standard Mathematics Core course for students majoring in humanities involve a social justice service-learning component with university students tutoring math to elementary school students. The course was first developed and implemented by Allison Henrich, who demonstrated a decrease in math anxiety in participating university students (Henrich & Lee, 2011). Seattle University mathematics professor Leanne Robertson runs Seattle University Math Corps: a highly successful mathematics outreach program that employs mathematics majors and provides tutoring to local elementary school students. No STEM-related courses at the Mathematics Department have yet involved an academic service learning component as part of mathematics course curriculum except for the Precalculus course described below.

Experimental Design

The setting for our study was two sections of a standard Precalculus course that served as a pre-requisite for the science and engineering track Calculus sequence. The course focused on algebra material and served as a mathematics refresher for students whose ACT and SAT scores would not allow them to be placed directly into a Calculus I course.

Topics covered in the course included quadratic, radical, and rational equations and inequalities, functions, inverses, and graphing, roots of higher-degree polynomials, exponential and logarithmic functions, as well as applications of all the above topics. Course audience in both sections consisted of first- and second-year university students of traditional age, majoring in a variety of fields: engineering, nursing, biology, business. Students' prior mathematical background varied: some students may have difficulty with basic exponents, while others are comfortable solving standard radical and rational equations.

Both experimental and control sections numbered 21 students each. The experimental section of the Precalculus course involved a service-learning component, while the other section served as control and consisted of standard in-class instruction only.

All students in the experimental section took part in the SU Service Learning program. Over the course of the quarter, they spent 2-3 hours per week tutoring basic algebra and sometimes arithmetic to middle school and high-school students, as well as adults returning to obtain their GED. The students put in a total 18-21 hours of tutoring work over the duration of the quarter. Service-learning component constituted 10% of students' grade, while homework, web-work, tests, and in-class participation made up the remaining 90%. On the first day of class, students in the experimental section received an overview of the service experience as community partners introduced their respective organizations and described the target populations that used their services. Representatives from Seattle

University Center for Community Engagement distributed necessary paper-work, explained the process and the timeline of contacting community partners, and conducted an interactive presentation on aspects of cultural competency that could be relevant in work with diverse populations.

During the first week of the quarter, students in the experimental section contacted the community partners of their choice, filled out various clearance documents, received training, and started tutoring. In the meantime, mathematical part of the course started as it usually would, with lectures, homework, and class-work. Tutoring continued through the quarter, and ended on the last week of the course, to give students time to prepare for the final exams.

The control section was identical to the service-learning section in every aspect of the course syllabus, such as the topics covered, the number of exams, attendance and make-up policy, etc. The only difference was the lack of the tutoring component in the control section, as well as a slight difference in grading weights assigned to exams and homework. The control section also received some amount of extra homework intended to balance the additional workload faced by experimental section.

Diagnostic test

We established a baseline of the students' prior knowledge and preparation by using a diagnostic pre-test. The pre-test was given to both sections on the first day of class and covered the pre-requisite material including arithmetic with fractions and radicals, basic factoring, and solving basic equations. In order to make sure the task was taken seriously, the students received a small amount of credit for completing the pre-test.

Written Reflection

In order to fully benefit from the service-learning experience, students must have an opportunity to engage in structured reflection and connect for themselves the tutoring experience with the content of the course (Bringle & Hatcher, 1999). An integral part of the experimental service-learning section was a weekly guided reflective diary of tutoring experiences, helping students analyze mathematical, as well as social and pedagogical, aspects of their work with the students from the community.

Each journal entry included a mathematical and a non-mathematical component. Mathematical reflection helped the students engage with and analyze the mathematical aspect of the tutoring experience and reflect on the following and similar questions:

- What is the topic that your student is studying? With which component did the student need help?
- What problem did you discuss with your student? What mathematical concept did you address?
- What piece of knowledge was missing from the students' understanding that prevented them from moving forward?
- What method did you use to approach the solution and how did you explain the material?
- Can you predict which future mathematical topics will rely on the problem you discussed with the student?
- Did you discover any gaps in your own mathematical knowledge? What steps did you take to address them?
- Did you discover any parallels between the topics you tutored and our mathematical lectures and problems from class?

Mathematical part of the journal became a place for conceptual analysis of mathematics discussed in the tutoring sessions, as well as an aid in reflective teaching.

The free-form non-mathematical reflection was intended to help the students process the human aspect of their experience with service-learning and tutoring. The following guiding questions were suggested to the students: "You may think about your pedagogical observation of your student. What do you think is holding this student back and what can be done to help the student succeed? How was your tutoring week? What non-mathematical problems did you encounter? Any thoughts on what you are seeing and experiencing while tutoring? Any questions you would like to ask me, or your fellow students, or the management of the organizations where you tutor?"

The non-mathematical part of reflection was optional, reserved for the time when students wanted to share a personal experience, however, most of the time students chose to write down their observations of service sites and of people they met there. Service-learning journal served as an effective communication tool between students and instructor, and readily alerted the latter as to potential problems or issues at the service sites.

In-class reflection

In addition to completing reflective journal entries, the experimental section held two in-class reflection meetings. Students split in groups according to their respective tutoring sites, which gave them opportunity to meet others who worked at the same location, exchange information, and discuss common issues, such as transportation, time commitment, and communication with the community partners.

In-class reflection meetings offered the participants an opportunity to discuss the pedagogical issues raised by the students themselves, through the journals or in class: How do you motivate an unmotivated student? How do you explain a concept of a variable? Of an equation? How do you guide a student through word problems? How do you deal with discipline problems?

Everyone was asked to contribute, and students from different tutoring groups offered valuable pedagogical suggestions, arriving on their own at well-known mathematics education techniques: the need to build on the basics (arithmetic) before proceeding to algebra, to move from concrete to abstract in explanations of material, to ensure that the given and the unknown were clearly identified before a word problem can be solved.

> What I have noticed most after tutoring at Seattle Central is that if you truly want to succeed you need to master the basics if you even want to excel at the next level of math. Barely grasping the concept is not enough to get through that current level of math, never mind the next. (post-meeting journal entry)

Reflection meetings fostered a sense of a community in the classroom, brought out a shared experience, and helped create a common bond among the students.

Anonymous end-of-term reflection

At the end of the course, the students submitted a typed anonymous reflection where they were free to comment on any aspect of their experience, to offer suggestions for improvement, and to voice any additional concerns regarding the course. Results of anonymous reflection are discussed in the Results section below.

End-of-term examination.

At the end of the quarter, both sections took a standard final exam with identical questions. Relevant statistics were computed and compared for both sections. The number of fundamental mistakes (see Introduction) was determined via a special rubric, which we will describe in the Results section along with the data obtained from the exams.

Results

Our diagnostic pre-test indicated that the experimental and control sections were comparable in preparation and abilities and showed similar score distributions, with the experimental section showing a slightly better average, but the difference between the two sections not being statistically significant.

Research question 1 was answered affirmatively. We compared the number of fundamental mistakes in the final exams for both sections: there were only 13 fundamental mistakes made by the 21 students of the experimental section, while the 21 students of the control section made 32 fundamental mistakes. Thus, there were 59% fewer fundamental mistakes in the experimental section than in the control one.

The following error types were analyzed and recorded via a grading rubric. In the list below, the shorthand Error Code is followed by an explanation of the mathematical error.

Arithmetic:

- R1: Subtracting polynomials without use of parentheses
- R2: leave square roots unsimplified
- S1: signs remain unchanged when a term is moved from one side of equation to another
- P1: failing to distribute multiplicative term to all terms in parentheses
- C1: mistakes in longer arithmetic calculations performed without a calculator.
- R3: miscellaneous errors in calculating square roots of real numbers.
- E1: failing to apply laws of exponents correctly

Algebra:

- A1: distributing exponents over sum or difference; ex: square of sum equals sum of squares
- A2: miscellaneous errors in algebra of radicals.
- A3: splitting radicals over sum/difference
- A4: reluctance to use formulas: ex: foiling every time instead of applying formula: $(a+b)^2 = (a+b)(a+b) = a^2 + ab + ba + b^2$
- A5: fundamental misunderstanding of formulas and reasoning behind them, leading to algebraic mistakes: ex: in applying square of difference $(a-b)^2 = a^2 - 2ab + b^2$ to example $(a-3)^2$, b is assumed to be (-3), not 3. Applying above formula leads to result $(a-3)^2 = a^2 + 6a + 9$

The following table summarizes final exam errors from experimental and control sections:

Error Code	Error Type	Experimental Section	Control Section
Arithmetic			
R1	If y=a+b then x-y=x- a + b	3	4
R2	leave $\sqrt{\frac{125}{4}}$ unsimplified		1
S1	x= - 2 equivalent to x-2=0		2
P1	4(a-b)=4a-b	1	4
C1	Mistakes in longer arithmetic computations w/o calculator	5	7
R3	R^2=d^2/2		1
E1	3^2=8		1
Algebra			
A1	$(a+b)^2$=a^2+b^2	1	6
A2	Root misc	3	3
A3	$\sqrt{a+b}=\sqrt{a}+\sqrt{b}$		1
A4	Foil without use of formula: $(a+b)^2=(a+b)(a+b)=a^2+ab+ba+b^2$		1
A5	$(a-b)^2=a^2+2ab+b^2$		1
Totals:		**13**	**32**

The course average for the experimental section was higher, due to the difference in the weights given to individual course components.

Data from the reflective diaries and the end-of-term anonymous reflection indicate that the answer to the second research question was also positive: out of 21 submitted anonymous reflections, 20 were positive, reflecting a sense of accomplishment and a clear understanding of the privilege of university education, as well as the appreciation of new friendships.

> I learned how to better explain everything and how important it is to verbalize the why and not just the how. I also think learning how to deal with children, and learn patience is essential in growing up. (end-of-term anonymous student-tutor reflection)

Similarly to Butler (2013), we observed a number of service-learning benefits that went far beyond the original goal of the project, including an increased level of confidence in oral communication skills mentioned by the international students. After the quarter ended, several students voluntarily continued their work with the community partners.

In their diaries, the students frequently pointed out multiple connections between the mathematical concepts covered in class and the topics they had explained in the tutoring sessions.

> Another mathematical challenge he is facing is remembering to flip the sign of the inequality when you multiply or divide by a negative number. This draws incredible parallels to what we are studying in class. Just today we learned the exact same thing while working with inequalities except that we use non-linear equations and my student only uses linear equations. Sometimes I have to be careful to remember this step as well so it is good practice for both of us. (weekly student-tutor journal entry)

> …The topic of focus for the week was helping Alfred understand the variables of a linear equation. We covered what "y = mx + b" represented (a linear graph) and discussed how to find the slope and y-intercept using just two coordinate points. This was actually coincidentally perfect, because we have been learning about the coordinate plane in our pre-calculus class. Although I was already very familiar with linear equations and the coordinate plane, analyzing the comparisons and contrasts between how I had explained it to Alfred versus how professor explained it to the class helped me to gain perspective. In listening to your explanations and taking notes, I was able to learn better ways that I could explain the linear graph to help Alfred understand it better. So, I started out by presenting him with the definition that "a graph is a drawing that represents all of the solutions of a linear equation."… Since Alfred already knew the basics of drawing a graph, we used his graphing abilities to understand how the slop (m) of the equation corresponded to the rise over run of the graph and how the y-intercept represented the "b" variable. (weekly student-tutor journal entry)

Many students rediscovered for themselves that the underlying concepts and definitions were in fact the same for the polynomial graphs and the radical equations covered in class, and the basic linear graphs and equations their tutees had studied in middle school. As Roscoe and Chi (2007) point out, peer tutors manifest highest levels of tutor learning as a result of explaining conceptual rather than process-based questions to the tutees. In our case, reflection diaries worked as a tool to reinforce mathematical knowledge gains made by the tutor as a result of the tutoring session.

In the non-mathematical section of their journals, the students often spoke of various life circumstances faced by their tutees and were able to directly observe complex interplay of social position and education and their often cyclical effect on each other.

> One thing that is clearly holding my student back in math is a language barrier. Since English is not his first language, it is difficult for him to read and interpret the word problems in his math workbook. It is also sometimes challenging for him to read explanations of how to do problems in the book. This makes the process of learning math more laborious and time consuming. On a positive note though, my student seems very determined to learn math so that he can pass his GED test and hopefully go to college later. He is very optimistic and likes to practice math on his own outside his tutoring sessions. I am excited to help him as much as I can next week. (weekly student-tutor journal entry)

Another Precalculus student watched her adult pupil miss more and more tutoring sessions and classes due to family problems, putting the GED further out of reach.

> Tutoring at Seattle Central Community College is really eye opening. It's really made me think of my blessings and truly be grateful for them…Recently one of the people I tutor (let's call her Hana) has dropped out of school. This was a woman that I admired and looked up to from the moment I met her. I took a while to really put my thoughts together on how to go about to write about this because when she told me of her decision over the

> phone, I was surprised by how big the blow was to me. She was a 57 year old woman aiming to be a nurse. She showed me how determined a person can be even when the odds and/or circumstances are against them. She had dropped out due to medical reasons which really upset me due to how well she was picking up on her material in class. (weekly student-tutor journal entry)

As instructor, it was a privilege to be part of student experience and to hear their thoughts and impressions of what they saw and experienced. Running a service-learning course can be an extremely rewarding experience for all involved, however, instructor who wishes to implement such an experiment, should be aware of potentially high time commitment that is involved. Seattle University faculty is deeply grateful for the work of the Center for Community Engagement that establishes contacts with community partners, finds placements for students, and largely takes care of the administrative overhead involved.

Ongoing project work

At the time of writing, academic service-learning experiment in pre-calculus course is ongoing, and we are still collecting and analyzing data.

Further projects may include:

- Introducing optional service-learning component into a regular pre-calculus course, with student given an option to participate. Instructor will introduce advantages of service-learning to mathematical knowledge. Course grade for participating students will be computed using a different grading scheme, giving students an option to earn a higher mark.
- Repeating service-learning experiment in pre-calculus, with university students tutoring elementary school kids. Difference in mathematical gains will be compared with the present results.
- Repeating Allison Henrich' service-learning experiment in the mathematics course for students of humanities (Henrich & Lee, 2011) and measuring mathematical gains of students involved.
- Measuring potential shift in attitudes towards mathematics in all participating and non-participating university students.

Conclusions and Implications for Mathematics Education

Our research statistically establishes a number of tangible benefits of service-learning to students' mathematical performance in class. The non-mathematical benefits have been widely explored, and they are confirmed by our study. Our research opens venues to further exploration of the long-term academic and non-academic benefits of service-learning to the university students, as well as to students from the community. Service-learning requires commitment of time and sometimes additional funding: our findings may encourage Mathematics departments and university administration to promote service-learning in mathematics courses.

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Eliciting mathematicians' pedagogical reasoning

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Given the prevalence of work in the RUME community to examine student thinking and develop instructional materials based on this research, we argue it is important to document the ways in which undergraduate mathematics instructors make sense of this research to inform their own teaching. We draw on Horn's notion of pedagogical reasoning in order to analyze video recorded conversations of over twenty mathematicians who elected to attend a workshop on inquiry-oriented instruction at a large national mathematics conference. In this context, we examine the questions: (1) How do undergraduate mathematics instructors engage in efforts to make sense of inquiry-oriented instruction? (2) How does variation in facilitation relate to instructors' reasoning about these issues? Preliminary findings suggest that differences in facilitation relate to how participants engaged in the mathematics, and that the nature of participants' engagement with the mathematics was related to their subsequent pedagogical reasoning.

Key words: mathematicians, pedagogical reasoning, inquiry-oriented instruction

Given the prevalence of work in the RUME community to examine student thinking and develop instructional materials based on this research (e.g. Wawro et. al., 2013; Larsen, Johnson & Bartlo, 2013; Rasmussen & Kwon, 2007), we argue it is essential that our community consider issues related to the dissemination and use of findings from our research. This work aims to serve that goal by examining our efforts to engage practitioners (instructors of undergraduate mathematics) in thinking about research-based, inquiry-oriented instructional materials for undergraduate mathematics courses. In this preliminary report, we begin to explore two research questions: (1) How do instructors of undergraduate mathematics (who are interested in inquiry-oriented instruction) reason about instructional issues, particularly in the context of inquiry-oriented mathematics instruction? (2) How does variation in facilitation relate to the ways in which instructors engage in reasoning about these instructional issues?

Theoretical Framing and Literature

National organizations have called for instructional change in undergraduate STEM courses, relating poor instructional quality to a lack of student interest and persistence (e.g., Fairweather, 2008; PCAST, 2012; Rasmussen & Ellis, 2013). Researchers from a range of STEM fields have developed and documented student-centered instructional approaches that result in greater conceptual learning gains and student attitudes when compared with classes in which lecture is the dominant form of instruction (e.g. Kogan & Laursen, 2013; Kwon, Rasmussen, & Allen, 2005; Larsen, Johnson, & Bartlo, 2013). While it is well-documented that instructional change is difficult to achieve at scale (Henderson et. al., 2011), there is evidence that suggests a sizable number of faculty in undergraduate STEM fields are making efforts to offer their students the kinds of student-centered learning experiences supported by these studies. Indeed, though 61% of STEM faculty report they use extensive lecturing when they teach, a full 49% of STEM faculty report they incorporate cooperative learning into their courses (Hurtado et. al., 2012). Given the rates at which STEM faculty now report use of cooperative learning in their instruction, we argue that there is a pressing need to document and leverage the pedagogical reasoning of those faculty who are working to implement these kinds of instructional approaches.

In this work, we follow Rasmussen & Kwon's (2007) characterization of inquiry-oriented instruction in which students are actively inquiring into the mathematics (e.g. by developing, justifying, and generalizing their own solution methods to open ended problems) and instructors are actively inquiring into students' thinking about the mathematics so as to build on students informal and intuitive ideas to help them make sense of and engage in more formal and conventional forms of mathematical reasoning.

We take a situated perspective, in which we view knowledge and learning to be evidenced in the interactions among members of a community (Lave & Wenger, 1991) – in this case, the community of instructors of undergraduate mathematics. As such, we look to document mathematicians' pedagogical reasoning by examining their conversations about instruction. We follow Horn's (2007) characterization of pedagogical reasoning, considering it to be instructors' reasoning about issues or questions about teaching "that are accompanied by some elaboration of reasons, explanations, or justifications" (p. 46). Analytically, we draw on the vertices of the instructional triangle (teaching, students, and mathematics) as a conceptual tool for organizing our analysis of these conversations about instructional issues.

Data Sources and Methods of Analysis

The data under consideration in this study were collected from a workshop conducted as part of a national mathematics conference, and these data are part of a broader project that is developing and analyzing a set of instructional supports for undergraduate mathematics instructors interested in implementing inquiry-oriented instruction. The workshop focused on implementing inquiry-oriented instruction, and was organized around research-based curricula that have been developed in the areas of linear algebra, abstract algebra, and differential equations. The workshop lasted a total of four hours, which was split across two 2-hour sessions on consecutive days. On each day, about half of the time was devoted to content-specific work in breakout groups (self-selected by the participants), and the other half of the time was spent discussing issues of inquiry-oriented instruction that cut across all three curricula. On Day 1, facilitators planned to engage participants with an overview of inquiry-oriented instruction, followed by time to engage in mathematical tasks from the curricula in the area of their selected breakout group. On Day 2, the focus was to be on student thinking related to the day 1 tasks and instructional moves designed to help instructors implement inquiry-oriented curricula.

The workshop included 25 participants, 21 of which responded to a workshop pre-survey that provided us with information about their background and home institutions. All participants except one were housed in Mathematics departments, and the group represented a diverse group of institutions and positions (see Figure 1). Less than a third of survey respondents reported that they prefer to lecture most of the time, and more than 70% reported that they like to have students work in groups on problems in class, and more than 60% report they frequently ask students to explain their thinking to the whole class when they teach. This was significant to our research because it suggests our sample is part of the sizable subset of undergraduate STEM faculty working to teach in student-centered ways, and the choice to attend the workshop also points to an interest, outside the RUME community, in research-based instructional approaches.

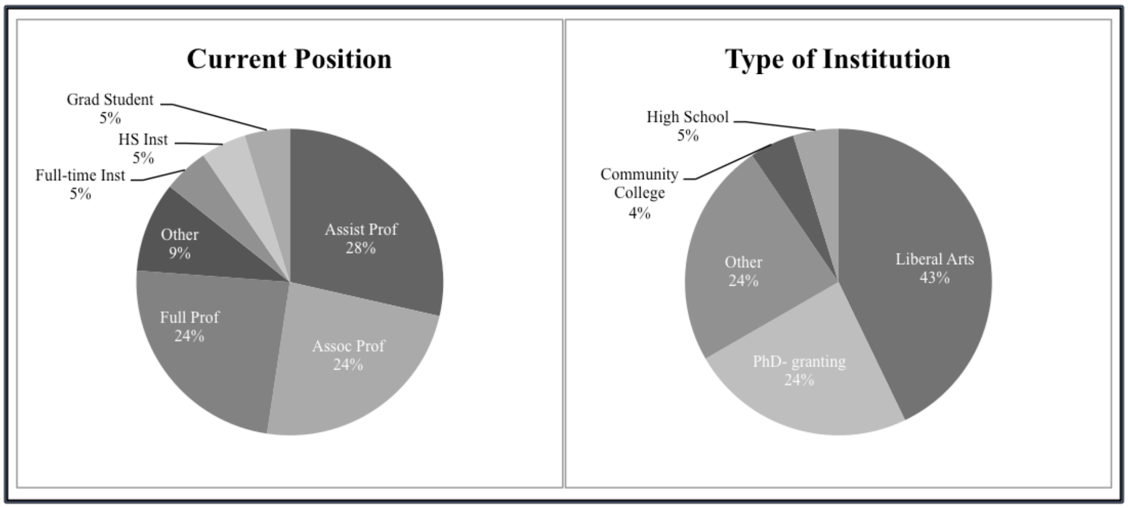

Figure 1. Position and Institution Types of Survey Respondents

In each of the two 2-hour workshop sessions, all whole-group and breakout segments were video- and/or audio-recorded for subsequent analysis. We began analyzing this recorded data by generating content logs to document the sequence of events in each segment of the workshop. Each content log was organized in a table with four columns: timestamp, description of events, focus of talk, and other comments. The 'focus of talk' column aimed to help us track whether the focus of talk was on the mathematics (M), the teacher (T), or students (S), and whether it was the facilitator or participants who were doing that talking. From these content logs, we generated summaries of each session to describe the focus and use of the time along with initial characterizations of the participants' pedagogical reasoning. We noted stark differences in the conversations of the linear algebra breakout group as compared to those in the abstract algebra group, so we decided to conduct our analysis as a comparative case study of these two groups (Yin, 2003). Content logs were then used to generate codes for the focus of participants' talk during the session and to identify conversational moments (selectively transcribed for closer analysis) when participants were engaged in pedagogical reasoning.

Findings

Our preliminary findings are two-fold: First, differences in facilitation appear to have played a role in how participants engaged in the mathematics on the first day of the workshop. Second, the nature of participants' engagement with the mathematics was related to their subsequent pedagogical reasoning. In this preliminary report, we focus primarily on the different ways in which participants in two of the breakout groups engaged in the mathematics on the first day of the workshop, and the variation in facilitation that may help explain those differences in engagement. In our presentation, we will provide further elaboration on differences in subsequent pedagogical reasoning of the two groups.

The facilitators of both the linear algebra and the abstract algebra group intended for the entire hour of the first day's breakout group to be focused on working through the mathematics in the respective task sequences. However, our content logs revealed the Abstract Algebra group spent a much larger portion of their hour-long breakout session on the first day of the workshop working through the math (86% of the time spent working through the math) than did the Linear Algebra group (30% of the time spent working through the math). Additionally, the nature of the mathematical talk of the two groups differed in that the abstract algebra group appeared to engage in the mathematics much more deeply than did the linear algebra group. We coded talk into six categories: logistics (e.g. "Does everyone have a handout?"), introductions, implementation questions (e.g. "How many times a week does your class meet?"), pedagogical moves (e.g. "I have them present their work as soon as we finish a task"), discussing mathematics (e.g. "They [students] came to different conclusions based on whether or not they considered all linear combinations"), and doing mathematics (e.g. "We want to show the additive inverse we'd expect from the big group stays in the small group."). Table 2 summarizes each breakout group's conversational focus on day 1 according to these categories.

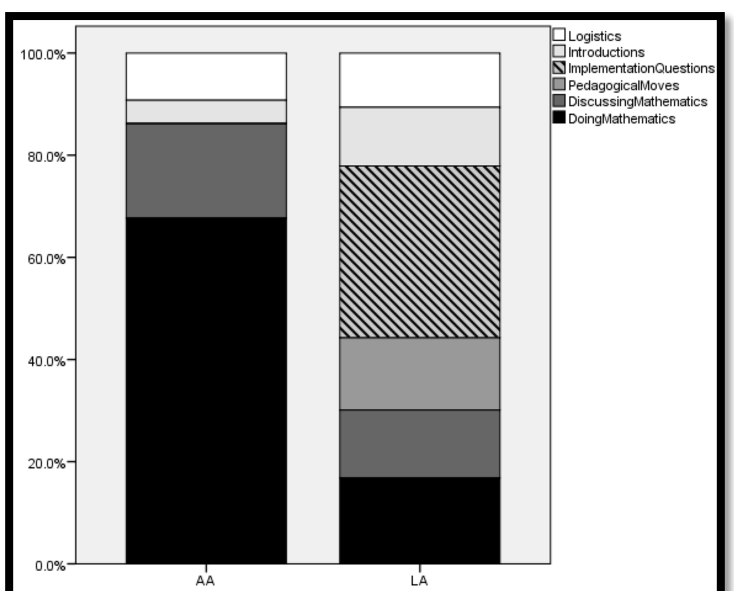

Figure 2: Conversational focus during Day 1 Breakout Groups

It was initially unclear why such differences in participants' engagement in the mathematics were observed. Analysis of the facilitation across the two breakout groups points to several factors that help explain these differences.

At the outset of the Abstract Algebra breakout session, the facilitator handed out the task statement and asked participants to turn and work together in small groups. As participants begin working, she circulated amongst them, listened, and occasionally chimed in with questions about their thought processes, methods, or mathematical assumptions. One participant later described the work of this breakout group, saying that the facilitator "modeled some instructor behaviors… pressing us on why we were thinking how we were thinking, pressing toward the subtlety without leading. We realized that because we understand abstract algebra, there are a bunch of similar ways to work with the definitions that are not obviously equivalent to a new user."

In the Linear Algebra session, the facilitator (who is one of the authors) also distributed the mathematical tasks for participants to work on, but devoted several minutes at the beginning to introductions in an effort to help create a safe, collaborative environment for sharing. However, this opportunity for participants to contextualize their institution (size and type of college, which students take linear algebra, etc.) also seemed to encourage speakers to bring up topics not directly related to the mathematics at hand (airing concerns about content coverage, logistics, student prerequisites, etc.). Additionally, there were moments in which the facilitator -- endeavoring to strike a balance between math instructor and professional development leader, and to respect the assumed content knowledge of her group members – inadvertently set norms that discouraged participants from fully engaging with the mathematics. Specifically, the facilitator began small group work by asking participants to begin by working on the second task in a sequence of related tasks, rather than starting with the first task, positioning the initial mathematical work as "easy." We posit that this created a setting with implicit professional risk: the participants were more or less strangers to one another, and several expressed feeling out of practice with the subject, so this potentially increased the pressure on those already feeling vulnerable. Later, in an attempt to refocus the group on the math, the facilitator suggested working through a task from the vantage point of a "typical student." Interestingly, instead of easing the mathematical pressure, this also created a barrier, with at least two participants later remarking that they didn't know how their students might approach the task. Finally, one of the participants had previously used the materials and was introduced as a potential resource for other instructors. Many of the digressions from the mathematics in the Linear Algebra group came in the form of specific questions directed to this participant.

Initial analysis suggests that this set-up up participants' mathematical engagement on the first day of the workshop was consequential for participants' subsequent pedagogical reasoning; in our presentation we will provide more detail on the differences in this subsequent reasoning.

Questions for Audience

- Might different content lend itself to participants engaging differently (e.g. linear algebra vs. abstract algebra)? If so, how can we account for this?
- Can we define or operationalize the kinds of tasks or activities that are productive for advancing participants' pedagogical reasoning?
- In what ways is it (or might it be) important for the facilitator to be inquiry-oriented in their stance toward participants' pedagogical reasoning (during workshop facilitation)?

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Active Learning in Undergraduate Precalculus and Single-variable Calculus

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The study presented here examines the active learning strategies currently in place in the Precalculus through single variable calculus sequence. While many lament the lack of active learning in undergraduate mathematics, our work reveals the reality behind that feeling. Results from a national survey of mathematics departments allow us to report the proportion of courses in the mainstream sequence utilizing active learning strategies, what those strategies are, and how those strategies are being implemented.

Key words: Census Survey, Precalculus, Calculus, Active Learning, Instructional Methods

Research suggests experiences in introductory mathematics courses can significantly influence student persistence within the STEM fields (Bressoud, Mesa, & Rasmussen, 2015; PCAST, 2012; Seymour & Hewitt, 1997; Wake, 2011). In the United States, the Precalculus to Calculus 2 (P2C2) sequence often serve as key prerequisite courses for students intending to pursue degrees in STEM fields, and difficulties in these courses often prevent students from continuing on in STEM. One approach that shows great promise for improving student success in the P2C2 sequence is the use of student-centered instruction. Recent studies have highlighted the educational benefits of using active learning strategies in post-secondary mathematics classrooms, including improved STEM retention rates (Ellis, Kelton, & Rasmussen, 2014; Ellis, Rasmussen, & Duncan, 2013; Rasmussen & Kwon, 2007; Seymour, & Hewitt, 1997) and narrowing achievement gaps (Kogan & Laursen, 2014; Laursen, Hassi, Kogan, & Weston, 2014). The use of such practices was also identified as a characteristic of programs with successful Calculus I programs (for a more detailed discussion see Bressoud & Rasmussen, 2015). Despite these educational benefits, Kuh (2008) points out that these instructional practices are currently not the norm in higher education.

The study presented here reports preliminary findings from data collected as part of a larger study, *Progress through Calculus* (PtC). The main research question addressed in this report is *how, and to what extent, are active learning strategies being implemented in the P2C2 sequence?* Narrowing down on this broad question, this presentation addresses the following questions:

1. How prevalent are active learning strategies in the P2C2 sequence?
2. What particular active learning strategies are being used, and how common are they?
3. What (if any) institutional factors relate to the use of active learning strategies?

Methods

The data reported here comes from a census survey undertaken as part of a larger, multi-phase project studying departmental and institutional factors that influence student success through the P2C2 sequence. The survey was administered to the 341 departments across the country that award graduate degrees in mathematics. The survey closed with an impressive overall response rate of 68%. The survey elicited information about many aspects of each department's implementation of the P2C2 sequence as a whole, informed by the results of the CSPCC study (Bressoud, Mesa, & Rasmussen, 2015; Bressoud & Rasmussen, 2015). This presentation reports on a subset of the data wherein participants provided detailed information about the individual courses that make up the mainstream P2C2 sequence.

The details requested about these mainstream courses included questions about course delivery to ascertain the primary format for regular class meetings and recitation sections (when applicable). 201 institutions completed these details, giving us information on 904

P2C2 courses. The following section includes descriptive statistics (frequencies, proportions, etc.) gathered from these responses. Ongoing analysis is examining the relationships between the use of active learning strategies and other factors (e.g., importance of active learning, DFW rates, instructor type).

Results

To frame our findings, we begin by noting that 44% of all mathematics departments responding to this survey reported that active learning strategies are "very important" for successful P2C2 courses, but *75% of those* reported that they are not very successful at implementing those strategies. Overall, 14% of institutions report being "very successful" with active learning strategies, 60% report being "somewhat successful," with the rest marked "not successful" or "not applicable."

The first question we answer relates to the usage of active learning strategies in introductory undergraduate mathematics courses across the country. Participants were asked to identify the primary instructional format for regular class meetings of each course in their department's mainstream P2C2 sequence. The proportions that follow in this section refer to the proportion of *courses* taught in a certain way, not the proportion of institutions offering such structures. Perhaps unsurprisingly, over 60% of courses are primarily taught in a lecture format. Of particular interest to us is the discovery that approximately 16% of courses incorporate some active learning techniques, while a further 3% are taught using mainly active learning techniques. Further investigation revealed that these proportions fall off through out the P2C2 sequence. While nearly 26% of precalculus courses incorporated at least some active learning techniques, this drops to 20% in first calculus courses, and drops again to 13% in secondary calculus. These proportions did not vary significantly between MA- and PhD-granting institutions.

Data was also collected regarding the instructional format of recitation sections (or labs) when applicable. Approximately one-third of the reported courses have recitation sections, and 15% *of these* use active learning strategies (5% of all courses). Overall, we found that 22% of reported courses have some active learning in the instructional approach in regular course meetings, recitation sections, or both.

Our data also allows us to look at what "active learning strategies" mean when usage is reported. Note that the responses to this question were not exclusive, as a course might utilize several different strategies. The most prevalent technique was group work, reported in 78% of courses and 92% of recitations that included active learning. IBL, clicker surveys, and flipped classes were each reported in 15-20% of lectures, and a sizable proportion indicated that they use something other than the options provided. The story is similar in recitation sections, except for the use of clicker surveys (only appeared in 2% of "active" recitations). The patterns of usage do not change across courses in the P2C2 sequence.

Conclusion & Future Directions

Our current quantitative analysis reveals some of the patterns of active learning in the P2C2 sequence. Only 22% of P2C2 courses incorporate at least one active learning component, but we also found that 40% of institutions have an active learning component in at least one mainstream P2C2 course. This suggests that active learning strategies are not being used consistently through the P2C2 sequence, but only in select courses. However, in courses that implement active learning strategies, we see a consistent trend in *the types* of active learning strategies being implemented across the P2C2 sequence. At this time, analysis of our existing data is still ongoing. Further analysis of this data will link active learning strategies and implementation to institutional factors (e.g., understanding of importance, school size) as well as student factors (e.g., DFW rates, target audience), thus providing a more complete picture of the role active learning plays in the P2C2 sequence.

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Talking about teaching: Social networks of instructors of undergraduate mathematics

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The RUME community has focused on students' understandings of and experiences with mathematics. This project sheds light on another part of the higher education system – the departmental culture surrounding undergraduate mathematics instruction. This paper reports on the interactions of members of a single mathematics department, centered on their conversations about undergraduate mathematics instruction. Social network analysis of this group sheds important light on the informal structure of the department.

Keywords: *Social networks, instructors, organizational culture, community*

It is widely known that experiences with introductory undergraduate mathematics courses are a significant factor affecting retention rates in STEM majors (Bressoud, Mesa, & Rasmussen, 2015; PCAST, 2012; Seymour & Hewitt, 1997). This has led to increased research and attention to these introductory courses. Very little of that research, however, uses a systems-level approach. In thinking about undergraduate mathematics education, we must consider the entire system at work and the cultures and communities at play at each level. Students and instructors function as individuals embedded in a variety of cultures and communities, each with their own pressures, values, beliefs, assumptions, and practices.

Focusing on the department as a unit of analysis makes particularly good sense when considering introductory mathematics courses. Many institutions offer multiple sections of courses such as Calculus I each term, taught by a range of instructors. The potential variation in experiences at a single institution is remarkable, and so case studies of individual classrooms do not capture the entire picture. This position is supported by the findings of the *Characteristics of Successful Programs in College Calculus* (CSPCC) study, wherein a coordination system was found to be one of the seven key features of successful programs (Bressoud & Rasmussen, 2015). Another reason to take a department-level approach is the potential of the department as a unit of change (e.g., Gibbs, Knapper, & Piccinin, 2008; Wieman, Perkins, & Gilbert, 2010). Work in education and organization science has shown that change is a social construct, best effected and sustained by a group rather than an individual (Corbo et. al., 2015; Daly, 2010).

Methods

Social network surveys were distributed to 61 individuals in the mathematics department at a large research university, one that was identified in the CSPCC study as being relatively more successful at implementing Calculus I. Network questions were used to ascertain the ties that exist between members of the community of calculus instructors, as well as the strength of those ties, and a variety of Likert scale and demographic questions were used to characterize the actors between whom ties do or do not exist (Coburn & Russell, 2008). Five relational networks were measured: advice about teaching (R1); sharing of instructional materials (R2); discussions about teaching (R3); friendship (R4); and influence on instruction (R5). The survey also included Likert scales designed to characterize the individuals, subgroups, and the larger community in terms of trust, innovative climate, professional learning community collaboration and involvement, as well as mathematical affect and beliefs.

Findings

Looking at the different networks, I note differing levels of inclusivity, from a high of 85% included (R3) to a low of 52% (R5). I further note the split, in terms of inclusivity, of the networks into R1, R2, and R5 vs. R3 and R4. This indicates that more actors are involved in discussions about instruction and friendship within the department than the sharing of advice, instructional materials, or influence. One possible interpretation of this is that R3 and R4 are more general relations than the others. Another is that R1, R2, and R5 all seem to involve acknowledging another as "expert" at something, while R3 and R4 may be relations between equals.

Instructors of the Precalculus through Calculus 2 (P2C2) courses are disproportionately active in the networks, especially in R1, R2, and R5. This is gauged by looking at the makeup of the main component of each relationship graph (in each case the only component) and how many of each instructor type are included in that component (Table 1). In R1, R2, and R5, P2C2 instructors account for significantly more of the graph component than their overall representation. In R3 and R4, the distribution of P2C2 and non-P2C2 is close to their overall distribution (within 3 people). The coordination of superficial aspects of P2C2 course structure (e.g., textbook, exams) seems to explain the over-representation of P2C2 instructors in the materials network (R2), but it does not directly explain their over-representation in advice (R1) and influence (R5). These network results seem to indicate that there is more to this coordination system than simply shared course elements.

Table 1: Components of relational networks, including P2C2 instructor breakdown.

Relation	Component (V, E)	Proportion of component that is P2C2 instructors	P2C2 instructors in component (n=23)	Non-P2C2 instructors in component (n=38)
R1	(38, 83)	0.500	0.826	0.500
R2	(36, 65)	0.528	0.826	0.447
R3	(52, 120)	0.385	0.870	0.842
R4	(51, 138)	0.431	0.957	0.763
R5	(32, 55)	0.500	0.696	0.421

Given the network investigations under investigation, it is natural to look for individual actors who are the "most" at something: Who asks for advice the most? Who is asked for advice the most? Who is the most influential? When looking for standout actors, we turn to their degree, the number of ties attached to their node. By asking about *in-degree, out-degree*, and *total degree*, we can begin build a rough picture of important actors. For sake of brevity, this proposal attends only to the advice network (R1) while the presentation will attend to all five. Total degree had mean 2.7 and standard deviation 4.7; in-degree had mean 1.4 and s.d. 3.7, and out-degree had mean 1.4 and s.d. 1.9. There is more variation in actors' out-degrees than in-degrees, which implies that while actors in the network seek different amounts of advice, they seek that advice from a select few. There is a clean break in the in-degree distribution separating three actors from the rest by more than two standard deviations.

Discussion

Since the data collected represents a snapshot of the department in its current state, it is impossible to establish causality between the coordination system in place and the social relations measured in this study. One explanation is that this department is made up of community-minded faculty members, the most communicative of whom are teaching the coordinated P2C2 courses. Another explanation is that the coordination system and the coordinators have developed a sense of community and shared responsibility for teaching these introductory courses, leading to an increase in communication about instruction. The discovery that the coordinators, who are formally in charge of P2C2 instruction, are also informal community leaders confirms Rasmussen and Ellis's (2015) finding that coordinators do more than simply manage the uniform elements of courses – they are central to active communities of instructors engaged in teaching mathematics.

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Ways in which engaging in someone else's reasoning is productive

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Typical goals for inquiry-oriented mathematics classrooms are for students to explain their reasoning and to make sense of others' reasoning. In this paper we offer a framework for interpreting ways in which engaging in the reasoning of someone else is productive for the person who is listening. The framework, which captures the relationship between engaging with another's reasoning, decentering, elaborating justifications, and refining/enriching conceptions, is the result of analysis of 10 individual problem-solving interviews with 10 mathematics education graduate students enrolled in a mathematics content course on chaos and fractals. The theoretical grounding for this work is that of the emergent perspective (Cobb & Yackel, 1996).

Keywords: Decentering, Argumentation, Social Norms, Fractals, Paradox.

Typical goals for inquiry-oriented mathematics classrooms are to foster particular social norms, such as students explaining their reasoning, listening to others' reasoning, and making sense of that reasoning (Yackel & Cobb, 1996). Indeed, such goals for student participation have been central to a long line of recommendations in the United States (National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). The purpose of this paper is to offer a framework for understanding the various ways in which engaging in the reasoning of someone else is productive for the person who is listening to and attempting to make sense of this reasoning. Prior research has documented ways in which teachers can initiate and sustain such norms for participation (e.g., Lampert, 1990; Stephan & Whitenack, 2003), but most research into the benefits of such engagement focuses on the *students'* thinking, not that of the one engaging in the other's reasoning (e.g., Teuscher, Moore, & Carlson, 2015). While there has been some research into mutual intellectual benefit stemming from peer-to-peer engagements (e.g., Kieran & Dreyfus, 1998), it has not been at the collegiate level. Our work contributes to this surprisingly sparse literature, extends notions identified in disparate settings, and adds nuance to existing notions of engaging and decentering.

The theoretical grounding for this work is that of the emergent perspective (Cobb & Yackel, 1996), which coordinates the individual cognitive perspective of constructivism (von Glasersfeld, 1995) and the sociocultural perspective based on symbolic interactionism (Blumer, 1969). A primary assumption from this point of view is that mathematical progress is a process of active individual construction and a process of mathematical enculturation. The interpretive framework, shown in Figure 1, lays out the central constructs in the emergent perspective. The within row relationships between respective collective and individual constructs is said to be reflexive, meaning that they are mutually constitutive, evolving together in a dynamic system. For example, (Yackel & Rasmussen, 2002) analyze individual students' evolving beliefs about their and others' role in relation to evolving classroom social norms. This work speaks to one way in which engaging in the reasoning of others (a social norm) is productive for the individual; namely doing so positively shapes beliefs.

Collective Perspective	Individual Perspective
Classroom social norms	Beliefs about own role, others' roles, and the general nature of mathematical activity
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions and activity

Figure 1. The interpretive framework

In furthering the relationships between the constructs in Figure 1, we argue for *across* row relationships. In particular, we take the stance that classroom social norms are also inextricably intertwined with individual mathematical conceptions and activity. In so doing we make an empirically grounded argument for a theoretical connection between the upper left hand cell of the interpretive framework and the bottom right hand cell.

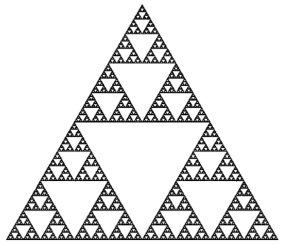
In our broader research program (Hershkowitz, Tabach, Rasmussen, & Dreyfus, 2014; Tabach, Rasmussen, Hershkowitz, & Dreyfus, 2015), we are investigating the coordination between individual and collective processes. In this report, however, we focus on analyzing individual mathematical conceptions and activity in an individual interview setting, with the subsequent goal of coordinating this analysis with an analysis of classroom video-recordings. This report lays a foundation for this subsequent analysis, but the framework for ways of engaging in someone else's reasoning is potentially significant on its own.

Methodology

The methodological approach for the larger study falls under the genre of "design-based research" (Cobb, 2000; Design-Based Research Collective, 2003). The study took place in an intact graduate level mathematics course about chaos and fractals with 11 students (10 of whom agreed to participate in individual interviews). Students were (or intended to be) secondary school teachers or community college instructors. Their masters degree program required a substantial component of mathematics, and the chaos and fractals course qualified as one of their mathematics courses. The course was taught by one of the research team members. Data collected as part of the larger study included video-recordings of each class session, individual problem solving interviews conducted at the middle and end of the semester, and copies of all student work. In this paper we report on an analysis of the 10 individual, mid-semester problem-solving interviews.

The following question from the mid semester interview is the focus of this analysis:

In class, we discussed the Sierpinski Triangle. How do you think about what happens to the perimeter and the area of the Sierpinski Triangle as the number of iterations tends to infinity?



Follow-up questions:

a. One a scale from 1 to 10 with 10 being the most confident, how confident are you about what happens to the area? Can you say more about why you said [confidence number]?
b. On a scale from 1 to 10, with 10 being the most confident, how confident are you about what happens to the perimeter? Why do you say [confidence number]?
c. A student named Fred claimed the following. Please read it out loud. What do you think about his argument? (Please explain)

Fred's Argument: The computation shows that the perimeter goes to infinity because the perimeter is given by $3\times({}^{3}/_{2})^{n}$ which increases to infinity as n tends to infinity. But, the perimeter can't *really* be infinitely long, because there is nothing left to draw a perimeter around, since the area goes to zero.

The Sierpinski Triangle is a fractal, and is the result of an infinite iterative process that begins with an equilateral triangle. Connecting the midpoints of its sides results in another equilateral triangle with sides half the length of the original's and area that is one-fourth of the original's, which is then removed. Repeating this process, *ad infinitum,* results in the Sierpinski Triangle. At each step of the process, the area of shrinks by a factor of ¾ and the perimeter grows by a factor of ${}^{3}/_{2}$. The perimeter of the Sierpinski Triangle can be described by the limit, as $n\to\infty$, of $P_0\times({}^{3}/_{2})^{n}$, and the area by the limit, as $n\to\infty$, of $A_0\times({}^{3}/_{4})^{n}$, where P_0 and A_0 refer to the perimeter and area of the starting triangle. Thus the Sierpinski Triangle has a perimeter of infinite length and an area of zero. This apparent contradiction comes from the fact that it is a fractal with Fractal dimension $\log_2(3)$, putting it between one- and two-dimensions.

The question was structured so that we would first gain insight into students' own reasoning about the area and perimeter of the Sierpinski triangle, followed by an opportunity for them to engage in the hypothetical reasoning of "Fred." The basis for Fred's reasoning was taken from a whole class discussion that took place several weeks before the interview. Thus, Fred's reasoning is authentic to the students and provides an ideal opportunity for us to subsequently coordinate individual and collective analyses.

While most studies of decentering and engagement have involved interactions between two or more people, ours involves one person interacting with the work of another, who cannot respond. While this setup potentially restricts the ability of interviewees to engage with Fred and his argument, it has certain affordances as well. One affordance is that all interviewees reacted to the *same* statement, allowing us to make direct comparisons. This setup also controls for a variety of other features, such as personal histories, that may influence how people react to each other in face-to-face settings.

The transcripts and student work produced during the interview were open coded using methods from grounded theory (Strauss & Corbin, 1998). This open coding, which was conducted collaboratively by the authors to minimize bias and ensure interpretations were grounded in the data, was informed by literature on student thinking about infinity, and in particular infinite iterative processes (Núñez, 1994; Mamolo & Zazkis, 2008), but did not rely on an a priori coding scheme.

The initial open coding of these interviews revealed differences between students' initial responses and those that followed reading Fred's argument. It also revealed a variety of ways of engaging and responding to Fred. We then supplemented our initial coding, using Toulmin's argumentation scheme (Toulmin, 1969) to analyze the pre- and post-Fred arguments presented by the students. Finally, each transcript was distilled into an argumentation log (Rasmussen & Stephan, 2008), coupled with the primary ways of reasoning being used in each argument and instances of engagement, and supplemented by statements about the mathematics that were not necessarily part of a coherent argument. Again using grounded theory, these were analyzed for shifts and relationships.

Results

Our analysis of students' responses revealed that responding to Fred's argument was a productive experience for most students. There was variation across students with regards to both the extent and nature of their engagement and growth, but we note two major categories of productivity that stemmed from an ability to engage in Fred's reasoning and decenter from their own: elaborating justifications and refining/enriching conceptions of particular mathematical ideas. Figure 2 is intended to capture the relationship between engaging with another's reasoning, decentering, elaborating justifications, and refining/enriching conceptions. Specifically, engaging with another's thinking can be foundational for (re)engaging with one's own thinking. That is, the act of decentering provided the means for elaborating justifications and refining one's thinking. The intersecting ovals in Figure 2 for these two acts signify the reciprocal relationship between justifying and refining conceptions.

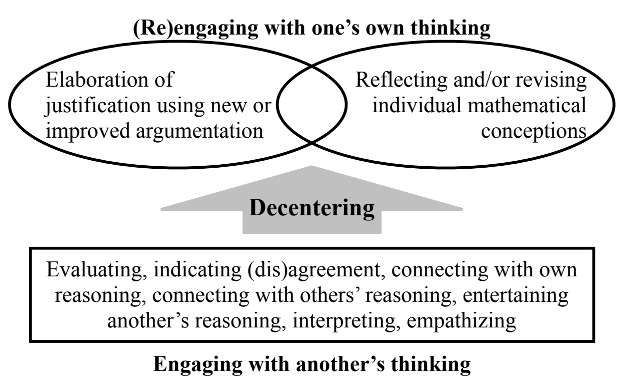


Figure 2. Productivity of engaging in another's reasoning

Since all of the interviewed students were or intended to be teachers at the secondary or postsecondary level, it is particularly interesting to look at their ability to engage with another's thinking. Doing so is foundational to teacher noticing (Jacobs, Lamb, & Philipp, 2010) in which teachers can instructionally build on student thinking. We found that all of the interviewees exhibited the ability to engage with Fred's thinking. We identified the following ways that interviewees engaged in Fred's reasoning: a) evaluating (with or without justification); b) indicating (dis)agreement (with or without justification); c) making connections to their own reasoning; d) making connections to classmates' reasoning; e) entertaining Fred's reasoning; f) interpreting Fred's reasoning; g) diagnosing Fred's reasoning; and h) empathizing with Fred. These ways of engaging provide an opportunity for the individual to decenter. By decenter, we mean putting aside one's own reasoning in an attempt to understand another's reasoning (Steffe & Thompson, 2000; Teuscher, Moore, & Carlson, 2015). Many interviewees, through decentering, engaged or re-engaged with their own thinking in a way that furthered their own thinking. This analysis lays the groundwork for coordinating individual and collective ways of participating in discourse since evaluating (with justification) and indicating (dis)agreement connect strongly to foundational classroom social norms.

In this paper we give a few brief examples of engaging and decentering. Most students gave some indication of agreement or disagreement with Fred's argument, e.g. "I agree with him that the perimeter increases to infinity […] but I disagree with his second line." This example shows a fairly superficial engagement in which the interviewee attended to Fred's reasoning but viewed it from her own point of view. Other students went further, e.g. "I disagree because we thought about it in terms of fencing […] so eventually it's all fence." The second student's explanation makes it clear that while she has not necessarily built a model of Fred's line of reasoning, she is aware of her own model and believes Fred's is different. This second student then elaborated and improved upon her original argument.

Interviewees also demonstrated a range of depth when engaging with Fred by interpreting his reasoning. Some interpreted Fred's thinking from their own point of view, but others made clear attempts to deduce Fred's reasoning from *his* point of view – in one case an interviewee requested more information about Fred's argument before settling on an interpretation. We saw evidence, across all interviews, that each act of engaging functioned as a potential stepping-stone to decentering, an opportunity that some students took up while others did not. We saw that students who engaged deeply with Fred's thinking and decentered from their own point of view appeared to (re)engage with their own thinking.

As a consequence of decentering, many of the students clarified and even advanced their own lines of mathematical reasoning as expressed by Figure 2. As Fred's argument was in response to a question they had already answered, many reacted by re-explaining or expanding their initial justification. Within mathematical thinking we observed two main subcategories: the elaboration of justification for their claims and the expansion of their thinking regarding the mathematical concepts involved in the task. By elaboration of justification, we mean that students were observed adding new or improved warrants and backings to strengthen their argument or even providing entirely alternative explanations. As an example, one student, Sandor, reacted to Fred's argument by noting that that it is *because* the area of the Sierpinski Triangle goes to zero that the perimeter goes to infinity, and explicitly connected the removal of triangles at each recursive step to adding the perimeter of these triangles to the total perimeter. Prior to engaging with Fred's statement, he had treated the two results as essentially separate features of the process – the connection between the two had gone unnoticed or at least unexplained.

With regards to the underlying mathematical concepts, we observed students exploring the nature of infinity, perimeter, and the Sierpinski Triangle itself in greater depth than they had in their initial arguments. Some students appeared to become aware of a distinction between potential infinity (the unending process) and actual infinity (the final resultant state) in their attempts to clarify their reasoning. Many students took the opportunity to define, or re-define, the perimeter of an object. Students also reflected on the fractal nature of the Sierpinski Triangle, noting that it exists "between" dimensions and therefore does not act in the way that a "normal" one- or two-dimensional object might, and that therefore traditional thinking about a perimeter enclosing area is not necessarily valid in this context.

While we identify decentering and mathematical thinking as distinct, we note that they are not disjoint. All of these examples of expanded thinking and reasoning occurred to some extent as a reaction to the thinking of someone else. We posit that decentering functioned as a catalyst for this process. Seeing Fred's argument, interviewees demonstrated a variety of strategies for engaging with student reasoning, which were taken up with varying depth. Deeper engagement took the form of decentering, which predicated (re)engagement with and growth of their own reasoning. That is to say, the greater the extent to which students engaged with Fred and decentered, the more productive the experience was with regards to their own thinking.

The Case of Curtis

To clarify the constructs and interpretations outlined above, we present the case of a single student, with pseudonym Curtis. We choose this student as an example because of the brevity and clarity of this portion of his interview, as well as the range of constructs identified in his experience with Fred. Figure 3 shows Toulmin analyses of Curtis's pre- and post-Fred arguments, as well as his comment about infinite processes.

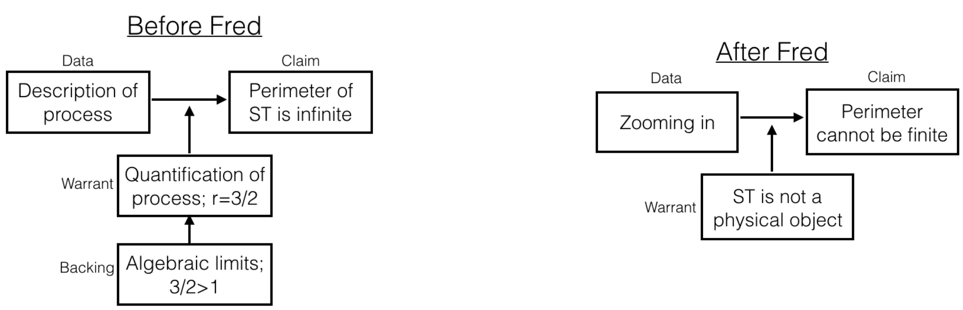


Figure 3. Toulmin analysis of Curtis' arguments.

Toulmin analysis of Curtis' pre- and post-Fred argumentation revealed shifts and changes. A small shift occurred in Curtis' claim: initially he showed that the perimeter is infinite, afterward he showed it could not be finite. This new claim is drawn from different data and is supported by a different warrant. Where initially Curtis used formal/symbolic reasoning, his second argument draws on heuristics and a sense that the Sierpinski Triangle is not a real object. He also brings up the fact that infinite processes do not have a 'final step' after which they reach their final state, something that was not mentioned prior to Fred.

Retracing the emergence of new topics for Curtis, we found that they were directly linked to his engagement with Fred's reasoning, and in particular resulted from his ability to decenter and look at Fred's reasoning in ways not related to his own. Curtis comments that Fred's "logic doesn't work," addressing more than just his faulty claim. The new warrant that Curtis provides, that the Sierpinski Triangle is not a physical object but rather "kind of just a concept," directly addresses an unspoken assumption on Fred's part. It seems that Curtis has identified and reacted to an implicit backing in Fred's argument – that the Sierpinski Triangle is a geometric object that obeys two-dimensional rules. Curtis' diagnosis of a misconception underlying Fred's reasoning implies that he has considered Fred's argument from a different viewpoint, effectively trying to put himself in Fred's shoes and understand fully his reasoning.

In addition to presenting a new argument, Curtis presents it in a new style. While his original argument was based in formal limits and notation, his new argument adopts some of Fred's informal, heuristic, and geometric language. Again, this supports the idea that Curtis is working from Fred's point of view, rather than his own.

Finally, Curtis' added commentary about infinite processes comes from his interpretation of Fred's argument. He says that Fred's argument is equivalent to there being a final step, a point where something is taken away and the area becomes zero, and notes that this is not how infinite processes work. This seems to address Fred's data, that the object becomes something with no area.

Altogether, we see that Curtis addresses all the pieces of Fred's argument (not just the claim) by thinking through Fred's reasoning (not just comparing it to his own). This includes an implicit backing that Fred does not explicitly state. He does so using Fred's style of reasoning, and (re)engages with his own reasoning to present a second argument and an observation about infinite processes. Throughout his response to Fred, Curtis addresses Fred's reasoning and explains why it does not work, rather than simply asserting that his own original ideas are correct.

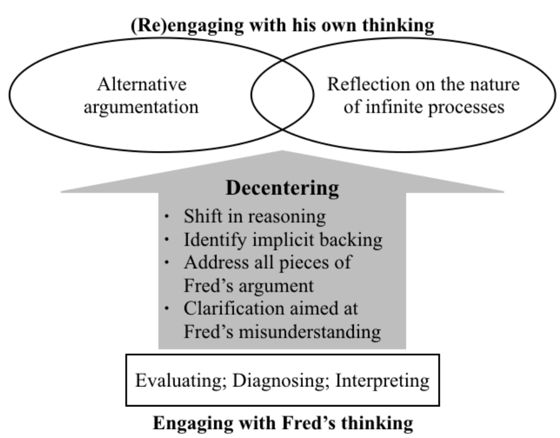


Figure 4. Curtis' productivity from engaging with Fred's reasoning.

Conclusion

In conclusion, we return to classroom social norms and the ultimate role we envision for our framework. We argue that the ways of engaging we observed in these interviews are closely related to particular classroom social norms. The relevant social norms related to engaging in others' reasoning include listening to others' reasoning, attempting to make sense of this reasoning, and indicating agreement or disagreement, with reasons. Moreover, acting in accordance with these norms led, through decentering, to enriched and refined mathematical conceptions and activity. The case of Curtis illustrates that decentering is an individual cognitive mechanism triggered by engaging with another's reasoning.

Prior work posits a reflexive relationship between engaging in others' reasoning (i.e., social norms) and individual beliefs. In Figure 1, this relationship coordinates the cells in the top row of the interpretive framework. As far as we are aware, the analysis in this paper is the first to coordinate social norms and individual mathematical conceptions and activity. That is, we provide evidence for a relationship between social norms (upper left hand cell of the interpretive framework in Figure 1) and individual conceptions (bottom right hand cell). This importance of this work lies in coordinating different analytic tools that separately address collective and individual phenomenon. Thus, our framework not only contributes to a nuanced understanding of engaging and decentering with another's reasoning, but also leads to links between individual mathematical conceptions and social activity.

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Students' understanding of mathematics in the context of chemical kinetics

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Abstract: This work explores general chemistry students' use of mathematical reasoning to solve quantitative chemical kinetics problems. Personal constructs, a variation of constructivism, provides the theoretical underpinning for this work, asserting that students engage in a continuous process of constructing and modifying their mental models according to new experiences. The study aimed to answer the following research question: How do non-major students in a second-semester general chemistry course and a physical chemistry course use mathematics to solve kinetics problems involving rate laws? To answer this question, semi-structured interviews using a think-aloud protocol were conducted. A blended processing framework, which targets how problem solvers draw from different knowledge domains, was used to interpret students' problem solving. Preliminary findings describe instances in which students blend their knowledge to solve kinetics problems.

Keywords: Rates, Problem Solving, Blended Processing, Chemistry, Kinetics

Understanding fundamental concepts in chemistry is intrinsically tied to understanding mathematical symbolism and operations, as well as translating between equations and physical realities. Because of this reality, studies in science education have begun to focus on students' understanding of and use of mathematics in scientific contexts (e.g. Becker & Towns, 2012). Findings from such studies allow researchers and practitioners to find ways to enhance students' abilities to interpret and use mathematical expressions in conjunction with conceptual understanding, rather than blindly applying routine mathematical procedures.

Research on quantitative problem solving investigates students' abilities to solve the problem correctly (e.g. Wilcox, Caballero, Rehn, & Pollock, 2013), to understand and set up the problem (e.g. Bodner & McMillen, 1986), or to execute problem-solving steps (e.g. Reif & Heller, 1982). However, such studies rarely examine how individuals *use* equations (Kuo, Hull, Gupta, & Elby, 2013). Because of the great importance of mathematics in chemistry, it is of the utmost importance to understand *how* equations are used and understood by chemistry students. Kuo et al. (2013) propose that equations could be used in two ways, where the second is more sophisticated and expert-like: 1) as computational tools to obtain an answer or 2) as holding meaning when blended with conceptual understanding.

This study explores undergraduate chemistry students' quantitative problem solving in the context of chemical kinetics because it is an anchoring concept of the undergraduate chemistry curriculum that requires the use of mathematics to understand and solve problems (Holme, Luxford, & Murphy, 2015; Holme & Murphy, 2012; Murphy, Holme, Zenisky, Caruthers, & Knaus, 2012). It has the power to provide insight into the nature of chemical reactions and processes, because it ties observable phenomenon with theoretical aspects of chemistry that are modeled mathematically (Çakmakci, Leach, & Donnelly, 2006). In addition, studies in this content area are understudied when compared to other topics in chemistry education research (CER) (AUTHOR, 2016, submitted).

The aim of this study is to identify how undergraduate chemistry students understand and use equations to solve kinetics problems. The guiding research question for this work is: How do non-major students in second-semester general chemistry and a non-majors physical chemistry course understand and use mathematics to solve kinetics problems involving rate laws? This study will provide insight into the mathematical processing stage of quantitative problem solving, providing instructors with an understanding of how students studying kinetics understand and use both the concepts and mathematics involved.

The theoretical framework for this study is personal constructs, a variation of constructivism, a framework that presents individuals as making sense of their experiences by inventing knowledge constructions and continually modifying them as they encounter more experiences (Bodner, 1986; Bodner, Klobuchar, & Gleelan, 2001). Specifically, Kelly's (1955) theory of personal constructs, a combination of personal and social constructivism, argues that while individuals differ in their knowledge constructions, one individual's constructs can be similar to another's, due to social interaction. A cognitive framework called blended processing is used to help describe and analyze problem solving. Blended processing describes a cognitive process that explores and models human information integration (Coulson & Oakley, 2000; Fauconnier & Turner, 1996, 1998, 2002). It provides a way to describe and understand individuals' mental spaces (or knowledge constructions) and their interactions (Bing & Redish, 2007; Hu & Rebello, 2013). In the context of science education research, blended processing can describe the "opportunistic *blending* of formal mathematical and conceptual reasoning *during* the mathematical processing stage" (Fauconnier & Turner, 2002; Hull, Kuo, Gupta, & Elby, 2013; Kuo et al., 2013; Sherin, 2001).

The primary data source for this study is individual semi-structured interviews with undergraduate chemistry students, which are conducted using a think-aloud protocol (Becker & Towns, 2012). This interview technique has students perform a task while explaining their thought process out loud. During these interviews participants solve kinetics problems involving rate laws, tables of data, and graphs. The written work is recorded physically on Livescribe™ paper and digitally by a Livescribe™ smartpen that captures both audio and writing in real time. The protocol is adapted from Kuo et al. (2013) to use a chemical kinetics context. It contains equations that the participants are asked to explain and problems they would be asked to solve in a general chemistry or upper-level undergraduate physical chemistry course.

The participant sample was selected using a homogenous sampling technique (Patton, 2002). Student participation is voluntary. Fall 2015 data collection yielded 21 individual interviews with second-semester general chemistry students. Spring 2016 data collection is ongoing with both second-semester general chemistry students and physical chemistry students. For completing the interview, students are compensated with a $10 iTunes gift card.

Audio data is transcribed verbatim following the interviews. To condense our data in a way that is conducive to answering our research question, we organized interviews into problem solving maps. To make the maps, we identified problem solving "steps" in large tables, where all data from the interview corresponding to each step were categorized with a brief descriptor, such as "highlights purpose of the equation." Keeping in mind a conceptual framework of blended processing, an open coding approach was used to analyze the problem solving maps. Frequently, codes were assigned to excerpts of data as they were organized into steps in the map, which meant that a problem-solving step received one code. However, there were also instances where multiple codes were assigned to all the data in one step or different codes were assigned to different parts of the data in one step. Preliminary thematic findings will be presented and discussed. Evidence of blended processing will be explored, in conjunction with evidence of other modes of reasoning.

This study holds the promise of developing a better understanding of how non-major chemistry students understand chemical kinetics, but more importantly how they use and understand mathematics in chemistry contexts.

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Using learning trajectories to structure teacher preparation in statistics

Abstract

As a result of the increased focus on data literacy and data science across the world, there has been a large demand for teacher preparation in statistics. However, exactly how this preparation should be structured remains an open question. The purpose of this paper is to report on the NSF-funded Project-XXX professional development program. Project-XXX provided professional development to enhance teachers' statistical knowledge for teaching. The project constructed two hypothetical learning trajectories for teacher learning and subsequently used the hypothetical learning trajectories to structure the professional development curriculum. This main goal of this paper is to illustrate how the utilization of the learning trajectory structure to design professional development curriculum allowed participating teachers to develop several aspects of Statistics Knowledge for Teaching (Groth, 2013).

Introduction

Large-scale research provides some indication of the key characteristics of effective teacher training (Doerr, Goldsmith, & Lewis, 2010; Garet, Porter, Desimone, Birman, & Yoon, 2001; Heck, Banilower, Weiss, & Rosenberg, 2008; Gersten et al., 2014). These include a focus on content knowledge, opportunities for active learning, and coherence with other learning activities (Garet et al., 2001).

The purpose of this article is to report on how using teacher learning trajectories to design teacher training offers a structure to develop teacher content knowledge in deep and meaningful ways. This study is based on the implementation of Project-XXX, a project funded by the National Science Foundation to develop professional development curriculum materials to enhance teachers' content knowledge of two fundamental statistics topics – sampling variability and regression. Project-XXX first developed hypothetical learning trajectories for teacher learning for these two topics and then designed a professional development curriculum around these learning trajectories. In exploring the results of the implementation of this professional development curriculum, we recognized the important role that the learning trajectories played in creating opportunities for teacher participants to achieve key developmental understandings in relation to knowledge of the content.

We aim to answer the following research question: *How did the use of learning trajectories to design professional development curriculum support the development of teachers' statistics knowledge for teaching* The findings suggest that learning trajectories offer a promising structure for aiding teacher professional development curriculum design.

Learning Trajectories

Given a specific topic, a learning trajectory (or learning progression) for that topic can be thought of as a framework that serves as a map for how to achieve different cognitive learning levels or learning outcomes. Learning trajectories (LTs) represent recent advances in instructional and curriculum design (Duschl, Schweingruber, & Shouse, 2007; Mohan, Chen, & Anderson, 2009; Stevens, Delgado, & Krajcik, 2010) that promote deep and integrated understanding of target topics by providing a model for the successive and gradual thinking about a topic one must go through to achieve depth of understanding. However, while learning trajectories essentially organize learners' thinking and learning, how a learning trajectory is built and the scope for which it is used differs in the literature.

Some authors build learning trajectories as a descriptive tool formed from extensive descriptive research and observations of students' learning in various settings. In this sense, learning trajectories are a way to synthesize how learners acquire increasingly advanced ways of thinking independent of instruction of a particular topic. For example, learning trajectories can be viewed as incrementally more sophisticated ways to think about a concept that emerge naturally while one moves toward expert-level understanding (Stevens, Shin, & Krajcik, 2009).

Another way learning trajectories can be built is through the consultation of expert opinions of how learners acquire advanced ways of thinking given a particular instructional treatment. For example, Confrey and Maloney (2010) describe LTs as "a researcher-conjectured, empirically-supported description of the ordered network of constructs a student encounters through instruction (i.e. activities, tasks, tools, forms, of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time" (Confrey, 2008; Confrey et al., 2008, 2009).

Project-XXX focuses on developing learning trajectories for teachers' thinking and learning. *The Project-XXX learning trajectories are defined as a curricular map for sampling variability and regression for teachers.* The trajectories are based on the analysis of statistical content and consultation of previous literature on the teaching and learning of sampling variability and regression along with a team of expert opinions of sensible sequences of this content. The trajectories therefore describe how teachers can acquire advanced ways of thinking about sampling variability and regression given a particular instructional path.

Theoretical Framework

We employed LTs as a map for *teacher* learning. *In this sense, the LTs were used to create a teacher curriculum, through a series of instructional tasks and assessments, for teacher learning of sampling variability and regression.* The LTs in this study therefore served to guide, develop, and order a sequence of instructional tasks for teacher training. In applying the LTs to teacher learning, we intersect LTs with frameworks of teacher knowledge. In particular, we draw upon the Mathematical Knowledge for Teaching (MKT) framework and the Statistical Knowledge for Teaching (SKT) framework (Groth, 2013).

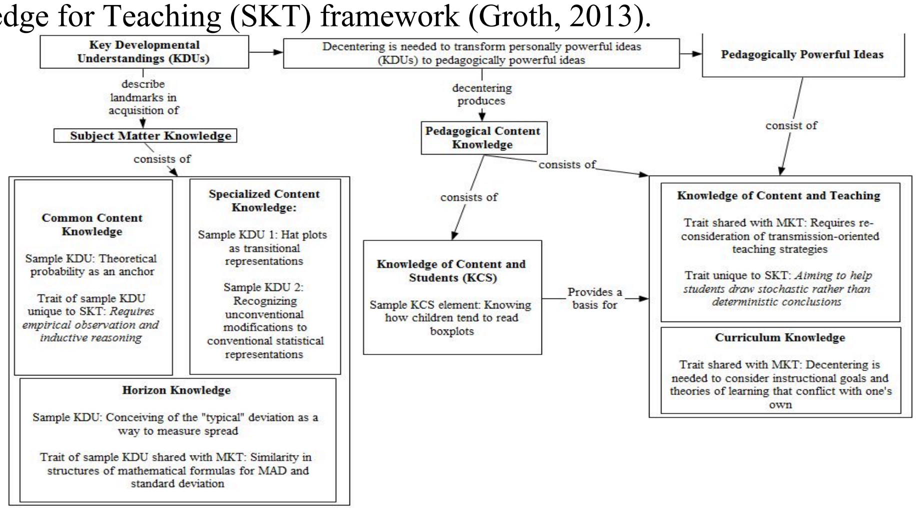

Figure 1 SKT Framework (Groth, 2013, pp. 143)

Groth (2013) identified Key Developmental Understandings (KDUs) as landmarks in the teachers' development of subject matter knowledge. Building from the work of Simon (2006), Groth describes KDUs as significant conceptual shifts. According to Groth, these landmarks or conceptual shifts can occur in each of the three types of subject matter knowledge in his framework (common content knowledge, specialized content knowledge, and horizon content knowledge). Groth's SKT framework also incorporates ideas outlined by Silverman and Thompson (2008) regarding the development of pedagogical content knowledge. In particular, Silverman and Thompson assert that teachers' development of KDUs with regard to subject matter knowledge are a necessary, but perhaps insufficient, first step with regard to improving student learning.

The primary purpose of the Project-XXX professional development was to develop teachers' SKT. Prior to the design of the professional development course, as part of the project, teacher LTs were constructed for the topics of sampling variability and regression. These LTs served as guides for the structure of the professional development. Figure 2 represents the relationship between teacher learning trajectories, professional development, and SKT. This study focuses on understanding how this process might work.

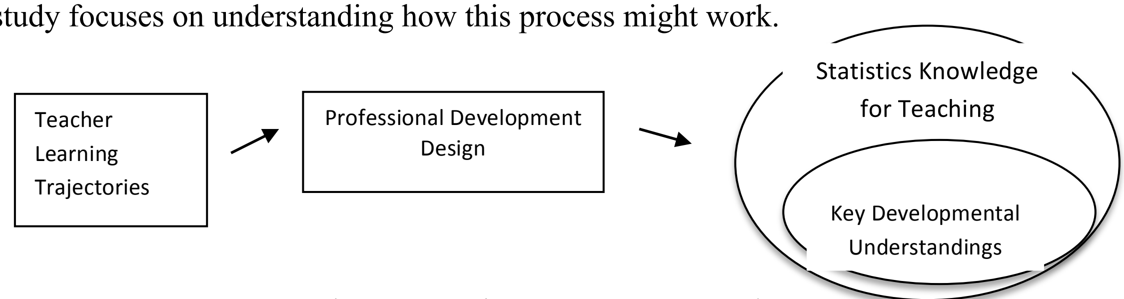

Figure 2. Project-XXX Conceptual Framework

Methods

Participants

Nine secondary teachers completed the entire pilot course. Seven of the 9 teachers taught in the local public school district. Two of the teachers taught in a private school within the city. Their average number of years teaching statistics was 2.4.

Data Sources

End-of-Loop Assessment Tasks. Assessment tasks were completed by the teachers at critical points of the LT in order to measure understanding with respect to the content included in the LT. The scoring of each part was modeled after the AP Statistics scoring of: E (Essentially Correct); P (Partially Correct); or I (Incorrect). The assessment tasks were scored each week by two scorers who were part of the research team but not present during the professional development session. The scorers graded the papers separately and then discussed their scores to come to a consensus on the final scores.

End-of-LT Assessment. At the completion of the content of each LT, teachers were assigned as homework an assessment intended to bring together the content of the entire trajectory.

Video of Class Sessions. Each class session of the professional development was videotaped. Outlines of the videos were created and portions of the videos were transcribed. The videos provide a means to confirm and elaborate on the observed patterns of teacher learning documented from the teachers' written work.

Analysis

A two-phase process was used to investigate how the use of LTs supported the development of Statistical Knowledge for Teaching (SKT), with a particular focus on Key Developmental Understandings (KDUs). The first phase took place during the analysis of teachers' written work on the End-of-Loop Assessment Tasks and the end-of the LT projects. This analysis provided insight into which ideas were pivotal to teacher understanding thus permitting the research team to identify a preliminary list of KDUs.

In the second phase of the data analysis, these prospective KDUs were then examined through the analysis of classroom interactions. During this phase, the videos were examined in order to determine how teachers' Statistical Knowledge for Teaching developed as they progressed through the LTs. Results of this analysis will be presented in the conference talk.

Results

We present illustrative examples of the KDUs that we identified along with supporting evidence consisting of samples of teacher work or transcript segments. While analyzing the teacher written work, several points were identified as being pivotal to teachers' development of SKT. Such ideas were those that repeatedly surfaced in multiple teachers' written work throughout the PD as well as in-class discussions.

Example 1 Common Content Knowledge KDU: Sample Size and the Sampling Distribution

One of the most persistent ideas that surfaced in teachers' work and discussion involved the relationship between sample size and the shape and spread of the sampling distribution. We have identified this as a KDU reflecting Common Content Knowledge in Groth's framework insofar as this is not a concept specific to the domain of teaching.

Teachers repeatedly made statements alluding to the fact that when repeated samples were taken and a sample mean was computed, then the shape of the sampling distribution should become more bell-shaped and the variability of the sampling distribution should decrease. For example, an assessment task for sampling variability asked teachers to compare three different approximate sampling distributions taken with samples of n=5, 15, and 30 according to their shape, variability, and center. There is evidence at this point that the nine teachers developed an understanding of the effect of the sample size on the spread, even if they were not yet clearly articulating the relationship to shape. Three samples of teacher responses are provided:

Assessment task question: Compare the three distributions that you constructed. What can you say about the shape of the distribution as the sample size, *n*, increases? What can you say about the mean? What can you say about the standard deviation?

Teacher Example 1: As *n* increases, the data gets more "compact" around the population mean $27,000 – thus the variation decreases. The mean income was closer to the population mean when n=15 than when n=30, but still both close to $27,000. The standard deviation decreases as *n* increases.
Teacher Example 2: As the sample size increase, the distributions are becoming less spread out. There is less variability in the distributions as the sample size increases.
Teacher Example 3: The shape of the distribution is more unimodal and symmetric, becoming more approximately normal. There is less variation as the sample size n increases. The mean got smaller and then bigger as the sample size increased. However, the mean stayed in the same interval between 25,000 and 30,000. The standard deviation decreased as the sample size n increased. I would expect that trend to continue, but the standard deviation was decreasing at a slower rate as n increased.

The videos of the class sessions provide further support for the assertion that this was a KDU for teachers that the loop design of the LTs fostered. After investigating these ideas over a period of time, teachers expressed different "aha" moments around the effect of the sample size. For example, the activity for Loop 3 engaged the teachers in sampling from four populations with vastly different distributions (bimodal distribution, skewed distribution, roughly normal distribution, and scattered distribution). The teachers took samples of size n=5, n=10, and n=25 and generated approximate sampling distributions for each sample size. They compared the sampling distributions of different sizes and noted the similarities in the effect of the sample size on the sampling distribution.

> Instructor: So, tell us what you've got there from Jamal [the bimodal distribution] and what happened with the samples of different sizes?
>
> Teacher 6: As you can see, as you increase the sample size, the variability gets smaller and smaller, if you look you are going from 20 to about 110, versus the spread going from 20 to about 80 where for size 10 you are going about 52 to 66, your variability is decreasing.

As the discussion continued, several teachers also compared the sampling distributions to the corresponding population distributions, noting the way in which the distributions with the larger sample sizes behaved in a similar manner, regardless of the distribution of the population. In particular, even when the population had a non-symmetric or bimodal shape, as the sample size increased, the variability of the sampling distribution decreased. During this comparison of the different distributions, one of the teachers, focused on the behavior of sampling distributions of the bimodal population distribution, began to talk through the reasons behind what she had observed.

> Teacher 5: No matter what the population looked like, there was a mean. And our data, or our samplings, were samplings of the average. So, they all should have been near the average of the population. No matter what [the population] looked like.

Another teacher builds on this idea and offers an argument for why the variation of the sampling distribution should logically decrease with an increase in the sample size:

> Teacher 7: I guess they can't do this because they are obviously…cards, but if we had done N=60, a.k.a. all the cards, it would have just been a straight line at 60… So that like N = 60 literally is just 60, 60, 60, 60 [the mean of the sample] over and over again, just a straight line of 60, *but that would have been a good thing to compare to n = 4, n=10, n = 30*. [Note: The example to which she was referring had a population mean of 60 and a population size of 60.]

In this segment, the teachers appear to not only recognize the effect of the sample size on the spread of the sampling distribution (as with their written work, their descriptions of the shape are not as explicit at this point). They also appear to be creating corresponding mental images of why this makes sense, no matter what the shape of the population is.

The significance of this idea as a KDU is reflected in their own comments a few minutes after the observations from Teacher 5 and 7 described above.

> Teacher 3: You know…I'm not sure that I ever understood that…I'm serious.
>
> Teacher 5: The light did come on, in terms of understanding what was happening in this activity.

Example 2 Specialized Content Knowledge KDU: Line of Best Fit Counterexamples

Specialized content knowledge, defined in the SKT framework as knowledge of content needed in the practice of teaching, may include teachers' ability to comment on student work and strategize ways they can address student errors. One way to illustrate to students their errors would be to provide students with examples for which their solution will not work. The ability to develop such counterexamples is knowledge specific to teaching.

In the regression LT, the teachers were asked to examine a scatterplot of drop heights versus bounce heights of a golf ball and place a piece of spaghetti on the scatterplot in such a way that they believed represented a line of best fit. Teachers also had to explain what their criterion was for the placement of the line and why they chose to place it there. This same activity had been given to 8th grade students. A second component to the teacher activity was then for the teachers to comment on the 8th grade students' work and, if the work showed a misunderstanding, then provide a counterexample scatterplot that would illustrate to the student that their placement criterion would not be successful in general.

During part one of the activity, all of the teachers created criteria that matched that of the previously collected student data. For example, one teacher asked "do you assume it [the spaghetti] goes through (0,0)?" She noted that in the context of the problem, dropping balls, if you dropped the ball from 0 height, you would get a 0 bounce height. She thus concluded that her line of best fit must go through the origin. This same reasoning was also seen in the student work. Another teacher stated that she placed her line in such a way that "there are 4 dots above and 4 dots below and so it is in the middle." Again, similar reasoning was uncovered in the student work with a student stating that they wanted to "split" the points.

When teachers were given the student results to analyze, they were asked to evaluate whether the criteria the students used to place their line was one that would work for *any* data set. If not, then the teachers were to give an example scatterplot for which their student criteria would not work. This proved difficult for the teachers. For example, one student had the origin criteria similar to one of the teachers. Looking at the student work, she stated: "I think that is a good idea." However, teacher 8 responded by saying "in this case, it [going through the origin] is ok but not all the time." At this point, a conversation emerged as to whether the criterion the student applies must work for any set of linear data or just the golf ball data in front of them. After a short deliberation about what defining a criterion means and how it should be applied, it was accepted that a criterion must work for any set of data. Then, the teachers created the counterexample of a data set that had a negative association and thus would require a line to have a negative slope so it would not enable the line to go through the origin. Although the teachers were able to develop counterexamples to help guide student misunderstandings in the context of the line of best fit, the work was non-trivial.

Discussion & Conclusion

As noted by Simon (2006), for someone to develop a Key Developmental Understanding, one must have repeated exposure to the concept. Additionally according to Simon, students without a KDU "do not tend to acquire it through explanation or demonstration" (p. 362); instead a KDU must emerge through discovery. In this way, a person would be able to shift their understanding and gain a Key Understanding. We gathered evidence to show that the Project-XXX LTs offered a platform for teachers to develop KDUs by scaffolding more complex ideas and repeatedly looping for each topic. Due to limitations in space, only two examples were presented above. We assert that the design of the Project-XXX activities to progress teachers through the LT facilitated the development of Statistical Knowledge for Teaching and the emergence of KDUs. In addition, the LT also allowed for the conceptual unpacking necessary to develop teachers' knowledge.

The LT's mapping created clearly-defined conceptual boundaries that allowed us to recognize when inadequate connections were begin made to horizon content knowledge. Thus, although this is a small study with nine teachers, we see great promise in the use of LT's in the design of teacher preparation curriculum to support the growth of teachers' knowledge. The SKT construct asserts that teacher statistical knowledge for teaching consists of both content knowledge and pedagogical knowledge. We saw evidence in this small scale study that building professional development using LTs shows promise in helping teachers advance both their content knowledge and their pedagogical knowledge. In addition to the development of subject matter knowledge, we also saw evidence of the translation of this subject matter knowledge into pedagogically powerful ideas.

The goal of this paper was to analyze the affordances of LTs in the design of professional development. In particular, we sought to understand how the use of LTs might support the development of Statistics Knowledge for Teaching, with a particular focus on the key developmental understandings that emerged. Doerr et al. (2010) have identified this type of small-scale study as making important contributions to our understanding of professional development. In particular, through the description and analysis of the "critical elements" (Borko et al., 2008) of the program, it is possible to better understand the teacher learning process and the potential of the program for sustained success.

We view the findings in this paper contributing to the advancement of knowledge and literature base in three ways. First, this study provides small-scale evidence that learning trajectories can not only be used to map student curriculum and learning, but also can be used as maps for teacher curriculum and learning. Prior to this study, learning trajectories have been used in the professional development only as tools to discuss student learning. However, this study illustrates several advantages that learning trajectories can offer to actually build, organize, and structure teacher learning. The LTs in this study were used to guide the nature and order of the sequence of content.

A second contribution of this study is the connection of learning trajectories to existing teacher learning constructs such as Statistical Knowledge for Teaching, Key Developmental Understandings, and Pedagogically Powerful Ideas. This study provides evidence that learning trajectories offer a means to observe and develop such constructs with teachers.

An additional contribution of this study is the presentation of two teacher learning trajectory maps for the challenging statistical topics of sampling variability and regression. These trajectories and accompanying materials can be utilized by others to teach teachers these topics.

The analysis of the use of the hypothetical LTs as a "critical element" of the program suggests that LTs can offer similar structures for teacher learning that mirror those previously documented for student learning. In particular, the LTs offered a framework for identifying and achieving KDUs and making instructional decisions based on the KDUs. The LTs gave the research team a way to see how KDUs were directly related to the development of SKT. Furthermore, the LTs provided a means for teachers to achieve KDUs due to their repeated exposure while moving through the loop structure of an LT. By construction, the LTs provided scaffolding for KDU development. The repeated exposure illustrated when cognitive shifts were occurring in teachers' thinking. In addition, this repetition allowed teachers to transform KDUs into pedagogically powerful ideas.

While Project-XXX has a specific focus on teachers' statistics knowledge, we suggest that the implications for mathematics teacher training are broader than this teacher population. In particular, by focusing on a "critical element" of Project-XXX– the use of LTs for teacher knowledge – we submit that the model has potential for other content within mathematics teacher professional development. One of the goals of Project-XXX was to focus on statistical content. While sampling variability and regression represent two important statistical content topics, there are various other topics about which teachers are likely to have had limited opportunities to develop knowledge for teaching (e.g., transformational geometry). The use of LTs for teacher learning offers a potentially powerful strategy for developing teachers' knowledge of these concepts, as well as others within the larger mathematics curriculum.

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A case study of developing self-efficacy in writing proof frameworks

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This case study documents the progression of one non-traditional individual's proof-writing through a semester. We analyzed the videotapes of this individual's one-on-one sessions working through our course notes for an inquiry-based transition-to-proof course. Our theoretical perspective informed our work with this individual and included the view that proof construction is a sequence of (mental, as well as physical) actions. It also included the use of proof frameworks as a means of initiating a written proof. This individual's early reluctance to use proof frameworks, after an initial introduction to them, was documented, as well as her later acceptance of, and proficiency with, them. By the end of the first semester, she had developed considerable facility with both the formal-rhetorical and problem-centered parts of proofs and a sense of self-efficacy.

Key words: Transition-to-proof, Proof Construction, Proof Frameworks, Self-efficacy

This case study concentrates on how one non-traditional mature individual, in one-on-one sessions, progressed from an initial reluctance to use the technique of proof frameworks (Selden, Benkhalti, & Selden, 2014; Selden & Selden, 1995) to a gradual acceptance of, and eventual proficiency with, both writing proof frameworks and completing many entire proofs. This case study further illuminates the well-known, and documented, tendency of students to write proofs from the top-down, and consequently, to be unable to develop complete proofs. (See the case of Willy, who focused too soon on the hypothesis, in Selden, McKee, & Selden, 2010, pp. 209-211). We also consider how this approach to proof construction helped this individual gain a sense of self-efficacy (Bandura, 1994, 1995) with regard to proving.

Theoretical Perspective

In our analysis and in our teaching, we consider proof construction to be a sequence of mental and physical actions, some of which do not appear in the final written proof text. Such a sequence of actions is related to, and extends, what has been called a "possible construction path" of a proof, illustrated in Selden and Selden (2009a). For example, suppose that in a partly completed proof, there is an "or" in the hypothesis of a statement yet to be proved: "If A or B, then C." Here, the situation is having to prove this statement. The interpretation is realizing the need to prove C by cases. The action is constructing two independent sub-proofs; one in which one supposes A and proves C, the other in which one supposes B and proves C.

When several similar situations are followed by similar actions, an *automated link* may be learned between such situations and actions. Subsequently, a situation can be followed by an action, without the need for any conscious processing between the two (Selden, McKee, & Selden, 2010). When students are first learning proof construction, many actions, such as the construction of *proof frameworks* (Selden, Benkhalti, & Selden, 2014; Selden & Selden, 1995), can become automated. A *proof framework* is determined just by the logical structure of the theorem statement and associated definitions. The most common form of a theorem is: "If P, then Q", where P is the hypothesis and Q is the conclusion. In order to construct a proof framework for it, one takes the hypothesis of the theorem, "P", and writes, "Suppose P" to begin the proof. Immediately afterwards, one takes the conclusion of the theorem, "Q", skips towards the bottom of the page, and writes "Therefore Q", leaving enough space for the rest of the proof to emerge in between. This produces the *first level* of the proof framework. At this point, one should focus on the conclusion and "unpack" its meaning. It may happen that the unpacked meaning of Q has the same logical form as the original theorem, that is, a statement with a hypothesis and a conclusion. In that case, one can repeat the above process, providing a *second level* proof framework in the blank space between the first and last lines of the emerging proof. (For some examples, see Selden, Benkhalti, & Selden, 2014).

Prior Research

While studies on students' learning to write proofs have been made before, they have not so specifically focused on proof frameworks. Hazzan (1999) has written about how students cope with the transition to upper level proof-based mathematics, specifically when they take their first undergraduate abstract algebra course. Dahlberg and Housman (1997) were interested in how a student develops his/her concept image when learning a new mathematical concept. They found that students who engaged in example generation and reflection during the study of definitions were able to attain a more comprehensive understanding. Housman and Porter (2003) found a correspondence between students who used transformational proof schemes and those who successfully generated examples when asked to do so. Selden, McKee, and Selden (2010) reported instances of students' tendencies to write proofs from the top down and their reluctance to unpack and use the conclusion to structure their proofs. This study extends that work.

Methodology: Conduct of the Study

We met regularly for individual 75-minute sessions with a mature working professional, Alice, who wanted to learn how to construct proofs. Alice followed the same course notes previously written for an inquiry-based course used with beginning mathematics graduate students who wanted extra practice in writing proofs. The sessions were almost entirely devoted to having Alice attempt to construct proofs in front of us, often thinking aloud, and to giving her feedback and advice on her work. The notes had been designed to provide graduate students with as many different kinds of proving experiences as possible and included the kinds of proofs often found in typical proof-based courses. They covered some sets, functions, real analysis, and algebra, in that order.

Alice had a good undergraduate background in mathematics from some time ago and also had prior teaching experience. She only worked on proofs during the actual times we met. While she usually came twice a week to see us and work on constructing proofs, sometimes when her paid work got a bit overwhelming, she would take a week off. Thus, unlike the graduate students who took the course as a one-semester 3-credit class, Alice worked with us on our course notes for two semesters at her own pace and did not want credit.

We met in a small seminar room with blackboards on three sides, and Alice constructed original proofs at the blackboard, eventually using the middle blackboard almost exclusively for her evolving proofs. After several meetings, she began to use the left board for definitions and the right board for scratch work. She did not seem shy or overly concerned with working at the board in front of us, and from the start, we developed a very collegial working relationship. She seemed to enjoy our interactions as she worked through the course notes. Thus, we gained greater than normal insight into her mode of working. We videotaped every session and took field notes on what Alice wrote on the three boards, along with her interactions with us. For this particular study, we reviewed the first semester videos and field notes several times, looking for signs of progression in Alice's approach to constructing proofs.

The Progression

Our First Meeting with Alice

We introduced Alice to the idea of proof frameworks and explained in detail how and why we use them. We also introduced her to the idea of unpacking the conclusion and mentioned that proofs are not written from the top down by mathematicians. With guidance, she was able to prove "If $A \subseteq B$, then $A \cap C \subseteq B \cap C$." In addition, she worked three exercises on writing proof frameworks--one on elementary number theory and two on set equality. Near the end of this meeting, Alice produced a proof framework for the next theorem in the notes. We felt that she not only understood our reasoning for using proof frameworks, but also how to construct them.

Our Second Meeting with Alice—Her Reluctance to Use Proof Frameworks Surfaces

At the beginning of the second meeting, Alice went to the middle board and produced the same proof framework, as she had done five days earlier at our first meeting (Figure 1).

Theorem: Let A, B, and C be sets. If $A \subseteq B$, then $C - B \subseteq C - A$. Proof: Let A, B, and C be sets. Suppose $A \subseteq B$. Suppose $x \in C - B$. So $x \in C$ and x is not an element of B.	Thus $x \in C$ and x is not in A. Therefore x is in $C - A$. Therefore, $C - B \subseteq C - A$.

Figure 1. The proof framework that Alice produced on the middle blackboard.

Then Alice stopped and after a long silence of 65 seconds, much to our surprise, said, "I have a question for you. I find it very difficult to see the framework. Let me show you how I do it, because somehow I get confused with the framework." We asked her what it was about the framework that was confusing, but she seemingly could not put it into words. So we encouraged her to write the proof the way she preferred. Thus, on the left board, Alice began to write the proof in her own way in top down fashion (Figure 2). She then paused for 15 seconds, and said, "We need to have one more," and wrote into her proof attempt, **"and $x \in A$"** immediately below **"$x \in C - B$"**, indicating with a caret that **"and $x \in A$"** was also part of her supposition (Figure 3). Then, after a 35-second pause, she added to her proof attempt, "**Since $x \in A$ and A is a subset B. Then $x \in B$.**" Shortly thereafter, Alice quietly said, "Oh, a contradiction". This was followed by, "Yeah, 'cause x doesn't belong to B. Yeah, problem here."

Theorem: Let A, B, and C be sets. If $A \subseteq B$, then $C - B \subseteq C - A$. Proof. Let A, B, C be sets. Suppose A is a subset of B. We need to prove that $C - B$ is a subset of $C - A$. Suppose $x \in C - B$. We need to prove that $x \in C - A$.

Figure 2. Alice's attempt at constructing a proof in her own way.

Then, after a ten second pause, Alice said, "The problem is right here, isn't it?" pointing and underlining "***B***" and the statement "**and $x \in A$**." We asked, "And what do you think that problem is?" Alice replied, "I assumed that [pointing to **"and $x \in A$"**], but I do not know. I only know this [pointing to **"A is a subset of B"**]. We replied, "So that's a good point you've made."

Theorem: Let A, B, and C be sets. If $A \subseteq B$, then $C - B \subseteq C - A$. Proof. Let A, B, C be sets. Suppose A is a subset of $\underline{\boldsymbol{B}}$. We need to prove that $C - B$ is a subset of $C - A$. Suppose $\boldsymbol{x \in C - B}$. We need to prove that $x \in C - A$. **<u>and $x \in A$</u>.**

Figure 3. Alice's adjustments to her proof attempt, done in her own way.

After that, for a few minutes, we talked about the structure of proofs, and why we use proof frameworks. Then we asked Alice to elaborate on why **"and $x \in A$"** is a problem. She said, "I didn't write it right. I should have said here [pointing to the blank space to the left of **"and $x \in A$"**] I'm going to make an assumption like **'Suppose x belongs to the A'**, and then since x belongs to the A and I know that A is a subset of B, **then the x will belong to the B**." She continued, "I also know that x belongs in the $C - B$, because I said it earlier. Then x belongs to the C but **x does not belong to the B**." To which one of us replied, "And then you said something. I thought I heard you say the word **'contradiction'**." Alice explained, "Yeah, I got a contradiction because then I'm saying here [pointing to the board] the x belongs to the B, and the x doesn't belong to the B." We agreed, and she offered, "That assumption [pointing to **"and $x \in A$"**] was bad." We then reiterated why proof frameworks are structured the way they are, and suggested that we could take Alice's original framework (Figure 1) and what Alice had written on the left board (Figure 3), and change the order to write a proof. We proceeded to help Alice do this.

Subsequent Meetings with Alice

As the meetings went on, we observed that Alice became very methodical in her approach to proving, and also somewhat more accustomed to writing proof frameworks. We hypothesize this was because of her technical work experience and perhaps because of her natural tendencies. By the 12th meeting, Alice had developed the following pattern of working: She would write the statement of the theorem to be proved on the middle board, then look up in the course notes the definitions of terms that occurred in the theorem statement, write them exactly as stated on the left board, and use the right board for scratch work. Indeed, during the 12th meeting, when she got to the theorem, "Let X, Y, and Z be sets. Let f: $X \rightarrow Y$ and g: $Y \rightarrow Z$ be 1-1 functions. Then $f \circ g$ is 1-1," she wrote the first and second level frameworks ostensibly on her own, and with some guidance from us, completed the proof and read it over for herself aloud.

By the 19th meeting at the end of the first semester, Alice was more fluent with writing proof frameworks than on the 12th meeting, and she had adopted the technique of writing definitions on the left board and changing the variable names to agree with those used in the theorem statement – all without prompting from us. This is remarkable as our experience has been that many students do not change variable names in definitions even when we suggest doing so, and this can often lead to difficulties. Alice continued meeting with us and working on the course notes at her own pace during the second semester. We plan to continue our analysis of those videos for Alice's continued progression.

Summary of Results

Alice came to us apparently with a reasonable undergraduate mathematics background, some of which she had forgotten. At the first meeting we explained the use of proof frameworks and our rationale for using them, and she practiced producing several of them. However, at the second meeting she told us that she found this way of working confusing. When she attempted her own alternative method of proving, she got into difficulty, and as a result, was more willing to try using proof frameworks again. Over the course of our subsequent meetings that semester, Alice became fluent with writing both first and second level frameworks, and adopted a methodical way of working. As time went on, she was able to complete proofs with less guidance from us. Indeed, she often mainly required some help with the problem-centered parts of proofs. In the following semester, she continued meeting with us and working on the course notes. We feel that, by the end of the second semester, she had developed a sense of self-efficacy (Bandura, 1994, 1995) regarding her proving ability and expect to document that further.

Implications

The initial tendency of many university students to write proofs in a top-down fashion tends to fade after sufficient exposure to writing proof frameworks. One might ask where this tendency comes from. According to Nachlieli and Herbst (2009), it is the norm among U.S. high school geometry teachers to require students, when doing two-column proofs, to follow every statement immediately by a reason. This implies top-down proof construction. However, as noted previously, automating the actions required to write the formal-rhetorical part of a proof can allow students to "get started" writing and exposes the "real problem" to be solved in order to complete the proof (Selden & Selden, 2009b). For this, persistence and self-efficacy are needed.

Discussion Questions

1. What more should we look for when we analyze the second semester videos?

2. In our experience, mathematicians just know how to structure proofs (e.g., including how proofs can begin and end). Apparently, they have tacitly learned this, as well as the importance of "knowing where they are going" (e.g., unpacking the conclusion). How and when do mathematics majors learn this, when not introduced to doing so explicitly via an inquiry-based course like ours?

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RUME- and Non-RUME-track students' motivations of enrolling in a RUME graduate course

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The purpose of this ongoing study is to investigate students' motivations in taking a graduate-level RUME course. Seven individual semi-structured interviews were conducted with graduate students enrolled in a RUME course at a large Midwestern university that has a RUME Ph.D. option in the mathematics department. Our analysis of those interviews utilized two theoretical frameworks: Self-Determination Theory (Ryan & Deci, 2000) and Hannula's (2006) needs and goals structure. Preliminary analysis of the interviews indicates that non-RUME-track students are extrinsically, need-motivated, while RUME-track students are intrinsically, goal-motivated when taking a RUME course. The researchers conjecture that knowing what influences non-RUME-track students may aid in closing the gap between the mathematical and RUME communities.

Introduction

There is a limited amount of research with regard to motivation in mathematics education (Wæge, 2009). Particularly, in our search, there appears to be little research regarding motivation of future mathematics educators at the tertiary level (e.g., Herzig, 2002). Hannula (2004) defined motivation to be "a potential to direct behaviour that is built into the system that controls emotion. This potential may be manifested in cognition, emotion and/or behaviour." According to Wæge (2009), motivation can be influenced by teachers: "students' motivation for learning mathematics, although it is considered relatively stable, can be influenced by changes in the teaching approach" (p. 90).

Pedagogy courses are sometimes offered within mathematics departments to help shape and educate graduate students who will go on to teach at the college level. These courses are sometimes referred to as Introduction to RUME courses. The purpose of this ongoing study is to investigate students' motivations in taking a graduate-level RUME course. In particular, our research goals for the current study are (a) to classify students' motivations for enrolling in and participating in the course, and (b) to investigate what the students took away from the course with respect to their academic goals, their career goals, their future course selections, and/or their current teaching practices. We are especially interested in the similarities and differences between RUME- and non-RUME-track students with respect to these questions.

Theoretical Perspective

Self-Determination Theory (SDT) is a general motivation theory that focuses on psychological needs (Ryan & Deci, 2000). They distinguish between psychological needs by looking at intrinsic and extrinsic motivation. Intrinsic motivation is defined as "doing an activity for the inherent satisfaction of the activity itself" while extrinsic motivation is "the performance of an activity in order to attain some separable outcome" (Ryan & Deci, 2000, p. 71). From their studies, Ryan & Deci developed the Self-Determination Continuum (see Figure 1).

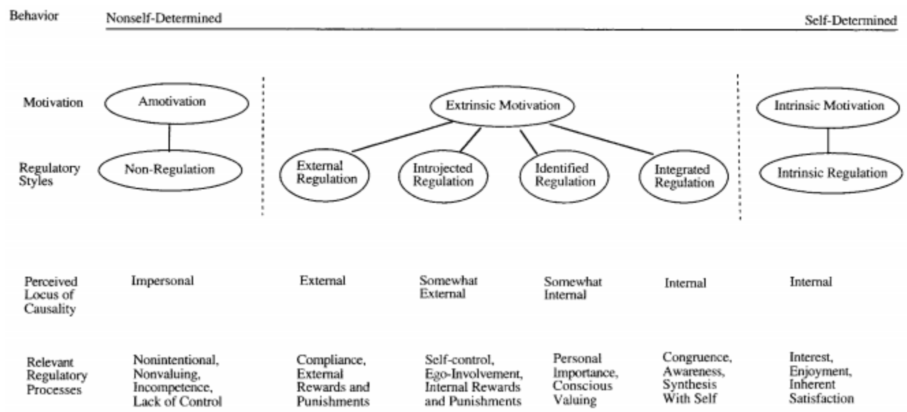

Figure 1; The Self-Determination Continuum (Ryan & Deci, 2000, p. 72)

Hannula (2006) says that "as a potential, motivation cannot be directly observed. It is observable only as it manifests itself in affect, cognition, and behaviour" (p. 175). He gives the examples of motivation being observable through beliefs, values, and emotional reactions. Needs (e.g., "autonomy, competency, and social belonging" (Hannula, 2006, p. 167)) which may transfer to goals, are used to structure this potential. Hannula's (2006) needs and goals structure, combined with Ryan & Deci's (2000) intrinsic and extrinsic framework, together will inform our analysis of the data.

Methodology

Seven individual semi-structured interviews up to 30 minutes in length were conducted. The individuals were students from a large Midwestern university who were enrolled in an Introductory to Math Pedagogy Research course. Four participants were interested in a RUME-track Ph.D. in mathematics. The class met for fifty-minute class periods three days a week. A research team member who was not enrolled in or teaching the course recruited the participants and conducted the interviews. In these interviews, students were asked about their motivations for taking the course, how those motivations have changed over time, how mathematics and RUME are related, and what they expect to leave the course with.

Preliminary Results

"I will admit that one primary reason is for the RUME teaching certificate that you get when you get your Ph.D. However, I am interested in the way that students learn math." –Student G (Non-RUME-track)

"My real passion is in teaching and in not so much teaching but in researching ways to create better teachers." – Student C (RUME-track)

Student G is expressing intrinsic motivation with the potential being rooted in needs while Student C is expressing extrinsic motivation with the potential being rooted in goals. A preliminary conjecture is that the non-RUME-track students' motivations are extrinsic, while the RUME-tracked students are more intrinsic. The potential motivators between the RUME-track students might tend toward goals, while the non-RUME-track students might tend toward competence, i.e. needs. Through this study, the researchers hope to gain a better understanding of why students take RUME courses and what they get out of them. Specifically, knowing the influences of non-RUME-track students can aid in closing the gap between the mathematical and RUME communities. Future research should include why non-RUME track students *do not* take RUME or pedagogy courses when given the opportunity.

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How Calculus students at successful programs talk about their instructors

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The CSPCC (Characteristics of Successful Programs in College Calculus) project was a 5-year study focused on Calculus I instruction at colleges and universities across the United States with overarching goals of identifying the factors that contribute to successful programs. In this poster, we draw from student focus group interview data collected from schools that were identified by the CSPCC project as being successful. The analyses we will present in this poster will characterize the ways in which calculus students talk about their instructors in an attempt to understand how their perceptions shape their experience.

Key words: Calculus, Student Perception, Thematic Analysis, Instructors

Over the past decade, numerous reports point to the need for national efforts to increase the number of students pursuing and professionals with degrees in science, technology, engineering, and mathematics (STEM) fields (see for example NSB, 2007; PCAST, 2012; Thomasian, 2011). According to the PCAST report (2012) increasing the retention rate of the students who enter college intending to major in a STEM field has the potential to significantly decrease the gap between the number of STEM degrees produced and the projected number of STEM degrees needed to sustain the United States position in the global market. While there are many reasons students leave STEM fields, there is a growing body of research that suggests that intending STEM students are switching out of STEM fields due to experiences in their introductory mathematics courses (Ellis, Kelton, & Rasmussen, 2014; PCAST, 2012; Rasmussen & Ellis, 2013), including experiencing poor instruction (Bressoud, Mesa, & Rasmussen, 2015; Seymour & Hewitt, 1997). In the United States each year over 300,000 students enroll in tertiary Calculus, many of which are just beginning their post-secondary education (Blair, Kirkman, Maxwell, 2013; Bressoud, Carlson, Mesa, & Rasmussen, 2013). To this end, we seek to better understand student experiences in successful Calculus courses by answering the question, *how do students in successful Calculus programs talk about their instructors?*

Methods

The CSPCC (Characteristics of Successful Programs in College Calculus) project was a 5-year study focused on Calculus I instruction at colleges and universities across the United States with overarching goals of identifying the factors that contribute to successful programs. The study consisted of a national survey conducted in fall 2010, followed by explanatory case study visits at seventeen institutions that were identified as successful because of student persistence (continuing to the next course in the calculus sequence) and reported increases in students' interest, confidence, and enjoyment of mathematics as a result of taking Calculus 1.

During site visits the research team conducted semi-structured student focus group interviews with current Calculus students in which they were given an opportunity to discuss various course components, their instructor, and overall course experience. We began data analysis by reading the interviews in their entirety and then choosing a subset of interview questions we felt were most relevant to our research goal. This subset of questions included:

- What types of things happen in class that help you learn calculus content?
- What would you say is your instructor's attitude towards calculus?
    - Does your teacher seem to care about your learning?
    - Does your teacher think students are capable of understanding calculus?

  - Do you think that this is typical of teachers in this math department?
- What do you think makes this program special?

Ongoing thematic analysis (Braun & Clarke, 2006) is being conducted on student responses from this subset of questions to identify overarching ways in which students at these institutions talk about their instructors. In the following section we highlight some initial themes that have emerged from our analysis.

Initial Findings

Currently our findings include three distinct perceptions of calculus instructors and their roles/characteristics in the classroom: (1) Students report instructors overwhelming *helpfulness* as an attempt to directly aid in students academic success; (2) a generally *friendly* demeanor; and (3) the instructor promoted an encouraging *atmosphere* in the classroom where students can interact with mathematics. To illustrate these findings we present experts from student interviews in Table 1.

Table 1

Emerging Themes Regarding How Students Talk About Instructors

Preliminary Theme	Institutional Level	Excerpt
Helpfulness	Bachelors	She's willing to help you as much as she possibly can if you're willing to try.
	Masters	She actually loves math so she wants to do everything possible for us to love math. She tries absolutely as hard as she can.
Friendliness	Bachelors	She's never condescending.
	Doctoral	I went to his office hours and he's really friendly and it makes it a lot easier to actually enjoy doing the math.
Great atmosphere	Bachelors	… Ms. M is interested in us doing well so it's a great atmosphere. That really helps… I mean you definitely have down there that the teacher definitely helps to make the experience, right?
	Bachelors	He creates a very comfortable environment and he (has) a really cool way of putting concepts together and making it connect with everything.

Conclusion

Overall students in successful Calculus programs speak highly of their experiences in the classroom and with the instructor. While analysis is still ongoing, one particularly interesting finding is the difference in the manner in which students at various institutional levels speak about their instructors. For instance, at bachelors granting institutions students tend to speak about their instructors with a very familiar tone while students at doctoral granting institutions give a real sense of distinct between them and their instructors, both physically and personally. Through ongoing analysis we hope to further develop current themes, illuminate more themes, and continue to investigate differences across institutional levels.

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Student problem solving in the context of volumes of revolution

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The literature on problem solving indicates that focusing on strategies for specific types of problems may be more beneficial than seeking to determine grand, general problem solving strategies that work across large domains. Given this guideline, we seek to understand and map out different strategies students' used in the specific context of volumes of revolution problems from calculus. Our study demonstrates the complex nature of solving volumes of revolution problems based on the multitude of diverse paths the students in our study took to achieve the desired "epistemic form" of an integral expression for a given volume problem. While the large-grained, overarching strategy for these students did not differ much, the complexity came in how the student carried out each step in their overall strategy.

Key words: problem solving, calculus, volumes of revolution, epistemic games

Introduction

Helping students become proficient in problem solving, or the ability to complete a task where a solution is not immediately known, is an important goal in mathematics education (Lester, 2013; National Research Council, 2001; Schoenfeld, 1992). In the last several decades, researchers have studied problem solving and students' problem solving abilities (Lesh & Zawojewski, 2007; Schoenfeld, 1992), and one critical finding has been that it is difficult to determine general problem solving strategies that are useful for any situation. Researchers argue that coming up with a list of strategies for problem solving is difficult because lists are either too small and cannot account for all situations, or too broad such that students are left without a sufficient guide for which strategies to use (Lesh & Zawojewski, 2007). While it is difficult to find strategies to solve problems generally, some researchers have suggested that it may be more beneficial instead to develop strategies for particular types of problems (Lesh & Zawojewski, 2007; Schoenfeld, 1992).

Calculus problems involving volumes of revolution may provide just such a context, since they are not so broad that it becomes impractical to develop general problem solving strategies, yet are complex in that there is no single set procedure to find a solution and students cannot simply memorize one template to apply to all problems. Students need to choose between using the disk, washer, and shell methods and must choose whether to integrate with respect to *dx* or *dy*. Students must also be able to recall and apply knowledge from a range of different mathematical topics (e.g. integration techniques, geometry, and solving equations). Also, we have found no literature on student understanding of volumes of revolution, meaning that it is an area in need of exploration. Thus, in this paper we examine the strategies students use to solve volumes of revolution problems. In particular, this paper is meant to address the following questions: (1) What strategies did students use when solving volumes of revolution problems? (2) What particular features of the problems guided or focused the students in their problem solving strategies?

Epistemic Forms and Games

In investigating problem solving strategies for volumes of revolution, we employ the lens of epistemic games (Collins & Ferguson, 1993). Epistemic games provide a useful language for describing how students go from a *starting condition*, which in our case is the initial volume of revolution problem, to a desired outcome called an *epistemic form*. An epistemic form consists of "an external structure or representations and the cognitive tools to… interpret that structure" (Redish, 2004, p. 30). Collins and Ferguson provided examples of possible epistemic forms like lists, charts, and diagrams, and Redish added to the list things like an abacus or a graph. In our study, the specific "external structure" that makes up the epistemic form is an integral expression that is set up to match the volume of the particular solid given in a problem.

In order to advance from the starting condition to the epistemic form, one must make *moves*, which in our case consists of actions (mental or physical) that a student performs to achieve the desired form. The overall activity of taking the starting condition, making moves, and arriving at an epistemic form is called an *epistemic game* (Collins & Ferguson, 1993). Recently researchers in physics education have used epistemic games as a means to analyze how students problem solve in physics tasks (Black & Wittmann, 2007; Tuminaro & Redish, 2007), and we extend this to investigate students' problem solving strategies in mathematics contexts as well. Depending on the grain size one uses in analyzing the moves that constitute an epistemic game, the description one provides of a student's epistemic game can vary (Black & Wittmann, 2007). In this study, we examine both larger-grained and smaller-grained games, and we define a *global game* as one that describes the general moves a student makes and a *local game* as one that is played out within each of the moves of a global game.

Methods

For this study, eight students from a second semester calculus course were invited to participate in a one-hour interview regarding volumes of revolution problems. The students came from a course in which the instructor had attended to a conceptual development of the disk and shell methods and had also given examples of some cases where each method would not work, given the techniques available to the class. In order to get a range of participants, students were chosen based on their responses to four quiz problems given in class. Three students (Gabi, Doug, Trevor) got all four quiz problems correct, two students (Sarah, Frank) got three correct, two students (Bryan, Claire) got two correct, and one student (John) got only one correct. In the interview, the students were asked to set up integrals for four standard, textbook-style volumes of revolution problems, where a region bounded by certain curves is rotated around either the *x*- or *y-axis* (such as in Stewart, 2012, p. 430-445). The problems indicated in words to take the region bounded by the following curves and rotate it around the specified axis, and to determine the volume of the generated solid: (1) $y=\sqrt{1-x}$, $y=2x$, $y=0$ around *x*-axis; (2) $y=\sqrt{1-x}$, $y=2x$, $y=0$ around *y*-axis; (3) $f(x)=4x-\frac{1}{2}x^4$, $y=0$ around *y*-axis; (4) $f(x)=4x-\frac{1}{2}x^4$, $y=0$ around *x*-axis. Graphs of the functions in each problem were also provided to the students. The students were asked to discuss their thinking aloud and the interviewer asked clarifying questions while the student worked.

The videotaped interviews were first watched to record the steps the individual students took in setting up the integrals. These steps were used to define a set of "moves" for the global epistemic games the students played. Once these moves had been described, the videos were watched again in their entirety to code each student's work according to these moves. These gave rise to patterns in the types of global games the students played. Next, each global move was analyzed to determine the specific local game each student played within each move. The local games were then compared and contrasted across students.

Results

In the following subsections we discuss the global games we observed and then present our findings on the local games played within each global move. We acknowledge here that slight differences occurred among some students, but the following is an attempt to synthesize the main patterns found in the data.

Global games

Six of the eight students played quite similar global games, with two variations (compare Figures 1a and 1b). The main difference seemed to be whether the students began setting up the integral before finding the bounds for the integral, or vice versa. Each move within these global games are explained in the following subsections as we examine each move more closely. Two students played different global games, that we do not portray in Figure 1. When solving shells problems, Gabi played the game in Figure 1a, but when solving disk/washer problems, she skipped the "visualizing slice" and "finding volume of slice" moves, since she seemed to have the related formulas memorized and could use them effectively. The other student, Bryan, rarely got past the visualzing volume move and had difficulty in making further progress.

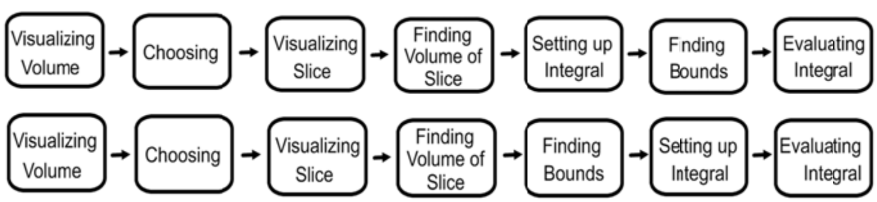
(a)
(b)


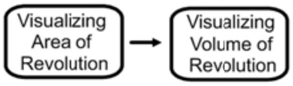
Figure 1. Global games played by (a) Claire, Frank, Gabi, John, Sarah, and (b) Doug, Trevor

Local games

While the majority of students had very similar global games, variations of students' problem solving strategies were more evident in the local games played within each move, especially as seen in the *choosing* game. In the following subsections, we explore each move in more detail.

Visualizing volume

The visualizing move involved an attempt to visualize the three-dimensional object whose volume was being calculated. There seemed to be two types of visualizing local games played by the students. All of the students started this local game by visualizing the area that would be revolved around one of the axes. For most students, this involved determining the curves that bounded the region and shading the region between the curves. All but two students then proceeded to visualize the revolution of this region in some way. The majority of the students drew a reflection of the area on the other side of the axis of rotation and gave descriptions of what the solid would look like. It is interesting to note, however, that the two students (Frank and Trevor) made no attempt to visualize the region's actual revolution. These two students also had more difficulty in setting up the integral expressions than some of the other students. This suggests that visualizing the rotated solid may be a factor that influences students' ability to solve these types of problems.

(a) 

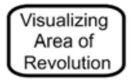
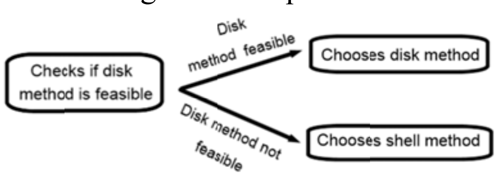
(b) 

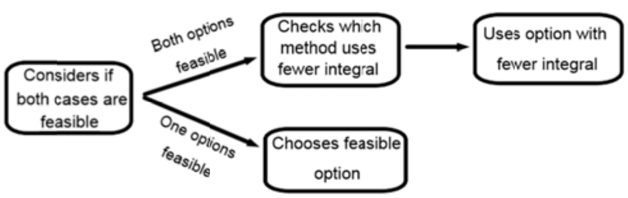
Figure 2. Visualizing game for (a) Bryan, Claire, Doug, Gabi, John, Sarah, and (b) Frank, Trevor

Choosing

The greatest variety among the students occurred in the *choosing* move, which involved determining whether to use disks, washers, or shells methods. While many of the students shared similar reasonings for choosing to use one method or another, each student seemed to play their own unique local game during this move. The most common move in the choosing local game was checking the feasibility of the disk/washer or shell methods, or in other words checking whether the resulting integral could *not* be evaluated given the techniques available to the students. Six of the eight students made the move "check feasibility" during the choosing local game, with four of the students' making it their first move (see Figure 3). In fact, Trevor based his entire decision solely on whether the disk/washer method was feasible or not (see Figure 3a). Claire also started off checking feasibility, but if both methods were feasible she would also check the number of integrals each method used (i.e. whether the region had to be split up due to intersection points or other features). In Doug's game, if both methods were feasible, he would look check to see if there was a square root term, and if so, he said he would always choose disks/washers, since squaring the function would eliminate the square root. If there was not he would choose the method that requires the fewest integrals (see figure 3c). Sarah had the most complicated choosing game (see figure 3d). If both methods were feasible she would randomly choose a method and keep going with that method until she ran into a problem. Like Doug, the number of integrals and square root terms seemed to be important in her choosing local game.



(a)



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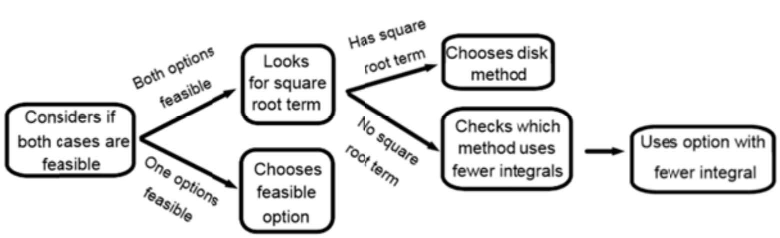

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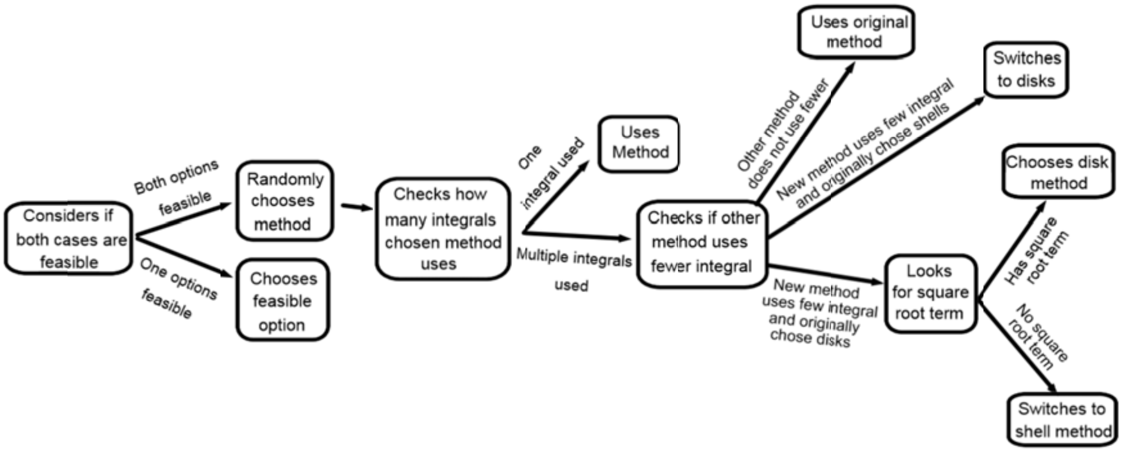
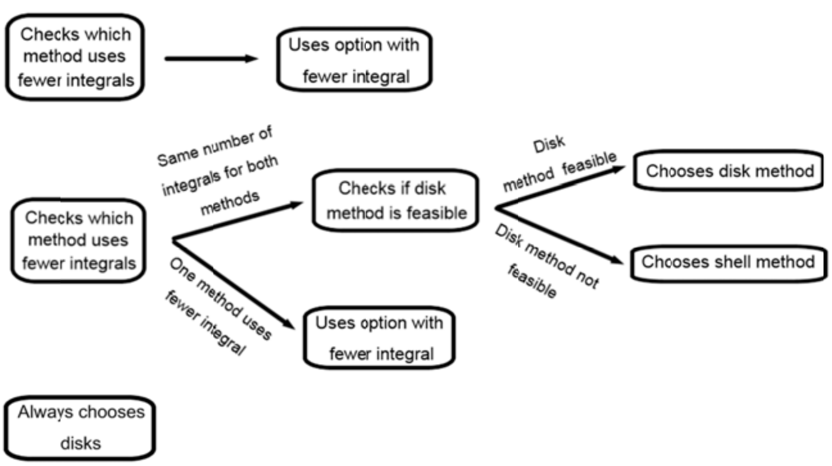


(d)
Figure 3. Choosing local game for (a) Trevor, (b) Claire, (c) Doug, and (d) Sarah

As mentioned in the previous paragraph, the number of integrals required (i.e. whether the region had to be split up) was a defining factor for many students in deciding whether to use the disk/washer or shell method. Five of the eight students made the move "check number of integrals," with two of them making it their first move within this local game (see Figure 4). In fact, determining "number of integrals" was the *only* move made by Gabi in this local game. If disks/washers or shells resulted in the same number of integrals, Frank would always default to the disk/washer method (see figure 4b), using the shell method only if the disk/washer method was not feasible. Another student who preferred disks/washers to shells was Bryan. In fact, when asked what method he would likely use in solving the problems, he indicated he would *always* choose disks or washers (see Figure 4c) because he was more comfortable working with them.



(a)
(b)
(c)
Figure 4. Choosing local game for (a) Gabi, (b) Frank, and (c) Bryan

John had a rather unique choosing local game (see Figure 5). His initial strategy depended mainly on whether the solid of revolution had a "hole" through the middle of it or not. If the solid did have a "hole" in the middle he would always try to use the shell method first. If not, then he would always try using the disk method first.

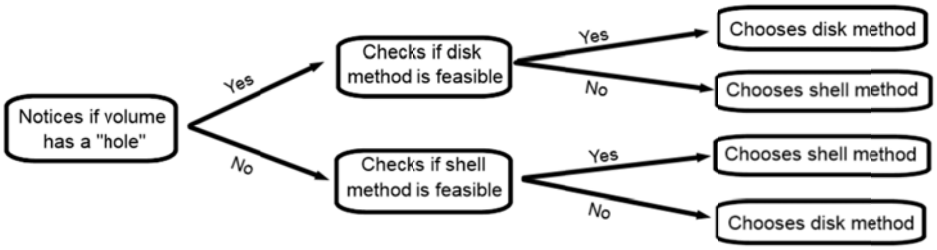


Figure 5. Choosing local game for John

Visualizing and finding the volume of one slice (disk/washer)

In the "finding the volume of one slice" game for disks/washers, which consisted of determining the integrand and differential to use for the integral, several students seemed to take the same general approach, though some students skipped some of the steps that others explicitly took. Sarah and Doug explicitly found all of the parameters needed for obtaining the volume of the disk, including the radius, the area of the disk's face, and the thickness/differential. They then put all this information together in a volume formula for one disk. Trevor also focused on finding different parameters involved in find the volume of the disk, but never explicitly attended to finding the thickness/differential of the disk, only adding the differential in later when he began to write the formula. Four of the students did not explicitly attempt to find any of the parameters, but would begin directly applying a memorized formula for the volume of disks or washer. These are summarized in Figure 6.

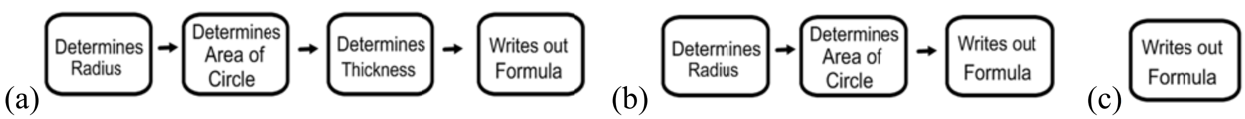


Figure 6. Finding volume of one slice local game for (a) Doug, Sarah, (b) Trevor, and (c) Claire, Frank, Gabi, John

Visualizing and finding the volume of one slice (shells)

The students all followed essentially the same basic local game when determining the integrand and differential for the shell method (see Figure 7). They would draw a generic shell and then draw a rectangular prism that represented a shell being cut and flattened, as had been shown in class during the conceptual development of the shell method. Students would then find the radius, circumference, height, and thickness/differential (although the ordering of these differed from student to student) and put these together to write down the integrand and differential. The fact that students always seemed to explicitly find the different parameters associated with shells, while being able to skip steps or use the formulas for disks, leads us to conclude that students were generally more familiar or comfortable with the disk method.

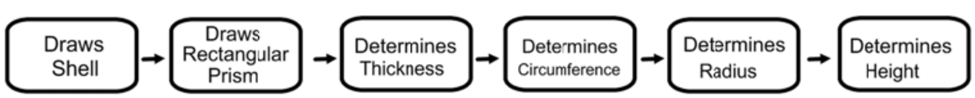


Figure 7. Finding volume of one shell local game played by most students

Setting up the integral and finding bounds

The setting up the integral and the finding bounds local games, which we discuss together due to space limitations, consisted of writing out the actual integral expression for the volume of revolution problem, with some moderate variation existing in how the students played this local game (see Figure 8). Four students first found all the information they needed to set up the integrand, began to write the integral expression, and then would return to the problem to find limits of integration. Sarah, however, began setting up an integral *before* determining any of the parameters she would needed and would then fill in the integral exprsesion as she found more information. Doug and Trevor on the other hand waited until *all* necessary information had been found before they began working on writing out the integral expression.

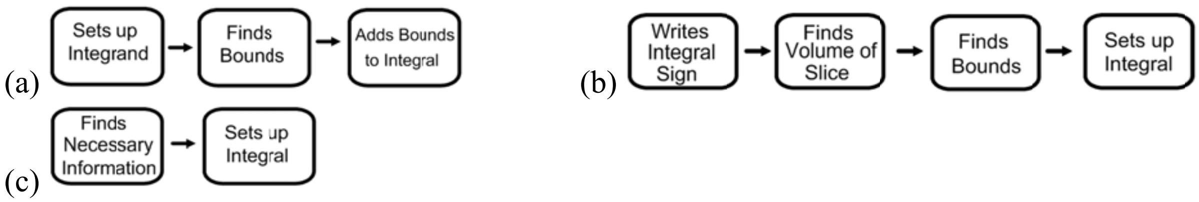


Figure 8. Setting up the integral local game played by (a) Claire, Gabi, Frank, John, (b) Sarah, and (c) Doug, Trevor

Discussion

In this study, we have analyzed and mapped out different epistemic games that students played in the volumes of revolution problems. We identified both global games for the entire problem solving process, as well as the local games that made up the individual moves of the global games. In this way, we have attended to both the larger-grained and smaller-grained strategies enacted by the students in this study as they solved volumes of revolution problems. The most striking outcome of this analysis is how it complexifies, to us, the nature of solving volumes of revolution problems. What could easily be viewed as a problem solving context with only six particular outcomes (one each for associating the disk, washer, and shell methods to a choice of *dx* versus *dy* integration), has been shown to have a much richer variety. The students in this study employed a range of moves to create the desired epistemic form of a completed integral expression. Especially in the *choosing* local game, we were surprised by the amount of diversity and complexity in how the students made moves toward the epistemic form.

Interestingly, however, this range did not seem to occur at the global level, though we acknowledge that this may be due to the common classroom experience for the eight students. Our study found that six of the the students played global games with essentially the same moves and with essentially the same move ordering. The only real difference was whether students would find all necessary information to set up the integral first, or whether they would begin to set up the integral and then go back to the problem to find missing information. We note that this type of overall global game resembles the *pictoral anaylsis* epistemic game described by Tuminaro and Reddish (2007) in physics problem solving contexts. As such, it is possible that learning to solve volumes of revolution problems could have potential effects on some aspects of students' physics problem solving.

We also conclude that solving volumes of revolution problems may be at the right level of complexity according to the problem solving literature, which suggests that contexts that are neither too broad nor too narrow may provide the best setting for developing problem solving strategies (Lesh & Zawojewski, 2007). The volumes of revolution problem context was clearly shown in this paper to be anything but trivial, but is also confined to a more narrow domain, so that strategies *can be* (and were!) developed by students. No student had a memorized template for all volume problems, meaning that they all engaged in problem solving at some level.

In regards to being able to solve volumes of revolution problems specifically, our data suggests that some moves may be especially useful, including visualizing the volumes of revolution, being equally comfortable with all available methods, checking to see which methods are feasible, determining which method uses fewer integrals, and examining which methods involve simpler algebra. Consequently it may be useful for calculus instructors to spend time developing some of these local games. For instance, an instructor may wish to have their students draw out solids of revolution so that they can become comfortable visualizing the desired object, or to have students examine the number of integrals required to work out a problem with respect to *dx* versus *dy*. While this list certainly does not contain all useful strategies in solving volumes of revolution problems, we believe that understanding which moves are useful can help students develop flexible problem solving strategies regarding volumes of revolution.

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Equity in Developmental Mathematics Students' Achievement at a Large Midwestern University

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With so many students entering college underprepared for the mainstream sequence of mathematics courses, mathematics departments continue to offer developmental or remedial courses with innovative methods of delivery. In order to support students in their college education, researchers continue to investigate the effectiveness of undergraduate remediation programs with mixed results. This paper provides quantitative data from an NSF-funded project from a large Midwestern university over three years of a developmental mathematics course. Pre- and post-measures show that both urban and African-American students benefited the most from supplemental instruction in contrast to the online-only format. Based on these results, I offer recommendations for undergraduate mathematics departments to support equitable opportunities for marginalized students ensuring a successful developmental mathematics program.

Keywords: Developmental mathematics; Equity and diversity

Mathematics departments across the country offer developmental courses or remediation to support the entering students they deem unprepared to meet the entry standards of their introductory courses. Researchers estimate more than a third of all incoming freshman sign up for a developmental course upon entering college often resulting in an over-abundance of students enrolling in developmental courses (Bettinger et al., 2013; Scott-Clayton, Crosta, & Belfield, 2014). With the cost of these programs nationwide for institutions of higher learning in the billions of dollars (Bettinger, Boatman, & Long, 2013), mathematics departments search for innovative solutions to ensure they can afford to support the education of as many students as possible.

The variety of delivery methods that mathematics departments use to provide content and instruction often lack in research-based teaching methods or resources creating an inequitable environment in terms of educational opportunity in developmental programs. TAs or faculty with little to no training in teaching strategies typically instruct these courses coupled with the over-representation of minorities in developmental courses can potentially cause students more harm in the first years of college (Attewell et al., 2006; Larnell, 2013). Designed as gateways to future mathematical success for all students, Bonham and Boylan (2011) acknowledged that "developmental mathematics as a barrier to educational opportunity represents a serious concern for the students as well as higher education policy makers" (p.2).

Considering these concerns, our NSF project[1] team reviewed three years of quantitative data collected from the developmental mathematics program at a large Midwestern university to compare the effects of various instructional methods between groups of students with similar backgrounds. Results in this paper compare how urban, low-income, and African-American students (who made up approximately one-eighth of the entire population of students enrolled in the online math course over the three year period) faired in an online-based tutoring program

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with or without additional face-to-face instruction faired compared to each other. The success of the urban and African-American students with additional face-to-face instruction compared to their peers provides a window into how researchers can evaluate and mathematics departments make changes to their developmental program to ensure more equitable opportunities for all students. I will first provide some background literature to further situate this study.

Literature Review

Current issues surrounding developmental mathematics consist of how mathematics departments select students and deliver content to ensure students' successful completion of their degree requirements and how researchers evaluate the effectiveness of programs providing reliability and generalizeability. Scott-Clayton, Crosta, and Belfield (2014) found that placement methods, typically involving the use of an exam, fail to correctly identify the students who need remediation. Other community college programs have experimented with incorporating high school transcript data (Jackson & Kurlaender, 2013); however, Bettinger et al. (2013) acknowledged that tests and high school performance still ignore the unseen qualities of students that influence their success in their first year of college, e.g. study habits and perseverance. Researchers continue to search for the right combination of indicators for placement and eventual success in developmental mathematics course (Scott-Clayton et al., 2014).

Mathematics departments are often limited by their resources and materials to offer their students research-based mathematics resources and instruction. Bonham and Boylan (2011) argued that successful programs incorporate technology and innovative materials for the classroom, extracurricular resources for students, and professional development for instructors. Since not all institutions can provide these opportunities for their students, innovative solutions arise that include online resources or tutoring programs paid by student tuition money. Researchers continue to search for opportunities for mathematics departments to provide resources and instruction to support the students who need them most.

In addition to the selection of students and the delivery methods, scholars have discussed methods to evaluate developmental mathematics programs to ensure reliability and provide higher education policy makers justification to enact institutional changes. Bettinger et al. (2013) noted that the amount of variation due to geography, student backgrounds, and other factors that currently go unseen that cannot be measured quantitatively or provided on a high school transcript and called on more studies to explore this variability. In providing data from particular geographic areas and groups of students, researchers can begin to explore similarities and differences to facilitate discussion around a complex solution to the complex problem of inequitable opportunities in developmental mathematics education.

Considering these discussions, the NSF-funded project investigated quantitative data from a census of all students in three years of the developmental mathematics course at a large Midwestern university to answer the following question:

- What effects does a supplementary face-to-face instruction in a developmental mathematics course have on different subpopulations of students' performance and future participation in mathematics?

Methods

I present quantitative data over three years of an NSF-funded project comparing the various methods of delivery of a developmental mathematics course at a large Midwestern university. Approximately 800 students each year enrolled in the online version of the developmental mathematics course, as determined predominately by placement exam score. The program ALEKS is the curriculum for the online course. Freshman identified by advisors as at-risk for failing first-year courses enroll in groups of around 15-20 students in a supplementary face-to-face section that meets twice a week with each class lasting two hours. Each class is taught by mathematics graduate students with the exception of one section taught by senior pre-service mathematics teachers as part of an NSF-funded project in concert with the teacher education department. While the sections led by mathematics graduate instructors engaged in material directly supporting the students' work on the ALEKS program, the seminal section with pre-service mathematics teachers engaged in a curriculum and instructional methods grounded in mathematics education research.

The project investigated the effect of taking any of the supplementary face-to-face sections on students' success in the online developmental course and on their performance in subsequent math courses. In this paper, I present comparisons of the percent difference between the means of quantitative pre-measures (ACT mathematics score, university placement exam, ALEKS pre-score) and post-measures (ALEKS post-score, final exam, final grade) to determine differences in outcomes between African-American, urban, and low-income[2] students who enrolled in the online-only version of the course and those who enrolled in the supplementary face-to-face sections. These particular students were selected based on the large percentage of students self-reported as African American and low-income hailing from the large urban area nearest to the university (75% and 90% respectively) and the teacher education department's interest in potential future summer enrichment programs for students from this area. I also provide similar comparisons between the enrollment and grades in the pursuant credit-bearing mathematics course for the students in the first year of the data set.

Results

The data for each of the three years demonstrates that African-American, urban, and low-income students who took the supplementary face-to-face course made significant gains in the online course compared to their peers who were not enrolled in the face-to-face enrichment course.

Table 1 includes the percent difference between the face-to-face and online only students within the subpopulations of students from the large urban area, low income, and African-American separately. Overall the data demonstrates that the face-to-face students started slightly behind and finished significantly ahead in both the urban and African-American groups and started significantly behind and finished slightly ahead in the low-income group. Considering this data also represents census data for a university over three years, this data shows the supplementary face-to-face instruction associates with a significant gain in post-measures across each of these subpopulations.

[2] Ethnicity was self-reported while the urban and low-income information was provided by researchers' map of the state near the urban area in question in conjunction with income information by zip code provided by the University of Michigan's Institute for Social Research: http://home.isr.umich.edu/.

Table 1 Percentage Difference between Face-to-Face and Online Group

	Pre-measures			Post-measures		
Urban	ACT Math	Placement	ALEKS Pre	ALEKS Post	Final Exam	Grade
2012	-2.6%	-6.4%	2.5%	***13.7%*****	***26.9%*****	***21.5%*****
2013	***-6.8*****	-6.3%	-11.3%	-2.2%	***16.7%*****	***10.6%****
2014	-3.7%	-2.0%	-17.4%	2.6%	9.3%	***9.7%****
Low income	ACT Math	Placement	ALEKS Pre	ALEKS Post	Final Exam	Grade
2012	***-5.6%*****	***-10.0%****	3.3%	5.5%	8.1%	7.6%
2013	***-6.2%****	-4.3%	-6.3%	0.4%	***21.2%*****	***12.5%*****
2014	***-4.5%****	1.0%	***-17.0%****	2.2%	2.4%	6.5%
African-American	ACT Math	Placement	ALEKS Pre	ALEKS Post	Final Exam	Grade
2012	0.7%	0.5%	-2.2%	***11.1%*****	***15.8%*****	***12.5%****
2013	***-5.5%****	***-12.9%*****	-14.3%	-2.4%	***15.3%*****	***9.1%*****
2014	-1.7%	-0.5%	-5.4%	7.0%	***17.9%*****	***14.9%*****

Note: $* p < 0.05$, $** p < 0.01$

Table 2 Percent Enrolling and Passing Next Math Course in 2012

		N	N enroll	% Enroll	N pass	% Pass
Urban	Face to face	47	35	***74.5%*****	18	***38.3%****
	Online	55	26	47.2%	14	25.5%
Low income	Face to face	75	52	***69.3%*****	26	34.7%
	Online	122	69	56.6%	41	33.6%
African American	Face to face	84	64	***76.2%*****	31	***36.9%*****
	Online	113	63	55.8%	29	25.7%
All students	Face to face	171	130	***76.0%*****	68	39.8%
	Online	619	394	63.7%	221	35.7%

Note: $* p < 0.05$, $** p < 0.01$

Table 2 shows the percent of students who enrolled and passed the proceeding mathematics course offered by the mathematics department within each of the subpopulations. Overall the data demonstrates that a significantly higher percentage of students enrolled in the next math course across not only all subpopulations but also the entire population of students. In addition, significantly higher percentage of students in the urban and African American subpopulations passed the course (receiving GPA>2.0).

A common theme of both of these tables indicates that the supplementary face-to-face section benefits not only students' performance in the online course for these subpopulations, but also potentially contributes to success in the next math course.

Discussion and Future Directions

These results provide an example of Bettinger et al.'s (2013) call for researchers to compare groups of students with similar backgrounds demonstrating how characteristics through information provided by the registrar can provide an avenue for a positive change in providing students opportunities in developmental programs. Factors to consider for determining students placement that include test scores and high school transcript information (Jackson & Kurlaender, 2013) can potentially include demographic information as well.

Providing students with face-to-face instruction increased the performance of the urban and African-American subgroups in this student population. Universities that offer online-based opportunities could experience strong gains in performance by providing supplementary face-to-face instruction sections to underprivileged students. This is not to say that engaging students in these opportunities is a panacea as students come from diverse backgrounds with a variety of different ways of learning and knowing. Even with the variety of student backgrounds, the data demonstrates sub-populations of student based on demographics that benefited the most from the resources offered by their institution. Although providing students with these sections could improve gains in performance, other mathematics departments should tread carefully and provide instruction that improves students' mathematical proficiency and not knowledge of correct procedures alone (Larnell, 2013; Kilpatrick, Swafford, & Findell, 2001).

As "success" in a mathematics course goes beyond just performance on course exam, future studies could dig deeper into how the students experienced the developmental mathematics program as well as track students' success longitudinally. Although the selection of students attending this institution was not a random sample of the nation, the scale of this case provides a window into a single university over three years to anticipate similar results for other large, public universities. Other mathematics departments could then take their own nuanced steps based on their results to ensure more equitable opportunities for their students' education and ameliorate the inequities in the first year of undergraduate mathematics. Further questions continue to remain to continue the goal of spurring the growth of developmental mathematics programs across the country to meet the needs of all students entering their respective universities:

- What makes face-to-face supplemental courses more successful for students than online-only courses?
- Which universities have similar demographics as the one referenced in this report?
- How can we motivate mathematics departments to make changes supported by evidence to invest in students' developmental mathematics programs?

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Exploring student understanding of the negative sign in introductory physics contexts

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Recent studies in physics education research demonstrate that although physics students are generally successful executing mathematical procedures, they struggle with the use of mathematical concepts for sense making. In this poster we investigate student reasoning about negative numbers in contexts commonly encountered in calculus-based introductory physics. We describe a large-scale study (N > 900) involving two introductory physics courses: calculus-based mechanics and calculus-based electricity and magnetism (E&M). We present data from six assessment items (3 in mechanics and 3 in E&M) that probe student understanding of negative numbers in physics contexts. Our results reveal that even mathematically well-prepared students struggle with the way that we symbolize in physics, and that the varied uses of the negative sign in physics can present an obstacle to understanding that persists throughout the introductory sequence.

Introduction

Signed numbers carry rich information about physics contexts. A confounding feature in physics is that the operations of addition and subtraction (represented by the symbols "+" and "–") can easily be confused with the descriptors, positive and negative, that can characterize the opposite natures of some physical quantities (position, charge, velocity, etc.)

Developing flexibility with negative numbers is a known challenge in math education. Vlassis[(1)] used written diagnostic questions and interviews to investigate the understanding of negative numbers by Belgian students taking algebra. She found that in order to fully understand the concept of a negative number, students had to develop a flexibility with the various ways in which negative numbers are used in context. The most challenging context is common to physics – quantifying opposites.

Sherin[(2)] refers to quantifying opposites in physics as the symbolic form "competing terms cluster," which includes the notion of zero to represent balance, and positive and negative quantities as competing terms in an expression. This cluster is built on a stable set of coordinated resources that includes a conceptual understanding of signed numbers and zero. He observes that flexibility with this symbolic form is a feature of expert problem solving in introductory physics.

Bajracharya, Wemyss, and Thompson[(3)] investigated student understanding of integration in the context of *P-V* diagrams in introductory physics. Their results suggest an incomplete understanding of the criteria that determine the sign of a definite integral. Students struggle with the concept of a negative area, and with the concept of positive and negative directions of integration. Even for students in calculus-based physics, negative quantities pose challenges.

Experimental Design

We administered a set of three questions at the end of the fall 2015 semester in the calculus-based introductory courses in Mechanics, and in E&M; a portion of each class was given a MC version of the questions while the rest were given an open ended version and asked to explain their reasoning. Each set probes the use of a signed quantity: 1) to represent a component of a vector quantity in 1-D, 2) to quantify opposites, and 3) to represent a difference of a position dependent quantity measured at two different locations (see Fig. 1).

Discussion

Our results show that engineering students really struggle to make sense of the physics use of the negative sign in almost every context except the one that is familiar from math class.

Surprisingly after a semester of calculus-based physics, one-third of the engineering students fail to recognize the context in which they learned about negative numbers - the position on a number line (see Fig 1, Mech 3)

Figure 1: Assessment items, "Mech" was administered in the Mechanics course and "EM" was administered in the E&M course.

Mech 1: An object moves along the x-axis, and the acceleration is measured to be $a_x = -8$ m/s^2. Consider the following statements about the "–" sign in "$a_x = -8$ m/s^2". Pick the statement that best describes the information this negative sign conveys about the situation.

a. The object moves in the negative direction
b. The object is slowing down
c. The object accelerates in the $-x$-direction
d. Both a and b
e. Both b and c

Mech 2: A hand exerts a force on a block as the block moves along a frictionless, horizontal surface. For a particular interval of the motion, the hand does W = – 2.7 J of work.
Consider the following statements about the "–" sign in the statement W = –2.7 J. The negative sign means:

I. the work done by the hand is in the negative direction
II. the force exerted by the hand is in the negative direction
III. the work done by the hand decreases the mechanical energy associated with the block

Which statements are true?

a. I only
b. II only
c. III only
d. I and II only
e. II and III only

Mech 3: A cart is moving along the x-axis. At a specific instant of time the cart is at a position x = –7 m. Consider the following statements about the "–" sign in "x = –7 m". Pick the statement that best describes the information this negative sign conveys about the situation.

a. The cart moves in the negative direction
b. The cart is to the negative direction from the origin
c. The cart is slowing down
d. Both a and b
e. Both a and c

EM 1: At a location along the x-axis, the electric field is measured to be $E_x = -10$ N/C. Consider the following statements about the "–" sign in "$E_x = -10$ N/C". Pick the statement that best describes the information this negative sign conveys about the situation.

a. The test charge is negative
b. The field is being created by negative charge
c. The field points in the $-x$-direction
d. Both a and b
e. Both b and c

EM 2: Valeria combs her hair in the winter and there is a transfer of charge such that $DQ_{comb} = -5$ mC. Consider the following statements about the "–" sign in the mathematical statement $DQ_{comb} = -5$ mC. The negative sign means:
I. negative charge was added to the comb. II. charge was taken away from the comb.
III. all of the electric charge in the comb is negative
Which statements could be true?

a. I only
b. II only
c. III only
d. I and III only
e. II and III only

EM 2: In physics lab, a student uses a voltmeter to measure the voltage across the terminals of a battery. The voltmeter reads –5V.
Consider the following statements about the "–" sign in the voltmeter reading "– 5V". Pick the statement that best describes the information this negative sign conveys about the situation.

a. the voltage is in the opposite direction as the current
b. there are 5V of negative charge in the battery
c. the voltage is in the negative direction
d. the voltage at one terminal is 5V less than the voltage at the other terminal
e. this battery has negative voltage

Table 1: Response rates, Mechanics(n=310) and E&M (n=402). Correct response rate is in bold type

Choice	*Mech 1*	*Mech 2*	*Mech 3*		*EM 1*	*EM 2*	*EM 3*
a	8%	17%	6%		16%	**33%**	32%
b	26%	17%	**67%**		21%	28%	14%
c	**26%**	**23%**	6%		**36%**	18%	18%
d	6%	29%	19%		12%	15%	**33%**
e	34%	14%	2%		14%	5%	3%

Explanations (from open-ended responses, Mechanics n=85, E&M, n=138)
Excerpts to be included in poster

1. Vlassis, J., “Making sense of the minus sign or becoming flexible in ‘negativity’”, Learning and instruction, 14, pp.469-484, (2004).

2. Sherin, B. L., “How students understand physics equations”, Cognition and instruction, 19, pp.479-541, (2001).

3. Bajracharya, R. R., Wemyss, T. M., and Thompson, J. R., “Student interpretation of the signs of definite integrals using graphical representations”, AIP Conference Proceedings 1413, pp.111, (2012).

Mary, Mary, is not quite so contrary: Unless she's wearing Hilbert's shoes

Researchers (Leron, 1985; Harel & Sowder, 1998) have argued that students' lack a preference for indirect proofs and have argued that the lack of preference is due to a preference for constructive arguments. Recent empirical research (author, 2015), however, which employed a comparative selection task involving a direct proof and an indirect proof of the contraposition form, found no evidence of a lack of preference for indirect proof. Recognizing that indirect proofs of the contradiction form may differ from those that employ the contraposition, this study documents students' proof preferences and selection rationales when engaging in a comparative selection task involving a direct proof and an indirect proof of the contradiction form.

Key words: Indirect proof, Proof preferences, Proof by contradiction

It has been argued by many that indirect proofs, that is, proof by contraposition and proof by contradiction, are particularly difficult for students (Tall, 1979; Robert & Schwarzenberger, 1991) and that students' difficulties are related to a lack of preference for these forms of proof (Leron, 1985; Harel & Sowder, 1998). Several reasons for students' difficulties and lack of preference have been proposed. Tall (1979) conducted an empirical study of 37 students' levels of confusion in relation to proofs by contradiction of the irrationality of the $\sqrt{2}$ using an instrument that included the standard proof and two alternative proofs. He found that students experienced significantly lower levels of confusion with one of the alternative forms; namely, that which employed generic structures (i.e., proof structures that were not specific to the numbers used). Tall argued that use of generic proofs will aid students' understanding of indirect proofs. In a reflective account of multiple teaching experiments, Leron (1985) noted that not only are students perplexed by proofs by contradiction but that such proofs stand in contrast to much of students' mathematical activity, for they call on students to not only build up a "false world" but to destroy this world. Hence, according to Leron, students' difficulties are related to the coupling of non-constructive reasoning and a detachment from one's "real" mathematical world. Using the standard proof of the infinitude of primes, Leron reported that constructive approaches, which explore and analyze mathematical objects in their own right prior to their use as tools for obtaining contradictions, may enhance students' understanding of proofs by contradiction. Harel and Sowder (1998) have also argued that students are not convinced by proof by contradiction and lack a preference for this form of proof. Drawing of data from 6 teaching experiments they argue that students' dislike of indirect proofs represents a particular manifestation of the constructive proof scheme: a scheme in which "students' doubts are removed by actual construction of objects – as opposed to mere justification of the existence of objects" (p. 272). Lastly, Antonini and Mariotti (2008) studied students' views and production of indirect proofs. Drawing on the theory of *Cognitive Unity* and a specific characterization of *Mathematical Theorems* (Mariotti, Bartolini Bussi, Boero, Ferri & Garuti 1997), this research has sought to explore: (a) linkages between students' informal, indirect geometric arguments in technological environments and their production of proofs by contradiction; and, (b) the nature of students' difficulties with indirect proofs. Specifically, within their work a distinction is made between mathematical theories (e.g., Euclidean geometry; Riemannian geometry; Number theory) and metatheories (e.g., Standard logic, Constructive logic). Drawing on interviews with university students, Antonini and Mariotti demonstrated that students' difficulties with indirect proof may be tied to students' lack of acceptance of metatheorical properties (e.g., $P \rightarrow Q \equiv \sim Q \rightarrow \sim P$). For instance, when presented with a proof by contraposition of the statement, "If n^2 is even then n is even," students readily accepted the contrapositive proof as a proof of the statement, "If n is odd then n^2 is odd" but struggled to accept the proof as a proof of the original statement. Speaking to this issue, a student remarked, "… The problem is that in this way we proved that n is odd implies n^2 is odd, and I accept this; but I do not feel satisfied with the other one" (p. 407). Antonini and Mariotti's work is novel, for their work is the only research that proposes students' lack of acceptance of indirect proofs may be due to metatheoretical issues.

Four aspects of research on students' difficulties and lack of preference for indirect proof are noteworthy. First, research on students' difficulties with indirect proof is unique in that it is the only area of research within the broad spectrum of research on students' difficulties with proof in which researchers have linked students' difficulties to a lack of preference for that form of proof. Second, while researchers (Tall, 1979, Healy & Hoyles 2000, Knuth, 2002) have routinely engaged students in comparative selection tasks to determine which form of proof students' find most convincing, researchers have not examined students' preferences (or lack of preference) for indirect proofs using comparative selection tasks involving a direct and an indirect proof. Indeed, there is a scarcity of empirical evidence to support current claims regarding students' lack of preference. Third, while Antonini and Mariotti (2008) have provided evidence of students' lack of acceptance of metatheoretical statements there is the question of whether it is a lack of acceptance or a lack of recognition of these statements that is prevalent and at the root of students' difficulties. Fourth, current accounts of students' dislike of indirect proofs and preference for constructive and generic proofs have ignored the fact that these reactions may be the result of the mathematics community's practices related to introducing novices to indirect proofs and the discourse that occurs around such proofs. For instance, in *How to Solve It*, a famous problem solving text by Polya (1957), the section on *reductio ad absurdum* and indirect proof[1] concludes with a section titled "Objections," in which Polya states:

> We should be familiar both with 'reductio ad absurdum' and with indirect proof. When, however, we have succeeded in deriving a result by either of these methods, we should not fail to look back at the solution and ask: Can you derive the result differently (p. 169).

Arguably, Polya's remarks do not provide the reader with a strong endorsement of either method. Moreover, such sentiments are not difficult to obtain as illustrated by the textbook excerpts shown in Figure 1.

[Concluding remarks, section on proof by contradiction] Many mathematicians feel that if a result can be verified by a direct proof, then this is the proof technique that should be used, as it is normally easier to understand. Text: *Mathematical Proofs: A Transition to Advanced Mathematics* (Chartrand et al., pg. 132)
A proof by contradiction is often easier, since more is assumed true; you are able to assume both the hypothesis and the negation of the conclusion. On the other hand, a proof by contradiction is likely to be less elegant than a proof by contrapositive. In any case, for elegance and clarity, it is better to choose a direct proof over an indirect proof whenever possible. Text: *Introduction to Advanced Mathematics* (Barnier & Feldman, 2000, p. 43).
There are times when it is not easy to see how to prove a mathematical statement, say ψ. When this happens one should try the strategy called proof by contradiction. This strategy is perhaps the strangest method of proof. Text: *A Logical Introduction to Proof* (Cunningham, 2012, p. 93).

Figure 1. Proof by Contradiction Text Excerpts

[1] Polya refers to proof by contraposition as indirect proof and proof by contradiction by its Latin name, *reductio ad absurdum*.

[2] Hardy referred to reductio ad absurdum (proof by contradiction) as a mathematician's "finest

These excerpts are not meant as backing for the claim that the mathematics community as a whole has exhibited a lack of preference. Indeed, the writings of Hardy (1940/2005), Euclid, Archimedes, and many contemporary mathematicians, as well as the famous proofs by contradiction of Hilbert (*cf*. Hilbert, 1890), stand in contrast to the remarks shown above.[2] Instead, the excerpts illustrate how a lack of preference might be due to various enculturative acts rather than an attribute of students. Yet, much of the research on indirect proof has ignored students' rationales for either preferring or exhibiting a lack of preference for such proofs. To be certain, there is a need for research that not only documents students' comparative preferences but also students' selection rationales; that is, their reasons for choosing a particular proof form.

In (author, 2015), a study was reported in which 53 mathematics majors were surveyed using a comparative selection task (see Figure 2) involving a direct and a (contraposition-form) indirect proof of the following theorem: *Suppose a set A has the property, for any subset B, A* $\subseteq$ *B, then A* = $\varnothing$*.* The proofs were presented side-by-side and students were asked, "Which proof, in your opinion, is the most convincing? In other words, which proof better persuades you of the truth of the theorem" and "Please explain your selection." The two proofs in the selection task were designed so as to control for various proof features; namely, the proofs were similar in length, and designed with the intent to be equal in their level of familiarity and complexity. For instance, complexity was equated by the prevalence of the proofs' content in the same textbook chapters in multiple texts. These controls were employed because pilot work had shown that when either complexity or familiarity were not equated, each were individually predictive of students' selections regardless of the proof type (i.e., direct or indirect).

Theorem 3: Suppose a set A has the property, for any subset B, A $\subseteq$ B. Then, A = $\varnothing$.	
Proof A:	**Proof B:**
Suppose A $\neq \varnothing$. Then there exists an *a*, such that $a \in$ A. Hence, A$\not\subseteq \varnothing$. Thus, there exists a subset B for which A$\not\subseteq$ B.	Assume A has the stated property. Recall, that $\varnothing$ is a subset of every set. Thus, $\varnothing \subseteq$ A. By the given property, A $\subseteq \varnothing$. Thus, A = $\varnothing$.

Figure 2. Comparative Selection Task

Surprisingly, the survey results indicated that the *indirect:direct* selection ratio for the Theorem 3 proof comparative selection task was 27:26. Thus, no evidence of a lack of preference was found. This finding lies in contrast to the findings of prior research and raises several questions: (1) to what extent is a lack of preference prevalent; and, (2) if prevalent, what are the characteristics of contexts in which a lack of preference is manifested? Furthermore, analyses of the students' selection rationales demonstrated that students' primary rationales were *certainty* and *complexity*. *Certainty* refers to the degree to which a student is certain of his/her understanding of the given proof and *complexity* refers to students' identification of one proof as being more complex than the other. What is of particular interest is that students' rationales did not identify a "more complex" proof nor were students more certain of one proof than the other. Instead students' responses demonstrated that complexity and certainty were subjective; that is,

[2] Hardy referred to reductio ad absurdum (proof by contradiction) as a mathematician's "finest weapon."

dependent on the individual and his or her understanding of the content employed. Drawing on Balacheff's cK¢ theory, (author) argued that preferences are mediated by students' conceptions.

While providing grounds for questioning the extent to which a lack of preference is prevalent among undergraduate students, reasons to continue investigating students' preferences remain. To begin, proof by contradiction and proof by contraposition differ at the metatheoretical level, with contraposition proofs requiring a direct proof of the contrapositive statement and use of the logical equivalence, $(P \rightarrow Q) \equiv (\sim Q \rightarrow \sim P)$, while proofs by contradiction require learners to not only negate a conditional statement (which is arguably more difficult than negating a premise and a conclusion separately) but also to produce an unspecified contradiction and to correctly interpret the ramifications of that contradiction (e.g, as the negation of a negated statement rather than as an error). In the previous study, the Theorem 3 selection task engaged students in a *direct:indirect* proof selection involving a *direct proof:proof by contraposition* comparison. Hence, there is reason to question if the lack of definitive preference, as evidenced by the students' selection ratio, is predictive of students' preferences in comparisons involving a proof by contradiction; especially, given the differences cited above. With this said, there are cultures in which the two forms of proof (contraposition and contradiction) are not distinguished at a nominal level, *e.g.*, in Italian (*cf.* Antonini & Mariotti, 2008). Moreover, pilot data showed students' may categorize a proof by contraposition as a proof by contradiction. Consequently, it may be that students' lack of definitive preference when engaging in *direct proof:proofs by contraposition* comparisons is predictive of students' preferences during *direct proof:proof by contradiction* comparisons. Certainly, more research is needed. The aim of this study is to address this need by pursuing the following research questions:

1. Do undergraduate mathematics students exhibit a lack of preference for indirect proof, when engaging in comparative tasks involving both a direct proof and proof by contradiction?
2. Which rationales do students provide for their selection of the most convincing proof, when engaging in comparative tasks involving both a direct proof and proof by contradiction?

The Study

The research reported in this paper is part of a larger research program generally focused on undergraduate mathematics students': (a) development of hypothetico-deductive reasoning (Piaget, 1968/1964); and (b) emerging *conceptions* of indirect proof, where conception is used in the sense of Balacheff's cK¢ model (2010; 2013). To investigate students' preferences, as these relate to selecting the most convincing proof, 85 mathematics students were recruited and given a paper survey containing Theorem 3 and two proofs of the statement, which were a direct and an indirect proof of the contradiction form (see Figure 3). The form was similar to that used in the previous study, with two exceptions; namely, the indirect proof form and a slight adjustment to the wording of the direct proof so as to produce proofs with equated lengths, (as determined by word counts of 40 and 41 words). As was the case in the prior study, complexity and familiarity were viewed as equated due to the content occurring in the same chapter in multiple introduction to proof texts.

The surveys were administered in either an abstract algebra or analysis course. Students completed the surveys under the supervision of the researcher and returned the surveys directly to the researcher. Proof order was randomized to avoid a priming effect. Analyses of the data involved the determination of selection ratios and the coding of students' rationales using a constant comparative methodology (Creswell, 1994). Multiple codes were employed when multiple rationales were provided by the students.

Theorem 3: Suppose a set A has the property, for any subset B, $A \subseteq B$. Then, $A = \varnothing$.	
Proof A	Proof B
Assume A has the stated property. Recall, that $\varnothing$ is a subset of every set. Thus, $\varnothing \subseteq A$. By the given property, since $\varnothing$ is a subset of A, $A \subseteq \varnothing$. It follows from that, $A = \varnothing$.	Assume A has the stated property and $A \neq \varnothing$. If $A \neq \varnothing$ then $A \not\subseteq \varnothing$. By the given property, $A \subseteq \varnothing$. Since, it cannot happen that $A \not\subseteq \varnothing$. and $A \subseteq \varnothing$, it follows that, $A = \varnothing$.

Figure 3. Theorem 3 Contradiction-Form Comparative Selection Task

Results

Data from the Theorem 3 comparative selection task indicate that students found the direct proof more convincing than the proof by contradiction with a *direct:indirect* selection ratio of 56:29. Thus, the students' preferences differ considerably from those observed in the *direct:indirect* comparative tasks involving a proof by contraposition (*direct:indirect* selection ratio of 26:27). Analyses of students' selection rationales also indicate differences in students' comparative assessments. Specifically, while the vast majority of students' rationales focused on *certainty* and *complexity* when engaging in the contraposition comparison, the selection rationales for the contradiction comparison were more varied. Indeed, six rationales were present in students' written remarks, which are reported with students' selection ratios in Table 1.

Selection Rationale	Selection Ratio (Contra-d: Direct)	*n*	Percent of Students
Simplicity / Ease	7:21	28	32.9%
Error (in Alternative)	4:14	18	21.2%
Directness / Straightforward	4:23	20	31.8%
Matched My Thinking	9:8	17	20.0%
Familiarity	1:12	15	15.3%
Stronger Argument	6:4	12	11.8%

Table 1. Students' Selection Rationales

Due to space limitations, examples of students' rationales will be restricted to: *simplicity*, *error*, and *matched my thinking*. Though *directness* was common, it is not included as it is self-evident in meaning. Below (Figure 4) are two examples of students' rationales coded as *simplicity*.

[Example 1.] The second is a proof by contradiction. I tend to find these proofs easier to follow. Proof A is not hard to follow as well but I think, in general, proof by contradiction is easier. (Selection: Indirect)
[Example 2.] Proof A is simpler. Proof B forces the reader to think about it more deeply. (Selection: Direct)

Figure 4. Simplicity Rationales

As can be seen by these remarks, students viewed both proofs as simple. However, as indicated by the *direct:indirect* selection ratio of 21:7, the simplicity rationale was more prevalent among students who selected the direct proof. The code *error* was used to denote student rationales that indicated a proof contained an error or that there was a statement that the student was uncertain about. Below (Figure 5.) are two examples of student rationales coded as *error*.

[Example 3.] Proof A stated the property that we need to prove and we cannot do that. (Selection: Direct)
[Example 4.] Proof A states that condition as an assumption which immediately made me question the validity of the proof. Proof B follows a standard version of a proof and make more sense than A. (Selection: Direct)

Figure 5.

As indicated by these students' rationales, there was a tendency among some students to view the contradiction argument as flawed. Indeed, the data indicate that 14 students (16.5%) selected the direct proof and provided this rationale. This finding suggests that rather than lacking a preference for indirect proof, students may have difficulty comprehending and/or validating indirect proofs of the contradiction-form.

The code *matched my thinking* was used for rationales that focused on students' statements of an alignment between their own approaches to proving and that taken in the selected proof. Four examples, which illustrate students' remarks, are provided in Figure 6.

[Example 5.] While thinking about how I would prove this theorem, Proof B seemed to match what I would have said. (Selection: Direct)
[Example 6.] Personally, I like working with direct proofs rather than contradictions. In Proof B the logic makes sense. (Selection: Direct)
[Example 7.] It was contradiction and I like to use contradiction to solve proofs. (Selection: Contradiction)
[Example 8.] When something seems obvious or believe it's true, it's easier for me to assume not and follow that way. (Selection: Contradiction)

Figure 6.

These responses, accompanied by a *direct:indirect* selection ratio of 8:9, suggest that the 20% of students who attended to their own approaches to the theorem (i.e, their habits of reasoning) did not demonstrate a preference for the direct proof by rather lacked a dominant preference. While the sample size for this rationale (and the others) is small, one must question if a preference would be evident in a larger data set. Nevertheless, it is particularly interesting that those who attended to their own approaches did not demonstrate a direct proof preference. Lastly, a note regarding the familiarity code is warranted. This code was used to indicate rationales focused on students' who cited familiarity with containment arguments (*e.g*., $A \subseteq B$, $B \subseteq A$, thus $A = B$) and their recognition that the direct proof employed a known proof technique.

Discussion

Findings from the survey suggest, as indicated by prior research (Leron, 1985; Harel & Sowder, 1998), that students may lack a preference for indirect proofs of the contradiction-form. Moreover, when these findings are considered in relation to the contraposition-form results, were no preference is evident, it appears that the two forms of proof are not the same in the eyes of undergraduate mathematics students. With this said, there are reasons that claims related to a lack of preference should be stated with caution. First, while the *indirect contraposition-form:direct proof* comparative selection task did not elicit the *error* rationale, this rationale was proposed by 16.5% of students in relation to the contradiction proof during the *indirect contradiction-form:direct proof* comparative selection task. Thus, it may be the case that students are more prone to comprehension difficulties with contradiction proofs rather than lack a preference for this form of proof. Second, while Tall (1979), Knuth, (2002), and Healy & Hoyles (2000) all reported results in which students' proof selections were impacted by familiarity, it was only the latter experiment, where a *contradiction-form:direct proof* comparison selection task was used, that students employed a *familiarity* rationale and stated that the direct proof employed a known technique. Familiarity is interesting in that while cognitive psychologist have argued that familiarity can create an immediate "feeling of rightness" it may also be the case that familiar proof forms are selected because students know those proof forms are accepted by the mathematics community or believe that the alternative is less favorable – a belief that could arise from reading texts like those in Figure 1. Thus, it is unclear if students' task-specific inclusion of the *familiarity* rationale is due to students' seeking "feelings of rightness," a type of deference to the community's argumentation norms, or something else. Certainly, more research is needed.

Furthermore, since familiarity strongly influences preferences and claims of preference have been predicated on assumptions of comprehension, it is worth examining those students who neither viewed the contradiction-form proof as flawed (*error* rationale) nor cited *familiarity*. Among the 85 students surveyed, 12 reported familiarity with containment arguments as their primary rationale and 1 reported familiarity with the contradiction proof. Additionally, while 14 reported an error in the contradiction argument, only 4 students viewed the direct proof as flawed. Removing these two categories of students from the population reduces the *direct:indirect* selection ratio[3] of 56:29 to a selection ratio of 31:24. Thus, the proportion of students selecting the direct proof (0.56) is not statistically significantly different from a 0.5 proportion ($z = 0.067$, $p < 0.05$). To be certain, among those who did not demonstrate a lack of comprehension and who did not defer to a "known technique," there is little evidence of a preference for the direct proof. Thus, while far from providing a definitive conclusion, this research raises multiple questions regarding students' preferences for or against proof by contradiction – perhaps with the exception of those who, like Hilbert, developed contradiction as a habit of reasoning.

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[3] This ratio is adjusted for the multiple codes used when students provided multiple rationales. In other words, there is no double counting of students in the adjusted preference ratio.

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When nothing leads to everything: Novices and experts working at the level of a logical theory

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Building on Antonini and Mariotti's (2008) theorization of mathematical theorem and research on students' meta-theoretical difficulties with indirect proof, this study examines mathematics majors' and mathematicians': (1) responses and approaches to the validation tasks related to the assertion $S^* \rightarrow S$, when given a primary statement, S, of the form $\forall n, P(n) \Rightarrow Q(n))$ and a secondary statement, S, of the form used in proofs by contradiction; namely, $\nexists n, P(n) \wedge \sim Q(n))$; and, (2) selection of a statement to prove given the choices S* and S. Findings indicate that novice proof writers' responses differ from advanced students' and mathematicians' both in their approaches and selections, with novices tending to become entangled in natural language antonyms and engage in the chunking of, rather than parsing of, quantified statements.*

Key words: Indirect proof, Proof by contradiction, Meta-theoretical Difficulties

How does one know that a mathematical theorem is true? Mariotti (2006) has proposed that to *know* truth in a mathematical sense requires not only a mathematical theory but also a logical theory:

> In their practice, mathematicians prove what they call 'true' statements, but 'truth' is always meant in relation to a specific theory. From a theoretical perspective, the truth of a valid statement is drawn from accepting both the hypothetical truth of the stated axioms and the fact that the stated rules of inference 'transform truth into truth' (p. 184).

These remarks align with definitions of proof, which focus on the elements of proofs and their use of logic; such as that proposed by Akin (2010), "Proofs are sequences of statements which can, in theory, be reduced to: (1) axioms, definitions, and previously proved results; or (2) statements obtained from earlier statements by the (formal – logical) rules of inference. To be certain, this definition points to the fact that without a mathematical theory from which to draw axioms, definitions, and previously proved results and without a logical theory to guide inferences, one cannot produce mathematical proofs. Indeed, we would obtain quite distinct results working in Riemannian rather than Euclidean geometry; especially if we were to use intuitionistic logic, such as that used by Brouwer, rather than the standard logic of mathematics. Working to clarify the systems that make 'truth' possible in mathematics Mariotti, Bartolini Bussi, Boero, Ferri & Garuti (1997) have argued that what characterizes a *mathematical theorem* is the triplet (statement, proof, reference theory), where reference theory is used to describe "a system of shared principles and deduction rules" (p.8).

Building on this characterization of mathematical theorem, Antonini and Mariotti (2008) examined a form of proof researchers (Robert, & Schwarzenberger, 1991) describe as highly problematic for students; namely, indirect proof. Drawing on data from interviews with tertiary students, Antonini and Mariotti demonstrated that while students may gain conviction from indirect proofs, this conviction is tied to the specific statements proved, as opposed to their logically equivalent statements. For instance, given a proof of $\sim Q \rightarrow \sim P$, students may gain conviction but fail to do so in relation to the statement $P \rightarrow Q$. Drawing on these findings, Antonini and Mariotti argue that when examined through the lens of that which characterizes mathematical theorems, indirect proofs are unique for they call on learners to employ theorems not only within the mathematical theory but also within the logical theory. Consequently, Antonini and Mariotti proposed a refinement of the (statement, proof, reference theory) triplet for indirect proofs, arguing that indirect proofs involve "the pairing of the sub-theorem (S*, C, T) and the meta-theorem (MS, MP, MT)" (p. 405). Meta-statement (MS) refers to statements such as "$S \rightarrow S^*$" where S refers to a primary statement (e.g., $P \rightarrow Q$) and S* refers to a secondary statement, such as the contrapositive of S (i.e., $\sim Q \rightarrow \sim P$). Meta-proof (MP) refers to the proof of $S \rightarrow S^*$ within the meta-theory (MT), i.e., the logical theory. Drawing on this model, Antonini and Mariotti argue students' difficulties with indirect proofs are metatheoretical; that is, due to a lack of acceptance of the meta-theorems employed.

Antonini and Mariotti's (2008) account of students' difficulties with indirect proof is unique among accounts of students' difficulties. Indeed, while researchers have argued that part of students' difficulties with indirect proof arise from difficulties negating statements (Wu Yu, Lin & Lee 2003; Antonini 2001, 2003; Thompson 1996), it is also the case that few have attended to the role of logic in such proofs. In Tall's (1979) study of students' levels of confusion related to indirect proofs of the irrationality of $\sqrt{2}$, students' confusion was attributed to the lack of generic structures in standard contradiction proofs rather than to difficulties at the level of the logical theory. One reason for this may be that the particular statement Tall (1979) examine was not a compound statement. Hence, the difficulties associated with negating conditional statements were not relevant. Drawing on reflective accounts of multiple teaching experiments, Leron (1985) explored students' responses to standard proofs of the infinitude of primes – that is to a mathematical theorem which did not take the form of a compound statement – and found that students' experience difficulties with the destructive, as opposed to constructive, nature of the indirect proof, as well as the "negative stretches" that accompany working in a "false world." Hence, working within the logical theory was not a primary source of difficulties. Through analyses of data from multiple teaching experiments, Harel and Sowder (1998) examined students' reactions to contradiction proofs and found that students lack a preference for this particular form of proof. They argued, like Leron (1985), that students' prefer constructive approaches and that such preferences are indicative of a constructive proof scheme. Beyond the rationales of Tall (1979), Leron (1985), and Harel and Sowder (1998), are Reid and Dobbin's (1998) emotioning rationale and Thompson's (1996) argument that indirect proof instruction tends to lack connections to students' informal reasoning. Indeed, multiple rationales have been proposed, though few take into account the specific meta-theorical nature of indirect proofs. This has been the case even though Goetting (1995) noted in relation to proof by contraposition, students were "wary of the validity of the 'backwards' arguments" (p. 124) and Leron (1985) noted in the case of proofs by contradiction, "we must be satisfied that the contradiction has indeed established the truth of the theorem (having falsified its negation), but psychologically, many questions remain" (p. 323) – remarks suggestive of attention to metatheoretical issues.

Arguably, Antonini and Mariotti's (2008) metatheoretical rationale represents a potentially critical advance in research on indirect proof in that it offers a route by which to explore aspects of indirect proof that are both unique and essential to that form of proof. With this said, little is known about tertiary students' metatheoretical reasoning. That this is the case may be due to the fact that there are a plethora of studies that focus on the development of proof through the refinement of students' informal arguments and that argues against the direct transition to formal proof and, consequently, do not advocate approaches involving training in logic (Jahnke, 2010; Maher & Martino, 1996). Moreover, many "Introduction to Proof" texts include instruction on logic but focus on building students' understandings through natural language activities rather than through instruction on logic as a theory (*cf.* Chartrand et al., 2013). Thus, one could conclude that there is little interest in metatheoretical issues for they run counter to current perspectives on productive approaches to proof and pedagogical practices in commonly used texts. On the other hand, the lack of research on metatheoretical reasoning is surprising since, as evident in Akin's definition, our basic definitions of proof rely on the existence of a logical theory. Furthermore, there is a profusion of research from cognitive psychology demonstrating that humans' ways of reasoning do not fully align with the forms of reasoning used in standard logic (*cf.* Oaksford & Chater, 2010). Specifically, general tendencies for interpreting conditional statements ($P \rightarrow Q$), do not align with those used in mathematics, with the exception of direct reasoning processes; e.g., accepting $P \rightarrow Q$ when P and Q are true. Thus, there are grounds for questioning pedagogical approaches premised on the idea that the logical theorems employed in mathematics will be readily employed by students. To be certain, there is reason to argue that further research is needed on students' approaches to and extent of success with metatheoretical work, especially in relation to those forms of proof for which such work is essential; namely, indirect proofs.

The Study

The purpose of this study is to address the need for research on students' approaches to and extent of success with metatheoretical work, by exploring three research questions:

1. To what extent are mathematics majors and mathematicians successful, when answering questions regarding the validity of $S^* \rightarrow S$; that is, when asked if a secondary statement S^* is sufficient to prove a primary statement S?
2. What are the similarities and differences observed among these populations when approaching questions regarding the validity of $S^* \rightarrow S$?
3. Which formulation, the statement or the secondary statement, do mathematics majors and mathematicians prefer, when asked to select a statement to prove?

These questions are of interest for they provide various avenues with which to explore the issue of students' potential difficulties working at the level of a logical theory. Indeed, if experts (mathematicians) are able to successfully verify metatheoretical statements, while novices struggle, then the findings will provide further evidence of Antonini and Mariotti's (2008) claim; whereas, if neither experience difficulties then an alternative to the metatheoretical hypothesis may be needed. Furthermore, if experts and novices avoid selection of secondary statements, then such data will provide further evidence of the preference hypotheses generated by Leron (1985) and Harel and Sowder (1998). In contrast, if no preference is evident then there will be cause to question the preference hypothesis.

Methods

To investigate mathematics students' and mathematicians' metatheoretical reasoning, three stages of data collection occurred. In the first stage, electronic surveys were sent to undergraduate mathematics majors who were within one year of completion of an introduction to proof course. The students were provided with Theorem 5 and Statement A (see Figure 1.) and asked to indicate if the following statement was true or false, "You can prove Theorem 5 by proving Statement A." Following this prompt, students were queried "If you were asked to prove Theorem 5 which would you pursue first?" and given the choices Theorem 5 and Statement A.

Theorem 5. For all positive integers n, if $n$mod(3) ≡ 2 then n is not a perfect square.
Statement A. There exists no positive integer n such that $n$mod(3) ≡ 2 and n is a perfect square.

Figure 1. Theorem 5 Task

In stage 2, clinical interviews were conducted with 21 mathematics majors, who met the criteria described above. In stage 3, clinical interviews were conducted with 6 mathematicians. In all of the clinical interviews, the manner in which the Theorem 5 task was posed matched that of the electronic survey. Interview responses were analyzed to gather categorical data for the validation and selection questions, as well as data regarding participants' approaches to the validation question. Using a constant-comparative methodology, descriptive codes were generated and used to further characterize participants' video-recorded responses.

Theorem 5 Task Design

The Theorem 5 validation task was designed with the assumption that at the conclusion of a proof by contradiction the prover would need to recognize that having shown Statement A, he or she can conclude that Theorem 5 was proven. This assumption is predicated on the following sequence of proving actions. First, to prove Theorem 5 using a proof by contradiction, one begins by assuming the negation of a statement of the form $\forall n \in \mathbf{Z}^+$, $P(n) \Rightarrow Q(n)$. Thus, by assuming $\sim(\forall n \in \mathbf{Z}^+, P(n) \Rightarrow Q(n))$, which is logically equivalent to $\exists n \in \mathbf{Z}^+$, $P(n) \wedge \sim Q(n)$. Second, the prover must arrive at a contradiction to an axiom, definition, previously proved theorem or an existing assumption (*i.e.*, something within the mathematical theory) and conclude from this contradiction that the statement $\exists n \in \mathbf{N}$, $P(n) \wedge \sim Q(n)$ is false; *i.e.*, $\sim(\exists n \in \mathbf{N}, P(n) \wedge \sim Q(n))$. While formally, one might say "It is not the case that there exists a positive integer such that …", informally one might argue, "no such n exists" or the more common, though grammatically more awkward, "there exists no positive integer n such that …". Lastly, one must recognize that the proven statement is sufficient to prove the original statement – in Antonini and Mariotti's terms, that S* proves S. Hence, the Theorem 5 validation task was designed with the last phase of this sequence in mind; that is, to explore mathematics majors' and mathematicians' perceptions of validity related to a standard phrasing of the result of a proof by contradiction. Furthermore, as was the case in Antonini and Mariotti's work, the task focused on the inferences to be drawn rather than the, perhaps technically more appropriate, determinations of equivalence.

Findings

In regard to the electronic survey, 35 mathematics majors who were advanced in terms of course work; that is, who had completed not only an introduction to proof course but also real analysis and abstract algebra courses, responded to the survey. The categorical responses are shown in Table 1 and indicate that: (a) the majority of advanced students were successful at recognizing the validity of the metatheoretical statement S* → S; and, (b) demonstrated only a slight preference for the primary (Theorem 5) over the secondary statement (Statement A), which was not significantly different from a near even split ($\chi^2 = 0.458$; $p = 0.499$, χ^2 *Good of Fit test*).

Prompt 1: *You can prove Theorem 5 by proving Statement A.*		
Response	**N**	**%**
True	29	83
False	6	17

Prompt 2: *If you were asked to prove Theorem 5, which would you pursue first?*		
Responses	**N**	**%**
Theorem 5	20	57
Statement A	15	45

Table 1. Advanced Mathematics Students' Survey Responses

While the surveys were predominantly taken by advanced students, the majority of interview volunteers were novice proof writers; *i.e.*, they had recently completed an introduction to proof course, had not completed both real analysis and abstract algebra courses but rather were enrolled in at most one of these courses at the time of the interviews. Thus, the interviews were conducted with a different population in terms of prior coursework. Findings from the interviews indicate that the novices' validation responses fell into four categories: *don't know*, *no*, *yes-no-yes*, and *yes*. For clarification, *don't know* was used for students who after deliberating indicated they were unable to determine if the statement "You can prove Theorem 5 by proving Statement A" was true or false. *Yes-no-yes* refers to responses in which the student initially articulated an intuitive response, sought to validate their intuition, decided "*no*" and then through further analyses decided (often with uncertainty) that their response was, "Yes, it's true." As seen in Table 2, where the novice proof writers' responses are shown by category, the most frequent response was *no* at 42.8%, with the majority of these students (67%) arguing that Statement A was the negation of Theorem 5. However, if the *yes* (28.6%) and *yes-no-yes* (23.8%) response categories are collapsed, then roughly half of the students were able to correctly respond to the validation statement. It is interesting to note that a secondary analysis of the sample's verbal responses, which coded students' responses for multiple instances of expressed hesitancy, equivocation, or doubt, found that 16 of the 21 students (76%) repeatedly articulated uncertainty. Lastly, interview participants overwhelming selected Theorem 5 for their "statement to prove."

Prompt 1: *You can prove Theorem 5 by proving Statement A.*		
Response	**N**	**%**
True (Yes)	6	28.6%
True (Yes-no-yes)	5	23.8%
False (No)	9	42.8%
Uncertain (Don't Know)	1	4.8%
Prompt 2: *If you were asked to prove Theorem 5, which would you pursue first?*		
Responses	**N**	**%**
Theorem 5	16	76.2
Statement A	2	9.5%
No Response	3	14.3%

Table 2. Novice Proof Writers' Theorem 5 Task Responses

Not surprisingly, the 6 mathematicians who were interviewed were, without exception, successful at the Theorem 5 validation task. Moreover, like the advanced students, their selection of a statement to prove was more balanced than that of the novices, with some expressing the selection "*either*." Data for the mathematician sample is provided in Table 3.

Furthermore, like the novices, 4 of the mathematician's (75%) expressed hesitancy or equivocations even though they did not shift their response to Prompt 1. However, unlike the novices, who doubted their responses, the mathematicians' expressions of uncertainty tended to be geared towards their own reasoning at the time of the interview; e.g., one remarked "I'm doubting myself right now for some reason" and another mentioned that it was "too early" in the morning for such questions.

Prompt 1: *You can prove Theorem 5 by proving Statement A.*		
Response	**N**	**%**
True	6	100
False	0	0
Prompt 2: *If you were asked to prove Theorem 5, which would you pursue first?*		
Responses	**N**	**%**
Theorem 5	2	33.3
Statement A	2	33.3
Either	2	33.3

Table 3. Mathematician's Theorem 5 Task Responses

Beyond the observed similarity of expressed hesitancy, the novices and mathematician responses were quite dissimilar. Indeed, while 5 of the 6 mathematicians approached the question of proving Theorem 5 by proving Statement A semantically, none of the novices were observed using a semantic approach. Instead, the novice proof writers tended to move away from the linguistic statements and work at a symbolic level, with many moving to truth tables to prove various equivalences. While it might seem that working at a symbolic level leads to a higher error rate, it appears that this is not the case. The same percentage of students who worked symbolically immediately responded *yes*, as did those who argued *no;* approximately 67% of each cohort. With this said, it is interesting to note that all (100%) of those who responded "*yes-no-yes*" worked symbolically. Since several students expressed a lack of comfort with the content as a rationale for moving to symbols, it may be that the mathematicians' greater content expertise played a role in their approaches to the Theorem 5 validation task. This finding raises questions about the skills novices need to evaluate statements with unfamiliar content.

A second important distinction between the novice proof writers' and mathematicians' approaches concerns how the two cohorts went about understanding Statement A. Specifically, the majority of novices (14 of 21; 67%) tended to engage in *chunking*; that is, they tended to break Statement A into two chunks, with the first containing the quantifying phrase, "there exists no positive integer," and the second chunk containing the open sentence, "$n$mod(3) ≡ 2 and n is a perfect square." Hence, when responding to Prompt 1, they sought relationships between the quantifying chunks of Statement A and Theorem 5 and then the open sentences, rather than holistically comparing the quantified statements, as illustrated in the transcript excerpt below.

Linda: This (Statement A) is the negation of this (Theorem 5) because this one says *for all* and this one says *there exists no* and this one is *if P then Q* and this one is *P and not Q*.

Figure 2. Chunking Transcript Excerpt

In contrast, the mathematicians and 4 of the successful students (3 who replied yes and 1 who responded *yes-no-yes*) engaged in *parsing*; that is, consideration of the various components of the quantified compound statements <u>and</u> their logical relations to each other. In other words, quantifiers were considered in relation to the open sentences they quantified rather than as separate sentence components. This approach is illustrated in the interview excerpt in Figure 3, where the student considers the negation of the quantifier in relation to the modified statement.

Nicolas: Because when you say "there exists no," that's a "for all" but then … but then you have to negate … (places fingers on P and ~ Q statements) … you, you have to negate Q.

Figure 3. Parsing Transcript Excerpt

Similarly, several mathematicians spoke aloud while responding to the Theorem 5 task, with many providing comments akin to the following mathematician's remark, "there exists no … so there is nothing that satisfies this (points to "$n\text{mod}(3) \equiv 2$ and n is a perfect square")."

Lastly, those who were observed *chunking* Statement A; that is, isolating the quantifying phrase "there exists no …" where also frequently observed becoming entangled in the natural language meanings and linguistic antonyms of "for all" and "there exists no." For instance, Patrick argued, as did others, that "for all" means "everything," "there exists no" means "nothing," and that "the opposite of nothing is everything." Thus, for some students, natural language functioned as an obstacle to validating the claim $S^* \rightarrow S$.

Discussion

The findings of the electronic survey of advanced mathematics students and the clinical interviews with novice proof writers and mathematicians indicate that while advanced students and mathematicians are quite successful validating the claim "You can prove Theorem 5 by proving Statement A," these determinations are quite difficult for novice proof writers who may engage in potentially unproductive chunking practices and employ inappropriate linkages between mathematical statements and natural language. While it is easy to argue that the majority of novices were simply weak in the content area of quantifiers and did not interpret them appropriately, for they neither engaged in valid parsing practices nor did they interpret the terms in a logically appropriate manner, there is reason to caution against this reaction. Many "introduction to proof" texts use natural language when introducing students to quantifiers and their negations. Moreover, in a review of these texts, it was found that none addressed the ambiguities that arise from natural language in relation to quantification (*cf.* Epp, 2003). Indeed, as was evident from a survey of dictionaries, and is illustrated in Figure 4, neither is there a transitive property for antonyms nor is there a lack of ambiguity to natural language when it comes to quantifying terms such as *for all* (everything), *exists no* (nothing) and *exists* (something; some), when expressed and negated using natural language antonyms.

Figure 4. Natural Language Examples of Quantified Terms

Finally, the novices "statement to prove" selections, which indicated a strong preference for Theorem 5, stand in contrast to those of advanced students and mathematicians, with neither group demonstrating a preference. Given that many novices had difficulty parsing Statement A this result is not surprising. With this said it appears that, at least for novices, the metatheoretical issues described by Antonini and Mariotti (2008) may play a role in students' interpretations and sense of certainty in the context of the results of a proofs by contradiction. In particular, given that in response to Prompt 1, 42.8% of novices replied *false* and only 28.6% replied *true*, there is reason to believe that – as Antonini and Mariotti have argued – students experience difficulties at the metatheoretical level in relation to the theorem $S^* \rightarrow S$, with the current work indicating that a potential source of these difficulties may be validating the relationships between S* and S.

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Students' explicit, unwarranted assumptions in "proofs" of false conjectures

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Although evaluating, refining, proving, and refuting conjectures are important aspects of doing mathematics, many students have limited experiences with these activities. In this study, undergraduate students completed prove-or-disprove tasks during task-based interviews. This paper explores the explicit, unwarranted assumptions made by six students on tasks involving false statements. In each case, the student explicitly assumed an exact condition necessary for the statement in the task to be true although it was not a given hypothesis. The need for an ungiven assumption did not prompt any of these students to think the statement may be false. Through prompting from the interviewer, two students overcame their assumption and correctly solved the task and two students partially overcame it by constructing a solution of cases. However, two other students were unable to overcome their assumptions. Students making explicit, unwarranted assumptions seems to be related to their limited experience with conjectures.

Key words: Conjectures, Unwarranted Assumptions, Mathematical Proof, Task-Based Interviews

The proving process is a complex combination of creativity and rigor that encompasses a multitude of activities including analyzing and identifying patterns and relationships, generating conjectures and generalizations, and evaluating, refining, proving, and refuting mathematical conjectures (Committee on the Undergraduate Program in Mathematics (CUPM), 2004; de Villiers, 2010; Durand-Guerrier, Boero, Douek, Epp, & Tanguay, 2012). However, many students have limited experience with the activities in the proving process that involve uncertainty and decision-making, such as exploring conjectures (Alibert & Thomas, 1991; de Villiers, 2010; Durand-Guerrier et al., 2012). This limited experience may inhibit students' development of "an attitude of reasonable skepticism" with respect to mathematics (Alibert & Thomas, 1991; de Villiers, 2010; Durand-Guerrier et al., 2012, p. 357).

Prior research has shown that high school and undergraduate students make unwarranted assumptions in proofs (Dvora, 2012; Selden and Selden, 1987; Weiss, 2009). In these cases, the students seem to be either unaware they had made an unwarranted assumption or the assumption was based on their perception of a geometric figure and was unrecognized as unwarranted. But what leads students to knowingly make unwarranted assumptions in a non-geometric proof context? Especially when the truth value of the statement is unknown? What makes a student explicitly assume an ungiven assumption rather than consider a statement may be false? These are the questions I investigate in this paper, but my actual research questions are: Why do students explicitly make unwarranted assumptions on prove-or-disprove tasks? What types of explicit, unwarranted assumptions do students make? Under what conditions do students overcome their explicit, unwarranted assumptions?

Literature Review

"One of the most important steps in [mathematical] research is to conjecture what is the truth and then attempt to verify it by hunting down a proof" (Burger, 2007, p. xii). Suppose a mathematician believes a certain conjecture is true, but while attempting to prove it, the mathematician needs an assumption that is not a hypothesis? There seem to be three reasonable courses of action commonly practiced by mathematicians: (a) consider that the conjecture may be false and search for a counterexample, (b) add the assumption to the hypotheses and prove a weaker conjecture, or (c) assume the needed assumption and justify it later (Burger, 2007; Selden & Selden, 1987; Weiss, Herbst, & Chen, 2009). Although (a) and (c) should lead to a decision on the truth value of the conjecture, in (b), the conjecture has been weakened and there is no verification of the truth value of the original conjecture.

In order for students to experience mathematics the way mathematicians do, they need to be engaged in exploring, proving, and refuting conjectures. CUPM (2004) suggests that students majoring in the mathematical sciences "learn a variety of ways to determine the truth or falsity of conjectures…to examine special cases, to look for counterexamples," and to analyze "the effects of modifying hypotheses" (p. 45). In his article on teaching proving, Dean (1996) suggests that when students are exploring a conjecture, "if little progress is being made, the student might add an additional hypothesis and see if this leads anywhere" (p. 53). In Burger's textbook, *Extending the frontiers of mathematics: Inquiries into proof and argumentation* (2007), directions for each problem statement in the text are 'Prove and extend *or* disprove and salvage' (p. xii). Burger (2007) offers many suggestions for extending a proven conjecture or salvaging a refuted conjecture, including weakening or adding to the hypotheses, respectively. Lastly, some high school teachers believe allowing students to make an assumption with the caveat that they must return and justify it later is a valuable instructional strategy (Weiss et al., 2009).

Despite the recommendations of CUPM (2004), many students have limited experiences exploring and refuting conjectures (Alibert & Thomas, 1991; de Villiers, 2010; Durand-Guerrier et al., 2012). In particular, "students are rarely, if ever, presented with false mathematical statements and asked to determine whether or not they are true" (Durand-Guerrier et al., 2012, p.357), and high school "students are rarely held accountable for finding the conditions under which a claim could be true (Herbst & Brach, 2006)" (Nachlieli, Herbst & Gonzalez, 2009, p. 432).

High school and undergraduate students' limited experiences may partially account for the difficulties they have studying conjectures. Students struggle with (a) knowing how to begin an exploration, (b) formulating ideas and opinions about the truth of a conjecture, and (c) connecting ideas and opinions to proofs or counterexamples (Alibert, 1988). Durand-Guerrier and Arsac (2005) suggest that students' difficulties determining the truth value of conjectures may stem from their narrow collection of possible counterexamples and limited mathematical knowledge as novices. In geometric contexts, high school students often make unwarranted assumptions based on geometric figures or diagrams even though they are taught not to do so (Weiss, 2009).

Other difficulties students have may be related to the inappropriate use of the strategies used by mathematicians and suggested by educators for exploring conjectures. Some high school and undergraduate students unknowingly make unwarranted assumptions that reduce a general conjecture to a special case (Selden & Selden, 1987; Weiss, 2009). Although mathematicians examine special cases when exploring conjectures, they do so knowingly and realize the general conjecture still needs to be considered (de Villiers, 2010). Weiss et al. (2009) reported on high school teachers' reactions to a video episode of a teacher allowing a student to make an unwarranted assumption in a proof under the condition that the student returned to justify the assumption later. Some teachers expressed concern that students would distort this practice (common of mathematicians) by developing a habit of making unwarranted assumptions and failing to return to them (Weiss et al., 2009).

Method of Inquiry

The data in this paper come from a larger study that (a) examined the reasoning students use to evaluate conjectures, (b) identified systematic errors students make during the proving process, and (c) investigated cognitive unity between students' evaluation of conjectures and construction of associated proofs and counterexamples.

Participants

The participants were twelve undergraduate students from a public university in Ohio who had passed at least one proof-based mathematics course with a grade of B or better. Ten students were in their fourth year of undergraduate study, and eleven students were mathematics or secondary mathematics education majors.

Procedures

I conducted two task-based interviews with each participant which were audio-recorded and transcribed. Participants were asked to think aloud during the completion of four tasks and to clarify or expand on their thinking as necessary. Each task was provided one at a time on a separate sheet of paper. Participants were provided with a list of definitions of terms in the tasks, but no other materials were allowed. Participants used a LiveScribe Pen and paper that recorded synchronously audio and writing. After each task, I asked follow-up questions on the participants' work on the task. Upon completing all tasks, I asked each participant general questions about their approaches to and understanding of proof and disproof.

Tasks

Each task required the participants to evaluate a conjecture and prove or disprove the conjecture accordingly. The tasks involve basic properties of functions and were chosen to be accessible to the participants. In line with Alcock and Weber (2010), each task referred to general objects and their properties and should have been approachable with either semantic or syntactic reasoning. The following three tasks will be discussed in this paper:

Injective Function Task: Let $f: A \to B$ be a function and suppose that $a_0 \in A$ and $b_0 \in B$ satisfy $f(a_0) = b_0$. Prove or disprove: If $f(a) = b$ and $a \neq a_0$, then $b \neq b_0$.

Monotonicity Task: Prove or disprove: If $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are decreasing on an interval *I*, then the composite function $f \circ g$ is increasing on *I*.

Global Maximum Task: Prove or disprove: If f is an increasing function, then there is no real number c that is a global maximum for f.

Each statement in these tasks is false. Any noninjective function is a counterexample for the Injective Function Task. A counterexample for the Monotonicity Task requires a function g with outputs that are not elements of the chosen interval *I*. Finally, any increasing function defined on a closed interval serves as a counterexample for the Global Maximum Task.

Analysis

I identified all errors made by participants in the proving process. Instances in which participants made assumptions that were not given hypotheses in the task were classified as *unwarranted assumptions*. An unwarranted assumption was further categorized as *explicit* if the participant expressed awareness of making it.

Results

Six of the twelve students in this study made an explicit, unwarranted assumption. Each of these students did so on exactly one task. In each case, the assumption the student made was exactly what was needed to make the statement in the task true, but was not a given hypothesis. Additionally, in the face of the needed assumption, no student considered the possibility that the statement may be false without prompting from the interviewer. In this section, I describe these students' explicit, unwarranted assumptions and the extent to which they overcame them. First, I discuss Edward and Jalynn, each of who overcame their explicit, unwarranted assumptions and correctly solved the associated tasks. Next, I present Evan and Inigo, who partially overcame their explicit, unwarranted assumptions by constructing task solutions involving cases. Lastly, I discuss Aurelia and Jay who failed to overcome their explicit, unwarranted assumptions and incorrectly solved the associated tasks.

Edward and Jalynn

Edward and Jalynn each made an explicit, unwarranted assumption while attempting to prove the statements in the Monotonicity and Injective Function Tasks, respectively. With prompting from the interviewer, they eventually realized that their assumptions were problematic and correctly decided the statements were false.

Edward

Edward decided that the statement in the Monotonicity Task was true and constructed a proof for it. Within his proof, Edward made the explicit, unwarranted assumption that the range of the function g was in the interval *I*. Upon completing his proof, he noted, "I'll say it's increasing on *I*. Although I didn't do a good job at all of proving where *I* is or working with where *I* is." I asked him how concerned he was about that, and he said:

> If they are both decreasing on an interval *I*, that doesn't necessarily mean the intervals overlap…Because we would need the range of g to be in *I*…we would need the domain of f to be the same decreasing interval as the range of *g,* and we'd need the domain of *g* to be decreasing. So, and I didn't prove that connection. I should have.

I inquired, "Does that invalidate your proof?" He responded, "Yes. I would not necessarily believe this proof because I didn't match up the range to the domain."

I pressed further regarding this assumption in his proof, and he indicated that it was a necessary but unwarranted assumption: "If I make that assumption,…it does work…But without making that assumption, I don't think it holds….I don't think that's an assumption I can legitimately make." Upon making sense of why the assumption was necessary for the statement to be true, Edward finally decided that the statement was false. He concluded:

> Without this [the assumption], f…could be increasing or decreasing on *I*. I mean, depending on where the range of *g* is mapped onto the domain of f and what, whether it's increasing or decreasing at that interval…'cause the interval…doesn't necessarily line up at f and g. That makes this statement false.

Thus, through interviewer prompting and analysis of the necessity of his assumption, Edward realized that he could not justify his assumption and the statement was false.

Jalynn

Jalynn knew that the Injective Function Task was related to the concept of one-to-one, but was confused by the notation $f\colon A \to B$, wondering whether it only indicated the domain and range of the function or if it also implied that the function was onto or one-to-one. After she began her proof, she realized she needed the assumption that the function f was one-to-one and said, "I can assume that it's one-to-one….There would just be a condition for it then." With this explicit, unwarranted assumption, Jalynn constructed a proof for the statement.

After she completed her proof, I asked Jalynn if she thought that the assumption that f was one-to-one was a necessary condition for her proof. She said that she was unsure because she was still confused about whether the notation indicated that the function was one-to-one. So, I asked her what she thought if we just assumed that the notation only indicated the domain and range of the function, and she replied "[that] probably would change it, but, I'm just trying to think of an example." She wrote $f(x) = x^2$, and showed that $f(3) = f(-3) = 9$. She indicated that this function was not one-to-one and that f being one-to-one was a necessary condition for this statement to be true.

Finally, I asked Jalynn to clarify whether she thought the statement was true or false, and she replied "it's true if it's one-to-one and it's false if. Overall it would be false in any case, just like how here [referring to her counterexample $f(x) = x^2$]…I guess it just asks for the general case." Like Edward, through prompting to consider the necessity of her explicit, unwarranted assumption, Jalynn analyzed it in the context of an example and realized she needed to consider the general case in which the statement was false.

Inigo and Evan

Inigo and Evan each made an explicit, unwarranted assumption while proving the statements in the Injective Function and Global Maximum Tasks, respectively. They were able to partially overcome these assumptions by constructing cases–one with and one without the assumption–for their solutions to the tasks. However, neither student realized that only one of the cases applied to the given task.

Inigo

Inigo assumed the statement in the Injective Function Task was true. While constructing his proof, in order to claim $f(a) = f(a_0)$ implies $a = a_0$, Inigo said he needed to assume f was one-to-one. He did so, making an explicit, unwarranted assumption, and completed his proof. He then said, "I know there's a flaw in some logic there because of this [underlining his assumption that f is one-to-one], but I'm finished." Inigo was content to stop with an invalid proof, but I was not willing to let it stand. I asked him if he could tell me why he thought it was wrong, and he said "I am assuming that this is one-to-one. And it's not necessarily one-to-one….And I know you can't actually make that assumption here". Inigo then realized that $f(x) = x^2$ served as a counterexample and said "So when it's one-to-one, that holds [indicating his proof]; and then when it's not, there [underlining his counterexample]….I broke this into cases." Thus, through prompting, Inigo only partially overcame his explicit, unwarranted assumption, deciding that a complete solution to the task included two cases. He did not realize that only the case without the assumption applied to the statement in the given task.

Evan

On the Global Maximum Task, Evan thought mistakenly that the given statement said the function *did* have a global maximum rather than saying it did *not* have a global maximum. Thus, Evan decided the statement was false and constructed a proof by contradiction to disprove the statement (proving the function did not have a global maximum). However, this proof included the implicit, unwarranted assumption that the domain of the function f was $\mathbb{R}$.

Because Evan had misread the statement, I confirmed with him that he thought the statement was false and asked him to reread the statement to ensure he was saying what he wanted to say. Upon looking back at his disproof, he realized he made the assumption that the domain of f was $\mathbb{R}$ and said he would "add a disclaimer" to his proof. He included his assumption in his disproof which made it an explicit, unwarranted assumption. Additionally, Evan wrote a second case in which the domain was a closed interval and proved the statement was true in that case. Like Inigo, Evan concluded he had two cases, but did not realize only one actually solved the given task.

Aurelia and Jay

Aurelia and Jay each made an explicit, unwarranted assumption while proving the Global Maximum and Injective Function Tasks, respectively. Both students failed to overcome these assumptions and incorrectly solved the tasks.

Aurelia

Aurelia struggled to determine the truth value of the Global Maximum Task. Upon first reading the statement, Aurelia said "So, I'm assuming that means if f is increasing throughout the whole entire function? So, this is obviously not true if you have…[a] function that stops at a certain point." However, she questioned whether a function could have a restricted domain. She drew a graph of $f(x) = x^2$ restricted to $[0,2]$ and asked herself "is that considered a function?" She was uncertain whether it was a function, but decided to assume that it was not a function because she thought I was "not trying to trick [her]". Thus, she made the explicit, unwarranted assumption that a function cannot have a restricted domain. This allowed her to assume that the function in the task was defined on $\mathbb{R}$, and she used this assumption to incorrectly "prove" the false statement.

Jay

Jay assumed the statement in the Injective Function Task was true and constructed a proof in which he made the explicit, unwarranted assumption that the function f was one-to-one. After he completed his proof, I asked him what the key step was in his proof, and he said "Well, just, for me, the idea since $a \neq a_0$, then, I, sort of, made a jump and assumed that $f(a)$ then is not equal to $f(a_0)$." I inquired about making this "jump", and he replied "That'll only be true if the function was one-to-one, but from just the given information, I don't know exactly if it is one-to-one." I continued attempting to draw information out of him about his use of one-to-one despite being uncertain whether f was one-to-one, but I was unable to get him to reconsider his assumption. He repeated that his proof would work if he knew the function was one-to-one, but he never indicated decisively whether he knew this. Despite my pressing, Jay was unable to overcome his explicit, unwarranted assumption and was satisfied with his "proof" for this false statement.

Discussion

Consistent with prior research with high school and undergraduate students, the students in this study seemed to lack key strategies for thinking about and identifying false statements. Some students made explicit, unwarranted assumptions rather than consider a given conjecture was false. In each case, the student completed a "proof" of a false statement that relied on and included the explicit, unwarranted assumption. Multiple students in this study, including Edward and Inigo, said they are rarely asked to consider statements in which the truth value is unknown. Inigo noted, "All throughout math classes, we're bombarded with what's true and not with what's false." It seems possible that limited exposure to conjectures may have inhibited these students' development of a healthy skepticism toward mathematics, as has been suggested in the literature (Alibert & Thomas, 1991; de Villiers, 2010; Durand-Guerrier et al., 2012). This may have led the students to do whatever it took to prove the statements rather than consider their potential falsity. This suggests students need more opportunities to engage in evaluating, refining, and refuting conjectures.

Another possible explanation for the students' behavior, as indicated by the concerns voiced by the high school teachers in Weiss et al.'s (2009) study, is that these students were misusing a common technique practiced by mathematicians. Edward was the only student who indicated he knew he should have returned to his assumption to justify it. The other students seemed content with simply adding to the hypotheses, even though some expressed concern over doing so. It is possible that these students were misusing a legitimate strategy they had seen mathematicians use, which may account for their uneasiness. However, these students also expressed a clear understanding of the logical nature of proofs during follow-up questioning, so perhaps their concern resulted from their knowing the assumptions were unwarranted, but not knowing what else to do. This would suggest again that the students' struggles were related to their limited experience with statements of unknown truth value.

Each explicit, unwarranted assumption made by the students in this study was an ungiven hypothesis that was necessary for the statement to be true. Thus, the need for each assumption should have indicated the potential falsity of each statement as well as exactly what was needed in a counterexample. On the Monotonicity Task, Edward assumed the range of the function $\boldsymbol{g}$ was in the interval I. On the Injective Function Task, Jalynn, Inigo, and Jay each assumed the given function $\boldsymbol{f}$ was one-to-one, and on the Global Maximum Task, Evan and Aurelia assumed the domain of the function $\boldsymbol{f}$ was $\mathbb{R}$. For each task, the fact that these assertions are not necessarily true is precisely why the statements are false. If students were accustomed to Burger's (2007) instructions to 'prove and extend or disprove and salvage,' the realization that they needed these assumptions to prove the statements should (a) indicate the statements are false, (b) provide the necessary conditions for a counterexample, and (c) specify an assumption to add to the hypotheses to salvage the statements. This would make the need for the assumption in a proof attempt a powerful tool in solving the task. However, it does not seem as though the students in this study were trained to recognize this power of needed assumptions.

Despite some of the students' concerns regarding their assumptions, prompting from the interviewer to reconsider their "proofs" or assumptions seemed necessary for them to overcome or partially overcome their explicit, unwarranted assumptions. However, this did not work in all cases as Aurelia and Jay were unable to overcome their assumptions. Interestingly, Inigo, Evan, and Jay each indicated during follow-up questioning that prove-or-disprove tasks are more difficult than prove tasks because if they got stuck in the middle of a proof, they would have to question whether they were trying to prove a false statement and consider looking for a counterexample. However, none of these students did this when confronted with the need for an ungiven assumption. Perhaps they did not consider or recognize this as a form of 'getting stuck.' Either way, it seems they possessed knowledge of an appropriate strategy to use the situation, but failed to use it.

The results of this study suggest a couple ideas for dealing with students making explicit, unwarranted assumptions. First and foremost, engage students in evaluating conjectures, including false conjectures. And so often. If students are rarely faced with conjectures, then it will be difficult for them to develop and use appropriate strategies for dealing with situations that are common in conjecturing contexts but not in contexts in which the truth value of a statement is known. Additionally, it seems as though students may not be inclined to question explicit, unwarranted assumptions on their own. They may need prompting from their instructors in order to recognize that needing an ungiven assumption means that they are 'stuck.' And we need to encourage students to explore this special type of being 'stuck' because of its potential power to indicate why a statement is false, what is needed for a counterexample, and what is necessary to make it true. Engaging students in evaluating conjectures and helping them recognize the potential power of a needed assumption may allow them to move ever closer toward thinking like mathematicians think.

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Limitations of a "chunky" meaning for slope

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This paper will investigate the question "What mathematical meanings do high school mathematics teachers hold for slope and rate?" It will also investigate to what extent these meanings for slope and rate are multiplicative, that is built on an image of quotient as a measure of relative size. A multiplicative meaning for rate of change is powerful because it allows the teacher to better differentiate between additive and multiplicative situations. The data comes from the administration of the diagnostic instrument named "Meanings for Mathematics Teaching Secondary Math" (MMTsm).

Key words: Secondary Teacher Preparation, Slope, Rate, Diagnostic Instrument

Coper-Gencturk (2015) followed 21 K-8 teachers for three years to determine how their mathematical knowledge and teaching changed over time. She found that the improvement in teachers' mathematical knowledge as a result of the master's program and their overall level of mathematical knowledge played significant roles in "indicating the extent to which teachers were successful in constructing meanings for mathematical rules and articulating what mathematical ideas students were supposed to learn" (Coper-Gencturk, 2015, p. 314). Teachers with lower content knowledge for teaching made superficial changes to their instruction such as putting students in groups to discuss procedures, or adding real-world or hands-on activities that were not clearly connected to the mathematical ideas being taught (Coper-Gencturk, 2015). It is important to understand and address mathematical weaknesses of teachers to help them implement meaningful changes in their classrooms.

Studies that employed time intensive methods such as interviews to study small samples (less than 10) of teachers described teachers who conveyed computational or additive meanings for slope (Coe, 2007; Stump, 1999). Project Aspire developed the instrument named *Mathematical Meanings for Teaching Secondary Mathematics (MMTsm)* to help professional developers and researchers more quickly and meaningfully diagnose teachers' mathematical thinking. This study builds on prior researchers' understandings of teachers' meanings for slope and rate, by investigating the following questions in a much larger sample of teachers using items related to slope and rate of change from the *MMTsm*.

1. What meanings for slope and rate might teachers' responses convey to their students?
2. To what extent do these meanings build on an image of quotient as a measure of relative size?

Theoretical Perspective

Thompson's (2013, 2015) work on meaning is the theoretical foundation of Project Aspire. Thompson defined "meaning" in the context of earlier research on the development of children's mathematical schemes. Harel and Thompson used the Piagetian notion of scheme to define a *stable* meaning as the "the space of implications that results from having assimilated to a scheme. *The scheme is the meaning*" (Thompson, Carlson, Byerley, & Hatfield, 2014, p. 13). Glasersfeld (1995) identified the three parts of schemes as follows:

1. Recognition of a certain situation.
2. A specific activity associated with that situation.
3. The expectation that the activity produces a certain previously experienced result (p. 65).

A person's meaning for a mathematical idea includes both what comes to mind when they encounter an idea and what is immediately implied by whatever comes to mind—what might come to mind easily next.

Literature Review

The explanations of constructs will use examples from interviews with teachers conducted in prior qualitative research. We will explain one non-multiplicative "chunky" way of thinking about slope and the limitations of this way of thinking. According to Castillo-Garsow (2010; 2012) a "chunky" way of thinking about quantities changing entails imagining completed chunk, that is an unit chunk. Thus, an individual using a "chunky" way of thinking is likely to imagine only changes in chunks instead of continuous change. Stump (2001) interviewed pre-service teachers named Joe, Tracie and Natalie and observed their teaching as part of a study on pre-service teachers' understandings of slope and how they expressed their meanings in the classroom. Joe planned and taught lessons on slope after discussions in a methods course designed to help teachers develop stronger meanings for slope. "Joe eventually defined slope as 'vertical change/horizontal change,' and presented a graph of the line passing through the points (0,0) and (3,2). He emphasized that the slope as a fraction, 2/3, up 2, over 3" (Stump, 2001, p. 216). One student in Joe's class "was having difficulty understanding how the two fractions 5/-6 and -5/6 could both represent the same slope. Although at the time Joe struggled in vain to help her understand, he later described her difficulty with the following insight: 'They think you are describing a movement as opposed to you describing a number, a measurement'" (Stump, 2001, p. 216). Although Joe's personal meanings were sufficient to allow him to see '5/-6' and '-5/6' as the same slope, the meaning for slope he conveyed to the student (namely, slope tells us how to go up and over) limited the student's ability to use slope productively. Further, the meaning for slope Joe conveyed to this student was strongly connected to the conventional Cartesian coordinate system and the act of moving over and up in chunks of 2 and 3. His meaning for slope could not be applied to polar coordinate systems or real world situations where two quantities change together, but do not move horizontally and vertically.

Joe conveyed a chunky, non-multiplicative meaning for slope because he did not say for any size change in x the change in y is 2/3 as large. Other teachers also did not strongly connect the idea of slope to the notion of a quotient as a measure of the relative size of the change in x and the change in y. Coe (2007) asked Peggy "why do we use division to calculate slope?" and she replied that she didn't know because "she never really thought of it as the division operation" (p. 207). Even though Peggy realized that there is a division symbol in the formula for slope she seemed not to have questioned how it related to her meanings for division.

Some teachers' tendency to avoid using a multiplicative meaning for quotient in explanations of slope may be because their meanings for quotient are weak. McDiarmid and Wilson (1989) gave a written instrument to 55 alternatively certified secondary teachers with mathematics degrees. He presented them with four story problems and asked them to choose which story problem could be solved by dividing by ½. Only 33% were able to identify quantitative situation that involved division by a fraction. In interviews by McDiarmid and Wilson (1989) some alternate route secondary teachers could see no real world application for division by fractions.

Ball (1989) asked prospective teachers "to develop a representation—a story, a model, a picture, a real-world situation—of the division statement $1\frac{3}{4} \div \frac{1}{2}$" (p. 21). Five out of 9 prospective secondary teachers and 0 out of 9 elementary teachers were able to generate an appropriate representation (p. 22). Byerley and Hatfield (2013) asked 17 pre-service secondary teachers to draw a picture representing a particular division problem. Six out of 17 were able to represent the relative size of 7.86 and .39 in an image to explain the meaning of a quotient (Byerley & Hatfield, 2013). Without an image of quotient as a measure of relative size, it is hard to build a meaning for slope as a measure of the relative size of the change in x and the change in y.

Item Development

The motivation for Project Aspire was to design items and scoring rubrics that allow researchers and teacher educators to categorize teachers' meanings with a written diagnostic instrument. Thompson (2015) summarized the process of creating items and rubrics for the *MMTsm*:

> (1) Create a draft item, interview teachers (in-service and pre-service) using the draft item. A panel of four mathematicians and six mathematics educators also reviewed draft items at multiple stages of item development. In interviews, we looked for whether teachers interpret the item as being about what we intended. We also looked for whether the item elicits the genre of responses we hoped (e.g., we do not want teachers to think that we simply want them to produce an answer as if to a routine question); (2) Revise the item; interview again if the revision is significant; (3) Administer the collection of items to a large sample of teachers. Analyze teachers' responses in terms of the meanings and ways of thinking they reveal; (4) Retire unusable items; (5) Interview teachers regarding responses that are ambiguous with regard to meaning in cases where it is important to settle the ambiguity; (6) Revise remaining items according to what we learned from teachers' responses, being always alert to opportunities to make multiple-choice options that teachers are likely to find appealing according to the meaning they hold; (7) Administer the set of revised items to a large sample of teachers.

We designed the item in *Figure 1* to reveal teachers' meanings for slope in the context of teaching. The inspiration for the name of the item came from Coe (2007) and his observations that the three teachers he interviewed did not connect the idea of slope with a measurement meaning of division. We designed Part B to prompt teachers to reflect on the relationship between any size change in x and the associated change in y because we anticipated many teachers would give responses to Part A that were similar to the student's explanation in Part B. We wanted to see if teachers could move beyond thinking of slope in terms of one-unit changes in x. Part B of "Slope and Division" gives teachers a chance to extend their meanings for slope to situations where x does not change by one, or alternatively reveal the limitations of their meanings for 3.04.

Mrs. Samber taught an introductory lesson on slope. In the lesson she divided 8.2 by 2.7 to calculate the slope of a line, getting 3.04. Convey to Mrs. Samber's students what 3.04 means. **Part B.** Mrs. Samber taught an introductory lesson on slope. In the lesson she divided 8.2 by 2.7 to calculate the slope of a line, getting 3.04. A student explained the meaning of 3.04 by saying, "It means that every time x changes by 1, y changes by 3.04." Mrs. Samber asked, "What would 3.04 mean if x changes by something other than 1?" What would be a good answer to Mrs. Samber's question?

Figure 1. The item "Slope and Division" was designed to reveal meanings for slope.

Rubric Development

After the first round of data collection from 144 teachers in Summer 2012 we categorized the thinking revealed in the items using a modification of a grounded theory approach (Corbin & Strauss, 2007). The modification is that we began our data analysis with strong theories of understanding magnitudes and rates of change, and of the nature of mathematical meanings and of characteristics that make them productive in instruction. We developed rubrics by grouping grounded codes into levels based on the quality of the mathematical meanings expressed.

We read the teacher's response literally, asking, "If this is what they said to a class, what meanings for the mathematical idea might students' learn?" During team discussions of rubrics and responses, we continually asked ourselves. "How productive would the teacher's response be for a student if this is what she or he said while teaching?" and, "How might students understand what the teacher said were they to take it at face value?"

The summary rubric for Slope and Division is given in Table 1. The rubric was refined many times as the project team conducted multiple rounds of scoring on data collected in Summer 2013.

Table 1. Rubric for Part A of "Slope and Division."

Level A3 Response:	The teacher conveyed that x can change by any amount and that y changes by 3.04 times the change in x.
Level A2a Response:	***Any*** of following: – The teacher wrote that for every change of 1 in x, there is a change of 3.04 in y. – The teacher wrote that for every change of 2.7 in x, there is a change of 8.2 in y. – The teacher wrote that a difference in x values is compared to a difference in y values.
Level A2b Response:	The teacher conveyed in words or graphically that the slope gives information about how to move horizontally and vertically. For example: – If x moves to the right 1 space, y moves up by 3.04. – If x runs by 2.7, y rises by 8.2. – The slope tells us to move horizontally by one and vertically by 3.04.
Level A1 Response:	***Any*** of following: – The teacher conveyed that 3.04 is the result of a calculation. – The teacher used a phrase such as "average rate of change", "constant rate of

	change" or "slantiness" without addressing the question of how 3.04 relates changes in x and changes in y. – The teacher simply stated the idiom "rise over run" without describing the changes.

Level A3 responses convey a multiplicative meaning for slope. A multiplicative meaning for slope builds on the meaning for quotient as a measure of relative size. Level A2a and level A2b responses convey an additive or chunky meaning for slope. Level A2a responses are considered slightly more productive for students than A2b responses because the meaning of slope in Level A2a responses is not constrained to horizontal and vertical motion on a Cartesian graph, but could be used productively in real world situations. Level A1 responses on our rubric represented more than one possible meaning for slope, but each of these meanings are similar in the sense that they convey that the meaning of slope is something to memorize. We scored responses that did not fit any other category at level A0. In cases where teachers responded with multiple meanings for slope in one response we decided to categorize their response according to the highest level meaning they conveyed.

Table 2. Categorization for Part B of "Slope and Division."

Gave reasonable meaning for 3.04	The teacher gave a mathematically reasonable explanation of what 3.04 means. For example "3.04 is the ratio" or "3.04 tells us how many times as large Δy is as Δx."
Gave explicit computations to find Δy	The teacher gave a clear instruction to find the change in y, the change in x should be multiplied by 3.04.
Gave vague computations to find Δy	The teacher answered the question "how to you find the change in y?" but does so without explicitly mentioning the change in y. For example they said "multiply it by 3.04."

The purpose of Part B is to allow teachers to think about the change in x varying continuously instead of in jumps of a fixed amount.

Briefly, common responses to Part B included explaining what 3.04 means, explaining how to find the change in y given an arbitrary change in x, or giving an example of how much y would change by if x changed by two. The quality of responses in each category varied from teachers who gave clear and understandable explanations of the meaning of 3.04 to those who explained what 3.04 meant by saying only "multiply it by 3.04." In scoring Part B we noted mathematical mistakes such as confounding y with Δy in a separate score not reported here. Although a portion of the responses at each level do contain mathematical errors, we categorize responses by the primary meaning conveyed ignoring mathematical mistakes. The categorization in Table 2 is based on our rubric for Part B. We will give the rubric for Part B and data in the longer paper.

Administration and Scoring

We administered the *MMTsm* to 157 high school teachers in two different Southwestern cities in Summer 2014. The high school teachers took the diagnostic exam at the beginning of professional development programs. The first author scored all responses to "Slope and Division." To estimate interrater reliability (IRR) an outside collaborator scored 50 overlapping responses. Scorers had perfect agreement on 84% of responses to Part A and 72% on responses to Part B. Non-perfect agreement was scored as disagreement. These IRR scores are lower than most other items on the *MMTsm* due to the complexity of teachers' responses. Responses were often not written in complete sentences and used pronouns with unclear antecedents so it was difficult to determine whether or not a student could make sense of the teacher's explanation. Because of the probability that the scorers might pick the same level by chance we also computed Cohen's Kappa for Part A (.773) and Part B (.621).

Results

The most common meaning conveyed in our sample was a chunky, additive meaning for slope (See Table 3).

Table 3. Responses to Part A "Slope and Division."

Response	Math Majors	Math Ed Majors	Other Majors	Total
A3-relative size	0	0	3	3
A2a-chunky	12	8	21	41
A2b-chunky graphical	19	29	30	78
A1-memorized	4	11	13	28
A0-other/IDK	1	1	2	4
No response	0	0	3	3
Total	36	49	71	157

Only three teachers out of 157 used a multiplicative meaning for quotient in explanations of slope in Part A. Approximately 76% of teachers showed a chunky or additive meaning for slope. Interestingly, about 86% of teachers who majored in mathematics and 82% of teachers who majored in mathematics education answered a chunky, additive meaning. Although chunky meanings for slope can be used productively in some situations, the responses to Part B often indicated that the teachers struggled to extend their chunky meaning to situations where the change in x is not equal to one. The response in Figure 2 conveys that slope gives information about how to move vertically and horizontally on a graph. The response conveys a chunky meaning for slope because the changes occur in chunks of one and 3.04. The meaning of 3.04 seems to be tightly tied to the change in y and only loosely connected to the associated change in x.

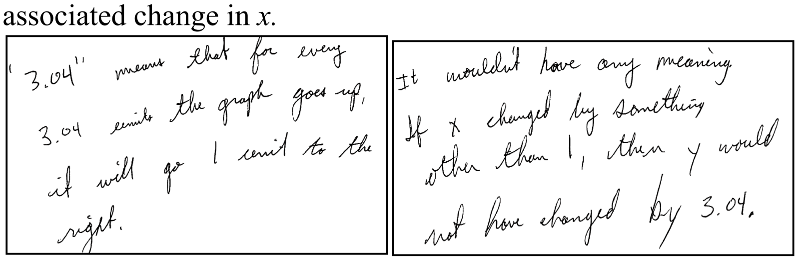


Figure 2. One teacher's "chunky" response to Part A and B.

The Part B response provides confirmation that for this teacher 3.04 is more strongly associated with the change in y, then a comparison of the relative size of the change in y and a change in x. It might convey to students that the slope gives information about vertical and horizontal motion on the graph and that the number 3.04 is only associated with the change in y and not with a comparison of changes in x and y.

The response in Figure 3 conveys that the slope is strongly associated with the change in y. In this case, the response incorrectly confounds the change in y with the slope. When the meaning for slope conveyed emphasizes that x changes by one the value of the slope and the change in y are identical and it becomes easier to confuse the two concepts.

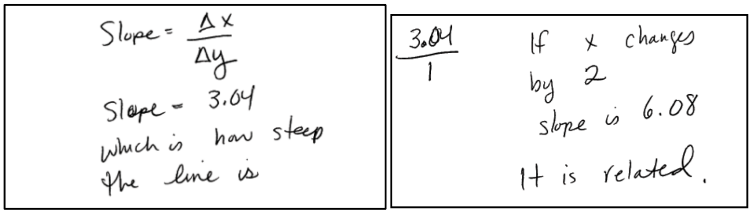


Figure 3. One teacher's response to Part A and associated chunky Part B response.

Some chunky responses conveyed that the only points on the line that "mattered" were the points obtained by the process of moving over and up in fixed chunks (see Figure 4). This response is not consistent with imagining that between any two points on the line there are infinitely many points.

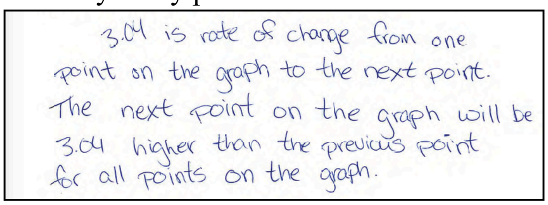


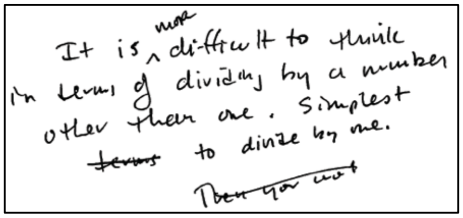
Figure 4. Chunky response conveying that the points on the line only occur at fixed intervals.

There are a variety of consequences of conveying that points on the line only occur at fixed intervals. If points only occur at fixed intervals it is possible to conceptualize slope as the distance between two points on a line. Some teachers in our sample explicitly responded that the slope is a distance between two points and some Calculus students who were interviewed on "Slope and Division" also told us explicitly that slope is the distance between the two points used in the slope formula. After confirming that, to the student, slope is a length, the interviewer asked the student, "Why do you divide the change in *y* and the change in *x* to get a length?" The student responded, "Because, it's you've got the one *x* here and the other one here and so you are trying to find the way which they both get to each other basically."

Table 4. Responses to Part A and Part B of "Slope and Division."

		Part B Response					
		Gave reasonable meaning for 3.04	Gave explicit computations to find Δy.	Gave vague computation s to find Δy.	Other	IDK Blank	Total
Part A	A3-relative size	1	2	0	0	0	3
	A2a/A2b chunky	30	31	18	40	0	119
	A1-memorized	2	8	6	8	4	28
	A0-other	2	0	0	0	1	3
	IDK/blank	0	0	1	0	3	4
	Total	35	41	25	48	8	157

In the case of the 48 responses categorized as "other" it was clear that the teacher struggled to respond to a situation with a change of *x* not equal to one. Note that 40 out of the 119 teachers who conveyed a chunky meaning in Part A were unable to cope with Part B in even a limited way.



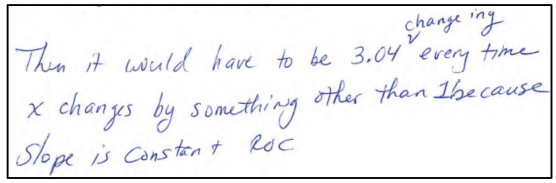


Figure 5. Two teachers (who conveyed chunky meanings in Part A)' responses to Part B

The two teachers in Figure 5 wrote chunky meanings in Part A and had difficulty explaining what 3.04 means when x changes by something other than 1. This is evidence that holding a chunky meaning for slope does not necessarily enable a teacher or learner to understand the proportional relationship between changes in x and changes in y.

Conclusions

The results show that many teachers have chunky meanings for slope that do not appear to be connected to an image of the relative size of Δx and Δy. If their meaning for slope was based on an understanding of the relative size of Δy and Δx, it should be easy to note that Δy is always 3.04 times as large as an arbitrary Δx in Part B. An inability to deal with an arbitrary sized Δx is problematic because in Calculus Δx becomes arbitrarily small yet retains a relationship of relative size with Δy. Although the results are not from a nationally representative sample of teachers, the sample size is large enough to strongly suggest that Stump's (1999;2001) and Coe's (2007) descriptions of a few teachers' meanings for slope are apparent in a much larger sample of teachers. Further investigations could use this instrument to research a nationally representative sample of teachers.

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Separating issues in the learning of algebra from mathematical problem solving

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Students' difficulty in learning school algebra has motivated a plethora of research on knowledge and skills needed for success in algebra and subsequent undergraduate mathematics courses. However, in gateway mathematics courses for science, technology, engineering, and mathematics majors, student success rates remain low. One reason for this may be to the lack of understanding of thresholds in student mathematical problem solving (MPS) practices necessary for success in later courses. Building from our synthesis of the literature in MPS, we developed Likert scale items to assess undergraduate students' MPS. We used this emerging assessment and individual, task-based interviews to better understand students' MPS. Preliminary results suggest that students' issues in algebra do not prohibit them from using their typical problem solving methods. Thus, the assessment items reflect students' MPS, regardless of possible misconceptions in algebra, and provide a mechanism for examining MPS capacity separate from procedural and conceptual issues in algebra.

Keywords: college algebra, problem solving, algebra learning

Issues in Algebra

Research shows several sources of difficulty in learning algebra. For example, students struggle in understanding the meaning of variables. In algebra instruction, x is frequently called a *variable*, accompanied with statements such as "x can be anything." But this conception is particularly misleading for equations such as $2x + 7 = 13$, in which x is actually an unknown quantity (Kieran, 2007). Additionally, in functions, variables stand for inputs and outputs. Many students use x and y to write equations and functions but do not actually attend to the meaning of those symbols; Students use these letters solely as a placeholder in equations and functions to replicate examples they have seen (Chazan, 2000). This confusion increases student difficulty converting word problems into equations (Kieran, 1992).

Algebra students also struggle with the meaning of the equal sign. Although the equal sign is often used to indicate a relationship between two quantities, for a function, the equal sign represents a naming of an object. Further, many students view the equal sign as a connector or operation, with little meaning beyond indicating the direction of the solution path (Schoenfeld & Arcavi, 1988). This connector usage leads to student concatenating operations using an equals sign as if they are using a calculator (i.e. 4 + 7 = 11 + 3 = 14). In elementary school, students use a "guess and check" method of solving equations. However, formal procedures taught in algebra can be difficult for students to internalize, as they have not previously needed to maintain symmetry across the equal sign (Kieran 1992). Further, in solving $2 + \_\_ = 5$, elementary students place 3 in the blank, which appears to add 3 to only one side of the equation.

Although Kieran (2007) asserts that the use of technology in the algebra classroom improves student understanding of functions, technology can also lead to some misunderstandings about equations. Though the equations $y = 2x + 8$ and $-2x + y = 8$ are equivalent, if such an equations are part of a system of equations, x and y correspond to specific values in a solution set rather than inputs and outputs (Chazan & Yerushalmy, 2003).

Studying mathematical problem solving

Nationwide, more than 40% of undergraduates pursuing science, technology, engineering, and mathematics (STEM) majors failed to complete their degrees (President's Council of

Advisors on Science and Technology, 2012), and for many students, progress is blocked by their lack of success in foundational mathematics courses. For example, as few as 10% of calculus-bound STEM intended College Algebra students reach calculus (Dunbar, 2005). Though important skills and procedures needed for success in calculus are identified in the research (e.g., Carlson, Oehrtman, & Engelke, 2010), the specific knowledge and skills emphasized in gateway mathematics courses seems insufficient for students' progression in STEM majors. Students appear to lack the necessary mathematical problem solving (MPS) skills and reasoning to be successful. MPS is at the forefront of instructional goals in mathematics education (e.g., National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; National Council of Teachers of Mathematics, 2000); however, little is understood the thresholds students must meet at various levels to ensure success in subsequent courses.

Campbell (2014) synthesized the research literature in MPS and separated MPS into five components. These components are sense-making, representing/connecting, reviewing, justification, and challenge/difficulty. Building from this work, we developed items to assess student's MPS in these areas. The MPS components or domains attend to three of the six problem solving components identified by Jonassen (1997) and the other components are controlled for in the design of the items. The roles of different components of MPS are largely unstudied due to the time and effort that must be invested to use existing tools (e.g. Oregon Department of Education, 2000; Dawkins & Epperson, 2014). By contrast, the goal of the emerging tool is to create a set of problem solving items that can be machine scored to quickly learn about students' problem solving techniques and practices. In the instrument, students complete a series of problems and then items targeting specific components of MPS. The items and instrument development are explained fully in Epperson, Rhoads, and Campbell (in press).

Student understandings in MPS and Algebra

This research takes place at a large, public university in the Southwest. We administered the MPS instrument to 70 (calculus-bound) College Algebra students and selected 11 for individual, one-hour problem-solving video-recorded interviews. In an interview, the researcher asks the student to explain his or her usual problem solving approaches and the specific MPS used on the problems and items from the assessment. The interview participants also complete a new problem and associated items. The recorded interviews were transcribed for analysis. The research adopts a mixed grounded theory approach to characterize the MPS used by the participants (Corbin & Strauss, 2008; Charmaz, 2006).

Interviews show interesting trends in students' MPS. First, participants only used diagrams and representing/connecting practices at the beginning of the problem solving process and did not incorporate them later. In addition, participants fixated on the problem statement, spending extra time rereading or rewriting the problem statement before attempting to solve the problem. These activities aligned with MPS capacity identified by their work on the MPS items. Participants' difficulties in algebra arose in the interviews. Students used confusing language pertaining to variable or unknown, such as "running through variables" to mean checking multiple values. A student also suggested that the needed *function* was an *inequality*. Despite these difficulties with algebra, students showed reluctance to use less analytical approaches. The students desired elegant functions or equations even if they proposed adequate solution paths using other logical means. However, many eventually used their less-preferred approach. These results indicate that students' limitations in algebra do not necessarily halt their problem solving practices. The implications of equal sign confusion are also under investigation. Separating the challenges of algebra learning from problem solving can provide a window into aspects of MPS necessary for student success in gateway mathematics courses for STEM.

Acknowledgment

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Calculus students' deductive reasoning and strategies when working with abstract propositions and calculus theorems

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In undergraduate mathematics, deductive reasoning is an important skill for learning theoretical ideas and is primarily characterized by the concept of logical implication. This plays roles whenever theorems are applied, i.e., one must first check if a theorem's hypotheses are satisfied and then make correct inferences. In calculus, students must learn how to apply theorems. However, most undergraduates have not received instruction in propositional logic. How do these students comprehend the abstract notion of logical implication and how do they reason conditionally with calculus theorems? Results from our study indicated that students struggled with notions of logical implication in abstract contexts, but performed better when working in calculus contexts. Strategies students used (successfully and unsuccessfully) were characterized. Findings indicate that some students use "example generating" strategies to successfully determine the validity of calculus implications. Background on current literature, results of our study, further avenues of inquiry, and instructional implications are discussed.

Key words: Logic, Implication, Calculus, Theorems, Conditionals

Background and Research Question

Calculus plays a fundamental role in many science, technology, engineering, and mathematics (STEM) areas such as physics and engineering. Thus, many STEM majors will take at least one semester of calculus as part of their major, during which they will encounter propositions, lemmas, and theorems. For example, students encounter the "If a function is differentiable at a point, then it is continuous at that point" theorem. Students must then apply this theorem in a variety of situations, such as when they are given a function that is differentiable or when they are given a function that is continuous. This deductive process, characterized by logical implication, is a hallmark of mathematical thinking. It seems natural to assume that to use a theorem effectively, a student must comprehend logical implication, which requires the understanding of the four classic reasoning patterns. These patterns are provided below with the assumption that the rule "A implies B" holds.

Modus ponens: Suppose A is True. Then B is True.
Inverse: Suppose A is False. Then it is not known whether B is True or False.
Contrapositive: Suppose B is False. Then A is False.
Converse: Suppose B is True. Then it is not known whether A is True or False.

Applying this reasoning can enable a student to know, for example, that a function being continuous at a point does not necessarily imply that it is differentiable at that point.

It is well-established that both children and adults struggle with these kinds of logical reasoning tasks (O'Brien, Shapiro, & Reali, 1971; Wason, 1968). However, it appears that people are more successful when the questions are posed in a context (as opposed to abstractly) (Stylianides, Stylianides, Philippou, 2004). Also, it is well known that students struggle with calculus ideas such as limits, differentiation, and integration (e.g., Carlson & Rasmussen, 2008; Tall, 1993; Orton, 1983; Zandieh, 2000). The instruction students receive about these key calculus ideas often includes theorem or theorem-like statements and students are expected to reason logically from them. Although much work has been done separately on the issues of logical implication and calculus learning, we know little about how students engage with logic tasks that are set in a calculus context. In particular, we were interested in whether calculus students had the same kinds of difficulties with calculus-based tasks as they did with the purely abstract tasks. In other words, are calculus theorems enough of a "context" to support students' productive reasoning or are those tasks treated in the same way as the classical, abstract tasks? This research project was designed to examine the following questions: How successful are calculus students with logical implication tasks set in calculus and abstract contexts? What strategies do students use when engaged in calculus theorem tasks involving logical implications? Answers to these questions can provide insights into student sense-making that can be then used to inform instructional design aimed at improving student understanding of theorems and definitions in calculus.

Research Methods

Similar to much of the prior work on student thinking about calculus, this study was done from a cognitive theoretical perspective and thus students' written and spoken statements were used as data on their thinking and understanding of the ideas. Surveys were given in a first semester differential Calculus I class at a university in New England near the end of the fall semester. There were a total of 52 participants. The surveys consisted of two parts. Part I consisted of calculus theorem tasks that were modeled after the four reasoning patterns on the previous page. In Part II, the same four tasks were given but presented in an abstract manner. Many of these tasks resembled syllogisms (e.g., All men are mortal. Socrates is a man. Therefore, Socrates is mortal) but were stated in a formal context using letters and symbols to represent statements. See Figure 1 for sample tasks. Although other researchers have established the difficulties students have with these kinds of abstract tasks, we sought to establish the extent to which these difficulties were apparent in the (relatively) less abstract context of calculus theorems.

To learn about student strategies, ten students were interviewed. During these clinical interviews (Hunting, 1997), participants were asked to work through a version the survey. They were also asked to explain the reasons for their answers. Interviews were recorded using LiveScribe technology to capture both their written work and spoken answers.

Theorem: ***For all functions f, if f is differentiable at a point x = c, then f is also continuous at the point x = c.***

2) Suppose h is a function that is continuous at $x = 7$. Then
a. h is differentiable at $x = 7$.
b. h is not differentiable at $x = 7$.
c. not enough information to decide whether or not h is differentiable at $x = 7$.

Explain the reason for your answer:

Proposition: ***For integers a and b, if a§b then aba§bab.***

8) Suppose (7)(4)(7)§(4)(7)(4) is true.
Then 7§4 is
a. True.
b. False.
c. Not enough information to decide if True or False.

Explain the reason for your answer:

Figure 1. (Left) A sample task from Part I. (Right) A sample task from Part II.

Data Analysis

Survey responses were coded as "correct" or "incorrect." In addition to coding interviewees' responses as correct or incorrect, during the initial analysis of the interviews, notes were taken concerning the manner in which interviewees explained their answers. The focus was on the kinds of strategies participants used when working through the problems. This phase of the analysis was informed, in part, by prior research on student thinking about implication and additional rounds of analysis utilized techniques from Grounded Theory

(Strauss & Corbin, 1990) to further characterize student strategies. Categories and sub-categories were developed to characterize these strategies. This work builds off a previous work (Case, 2015) and the primary, new contribution in this report is a detailed analysis of the interviewee strategies for carrying out the tasks.

Survey Results

Consistent with prior research, students had difficulties with the abstract tasks. However, as Figure 2 shows, students were more successful on the calculus tasks than on the abstract tasks. On the calculus tasks, 63% answered at least three of the four tasks correctly and 33% answered all four correctly. In contrast, only 8% of students produced correct answers for at least three of the abstract tasks and none got all four correct. These differences between the calculus and abstract consistency percentages were statistically significant, suggesting that the context of calculus prompts students to engage differently with the calculus tasks than with the abstract tasks.

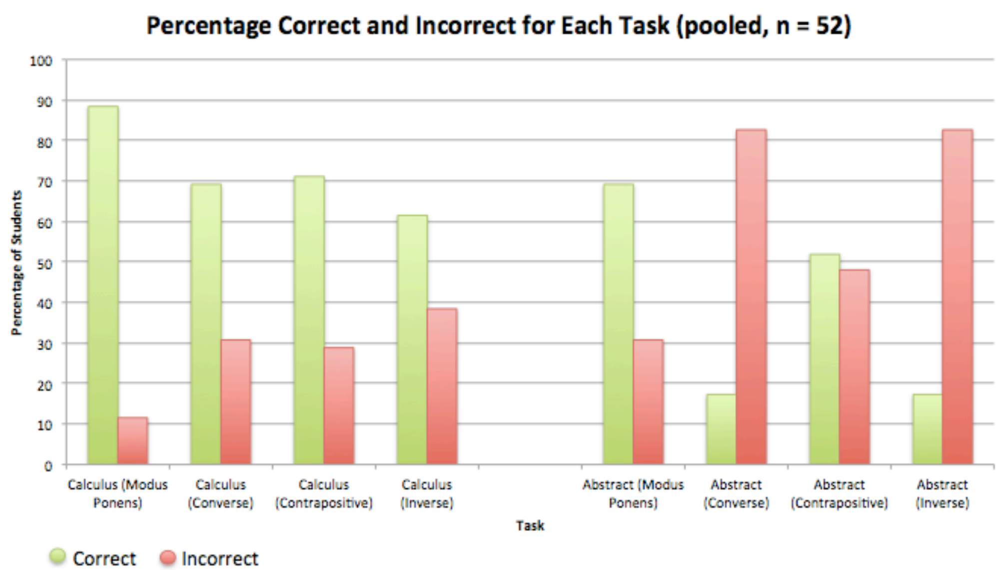

Figure 2. Student Performance on Calculus and Abstract Tasks from Survey Data.

We were also interested in potential relationships between success on one type of task and success on the other. For example, given that a student identified the correct answer to an abstract task, what is the conditional probability that they also answered the calculus version of that same task correctly? Given that a student did not correctly answer an abstract task, how likely are they to answer the calculus version of that same task correctly? The results (see Table 1) show that, for the *modus ponens*, *converse*, and *inverse* tasks, using a 2-proportion z-test, there was no statistically significant advantage when answering the calculus version of a task given a correct answer on the abstract version. However, for the *contrapositive* task, there does seem to be an advantage. Overall, these probabilities suggest that students who answer an abstract task correctly may not necessarily be more likely to answer the calculus version correctly. Stated differently, students can make sense of calculus theorems/definitions whether or not they are able to answer abstract logical reasoning tasks.

Interview Results

Although analyses of the survey data provided some insights (e.g., the calculus context seems to make some of the reasoning patterns easier for students to understand, the abstractly stated tasks are generally much more difficult for students, etc.), we wanted to understand more about student thinking concerning the inferences to gain further insight into the findings from the survey data analyses. From analysis of the interview data, we identified several different ways in which students approached the tasks. As displayed in Figure 3, there were three main ways of thinking (plus “other’), some of which had sub-categories that characterized the thinking at even finer levels of detail.

	Probability of Correct Calculus Answer Given a Correct Abstract Answer	Probability of Correct Calculus Answer Given an Incorrect Abstract Answer	p-value
Modus Ponens	.89	.88	$p > 0.05$
Converse	.89	.65	$p > 0.05$
Contrapositive	.85	.56	$0.01 < p < 0.05$*
Inverse	.56	.63	$p > 0.05$

Table 1. Conditional Probabilities of Answering Calculus Tasks Correctly (* indicates statistical significance with $\alpha = 0.05$)

We first consider the strategy located on the left-most branch. Interviewees who responded with “Child’s Logic” (O'Brien, Shapiro, & Reali, 1971) tended to match truth-values (that is, they responded with “True” given a true premise and responded with “False” given a false premise). This strategy generates correct answers to two of the four tasks. Responses based on some formal knowledge of conditionals were also given a category. Here, participants explained their work by following some rule(s) (e.g., the converse of a conditional statement does not necessarily hold). Responses were also provided that involved the generation of examples. For example, some interviewees drew graphs or verbalized a particular mathematical scenario. Finally, some responses were difficult to categorize and/or did not seem to fit the previous three categories.

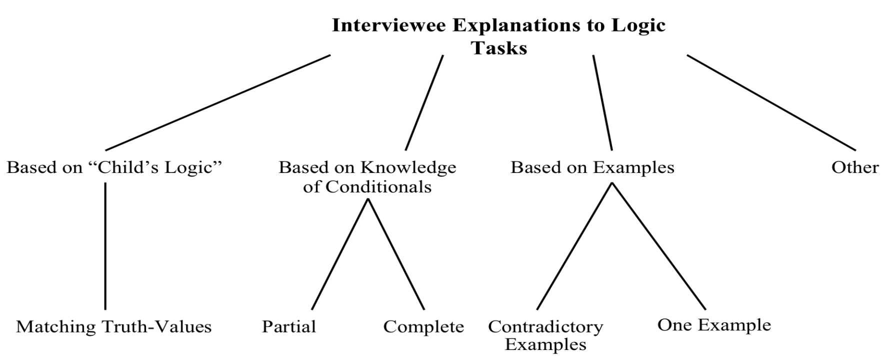

Figure 3. Types of Reasoning Exhibited by Interviewees.

Although each of the strategies provided insights into student thinking, here we discuss just one in detail. This strategy involves generating contradictory examples or situations in order to deduce the correct answer. This method was most often used on the calculus *converse* task and the calculus *inverse* task and it generated rich data on student thinking, and has potentially useful instructional implications (discussed later). We now examine a transcript excerpt that illustrates this kind thinking.

> **Jack:** *So, it’s just like* [pauses to draw axes and says something inaudible] *and something goes like…this* [draws a continuous function with a sharp corner]. *And I mean you could define it as maybe two different line segments and try to do it that way, but the function itself isn’t continuous* [we suspect, from the context, that he

meant “differentiable”] *because at that point there’s no specific, um, rate of change. However, for “b”, um…a function…very well could be not continuous and not differentiable. Say the function just* [draws a linear function with a hole]*…so you have some function that just has a hole in it. It’s not continuous and it’s not differentiable.*

Figure 4. Jack’s Contradictory Examples to the Calculus *Inverse Task.*

Here Jack produces two function graphs that invalidate two of the multiple-choice options (“*f* is continuous at the point” and “*f* is not continuous at the point”) in order to infer the correct answer: “not enough information to decide.” This strategy allows participants to take advantage of the familiar calculus materials presented in the problem so that the correct answer becomes clear. Five interviewees used examples at some point during the calculus portion of the interview. Four out of these five interviewees used contradictory examples.

On the abstract portion, only one student tried to answer a task with a generated example. As discussed above, survey participants did not perform as well on the abstract tasks. This may be in part because they are unable to create scenarios based on the task that they can work and reason with.

Implications and Further Avenues of Inquiry

Not surprisingly, our findings corroborate the established claim that students find abstract logic tasks challenging. However, in contrast, students responded to the calculus tasks in ways similar to how others have responded to logic tasks set in familiar contexts. In other words, although calculus ideas can be consider quite “abstract,” students manage calculus-based tasks in ways that suggest that the context enables them to reason more productively in comparison to the purely abstract tasks. On one hand, these results suggest that calculus students may need more preparation in formal logic, however, even without complete command over formal logic, they are still able reason appropriately when calculus ideas are involved and when they utilize “example generating” strategies. This suggests that it might be useful for instructors to help students develop this strategy. For example, when introducing a theorem such as “differentiability implies continuity”, instructors can model the “example generating” strategy while working through the various cases that might come up when faced with different functions. Some students may believe that they should just know answers to these kinds of tasks and by modeling how to reason through them with examples, instructors can strengthen students’ problem-solving skills. These findings also generated new questions for further research. It would be productive to investigate whether the wording of the theorem and theorem premise affect participant performance (for example, how would participants work through the four tasks if the given theorem structure resembled “if not A then B” rather than “if A then B?”). It might also be useful to examine the impact of instruction about “example generating” (and other) strategies on performance with the goal of enhancing students’ abilities to make sense of the theorems and definitions that are such an essential part of calculus. *Questions posed to the audience will include:* What other theorems or propositions might be worth examining in a study like this? Are there any other teaching implications that may be potentially derived from this study?

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Use of strategic knowledge in a transition-to-proof course: Differences between an undergraduate and graduate student

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The ability to construct proofs has become one of, if not the, paramount cognitive goal of every mathematical science major. However, students continue to struggle with proof construction and, particularly, with proof by contradiction construction. This paper is situated in a larger research project on the development of an individual's understanding of proof by contradiction in a transition-to-proof course. The purpose of this paper is to compare proof construction between two students, one graduate and one undergraduate, in the same transition-to-proof course. The analysis utilizes Keith Weber's framework for Strategic Knowledge and shows that while both students readily used symbolic manipulation to prove statements, the graduate student utilized internal and flexible procedures to begin proofs as opposed to the external and rigid procedures utilized by the undergraduate.

Key words: Mathematics Education; Strategic Knowledge; Proof by Contradiction

Introduction and Overview

The ability to construct proofs has become one of, if not the, paramount cognitive goal of every mathematical science major (Schumacher & Siegel, 2015). However, students at all levels struggle with proof construction (Stylianou, Blanton, & Rotou, 2014), and in particular struggle with constructing proofs by contradiction (Brown, 2013). The purpose of this paper is to report on the results of a pilot study on student's understanding of proof by contradiction in a transition-to-proof course. In particular, this paper will address the following research question: Is there a difference in proof by contradiction strategies between two students, an undergraduate and a graduate student, enrolled in the same transition-to-proof course? The Strategic Knowledge framework, outlined in Weber (2004), will be used to analyze the strategies these students utilized in constructing proofs. The following section will give a brief overview of the Strategic Knowledge framework.

Strategic Knowledge Framework

Weber (2004) developed a framework for describing undergraduate proof construction processes based on the observations of 176 undergraduate students' proofs over multiple studies. This framework classified the types of proofs produced as one of the following: procedural, syntactic, or semantic.

In a proof using a procedural method, "one attempts to construct a proof by applying a procedure, i.e., a prescribed set of specific steps, that he or she believes will yield a valid proof" (Weber, 2004). The procedure can either be an algorithm or a process. Algorithms are characterized as external and highly mechanical to the student, whereas a process is internal and flexible. By external, it is meant the procedure came from outside of the student, such as from an instructor. By internal, it is meant the procedure has been interpreted and constructed by the individual.

In a proof using syntactic methods, "one attempts to write a proof by manipulating correctly stated definitions and other relevant facts in a logically permissible way" (Weber, 2004). Proofs of this form are no more than unpacking definitions and using tautologies to manipulate symbols to achieve the desired conclusion. Students using this method do not need to consider the meaning of their syntactic statements.

In a proof using semantic methods, "one first attempts to understand why a statement is true by examining representations (e.g., diagrams) of relevant mathematical objects and then uses this intuitive argument as a basis for constructing a formal proof" (Weber, 2004). Very few undergraduate research subjects, if any, attempted semantic proofs; 0 of 56 proofs in abstract algebra and 17 of 120 proofs in real analysis.

Methodology

This case study is situated in a larger research project on the development of an individual's understanding of proof by contradiction in a transition-to-proof course. *Bridge to Higher Mathematics/Thinking Mathematically: Intro to Proof* is the first course in which students are formally introduced to mathematical proofs and their accompanying methods at a large, public university in the southeastern United States. Data for this report consists of written student attempts to prove three number theory statements[1] as well as individual interviews detailing their thought process while constructing the proofs.

Two students volunteered to be interviewed in Spring 2015: one undergraduate and one graduate student. The undergraduate, James, is a double major, in Computer Science and Mathematics, while the graduate, Frank, is an Economics major. Despite the difference in degree program, both James and Frank have completed similar mathematics courses and can be considered to have similar mathematical backgrounds.

Data Analysis

A problem-by-problem analysis of the two interviewees using the Strategic Knowledge framework follows. This analysis will begin with an overview of their exhibited proof strategies for the problem, followed by a copy of their written proof for the problem, and ending with an in-depth analysis utilizing the participants' responses during the interview. Due to page limitations, analysis of only two of the three proofs will be provided.

To code proof methods, the following guidelines were used. First, any mention or consideration of the meaning of a mathematical statement was coded as "semantic". If there was no mention or consideration of the meaning of the mathematical statements and the proof was primarily written with symbolic manipulation, the proof was coded as "syntactic". For the remaining methods, rigid (i.e. specific to the particular problem) and external (i.e. rules set by another party) methods were coded as "algorithmic" and flexible (i.e. adaptable to a range of problems) and internal (i.e. synthesized rules for the individual) methods were coded as "process".

Problem 1: If *a* is an irrational number, then *a+2* is an irrational number.

For problem one, James began with an algorithmic approach to the proof. Once he converted the statement to symbolic notation, he then primarily used a syntactic approach to complete the proof. At no time during the proof did he exhibit or profess a semantic approach to the statement. James' written work for problem one is displayed in Figure 1 below.

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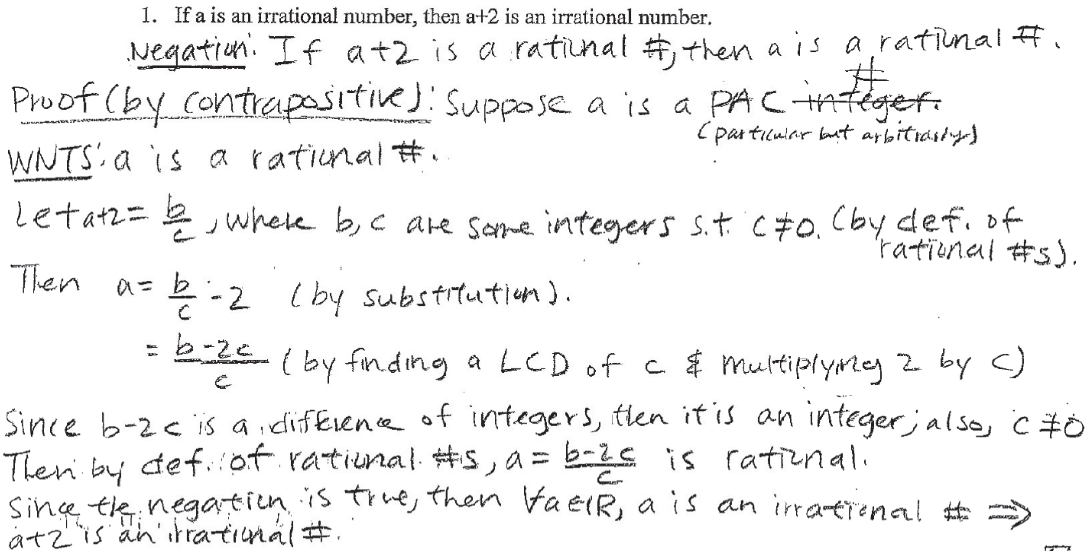
[1] All three statements could be proved by contradiction, though contradiction was not necessary.



1. If a is an irrational number, then a+2 is an irrational number.

Negation: If $a+2$ is a rational #, then a is a rational #.

Proof (by contrapositive): Suppose a is a PAC ~~integer~~ # (particular but arbitrary)

WNTS: a is a rational #.

Let $a+2 = \frac{b}{c}$, where b, c are some integers s.t. $c \neq 0$. (by def. of rational #s).

Then $a = \frac{b}{c} - 2$ (by substitution).

$= \frac{b-2c}{c}$ (by finding a LCD of c & multiplying 2 by c)

Since $b-2c$ is a difference of integers, then it is an integer; also, $c \neq 0$

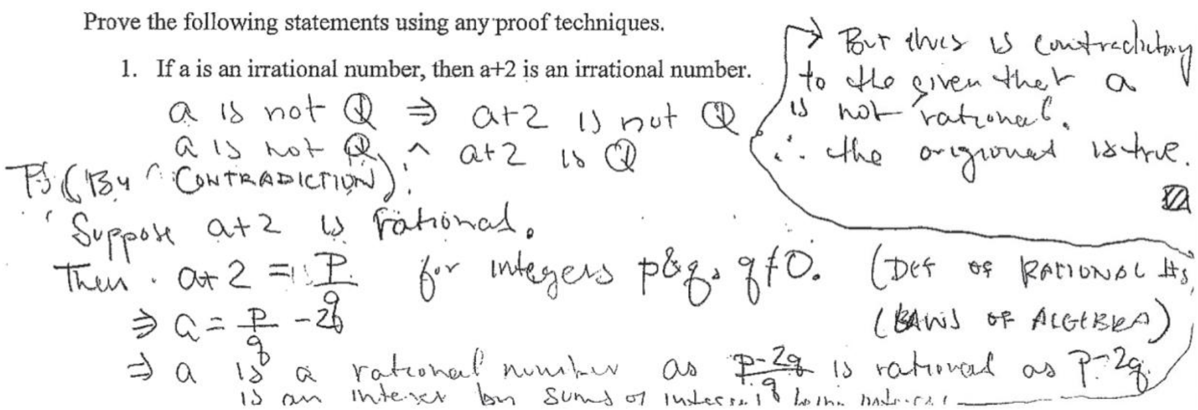
Then by def. of rational #s, $a = \frac{b-2c}{c}$ is rational.

Since the negation is true, then $\forall a \in \mathbb{R}$, a is an irrational # $\Rightarrow$ $a+2$ is an irrational #.

Figure 1: James' Proof for 1st Statement

When asked how he started the proof, James stated "So I guess I did more practice on them [proofs by contrapositive], during Discrete and Bridge, that's where I got used to it." James' use of the phrase "I got used to it" indicates a passive and external role in writing proofs by contrapositive. When asked why he chose contrapositive, he continued to repeat that he uses contrapositive with "these types of proof"; his inability to articulate exactly what this type of proof was illustrates the external nature of why he completed the proof as he did.

For problem one, Frank utilized a syntactic method to write the proof by converting the statement to symbolic notation, after which he manipulated the symbols to complete the proof. He also displayed a flexible procedure for proof by contradiction, though at no time during the proof did he exhibit or profess a semantic approach to the statement. Frank's written work for problem one is displayed in Figure 2 below.



Prove the following statements using any proof techniques.

1. If a is an irrational number, then a+2 is an irrational number.

a is not $\mathbb{Q} \Rightarrow a+2$ is not $\mathbb{Q}$

a is not $\mathbb{Q} \wedge a+2$ is $\mathbb{Q}$

Pf (By CONTRADICTION):

Suppose $a+2$ is rational.

Then $a+2 = \frac{p}{q}$ for integers p & q, $q \neq 0$. (DEF OF RATIONAL #s)

$\Rightarrow a = \frac{p}{q} - 2$ (LAWS OF ALGEBRA)

$\Rightarrow a$ is a rational number as $\frac{p-2q}{q}$ is rational as $p-2q$ is an integer by sums of integers being integers

But this is contradictory to the given that a is not rational. $\therefore$ the original is true. ∎

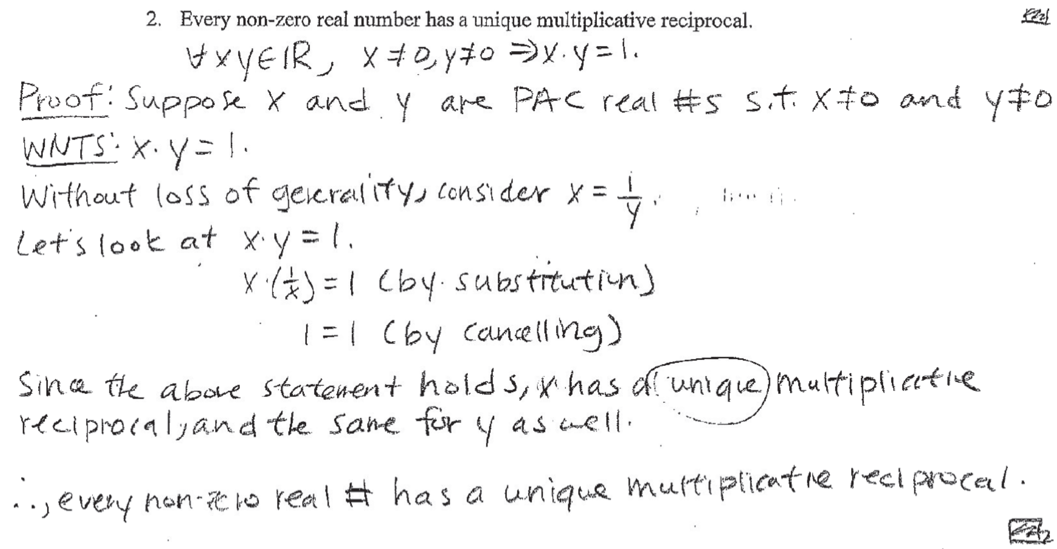
Figure 2: Frank's Proof for 1st Statement

When asked how he started the proof, Frank stated "I basically set it up so that I could say that *a+2* is rational and solved it out and said that by subtracting the two to the other side, you would still get a rational number and then you would get *a* is rational, which is not true because of the givens." This flexible overview of his proof is evidence of procedural knowledge and, in particular, a process for proving statements by contradiction.

As evidenced above, Frank began the proof by converting the statement to be proven into prepositional logic notation and mainly uses syntactic methods to continue in the proof. He does not consider the meaning of the statement, evidenced by his explanation: "But I think once I got here [Suppose *a+2* is rational], it was very obvious that I could just solve it out."

Problem 2: Every non-zero real number has a unique multiplicative reciprocal.

For problem two, James utilized a syntactic approach for the entire proof. However, he showed a procedural approach to the proof in general through his structure and reliance on definitions to fill the holes of the syntactic method. During the discussion of his proof, James showed he explicitly did not use a semantic approach to the statement. James' written work for problem two is displayed in Figure 3 below.


2. Every non-zero real number has a unique multiplicative reciprocal.

$\forall x y \in \mathbb{R}$, $x \neq 0, y \neq 0 \Rightarrow x \cdot y = 1.$

Proof: Suppose x and y are PAC real #s s.t. $x \neq 0$ and $y \neq 0$.

WNTS: $x \cdot y = 1$.

Without loss of generality, consider $x = \frac{1}{y}$.

Let's look at $x \cdot y = 1$.

$x \cdot (\frac{1}{x}) = 1$ (by substitution)

$1 = 1$ (by cancelling)

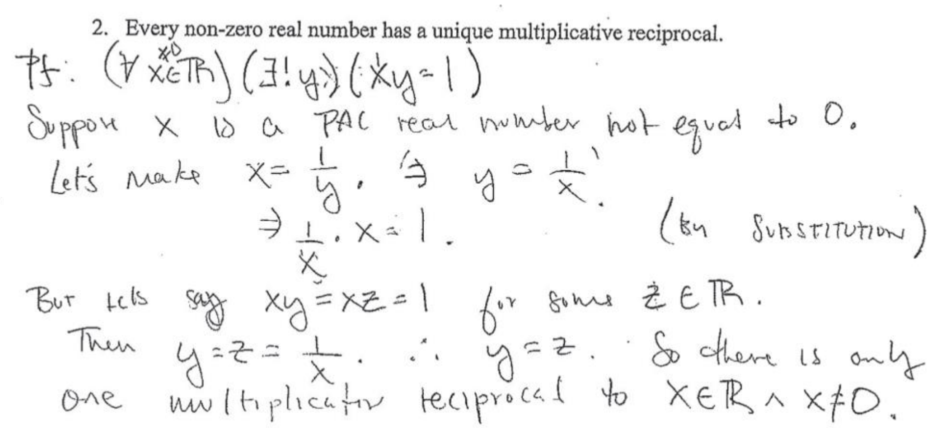
Since the above statement holds, x has a unique multiplicative reciprocal, and the same for y as well.

∴, every non-zero real # has a unique multiplicative reciprocal.

Figure 3: James' Proof for 2nd Statement

James' structure of proof highlights an external procedure to proving the statement. When James cannot prove a statement by symbolic manipulation, he relies on definitions. For example, in the proof above, James makes no justification as to why this reciprocal is unique. When probed whether he used the multiplicative inverse of *x* is *1/x* by definition, he says "Is that a definition? That's not a definition, is it? I don't think it is a definition, in my opinion." However, when probed specifically about why the reciprocal is unique, he states "Because *x* is unique, right? So it is a unique, a unique multiplicative inverse." As no other justification was conveyed, it must be by definition of a multiplicative inverse. This reliance on definitions can thus be seen as an external rule to justifying a statement when a justification is unknown.

For problem two, Frank utilized a syntactic approach for nearly the entire proof. However, he showed a procedural approach to the existence statement. At no time during the proof did he exhibit or profess a semantic approach to the statement. James' written work for problem two is displayed in Figure 4 below.


2. Every non-zero real number has a unique multiplicative reciprocal.

Pf: $(\forall x \in \mathbb{R}, x \neq 0)(\exists! y)(xy=1)$

Suppose x is a PAC real number not equal to 0.

Let's make $x=\frac{1}{y}$. $\ni$ $y=\frac{1}{x}$.

$\ni \frac{1}{x} \cdot x = 1$. (by substitution)

But lets say $xy=xz=1$ for some $z \in \mathbb{R}$.

Then $y=z=\frac{1}{x}$. $\therefore$ $y=z$. So there is only one multiplicative reciprocal to $x \in \mathbb{R} \wedge x \neq 0$.

Figure 4: Frank's Proof for 2nd Statement

Frank began his proof by rewriting the statement in symbolic notation, just as he did in problem one. When explaining how he solved the proof, he stated "For number 2, I … basically put it into a more mathematical format. And then I … did some scratch work to solve for what the multiplicative reciprocal would be." Again, Frank relies on symbolic manipulation to proceed in the statement. However, when asked what type of proof this is, Frank said it was a direct proof. While it was suggested that multiple proofs could be combined, Frank used process of elimination to say the proof did not use contradiction, contrapositive, or induction. Since Frank still successfully proved that the multiplicative inverse is unique with a proof by contradiction (not explicitly), it can be said he has an external procedure to prove the existence of a mathematical object.

Discussion

While both James and Frank used syntactic methods to prove this statement, James relied on rigid, external procedures to each problem to begin proofs, whereas Frank relied on flexible, internal approaches to begin proofs. When possible, both participants utilized symbolic manipulation, and thus exhibited a (productive) use of syntactic knowledge. Furthermore, since participant thought about the meaning of the mathematical concepts in the statements, we conclude that neither used semantic knowledge in their proof constructions.

This case study of two students, one undergraduate and one graduate, builds on the results of Weber (2001), in which Weber interviewed four undergraduate and four doctoral mathematics majors to examine differences in their proof construction. In this case study, the students have similar mathematical preparation and yet, the graduate student utilizes processes exhibited by the doctoral mathematics students in Weber's research. While it is reasonable to expect a difference in proof construction between students with different mathematical background, it is not clear why there should be a difference between students with the same mathematical background and different levels of program. Therefore, more research is needed to examine the differences between undergraduates and graduates with the same mathematical background with respect to their proof construction.

Questions for the Audience

- How does major affect the types of strategic knowledge used to construct proofs?
- How does strategic knowledge fit within a student's proof schema?
- How much of an issue is the concept of infinity when students write contradiction proofs?

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Root of Misconceptions – the Incorporation of Mathematical Ideas in History

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The evolution of a mathematical concept in history has been the process of merging different ideas to form a more rich, general, and rigorous concept. Ironically, students, when learning such well-developed concepts, have similar difficulties and make the same misconceptions again and again. To illustrate, despite the well-developed and defined concept of real numbers, many students still have difficulties in comparing fractions or doing basic operations on irrational numbers. In this poster, the incorporation of different ideas to form a general and rigorous mathematical concept in history is examined. Students' struggles and misconceptions in learning the concepts are investigated from the perspective of the incorporation process. Finally, a model for differentiating and validating the variations of a general mathematical concept is suggested for resolving learning difficulties and misconceptions.

Key words: Misconceptions, History of Mathematics, Formation of Concepts.

Misconceptions due to no differentiation

Things don't always turn out the way you want, and don't always work the way you expect. One common kind of mathematical misconception is no differentiation (Schechter, 2009), for example, adding variables and numbers together (e.g., 5x+3=8) or adding fractions like integers (2/3 + ½ = 3/5). Some no differentiation cases are about properties. For example, everything is additive (e.g., 1/(x+y) = 1/x + 1/y; $\sqrt{(x+y)} = \sqrt{x} + \sqrt{y}$; sin(x+y)=sinx+siny). Everything is commutative (e.g., log2x=2logx, sin2x=2sinx). It seems students are lost in the bigger misconception of "general" in mathematics and overlook the variations of operations, properties, or methods embedded in a "general" mathematical concept.

The possible reason of the misconceptions

Generality is emphasized in mathematics. For example, mathematics is applicable to different fields or mathematical methods work for all cases. However, rather than generality, different mathematical ideas were incorporated in history in terms of extending or modifying existing mathematical structures, or creating general construct to encompass different ideas, for instance, the real number line and the concept of function (Benson, 2003; Kleiner, 1989; Kline, 1972; Ponte, 1992). The incorporation of different ideas to form a general and consistent concept or system may make it difficult for students to differentiate differences. For example, there is rich mathematics education literature about students' struggles in differentiating different operation rules regarding to different kind of numbers (whole numbers, fractions, irrational numbers). Therefore, knowing how different ideas were incorporated in history may help recognize the threads of differences embedded in a general concept of mathematics (e.g., different rules, properties, or methods).

The formation of a mathematical concept in history

In this study, the formation of the concept function and real number line in history were examined. In particular, the original meanings of real-life contexts, operations, and methods that have been lost or hidden in the current meaning of the two concepts were examined. In history, the concept of function started with "tables" (e.g., the values of square roots). The concept of function then was developed as the corresponding values on a graph in analytical geometry in

16th and 17th centuries. After 17th century, with the development of algebra, the focus of function was shifted to analytical expressions (e.g., algebraic expressions), departed from graphs. Since functions as analytical expressions are only a small subset of all functions, the idea of function was gradually changed to the correspondence between sets, numerical or non-numerical, to replace the perspective of analytical expression.

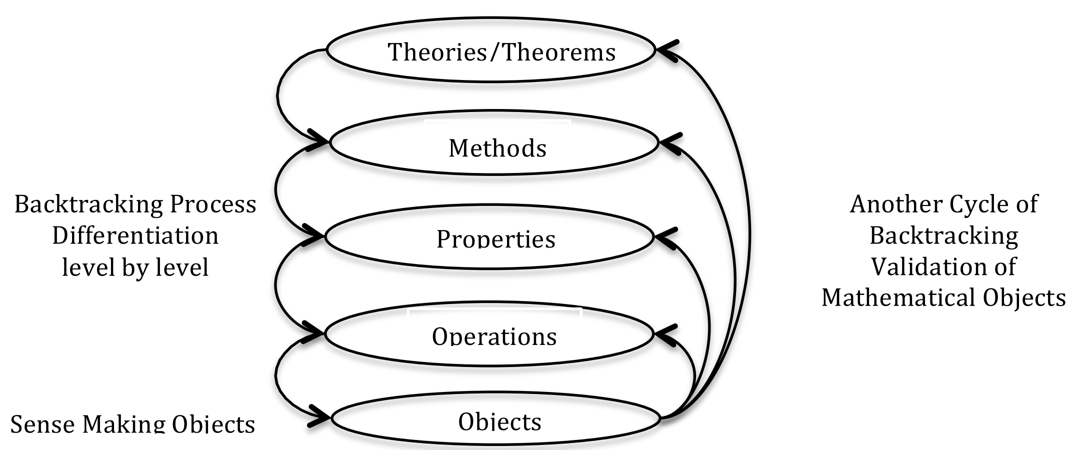
Fractions were invented as a method for dealing with real-life problems (taxes, commercial exchange) in ancient Egypt and China. A fraction was not a number, but a method. Every natural thing exists in the form of natural numbers. The operations on fractions (e.g., adding or multiplying fractions) were based on the idea of ratio, not "numbers". For example, Pythagoreans took fractions as commensurable ratios. Moreover, a square root of a number was not a "number", but a magnitude that could not be explained for a long time in history.

In summary, there are different contextual meanings, rules, and properties incorporated to the concept of function and real numbers, as we have seen in the history. Students' difficulties and misconceptions regarding to the concepts, to some extent, are related to the incorporation process.

The model of differentiation and validation

A model was constructed based on the incorporation of ideas in history. The model (See Figure below) is the backtracking process regarding to the five levels of mathematical entities: (1) mathematical object (2) operation (3) property (4) method (5) theorem/theory. There are different purposes in the backtracking process for avoiding or correcting misconceptions. The special feature of multiple cycles of backtracking process is possible if needed.

Misconceptions on the mathematical object level (e.g., negative numbers, irrational numbers) were corrected by backtracking to the physical world or the existing mathematical models to search for meanings or explanations for new mathematical objects. Misconceptions on the property level (e.g., additive or commutative property) were corrected by backtracking to the mathematical object level to validate new mathematical objects (e.g., quaternions, matrices), and to differentiate new mathematical objects and their operational rules from the existing ones. Misconceptions on the method level (e.g., an infinitesimal is a fixed number) were corrected by backtracking to the operation level (e.g., a variable approaching to a point) and property level (e.g. a continuous function) to refine and replace the idea of infinitesimal. Misconceptions on the theorem/theory level (e.g., continuous functions are not differentiable only at some points) were corrected by counterexamples, which were new mathematical objects with new properties (e.g., a continuous function with nowhere differentiable, concave polyhedrons).



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The Effect of Mathematics Hybrid Course on Students' Mathematical Beliefs

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Computer-based courses (e.g., online or hybrid) have significantly changed the design of pedagogy and curriculum in the past decade, which include online teaching and learning on mathematics education. As beliefs play an essential role on achievement, the impact of computer-based courses on mathematical beliefs is still underdeveloped. In particular, we are interested in whether mathematics hybrid class (blend of online and face-to-face) has different impact on students' mathematical beliefs compared to regular face-to-face class. A two-by-two design of instruction method (hybrid vs. regular) and mathematics performance (high vs. low) was employed. The results showed that both hybrid and regular class students believed understanding and memorization were equally important in mathematics learning. Hybrid class students showed more flexibility in selecting solution methods compared to regular class students on their beliefs about problem solving.

Key words: Hybrid Course, Mathematical Beliefs, Quantitative Research, Developmental Mathematics

Introduction

Computers have been used comprehensively in education in the past decade. The major computer-based course designs have been developed in the form of online (or internet) or hybrid (blend of online and face-to-face). Such course designs make it possible for students to learn any time anywhere (Lemone, 1999; Kadlubowski, 2001). The pedagogy has also been changed significantly (Czerniewicz, 2001; Macdonald et al., 2001) compared to the traditional one. For example, in mathematics, students can watch video lessons, follow step-by-step interactive tutorials, communicate through Internet, and do homework and tests online. Teachers are no longer troubled by pile of homework assignments and tests for grading (Engelbrecht & Harding, 2005; Juan et al., 2011)

A growing research about computer-based mathematics courses addressed a variety of issues including online curriculum/course design (Lee, 2014; Wenner, Burn, & Baer; 2011), factors related to online course achievement (Kim & Hodges, 2012; Kim, Park, & Cozart, 2014; Wadswarth et al., 2007), teaching (Cavanagh & Mitchelmore, 2011; Engelbrecht & Harding, 2005; Juan, Steegmann, & Huertas, 2011), and assessment (Engelbrecht & Harding, 2004; Groen, 2006). However, as mathematical reasoning and problem solving are the core of mathematics practice (Polya, 1954; Schoenfeld, 1992), a Google scholar and Eric Index search find little study about the effect of computer-based courses on students' mathematical beliefs about mathematical reasoning and problem solving methods and strategies. Mathematical beliefs play an essential role in the learning process of mathematics (Schoenfeld, 1983, 1985, 1989) or even academic performance (Carlson, 1999; Schommer-Aikins, 2002). Briefly speaking, one's mathematical beliefs affect his/her ways of thinking and mathematical practice, and ultimately affect his/her mathematics performance.

Therefore, the purpose of the study is to examine students' mathematical beliefs to explore the impact of mathematics hybrid course. In particular, we are interested in the differences of mathematical beliefs between hybrid (blend of online and face-to-face) and regular (face-to-face) class students. This study may contribute to the underdeveloped literature about the impact of online learning on students' mathematical beliefs.

Literature Review

This section will review mathematical beliefs, particularly on mathematical reasoning and problem solving.

Mathematical Reasoning

According to the literature in mathematics and mathematics education, mathematicians or experts emphasize on reasoning and understanding while novice mathematics learners emphasize on memorization and replication. Polya (1954) noted: "The result of mathematician's creative work is demonstrative reasoning, a proof; but the proof is discovered by plausible reasoning…" (p. vi). Ross (1998) noted: "It should be emphasized that the foundation of mathematics is reasoning. While science verifies through observation, mathematics verifies through logical reasoning" (p. 254). Mathematical reasoning includes the sense making of numbers and symbols, the derivation of rules, properties, and theorems, the emergence and utilization of mathematical methods, and the logical connection and analysis of mathematical statements. Rather than focusing on reasoning and understanding, novice mathematics leaners tend to memorize solution procedures and replicate them in problem solving. For example, a learner may be able to apply the method of isolating the variable for solving linear equations without understanding the equivalent relationship between the right and left side of an equation, or the equivalent equations that are transformed from the original equation. In other words, students may be able to apply the addition and multiplication properties by rote in solving equations without knowing the properties. Ross (1998) mentioned: "if reasoning ability is not developed in the student, then mathematics simply becomes a matter of following a set of procedures and mimicking examples without thought as to why they make sense." (p. 254)

Problem Solving

Mathematicians or expert problem solvers try to solve difficult or challenging problems, while novice mathematics learners tend to master routine problems. Schoenfeld (1992) mentioned: "The unifying theme is that the work of mathematicians, on an ongoing basis, is solving problems - problems of the "perplexing or difficult" kind" (p. 15). He noted Halmos' argument that "students' mathematical experiences should prepare them for tackling such challenges. That is, students should engage in "real" problem solving, learning during their academic careers to work problems of significant difficulty and complexity." (p. 16). Indeed, solving difficult or perplexing problems enable one to think, overcome obstacles, and come up with some ways to apply mathematical methods to work out the answer. Novice mathematics learners tend to practice and master certain types of routine problems, which help focus on memorizing and replication. For example, mastering problem types and their corresponding procedures helps one identify similar problems and replicate solution procedures.

While novice mathematics learners try to memorize problems/problem types and their corresponding solution procedures, expert problem solvers try to see the way general rules can be used and worked out in solving challenging problems. As Carlson (1999) noted, an expert in mathematics is the one who "needs to concentrate more on the systematic use of general thought process rather than on memorizing isolated facts and algorithms" (p.247) Indeed, Polya emphasized general rules as “one must have them assimilated into one’s flesh and blood and ready for instant use” (Pólya and Szegö, 1925, preface, p. vii.)

The use of general rules does not mean to use only one way for solving a problem. Multiple ways of solving a problem means to try different plans or strategies, which may involve the same or different general rules, or different ways general rules or properties are applied. Star and Rittle-Johnson (2007) noted Dowker’s (1992) study that “expert mathematicians know and use more strategies than novices, even choosing to use different strategies when attempting identical problems on different occasions”. In other words, experts in mathematics may try different ways in exploring a problem instead of seeking an authoritative way for solving problems. As Carlson (1999) noted, the expert view in mathematics “examine situations in many ways…rather than follow a single approach from an authoritative source”. (p. 247)

Theoretical Framework

Four pairs of contrasting mathematical beliefs are constructed based on the above literature review on mathematical reasoning and problem solving. The first pair of contrasting beliefs is about mathematical reasoning. The rest three pairs are about problem solving. The first pair of contrasting beliefs is “understanding versus memorization”. The second pair is “solving challenging problems versus solving routine problems”. The third pair is “using general methods versus using case-based methods”. The fourth pair is “using flexible methods versus using authoritative methods”.

The four pairs of contrasting beliefs, according the literature review above, can be characterized as the contrasting of expert beliefs versus novice beliefs. In the study, experts and novice learners were characterized as high and low performance students. Combined with the two types of Hybrid and Regular instruction methods, a two-by-two design of instruction method (hybrid vs. regular) and mathematics performance (high vs. low) was employed. The two-by-two table is illustrated below (see Table 1). The two independent variables are the instruction method (Hybrid, Regular) and student performance (High, Low). The dependent variable is student’ mathematical belief scores.

Table 1: The Two-by-Two Research Design

	Hybrid	Regular
High	Four beliefs	Four beliefs
Low	Four beliefs	Four beliefs

The Likert scale of 5 points has been widely used in the literature (Mason & Scrivani, 2004; Schommer-Aikins, Duell, & Hutter; 2005) for measuring mathematical beliefs. A Liker-type item contains the following features: (1) response levels are arranged horizontally; (2) response levels are anchored with consecutive integers; (3) response levels are also anchored with verbal labels which connote more-or-less evenly-spaced gradations and (4) verbal labels are bivalent and symmetrical about a neutral middle (Kislenko & Grevholm, 2008; Uebersax, 2006). A common Liker-type item for measuring mathematical belief ranges from 1 (= totally disagree) to 5 (= totally agree). However, to measure a pair of contrasting beliefs, the points will be arranged in the way that the two ends (1 and 5) mean strongly agreeing to each of the two contrasting beliefs. The two points (2 and 4) close to the middle point (3) mean somewhat agreeing. The middle point (3) remains neutral.

Method

The two research questions of the study are: (1) What are the differences between high and low performance students about their mathematical beliefs? (2) What are the differences between hybrid and regular class students about their mathematical beliefs? In the study, the mathematical beliefs refer to mathematical reasoning beliefs (mathematical understanding vs. memorization) and problem solving beliefs (challenging problems vs. routine problems, general methods vs. case-based methods, and flexible methods vs. authoritative methods).

Participants and Procedure

Students who took developmental mathematics courses (e.g., Foundations for Algebra, Introductory Algebra, or Intermediate Algebra) at a university in the west region of the U.S were invited to participate in this study. The students were given a questionnaire two weeks before the finals week in the Spring Semester 2013. There were 229 students involved in this study, where 204 student data were valid. Among the 204 students, 60 students were from 7 hybrid classes and 144 students were from 11 regular classes. Students' enrollment in either regular or hybrid class was based on their own will. They were not assigned to the classes.

Instruments

The questionnaire contains 15 questions based on the four types of contrasting mathematical beliefs, as described in the framework. In particular, there were two questions (#5, #14) about mathematical understanding (understanding vs. memorization), two questions (#4, #12) about challenging problems (challenging vs. routine problems), four questions (#1, #6, #7, #13) about generality (general methods vs. case-based methods), and three questions (#2, #3, #11) about flexibility (multiple methods vs. authoritative methods).

A Likert-type item was used for each question in the questionnaire (see Table 2 for an example). The scale is from 1 to 5 where 4 and 5 mean the answer is toward (a) (e.g., somewhat (a) for 4 and far more (a) for 5), and 2 and 1 mean the answer is toward (b) (i.e., somewhat (b) for 2 and far more (b) for 1). The number 3 on the scale means equally (a) and (b).

Table 2: An Example of the Likert-type Question

My confidence in preparing for mathematics exams depends on
(a) how many problems I attempted
(b) how many challenging problems I attempted

Far More (a)	Somewhat More (a)	Equally (a) & (b)	Somewhat More (b)	Far More (b)
1	2	3	4	5

The following Table 3 shows one example question for each of the four mathematical belief categories.

Table 3: Categories and Example Questions of the Mathematical Belief Questionnaire

Category	Example Question
Mathematical understanding	When studying mathematics in a textbook or in course materials: (a) I find the important information and memorize it the way it is presented. (b) I organize the material in my own way so that I can understand it.
Challenging	My confidence in preparing for mathematics exams depends on:

problems	(a) how many problems I attempted. (b) how many challenging problems I attempted.
Generality	To me, it is important to: (a) find one method that can be used to solve many problems. (b) memorize different methods for solving different problems.
Flexibility	For learning to solve problems, it is important to: (a) follow the way my teacher teaches or the textbook suggests. (b) find the way I feel like and/or comfortable with.

The questionnaire items were designed in the way that some items had higher score (e.g., 4 or 5) for expert beliefs, and some items had lower score (e.g., 1 or 2) for expert beliefs. This design was to prevent students from seeing any pattern in answering the questionnaire.

Analysis

Students' performance levels (High and Low) were characterized by their final letter grades. Grades A, A– or B+ were grouped as high performance. Grades B or below (including no pass) were grouped as low performance. Students received B+ if their final number grades were 87 or above.

The scores of some Liker-items of the questionnaire were transformed to match the score distribution of 1 to 5 from novice beliefs to expert beliefs.

Four 2-by-2 two-way ANOVA tests were conducted for the average mean of each of the four belief categories – mathematical understanding, challenging problems, generality, and flexibility. Each test contained two independent variables (instruction method, student performance) and one dependent variable (mean score of mathematical beliefs).

Results

Beliefs about Mathematical Understanding

A two-way ANOVA test of instruction method (Hybrid, Regular) and student performance (High, Low) on beliefs about mathematical understanding showed a significant main effect for student performance. The high performance students significantly recognized the importance of understanding (M=3.21) in mathematics learning compared to the low performance students (M=2.96). However, there was no main effect for instruction method. Both hybrid (M=3.00) and regular (M=3.08) class students' belief about mathematical understanding are close to neutral (M=3.00). All hybrid and regular class students believed understanding and memorization were equally important. The statistics of the ANOVA test are shown in Table 4.

Table 4: The Statistics of ANOVA Test on Beliefs about Mathematical Understanding

	M	SD	*F*-score	Interaction	*p*-value
Instruction Method			$F(1, 191)=0.05$	No	0.819
Hybrid	3.00	1.05			
Regular	3.08	0.95			
Student Performance			$F(1, 191)=4.40$	No	0.037*
High	3.21	0.93			
Low	2.96	1.00			

Note. *p<.05. **p<.01

Beliefs about Challenging Problems in Problem Solving

A two-way ANOVA test of instruction method (Hybrid, Regular) and student performance (High, Low) on beliefs about challenging problems showed a significant main effect for students' performance. High performance students significantly recognized the value of challenging problems (M=3.39) in learning to solve problems compared to low performance students (M=3.00). Low performance students took both of challenging problems and the amount of problems equally important in preparing for a test. There was no significant main effect for instruction method. Both hybrid (M=3.14) and regular (M=3.15) class students slightly favored doing challenging problems in learning to solve problems. The statistics of the ANOVA test are shown in Table 5.

Table 5: The Statistics of ANOVA Test on Beliefs about Challenging Problems

	M	SD	*F*-score	Interaction	*p*-value
Instruction Method			*F*(1, 190)=0.18	No	0.673
Hybrid	3.14	1.00			
Regular	3.15	0.86			
Student Performance			*F*(1, 190)=7.72	No	0.006**
High	3.39	0.86			
Low	3.00	0.96			

Note. *p<.05. **p<.01

Beliefs about Generality in Problem Solving

A two-way ANOVA test of instruction method (Hybrid, Regular) and student performance (High, Low) on beliefs about generality (regarding problem-solving methods) showed that there were no significant main effects for both of instruction method (M_{hybrid}=2.92, $M_{regular}$=3.13, p=0.136) and student performance (M_{high}=3.15, M_{low}=3.01, p=0.207).

Beliefs about Flexibility in Problem Solving

A two-way ANOVA test of instruction method (Hybrid and Regular) and student performance (High, Low) on beliefs about flexibility showed significant main effects. Hybrid class students were significantly more flexible in choosing problem-solving methods (M=3.33) compared to regular class students (M=3.11). High performance students were significantly more flexible (M=3.31) compared to low performance students (M=3.09) in choosing problem-solving methods. The statistics are showed below in Table 6.

Table 6: The Statistics of ANOVA Test on Beliefs about Flexibility

	M	SD	*F*-score	Interaction	*p*-value
Instruction Method			*F*(1, 191)=7.10	No	0.008**
Hybrid	3.33	0.71			
Regular	3.11	0.66			
Student Performance			*F*(1, 191)=7.19	No	0.008**
High	3.31	0.66			
Low	3.09	0.68			

Note. *p<.05. **p<.01

Discussions

The ANOVA tests show that there were no differences between hybrid and regular class students on the mathematical beliefs of understanding, challenging problems, and generality. For mathematical understanding, both of the hybrid and regular class students believed understanding and memorization were equally important in learning mathematics (i.e., the belief mean scores of the two groups are 3.00 and 3.08). A possible explanation could be that teachers in the face-to-face developmental mathematics lectures might not emphasize enough on mathematical reasoning (e.g., teaching why), but more on mathematical facts and procedural skills. For challenging problems, both of the hybrid and regular students slightly preferred doing challenging problems in problem solving (i.e., the belief mean scores of the two groups are 3.14 and 3.15). A possible explanation for no difference between the two groups could be the problem solving opportunities (e.g., solving difficult problems) for both hybrid and regular students were similar. Homework problems teachers assigned to the students might have similar level of difficulty for both hybrid and regular classes. This may be due to that all developmental mathematics courses had the same departmental final exam. For generality, a possible explanation for no difference between the two groups could be that the teachers focused mainly on (general) standard algorithms without enough introductions to multiple ways of solving problems. Both of the regular and hybrid students received help from their teachers about the benefit of general methods in solving problems (i.e., hybrid courses have face-to-face sections). For the students' beliefs about flexibility, the hybrid class students were significantly more flexible in selecting solution methods (M=3.33) compared to the regular class students (M=3.11). It is possible that the hybrid students received less authority or emphasis from their teachers (i.e., less face-to-face time) about the selection of solution methods in problem solving.

The ANOVA tests show that there were significant differences between high and low performance students on the mathematical beliefs of understanding, challenging problems, and flexibility. The results are consistent to the literature. High performance students show more appreciation on mathematical reasoning (Carlson, 1999). They are more willing to take challenge in problem solving (Schoenfeld, 1983, 1985). High performance students are also more flexible in the selection of problem-solving methods (Dowker, 1992). However, there was no significant difference between high or low performance students in the use of general or case-based solution methods. A possible explanation could be it is difficult to differentiate general methods from case-based methods in developmental mathematics. Another possible explanation could be that the idea of general methods might not be emphasized in teachers' teaching.

Implications

High performance students may hold more factors of success in computer-based learning context. They may also hold more potential of being enriched and promoted by the extended materials computer-based course can offer. The reasons are: first, hybrid or online classes significantly require self-efficacy and self-regulated learning ability (Hung et al., 2010; Smith, Murphy, & Mahoney, 2003). High performance students, according to this study, hold significantly more positive beliefs in mathematical reasoning and understanding compared to low performance students. The beliefs may strengthen students' confidence in self-regulated learning in understanding mathematics in computer-based learning context. Second, hybrid or online courses have more flexibility in offering students challenging problems or extended materials to challenge students compared to traditional face-to-face classes (Lin & Hsieh, 2001; del Valle & Duffy, 2009). High performance students, according to this study, are more likely to do challenging problems in learning to solve problems. Traditional face-to-face classes generally have more constraints on lecture time, lecture content, and homework assignments due to the different performance levels of students in a class. Hybrid or online mathematics courses may allow high performance students to move faster in watching lecture videos and doing homework, and therefore, to invest more time on self-regulated exploration as well as doing more challenging problems for extra credit. Third, hybrid or online mathematics courses provide more freedom and less authority in problem solving activities (Rosa & Lerman, 2011; van de Sande, 2011). According to the study results, the hybrid and high performance students were significantly more flexible in choosing solution methods. High performance students, therefore, may have more space to develop their own methods or knowledge in problem solving.

Finally, we hope this study sheds light on students' mathematical beliefs under computer-based learning context, and contribute to the effort of enriching online mathematics education.

Limitations

This study has two unfortunate limitations. The first limitation is that this study was not a pre-post research design. The initial design was to attain the gain scores of the pre-post belief items. This study was initiated to help the developmental mathematics department make decision about retaining or dropping hybrid courses. Due to the timeline of decision making as well the IRB (Institutional Review Board) process, the experiment was finally conducted as a one-shot experiment (i.e., questionnaires at the end of the semester for hybrid and regular students). The second limitation is the categorization of high/low performance students. Due to the need of quantitative study, the classification of mathematics performance based on grades was a practical way for this study, but it could result in debates due to different definitions on high/low performance (e.g., the cutting grade could be C for high/low performance). Sometimes, it is even acceptable to differentiate grade A students just because there are different types of grade A students. We understand the limitation, but it seems inevitable.

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On the axiomatic formalization of mathematical understanding

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This study adopts a property-based perspective to investigate the forms of abstraction, instantiation, and representation used by undergraduate topology students when acting to understand and use the concept of a continuous function as it is defined axiomatically. Based on a series of task-based interviews, profile cases are being developed to compare and contrast the distinct ways of thinking and processes of understanding observed by students undergoing this transition. A framework has been established to interpret the participants' interactions with the underlying mathematical properties of continuous functions while they reconstructed their concept images to reflect a topological (axiomatic) structure. This will provide insight into how such properties can be successfully incorporated into students' concept images and accessed; and which obstacles prevent this. Preliminary results reveal several coherent categories of participants' progression of understanding. This report will outline these profiles and seek critical feedback on the direction of the described research.

Key words: Continuous Functions, Topology, Axiomatic Formalism, Abstraction, Properties

Since Hilbert's program at the turn of the last century, modern mathematics has rested on the notion of an axiomatic system (Zach, 2015). These consist of collections of declarative statements, or axioms, whose interactions describe the properties and relationships of the primitive elements in the system. Other properties can be logically deduced from the axioms, without the need for intuition. Reasoning in this manner is considered the ideal goal for students of advanced mathematics, although it may not be natural for many at first (Freudenthal, 1991; Tall, 2013).

This research seeks to illuminate the transition that learners face when attempting to alter and embed their informal and more formal ways of understanding within axiomatic structures. By exploring the participants' transformative use of abstraction in the reconstruction of their concept images for continuous functions in axiomatic contexts, this study contributes to an emerging perspective on the construction of axiomatic mathematical understanding in general. The dual processes involved in the abstraction and instantiation of such properties should play an essential role in the development of axiomatic knowledge structures.

Background

Axiomatic mathematical understanding

Advanced mathematical thinking has been shown to be different from its earlier forms (Harel, 2000; Harel & Sowder, 2005; Sfard, 1994; Sierpinska, 1990; Tall, 2013). Students of advanced mathematics must revise their concept images for earlier ideas in ways that no longer rely on embodied metaphors and intuition about objects in the physical world (Sfard, 1994). Instead, mathematical properties gain importance as they are transformed from descriptions into definitions (Freudenthal, 1991; Tall, 2013). Eventually, the need for axiomatic understanding demands a complete reversal of the relationship between properties and mathematical objects (Freudenthal, 1991; Garcia & Piaget, 1983/1989).

The transition to axiomatic processes of understanding is fundamentally different than earlier transitions faced by mathematics students. It requires a substantial shift in the students' thinking—from descriptive activities concerning the properties of mathematical objects, to the construction of axiomatic systems and definitions based *a priori* on collections of those properties. As students are led toward increasingly abstract forms of thought that are less grounded in everyday experience, this can result in profound difficulties and misconceptions as they build their formal understanding of advanced mathematical topics (Freudenthal, 1991; Harel & Tall, 1991; Tall, 2013). Learners' abstractions, instantiations and representations of mathematical properties are therefore a vital research focus for the exploration of their transitions to axiomatic understanding. This is the primary unit of analysis in the study described here.

Cognitive structures in advanced mathematical thinking

Tall and Vinner (1981) use the term concept image to describe "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). In this definition, the "structure" of the concept image was left largely without description; used as a holistic notion to refer to its totality, rather than as an explicit description of its organization. However, several elements of that structure have since been outlined in detail, such as:

1. Basis for categorization—Whether students categorize pre-requisite concepts through exemplar representations, prototypical abstractions, or metaphorical comparisons is a significant factor in their ability to generalize continuous functions to broader contexts (Alcock & Simpson, 2011; Lakoff, 1987).
2. Defining activities—The properties that have been abstracted into a student's personal concept definition may not coincide with the formal definition, or even the examples they consider relevant to the concept. Analyzing participants' defining activities according to the DMA framework (Zandieh & Rasmussen, 2010; Dawkins, 2012) creates a context for the interpretation of the relationship between concept image and concept definition.
3. Example space structure—Several factors involved in the structure of a student's example space have been examined by researchers in recent years including: density, connectedness, and axiological nature (Sinclair, Watson, Zazkis and Mason, 2011); and its dimensions of variation and range of permissible change (Watson & Mason, 2005). The example space will be a key structural component in this analysis of the concept image.
4. Use of metaphor and embodiment—Many students will continue to use embodied metaphors and their physical intuition to guide their understanding, rather than axioms and definitions. Whether this is intrinsic to mathematical thought (Lakoff & Nunez, 2000) or an obstacle that may be overcome (Sfard, 1994), it remains a factor in any complete study of transitions in understanding.
5. Abstraction types—Hershkowitz, et al. (2001) defined abstraction as "an activity of vertically reorganizing previously constructed mathematics into a new mathematical structure" (p. 202). As participants reconstructed and reorganized their understanding of continuous functions, certain activities related to abstraction were observed and can now be analyzed. Piaget's four types of abstraction will play an important role here (as cited in von Glasersfeld, 1995), as well as Hamton's (2005) explanation of the importance of context in abstraction and instantiation.

While these constructs have been considered in isolation, there has been less work in seeking relationships between these separate elements of the concept image structure. The main contribution of the described research will be to explore a number of distinct cases of interaction between these five elements, forging the way toward a more complete understanding of how the concept image is structured and re-structured as the participants' transition to axiomatic formalism proceeds.

Significance

This study contributes to the theoretical knowledge about advanced mathematical understanding by: 1) providing insight into students' transitions to axiomatic content, especially in the important context of continuous functions; and 2) exploring the effect of shifting the focus of mathematical learning research onto learners' mental representations of mathematical properties rather than mental or mathematical objects.

The continuous function concept is of great importance, not only as a window into the transition of students' understanding toward axiomatic settings, but also in its own right. The long historical formulation of this idea led to the development of the important field of topology (Moore, 1995), and is central to the exploration of topological invariants through its role in the definition of homeomorphism, the defining criteria for the preservation of topological properties (Munkres, 1975/2000).

By studying the transition to an axiomatic system in the context of continuous functions, this research spotlights the participants' abstraction activities relating to important mathematical properties. Not only is continuity itself a property of the reified notion of a function (Dubinsky,1991; Sfard, 1994), but it is a complex of relationships between sub-properties, such as sets that are open or closed, sequences that converge, and images/preimages of a given function. These interactions generate the property of continuity at higher levels of abstraction, and are therefore worthy of investigation. The advanced study of properties and their relationships will be needed to effectively model students' transitions to axiomatic formalism in undergraduate and graduate mathematics classrooms.

There are a number of research perspectives about how students acquire mathematical understanding (Arnon, et al., 2014; Sfard, 1994; Tall, 2013) through the cognitive representations of actions, objects, or symbols. Alternatively, Slavit (1997) demonstrated a "property-oriented view" (p. 263) of students' understanding of functions, blending with Sfard's (1994) operational-structural perspective to "discuss how a student can reify the notion of function as a mathematical object that possesses or does not possess various functional properties" (Slavit, 1997, p. 263). This investigation aims to elaborate this perspective greatly, establishing a scheme to describe the structure of participants' concept images for continuous functions in terms of mathematical properties and students' mental actions upon them. The cases constituted in this study will enable future research on targeted instructional techniques to accommodate diverse profiles of student learning in axiomatic contexts.

Methods

This qualitative, case-oriented research was first informed by several cycles of grounded theory-building that occurred over three semesters at a large university in the southwest United States. The initial studies aided in the development of a categorization scheme for possible factors and obstacles involved in the development of an axiomatic understanding of continuous functions. These categories served as the basis for the constitution of archetypal cases, which came to be organized around learners' uses of abstraction and instantiation of mathematical properties in axiomatic contexts.

Research design

Grounded theory framework for preliminary data generation

The evolution of the theoretical background for this study relied on several iterations of applied grounded theory methodology, which provided enough initial data to extract meaningful dimensions for further study. Three semesters of preliminary interviews served as the ultimate basis for discovering categories in the emerging theoretical model, although those categories were also informed by a pre-existing theoretical framework derived from the research literature. This research guided the formation of interview tasks and questions, designed to elicit specific, observable acts of understanding. These tasks and questions were then modified based on participants' responses and the researcher's own reflective insights. Categories were formulated through this iterative process, which were then modified via reflexive feedback and sharpened into relevant dimensions for study.

Case analyses

The next phase of research used these theoretical categories to develop cases of particular ways of thinking and processes of understanding students use in combination to develop axiomatic understanding. Although it is not claimed that these cases are generalizable, they contribute to an understanding of the interaction among various types of abstraction and instantiation students might use at this stage in their mathematical development. This will be an essential first step to conducting further research in this area.

The choice of case-oriented research for this purpose was justified by the complexity of the phenomena being investigated. Whereas a variable-oriented approach presupposes a homogeneous population from which to select a randomized sample, case studies seek to draw out differences in the population and to explore complex relationships between conditions and outcomes. Through in-depth investigations of cases, Ragin (2004) explains that qualitative researchers can often account for "causal heterogeneity" and "conjunctural causation" (p. 135); providing models for phenomena with multiple factors that the analytic tools of variable-oriented researchers cannot manage.

Participants and Selection Criteria

Five participants from an undergraduate topology class of approximately 30 students were selected for profiling. They were chosen for their theorizing capacity based on their answers to a prerequisite knowledge assessment and brief interviews. Criteria for selection were divided into four categories relating to their understanding of the prerequisite concepts: 1) categorization schemes and types of abstract representations, 2) personal concept definitions and their alignment with the formal definition, 3) example space structure and coherence, and 4) use of metaphor, visualization and multiple representations.

Procedures

This investigation is a multiple-case analysis consisting of five distinct cases of participants' cognitive transformations as they reconstructed their concept images for continuous functions to reflect an axiomatic structure. The cases were chosen based on classroom observations, a preliminary assessment, and a brief interview with each individual in the sample pool. The theoretical criteria for the constitution of these cases came from the research literature and insights that have emerged from preliminary study data as described above, as well as a textbook and curricular analysis.

Textbook analysis

Twelve topology-related textbooks, used widely in introductory topology courses across the U.S., were analyzed in the preliminary data collection process. In particular, one textbook (Croom, 1989) was chosen by the participating professor as the course textbook for the semester of the study. The goal was to discern the intended learning that authors expect students to follow while transitioning to an axiomatic understanding of continuity. These sequences represent classical categorization schemes for the central notion of continuous functions and several pre-requisite and co-requisite concepts such as: functions, open and closed sets, sequences, and limits. Such schemes are the goal state for the structure of students' concept images and not representative of the natural categorization schemes that most students will adopt at first.

The approaches that were studied varied widely with respect to these topics, affected in some cases by the need to construct the concepts from prior knowledge, and in others by the authors' willingness to present an abstract definition without explicit motivation. Codes for each of the analysis categories reached saturation, with themes becoming redundant among the twelve textbooks. Nevertheless, these codes represent a large variety of potential didactical approaches to the wider subject of continuous functions. Different blends of the above approaches might be chosen by the professor, with more or less emphasis on examples, prototypical abstractions, categorization rules, or metaphors. Variations in the presentation of the content may influence students' approaches to understanding the topics, presenting possible future avenues for research.

Task-based interviews and artifact analysis

Since a learner's enactment of understanding is fluid and context dependent (Duffin & Simpson, 2000; Sierpinska, 1994) qualitative, task-based interviews were deemed the most appropriate manner of eliciting appropriate actions and capturing the evolving state of her or his cognitive structure. However, there are challenges involved in eliciting a full, reasoned solution or proof in a time-limited setting. Participants may demonstrate some of their reasoning processes in this way, but they cannot necessarily demonstrate their ability to formally produce a proof, or work through complex threads of logical reasoning. For this reason, classwork (e.g. quizzes, exams) and homework was also analyzed in order to gain insight into the participants' full range of mathematizing abilities.

Analysis was centered on participants' in-class work and the results of three rounds of task-based interviews held throughout the semester. These hour-and-a-half long interviews were focused on these three broad topics: 1) the description and use of open/closed sets, sequences, and real-valued continuous functions; 2) the description of continuous functions in abstract contexts; and 3) the use of continuous functions in abstract contexts. The interview questions were designed to elicit the participants' personal concept definitions and elements of their concept images for these topics, such as salient metaphors, the example space structure and the basis for their categorization schemes. Students were then tasked with reconciling their definitions to divergent elements of their concept images and/or the formal definitions for these topics. Further tasks were designed to provoke acts of abstraction from the participants as they tried to enact their understanding in proof and problem-solving contexts.

Questions for Audience

1. Might it be possible to find different cases of student thinking in different classroom contexts (e.g. metric space courses, geometrically-oriented introductions, or more abstractly presented material)?
2. To what extent are a student's uses of abstraction and instantiation related to each other? In other words, could we hope to predict how a student uses a mathematical concept by the process they used to define it?

3. In what other ways might the transition to axiomatic formalism reflect or contrast with earlier transitions?

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Calculus Students' Understanding of the Vertex of the Quadratic Function in Relation to the Concept of Derivative

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Abstract

The purpose of this study was to gain insight into thirty Calculus I students' understanding of the relationship between the concept of vertex of a quadratic function and the concept of the derivative. APOS (action-process-object-schema) theory (Asiala et al., 1996) was used in analysis on student written work, think-aloud, and follow up group interviews. Students' personal meanings of the vertex, including misconceptions, were explored, and how students relate the vertex to the understanding of the derivative. Results give evidence of students' lack of connection between different problem types which use the derivative to find the vertex. Implications and suggestions for teaching are made based on the results. Future research is suggested as a continuation to improve student understanding of the vertex of quadratic functions and the derivative.

Keywords: Quadratic function, Vertex, Derivative, APOS

It is well documented that students have trouble with quadratic functions (Afamasaga-Fuata'i, 1992; Eraslan 2008; Metcalf, 2007; Zaslavsky, 1997). It is also well documented that students have trouble with the concept of the derivative. As the derivative can be used to find the maximum or minimum of a quadratic function, also known as the vertex, this study aims to explore thirty *Calculus of Variable I* students' understanding of the relationship between the two concepts, vertex of a quadratic function and its' derivative. Understanding of a vertex by calculus I students is closely tied to students' understanding of a quadratic function for which the vertex is a particular point; a point with respect to which many algebraic and graphical properties of a quadratic function could be described (such as, the extreme value, axis of symmetry, increasing/decreasing values of a function). For this study, understanding the relationship between the vertex of a quadratic function and the derivative function includes recognizing that at the vertex of a quadratic function, the slope of the tangent line is zero, as well as being able to relate other properties between a quadratic function f and its derivative f', such as where the function is increasing or decreasing in relation to the values of the slope of the tangent line. This presents the following research question: How do Calculus I students perceive and relate the concept of the vertex of a quadratic function to the derivative in different problem situations?

There are some studies that specifically include a component on the vertex of a quadratic function (Borgen and Manu 2002; Ellis and Grinstead, 2008) Most of the studies that focus on the vertex of quadratic functions were done with students in classes that usually precede Calculus I. Studies with Calculus I students often involve concepts such as functions and variables (Vinner & Dreyfus, 1989), limits and continuity (Ferrini-Mundy & Graham, 1991), derivative (Maharaj, 2013; Orton, 1983a), and integrals (Orton, 1983b) without a specific focus on the vertex of a quadratic function. Asiala, Cottrill, Dubinsky, & Schwingendorf (1997) explored calculus students' graphical understanding of a function and its derivative and suggested that students who had an instructional treatment based on theoretical analysis may have more success in fostering an understanding of the graph of a function and its derivative versus those in traditional courses. Other studies, such as White and Mitchelmore (1996) found that calculus students had an “underdeveloped concept of a variable” (White & Mitchelmore, 1996, p. 88). However, most studies do not necessarily make an explicit connection between the derivative and the vertex of a quadratic function. This study aims to look at Calculus I students’ understanding of the relationship between these two concepts.

Analysis of student written work, think-aloud sessions, and follow up group interviews were done using APOS theory as an assessment tool to classify and make distinctions in students’ answers and reasoning on problems relating the vertex of a quadratic function and the derivative (Asiala, Brown, DeVries, Dubinsky, Mathews, & Thomas, 1996). APOS framework was most appropriate to analyze student perception and understanding of the concept of vertex of the quadratic function in relation to the derivative because of the theories ability to describe and analyze possible mental constructions representing different levels of students’ understanding. This theory has been proven useful for constructing a genetic decomposition of a function (Breidenbach et al., 1992), as well as a good model for studying about learning and teaching of other important mathematical concepts (Asiala et al., 1997; Clark et al., 1997; Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996). By attempting to characterize student understanding of the vertex of a quadratic function in relation to the derivative based on the action, process, and object levels, this framework proved to be a useful tool in interpreting students’ performance and understanding.

As part of a larger study, this poster presentation offers results from two questions from the think aloud interviews, an algorithmic problem and a real world application problem, used to determine if students could recognize the relationship between the vertex of a quadratic function and the derivative in two different problem contexts. Misconceptions of the vertex of a quadratic function, including misconceptions of the vertex as always being an intercept, misconceptions of the vertex as the origin, and misconceptions of the vertex as a point of inflection all contributed to student difficulty with answering and describing questions pertaining to the relationship of the vertex of a quadratic function and its’ derivative function. Many of the students work appeared to be consistent with action conception of understanding the vertex and its’ relationship to the derivative according to APOS. On the other hand, the more a student was able to talk and describe about the vertex accordingly, the more conceptually the student was able to talk about the concepts and the relationships between the vertex of the quadratic function and the derivative function. According to APOS, those students who could speak with meaning possibly exhibited at least a process level of understanding, as they could reflect and describe the reasons behind the steps.

Several implications for teaching are suggested based on the results and discussion. First, since many students appear to be performing at an action level of understanding, it is important to reassess how students are taught. It is understood that teaching goals should be to help students to develop their understanding of concepts beyond the action level. Pedagogical methods might include students talking out loud either in groups or as a class to explain reasons behind their procedures. A combination of individual and group activities on various APOS levels, class discussion, and individual exercises could help foster conceptual growth in students.

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A Study of Common Student Practices for Determining the Domain and Range of Graphs

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This study focuses on how students in different postsecondary mathematics courses perform on domain and range tasks regarding graphs of functions. Students often focus on notable aspects of a graph and fail to see the graph in its entirety. Many students struggle with piecewise functions, especially those involving horizontal segments. Findings indicate that Calculus I students performed better on domain tasks than students in lower math course students; however, they did not outperform students in lower math courses on range tasks. In general, student performance did not provide evidence of a deep understanding of domain and range.

Keywords: Graphs of functions, Domain and range, Cognitive research

Functions are important because they model quantitative relationships and serve as foundational notions for more advanced mathematics topics (Blair, 2006). However, the concept of a function, the different representations of functions, and how they are linked post challenges for students (Kaput, 1989; Kleiner, 2012; Sierpinska, 1992; Tall & DeMarois, 1996). Domain and range play key roles in understanding relationships between the two variables in a function (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). Yet, there has been little research on common practices students use to determine the domain and range, specifically for the graph of a function.

Common Practices: Strategies, Transitional Conceptions, Use of Representations

During the meaning-making process, individuals often rely on their own practices. These practices, which are based on conceptions that have developed around mathematical ideas, include strategies that individuals choose to employ to help develop understanding and solve items. Chiu, Kessel, Moschkovich, and Muñoz-Nuñez (2001) defined a *strategy* as "a sequence of actions used to achieve a goal, such as accomplishing a particular task or solving a particular problem" (p. 219). Following Smith, diSessa, and Roschelle (1993), Moschkovich (1999) defined a *transitional conception* as "a conception that is the result of sense-making, sometimes productive, and has the potential to be refined" (p. 172). To study individuals' meaning-making processes, it is crucial to consider their transitional conceptions along with the strategies they employ and the representations they use when engaged in mathematical tasks.

In previous research (Cho & Moore-Russo, 2014), ten common practices on tasks involving the domain and range of a function's graph were identified. Building on the findings from that study, this study considers the following research questions:

1. How do common student practices align with students' performance on tasks involving the domain and range of a function's graphical representation?
2. How do students in different mathematics courses perform on tasks involving the domain and range of functions in graphical form?

Methods

The study participants were students enrolled in one of three mathematics courses at a four-year college in the eastern United States. Algebraic Problem Solving (APS), Pre-calculus (Precalc), and Calculus I (Calc) courses were selected for this study, since these courses address both the concept of function, in general, as well as the notions of domain and range, in particular. Six of the courses were APS classes, two were Precalc classes, and three were Calc classes. In total, there were 219 participants in the study: 128 APS students (under four instructors), 54 Precalc students (under two instructors), and 37 Calc students (under two instructors).

The APS course, commonly known as College Algebra at other institutions, is open to all students, meets the basic mathematics competency requirement for the college, and introduces the ideas of function, domain, and range. In Precalc, instructors concentrate on how to identify the domain and range of the graphical representations of functions, and students work with a variety of functions, including piecewise functions. In the Calc course, students use the concept of domain and range on graphs, but instructors do not directly teach those concepts.

Data collection

The research team members had over 40 years combined experience teaching secondary and postsecondary courses. Based on their experience and previous research, the researchers developed a paper-and-pencil multiple-choice test that consisted of 20 graphs. The graphs consisted of a variety of functions and included both continuous and discontinuous piecewise functions. Odd numbered items required a response to a function's domain and even numbered items required a response to its range; hence, there were a total of 40 items. Each item had five options. The instrument reliability was acceptable (Cronbach's $\alpha = .69$).

To remind students of the concepts of functions, domain, and range, the definitions for all three were listed on the front page of the test. Students were motivated to complete this test as a means to check and develop their concepts of domain and range. They did not receive any compensation, and their participation was voluntary. After obtaining consent from student volunteers, the multiple-choice test was administered in class at a time most convenient for the instructors. All students completed the test within 20-30 minutes.

Data analysis

The data were analyzed using SPSS software. All statistical tests used $\alpha = .05$ when assessing statistical significant and were two-tailed (where appropriate). A MANOVA was used to analyze if students' performance on the domain and range tasks in the 40 items varied according to the college mathematics course in which they were enrolled. This was appropriate since domain and range task performance are both dependent variables in this study. In addition, choosing a MANOVA (as opposed to two separate ANOVAs) reduces the likelihood of committing a Type I error, as well as accounts for any correlation between the dependent variables. In addition, a series of Bonferroni-corrected post hoc comparisons were used to find which math courses differed in domain and range performance.

Results

The first research question examines how students' practices align with their performance on tasks involving the domain and range of functions in their graphical form.

Significant transitional conceptions or strategies

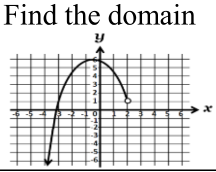
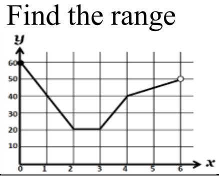
This study looks for relationships between how often students used common conceptions, strategies, or representations in light of their performance on the domain and range items. The research team used the most common student practices identified in previous research (Cho & Moore-Russo, 2014), which are listed in Table 1 with their codes.

Table 1
Common Practices Associated with Incorrect Responses

Abbreviation	Common Practice
EdptFocus	Focusing on the endpoints of a graph or the interval endpoints of a discontinuous graph
ConfuseDR	Confusing the domain and range
IntDescend	Representing an interval in set notation in descending order
NoOverlap	Not combining abutting or overlapping intervals
Intercept	Focusing on either x- intercept or y-intercepts
IntNotation	Confusing the notations for open ( ) and closed [ ] intervals
RangeLtoH	Treating the range as continuous from the lowest to the highest value for a discontinuous, piecewise function
OpenPoint	Not noticing or ignoring an open point

To show how items were coded for common practices, Table 2 provides examples of the coding used for two items. Note that for each of the 40 items, codes were not assigned to the item's correct answer nor were they assigned to option E, "None of the above."

Table 2
Examples of Codes Assigned to the Options of Selected Items

Test items	Multiple choice options	Assigned Codes
11. Find the domain	A) $(-\infty, 2)$ or $x < 2$	None (correct answer)
	B) $(2, -\infty)$	*EdptFocus, IntDescend*
	C) $\{-3.2\}$ or $x = -3.2$	*Intercept*
	D) $(-\infty, 6)$ or $x < 6$	*ConfuseDR*
	E) None of the above	None
24. Find the range	A) $[60, 20]\cup[20, 20]\cup[20, 40]\cup[40, 50)$	*NoOverlap, IntDescend*
	B) $[20, 60]$ or $20 \leq y \leq 60$	None (correct answer)
	C) $[20, 50) \cup (50, 60]$	*NoOverlap*
	D) $[60, 50)$	*EdptFocus, IntDescend*
	E) None of the above	None

The occurrences of the coded practices for each student were tallied and compared against the percentage of domain and range items the student had answered correctly. The correlation matrix in Table 3 displays those results. Results in Table 3 suggest that the practices students used did not have the same relationship with student performance on the domain and range items on the test. For example, more frequent use of the *Intercept*, *IntNotation*, and *OpenPoint* strategies tended to have more of a negative influence on performance on domain items than range items. Interestingly, neither the *Intercept* nor the *OpenPoint* strategy appeared to be related to students' performance on range items. There were also strategies that had a stronger, negative relationship with performance on range items. The more often students used the *EdptFocus*, *IntDescend*, or *NoOverlap* strategies, the fewer range items they answered correctly. It should be first noted that the *NoOverlap* strategy only applied to range tasks. Next, there does appear to be an issue of multi-collinearity between the *EdptFocus* and the *IntDescend* strategies (r = .94). This is most likely because these strategies were often present in many of the same items. In addition, the lack of a relationship between the *RangeLtoH* strategy with both performance on domain items (r = -.09) and range items (r = -.06) seem to suggest that the use of this strategy neither helps nor hinders a student's performance on domain and range tasks.

Table 3
Summary of Correlations between the Frequencies of Common Practices Associated with Incorrect Responses and Performance on Domain and Range Items

Variable	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
1. *% Domain Correct*	1									
2. *% Range Correct*	.63**	1								
3. *EdptFocus*	-.49**	-.66**	1							
4. *ConfuseDR*	-.52**	-.50**	.31**	1						
5. *IntDescend*	-.45**	-.66**	.94**	.34**	1					
6. *NoOverlap*	-.26**	-.46**	.50**	.11	.47**	1				
7. *Intercept*	-.28**	.00	.01	.05	.02	-.02	1			
8. *IntNotation*	-.54**	-.35**	.39**	.19**	.30**	.25**	-.07	1		
9. *RangeLtoH*	-.09	-.06	-.08	.15*	-.12	-.07	.02	.09	1	
10. *OpenPoint*	-.25**	.00	-.08	.04	-.19**	-.23**	-.12	.21**	.27**	1

Note. n = 219; *p < .05; **p < .01

Most difficult items for students

The six items with the lowest percentage of correct responses are displayed in Table 4. Items 27, 28, and 29 involved piecewise function graphs with several horizontal segments or open end points. Most students who selected incorrect options for these items either did not notice or ignored the open point to measure the domain or range. Many students also chose option E as the answer for these items, which might suggest they did not have a strategies for how to solve these tasks. Item 34 was a piecewise function graph whose output values overlapped. Most students who selected an incorrect option for this item either did not notice or ignored the overlapped portion. Item 38 was a piecewise function graph with one horizontal segment and two end arrows denoting that the function continued both as the inputs approached negative and positive infinity. Many students seemed to focus on this and selected option C, which stated "all real numbers". However, many students' transitional conceptions failed to take into account the vertical gap in outputs between the values of 2 and 3. The graph displayed in item 17 showed part of a parabola with two open points, one on the x-axis and one on the y-axis. Many students did not notice or ignored the open point on the y-axis.

Table 4
Occurences of Common Practices Associated with Incorrect Responses on Most Difficult Items

Item	Correct Responses	Common Practices Associated with Incorrect Responses						
		EdptFocus	*ConfuseDR*	*IntDescend*	*NoOverlap*	*IntNotation*	*RangeLtoH*	*OpenPoint*
27	20.55%		X		X		X	X
29	22.37%	X					X	X
34	24.20%	X			X		X	
38	24.20%	X	X	X		X		
17	26.03%	X	X	X				X
28	27.85%		X				X	X

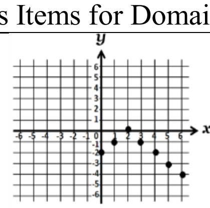
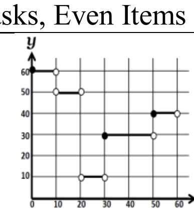
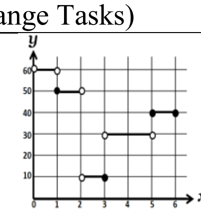
A matched pairs *t*-test was conducted to see if there was any statistically significant difference between how well students performed on the domain items as opposed to the range items. On average, students answered 52.26% of the domain items correctly (SD = 22.16%), whereas they only answered 43.54% of range items correctly (SD = 21.47%). This mean difference of 8.72% was statistically significant, $t(218) = 6.91$, $p < .01$, $d = .47$. On average, students performed nearly a half standard deviation better on the domain items, which is a moderate difference. This result concurred with the previous study's findings, which found that range items were more difficult than domain items for students (Cho & Moore-Russo, 2014).

Performance on items involving piecewise functions

Students seem to lack strategies, even ones related to transitional conceptions, to make meaning of piecewise functions when determining domain and range. Many participants selected option E indicating that none of the response options provided for an item was correct. However, the correct response was located in options A through D for each item on the test. The research team found that a higher percentage of students selecting option E came from piecewise functions that included horizontal segments or disconnected points in their graphs. In fact, the six items with the highest percentages of students choosing E were domain and range tasks related to three graphs of piecewise functions that included horizontal segments or disconnected points. Those items were item 21, 22, 27, 28, 29, and 30 (see Table 5). Recall that odds items involved domain tasks, and even items involved range tasks.

Table 5

Items with Highest Percentages of Option E Responses

		Graph on Test and Associated Items (Odds Items for Domain Tasks, Even Items for Range Tasks)		
		 Items 21 and 22	 Items 27 and 28	 Items 29 and 30
% of Option E Responses	Domain	20.55	19.63	24.20
	Range	23.74	24.20	23.29

The percentages of option E responses was 20.55% for item 21, 23.74% for item 22, 19.63% for item 27, 24.20% for item 28, 24.20% for item 29, and 23.29% for item 30. This could indicate that participants struggled to make meaning of the items involving piecewise function graphs and horizontal segments or points.

Student levels and performance

The second research question considered differences between students in different levels of courses and their performance on domain and range tasks for functions in their graphical forms. The research team found that the level of math had a significant effect on performance on the multiple-choice test, $\Lambda = .90$, $F\ (4, 430) = 5.74$, $p < .01$. From here, the research team decided to examine how class level related to domain and range performance individually. The math course level had a significant effect on domain performance, $F\ (2, 216) = 9.09$, $p < .01$.

Table 6
Results for Items Related to Domain and Range (Reported as Percentages)

Course	Domain		Range	
	M	SD	M	SD
APS	47.70	21.63	40.16	20.32
Precalc	54.91	21.73	49.07	22.13
Calc	64.19	19.91	47.16	22.66
Overall	52.26	22.16	43.54	21.47

As Table 6 illustrates, on average, students who enrolled in Calc had the best domain performance, while those in the APS course had the worst. A series of Bonferroni-corrected post hoc comparisons were run to find which math courses differed in domain performance, and found that the only statistically significant difference in domain performance (using a familywise α = .05) occurred between Calc students and APS students. On average, Calc students performed over three-quarters of a standard deviation better than the APS students (d = .77). The math course level also had a significant effect on range performance, F (2, 216) = 4.02, p < .05. On average, Precalc students performed the best on range items (doing better than Calc students), while those in the APS course performed the worst. With the series of Bonferroni-corrected post hoc comparisons, the researchers found that the only statistically significant difference in range performance (again, using a familywise α = .05) occurred between Precalc and APS students. As Table 6 illustrates, Precalc students, on average, performed almost 9% better than the APS students; this effect was moderate (d = .42).

Discussion and Conclusions

Learning about the domain and range of functions, including studying them in a function's graphical form, is common in many secondary and early postsecondary mathematics courses. Students in Calc should easily make meaning of tasks that involve these topics. Our results, on the other hand, suggest that this is not the case. Overall, many of the study participants seemed to have difficulty performing domain and range tasks on graphs of functions. As Table 6 indicates, the average participant's performance was only 52.26% for domain tasks and 43.54% for range tasks. These findings also confirm what had been noted in previous research that college students, in general, have more difficulty on range tasks than on domain tasks. While our sample illustrates the notion that students' performance on domain tasks tends to improve as they advance to higher level mathematics courses, we did not have statistical support to generalize these findings across the three courses involved. Even though Calc students, on average, had a better understanding of domain than students in the other two math courses; we only had enough evidence to support that students enrolled in Calc had a better understanding of domain than APS students. There was not enough evidence to support the claim that Calc students, on average, had a better understanding of range than APS students or Precalc students.

The relationship between student practices and performance were not the same for the domain and range tasks. As Table 3 suggests, more frequent use of the *Intercept*, *IntNotation*, and *OpenPoint* strategies tended to have a negative influence on performance for domain tasks, while the *EdptFocus*, *IntDescend*, or *NoOverlap* strategies tended to have a negative influence on performance for range tasks. We can conclude that students do not necessarily utilize the same strategies when solving for both domain and range tasks; rather, they discriminate the type of practice they use depending on the need to determine domain or range.

When the participants engaged in the domain and range tasks, many seemed to have traced the graph from the start to the end (i.e., left to the right or bottom to top). Even though the piecewise sections of the functions often abutted or overlapped in their intervals, students seemed to hyper-focus on the "micro" and not the "macro"– forgetting to look at the graph in its entirety and hence failing to combine abutting or overlapping intervals. They also struggled with piecewise functions in ways that suggest that students often fail to take into account the graph as a whole. Just as the saying that a "person can't see a forest for the trees" goes, students might get so involved in identifying notable aspects of a graph (particularly graphs that lack continuity) or tracing a graph through particular points that they fail to see the graph in its entirety. Participants struggled with range items when horizontal segments were part of a discontinuous function's graph. This finding corresponds with the results of previous study and seems to provide evidence that instructors need to recognize that some transitional conceptions students hold need to be revisited to help students make meaning of domain and range at both the micro and macro levels.

Limitations and Future Research

For each item, participants had five response options, including option E "none of the above." Many students selected this response, especially for items with horizontal segments. If the participants had been prompted to write in their answers when selecting the "E" response, they might have more insight to their strategies and transitional conceptions. In addition, we also note that multiple strategies could have been used when students selected a particular choice for each item. This was most likely the reason for the multi-collinearity witnessed in Table 3 between the *EdptFocus* and the *IntDescend* strategies. Hence, there are study limitations that result from the design of the instrument items. Repeated interviews over time with students or longitudinal studies involving pretests and repeated post-tests would provide more detailed insight on how students' transitional conceptions, strategies, and uses of representations develop or persist.

Another limitation relates to the follow-up contrasts conducted for the second research question. We used Bonferroni-corrected post hoc contrasts rather than assume which classes might differ in terms of performance. While our post-hoc contrasts allow us to examine differences between all three class levels, we had to control for the possibility of making a Type I error. Consequently, we may have been too conservative in our findings. Had we determined planned contrasts a priori instead, we might have found more significant differences between math class levels, as our alpha level would have been much higher for determining significance.

Suggestions exist on how high school teachers can emphasize connections while teaching functions (e.g., Moore-Russo & Golzy, 2005), and studies provide evidence that explicit presentation of multiple representations of mathematical ideas and reference to the connections between them using a multimodal approach are important instructional considerations (McGee & Moore-Russo, 2015; Moore-Russo & Viglietti, 2012; Wilmot et al., 2011). Research also suggests that the way ideas related to functions are taught in at the secondary level may vary from the way they are taught at the postsecondary level (Nagle, Moore-Russo, Viglietti, & Martin, 2013). A study of both high school and college instructors could help point out similarities and differences in methods for teaching domain and range and the connections explicitly made during instruction of these topics between the graphical representation and other representations.

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Struggling to comprehend the zero-product property

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The zero-product property (ZPP), typically stated as 'if ab=0 then a=0 or b=0,' is an important property in school algebra (as a technique for solving equations) and abstract algebra (as the defining characteristic of integral domains). While the struggles of secondary mathematics students to employ the ZPP are well-documented, it unclear how undergraduate students preparing to take abstract algebra understand the ZPP as they enter abstract algebra. To this end, this paper documents students' understanding of the ZPP while also investigating how students might be able to develop and harness their own intuitive understandings of the property.

Key words: abstract algebra, secondary algebra, zero-product property

Tall, de Lima, and Healy (2014) posed the following task to 77 high school algebra students (p. 7):

> To solve the equation $(x-3)(x-2)=0$ for real numbers, John answered in a single line that '$x=3$ or $x=2$.' Is his answer correct? Analyse and comment on John's answer.

Remarkably, only 30 out of the 77 students claimed that the solution was correct, and all of the students who attempted to find a solution distributed and applied the quadratic formula (sometimes incorrectly). Of particular importance, not a single student referenced the property that if a product of two real numbers is zero, one of the two numbers must themselves be zero (if $ab=0$, then $a=0$ or $b=0$), commonly referred to as the zero-product property (ZPP). Tall et al. suggested that this phenomena –students overlooking an efficient algebraic solution using the ZPP – resulted from an overemphasis on students learning to solve linear equations through "procedurally embodied symbol shifting" (p. 11). He concluded that such a blind focus on procedure can lead to even greater difficulty with subsequent content. In particular, Tall et al. reasoned that secondary algebra "[transforms] into an axiomatic formal world of set-theoretic definition and proof in university pure mathematics" (p. 12).

This is consistent with characterizations of abstract algebra elsewhere in the literature. Ideally, an abstract algebra course is "the place where students might extract common features from the many mathematical systems that they have used in previous mathematics courses" (Findell, 2001, p. 12). Indeed, much of school algebra involves solving equations, and the properties needed to solve the general linear equation $ax+b=c$, for example, are precisely the field axioms (Kleiner, 1999). These connections are completely natural and advantageous from an expert's perspective. However, there is substantial reason to believe that this is not the case for students (CBMS, 2001), who might see abstract algebra as "a completely different subject from school algebra" (Cuoco, 2001, p. 169). The ramifications of this disconnect are potentially dire for affected pre-service teachers, who "will understandably have a very limited notion of what algebra is about, and will be unequipped to address the curricular breadth now encompassed in school algebra" (CBMS, 2001, p. 109). At the very least, such teachers would likely be inhibited in their abilities to communicate the core ideas of the subject to their students.

The ZPP is uniquely situated to provide insight into such issues. Though many of these algebraic properties from school algebra have important uses in abstract algebra, the ZPP is of particular importance as the defining characteristic of integral domains – a fundamental ring-theoretic structure. The typical student preparing to take abstract algebra is likely to have completed a significant amount of mathematics courses, including the calculus course sequence and linear algebra. Moreover, the ZPP appears regularly in these courses in some form. Thus, before reaching any conclusions regarding undergraduate students' understanding of this property, additional research is needed, yet there are no studies that directly investigate undergraduate students' understanding of the ZPP. Thus, I designed the present study to investigate how undergraduate students' might intuitively notice, reason with, and understand the zero-product property.

Literature

There are several studies documenting student activity with algebraic structures containing zero-divisors. The students in each study generally do not recognize the implications for the zero-product property and continue to tacitly assume that the property holds.

Simpson and Stehlikova (2006) documented a case study of one student (Molly, who had completed abstract algebra) and her largely self-guided attempts to make sense of a disguised (isomorphic) rendering of the commutative ring $\mathbb{Z}_{99}$ in an independent research project over a period of several years. They noted that, early in the study (interviews 3 and 4), Molly's correspondence with the researchers revealed that she had made note that multiples of 3 "cause the equation $[ax = b]$ to have multiple solutions," and later cause "problems when cancelling" (p. 362). These early observations of the consequences of zero-divisors are notable because she later (interview 8) solves quadratic equations of the form $ax^2 + bx = 99$ by factoring into $x(ax + b) = 99$ and invoking "her knowledge from ordinary arithmetic that a product is zero if and only if one of the factors is zero" (p. 363). Despite her prior work demonstrating her awareness of elements that caused equations to have multiple solutions, she employed the ZPP and thus implicitly asserting that the only possibility was that one of the factors had to be equal to the additive identity. At this point, the authors observed that she had "no obvious sense of the presence of zero-divisors at this point," (p. 363). It was not until a subsequent revision of her work occurring long afterwards (approximately 25 weeks) that she began to show signs of implicit attention to the effects of zero-divisors and even longer (an additional 23 weeks) until she explicitly linked zero-divisors to the ZPP, which certainly suggests that this was not an obvious or intuitive connection for her. Other than to generally note that Molly struggled to focus her attention in productive ways (which was one of the principal findings of the study), the researchers did not hypothesize any specific explanations about her mathematical reasoning with zero-divisors and the ZPP.

Ochoviet and Oktaç (2009) reported similar findings from a study of task-based surveys with both secondary and undergraduate mathematics students focused on using the zero-product property in different algebraic structures. One such task presented secondary mathematics students with a multiplication table for $\mathbb{Z}_6$. After being asked to compute several calculations using the table and solve the equation $3 \cdot x = 0$, they were asked, "a and b are elements of the set A as above. It is known that $a \cdot b = 0$; based on this information, what can you say about a and b? Explain your answer." (p. 130). While specific data on the proportions of correct responses is not included, the researchers noted that "most" of the late secondary students concluded that either a or b must be zero. Students responded similarly to the same question for more familiar structures, including rings of functions and matrices.

These studies indicate that students struggle to reconcile and notice the ramifications of zero-divisors for the zero-product property. However, what is presently unclear is why students are reasoning in this way, as there are no empirically-based explanatory models of how students are thinking about zero-divisors and the ZPP in these situations. To this end, in this paper I attempt to explore this phenomenon by answering the following research question: Why do students struggle to make connections between zero-divisors and the zero-product property?

Theoretical Orientation

The research questions that frame this study are compatible with and supported by the theory of Realistic Mathematics Education (RME). A central tenet of RME is that the starting point of an instructional sequence should be *experientially real* to the student so that the mathematical activity becomes personally meaningful (Freudenthal, 1991). An RME design heuristic that informed the instructional design was *guided reinvention*, in which "the idea is to allow learners to come to regard the knowledge they acquire as their own private knowledge, knowledge for which they themselves are responsible" (Gravemeijer, 1999, p. 158). A point of clarification is in order here. The primary objective is not to have students reinvent the statement of the ZPP – the literature detailed above provides two instances in which the students are familiar with the statement of the property yet still struggle to employ it effectively. While the statement of the property itself is certainly important, the objective is rather to investigate how students might be able to reinvent or discover particular notions related to this property. Namely, how they might identify its potential uses, identify the connection with zero-divisors, or discern whether it holds for a particular algebraic structure.

As a means of executing these design heuristics, I adopted the teaching experiment methodology (Steffe & Thompson, 2000) in order to "begin making essential distinctions in students' ways and means of operating" (p. 275) while also "[experimenting] with the ways and means of influencing students' mathematical knowledge" (p. 274). In particular, the teaching experiment methodology provided a means to develop models students' intuitive understandings of the ZPP. The task design, following suggestions from the literature (e.g. Cook, 2012), centered on equation solving as an experiential real means for students to encounter the ZPP.

Methods

This preliminary proposal focuses on a pilot teaching experiment as part of a larger study to answer the stated research question. The two participants, Steve and Lindsey, were selected on the basis of mathematical preparation (successful completion of linear algebra and a transition-to-proof course, but no abstract algebra), and a willingness to participate in the study. Data was collected using a recording Livescribe pen, which synchronized students' written work with the corresponding audio.

I employed two methods for data analysis: ongoing and retrospective. The ongoing analysis occurred during and between sessions as I attempted to construct more stable models of student thinking in situations involving zero-divisors and the zero-product property. At the conclusion of data collection, I engaged in a retrospective analysis using Clement's (2000) interpretive analysis cycle for generative task-based interviews. Similar to Glaser and Strauss's (1967) constant comparative method from grounded theory, this interpretive analysis cycle was similar in nature to how I conducted the ongoing analysis, but allowed for the possibility of identifying common themes and observing conceptual change across the entire data set:

- Segmenting the data;
- Making observations of student behavior with zero-divisors and the ZPP from each segment;
- Hypothesizing models of mental processes that can explain the observations;
- Return to the data to refine and look for confirming or disconfirming observations; modifying and/or extending the models as needed.

Results

In this section I present themes emerging from the preliminary results in the pilot teaching experiment in this study:

Noticing zero-divisors

In an initial task in which Steve and Lindsey reviewed modular arithmetic (using clock arithmetic as a metaphor), they constructed both an addition and a multiplication table. Steve noticed that several elements have repeating patterns, and commented, "My initial thought was, it's the factor by which you are multiplying, um, a quantity. In this case, 5 is going to produce unique numbers. And that's how 4 over here produces, can produce, one of three numbers. So it won't produce anything unique." Essentially, he has noticed that 4 and 5 are fundamentally different types of elements, but his attention appears to focus on the pattern itself, and not the multiple ways to multiply to obtain 0. Soon after they solved $4(x-5)=0$ in $\mathbb{Z}_{12}$ by the same method they used to solve the equation in $\mathbb{R}$. Of course, in this case their second line (after distribution) read '$4x-8=0$,' leading to a solution of $x=2$. Neither Steve nor Lindsey noticed that $x=2$ was different than their solution to the same equation in $\mathbb{R}$, and also did not recognize that there were multiple solutions. Later, when I asked them to speculate about why $4(x-5)=0$ could have multiple solutions in $\mathbb{Z}_{12}$ but not in $\mathbb{Z}$, Lindsey responded: "because it keeps wrapping around."

Confusion with the converse

Towards the end of the teaching experiment, I asked Steve and Lindsey directly if the ZPP held in $\mathbb{R}$. Both students agreed that the property held, and justified their claim by asserting that $a \cdot 0 = 0$ and $0 \cdot a = 0$. While these are certainly true statements, they involve the converse of the ZPP instead of the ZPP itself (it should be noted that the converse of the ZPP is true in any ring). Curiously, though the same line of reasoning could easily have led them to incorrectly assert that it held in $\mathbb{Z}_{12}$ as well, they made explicit connections with pairs of zero-divisors, which they had not done while solving equations. Immediately after the task was posted, Lindsey asserted that the ZPP did not hold, and started citing a litany of counterexamples, including 2 times 6, 3 times 4, and 6 times 8. That their initial line of reasoning did not interfere might indicate that searching for counterexamples occurred prior to any attempt at an abstract argument.

Discussion Questions

(1) What other types of tasks might provide insight into student understanding of the ZPP?
(2) In what ways can tasks force students to confront their avoidance of the ZPP?

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An overview of research on the arithmetic mean in university introductory statistics courses.

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There is a dearth of research on the arithmetic mean at the university level. This poster will cover overlap of several studies (some unpublished) on university students' understanding of the mean and university statistics instructors' beliefs about their students' understandings of the mean.

Key words: Mean, Average, Arithmetic Mean, Introductory Statistics, Statistics

Children through grade 8 have shown the ability to calculate a mean, but have trouble thinking of the mean beyond the procedure of calculating it (Mokros & Russell, 1995). It is theorized that the early introduction of the procedure for calculating the mean may interfere with the ability to understand the mean conceptually as an object, as opposed to the result of a set of procedures, (Mokros & Russell, 1995) and that introducing the procedure early on would require a more difficult route towards a conceptual understanding (Cook & Fukawa-Connelly, 2012). This poster examines overlapping findings of university students' understanding of the mean at the, and how it relates to what is being taught in introductory statistics courses at the university level.

What Do University Students Understand About the Mean?

In one study on incoming mathematics majors' statistical knowledge, all participants believed they understood what the mean was and were confident in their responses when asked to describe the mean. However, each participant only thought of the mean as the result of a procedure, often described as "add up all the numbers and divide by the amount of numbers". In contrast, when asked about the standard deviation, each participant who had knowledge of the standard deviation was unable to describe the procedure; however, some of these participants did go on to describe it as a measure of spread. In the discussion of this paper, it was suggested that not knowing the procedure benefited the students as they had to think about standard deviation more conceptually, something they did not need to do with the mean (Cook & Fukawa-Connelly, 2015).

In a different study that examined student understanding of mean, median and standard deviation at the conclusion of a first university course on statistics, some students described the mean more conceptually in comparison to the study of incoming knowledge. In a survey, 28% of the respondents used the notion of center or representativeness in their descriptions of the mean, and nearly all respondents cited the calculating formula either alone or in conjunction with more conceptual descriptions. In contrast, very few respondents cited the calculating formula for the standard deviation and used phrases to describe smaller standard deviations as data sets with "more data in the middle" (Cook & Fukawa-Connelly, 2014), a finding consistent with a study exclusively on student understanding of the standard deviation (delMas & Liu, 2005).

This limited amount of research indicates that most students leave their first statistics course with a similar understanding of the mean as when they entered. This understanding are similar to the understandings held by the children in Mokros and Russell's study (1995).

What Do University Statistics Instructors Believe About their Students and the Mean?

The following data comes from a small online survey of current introductory statistics instructors. In the survey, 16% of respondents reported that they have no assumptions about their students' understanding of the mean and 63% assume their students can calculate the mean of a set of data. Additionally, 32% expect that their students will enter class with an understanding of what will happen to a mean if particular pieces of data are removed from the data, or new data points are included (ie get bigger, smaller or stay the same). In the study referenced above, less than half of students were able to answer a question of this type correct at the end of a statistics course, with many citing that they needed to know what all the data points were.

This survey also asked what aspects of the mean that they explicitly cover in class, with the most common responses being:

- Mean affected by outliers (100%)
- Mean is a measure of central tendency (95%)
- How to calculate a mean (89%)
- How a mean will change if data is added or removed (84%)

However, when asked how many minutes of class time they spend over then entire course covering the mean as it's own concept, the average amount of time was 15 minutes (CI: 9, 21), with 16% of respondents reporting they spend no time and 21% report spending under 5 minutes. Thus, despite 37% of respondents reporting they spend between 0 and 5 minutes teaching explicitly about the mean, over 80% report that the do explicitly teach the four items above. This seems to imply that instructors see teaching the mean explicitly as a brief review, or that students have a robust enough understanding coming in that they will easily be able to pick up important concepts related to data fluency and inference. This poster will also explore how university statistics text books introduce the mean and what conceptual aspect of the mean are explicitly covered.

Implications.

I believe that instructors and students have both miss-assessed student understanding of the mean as a trivial concept, and these assumptions (of both students and instructors) potentially hinder learning of more advanced statistical concepts. More research is required to defend this belief; however, these studies support that students rely on a procedural understanding of the mean and instructors believe they have (or can quickly develop) a conceptual understanding of the mean. The actual implications of this misalignment are unknown.

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Divergent definitions of inquiry-based learning in undergraduate mathematics

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Inquiry-based learning is becoming more important and widely practiced in undergraduate mathematics education. As a result, research about inquiry-based learning is similarly becoming more common, including questions of the efficacy of such methods. Yet, thus far, there has been little effort on the part of practitioners or researchers to come to a description of the range(s) of practice that can or should be understood as inquiry-based learning. As a result, studies, comparisons and critiques can be dismissed as not using the appropriate definition, without adjudicating the quality of the evidence or implications for research and teaching. Through a large-scale literature review and surveying of experts in the community, this study begins the conversation about possible areas of agreement that would allow for a constituent definition of inquiry-based learning and allow for differentiation with non-inquiry pedagogical practices.

Keywords: Inquiry, Inquiry Based Learning, Inquiry Oriented Learning, Definition

Over the past few years, a growing amount of literature has been published on undergraduate inquiry based mathematics education. This type of education puts the teacher in the place of a guide who has the role of asking thought provoking questions. From this, students learn by working through questions and frustrations to gain a deep understanding of a particular concept and reflect on what they just learned and what implications may be. The change in undergraduate teaching practices, in classes such as calculus and linear algebra that have traditional curricula, means that there is a concurrent growth in professional development, publications about teaching and curriculum, and research on inquiry-based instruction. Additionally, this increase in research, professional development and implementation has spurned an upcoming special interest group of the mathematical association of america in inquiry based learning.

However, despite the growth of published materials on inquiry, we argue that the term is not consistently defined, and some publications do not define it at all. The lack of a definition has been the source of some debate in the past. One published paper claimed that inquiry based learning does not work (Kirshchner, Sweller & Clark, 2006); however, criticisms of this paper centered on the fact that the authors misunderstood what inquiry based learning is and were over simplifying it as unguided discovery (Hmelo-Silver, Duncan, & Chinn, 2007). In this paper we survey the literature for uses of the term inquiry and survey current experts in undergraduate mathematics inquiry to learn how they define inquiry. In this survey some participants point out that people continue to over-simplify inquiry, a problem for the field if inquiry is going to be promoted as more effective than directive methods. This paper is the beginning of a discussion about defining inquiry based learning in an undergraduate mathematics classroom in order to allow for meaningful discussion and evaluation.

Background

Inquiry based mathematics education has become more popular in undergraduate settings. However, the term "inquiry" lacks a clear and concise definition. Instead, throughout recent literature, we identified six major themes when defining the term inquiry. These six themes are all distinct, and, they show up individually and in clusters in papers about IBL in undergraduate mathematics.

The first theme is student ownership of knowledge (student ownership), Johnson (2014) described the idea as "Learners regard the knowledge they acquire as their own personal knowledge they are responsible for". To add, in a separate paper, inquiry is defined as students being encouraged to create knowledge by themselves (Ko & Mesa, 2014).

The second theme is new knowledge building on existing knowledge (knowledge building). For example it is stated that "Inquiry mathematics allows students to… find new ways to use prior knowledge to understand equations" (Keene & McNeil, 2014).

The third theme is students participating in mathematics (doing math). Johnson added that inquiry is the "Expansion of what is experimentally real" while learning mathematical skills is "Synonymous with becoming a participant in the community" (2014).

The fourth theme is the importance of the student/instructor relationship (student/instructor relationship). This relationship, as many papers state, is crucially important because it "Enables instructors to have a deeper understanding of students and learning" (Ko & Mesa, 2014) and "teachers need to understand student thinking" in inquiry (Larson, Wawro, Zandieh, Rasmussen, Plaxco, Czeranko, 2014), a statement that arises from Rasmussen and Kwon's "Inquiry-Oriented Learning" (IOL), where the instructor inquiring into student learning is a key component (2007).

The fifth theme is the importance of student to student interaction (peer involvement). Rasmussen and Kwon explain how part of inquiry includes a student's' ability to "routinely explain and justify their thinking, listen to and attempt to make sense of others' ideas" (2007).

The sixth theme is better alignment with how people learn which leads to increased student success (student success). Overall, students who take inquiry based mathematics classes do better in other classes because they have gained the necessary tools to be able to decipher future problem sets (Mantini, Trigalet, Davis, 2014; Yoshinobu & Jones, 2012). A recent study showed some evidence that students in an IBL calculus class that covered fewer topics did at least as well as their directively taught peers when they took calculus II (Laursen et al, 2014).

These six themes appear to inform the concept of inquiry based mathematics education. With this, however, comes some debate on the definition of inquiry in regards to undergraduate mathematics education. In this study, our aim is to survey the entire undergraduate inquiry community to better understand how the community defines inquiry.

Methods

Literature search

To orient ourselves, we attempted to collect all papers about inquiry-based instruction, broadly defined, in undergraduate mathematics education. In order to do so, the second author searched a database mutli-search that included over 20 databases, including JStor, ERIC, Academic Search Premier and more using criteria such as, 'inquiry-oriented,' 'inquiry-based,''guided-discovery,' and 'realistic mathematics education' always in conjunction with undergraduate mathematics. Additionally, the second author searched the conference proceedings of the three most recent SIGMAA-RUME conferences. Moreover, we asked experts in the field for recommendations of other articles. For each identified article, the second author carried out two tasks; first identifying the author(s) and any instructors of inquiry-based courses and adding them to a list of undergraduate faculty who teach or do research on inquiry-based courses. The second action was to extract the definition of inquiry-based, inquiry-oriented, or guided-discovery (hereafter shortened to inquiry-based) that the author(s) gave in the paper. If the authors did not specify a definition, that was also noted. Based on the literature review, we noted that many papers took as unproblematic the definition of inquiry-based teaching. As a result, the first and second author also attempted to characterize the instruction that was described. There appeared to be significant differences in the range of practices that authors, even of research papers, described as inquiry-based instruction, meaning, there appeared to be little agreement about the defining features and the types of experiences that students might have, thus making large-scale evaluation of the efficacy problematic.

Participants

The population of this study is any author of a peer reviewed research article or conference proceeding dealing with inquiry in the undergraduate mathematics classroom, a person who has experience in professional development of inquiry in undergraduate mathematics classroom or a person who has authored a textbook or support materials with inquiry in the undergraduate classroom in mind that the first and second authors identified via their literature review and subsequent snowballing technique. The authors identified 67 persons who fit in one of these 3 categories in an attempt to be as exhaustive as possible. All 67 members of the population were invited to complete the survey and 18 persons participated. The participants included 10 mathematics educators, 6 mathematicians and 2 STEM educators, 4 of the mathematics educators were doctoral students while none of the mathematicians or STEM educators were.

Survey and Coding

The researchers developed a nine question survey with 2 demographic questions addressing employment position and asking each participant to identify themselves as a mathematician, math educator, scientist, science/STEM educator or other. There were 7 free response questions centering on various aspects and understandings of inquiry. Survey responses were then independently coded by each researcher using a method most closely associated with grounded theory (Strauss & Corbin, 1990). Each researcher read through the responses keeping in mind the 6 categories identified in the literature review, student ownership, knowledge building, doing math, student-instructor relationship, peer involvement and student success. The research team developed a coding manual to identify when a response invoked a particular code. For example, we coded a response as invoking the notion of student ownership when it included such phrases as:

- Students are doing the intellectual work of discovering
- Students should as much as possible be responsible for the acquisition of knowledge
- Investigation… generated by the learner
- Student/learner engagement via their own problem solving… and active involvement
- The problems are designed to encourage students to… contract their own justifications for their conclusions
- Developing their own ideas

In addition, researchers independently created their own codes for responses that did not fit in an identified category.

Results

We report three preliminary results. The first of which describes commonalities among the different definitions. In particular, the definitions that the respondents provided overwhelmingly focused on three particular ideas; that the instructor-student relationship is different than in a traditional class (sometimes described as a guide or facilitator), that student curiosity is important and should be nurtured, and that the classes include peer-to-peer interactions. Here we present a representative examples of such a definition;

- A mathematician offered the definition: I always go with the AIBL.org folks on this one: "What this means is that we define IBL broadly, and support the use of a wide range of teaching methods in mathematics courses consistent with courses where students are (a) deeply engaged in rich mathematical tasks, and (b) have ample opportunities to collaborate with peers (where collaboration is defined broadly)."

In this example the themes of instructor-student relationship, peer-to-peer discussion, and student curiosity are represented. Moreover, we note that the adjective ‘rich’ does not have a clear definition, meaning that different observers could come to different conclusions about whether or not students are engaged in such a task.

Most definitions, 17 of the 18 participants, gave relatively few criteria, typically two or three criteria. There were instances where a respondent gave multiple sub-criteria describing, for example, the notion of student mathematical responsibility such as the mathematician who offered the following definition:

- I believe that there are two essential elements to IBL. Students should as much as possible be responsible for:
    - 1. guiding the acquisition of knowledge and
    - 2. validating the ideas presented. (Students should not, that is, be looking to the instructor as the sole authority.)

In this case, the respondent gave two related descriptions of the student’s actions in class that both relate to the ‘responsibility’ code. Similar are descriptions of the student role that include ‘conjecturing’ and ‘questioning,’ although focused on different aspects of student activities. This trend of giving relatively few criteria for a definition was inclusive of all categories of respondents, including mathematics education researchers.

The single most common definition used in research, given by four respondents, all mathematics education researchers, was that “students are inquiring into mathematics and the instructor is inquiring into student thinking.” This definition is interesting in that the second clause gives some description about the instructor’s role in the class; that the instructor is to be doing on a daily basis; investigating and understanding the student thinking about the mathematics. In terms of what the students are to be doing, the phrase ‘students inquiring into the mathematics’ is open to a wide-range of interpretations such that people could plausibly argue that almost any mathematical activity done by the students is inquiry. As a result, it appears that with this definition, the actions that the professor takes are more important than anything the students do.

Less commonly, participants described what types of activities the students should engage in. Only 3 respondents did so. When they did, they gave responses similar to the below:

- A math educator offered the following, noting it was used in research and served as a personal definition: I consider inquiry to involve student/learner engagement via their own problem solving, problem posing, questioning, and active involvement...this is as opposed to students/learners being passive participants in their learning of mathematics.

In these instances, the participants used terms such as problem-posing, questioning, conjecturing, and introducing key mathematical ideas. This gives much more explicit description by which an observer might decide whether a particular class is engaging in IBL. Similar in tone were definitions that suggested that students should ‘regularly introduce key ideas.’ and one qualified the statement by writing that ‘as much as possible’ students should be the

A second preliminary result is that participants are largely in agreement that there is not much agreement in the details of IBL. Three participants said they had never seen a published definition of it, three others said that they have experience with it being defined as simply group work or active learning, but that is not enough. One participant stated that "definitions in the literature are all over the map." Another participant thought the definition was sometimes used without consideration of the instructor's role in the inquiry. In all, only 4 participants stated they had not come across a definition of inquiry that did not fit with their definition, with one of the four saying "(Some definitions) are not quite as detailed as mine, but the spirit is usually the same."

A third preliminary result is that inquiry involves "students doing meaningful work" or "being active participants in mathematics"; however the description of what is meaningful work differs from participant to participant if they describe it at all. One participant suggests that the instructor must "put students in direct contact with mathematical questions, objects, and phenomena", another offers more specific criteria stating "This involves working through mathematical activities and classroom discussions where knowledge of a mathematical concept is developed based on the students' prior knowledge. (Students) are expected to participate in the learning of a mathematical concept. Since the goal is to understand a mathematical concept, asking questions and making mistakes is viewed as part of the learning process.". In addition, similar phrases such as "students are inquiring into mathematics" were common in the data, but what it means to "inquire into math" is not clear. The authors can guess what is meant by it and it may be assumed by the inquiry community, but it is not enough to be definitional. Further investigation is required to understand what range of tasks would be considered in doing meaningful work and what it means to "inquire into mathematics".

Discussion/Future Directions

Given the most common aspects of these definitions there are a wide-range of pedagogical practices that can be described in these terms. If students commonly engaged in group work, no matter the tasks, even doing exercises, as long as the students talk to each other, the students express questions, and the professor is more conversational it would fit within the most commonly given definitional criteria. Moreover, if the professor inquires about student thinking it would possibly fit the most commonly given definition. As a result, it appears that there is no set of criteria that describe a classroom that would allow observers to reliably differentiate between an IBL class and one that is somehow not; that is, where can researchers agree to differentiate between a lecture-class and an IBL class?

A next step is member checking the codes we identified in the data. We will reach back out to the entire population and ask them for ranked input on the themes identified from their free responses to gauge how important these experts believe each theme is. After member checking the data, we hope to offer a community definition of what inquiry based learning is for an undergraduate mathematics classroom.

Preliminary Report Questions

1. In the data the phrases "Student Centered" and "Student Responsibility" are used in very similar ways. Are these different, or are people using two words (centered and responsibility) for the same meaning?
2. A commonly stated characteristic of inquiry is students "doing math" or "participating in math". What does it mean to "do math"?
3. Are there characteristics of inquiry we have not coded or found in the data and you believe are important in informing a definition of inquiry?

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Paper title: Using Reading Journals in Calculus

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We study the effects of using reading journals in a first semester calculus class. Students were given instructions on how to read the textbook before class, including keeping a reading journal. The student quiz scores were compared on weeks that journals were due and on weeks they were not due. We found inconclusive results, including some evidence suggesting that students score higher on quizzes when they are required to submit reading journals. Many students also indicated on surveys that the journals were beneficial to their learning and to their completion of pre-class reading assignments, although some felt otherwise.

Key words: Calculus, Quizzes, Reading, Journals

Introduction

This study explores the use of reading journals in a first semester calculus class. Our goals were to find a way to get calculus students to read the textbook before class, and to do so in such a way that requires more effort than just skimming the pages. In addition to studying whether students actually read the textbook when we ask them to, we were further interested in whether doing so has any effect upon their course performance. We identified the following research question: Does the use of reading journals affect student performance or experience in Calculus? We also gave students surveys and final course evaluations where they had a chance to offer their opinion on their experience with the journals and whether they felt that their use was beneficial for learning. This study is based on a pilot study done in 2009 at Vanderbilt University.

The Math Course and Teaching Methods

The course that we are implementing the journals in is Calculus I. This is the first semester of the calculus sequence. The course begins with the idea of average rate of change, and develops the concepts of limit, continuity, the derivative (instantaneous rate of change), antiderivative, and the definite and indefinite integral, including the Fundamental Theorem of Calculus and some applications of derivatives and integrals. The course is both a general education elective and a required course for many majors including math, and computer and natural sciences. The students at Hawaii Pacific University (HPU) are academically, culturally and ethnically diverse. HPU is one of the most culturally diverse universities in the world, with students from more than 80 countries. Huntingdon College is located in Alabama and has approximately 1100 students, many of whom are from Alabama. We will combine the data collected in Fall 2015 from parallel studies that took place at both schools.

Intervention: Journals

The purpose of the study is to determine whether pre-class reading journals have an impact on student performance. Every other week the students were given daily journal assignments. The journal was an aide to assist students in reading slowly and comprehending the textbook. The section in the textbook that was assigned reading as part of the journal was the section that would be discussed in the lecture during the class day that the journal was due. Therefore, the students were reading a preview of the material that they had not learned yet. The journal consisted of a reading assignment in the textbook, to be covered in class the next day. Additionally the journal assignment instructed the student to take notes on the reading, including writing down all the important points covered in the text. Lastly, the journal instructed students to answer an open-ended conceptual question about the reading. The students also had the opportunity to ask the instructor questions.

The instructor gave the students a verbal explanation on the first week of class about how to properly read a mathematics textbook. It was explained that it is different from reading a novel or a history book in that one cannot skim the pages and expect to understand them. The purpose of taking notes is to get the students to slow down and think about what they are reading. It was noted to the students that the journals can be useful later for studying for tests and quizzes, and it was suggested that students do not copy all the words and examples in the section, but rather just the most important formulas and definitions. A sample journal assignment was provided to students so they understood what was expected of them. After this initial explanation, no further instructions were given.

The journals had to be submitted on time or they were not accepted. They were graded on a scale of 0, 1 or 2. A 0 score meant the journal was not submitted. A score of 1 meant that the journal was incomplete. A score of 2 meant the entire journal assignment was completed. The score of 2 did not necessarily mean that the notes were high quality or that the answer provided to the conceptual question was correct. However, the students did receive minimal written feedback on the journals they submitted. If a student asked a clarifying question or mentioned that something confused them, the instructor provided a written response. Most student journals took up approximately one page of loose-leaf paper, double sided.

The journal assignments were given daily, but only every other week. On the weeks that no journal was assigned, students were not given any explicit instructions to read the textbook, but of course they were not prevented from doing so either. Every week in class the students took a quiz. The reason that we chose to compare quiz scores instead of scores on midterm and final tests had to do with the study design. We want to compare individual students to themselves and to distribute the difficulty of the materials relatively evenly. Sometimes the quiz and the journal questions were similar or related, but other times they were less so. The students were informed about what sections of the textbook would be on the quiz, the quizzes were announced ahead of time in the syllabus and during class, and again, the quiz questions were always similar to problems that were discussed in class and/or given on the online homework.

In addition to collecting journals and quizzes, we did administer intake and exit surveys. The intake and exit surveys contained some paired questions, and the exit survey contained additional questions specifically about the journals and the student experience reading mathematics. The questions were given on a Likert scale, from Strongly Disagree, Disagree, Agree to Strongly Agree. These responses were coded into numbers from 1-4 when entered into data. Students were also given the opportunity on the exit survey to respond to open-ended questions.

Conclusions and Implications

We will do an overall analysis of the data collected during 2015 in the parallel studies. In particular we will focus on analyzing whether student quiz scores appear to be improved by the use of journals to conclude whether this intervention is helpful to students and whether it should be adopted by calculus teachers to improve student learning. We will also analyze the results of the surveys to determine the student's perspective on how the journals aided their learning.

Why research on proof-oriented mathematical behavior should attend to the role of particular mathematical content

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Because proving characterizes much mathematical practice, it continues to be a prominent focus of mathematics education research. Aspects of proving, such as definition use, example use, and logic, act as subdomains for this area of research. To yield such content-general claims, studies often downplay or try to control for the influence of particular mathematical content (analysis, algebra, number theory etc.) and students' mathematical meanings for this content. In this paper, we consider the possible negative consequences for mathematics education research of adopting such a domain-general characterization of proving behavior. We do so by comparing content-general and content-specific analyses of two proving episodes taken from the prior research of the two authors respectively. We intend to sensitize the research community to the role particular mathematical content can and should play in research on mathematical proving.

Keywords: Proving, mathematical meanings, comparative analyses

Since at least the time of Euclid's geometry, proving has been understood to characterize mathematics as a discipline. Inasmuch as mathematics educators endeavor to engage students in authentic mathematical activity, they have expended much effort to provide students with meaningful proving experiences and document the emergence of proving as a mathematical practice among novices. While we certainly endorse this agenda for instruction and research, we are concerned that framing mathematical proving as a single, domain-general practice may inappropriately downplay the role particular mathematics content plays therein. We observe two trends in the research literature on mathematical proving: 1) making content-independent claims about mathematical proving using data from a particular mathematical context (i.e. analysis, algebra, number theory, geometry) or 2) eliciting student proving behavior in various mathematical contexts (and non-mathematical ones) to yield content-independent findings. In this paper, we consider the possible consequences for research on mathematical proving of downplaying the role of particular mathematical content. We do not at all intend to deny the validity or value of prior research framed in a content-independent manner (some of which we authored), but rather seek to sensitize the community to possible blind spots induced by common lenses applied to research data and to endorse a research agenda focused on the interplay between proving and particular mathematical content.

To portray such blind spots induced by a research lens, this paper presents dual analysis of two episodes taken from prior studies conducted by the two authors respectively. In each case, we compare 1) a content-independent analysis focused on common constructs from proof-oriented mathematics education research – example use, definition use, proof production, logic – with 2) a content-specific analysis focused on explaining students' proving behavior in situ.

Motivating Trends and Questions

It is common to frame both the research questions and findings using these content-independent constructs such that they form informal subdomains of proof-oriented research. One can find numerous examples of studies on proof-oriented mathematical activity that make content-independent claims about

- example use – Alcock & Inglis, 2008; Karunakaran, 2014; Sandefur, Mason, Stylianides, & Watson, 2013,
- definition use – Alcock & Simpson, 2002; Ouvrier-Buffett, 2011,
- proof production – Dawkins, 2012; Raman, Sandefur, Birky, Campbell, & Somers, 2009; Stylianides & Stylianides, 2009,
- logic – Epp, 2003; Selden & Selden, 1995, and
- understanding of proof – Sowder & Harel, 2003; Stylianou, Blanton, & Rotou, 2015.

It is not our goal to critique these studies per se, but rather to sensitize mathematics education researchers to the consequences of consistently investigating proving while downplaying the mathematical meanings that populate the arguments that students produce.

Why do many proof-oriented studies downplay mathematics content? Even if this question had one answer, no available evidence reveals it. Nevertheless, we proffer some possible explanations. One explanation is psychological. Proof's role in mathematics as a discipline and the mathematics education community's emphasis on mathematical process both lead researchers themselves to conceptualize proving in real analysis as one instantiation of a broader phenomenon. Because we as experts can see some uniformity across our broad experiences with proving, we assimilate instances of proving into our more general understanding. A second explanation involves empirical findings. The growing body of evidence of students' difficulties interpreting, producing, and assessing proofs compels mathematics educators to improve upon proof-oriented instruction. Students perceive the transition into proof-oriented courses as a difficult transition, so it seems natural to partition such courses apart from other aspects of the curriculum (though we agree with Reid's, 2011, argument that proving should become and is becoming integrated as a ubiquitous means of mathematical teaching and learning).

A third explanation relates to the analytic process itself. Mathematics educators frequently use localized data to make analytic generalizations (Firestone, 1993) by constructing frameworks and in-depth characterizations of relatively few cases. While such studies rarely make explicit claims to sample-to-population generalizations, it remains unclear how to situate the resulting theory. For instance, Antonini (2003) presented findings suggesting conditions under which students may produce proofs by contradiction, which previous studies reported as challenging. Antonini describes students' exploration of a geometric conjecture involving transversal configurations, but frames his research hypothesis in the following way:

> In task like "given A what can you deduce?" the conjecture can be produced via the analysis of a non-example. The argumentation that justifies the fact that the generated example is a non-example can be re-elaborated and become part of the argumentation of the conjecture. In this case, the argumentation takes an indirect form. (p. 50)

If patterns in students' proof-oriented behavior can be so characterized using content-independent language, when and why should research findings be framed within the content domain at hand (i.e. geometry or the planar geometry of lines)? Antonini's subjects' proofs (and the solution of the given task) depended upon characterizing pairs of lines as intersecting or parallel, which happen to be familiar definitions that are also negations one of another. We posit that this content-specific feature of the task likely contributed to the students' successful use of indirect proof. This raises the question, when and why should researchers emphasize the role of specific mathematical understandings and meanings in framing and explaining research findings on proving? By providing dual analyses (content-independent and content-focused) of two example episodes, this paper sets forth some answers to these questions.

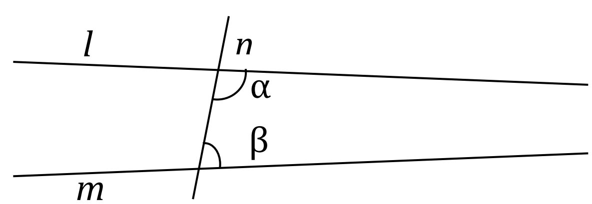
Comparative Analyses

The following sections set forth our two episodes and the dual analysis thereof. The first episode appeared during a sequence of task-based interviews as part of the first author's investigation of student learning of neutral, axiomatic geometry. Episode 1 features two undergraduate mathematics majors trying to prove the equivalence of Euclid's Fifth Postulate (EFP) and Playfair's Parallel Postulate (PPP). Analysis of Episode 1 also appeared in Dawkins (2012). The second episode appeared during a sequence of task-based interviews with expert and novice mathematics students conducted by the second author. Episode 2 features a graduate student in mathematics, designated an expert prover, attempting a novel analysis task about sequences. Analysis of Episode 2 also appeared in Karunakaran (2014). For the sake of brevity and clarity in this theoretical paper, we omit presenting the full methodologies of these studies, which are available in the cited references.

Episode 1: Proving the equivalence of geometric postulates

For reference, the students' statements and diagrams for EFP and PPP appear in Figure 1. As part of a homework assignment prior to the interview, Kirk and Oren had produced a proof of the equivalence of the two postulates using the auxiliary claim we shall call Theorem *, which states "Given two lines cut by a transversal, if the same side interior angles sum is 180, then the two lines do not meet on that side of the transversal." When asked to explain the postulates, the pair found themselves using language from each to explain the other. Oren noted this circularity and attributed it to the statements' mutual implication. Kirk rather explained that the statements "are the same." Oren alternatively explained the postulates' meaning by extending his forearms to represent parallel lines and noting that any amount of rotation from the parallel position would cause the lines to intersect.

Euclid's Fifth Postulate (EFP): "Given two lines cut by a transversal, if the two interior angles on one side of the transversal sum to less than 180°, then the lines will intersect on that side of the transversal."



Playfair's Parallel Postulate (PPP): "Given any line and a point not on that line, there exists only one line through the given point that

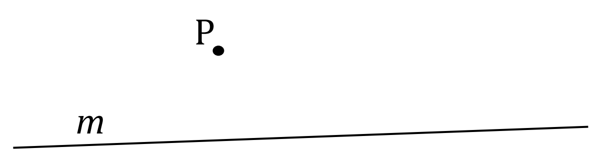


Figure 1: Kirk and Oren's statements and diagram for EFP and PPP

The students began the task intending to prove that EFP ⇒ PPP. The students' argument depended upon dividing the line arrangements into three cases, depending upon the angle sum $\alpha + \beta$. They successfully argued, using EFP and Theorem *, that:

- if $\alpha + \beta < 180°$, lines *l* and *m* meet on that side of line *n*,
- if $\alpha + \beta = 180°$, lines *l* and *m* do not meet,
- if $\alpha + \beta > 180°$, lines *l* and *m* meet on the other side of *n*.

Kirk considered this argument sufficient to prove PPP because it guaranteed that there was only one instance in which the lines *l* and *m* are parallel. He said, "Playfair's Postulate basically states that there's only one instance or case where the lines will not meet." Oren disagreed because he was concerned about how the choice of lines through point *P* (in PPP) corresponded to the angle sums (in EFP). Through their discussion, Kirk also became concerned saying, "It's just hard because Playfair's doesn't include this line *n*, so you are trying to find a way to go from having this line *n* to not having this line *n* in Playfair's." Ultimately, the interviewer invited the students to begin with the diagram for PPP to construct their argument. The pair was able to then use their three cases argument to complete the proof, and Oren correctly identified the need for warrants justifying the construction a transversal line *n* and guaranteeing that each line *l* through *P* corresponded to exactly one angle sum $\alpha + \beta$. Despite their work prior to the interview, Kirk and Oren's proof production took over 40 minutes.

Analysis 1 of Episode 1. The original study in which this episode occurred sought to investigate students' interpretation and use of conditional ("if…then…") statements. The first author used this task because EFP, PPP, and EFP ⇒ PPP can all be understood as conditional statements. Kirk and Oren's initial difficulties in proving EFP ⇒ PPP can be reasonably attributed to the logical structure of their argument, specifically the proof frame (Selden & Selden, 1995). Zandieh, Knapp, and Roh (2008) also reported on students' difficulties with this proof. They attribute this to the fact that students do not adopt a Conditional-Implies-Conditional (CIC) proof frame in which the proof proceeds from the hypotheses of the consequent statement (in this case the point and line arrangement of PPP) to the conclusions of that statement (exactly one parallel through *P*). Kirk and Oren displayed similar difficulty because they adopted the standard proof frame that begins with hypotheses (EFP) and ends with the conclusion (PPP). Kirk's overall argument could be framed by the valid syllogism "EFP (and Theorem *) ⇒ 3 Cases, 3 Cases ⇒ PPP, therefore EFP ⇒ PPP." However, this argument failed to prove that the conclusions of PPP are entailed in its hypotheses, as the CIC proof does. In Raman et al.'s (2009) language, Kirk understood the *key idea* of the proof (3 Cases argument), but lacked the *technical handle* (the proof frame) to construct a valid proof. Ultimately, the interviewer had to prompt the pair to begin with the diagram from PPP, which implicitly introduced the CIC proof frame. This modification allowed the students to produce a valid and more normative proof.

Analysis 2 of Episode 1. Several aspects of Kirk's behavior in the episode are not explained by the absence of an appropriate proof frame. For instance, why was Kirk convinced by his 3 Cases argument while Oren was not? Also, when Kirk described their intention to prove PPP from EFP, he appeared to metonymize (Zandieh & Knapp, 2006) the two statements by their diagrams. To get from EFP to PPP, one diagram needed to be transformed into the other, which required removing a transversal. We posit that a viable explanation for these phenomena requires attention to the geometric nature of Kirk's reasoning (in a visual-spatial sense). Much like Oren's explanation using his forearms to observe the possible arrangements of two lines, Kirk seemed to interpret the postulates as describing geometric possibilities in a quasi-empirical way. This explains why Kirk metonymized the postulates by their diagrams and said they were "the same" (rather than implied each other): the statements described the same set of geometric possibilities.

Analytically, this account of Kirk's reasoning suggests an alternative syllogistic model: "EFP (and Theorem *) ⇒ Only One Instance, PPP ⇒ Only One Instance, therefore EFP ⇒ PPP." Though this argument is invalid, it reflects Kirk's understanding that the statements are linked because they describe the same possible arrangements of lines. However, each implication in this syllogism is distinct in meaning. His explanation suggested that he viewed Only One Parallel as a paraphrase of PPP rather than a consequence of it. Furthermore, his initial argument did not suggest any directionality to his conclusion since the statements were "the same." Thus, Kirk's empirical reasoning convinced him that the 3 Cases argument proved that EFP ⇒ PPP. Oren, in contrast, seemed to interpret the task of proving in a more conventional hypothetical-deductive manner in which warrants justify inferences that form a chain from hypotheses to conclusions. In short, a researcher imposing a deductive frame on Kirk's reasoning easily misrepresents it.

Episode 2: Proving and disproving conjectures about sequences of real numbers

Upon being asked to validate or refute the mathematical statement given in Figure 2, Zander immediately stated, "So, the first thing that I would do is to see if [the series] obviously doesn't converge." He was asked to further talk about what he aimed to do, and Zander stated that he would search for a counterexample to the statement. That is, he would look for a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers satisfying the condition that $0 < a_n \leq a_{2n} + a_{2n+1}$, such that the series $\Sigma_{n=1}^{\infty} a_n$ does not converge.

TASK 1

Validate or refute following statement:

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $0 < a_n \leq a_{2n} + a_{2n+1}$, $\forall\, n \in \mathbb{Z}$ & $n \geq 1$. Then the series $\sum_{n=1}^{\infty} a_n$ converges.

Figure 2. The statement of the original Task 1 statement as presented to Zander.

Zander quickly generated the valid counterexample sequence $a_n = 1 \;\; \forall\, n$. At this juncture, the interviewer asked Zander to prove a slightly modified version of statement in Task 1. The modified statement read, "Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $0 < a_n \leq a_{2n} + a_{2n+1}$, $\forall\;\, n \in \mathbb{Z}$ & $n \geq 1$. Then the series $\Sigma_{n=1}^{\infty} a_n$ ***diverges***."

As before, immediately after being given the modified task statement, Zander stated, "Ok. Uh well … right so then I would have to find an example where it converges." The interviewer asked Zander to confirm whether this meant that he was looking for a counterexample to the modified statement, which he did. Also, Zander quickly considered and discarded the use of various tests for convergence and divergence (e.g. ratio test; comparison test) because he anticipated that none of the tests would "guarantee divergence."

Then, Zander recalled an example of a convergent series with which he seemed familiar: the series $\Sigma_{n=1}^{\infty} \left(\frac{1}{2^n}\right)$. He stated his intentions for choosing this example saying, "maybe we can find a way uh to make a sequence where a_n [term from the sequence described in the task] is equal to $\frac{1}{2^n}$ or smaller than or something like that. Cause then that would converge as well." However, he noted that the corresponding sequence does not satisfy the inequality condition $0 < a_n \leq a_{2n} + a_{2n+1}$. To work around this, he attempted to generate a counterexample by modifying the sequence $\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$ such that each of the terms repeat using the rule $a_{2n} = \frac{1}{2} a_n$ and $a_{2n+1} = \frac{1}{2} a_n$, and with $a_0 = a_1 = a_2 = a_3 = 1$. At this point he realized that "halving" the terms was the "best–case scenario" in order to satisfy the inequality since "it's sort of the cutoff I mean because if we take it to be any smaller a half, say like a tenth of a tenth and then it no longer fulfills this second inequality."

At this point, he stated that he now believed the modified task statement to be true. Zander then called on the harmonic series to attempt to prove that the modified statement is true, even though the harmonic series does not satisfy the inequality condition. He explained that he would like to show that the terms of the harmonic series (or some variant of it) would be a necessary lower-bound to the corresponding terms of the series in the task and thereby the series in the task would also have to diverge (using the comparison test).

Analysis 1 of Episode 2. The study in which this episode occurred focused on finding similarities and differences between expert and novice's proving behaviors. As such, the original analysis characterized Zander's proving behaviors across the various real analysis tasks provided. Zander used the strategy of searching for a counterexample on this and other tasks. When asked about why he did so, he replied, "Because the counterexample might tell you why it always diverges … or rather the inability to find a counterexample might tell you why it always converges." So, on multiple tasks Zander used this strategy of searching for a counterexample to either successfully find a counterexample invalidating the statement or to gain knowledge about why the statement is valid through the inability to find a counterexample. The interviewer also asked Zander why he called on the series series $\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$ and the harmonic series, even though neither one satisfies the inequality condition. He explained that he routinely looked for examples that was relevant to the task context and would provide him with "a picture" or a "prototypical" example that helped him understand the task better.

Thus, this episode supports the general claim that Zander's proving strategy often included searching for counterexamples (regardless of whether he believes one exists), which he perceived useful because he can either successfully find a counterexample or he would gain some insight into why the search for the counterexample is failing and that could tell him why the statement may be true. Furthermore, Zander's work within this episode also supported the claim that he routinely used what he considered "prototypical" examples or visualized "pictures" to gain insight into why a particular claim is true, consistent with previous finding associating visualization and examples with conviction and insight (e.g. Alcock & Simpson, 2004).

Analysis 2 of Episode 2. Even though we can make the content–general claims present in Analysis 1, this may not account for his "expertise" or his relative success on this task. We observe nuances within Zander's search of counterexample and his choice of example series ($\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$ and the harmonic series) that provide insights about his use of his content-specific knowledge about series. Throughout the task, Zander paid particular attention to the growth patterns of various series, which can rightly be considered a link between the inequality condition and the convergence of monotone increasing series. When Zander searched for a counterexample for the modified task statement, he called on the series $\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$ because he knew this to be a series that converged. However, he noted that this series did not satisfy the inequality condition, but by examining the rate at which the terms of the this sequence decreased, he switched his strategy to find "a way uh to make a sequence where a_n is equal to $\frac{1}{2^n}$ or smaller than or something like that … then that would converge as well." So, Zander deduced that "halving" the terms of the sequence would be the "best–case scenario" since,

> "if we take it to be any smaller a half, say like a tenth of a tenth and then it no longer fulfills this second inequality [and] if we take something that was bigger than a half then that's only more problematic because you're just throwing in bigger numbers into the sequence … I think this if I'm right in saying that this sequence always diverges this actually might be a key to the reason why."

In what ways was this scenario "best?" Zander wanted to find a series that converged, so the added terms must decrease, but the inequality limited the rate at which they decreased. Zander's modified example was his "best" possibility to have a minimal growth rate (so as to converge) while satisfying the inequality condition in the task. It seems that a pivotal reason for Zander beginning to believe that the modified statement is valid is because he noticed that the terms of any sequence that would satisfy the inequality condition would have to have a particular growth rate that was not too fast or not too slow. He called upon the harmonic series (even though it is not a series that satisfies the inequality condition) as a "prototypical" example of a divergent series with a small growth rate, since the sequence of terms added converges to 0. Part of what made Zander's proving successful (his "expertise") was his ability to interpret the conditions in the task as constraints on the growth rate of the series and call upon canonical examples that displayed particular growth behaviors. Both his knowledge and use of the prototypical examples point to his analysis-specific knowledge of series, growth rates, and comparison proof methods.

Discussion and Conclusions

We present dual analyses of these two brief proving episodes to portray the alternative insights gained by content-general analysis (of logic, argumentation, example use, etc.) versus content-specific analysis (of empirical or hypothetical/deductive reasoning, growth rates of sequences and series, etc.). Our two studies reflect common research paradigms within mathematics education: 1) task-based interviews intended to elicit instances of mathematical behavior related to a general topic of interest and 2) comparing and contrasting expert/novice mathematical behavior. While both studies employed grounded theory methods, affording these various analyses, these studies still began with guiding questions and theoretical framings (as no investigation can avoid being, on some level, theory-laden). Regarding Episode 1, it was only after attempts to generally characterize Kirk and Oren's interpretations of conditional statements failed that the author attended to the broader differences between the ways they interpreted the statements and the task at hand, which explain their very different assessments of their proving activity. Regarding Episode 2, the second author designed the study to include tasks in various mathematical contexts, but later refined the study tasks to only include real analysis tasks. While the content-general claims about Zander's proving expertise are supported by Zander's proving practice and his self-reflection, they may also hide the role and value of Zander's extensive experience with real analysis in his interpretation and progress on the task.

As we stated before, our goal is not to deny the value of content-general proof research, but rather to sensitize the mathematics education research community to the liabilities of such a research lens. When and why should researchers attend to the role of particular content in their findings? The first episode suggests that content-general models of student activity such as logic may be broadly applied, but may also be misleading or dishonest to a student's reasoning process. We maintain that the two syllogisms are, in some sense, viable renderings of Kirk's reasoning, but the non-uniqueness of such logical models of his reasoning is troubling. Also, the three "implications" in the latter model of his reasoning are all distinct in meaning and likely gloss over the nature of Kirk's inferences. To hazard an analytic generalization, researchers must be wary applying a content-general model to student reasoning, especially when the chosen model reflect the *researcher's questions* more than the *students' mathematical behavior*.

The second episode suggests that characterizations of "successful proving" or "expertise" must account for the fact that both proving behavior and expertise are highly multi-dimensional. Certainly example use is an important dimension of Zander's proving behavior, as evidenced by his own awareness and explanations thereof. However, the use of such content-general heuristics for further research and instruction necessitate awareness of how example use interacts with other elements of Zander's experience and understanding to afford the behaviors observed in Episode 2. In general, we encourage more research on proving behavior to attend to the role of mathematical meanings (Thompson, 2013). Furthermore, the growing presence of (content-general) introduction to proof courses (Selden, 2012) entails a great need for research on the existence and development of content-general proving behaviors and how they can be fostered within and across mathematical contexts.

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Mathematicians' rationale for presenting proofs: A case study of introductory abstract algebra and real analysis courses

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Proofs are essential to communicate mathematics in upper-level undergraduate courses. In an interview study with nine mathematicians, Weber (2012) describes five reasons for why mathematicians present proofs to their undergraduate students. Following Weber's (2012) study, we designed a mixed study to specifically examine what mathematicians say undergraduates should gain from the proofs they read or see during lecture in introductory abstract algebra and real analysis. Our preliminary findings suggest that: (i) A significant number of mathematicians said undergraduates should gain the skills needed to recognize various proof type and proving techniques, (ii) consistent with Weber's (2012) findings, only one mathematician said undergraduates should gain conviction from proofs, and finally (3) some mathematicians presented proof for reasons not described in Weber's (2012) study such as to help their students develop appreciation for rigor.

Key words: Proof, Purpose of proof, Proof presentation, Undergraduate mathematics

In upper-level mathematics courses, mathematicians regularly use proofs to convey mathematics to their students. As a result, mathematicians expect their students to gain some understanding from the proofs they present. A plethora of research suggest that student find the concept of proof problematic (Harel & Sowder, 1998; Inglis & Alcock, 2012; Moore, 1994; A. Selden & Selden, 2003). Research on undergraduates interaction with proofs suggests that undergraduates often times have difficulty with determining the validity of a proof and/or constructing a valid proof (Alcock & Weber, 2005; Inglis & Alcock, 2012, Selden & Selden, 2003; Weber, 2010). For instance, Selden and Selden (2003) argued that when reading proofs undergraduates tend to focus on surface features of mathematical arguments as opposed to its *global* feature. Participants in their study showed only limited ability to determine if a mathematical argument is valid or qualifies as a proof or not.

Empirical studies focusing on what mathematicians expect their upper-level undergraduates to gain from proofs are rare. In a semi-structured interview with nine mathematicians, Weber (2012) argued that most mathematicians present proofs mainly to facilitate their students' understanding of mathematical concepts and/or illustrate some proving techniques. Yopp (2011) also reports that in advanced undergraduate mathematics courses, mathematicians mainly present proofs to show their students how to prove theorems.

The extent to which students actually learn mathematical concepts from seeing proofs remains an open research problem. However, one can infer from existing research that undergraduates actually do not gain mathematical understanding from proofs (Conradie & Frith, 2000). Weber (2012) also evidenced that mathematicians rarely present proofs to convince their students that a theorem or a proposition is true; this is in contrast to the primary role of proof in mathematics scholarship (Hersh, 1993). Alternatively, Hersh (1993) maintains that in mathematics classroom, the primary goal of presenting proofs should be to provide an explanation for why a theorem is true. Interestingly, some participants in Weber's (2012) study expressed doubt if proof is indeed an effective way to convey mathematics to all their students. Our study contributes to the growing body of literature on the purpose of proof in undergraduate mathematics instruction by examining the following research question: What roles do proofs play in the teaching of introductory abstract algebra and/or real analysis courses? In what follows, we discuss the theoretical framework guiding this study.

Theoretical framework and literature review

The three most important roles of proofs discussed in the proof literature are: (1) *conviction* or *verification*, (2) *explanation, and* (3*) illustrating proving techniques*. *Convincing* is the idea that a proof demonstrates that a theorem is true. Although undergraduates and surprisingly mathematicians (e.g., Weber, Mejia-Ramos, & Inglis, 2014) are sometimes convinced without proof, De Villiers (1990) writes that "the well-known limitations of intuition and quasi-empirical methods" underscore the vitality of proof as a useful means of *verification* (p.19). Convincing is perhaps the primary goal of any proof. Indeed, some such as Hersh (1993) actually define proof simply as "a convincing argument, as judged by competent judges" (p. 389).

Convincing may be the primary goal of any published proof; however, there is a consensus that the functionality of a proof is not, and should not, be limited to verifying that a theorem is true (De Villiers, 1990; Hersh, 1993). The fact that we have different published proofs in peer-reviewed journals of a single known result inevitably leads us to believe that proofs are far more than a certificate of truth. Indeed, it appears that there is considerable interest in the insight that is gained from the reasoning utilized in a proof. For a mathematician, a proof—beyond convincing—also functions as an explanatory argument. To explain is to provide insight as to why a theorem is true (De Villiers, 1990; Hersh, 1993; Knuth, 2002; Thurston, 1995; Weber, 2002; Weber, 2008). Explanatory proofs are insightful precisely because they make "reference to a characterizing property of an entity or structure mentioned in the theorem, such that from the proof it is evident that the result depended upon the property" (Steiner, 1978). According to De Villiers (1990), explanatory proofs provide "psychological satisfactory sense of illumination" (p.19).

Mathematicians' desire for explanatory proofs is evident in the controversy surrounding Appel and Haken's joint proof of the Four-Color theorem (Thurston, 1995). Appel and Haken's joint proof heavily depended on a computer; for that reason, renowned mathematicians such as Paul Halmos showed dissatisfaction toward the proof, as it apparently did not provide any insight for why the theorem must be true. Stressing the importance of the explanation in a proof, Hanna (2000) writes: "[a proof] becomes both convincing and legitimate to a mathematician only when it leads to real mathematical understanding" (Hanna, 2000). In fact, all eight mathematicians interviewed in Weber (2008) claimed that the primary reason they read published proofs is to gain insight. In particular, in undergraduate mathematics education, Hersh (1993) argued that the primary role of proofs should be to offer insights and provide complete explanations why a given theorem is true. Harel and Sowder (2007) complement this when they say: "…mathematics as sense making means that one should not only convince oneself that the particular topic/procedure makes sense, but also that one should be able to convince others through explanation and justification of her or his conclusions" (p. 808-809). In addition, Hersh (1993) maintains that one should consider the explanatory power of a particular proof when making the decision whether or not a proof is worth presenting in class. Hersh (1983) writes: "proof can make its greatest contribution in the classroom only when the teacher is able to use proofs that convey understanding" (p.7). Therefore, it is important that instructors make use of more explanatory proofs in their instruction when possible.

Proofs, beyond convincing and explaining, can function as tools to communicate techniques or ways of reasoning that can later be used to tackle other problems. Thurston (1995) argued that mathematicians sometimes use proofs to communicate a developed body of common knowledge or new techniques in the case of truly novel proofs. For example, mathematicians interviewed in Weber's (2010) study stated that when reading a proof, they would hope to learn new techniques that might eventually help them prove conjectures or problems they have been thinking about in their research.

De Villiers (1990) proposes even more roles of proofs: proofs as a means of discovery and proofs as a means of systematization. He argues that proofs are our only tool "in the systematization of various known results into deductive system of axioms, definitions and theorems" (p.20). Take, for example, the proof of the intermediate value theorem for continuous functions; he asserts that the primary function of this proof is basically a systematization of continuous functions. Systematization, among other things, provides global perspective, simplifies mathematical theories, and enables us to identify inconsistencies, circular reasoning, and hidden assumptions (De Villiers, 1990). In addition, a proof enables us to explore, generalize, analyze, and discover mathematical ideas (De Villiers, 1990). For example, the invention of non-Euclidean geometries would have been completely unthinkable without our capacities of deductive reasoning and proof, since these ideas are unintuitive.

Research methodology

Fifteen mathematicians agreed to participate in our study. All participants were solicited from a large public university in the United States. The mathematicians come from a wide range of research interests including, but not limited to, analysis, algebra and topology. The lead author provided the mathematicians a written task asking them to briefly describe what they would hope an undergraduate student enrolled in introductory abstract algebra and/or real analysis would gain from reading or seeing proofs during lecture. Fourteen of the 15 mathematicians who agreed to complete the written task have at least seven years of teaching experience in tertiary institution. While a significant number of the participants taught at least two proof-based mathematics courses, four mathematicians said they have not taught any proof-based course at this institution.

We also conducted task-based interviews with three mathematicians (an algebraist, and analyst and a topologist). Two of the mathematicians who agreed to be interviewed did not complete the written task. The two algebraists and the one analyst that we interviewed have at least ten years of experience teaching introductory abstract algebra and real analysis respectively. During the interview we asked the mathematicians the following questions:

- Why would you present the proof of Lagrange's theorem?

- In general, what would you say is the purpose(s) of presenting proofs during lecture in undergraduate mathematics courses such real analysis (or abstract algebra if the interviewee is algebraist?
- Is there a proof that you would consider a 'must see' in your introductory real analysis (or abstract algebra if the interviewee is algebraist) (adopted from Weber's (2012) study)

Results and discussion

We present our preliminary results as follows. Recall that the main goal of this study is to explore what mathematicians hope their undergraduate students gain from the proofs they present in upper-level undergraduate courses such as abstract algebra and/or real analysis. Two researchers independently coded participants written response based on categories presented in Weber's (2012 study. As it is evidenced in Table 1, the majority of mathematicians (60%) said that they would hope undergraduates develop proficiency in recognizing proof type. This includes, but is not limited to, identifying whether the proof is a direct proof, a proof by contradiction, a proof by cases, or a proof by mathematical induction. We find this surprising because we were expecting that mathematicians would only say this for undergraduates in intro-to –proof courses. Consistent with Weber's (2012) study, we found that (1) a significant number of our participants (46.67%) said they would hope undergraduates would learn new proving techniques from seeing proofs during lecture, and (2) only one mathematician described conviction as an important role of proof for undergraduates. The following interview excerpt indicates that mathematicians present proofs to illustrate some proving techniques.

I: Is there a proof that you consider a must-see in your abstract algebra course?

P: A proof that I consider a must-see um there are a number of types of proofs that I think that they should see um for instance um when some either the uniqueness of the zero element, the uniqueness of inverses of elements, something to that effect. I think it's a must-see. Um um what other things? Uh um either the idea that a kernel of an image of a homomorphism is a subgroup

I: Why would you think that is a must-see or is important for them to see?

P: to see? Well because many constructions or many ideas that we use to study groups are based on the study of homomorphisms between groups.

Additionally, some mathematicians said they would hope that undergraduates would develop proficiency in logical inferences from seeing proofs presented in upper-level undergraduate mathematics courses. Also, a small percentage of mathematicians (13.33%) said they presented proofs so that students can appreciative the rigor that goes into writing proofs. We find this interesting because it has not been evidenced, to our knowledge, in any empirical study.

Table 1 Mathematicians' reasons for presenting proofs in upper level mathematics courses

Reason	Percentage of participants
To help students recognize proof type	60%
To illustrate proving techniques	46.67%
To develop proficiency in logical inferences	33.33%
To illustrate why a theorem is true	33.33%
To help students recognize proof type	20%
To develop appreciation for rigor	13.33%
To communicate mathematical ideas	13.33%
To establish that a theorem is true	6.67%

Discussion questions and implications for further research

We believe that our preliminary study contributes to the scarce literature on the role proofs play in undergraduate mathematics education. We plan to analyze our interview transcripts to examine if there are additional reasons why mathematicians present proofs in upper-level mathematics courses. During our presentation, we would like to get some feedback on the following questions.

(1) What methodological suggestions might you offer us to examine any non-mathematical benefits, assuming that there are some, that one can acquire from reading or seeing a proof during lecture, and to what extent do we care?

(2) Are there good reasons to believe that mathematicians present proofs in different classes for different reasons? How can we explore that?

In summary, we believe that our study provides further evidence for the claim that convincing should not be the primary goal of presenting proofs in mathematics instruction. Finally, we hope that further research such as interviewing more mathematicians can provide insight into additional roles that proof can play in undergraduate mathematics education.

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Undergraduate students proof-reading strategies: A case study at one research institution

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Weber and Mejia-Ramos (2013) identified five effective proof-reading strategies that undergraduate students in proof-based courses can use to facilitate their proof comprehension. Following their study, we designed a survey study to examine how undergraduate students' proof-reading strategies relate to what proficient learners of mathematics (mathematics professors) say undergraduates should employ when reading proofs. Our preliminary findings are: (i) Majority of the professors in our study claimed that undergraduates should use the strategies identified in Weber and Mejia-Ramos' (2013) study, (ii) Professors' response significantly differed from undergraduates' in only two of the five proof-reading strategies described in Weber and Mejia-Ramos' (2013) study (trying to prove a theorem before reading its proof and illustrating confusing assertions with examples), and finally (iii) Undergraduate students, for the most part, tend to agree with their professors' preferred proof-reading strategies.

Key words: Proof, Proof-reading strategies, Proof comprehension, Undergraduate mathematics

In upper level mathematics courses, mathematicians regularly use proofs to convey mathematics to their students. As a result, students in these courses are expected to spend sufficient time reading and writing proofs (Weber & Mejia-Ramos, 2014). Research on undergraduates' interaction with proofs suggests that undergraduates often times have difficulty with determining the validity of a proof and/or constructing a valid proof (Alcock & Weber, 2005; Inglis & Alcock, 2012, Selden & Selden, 2003; Weber, 2010). For instance, Selden and Selden (2003) argued that when reading proofs undergraduates tend to focus on surface features of a mathematical argument as opposed to its *global* feature. Participants in their study showed only limited ability to determine if a mathematical argument is valid or qualifies as a proof or not. Studies also suggest that undergraduates often do not actually gain understanding from the proofs they read (Conradie & Frith, 2000; Cowen, 1991). There is, however, very little research on how undergraduates read proofs with the intent of learning mathematics from them. In an effort to improve students' understanding of proof, Weber and Mejia-Ramos (2013) developed five proof-reading strategies that undergraduates can use to improve their proof comprehensions, which form the basis for this study.

Theory

We designed our survey study based on Weber and Mejia-Ramos' (2013) studies on effective proof-reading strategies. In a qualitative study, Weber and Mejia-Ramos (2013) observed four mathematics majors and prospective teachers read six proofs. The authors considered these students to be strong because they were successful in both their content-based mathematics courses and on the follow up proof comprehension test that the authors designed based on Mejia-Ramos et al's (2012) proof comprehension assessment model. Their analysis revealed five proof-reading strategies that the students used to facilitate their understanding of the proofs. These five strategies identified in their study are: (1) trying to prove a theorem before reading its proof, (2) comparing the assumptions and conclusions in the proof with the proof technique being used, (3) breaking a longer proof into parts or sub-proofs, (4) comparing the proof approach to the one's approach, and (5) using an example to understand a confusing inference. Weber and Mejia-Ramos (2013) followed up their qualitative study with a large-scale internet-based survey study that included mathematics majors and mathematicians from 50 large state universities in the United States. The purpose of their quantitative study was two-fold: (1) to explore whether mathematicians prefer mathematics majors to use these five proof-reading strategies and (2) to explore to what extent mathematics majors use these strategies. The main finding of their study is that the majority of mathematics major do not use these proof-reading strategies. This continues to be the case even though the majority of mathematicians believed that mathematics majors should use these strategies. This is an interesting finding since it sheds light on why undergraduate students often times gain little from proofs (e.g., Conradie & Frith, 2000; Cowen, 1991; Rowland, 2001). Our study examines whether these findings hold in one large institution.

Previous research on student comprehension of proofs

The literature on proof comprehension is relatively sparse. Some earlier studies on proof assessment indicate that mathematicians do not necessarily evaluate their students' understanding of a given proof effectively (Conradie & Frith, 2000, Weber, 2012). Conradie and Frith (2000), for instance, maintain that mathematicians' ways of testing their students' understanding of a proof usually require nothing beyond recalling the statements and its proof. The mathematicians interviewed in Weber's (2012) study also conceded this. In Weber's (2012) study mathematicians reported that they measured their students' understanding of proofs by (1) asking students to construct a proof for a similar theorem to the one that was proven in class, and/or (2) asking them to reproduce a proof; and some said they do not assess their students' understanding of a proof. Conradie and Frith (2000) maintain that students can pass simply by memorizing the statement and proof of each theorem as presented in class; this, however, as they point out, does not effectively reflect students' understanding.

There are fewer studies on what students do when they read proofs for understanding. For example, Inglis and Alcock (2012) conducted a study that compared and contrasted beginning undergraduate students' proof-reading habits to those of research-active mathematicians. By studying their participants' eye movement while reading a proof, they concluded that undergraduate students, compared to the experts in their study, spend more time focusing on the "surface feature" of a mathematical proof. Based on this observation, the researchers suggest that undergraduates spend less time focusing on the logical structure of the argument; this, in turn, seems to explain why students often have difficulty understanding the logical structure of a mathematical argument, as evidenced elsewhere in the literature (A. Selden & Selden, 2003).

There is a growing body of literature aimed at improving undergraduates' proof comprehension. Recently, Hodds et. al (2014) put forward a pedagogical technique known as *self-explanation training* that they argued can improve students' proof comprehension by improving their engagement with the proof. Additionally, Weber and Mejia-Ramos' (2013) study also describes five proof reading strategies that undergraduates can use to facilitate their understanding of proof. Our study contributes to the growing body of literature in proof comprehension by examining the following research questions: (1) To what extent do professors endorse the proof-reading strategies described in Weber and Mejia-Ramos' (2013) study? (2)To what extent do undergraduate students use proof-reading strategies described in Weber et.al's (2013) study?

Research methodology

The population for this study consisted of undergraduate students who have at least taken or enrolled in a transition-to-proof course, and mathematics professors. All participants were solicited from a large public university in the United States. Because we were investigating the relationship between professors' suggestions and students' uptake, we believed that asking both groups and attempting to relate them at the professor-and university-level is useful. We should note that although the majority of our undergraduate student participants were taking a transition-to-proof course, a significant number of them had at least two proof-based mathematics course, including, but not limited to, introductory abstract algebra and real analysis.

We replicated the survey items in Weber and Mejia-Ramos' (2013) study where they asked mathematics majors to indicate the extent to which the aforementioned proof-reading strategies are reflective of their own. For undergraduate students, one of the researchers visited all proof-based undergraduate mathematics courses offered at this institution at the time this research was taking place and asked the students to complete the survey. Nearly all undergraduate students (92) who were enrolled in at least one proof-based course completed the survey. Most undergraduates completed the survey in less than 10 minutes. Following Weber and Mejia-Ramos' (2013) study, we also disseminated the survey to mathematics professors in this institution. Fifteen mathematics professors agreed to participate. The survey questionnaires for the professors were virtually identical questions; however, they were directed to reflect undergraduate students' proof-reading experience as opposed their own. For instance, to examine to what extent undergraduate students employ proof-reading strategy #1 (trying to prove a theorem before reading its proof), we asked them to what extent they agreed with the following statement: When I read a theorem, I usually try to think about how I would prove the theorem before reading its proof. For professors, the item above was phrased as follows: when reading a theorem undergraduate students should usually try to think about how they would prove the theorem before reading its proof. All participants were asked to indicate their choice using a five-point Likert scale (strongly agree (5), agree (4), neutral (3), disagree (2), and strongly disagree (1)).We used to the statistical software JMP 12.1 Pro to determine if there is a statically significant difference between the two groups. We will present our findings in the next section.

Results

We organize our results based on the research questions. Recall that the main goal of this study is to explore to what extent undergraduate students in one large research institution employ proof-reading strategies that professors in that same institution find desirable. As it is evidenced in Table 1, the majority of professors claimed that undergraduate students should employ all five proof-reading strategies described in Weber and Mejia-Ramos' (2013) study. In particular, a significant number of professors (85.71%) strongly agreed or agreed that undergraduate students should use examples to verify the veracity of potentially confusing assertions in a proof; on the other hand, only 66.3% of undergraduate students claimed to employ this strategy. In fact, using Wilcoxon Each Pair Test we found that undergraduate students' response on this proof reading strategy is significantly different from professors with an alpha-level of 0.05.

Additionally, a large percentage of professors (73.33%) either strongly agreed or agreed that when reading a theorem, undergraduate students should attempt to prove the theorem before reading its proof (strategy #1), however, only 60.87% of undergraduate students claimed to have used this proof-reading strategy. Indeed, a Wilcoxon Each Pair Test revealed that undergraduate students' response statistically significantly differed from professors with an alpha-level of 0.05. This finding is consistent with that presented in Weber and Mejia-Ramos (2013) study where the majority of mathematicians (88%) agreed that mathematics majors should try to prove a theorem before reading its proof. Weber and Mejia-Ramos' (2013) study also evidenced that only 31% of mathematics majors in their study said they would attempt to prove a theorem before reading its proof (strategy #1). In our study, we have no evidence to support this claim; on the contrary, our study revealed that the majority of undergraduate students did in fact claim to use these strategies. We will present a plausible explanation for this discrepancy in the next section.

Table 1 Percentage of participants who strongly agree or agree on the survey items (see Appendix 1)

Strategy	Professors	Undergraduates
1. Trying to prove a theorem before reading its proof	73.33%	60.87%
2. Considering proof's frameworks	86.67%	88.04%
3. Comparing proof method with one's own approach	60%	63.04%
4. Breaking a long proof into parts	66.67%	69.57%
5. Illustrating confusing assertion with an example	85.71%	66.3%

Table 2 *p-values* in Wilcoxon Each Pair Test (based on Wilcoxon rank scores, also called Mann-Whitney test) using the statistical software JMP 12.1 pro

Participants	Strategy #1	Strategy #2	Strategy #3	Strategy #4	Strategy #5
Professors vs. Undergraduates	**0.0303**	0.3663	0.645	0.7095	**0.0471**

Discussion and implications for further research

On proof-reading strategy, in this paper, we argued that statistically significant difference between undergraduate students and professors existed only in the two of the five proof-reading strategies (strategies #1 and #5), suggesting that undergraduate students mostly claimed to employ desirable proof-reading strategies. We have also argued that undergraduate students' proof-reading strategy, for the most part, tend to agree with what their professors say undergraduates should do when reading proofs.

The level of agreement between undergraduates and professors on strategy #4 (breaking a longer proof into parts or sub-proofs) is encouraging. It is encouraging because they are using a reading strategy that is identified in the literature as effective for proof comprehension (Weber, 2015, Weber & Mejia-Ramos, 2013). At the same time, we are surprised by this result because Weber and Mejia-Ramos (2013) in their survey study found that only 38% of mathematics major claimed to have employed it. We believe there are several plausible explanations for this discrepancy. First, while our survey questions were identical to theirs, the choices our participants had were slightly different. In their study, participants were given two choices and asked to indicate if they agree or disagree; in contrast, in our study, participants were asked to indicate their choice on a five point Likert scale. Second, their Internet based survey included participants from 50 large institutions in the United States; on the other hand, our pool of participants comes from a single institution. Thus, it could be the case that mathematicians in this institution explicitly discuss these proof-reading strategies with their students. Finally, our undergraduate participants were different from theirs in the sense that our participants were not only mathematics majors, our study incorporated participants majoring in computer science, and secondary mathematics education. We plan to conduct further analysis of our data to examine if our preliminary results hold for mathematics majors only, prior to this we would like to use our presentation to receive feedback regarding the inconsistency of our result to that of Weber and Mejia-Ramos' (2013). In particular, we would like to focus on the following discussion questions: (1)To what extent do you agree or disagree with our potential explanation for inconsistency? (2) What further analysis of our survey data might explain the inconsistency?

In summary, our study provides further evidence that the strategies described in Weber et.al's (2013) are indeed considered effective in facilitating proof comprehension. We have also argued, contrary to Weber and Mejia-Ramos (2013) study, the majority of undergraduate report to use these effective proof-reading strategies. As a result, we believe this study is a welcome addition to the paucity of the literature in proof comprehension. Finally, we hope that further research such as interviewing these mathematicians might provide insight into the surprising level of agreement between them and their students.

Appendix 1. Survey Items (slightly modified from Weber and Mejia-Ramos (2013) study)
Strategy #1: When I read a theorem, I usually try to think about how I would prove the theorem before reading its proof.

Strongly agree Agree Neutral Disagree Strongly disagree

Strategy #2: When I read a proof of a theorem, I consider what is being assumed, what is being concluded, and what proof technique is being used.

Strongly agree Agree Neutral Disagree Strongly disagree

Strategy #3: When I read a proof, I compare how the methods used in the proof compares to the methods I would use to prove the theorem.

Strongly agree Agree Neutral Disagree Strongly disagree

Strategy #4: When I read a long proof, I try to break it into parts or sub-proofs.

Strongly agree Agree Neutral Disagree Strongly disagree

Strategy #5: When I read a new assertion in a proof that I find confusing, I sometimes check whether that assertion is true with specific example.

Strongly agree Agree Neutral Disagree Strongly disagree

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CLEARING THE WAY FOR MINDSET CHANGES THROUGH FORMATIVE ASSESSMENT

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One of the reasons for the exodus in STEM majors is the introductory calculus curriculum. Although there is evidence that curricula like CLEAR calculus promoted significant gains in students' growth mindset, it is unclear how this curriculum promotes mindset changes. The purpose of this case study was to investigate which features of CLEAR Calculus promoted positive changes in students' mindsets. After administering the Patterns of Adaptive Learning Scale to assess students' initial mindset in one section of calculus, four students were selected for interviews. Although participants were selected for maximal variation in their mindset at the beginning of the course, there were a lot of similar themes in their interviews. Students cited that CLEAR Calculus curriculum challenges them in ways that facilitates deeper comprehensive learning than that of a traditional calculus course.

Key words: Calculus, formative assessment, mindsets

Prospective STEM majors who declare a non-STEM major are most likely to do so after introductory calculus (Bressoud, Rasmussen, Carlson, & Mesa, 2014); students cite their lack of a perceived relationship with their instructor and the inability to seek help as primary reasons for switching (Ellis & Rasmussen, 2014). One possible solution is the use of formative assessments such as exit tickets; such assignments show promise in helping students to perceive their instructor as more approachable and caring about their success (Black & Wiliam 1998, 2009; Author 2, 2014).

However, the number formative assessments completed are a far stronger predictor of students' success than their weight in the course grade would indicate (Author 2, 2015). One possible explanation for this effect was that students who completed more post-labs had different mindsets about learning mathematics than those that did not. It has been noticed that mindsets play a significant role in the overall success of calculus students. Dweck (2006) defines mindset in two different ways: fixed mindset and growth mindset. Students classified under the fixed mindset, if not immediately successful in introductory calculus often leave the STEM field. However, growth mindset students can persist and succeed, even after failures as severe as failing a course (Dweck, 2007).

We examined how CLEAR Calculus supports positive mindset changes in students through a case study of four students enrolled in an introductory calculus class taught using CLEAR Calculus. This research will be guided by the question: What are the features of CLEAR Calculus that promote positive changes in students' mindsets? By understanding what makes this curriculum effective, interested practitioners who are not implementing CLEAR Calculus can learn what components to add to their classes if they would like to see a positive increase in their students' mindsets. We argue CLEAR Calculus supports positive changes in students' mindsets because the labs make challenge and conceptual understanding central components of the course, while the set routine of the class and the use of formative assessments helped to prevent students from feeling overwhelmed.

The theoretical perspective for this case study (Patton, 2002) was Dweck's (2006) mindsets. Participants attended a midsized rural regional university in the South, and were recruited from an introductory calculus course taught using CLEAR Calculus labs. These labs are built upon developing systematic reasoning about conceptually accessible approximations and error analyses but mirroring the rigorous structure of formal limit definitions and arguments (Oehrtman, 2008, 2009).

Students in the course took the Patterns of Adaptive Learning Scale (PALS) during the second week of the semester. Four participants participated in semi-structured interviews (Patton, 2002) to obtain a sample with maximum variation according to their mindset (Table 1). Author 1 observed the class and consulted with the instructor of the course for triangulation of the interview data. After the interviews were transcribed, the data was analyzed using standards of evidence derived from the literature.

Table 1
Participants

Pseudonym	**Year**	**Major**	**Mindset**
Ian	Junior	Math	Strong Growth
Roland	Freshmen	Biology	Weak Growth
Penelope	Sophomore	Biology	Weak Fixed
Steven	Freshmen	Math	Strong Fixed

Overall, participants found the feature of the CLEAR calculus that caused them to become more growth mindset-orientated was the presence of safe challenges. Although the labs were always challenging for students, the labs were also seen as the central feature in the course and the main difference between their current calculus experience and their previous mathematics classes, particularly those taken in high school. While the labs provided the challenge needed to help students begin to examine their belief systems, the formative assessments were also seen as a key feature of the course. Even though postlabs required little time, they were aware that questions on the postlab would be answered by the instructor. For students with a more fixed mindset, this help was available with minimal effort and without admitting the need for help in front of peers, which made seeking aid more palatable.

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What do students attend to when first graphing in R^3?

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This poster considers what students attend to as they first encounter R^3 coordinate axes and are asked to graph functions with free variables. Graphs are critical representations, yet students struggle with graphing functions of more than one variable. Because prior work has revealed that students' conceptions of multivariable graph are often related to their conceptions about single variable functions, I used an actor-oriented transfer perspective to identify what students see as similar between graphing functions with free variables in R^2 and R^3. I considered what students attended to mathematically, and found that they focused on equidistance, parallelism, and coordinate points.

Key words: generalisation, multivariable calculus, multivariable functions, graphing

Including multivariate topics in K-12 mathematics is one way to increase mathematical competence for all students (Ganter & Haver, 2011; Shaughnessy, 2011). Because multivariable topics share many similarities with their univariate counterparts, many researchers studying student learning of multivariable topics focus on how students generalise from the single- to multivariable context (e.g., Dorko & Weber, 2013; Kabael, 2011; Yerushalmy, 1997). This poster exhibits some initial findings from a longitudinal study that seeks to explore how calculus students generalise function and limit from the single- to multivariable context. Specifically, it considers what students attend to as they first encounter R^3 coordinate axes and are asked to graph functions with free variables.

Graphs are critical representations in calculus, yet students struggle with creating graphs of multivariable functions (Kabael, 2011; Martinez-Planell & Trigueros, 2012). Students' correct understandings about the shapes of graphs in R^2, for instance, may interfere with their learning about graphs in R^3. Some students graph $f(x,y) = x^2$ as a parabola rather than as a parabolic surface. Students may also draw $f(x,y) = x^2 + y^2$ as a cylinder or a sphere because they are accustomed to $x^2 + y^2$ representing a circle in R^2. These examples illustrate that part of students' thinking about multivariable functions' graphs comes from generalising the ways they think about graphs in R^2. I sought to further explore this, with the hypothesis that learning more about what students attend to when graphing can help instructors emphasize the productive connections students see across situations and target students' misconceptions. Toward that end, this poster focuses on the following research question: *what do students attend to as they first think about graphing multivariable functions with free variables?*

Theoretical Framework

I use an actor-oriented transfer lens to study student thinking about graphing. Actor-oriented transfer focuses on what students see as similar across situations, even if their perceptions of similarity are not normatively correct (Lobato, 2003). From this perspective, students' graphing activity in R^3, even if incorrect, *makes sense to them* for some particular reasons, and the goal is to uncover those reasons. In the two examples given above, students' reasons for drawing $f(x,y) = x^2$ as a parabola and $f(x,y) = x^2 + y^2$ as a cylinder or sphere might indicate that they are attending to the way similar equations, $f(x) = x^2$ and $x^2 + y^2 = r^2$, look in R^2. My use of an actor-oriented transfer perspective affords identifying more of these sorts of connections that students see and use as they think about what graphs of multivariable functions look like.

Methods of Data Collection and Analysis

I asked 12 differential calculus students about multivariable functions so that I could observe the initial sense making of students who had not yet received instruction regarding these functions. I hypothesised that this would allow me to observe students' abstractions in real time. This poster focuses on data from three tasks: students' graphs of $y = 2$ in R^2, $y = 3$ in R^3, and $f(x,y) = x^2 + 6$ in R^3. I asked follow-up questions such as "why did you draw a [line, plane, curved surface] here?" I analysed my data by first identifying instances of generalisation, defined as "the influence of a learner's prior activities on his or her activity in novel situations" (Ellis, 2007, p. 225). A colleague and I then reviewed and discussed those episodes, with the goal of characterising the nature of those generalisations. Specifically, we looked for (a) any references on the students' part to graphing or functions in R^2, which we coded using Ellis' (2007) generalisation taxonomy, and (b) what mathematical concepts or ideas students leveraged as they generalised.

Results

Due to space limitations, I focus on a particular student, Alex, and then give brief details about ways other students answered these tasks. Alex drew a correct graph of $y = 3$ in R^3 despite having seen R^3 coordinate axes for the first time in the interview. Alex's work is compelling because he gave two incorrect answers, then reasoned to a correct answer by *connecting back* (c.f. Ellis, 2007) to the graph of $y = 2$ in R^2 and attending to two mathematical properties: equidistance and parallelism. He generalised these from the univariate case to describe the graph of $y = 3$ in R^3 as a plane "that is 3 away from the plane that x and z creates":

Alex: Actually, $y = 3$ … would be an entire plane….It has to be parallel to x, and this has to be parallel to z, so it would be this plane right here that is 3 away from the plane that x and z creates… like for the last question when y is equal to 2, that is every value that is 2 away from $y = 0$, right? So I'm thinking that like $y = 0$ would be the same as this [shades xz plane]. So it's 3, it's 3 in the positive [y] direction, because it's a positive 3, it's y equals that… *Interviewer:* Tell me about this parallel, like you said it's going to be parallel to x and z? *Alex:* It's going to be parallel to x in the same way that this line right here [$y = 2$ in R^2] is parallel to the x, to the x-axis. So it's kind of the same thing except it's like, it would be like that if it was a plane."

The sketching activity, and connecting back to the graph of $y = 2$ in R^2, allowed Alex to generalize that $y = b$ is a line in R^2 and a plane in R^3. He drew two incorrect graphs before drawing the correct one ("actually, $y = 3$ would be an entire plane"), and it was in the process of creating and reviewing these graphs that he appeared to focus on using the equidistance and parallelism to arrive at the correct answer. Alex's thinking about these two ideas is representative of other students. Another, asked to graph $y = 3$ in R^3, said "so on an xy [R^2] graph at 3, would be going this way. So on the y, following the x. So [on R^3 axes] this would be on the y, this is the 3 point on the y, and it's following the x axis." This student *created a new situation* (c.f. Ellis, 2007) that he viewed as similar to the current situation, and generalised by thinking about parallelism, which he stated as "following." Other ways students thought about this question were in terms of plotting points; for instance, "$y = 3$ at all points on the graph, any point you evaluate, so if you say $z = 2$ and $x = 2$, it's going to be 3." Hence the initial data analysis suggests that as students generalise, some of the things they attend to equidistance, parallelism, and coordinate points.

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Investigating a mathematics graduate student's construction of a hypothetical learning trajectory

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This study reports results of how a teacher's mathematical meanings and instructional planning decisions transformed while participating in and then generating a hypothetical learning trajectory on angles, angle measure and the radius as a unit of measurement. Using a teaching experiment methodology, an initial clinical interview was designed to reveal the teacher's meanings for angles and angle measure and to gain information about the teacher's instructional planning decisions. The teacher participated in a researcher generated HLT designed to promote the construction of productive meanings for angles and angle measure and then constructed her own HLT for her students. The initial interview revealed that the teacher had several unproductive meanings for angles and angle measure that caused the teacher perturbations while participating in the tasks of the researcher generated HLT. This participation allowed her to construct different meanings for angles and angle measure which changed her instructional planning decisions.

Key words: Hypothetical Learning Trajectories, Trigonometry, Graduate Teaching Assistant Education, Mathematical Meanings

Students and teachers often have difficulty reasoning about topics related to trigonometric functions (Moore, 2010; Thompson, Carlson, & Silverman, 2007; Weber, 2005). Moore (2010) described several reasons that students may have difficulty reasoning about trigonometric functions including the approach that current curricular materials take when introducing the sine and cosine functions. Many teachers introduce trigonometric functions in both right triangle contexts and unit circle contexts, though they rarely make connections between the two. This approach hinders students' ability to develop coherent meanings for these functions. This has led researchers to start working on how students reason quantitatively and covariationally about trigonometric functions (Moore, 2010, 2012, 2014; Moore & LaForest, 2014). For students to develop coherent meanings for trigonometric functions, they must first develop meanings for angles, angle measure, and the radius as a unit of measurement. Moore (2009) investigated students' meanings for these concepts.

Teachers should strive to have their students build coherent mathematical meanings (Thompson, 2013). Simon (1995) shared three episodes from teaching that paint the picture of a teacher guided by his conceptual goals for his students' learning. A teacher's consideration of this learning goal, the learning activities, and the thinking and learning in which students might potentially engage in make up a hypothetical learning trajectory (HLT). The term refers to a teacher's prediction of the path by which learning may occur and characterizes expected tendencies of student learning. It is hypothetical in the sense that the actual learning trajectory of an individual is not knowable in advance. A teacher's HLT for her students has three parts: the teacher's goal for students' learning, the mathematical tasks used to promote student learning, and hypotheses about the process of the students' learning (Simon & Tzur, 2004). Simon and Tzur (2004) propose that having a teacher generate a HLT is a way for a teacher to teach based on her anticipation of how students might come to learn a particular concept, knowledge of what her students' current understandings are, tasks that she can use to promote learning of the concept, and her own understandings of the goal of the lesson. The generation of a HLT requires a teacher to think about what meanings she needed to know in order to build the proposed meanings. I hypothesize that the act of generating a

HLT serves as an impetus for getting a teacher to focus on mathematical meanings and to leverage student thinking when designing instructional interventions.

Methodology and Research Questions

The primary goal of this study was to explore how a teacher's mathematical meanings and instructional planning decisions change while participating in and then creating a hypothetical learning trajectory on angle, angle measure, and the radius as a unit of measurement. The study was conducted using a teaching experiment methodology (Steffe & Thompson, 2000). The subject is a graduate student in applied mathematics who was teaching *Pathways Precalculus* (Carlson, Oehrtman, & Moore, 2014) at the university level. I will refer to the subject as Lily. All sessions were videotaped and all written work produced was scanned and used for analysis. An initial clinical interview was conducted to build a model of the teacher's meanings for angle, angle measure, and the radius as a unit of measurement and to gain information about the teacher's instructional planning decisions for angle, angle measure, and the radius as a unit of measurement. The teacher then participated in two exploratory teaching sessions that were designed to resemble a hypothetical learning trajectory for a student's meanings for angles, angle measure, and the radius as a unit of measurement. During each session, I gave the teacher tasks that I designed to reveal and push the boundaries of the teacher's mathematical meanings. These tasks were designed before the initial interview and then modified to reflect the insights I gained from working with the teacher. The last part of the intervention was to have the teacher create a hypothetical learning trajectory for her students. The teacher was given a template for a HLT that was adapted from a Lesson Logic Form (Thompson, 2008). The HLT Lily created provided insight on how her meanings for angles and angle measure had changed as well as what meanings she wished her students to construct in class. The two research questions were "in what ways and to what extent does a teacher participating in and then generating a hypothetical learning trajectory on angles and angle measure affect the teacher's mathematical meanings for angles and angle measure?" and "in what ways and to what extent does a teacher participating in and then generating a hypothetical learning trajectory on angles and angle measure affect the teacher's instructional planning and decisions?"

Conceptual Analysis of Angles and Angle Measure

In order to produce a hypothetical learning trajectory for angles and angle measure, I needed to identify what meanings would comprise a propitious way of understanding of angles and angle measure. From these ways of understanding I identified six learning goals for students and then used prior research on students' meanings for angle measure (Moore, 2009) to design tasks that a teacher's use of would promote his/her students' construction of these desired understandings. Some of the tasks were adapted from the *Pathways Precalculus* curriculum (Carlson et al., 2014) as well as dissertation studies conducted by Moore (2010) and Tallman (2015).

An angle is a geometric object that consists of two rays that meet at a common endpoint, often called the vertex of the angle. A measurable attribute of an angle is its "openness." When a circle is centered at the vertex of the angle, one can quantify the measure of openness by measuring the length of the subtended arc in comparison to the length of either the radius or circumference of the circle, or, more generally, any unit of length that is proportional to the circle's radius or circumference. An angle can be measured by quantifying what percentage of the circle's circumference the subtended arc length is or by measuring the subtended arc length in units of the radius length. The six learning goals I identified for the researcher-generated HLT are students will understand:

1. …that an angle is an object that consists of two rays that share a common vertex.

2. …that the measurable attribute of interest of an angle is its “openness.”
3. …the “openness” of an angle in terms of the length of the subtended arc of the circle centered at the vertex of the angle.
4. …that any particular angle subtends the same fraction of the circumference of all circles centered at the vertex of the angle.
5. …that the unit of measure of this subtended arc length must be proportional to the circumference of the circle centered at the vertex of the angle so that the size of the circle does not matter.
6. …that angles measured in radians are measured by measuring the subtended arc length in units of the length of the radius of the circle centered at the vertex of the circle and that angles measured in degrees are measured by measuring the subtended arc length in units of $1/360$ of the circumference of the circle centered at the vertex of the angle.

With these learning goals in mind, I selected and/or designed seven tasks that could be used to promote the construction of these ways of understanding. In combination with the learning goals, these made up the researcher generated HLT that was used during the study.

Results

The initial clinical interview began with Lily creating a lesson plan for angles and angle measure. Lily’s lesson plan began with asking her students “What is an angle?” Lily’s answer to the question was that “an angle is an object that can be measured” and also that she “would love for them to relate it to a circle.” Next Lily planned to look at one picture and ask her students “How many angles can we measure in this picture?” Lily’s intended answer to this question revealed that Lily’s meaning for angle and angle measure was potentially different from the meanings I outlined in the researcher-generated HLT. When asked what she wanted her students to understand about angle measure, Lily drew a picture similar to the following picture (colors added to ease discussion of what she was referencing):

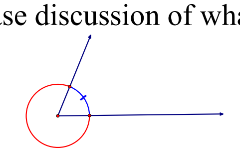

Figure 1: Lily’s initial image of two angles.

Initially Lily drew the part of the picture that is in blue, identifying that the blue arc and tick mark she had drawn indicated that students should be thinking about the interior space between the two rays. Then Lily added the red arc and said that she also wanted students to recognize that “this” was another angle. This revealed that Lily did not view the object of an angle as two rays that met at a common endpoint, but that some other aspect was also present in her scheme for angles. When asked how we measure an angle, Lily stated that we could measure an angle by comparing the subtended arc to the circumference or radius. Lily then wanted her students to imagine that every angle can be drawn inside a circle and then said that she would return to asking her students what an angle was. Finally, she would conclude her lesson by asking, “How can we measure the angle?” and brought up that she expected students to mention protractors, SOHCAHTOA, radians, and degrees. Then she would ask students what a radian was and what a degree was.

Following this, I proceeded to ask questions that I had designed to reveal more about Lily’s meanings for angles and angle measure. The first question I posed was “What is an angle?” Lily’s answer revealed that the word angle invoked a lot of meanings for her and that she had not made a distinction between an angle and an angle’s measure. Her mental image of an angle included rotations and a circle. She stated “we have this notion of going around a circle, which is where I naturally think about angles now.” She mentioned that something being 360 degrees was the same as something being 720 degrees, but did not mention what this “something” was. Lily was then presented with an image of an angle,

$\angle ABC$, and asked how many angles were pictured. Lily's answer was that there were infinitely many angles, depending on where you drew in an arc, or what you wanted to measure. This added evidence to the idea that Lily's meaning for angle consisted of more than just two rays that have a common endpoint.

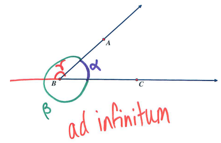

Figure 2: Lily's image of an infinite number of angles.

I then asked Lily what it meant to measure an angle. Lily stated, "we're looking at maybe a proportional relationship of what is cut off if we were to imagine the entire circle there. It's the relationship between this arc here and the entire circle." I then asked her to clarify what was proportional and she responded, "we're looking at the proportion of this arc to this radius." This revealed that Lily is able to think about angle measures as a ratio of the subtended arc length to the circumference and the ratio of the subtended arc length to the radius length, but that she was using these two ratios interchangeably. She was initially discussing the subtended arc length as a proportion of the circumference, but then drew a picture and defined an equation that found the proportion of the subtended arc to the radius. When asked what it means for two angles to have the same measure, Lily referred back to the ratio of subtended arc length to the length of the radius and stated that "two angles have the same measure if and only if s-one over r-one is equal to s-two over r-two where s-one and r-one are from angle one and s-two and r-two are from angle two." S-one and s-two stood for the subtended arc length and r-one and r-two stood for the radius length of each angle. When asked further questions about measuring an angle in degrees or in radians, her lack of distinction between the two ratios she had identified caused her problems when writing equations that described what it meant for an angle to have a measure of one degree or of one radian. In summary, the initial clinical interview revealed that Lily's definition on an angle was conflated with her process for measuring the angle. Lily did not make the distinction between the object of an angle and the measureable attribute of openness. Lily also had a strong conception of the measure of an angle being related to the portion of the circle subtended, though she used ratios of the subtended arc length to the radius and circumference interchangeably, and not always correctly. Identifying these meanings led to the researcher's modification of some of the tasks and questions to try and address what the researcher viewed as unproductive meanings that Lily had.

The next two sessions involved Lily working through 7 tasks with the researcher. I designed the first task to help Lily distinguish between an angle as an object and the measure of an angle as a quantity. I presented Lily with a Geometer's Sketchpad (Jackiw, 2011) file that had an angle pictured and asked her to describe the picture. She was then able to drag a point located on one of the rays of the angle, which changed the openness of the angle and traced out an arc of the circle the point was located on in red. Even though the full circle was not drawn, Lily imagined that she could think about continuing to trace out the subtended arc as a way to create a circle that would be related to the subtended arc, and described that she could measure the amount of openness between the two rays by creating a relationship between the portion of the circle the arc subtended. Her initial description involved creating a ratio between the subtended arc length and the circumference but then Lily described that we could measure an angle by relating the arc length to the radius length. Throughout the sessions, Lily identified two consistent ratios that can be used to measure an angle:

$\frac{\text{length of the subtended arc}}{\text{length of the circumference}}$ and $\frac{\text{length of the subtended arc}}{\text{length of the radius}}$. However, Lily did not distinguish between these two ratios, often citing one in an explanation, but actually using the other in her work.

In the second task, Lily used these ratios to again talk about how she could measure an angle. Lily was given that for a particular angle, the length of the subtended arc was 11.48 cm and the length of the circumference was 26.04 cm and asked if she would know what the length of the subtended arc would be if the circumference changed to 16.8 cm. She demonstrated fluency of using consistent ratios to find the missing subtended arc length. I then asked her if she was measuring the same angle in her picture. Lily's response was that no, she was not measuring the same angle but was instead measuring two angles that have the same measure. I took the opportunity to further probe Lily's definition of an angle (Excerpt 1). This showed a shift in Lily's definition of an angle from her initial clinical interview.

Excerpt 1

Interviewer: So why would there be different angles?
Lily: So again, we talked about that an angle is an object. So these are two different objects.
I: An object that consists of?
Lily: That consists of two rays meeting at a common point.
I: How many rays meeting at a common point have you drawn?
Lily: Well actually, I guess I'm thinking of line segments. If I were to think of it as rays, where rays go on forever, then they would have the same rays and so they would be the same.
I: So does changing the size of the circle you're looking at change the original object of the angle?
Lily: I'm going to go with no, because if you're thinking about rays, they go on forever.

I used the next two tasks to help Lily distinguish between the need for a unit of measure that would be used to measure an angle and a unit of measure that would be used to measure the subtended arc. Lily was asked to create a protractor that would measure an angle in gips, given that any circle is eight gips. Initially Lily talked about measuring the angle and measuring the subtended arc length interchangeably, but as we discussed what we were measuring, Lily identified that units of measure for those two things should not be the same since one was a length and the other was an amount of openness. I took the opportunity to ask Lily what the difference was between something that had a measure of one radian and something that had a measure of one radius length. Lily articulated that if we are measuring using the radius, we are measuring a subtended arc length. If we are measuring in radians, we are measuring an amount of openness. Throughout subsequent tasks, Lily still used the radians and radius lengths interchangeably, though when it was brought to her attention, she could identify which one she had actually meant. Lily stated, "a radian is an angle measure that corresponds to the number of radius lengths in the arc subtended by said angle."

During the last session, I presented Lily with a template for a HLT and asked her to plan a lesson for angles and angle measure for her students. Lily identified five learning goals:

1. An angle is formed when 2 rays meet at a common vertex.
2. How do we measure angles? (Determine the openness between the rays, use circles)
3. "openness" can be the larger or smaller value.
4. We measure angles commonly in units called radians. A radian is a unit equivalent to 1 radius length of the circle in question.
5. There are 2π radians in one circle.

Lily then identified that students would need to be familiar with circles, including the circumference formula and how it relates to radius length, prior to the lesson. Lily then started designing/selecting tasks that she could use to promote students' construction of the five learning goals she identified. As she went through this process, several of her learning goals evolved as she continued to think about them. She wanted her students to understand that when they are measuring an angle, they are describing the openness between the rays that form the angle and intended to do this by starting with an example of an angle with a measure of ninety degrees because her students would be familiar with this angle. Her goal was to have students make the connection that an angle with a measure of 90 degrees also cut off one-fourth of the circumference of a circle. This led to her changing her third learning goal to "understand that the subtended arc and circumference have a consistent relationship that can be used to measure angles," and the fourth goal became "a radian is a unit of measure often used for angles that is equivalent to one radius length on the subtended arc of the drawn circle." She later defined that "an angle that subtends an arc with a length of one radius (of the circle) is said to have a measure of one radian." Lily continued to select tasks that she would use in a lesson. She identified that she wanted to spend the first day of the unit focusing on the meaning of angles and angle measure and then spend a second day practicing these meanings in application problems.

Discussion and Implications for Future Research

Several changes occurred in Lily's mathematical meanings for angles and angle measure between the initial clinical interview and her generation of a HLT for her students. Initially, Lily's definition of an angle included more aspects than two rays that meet at a common endpoint. She believed that when she drew an arc between the two rays that the arc was part of the object of the angle, which meant that if she drew a circle centered at the vertex of the angle, she viewed this as creating two angles, where each subtended arc was part of a separate angle. This contributed to her unclear distinction between an angle as an object and an angle as a quantity. I hypothesize that this is because she viewed the subtended arc as part of the angle, and therefore the subtended arc was no longer a measurable attribute of an angle, but was instead part of the angle itself, thus making an angle a measureable attribute of another object, such as a triangle. Lily also used both radians and radius lengths when talking about measuring both an angle and an arc length. Lily was initially using these two units interchangeably. Lily also showed a tendency to talk about ratios whenever she was asked to explain the meaning or process of measuring an angle. At different times, Lily mentioned two consistent ratios when measuring an angle: $\frac{\text{subtended arc length}}{\text{radius length}}$ and $\frac{\text{subtended arc length}}{\text{circumference length}}$. Lily was aware that both of these ratios were consistent and on multiple occasions did not differentiate between which ratio she intended to use. Lily fluidly switched between saying that an angle will cut off the same portion of any circle's circumference and saying that the ratio of subtended arc length to radius length would be the same for any circle.

During the next three sessions, some of these initial meanings caused Lily perturbations while working through the tasks of the researcher generated HLT and when generating her own HLT. These perturbations caused Lily to make accommodations to her schemes for angle and angle measure. In the final session, Lily defined an angle as an object that is formed when two rays meet at a common vertex. This accommodation to her meaning for angles was a result of being confronted with tasks in which she was unable to assimilate the information in front of her to one of her already existing schemes. Specifically, this accommodation was a result of realizing that in order for you to be able to use a circle of any size to measure the same angle, you had to think of these different sized angles as still representing the same original angle.

By the end of the study Lily also had a clearer distinction between a radian as a unit of measurement and a radius length as a unit of measurement. While writing her HLT she used radian when she meant radius length once, but was able to identify that she had done so and made the change. Being asked to articulate what a gip measured in task 3 had required Lily to think about the difference between measuring a subtended arc and measuring an angle.

While working on creating her HLT, Lily was still inconsistent in her use of the two ratios she had identified as staying consistent for the same angle. The first time the researcher asked Lily if she realized she had been using two different ratios was during the final session while Lily was creating her HLT. Lily identified that they were two different ratios but that she hadn't really thought about that for herself. The opportunity for Lily to reflect on this distinction was lost because the researcher did not address this inconsistency in her use of the two ratios during the tasks of the hypothetical learning trajectory.

Together, the implications of the changes that occurred and the changes that did not occur provide evidence of the importance of the initial model the researcher created of Lily's mathematical meanings after the initial clinical interview. The researcher modified the questions asked during the tasks of the researcher-generated HLT to try to specifically address the meanings identified from the first interview. The activity of completing the tasks allowed Lily to make the necessary accommodations to her schemes for angle and angle measure. The only concept that is potentially still problematic for Lily is that she is inconsistent in her use of and meaning for the ratios of subtended arc length to radius length and to circumference length. This highlights the importance of identifying your student's meanings as a starting point for constructing a HLT. These results also highlight the effectiveness of the tasks in the researcher-generated HLT on providing the activity for a teacher to make accommodations to his/her schemes invoked by the tasks in the HLT.

These accommodations to Lily's scheme showed up in the second lesson plan she created. I hypothesize that Lily recognized the usefulness of the meanings she had constructed during the tasks of the previous two sessions and wanted to help her students construct these same useful meanings. Several of Lily's learning goals were meanings that she had either not had prior to the study, or had not been able to articulate. Lily's third learning goal is a reflection of a meaning that Lily had prior to the study. This shows that Lily also recognized the importance of her prior meanings, and did not only model her HLT after the accommodations she was aware of making. I include this to highlight that teachers do not start as a blank slate. Teachers are unable to help their students construct productive meanings if they do not have these meanings for themselves. Thus any sort of hypothesized intervention for improving teaching has to take this in to account. This study provides evidence that working through the tasks in a researcher-generated HLT and then creating your own HLT is one possible way to help teachers make accommodations to their schemes and then recognize the impact that these accommodations can have on their students.

Lily's reflection on the tasks she had completed and the accommodations she had made helped her identify different learning goals for her students. The new learning goals she identified make up a more robust understanding of angles and angle measure than what her initial lesson plan contained. Lily's second lesson plan also included specific activities and tasks that she intended to do with her students and conversations she hoped to have with her students, both of which were barely contained in the first lesson plan. This suggests that providing Lily with a template that specifically asked her to identify learning goals and tasks that would help students construct those understandings helped contribute to a more detailed and robust lesson plan.

The results of this study suggest that having Lily work through tasks in a researcher generated HLT caused changes in both her schemes for angles and angle measure as well as what she identified as being important to teach her students. The study also suggested that the use of a HLT provided a way to encourage a teacher to think about student thinking as she planned her lesson. A future study involving these ideas will allow the researcher to identify what aspects of participating in the HLT caused this effect.

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DOES IT CONVERGE? A LOOK AT SECOND SEMESTER CALCULUS STUDENTS' STRUGGLES DETERMINING CONVERGENCE OF SERIES

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Despite the multitude of research that exists on student difficulty in first semester calculus courses, little is known about student difficulty determining convergence of sequences and series in second semester calculus courses. In our preliminary report, we attempt to address this gap specifically by analyzing student work from an exam question that asks students to determine the convergence of a series and follow-up semi-structured interviews. We develop a framework that can be used to help analyze the mistakes students make when determining the convergence of series. In addition, we analyze how student errors relate to prerequisites they are expected to have entering the course, and how these errors are unique to knowledge about series.

Key words: Series, Framework, Convergence, Calculus, Undergraduate Mathematics

Researchers have noted that there is a lack of research in the area of infinite series (González-Martín, Nardi, & Biza, 2011). Moreover, the research that does exist does not focus on undergraduates in second semester calculus, but rather on undergraduates in real analysis (González-Martín, Nardi, & Biza, 2011; Alcock & Simpson, 2004; Alcock & Simpson, 2005), how graduate students understand series (Martínez-Planell, Gonzalez, DiCristina, & Acevedo, 2012), and humanities students' difficulty with the concept of infinity when dealing with series (Sierpińska, 1987).

In this study, we begin to fill a gap in the literature by developing a framework that can be used to analyze student errors that occur while solving problems in second semester calculus courses related to sequences and series. Moreover, since researchers have argued that first semester calculus students struggle because they lack the necessary prerequisite skills such as the function concept (Ferrini-Mundy & Graham, 1991; Carlson, Madison & West, 2010; Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997), we also look to see how the errors students make are related to prerequisite skills they should have acquired prior to entering their second semester calculus courses. In particular, we aim to (1) determine the errors students make when solving typical second semester calculus problems on series, (2) determine the relationship these errors have to prerequisite skills, (3) determine how the errors made are unique to series, and (4) develop a framework for analyzing student errors.

Research methodology

The targeted population for this study is undergraduate students enrolled in a second semester calculus course in a large public university in the northeastern United States. Fifty-five students in the course agreed to have their work on a sequences and series exam photographed. Thirty-four of these students also agreed to be interviewed working through problems on sequences and series similar to those seen on their exam, though only eight students responded to an e-mail to set up the interview with seven showing up for their interview.

Recall that the main research aims in this study restated from the introduction are to (1) determine what mistakes students make when they solve problems on series typically seen in a second semester calculus course, (2) determine how these mistakes relate to prerequisite skills students are expected to have prior to entering a second semester calculus course, (3) determine what knowledge of series aside from prerequisite knowledge students need to avoid the mistakes seen in (1), and (4) develop a framework for analyzing student mistakes determining the convergence of sequences and series.

This preliminary report focuses on student responses to one question on their exam, a problem that focused on student knowledge of comparison tests (or integral test) to determine the convergence or divergence of a series:

Determine whether the following series converges or diverges. Be explicit about any test you use to justify your response. Calculate the sum of any convergent geometric series. Justify your response by showing your work.

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2}$$

To address the aims in our study, we went through two rounds of coding. Since we could not find any theoretical work in this area, we opted for open coding. In the first round, we wrote a description of the type of error we saw. In the second round of coding, we came up with categories to fit our descriptions into. The categories, abbreviations, and an explanation of the categories are given below in table 1:

Table 1: Categories, Abbreviations, and Explanation Table

Categories, Abbreviations, and a Brief Explanation with an Example

Category	Abbreviation	Explanation
No Mistakes	NM	A completely correct answer
Notational Error	NE	A notational error. For example, a student says $\frac{1}{n}$ diverges without including the series symbol.
Algebra of Series	AS	Student splits up a series when one diverges. For example, he may write $\sum_{n=1}^{\infty} \frac{1}{n} + \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n} + \sum_{n=1}^{\infty} \frac{1}{n^2}$
Algebra	A	An algebraic error. A student might, for example, "plug in" infinity, or incorrectly simplify a rational expression by "cancelling" through a sum
Function Choice	FC	Wrong function choice when using a comparison test. For example, a student might try to make a comparison with $\frac{1}{n^2}$.
Unchecked Assumptions	UA	Student failed to check that the function satisfied the assumptions in the integral test.
Algebra error leading to Incorrect Test Choice	AITC	Student reaches a false conclusion (usually in the ratio test) because of an algebraic mistake. This mistake typically was cancelling through a sum.
Incorrect Test Choice	ITC	Student chooses an incorrect test, such as an nth term test, or a geometric test.

(continued)

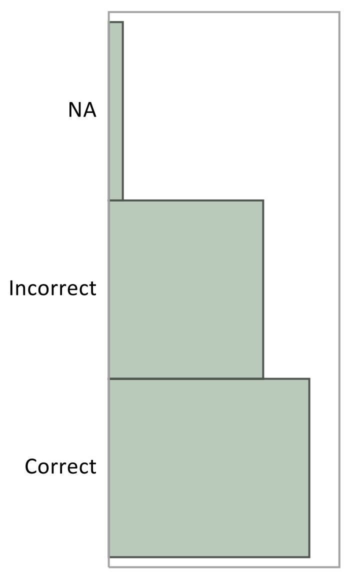
Table 1: Categories, Abbreviations, and Explanation Table (continued)

Category	Abbreviation	Explanation
Wrong Conclusion in Test #1	WCT1	Student uses a test other than the integral test or a comparison test, and reaches an incorrect conclusion from that test. For example, a student uses the ratio test and says that a value of 1 means the series converges.
Wrong Conclusion in Test #2	WCT2	The student correctly chooses a comparison test or the integral test, but reaches an incorrect conclusion using that test. For example, a student says the series converges because it is larger than the series $\frac{1}{n}$.

Preliminary results and discussion questions

In what follows we present the preliminary results of student responses to the question on the exam stated above. Figure 1 below shows that most students, about 54.5%, answered the question correctly. By a correct answer, we mean an answer that would have received full credit or only lost a point or two on the examination in the judgement of the authors of this paper, both of whom have experience teaching second semester calculus.

Figure 1: Correct/Incorrect

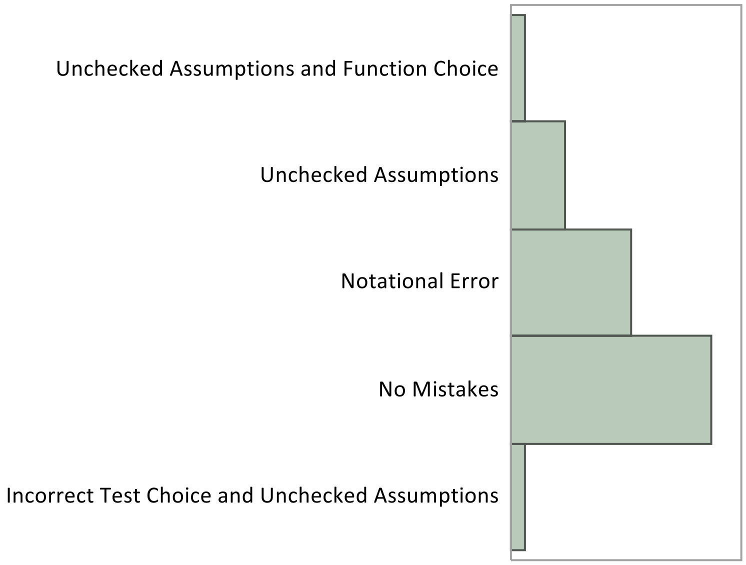


Frequencies

Level	Count	Prob
Correct	30	0.54545
Incorrect	23	0.41818
NA	2	0.03636
Total	55	1.00000

Half of the students that got the problem correct made no errors whatsoever, and another 30% only made notational errors. Figure 2 shows the types of errors made by students that answered the question correctly. Note that multiple errors were possible on the same problem.

Figure 2: Distributions of Correct Responses

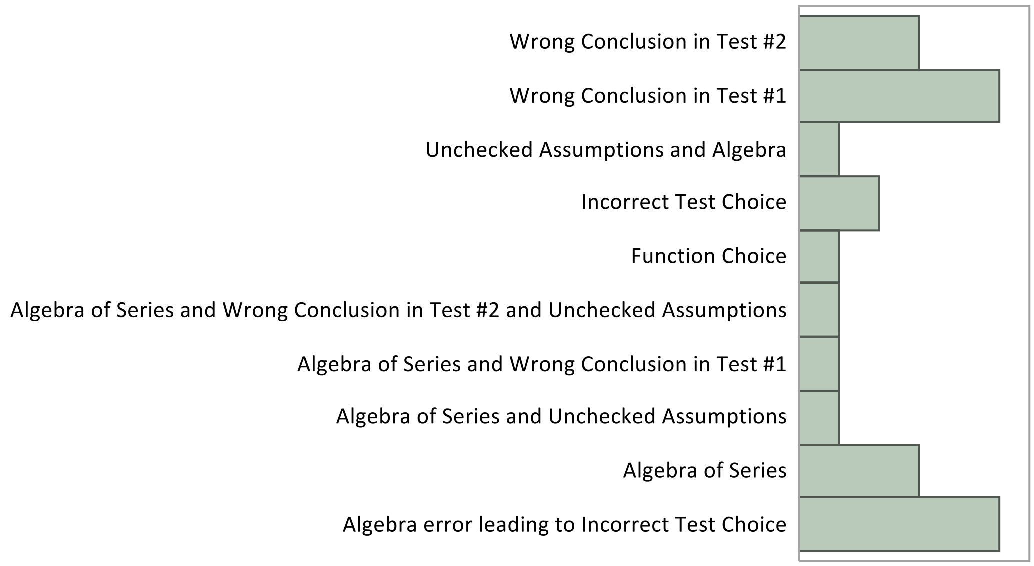


Frequencies

Level	Count	Prob
Incorrect Test Choice and Unchecked Assumptions	1	0.03333
No Mistakes	15	0.50000
Notational Error	9	0.30000
Unchecked Assumptions	4	0.13333
Unchecked Assumptions and Function Choice	1	0.03333
Total	30	1.00000

Finally, figure 3 shows the types of errors made by students that answered the problem incorrectly.

Figure 3: Distributions of Incorrect Responses



Frequencies

Level	Count	Prob
Algebra error leading to Incorrect Test Choice	5	0.21739
Algebra of Series	3	0.13043
Algebra of Series and Unchecked Assumptions	1	0.04348
Algebra of Series and Wrong Conclusion in Test #1	1	0.04348
Algebra of Series, Wrong Conclusion in Test #2, Unchecked Assumptions	1	0.04348
Function Choice	1	0.04348
Incorrect Test Choice	2	0.08696
Unchecked Assumptions and Algebra	1	0.04348
Wrong Conclusion in Test #1	5	0.21739
Wrong Conclusion in Test #2	3	0.13043
Total	23	1.00000

The preliminary data in the three figures above indicate algebraic manipulation as a prerequisite skill that causes student mistakes. While none of the students that answered the question correctly made an algebra mistake, 12 of the 23 students that answered the question incorrectly made some kind of algebra mistake.

Students also appeared to have difficulties that are somewhat unrelated to prerequisite knowledge. Nine students of the 53 failed to check the assumptions in the test they were using. For instance, they often did not check the continuity or the monotonicity of the function when using the integral test. Seven students chose the wrong test to use in this problem, and another 10 students made a mistake regarding the conclusion of their selected test.

Moving forward, we plan to continue our analysis of the other problems on the exam as well as analyze interview transcripts to get a better idea on why students might be making some of these errors. Prior to further analysis, we would like to use our presentation to receive feedback on the following questions:

(1) Which of the categories we have used might be unique to this particular problem and not appear when we look at other traditional series problems?

(2) What categories might we need to add to encompass mistakes that we might see in other problems that we did not see here, particularly for problems related to sequences?
(3) What other methodological suggestions might you offer us to examine the data further?

Implications for further research

Once we have a better understanding of the types of errors students are making and why these errors are being made, we can begin investigating teaching strategies to help students avoid these errors. In addition, by finding the most common prerequisite mistakes, we can investigate the curriculum and teaching of prior mathematics courses and help students be better prepared when entering second semester calculus courses. Finally, we can continue studying student errors and improving upon our framework.

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Learning to think, talk, and act like an instructor: A framework for novice tertiary instructor teaching preparation programs

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In this report I present a framework to characterize novice tertiary instructor teaching preparation programs. This framework was developed through case study analyses of four graduate student teaching assistant professional development (GTA PD) programs at institutions identified as having more successful calculus programs compared to other institutions. The components of the framework are the structure of the program, the departmental and institutional culture and context that the program is situated within, and the types of knowledge and practices emphasized in the program. In this report I characterize one of the programs involved in the development of the framework as an example of how it is used. In addition to characterizing existing programs, this framework can be used to evaluate programs and aid in the development of new novice tertiary instructor teaching preparation programs.

Keywords: Graduate student teaching assistant, professional development, pedagogies of practice, mathematical knowledge for teaching, framework

Theoretically driven research centered on teaching preparation of graduate students (and other novice tertiary mathematics instructors) pales in comparison to the literature related to professional development of K-12 mathematics teachers. While there are aspects of K-12 professional development (PD) programs that can be highly relevant and informative to the tertiary level, there are also many ways in which tertiary level teaching preparation should be examined as its own field. In this report, I introduce a theoretical framework that draws on K-12 PD literature and responds to the particular needs at the tertiary level, and use this framework to characterize one graduate student teaching preparation program as an example of its use.

The National Science Board (NSB) uses the term professional development to refer both to teacher preparations (i.e. the teaching of pre-service teachers, prospective teachers, and teacher candidates) and to the development of practicing teachers (i.e. in-service teachers and practicing teachers) (National Science Board, 2012). Novice tertiary instructors, especially graduate students, have commonalities with both categories of teachers: the training they receive for these roles is typically the first training to teach they will have received, however often they receive a large portion of this training while they are teaching. For many practicing tertiary instructors, any professional development related to teaching they may have received as graduate students or post-docs is likely to be their only formal training as mathematics *educators*, rather than as mathematics *researchers*, and can help enculturate graduate students into academia (Austin, 2002). Thus, the literature on professional development programs designed both for pre-service and in-service teachers at the K-12 level is relevant to tertiary teaching preparation. While there is extensive research into the professional development of teachers at the K-12 level, there is substantially less literature focusing on tertiary instructor teaching preparation, especially that is theoretically driven. A large portion of the studies focused on tertiary instructor teaching preparation report on the success of existing programs or needs (often unmet) of novice instructors (e.g. Hauk et al., 2009; Kung & Speer, 2009; Speer, Gutmann, & Murphy, 2005).

However, the body of research that connects aspects of these programs to identify commonalities and key features to consider when creating a new program is lacking.

Ten years ago, Speer and her colleagues initiated the conversation among mathematics education researchers interested in novice tertiary instructor teaching preparation, calling attention to what we could learn from K-12 PD, and identified a number of research directions to pursue (Speer, Gutmann, & Murphy, 2005). Many of these directions have been pursued directly by Speer and others since this call, and as a result there are more productive models of novice tertiary instructor teaching preparation programs in existence. In this paper, I develop a theoretically driven model that connects such productive programs. This framework may be used to better understand (and make improvements to) existing programs as well as to influence the development of a new program geared at preparing GTAs and other novice tertiary instructors.

Methods

As part of a large, national study focused on identifying elements present in successful calculus programs *Characteristics of Successful Programs in College Calculus (CSPCC)* (MAA, 2013), I studied the graduate student teaching preparation programs at four institutions with successful calculus programs where graduate students and post docs were involved in the teaching of calculus. Through analyses of survey data, the project team identified institutions that were more successful than comparable institutions, where success was viewed as a combination of retaining students' positive dispositions towards mathematics, retaining students' intentions to take Calculus II, and having a reasonable pass rate. We then conducted case studies (Stake, 1995) at these institutions to learn what they were doing in calculus that may be contributing to students' success, and how this success could be translated to other institutions. Robust novice instructor teaching preparation programs were one such element, and were then studied in depth in the national sample and at the case study institutions.

As part of the MAA study, an abundance of data was collected surrounding four PhD-granting institution's GTA PD programs. This included the collection of all documents related to the GTA PD, observations of the training when possible, observations of instructor meetings, observations of graduate students teaching and leading recitation section, and interviews with graduate students, administrators, PD facilitators, and students.

I drew on qualitative research strategies (e.g, Braun & Clarke, 2006; Miles & Huberman, 1994; Stake, 1995, 2005; Yin, 2003) and employed three specific techniques for analyzing this data: pattern matching, explanation building, and cross-case syntheses. Through pattern matching I developed systematic groupings of data using inductive thematic analysis (Braun & Clarke, 2006). Inductive thematic analysis is a bottom-up approach, where the themes are data-driven, though are not developed in an "epistemological vacuum" (p. 84). Through these analytic techniques I developed the framework for novice tertiary instructor teaching preparation programs, described below. While I attended to the ways in which these institutions prepared graduate students in their roles as instructors, these programs can be informative for preparing other novice tertiary instructors, such as post-docs, lecturers, and new tenure-track faculty.

Components of framework

The central component of this framework is the *structure* of the teaching preparation program; when it occurs, for how long, who participates, what is discussed, and how. Within this structural design, different aspects of knowledge are emphasized and to varying degrees, and participants engage in different practices to gain this knowledge and to varying levels of authenticity. This structure, with the various types of knowledge emphasized through different practices, is like the structure of a house. The design of any house is influenced and constrained by the environment (the square footage available, the zoning laws, the terrain of the land, etc.) and the designer(s) (the architect and possibly the new owners). Similarly, the structure of a teaching preparation program is influenced and constrained by the environment within which it is situated: the institution and the department.

The structure of the program is constrained, determined, and enabled by the surrounding environment. The *institutional and departmental context* and *culture* together comprise the environment within which the teaching preparation program exists. The institutional and departmental context guides the needs and capabilities of a teaching preparation program. For instance, the responsibilities of novice instructors are determined by (a) the number of graduate students, post-docs, and other novice instructors in the department in relation to the number of other faculty and in relation to the number of undergraduates served by the department, (b) the types of classrooms available (large lecture halls versus small classrooms), and other components of the context of the institution and department. The institutional and departmental culture shapes how the department responds to these needs and capabilities. For instance, whether graduate students serve as recitation leaders or course instructors will be shaped by (a) the institution and departments' views on class size, (b) their orientation toward optimal learning environments, (c) their aspirations for undergraduate instruction, and other components of the culture of the institution and department.

Within the structure of the program, different *knowledge* and *practices* are emphasized and in different ways. As part of developing as an instructor, one develops knowledge and practices surrounding instruction. Thus, the tertiary teaching preparation programs emphasize different types of knowledge and practices depending on the community and needs within than institution.

One way to characterize the types of knowledge needed to teach is the classic distinction by Shulman (1986), who differentiated between *pedagogical knowledge (PK), content knowledge (CK),* and *pedagogical content knowledge (PCK).* Pedagogical content knowledge is distinct from a blend of basic pedagogical knowledge and basic content knowledge and was introduced by Shulman in response to the wide-held belief that content knowledge alone was sufficient to teach. PCK is the particular form of content knowledge related to the aspects of content knowledge "most germane to its teachability", including ways of representing content so that it is understandable to others (Schulman, 1986, p. 9).

To characterize the practices graduate students can legitimately and peripherally engage in as they learn how to be tertiary instructors, I draw on Grossman et al.s' (2009) *pedagogies of practice.* Grossman and her colleagues (2009) identified three concepts for describing ways to teach practices in professional education: *representations of practice, decompositions of practice,* and *approximations of practice.* Representations of practice comprise different ways practice can be represented for novices. In teacher education, one may represent the practices of teaching through written case studies, Videocases, photographs of the classroom, narratives, lesson plans, technological reproductions, among many others. The authors note that "the nature of the representation determines to a large extent the visibility of certain facets of practice" (p. 2066) and thus different representations of the same practices have different affordances for the learner. Decompositions of practice break down a complex practice into its multiple parts, which has affordances as well as limitations. By decomposing a practice, it may remove the practice from the actual context within which it is situated (for an elaboration on this point see Putnam &

Borko, 2000) however it also enables the novice to focus on specific aspects of a practice without the complications of the actual context. Approximations of practice are activities that allow novices to engage in legitimate practices of a community in a peripheral way, meaning that they are "more or less proximal to the practices of a profession." These approximations may take the learner directly to the practice, as is done during student-teaching, or bring the practice to the learner through various representations, such as video or role-playing.

Teaching preparation programs provide many examples of representations, decompositions, and approximations of the practices of teaching with varying levels of authenticity. For instance, by watching Videocases, novice teachers are able to "enter" the classroom, observe student behavior and imagine how they would react as the teacher, without the actual responsibility of being in the classroom. This approximation of teaching has a low level of authenticity because real teachers do not have the opportunity to pause or rewind classroom activity in order to decide how to react or how to interpret the situation. Practice teaching is an example of an approximation of teaching with much higher authenticity. During practice teaching, novice teachers have limited responsibility in the classroom, but are able to experience it in real time and in a much more authentic way than by watching a video. Grossman and her colleagues (2009) highlight the benefits of representations, decompositions, and approximations of practice with varying levels of authenticity, which "quiet the background noise so that they can tune in to one facet of practice at a time" (p. 2083). As novices participate in the practices of a community (through approximations of practice, representations of practice, and/or decompositions of practice) they do not just develop the skills of the community, but also develop (to varying degrees) a shared knowledge base and shared dispositions. Figure 1 illustrates the relationships between them, and provides a visualize representation of the framework for novice tertiary teaching preparation.

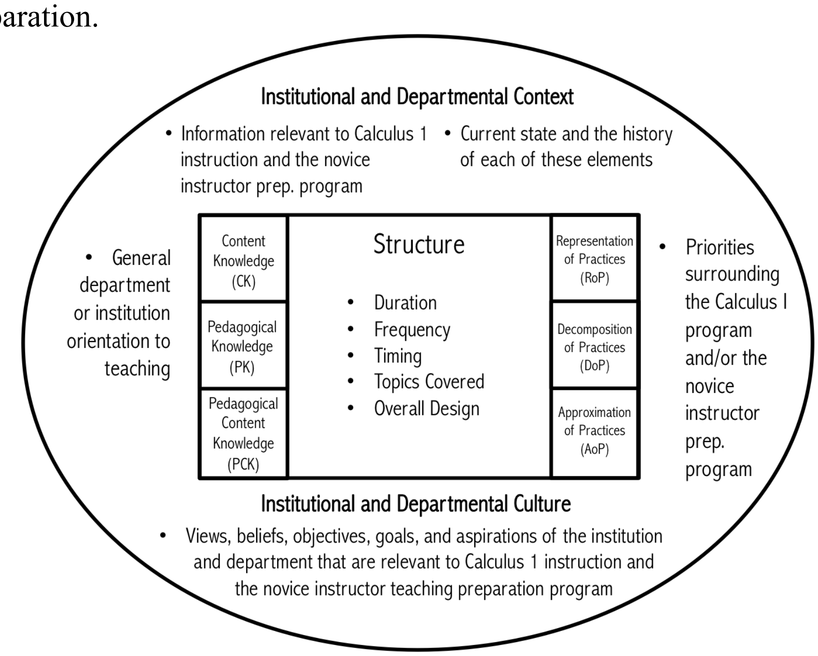

Figure 1 Framework of instructor teaching preparation programs

Different tertiary teaching preparation programs necessarily focus on different types of knowledge depending on their goals and guiding philosophies, as well as depending on the department's needs and the needs of the novice instructors. For instance, if novice instructors typically come into their role as instructors at a specific institution with extensive teaching experience but are less confident in their mathematical knowledge, a tertiary teaching preparation programs may emphasize content knowledge related to teaching more than pedagogical knowledge. If, instead, novice instructors typically come into their role at a specific institution with very strong mathematical content knowledge but little to no experience interacting with students, than tertiary teaching preparation programs may emphasize pedagogical knowledge and pedagogical content knowledge, but not content knowledge.

Within the structure of tertiary teaching preparation programs, different types of knowledge is emphasized to different degrees of depth and novices engage in different pedagogies of practices to varying degrees of authenticity. These varying degrees of depth and authenticity are represented in the framework by darker or lighter shading of the six smaller boxes in Figure 1, where darker represents knowledge emphasized more deeply or more authentic pedagogies of practice. These emphases are guided and constrained by the institutional and departmental environment that the program is situated within, and help to provide more information about the structure of the program. The level of shading was determined through the case study analysis.

An Example

Here I use the framework to visually represent one model of novice instructor teaching preparation programs, called the Apprenticeship Model. The Apprenticeship Model of novice instructor teaching preparation was enacted at a small university with around 5,000 undergraduate students, where fall enrollment in Calculus 1 is around 270 and class sizes are around 45. Graduate students, both Master's and Doctoral students, are involved in the teaching of Calculus I as teaching assistants, tutors, and course instructors. Post-docs are not involved in the teaching of Calculus I at this university.

The primary guiding philosophy behind the Apprenticeship model is the desire to transition graduate students into the role of instructor, both as part of their immediate role as GTAs and as their (potential) future role as undergraduate mathematics instructors. Embedded within this philosophy is the belief that people learning a new profession (who will develop a professional identity surrounding it) must participate in the practices of the profession with growing responsibility. This belief is in line with a perspective in which learning is viewed as the process of engaging a novice in the practices of the profession with legitimate but peripheral participation (Lave & Wenger, 1991). The term "peripheral" indicates that the practices novices are involved in are less central versions of the authentic practices, or are central practices with limited responsibility. As one clinical psychology professor involved in the Grossman et al. (2009) study said when describing how clinical psychologists are prepared, "if you're learning to paddle, you wouldn't practice kayaking down the rapids. You would paddle on a smooth lake to learn your strokes" (p. 2026). The main components of the Apprenticeship model are:

- A three-unit class, inspired by Lesson Study (Lewis, 2004), that takes place during the semester before the graduate student is placed as a course instructor.
- A mentor instructor for whom the mentee acts as a teaching assistant in the class they will be teaching during the semester before the graduate student is placed as a course instructor.
- Weekly course meetings once the graduate student is placed as a course instructor.
- Observations and feedback once the graduate student is placed as a course instructor.

Graduate students are required to participate in a number of teaching development activities, both prior to teaching and while they teach. All new GTAs must attend a one-day seminar led by the mathematics department, with some of this time spent doing practice teaching presentations. During the seminar faculty conduct workshops on topics including pedagogical basics, such as how to write well on the board, as well as more advanced pedagogical topics, such as how to implement cooperative learning. Additionally, all first-year GTAs are assigned a faculty mentor during the orientation session.

As shown in Figure 2, the framework representation of the Apprenticeship Model gives a clear overview of the structure and encompassing environment of the novice instructor teaching preparation program.

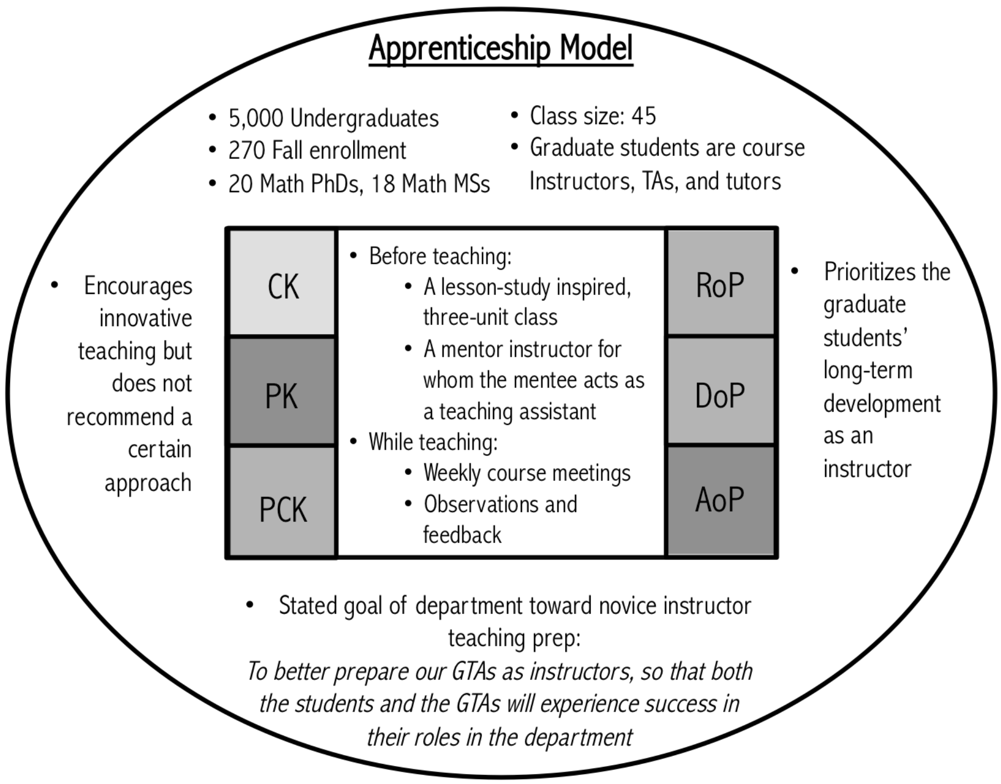

Figure 2 Apprenticeship model

The main structural components of the program are a lesson-study inspired course and mentoring that occur before the GGTA is placed as an instructor, and ongoing meetings and observations once the GTA is placed as an instructor. The shading provides a visual representation for the level of emphasis of the knowledge and the level of authenticity of the practices involved in the programs. Within this structure, pedagogical knowledge is emphasized more deeply than PCK or content knowledge, though PCK is emphasized through both the lesson-study inspired course and the mentoring. Content knowledge is potentially emphasized through the mentoring, although it is not a primary focus. During the lesson-study course, novice instructors participate in a number of pedagogies of practice to varying degrees of authenticity. Through the lesson-study-like iterations of developing, presenting, and refining lessons, graduate students engage in approximations of the practice of teaching to increasing degrees of authenticity. The practice of teaching is decomposed into planning, presenting, and refining through the lesson-study course with medium level of authenticity. Through both the lesson-study course and the mentoring, graduate students have multiple opportunities for teaching to be represented, by other graduate students, their mentor instructor, and by reading and watching cases. This program is situated within a small department that prioritized graduate students' long-term development as instructors and encourages innovative teaching but does not require a certain pedagogical approach.

Conclusion

While the framework representation does not give the rich detail of the program on its own, it provides information useful in comparing across models, and can be used to ask and answer questions regarding the evaluation or implementation of an individual model. In the presentation of this report, I will use the framework to compare two novice tertiary programs to highlight this affordance. The framework can also be used to evaluate a program or to help with the creation or improvement of a teaching preparation program. To aid in the evaluation of a program, a mathematics department may determine that their GTAs and post docs seem to know very little about how their students may think about mathematics, their difficulties, and how to explain problems so that they will better understand them. They could use this framework to describe their current program and identify that they are not, in fact, spending time during the teaching preparation discussing PCK. To aid in the development of a program, this framework can help direct attention to important components to consider. In many mathematics departments, a more robust novice teacher preparation program is developed based on the initiative of one or two motivated individuals – the change agents. Often, these change agents are not necessarily mathematics education experts, or have good ideas about what the novice instructors need at their institution but do not know how to go about setting up a new program. The framework introduced in this report provides an organized and systematic way to think about the components of a teaching preparation program.

Many institutions are currently seeking to make improvements to their GTA training programs – in fact, in a recent survey through the *Progress through Calculus (PtC)* project (an extension of the CSPCC project), the MAA has determined that 68 graduate degree granting mathematics departments are either currently implementing changes to their GTA PD program or are discussing changes for the future. In order to implement these changes, change agents at these institutions often draw from their own experiences as graduate students or knowledge of other programs to adapt to their institutions. One additional use for this framework would be to characterize a large number of programs and provide the visual representations to institutions looking to implement changes to their program. These condensed visual representations would enable the change agents to consider many different programs and compare specific aspects across the programs easily. Through the PtC project, we have collected data from 135 graduate degree granting mathematics departments regarding their GTA PD programs. A future stage of this work will be to use the framework discussed in this paper to characterize these programs to begin to create a visual library of novice tertiary instructor teaching preparation programs that can be then adapted by institutions for GTA PD or other novice teacher preparation.

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Gender, switching, and student perceptions of Calculus I

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We analyze survey data to explore how students' reported perceptions of their Calculus I experiences relate to their gender and persistence in calculus. We draw from student free-responses from several universities involved in a comprehensive US national study of Calculus I We perform a thematic analysis on the data, identifying numeric patterns via Dedoose, a mixed methods program, and inspecting student responses within identified themes. Our analyses indicate that female students report negative affect and a desire for authentic learning more often than males. Student preparation also plays a role in changes in confidence. We discuss potential factors that influence student persistence in calculus.

Keywords: Calculus, gender, persistence, affect, thematic analysis, mixed methods

Stemming from national need to increase persistence in Science, Technology, Engineering, and Mathematics (STEM), Ellis, Fosdick, and Rasmussen conducted a study focused on student persistence in calculus and investigated factors which may impact the likelihood of a student switching out of a STEM major (2015). They identified a striking relationship between gender, switching, and mathematical confidence. Specifically, females were significantly more likely to decrease their intentions to take Calculus II after taking Calculus I. When given a list of potential reasons for not continuing, female students cited that they, "do not believe [they] understand the ideas of Calculus I well enough to take Calculus II," with significantly greater frequency than their male counterparts. These results highlight the role that calculus is playing in students' decisions to leave STEM pursuits, and may help to explain the larger issue of the STEM Gender Gap (Eagan, Lozano, Hurtado, & Case, 2013; Seymour & Hewitt, 1997). This work motivated us to delve more deeply into student reports of their experiences in Calculus I. Specifically, we examine the relationships between students' description of their experience in Calculus I, their gender, and their decisions to persist in calculus.

Educators have long been interested in identifying factors that may contribute to the disparity in gender representation in STEM (Fennema & Sherman, 1976 & 1978; Griffith, 2010; Good, Rattan, & Dweck, 2012; Ellis, Fosdick, & Rasmussen, 2015). While there is consistent evidence against gender-based differences in mathematical ability (Fennema & Sherman, 1978; Islam, & Al-Ghassani, 2015; Lindberg, Hyde, & Peterson, 2010), there are clear distinctions between men and women in their persistence in STEM fields (Cunnigham, Hover, & Sparks, 2015; Eagan et al., 2013), and their self-reports of success in these fields (Griffith, 2010; Good, Rattan, & Dweck, 2012).

Researchers have begun to articulate factors related to persistence and the representation of females and other minorities in STEM majors (Ellis, Fosdick, & Rasmussen, 2015; Fennema & Sherman, 1976; Graham, Frederick, Byars-Winston, Hunter, & Handelsman, 2013; Griffith, 2010; Wolniak, Mayhew, & Engberg, 2012). Griffith found that certain environmental factors (such as the representation of females and minorities in graduate programs) can increase STEM participation and success by minorities (2010). Good, Rattan, and Dweck (2012) found that a sense of belonging was related to student persistence in math and that women who exhibited a fixed intelligence mindset coupled with gender stereotyping in the classroom experienced reduced sense of belonging (2012). Gender differences in confidence have also been identified as a possible factor to explain why women discontinue pursuing mathematics at a higher rate than men (Ellis, Fosdick, & Rasmussen, 2015; Fennema & Sherman, 1978).

Our research contributes to this literature by offering an inductive, qualitative analysis of student statements pertaining to their experiences in Calculus I. We draw on students' responses to an open-ended survey question from the *Characteristics of Successful Programs in College Calculus (CSPCC)* project.

In this report we address the following research question: *How do student characterizations of their experience in Calculus I relate to student gender and persistence in calculus?*

Methods

This work is embedded within a larger project aimed at investigating college calculus at a national level – the CSPCC project. The first phase of this work involved a survey of "mainstream" Calculus I students from a stratified random sample of colleges and universities. Two surveys were sent to students at the beginning and the end of the fall term. On the beginning-of-term survey, students were asked questions related to their demographics, previous mathematical experiences, affect towards mathematics, and career plans. On the end-of-term survey, students were asked questions related to their experience in Calculus I, affect towards mathematics, and career plans. At the end of the end-of-term survey, students were asked the open-ended question: "Is there anything else you want to tell us about your experience in Calculus I?" We analyze students' responses to this question in this report. The surveys provide us with information to distinguish students based on gender and whether they continued in calculus. Students who began Calculus I intending to take Calculus II and persisted in these intentions are referred to as *Persisters,* while students who began Calculus I intending to take Calculus II and switched these intentions are referred to as *Switchers.*

There were 522 students who provided a response to the open-ended question, reported their gender in the beginning of the term survey, and were coded as a Switcher or a Persister. To characterize the ways these students discussed their experiences in Calculus I, and to relate these characterizations to student gender and persistence, we employed thematic analysis (Braun & Clarke, 2006).

In this analysis we first familiarized ourselves with the student responses, blind to the gender and persistence of the students, though aware of the literature related to the STEM gender gap and, more specifically, aware of the relationship in this data set between gender, reported mathematical confidence, and persistence in calculus. We took an inductive approach, deriving themes from the data, but we brought to bear our knowledge of the literature in organizing these themes. The two authors each coded subsets of 50 student responses to develop and refine codes. The final codes, reported in Table 1, were finalized after multiple iterations of comparing codes and once 85% reliability was consistently achieved between researchers. One researcher then coded all responses, with a small percentage of questionable responses coded by both researchers. We weighted the codes on a scale of -1 to 1 to indicate a negative, neutral, or positive connotation. For each student response, we coded each sentence with as many codes as appropriate. The NA code was only used if the entire student response was irrelevant.

To frame this work we draw on literature surrounding affect. We define and understand affect according to Phillip's summary of research done on mathematical belief and affect from the years 1992 to 2007. By consolidating definitions from research, Phillip defines affect as "a disposition or tendency or an emotion or feeling attached to an idea or object. Affect is comprised of emotions, attitudes, and beliefs" (Phillip, 2007, p. 259). In our examination of students' open-ended responses about Calculus I, we analyze students' reported affect towards Calculus I. Nearly all students' responses could be viewed as affective statements. Thus, we narrowed our use of the "Affect" code to only capture statements about a student's emotions, attitudes, or beliefs towards the calculus course, oneself as a learner, or mathematics in general. For instance, "*This professor is pretty good at explaining the concepts*," is an example of a response that was coded as being about the teacher but not as a report of the student's affect. By contrast, "*I feel that I am loving math because my professor loves to teach it. She makes class so much fun and she believes in us*," is an example of a response that was coded with both the "Teacher" and "Affect" codes.

Table 1. Codes, code descriptions, and examples.

Code	Includes statements about…
Affect	Student's emotions, attitudes, and beliefs about (a) the calculus course, (b) mathematics, (c) themselves as learners. *For the first time in my life I really struggled in a math class.*
Assignments and assessments	Assignments, and both formative and summative assessments. *The exams tended to be stressful/time consuming.*
Pacing	The pacing of the course in general and of class sessions. *The length of class … didn't really allow for … anything rather than the 'spewing' of material*
Preparation	Preparation coming into the course and preparation going into the next course. *Taking calculus in high school helped me succeed in this class!*
TA	The TA and his/her aspects such as communication, availability, helpfulness, etc. *The help desk hours with the T.A. were great.*
Teacher	The teacher and his/her aspects such as communication, availability, helpfulness, etc. *I thoroughly enjoyed my professors teaching style and presentation of material.*
Teaching	Specific teaching practices and teacher-controlled aspects of class room environment. *My instructor …had no passion for learning and lectured instead of taught.*
Other	Other reasons and resources that may have impacted the student's success *Without the math tutoring lab there is no way I would do well in this class or even pass.*
Not applicable	Anything irrelevant to the calculus course. *If a teacher truly loves the subject, students can tell and learn to love it too.*

Once the responses were coded, we used a software program called Dedoose to identify patterns between the coding, student gender, and persistence. Dedoose allowed us to easily identify the prevalence of codes in our data set, check code co-occurrence, and view student descriptor information.

Results

To understand the relationships between students' responses, their gender, and their calculus persistence, we provide an overview of the distribution of the most prevalent codes among the four categories of students in Table 2. Of the 522 original student responses, 68 were coded as not applicable and were filtered out, leaving 454 relevant comments. Half of these comments came from Male Persisters, 9% from Male Switchers, 32% from Female Persisters, and 10% from Female Switchers. Among all students, the most frequent responses were related to Affect, the Teacher, Assignments and Assessments, and Preparation. However, the frequency of these responses within each student group varies; for instance, 37% of Male Persisters' responses were coded as Affect while 63% of Male Switchers' responses were coded this way.

Table 2. Prevalence of codes among four student groups.

	Male Persister *(n=268)*	**Male Switcher** *(n=43)*	**Female Persister** *(n=160)*	**Female Switcher** *(n=51)*
Affect *(n=238)*	37%	63%	51%	61%
Teacher *(n=182)*	32%	26%	43%	35%
A&A *(n=109)*	19%	16%	24%	24%
Prep *(n=82)*	12%	21%	19%	22%
Content *(n=78)*	16%	9%	14%	16%
Teaching *(n=76)*	15%	12%	13%	18%
Pacing *(n=18)*	3%	5%	4%	2%
Other *(n=54)*	7%	9%	16%	8%

Much research has been done on mathematical affect and its role in student persistence (Ellis, Fosdick, & Rasmussen, 2015; Fennema & Sherman, 1978; Good, Rattan, & Dweck, 2012). In our data, affect was the most pervasive code – and more so among Switchers than Persisters. As shown in Table 3, of the 238 responses coded with affect, the majority were coded with a positive weight (109), followed by 71 weighted negative, 62 neutral, and 8 responses coded with mixed affect, such as including both a positive affect statement and a negative affect statement.

Table 3. Prevalence of codes among four student groups.

Affect	**Male Persister** *(n=99)*	**Male Switcher** *(n=27)*	**Female Persister** *(n=81)*	**Female Switcher** *(n=31)*
Negative *(n=71)*	23%	37%	28%	48%
Neutral *(n=62)*	18%	26%	22%	29%
Positive *(n=107)*	56%	37%	43%	23%
Mixed *(n=8)*	3%	0%	6%	0%

Among the 99 Male Persisters' responses coded Affect, the majority (56%) were positive followed by 23% negative responses. Among the 27 Male Switchers, 37% of the Affect responses were negative and 37% were positive. Among the 81 Female Persisters, 43% were positive and 28% were negative. Strikingly, among the 31 Female Switchers, 48% of the Affect responses were weighted negative and only 23% were positive. These results indicate that Switchers, both men and women, are more likely to comment on their experience in Calculus I with an Affect statement (as shown in Table 2) and more likely for their Affect statements to be negative (as shown in Table 3). In order to better understand what aspects of their experiences in Calculus I the students expressed emotions, attitudes, or beliefs about we investigated the patterns that emerged within the Affect coding. Many interesting patterns emerged from the data within the coding, especially in the relationships among gender, persistence, and responses that were coded with Affect as well as something else. These responses point to aspects of Calculus I that students were especially emphatic about (either positively or negatively) and that possibly had a role in their decisions to persist in calculus. In this report, we focus on responses that are coded in such a way as to fall under the overlap of Affect and Teacher as well as Affect and Preparation. We investigated these relationships further by conducting a second level of thematic analysis on the responses.

Affect and Teaching

First, we analyzed the student responses coded with Affect and Teacher. Though thematic analysis, we refined the analysis further to uncover three subthemes related to affect – affect towards self, affect towards the course, and affect towards math. In this report, we focus on the most prevalent affect subtheme, affect towards self, and what role the teacher plays.

Statements coded with affect towards self entailed evaluations of personal learning and sometimes of self-worth. Usually, the comments merely focused on how well students learned in the course, but some comments were more personal and connected performance in the classroom to assessment of their intelligence or ability. This subtheme was weighted -2, 1, 0, 1, or 2 to designate a negative change in self-perception, negative self-perception, a neutral view of self, positive self-perception, and a positive change in self-perception, respectively.

As seen in Table 4, over half of Male Persister affect towards self comments were negative. This proportion grew for Male Switchers. Female Persister made more Affect towards self/ Teacher comments in general, and their responses were spread across the entire spectrum of the subtheme. They were the only student group to report a negative change in affect or positive affect. Female Switchers also had a high proportion of negative comments.

Table 4. Prevalence of Affect towards self/ Teacher subtheme and its weights among four student groups.

	Male Persister	**Male Switcher**	**Female Persister**	**Female Switcher**
Affect towards self *(n=37)*	9	3	20	5
Negative change (n=4)	0%	0%	20%	0%
Negative (n=16)	56%	67%	30%	60%
Neutral (n=10)	22%	33%	25%	40%
Positive (n=4)	0%	0%	20%	0%
Positive change (n=3)	22%	0%	5%	0%

An analysis of the quotes themselves were more revealing of some interesting patterns. For the male Persisters reporting a negative change in self-perception, they mentioned possible major or career changes rather than changes in self-worth. The female Persisters made more personal statements, such as, "This class really made me question my abilities in math through the instructor's poor teaching methods… I have a strong math background. However, this class completely destroyed that for me." Among the students reporting negative self-perception, females generally made more personal statements than males, i.e. to report feeling stupid as opposed to struggling in the class. All student groups reported problems in the teacher's ability to communicate. However, only female students reported that teachers were not personable. One female Persister commented saying, "If you asked the wrong question or gave the wrong response, he had a tendency to make you feel stupid." Female Switchers also described teachers who made them feel badly about themselves.

For affect towards self, the neutral to positive change comments only mentioned learning and not more direct evaluations of self. Interestingly, males only outnumbered females in making affective statements when reporting a positive change in self-perception. For positive comments, both male and female students frequently mentioned teacher helpfulness, saying that the teacher presented challenging problems but made an effort to equip students to succeed.

Affect and Preparation

After determining that a large group of students' responses were coded with an Affect code and a Preparation code, we isolated this group of 45 responses and did a second level of analyses. These responses were reviewed and grouped based on emergent themes. Four main themes emerged, and 37 of the 45 codes fit into one of these four themes. These themes related to previous mathematical/calculus experience and how this affected their college calculus experience. The responses that did not fit into one of these themes either mentioned how they felt about their preparation moving forward (4), mentioned that they were repeating college calculus (3) or did not make a clear enough statement about their preparation to code as one of the above four codes (1).

The first theme related to having taken calculus in high school and that college calculus is in comparison worse, labeled "Bad in comparison". These responses indicated that students entered college calculus with certain expectations of what calculus is and expecting to easily succeed in college calculus since they already took the course. Often these students blame the negative experience in college calculus, in comparison to high school calculus, for their dissuasion from pursuing more mathematics. The second theme related to having taken calculus in high school and glad that they are retaking it, labeled "Felt prepared". These responses indicated that the students were appreciative of previously taken calculus and recognize that there was more to learn in college calculus. The third theme related to not having taken calculus in high school and feeling less prepared than others in the class because of this, labeled "Not prepared". These responses indicated that students were aware that many of their classmates had previously taken calculus, and that they were at a disadvantage because they had not taken it before. The final theme related to not having taken calculus in high school and feeling empowered because of their success, labeled "Empowered". These responses indicated that students were aware that many of their classmates had previously taken calculus and that they were in the minority for not, and so their success in the course in spite of this gave them increased confidence.

As shown in Table 4, among Male Persisters the most common Affect/Preparation response fell into the "Bad in Comparison" theme, closely followed by the "Empowered" theme. Among Female Persisters, the most common Affect/Preparation response also fell into the "Bad in Comparison" theme, closely followed by the "Felt Prepared" and "Not Prepared" themes no responses in the "Empowered" theme. Male Switchers had one response in each theme except for the "Empowered" theme, with two responses. All of the Female Switchers fell into the "Bad in Comparison" theme.

Table 4. Prevalence of Affect/ Preparation subthemes among four student groups.

	Male Persister *(n=11)*	**Male Switcher** *(n=5)*	**Female Persister** *(n=14)*	**Female Switcher** *(n=7)*
Bad in comparison *(n=20)*	45%	20%	50%	100%
Felt prepared *(n=6)*	9%	20%	29%	0%
Not prepared *(n=5)*	9%	20%	21%	0%
Empowered *(n=6)*	36%	40%	0%	0%

While these numbers are small, the patterns are surprising: of the men whose responses who were coded as Preparation and Affect, 50% (n=8) mentioned having taken high school calculus. Of these, only two "Felt Prepared". Of the women whose responses who were coded as Preparation and Affect, 86% (n=18) mentioned having taken high school calculus. Of these, none of the women Switchers but four of the women Persisters "Felt Prepared". Of the 50% of the men who did not have calculus before entering college, 6 were in the "Empowered" theme while none of the women were. These numbers indicate that for the men in our sample, entering college without having already taken calculus could be an empowering experience; while for women this was not the case. Instead, the vast majority of women in this sample had previously taken calculus in high school, and only some of the Persisters felt that this improved their time in college calculus.

Discussion

This work was motivated by work that clearly linked gender to persistence in calculus, with a lack of confidence in mathematical ability as a major contributing factor for women's decisions to leave calculus but not men's. In this report, we further investigated aspects of male and female Calculus I students' reports of their experience in calculus to try to better understand the link between gender and persistence in calculus. Our analyses identified a number of aspects of the Calculus I experience as related to gender and persistence.

Women reported affect towards self in relation to the teacher more often than men. Females' negative comments tended to relate their success directly to their self-worth. This may indicate that women are holding to a fixed intelligence mindset (Dweck, 2008). Due to the stereotyped nature of people who should perform well in math, a fixed mindset is especially detrimental to women and minorities (Good, Rattan, & Dweck, 2012). Females complained that teachers are not personable, while males did not. Unfriendly teaching practices may be reinforcing a fixed mindset, since students do not feel safe to make mistakes. Males and females made positive comments about teachers who challenged them, yet demonstrated a desire to help them succeed and took actions such as demonstrating multiple techniques to solve a problem. These actions are more conducive to a growth mindset, which is beneficial for all students and could make a significant difference for underrepresented populations in STEM.

We also saw interesting differences in the way students discussed their preparation. Often, students who have taken calculus in high school are recommended to take Calculus I in college for a refresher, and easy introduction to what college math is like, or just to have an easier first semester. It seems that this advice may be frustrating many students, especially women, giving them an inflated perspective on their understanding of calculus, and results in them having a more negative experience in Calculus I.

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Supporting institutional change: A two-pronged approach related to graduate teaching assistant professional development

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Graduate students teaching assistants (GTAs) are responsible for teaching a large percentage of undergraduate mathematics courses and many of them will go on to careers as educators. However, they often receive minimal training for their teaching responsibilities, and as a result often are not successful as teachers. In response, there is increased national interest in improving the way mathematics departments prepare their GTAs. In this report, we share the initial phases of joint work aimed at supporting institutions in developing or improving a GTA professional development (PD) program. We report on findings from analyses of a baseline survey designed to provide insights into the characteristics of current GTA PD programs in terms of their content, format and duration. Results indicate that there are many institutions seeking improvements to their GTA PD program, and that their needs are in line with the change strategies that the joint projects are employing.

Key words: Graduate student teaching assistants, Professional Development, Institutional Change

It is well documented that graduate student teaching assistants and associates (GTAs) play a large role in undergraduate mathematics education (Belnap & Allred, 2009; Ellis, 2014), that GTAs often hold novice beliefs about the teaching and learning of mathematics (DeFranco & McGivney-Burelle, 2001; Gutmann, 2009; Hauk et al., 2009; Raychaudhuri & Hsu, 2012), have novice knowledge related to teaching (Kung, 2010; Kung & Speer, 2009; Speer, Gutmann, & Murphy, 2005), and yet are more open to student-centered teaching practices than more experienced mathematics instructors (Ellis, 2014; Seymour, 2005). It is also well documented that many GTAs are minimally prepared to teach, and that more robust teaching preparation can result in expert-like beliefs, knowledge, and practice (Alvine et al., 2007; Barry & Dotger, 2011; Hauk et al., 2006; Kung & Speer, 2009; Luft, Kurdziel, Roehrig & Turner, 2004).

For the above-stated reasons, GTAs and their preparation to teach can play important roles in the effective teaching and learning of undergraduate mathematics. In particular, recent findings suggest that the presence of a robust GTA professional development (PD) program is characteristic of departments with successful calculus programs (Ellis, 2015). The context of two projects (under the auspices of the Mathematical Association of America (MAA) and funded by the National Science Foundation (NSF)) provides opportunities to examine the state of GTA PD nationally and the ways in which such programs interact with departmental efforts to improve the teaching and learning of calculus.

The work reported on in this proposal is the first step in the larger and longer-term efforts to understand department change and GTA PD. Here we report on findings from analyses of data from a baseline survey that was designed to provide insights into the characteristics of current GTA PD programs in terms of their content, format and duration. In addition to being a basis for future comparisons, these data provide the mathematics community with information about the prevalence and features of currently-existing efforts to prepare graduate students for their teaching-related responsibilities.

As further context for this work we briefly describe the two projects and their goals related to institutional change and GTA PD. The first project, Progress through Calculus (PtC) (NSF DUE-1430540), aims to observe and facilitate institutional change related to the Precalculus-Calculus II sequence. This project is a continuation of the Characteristics of Successful Programs in College Calculus (CSPCC) study and is specifically focused on observing and supporting graduate-degree granting mathematics departments in implementing the characteristics found to be related to student success in calculus through the CSPCC project. As noted above, one such characteristic was robust GTA PD programs (Ellis, 2015). The second project, College Mathematics Instructor Development Source (CoMInDS) (NSF DUE-1432381), aims to support mathematics departments in developing and improving GTA PD programs by broadening access to resources related to GTA PD and to support for individuals and departments implementing these resources.

Together, these two projects aim to increase awareness of the need for GTA PD, help institutions think about how to implement robust GTA PD in relation to other needs of their departments, learn about different types of GTA PD programs, and have the resources to successfully implement such programs. As a first step in documenting and understanding departmental change, the two projects have collaborated to understand the current national landscape of existing GTA PD programs and the GTA PD-related needs of mathematics departments.

Theoretical Background

With the long-term goal of analyzing factors that influence how and why departments change, we approach this work with an eye towards *change strategies*. Henderson, Beach, and Finklestein (2010) conducted a large-scale meta-analysis of research on facilitating change in undergraduate science, technology, engineering, and mathematics (STEM) instruction. Through this work they determined four broad categories of change strategies: disseminating curriculum and pedagogy, developing reflective teachers, enacting policy, and developing shared vision. The change strategy of *disseminating curriculum and pedagogy* is focused on sharing experts' knowledge with individuals and encouraging the implementation of the strategy, such as through journal articles, workshops, and research presentations. The change strategy of *developing reflective teachers* is focused on encouraging and supporting reflective practices by individual instructors that lead to instructor-identified and defined change outcomes. The strategy of *enacting policy* is focused on prescribing a new environment that requires or strongly encourages new practices. The last strategy, *developing a shared vision*, is focused on empowering and supporting stakeholders to collectively develop a new environment that encourages instructional change.

The least successful change strategies were developing and testing "best practice" curricular materials and then making these materials available to other faculty and "top-down" policy-making meant to influence instructional practices. Successful strategies involve shifting the focus from strategies with exact intended outcomes before implementation to those that acknowledge that the final outcomes will be shaped by the individuals and/or environment involved in the system. The most effective change strategies were aligned with or sought to change the beliefs of the individuals involved, were long-term interventions, sought to understand the system that was trying to be changed and designed a strategy that is compatible with the system.

The larger PtC project involves identifying departments where changes are planned for their calculus sequence and documenting those efforts with a particular focus on the role that the creation or enhancement of GTA PD programs plays in those efforts. The larger CoMInDS project involves identifying individuals within departments who are implementing changes to their GTA PD programs and working with their perspectives towards GTA PD in ways compatible with their university systems. Both projects seek to work with the beliefs of the change agents involved and work within the larger systems to implement changes. The lens of *change strategies* serves as a guiding framework for our baseline survey instrument design and data analysis that supports both larger projects' goals.

Methods

A survey was sent to department chairs at all graduate-degree granting mathematics departments in the US (n = 341). The survey has three parts: Part I requested a list of all courses in the department's mainstream precalculus/calculus sequence, Part II asked about departmental practices in support of the precalculus/calculus sequence and contained 18 questions about GTA PD and Part III asked for enrollment data and other specific information about each of these courses. The questions related to GTA PD were jointly designed by members of the CoMInDS and PtC teams and were designed to provide insights into the following questions:

(a) *What GTA PD programs are currently being implemented across the country?; and*

(b) *What are the interests and needs of mathematics departments related to GTA PD?*

Department chairs were encouraged to have local experts in his or her department fill out the components of the survey with which they were most knowledgeable. For instance, many of the questions about GTA PD may not be known by the chair but instead by the facilitator of the GTA PD program, and so this person would hopefully be the one filling out this section of the survey. The survey was administered using Qualtrics and distributed by the MAA. Follow up emails and phone calls are ongoing to encourage full participation and response rate – currently, 56.3% (n=192) of all institutions have responded, 63% (n=114) of PhD-granting and 48.8% (n=78) of Master's-granting. The questions about GTA PD included multiple choice questions, Likert scale questions, and open-ended questions. In this report we discuss responses to the multiple choice and Likert scale questions.

Results

Results are reported from descriptive analyses of the survey response data that were aimed at addressing the two questions listed above.

What GTA PD programs are currently being implemented across the country?

There were eight questions on the survey to address various aspects of the structure and context of the department-lead GTA PD programs (we did not ask questions related to university-lead, non-mathematics specific GTA PD.) These questions addressed who the primary audience of the GTA PD is, how many GTAs participate and when, the format and activities included in the PD, the source(s) of the materials used in and who facilitated the PD. As shown in Table 1, three-quarters of PhD-granting institutions have department specific GTA PD, while only 35% of Master's-granting institutions do. For the remainder of this report, we attend to the PhD-granting institutions unless otherwise noted.

The primary audience for the department specific GTA PD was lead instructors (60%) and recitation leaders (59%). The majority of these programs were geared to all GTAs (61%), and most often before they teach the first time (67%) or during their first term of teaching (37%).

Almost half of these programs consist of a term-long course or seminar, while 28% involve a multi-day workshop and 17% involve a one- to four-hour long workshop or orientation. Although what is done during this time varies widely across institutions, it is uplifting to know that many departments have a specific course for preparing GTAs, and that it is possible to target efforts at improving these courses rather than convincing universities that such courses are helpful. The most common aspects reported as part of the programs were: Student evaluations of GTAs required by the institution or department (69%), GTAs observed by an experienced instructor while teaching in the classroom and receive feedback on their teaching (60%), GTAs are observed by a faculty member while teaching in the classroom (57%), GTAs practice teaching and receive feedback on their teaching (56%), GTAs develop lesson plans (35%), and GTAs learn classroom assessment methods (31%). The majority of departments use in-house materials for the teaching preparation (67%) while 31% use published materials. The majority of these programs are facilitated by one or more individuals for whom this is part of their official responsibilities for multiple years (61%).

Table 1. A sample of descriptive analyses related to the structure of the program

	Total (n=192)	**PhD Granting** (n=114)	**Masters Granting** (n=78)	**Minority Serving Institutions** (n=27)
Has a department specific GTA PD	66%	75%	35%	56%
Primary audience				
Recitation leaders	44%	59%	14%	30%
Primary instructors	53%	60%	28%	52%
How many GTAs participate?				
All	54%	61%	28%	44%
When				
Before teaching for the first time (e.g., pre-term orientation)	57%	67%	31%	44%
During their first term of teaching	32%	37%	15%	33%
Format				
Term-long course or seminar	37%	47%	13%	30%
Multi-day workshop	23%	28%	10%	19%
Short workshop or orientation (1-4 hours)	17%	17%	13%	15%
Activities				
Required student evaluations	60%	69%	31%	52%
GTAs are observed by an experienced instructor while teaching in the classroom and receive feedback on their teaching	52%	60%	27%	41%
GTAs practice teaching and receive feedback on their teaching	45%	56%	18%	33%
GTAs develop lesson plans	30%	35%	17%	33%
GTAs learn classroom assessment methods	27%	31%	17%	30%
Source of materials				
Created by the providers of GTA PD	59%	67%	32%	56%
Published materials	28%	31%	15%	15%

Who facilitates				
One or more individuals for whom this is part of their official responsibilities	53%	61%	27%	41%

What are the interests and needs of mathematics departments related to GTA PD?

There were four questions on the survey to understand the interests and needs of the mathematics departments related to GTA PD. Only 19% of PhD-granting institutions reported that their GTA PD is preparing GTAs "very well," while 41% reported they did well and 39% reported that they prepared their GTAs adequately. Over 60% of PhD-granting institutions report that the department is generally satisfied with the GTA PD program, and 33% responded that they were adequate but that there was room for improvement. It is these 38 institutions and the 5% that are not satisfied with their programs for whom we will target our improvement efforts. Over a quarter of the institutions report that changes to their GTA PD program have recently been or currently are being implemented, and almost 20% report that changes are being discussed. When asked what resources would be most helpful to them in strengthening their GTA PD programs, institutions most often marked: research-based information about best practices in GTA teaching preparation (64%), tools for evaluating effectiveness of GTA teaching preparation (55%), collegial conversations or mentoring for GTA teaching preparation staff with colleagues at similar institutions (54%), professional development for GTA teaching preparation staff (e.g., workshops, conference sessions) (45%), and online library of tested resources (41%).

Discussion and Next Steps

Results indicate that there are many institutions that are seeking improvements to their GTA PD program, and that their needs are in line with the change strategies that are part of the PtC and CoMInDS projects. These findings provide both the baseline data needed to document and analyze change and substantiate the claim that there are departments that can serve as the context for carrying out studies of departmental change.

Other sections of the survey aim to generate data on a different, but related topic – to situate the interests and needs of mathematics departments related to GTA PD in relation to the larger system of first and second year undergraduate mathematics instruction (often where GTAs are involved in the teaching). To address this goal, we will continue to analyze these results in relation to the other sections of the PtC census survey. We will specifically target institutions looking to make changes to their GTA PD program and investigate what other aspects of their programs they feel confident in and what aspects they are also looking to improve. For instance, are programs looking to improve their GTA PD programs also looking to improve the coordination of their precalculus through calculus sequence? If so, then we may look into ways in which we can capitalize on this relationship to better support these institutions. In doing so, we can develop data collection methods and analysis approaches that utilize the *change strategies* framework to understand specific institutions and to generalize across institutions. These efforts can then contribute to the mathematics education community's understanding of factors that support and/or inhibit change occurring as departments strive to improve the teaching and learning of undergraduate mathematics beyond GTA PD.

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Questions

- In what ways do you see this work translating to other goals of undergraduate mathematics education reform?
- Are there other characteristics of GTA programs or aspects of departmental culture that we should gather data on as we endeavor to understand the factors that enable and inhibit change?

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A framework for examining the 2-D and 3-D spatial skills needed for calculus

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Having well developed spatial thinking skills is critical to success in many STEM fields such as engineering, chemistry, and physics; these skills are equally critical for success in mathematics. We present a framework for examining how spatial skills are manifested in math problems. We examine established spatial skills definitions and correlate them with the spatial skills needed to successfully solve a standard calculus problem – find the volume of a solid of revolution. This problem is deconstructed into steps and analyzed according to what 2-D and 3-D spatial skills are necessary to visualize and solve the problem.

Key words: spatial skills, calculus, rotation about an axis, volumes of revolution

Introduction

Mathematics, especially areas like geometry and calculus, require both 2-D and 3-D spatial thinking skills. Spatial thinking skills can be learned and, as expressed by the National Research Council (2006), should be taught at all levels of the education system. These same spatial thinking skills, once acquired, can be applied in many areas of science and mathematics. Calculus, in particular, presents many situations that require students to move between 2-D and 3-D representations, such as when they are to determine the volume of a solid of revolution.

Having well-developed spatial thinking skills is directly linked to future success in engineering careers (Adánez & Velasco, 2002; Miller, 1996; Sorby, 1999). In searching the literature, most programs that aim to increase the spatial thinking skills of students seemed to be targeted at engineers (Sorby, 1999). It is appropriate that there would be an emphasis in this area, but the authors argue that other majors, specifically STEM majors such as physics, chemistry and mathematics, also need these spatial thinking skills to be successful in their future careers and would benefit from similar skill building activities. We identified the spatial skills necessary to complete a common calculus problem – compute the volume of a solid of revolution. This problem was chosen as it requires spatial thinking skills that are known to be troublesome for students: rotations and cross-sections.

Here, we provide applicable spatial skills definitions (focusing on 2-D and 3-D spatial skills) used by several authors. Next, we deconstruct a classic second semester calculus problem, identifying the requisite spatial skills. From this deconstructed problem, we construct a framework for analyzing spatial skills required for calculus problems.

Spatial Skills Definitions

Pittalis and Christou (2010) and Cohen and Hegarty (2012) define two spatial skills important to the study of calculus: spatial visualization and spatial orientation. *Spatial visualization* is defined as the ability to comprehend imaginary movements in 3-D space or the ability to manipulate objects in imagination. An example of the use of this skill would be imagining the 3-D cube that can be created from a 2-D net with the six faces of the cube outlined in a plane. *Spatial orientation* is defined as the ability to remain unconfused by changing the orientation in which a spatial configuration is presented. An example of the use of this skill would be orienting disks or washers within the 3-D object when deciding how to compute its volume.

Along with these definitions, Pittalis and Christou (2010) further define four other skills that are applicable to calculus and apply to 2-D and 3-D representations. These skills are representing objects, structuring, measurement, and mathematical properties. *Representing objects* is defined as manipulating forms of 2-D or 3-D objects and constructing a 2-D or 3-D model. An example of the use of this skill would be constructing a 3-D shape by rotating a 2-D shape around an axis. *Structuring* is defined as constructing partitions of 2-D or 3-D objects and manipulating the partitioning of 2-D or 3-D objects. An example of the use of this skill would be using a cross section (disk or washer) to partition a 3-D shape to find its volume. *Measurement* is defined as calculating and estimating. An example of the use of this skill would be calculating or estimating the volume of a solid. The last of the four, *mathematical properties*, is defined as realizing, identifying, and comparing structural elements. Examples of the use of this skill would be finding intersection points, finding limits of integration, and realizing the interior and surface of the constructed 3-D object.

Although there are many other spatial skills defined, we have chosen the ones deemed most applicable to the study of calculus, specifically 2-D and 3-D representations. In the next section we examine one such classic calculus problem and identify what spatial skills are being used to move toward the solution at each step in the process.

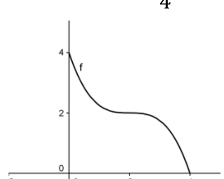
A Classic Calculus Problem

Below, we examine a solution to the problem:

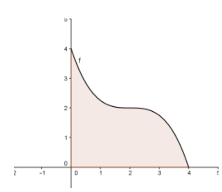
> Find the volume of the 3-D shape generated by rotating the region bounded by the function $f(x) = -\frac{1}{4}(x-2)^3 + 2$, the *x*-axis and the *y*-axis.

This is a typical second semester calculus problem. Upon close examination, it can be seen that there are many different spatial skills being used as a student proceeds toward a solution. Figure 1 presents the problem as a series of small steps that will later be classified using the suggested framework. While these steps are listed in the order a textbook or instructor might present them, they need not happen in this particular order. Rather, we are interested in capturing places where spatial skills might be useful to the problem solver.

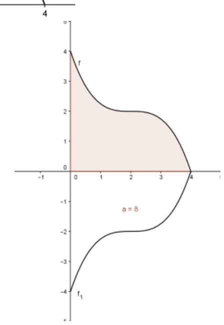
STEP 1:
Graph function: $f(x) = -\frac{1}{4}(x-2)^3 + 2$.



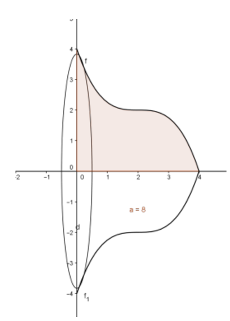
STEP 2:
Create 2D region with $f(x), x = 0, y = 0$.



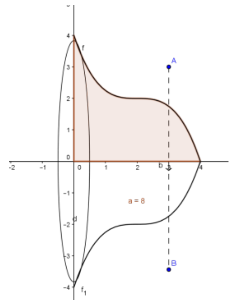
STEP 3:
Reflect the region about $y = 0$.



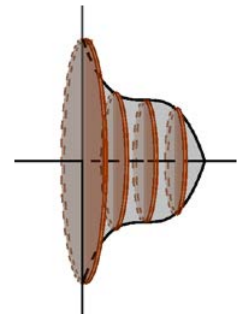
STEP 4:
Complete the 3D shape by rotating about the specified axis.



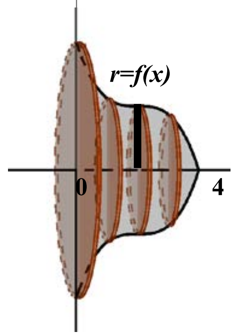
STEP 5:
Determine which way to slice:
horizontally or vertically?



STEP 6:
Position 3D disks.



STEP 7:
Draw radius and measure.
Determine limits of integration (0 to 4).



STEP 8:
Access volume of a cylinder formula:
$V = \pi r^2 h$

STEP 9:
Set up integral with boundaries.
$$\int_0^4 \pi(f(x))^2 dx = \int_0^0 \pi(-\frac{1}{4}(x-2)^3 + 2)^2 dx$$

*Figure 1***.** Spatial skills needed for rotating a region of the plane about an axis.

Proposed Framework: Spatial Skills for Calculus

Using the spatial skills definitions above, we constructed the framework in Figure 2 to classify the spatial skills needed to solve calculus problems. Figure 2 illustrates how the steps in the solution process in Figure 1 are mapped to one or more spatial skills.

	2 - D				***3 - D***			
	Representing	*Structuring*	*Measurement*	*Properties*	*Representing*	*Structuring*	*Measurement*	*Properties*
Visualization	Step 1: Draw 2D graph. Step 2: Identify region to be rotated.			Step 7: Determine the radius of a slice.	Step 4: Imagine the 2D region being rotated and the resulting 3D shape.	Step 6: Imagine slices filling up the figure.		
Orientation	Step 3: Identify the axis of rotation.		Step 7: Determine the radius of a slice.	Step 7: Determine the limits of integration.	Step 5: Determine which way to slice- horizontally or vertically?		Step 8: Determine the volume of a slice. Step 9: Set up and evaluate the integral.	Step 8: Identify the volume of a cylinder formula.

Figure 2. Framework to analyze spatial skills required in calculus problems. This figure illustrates the spatial skills required for the problem in Figure 1.

The row headings of visualization and orientation capture the broad spatial abilities of (1) being able to imagine and manipulate an object in 2-D or 3-D space, and (2) being able to remain unconfused when considering different perspectives of an object. There are two broad column headings of 2-D and 3-D, indicating that at various points in the problem solving process the student must think about either a 2-D object or a 3-D object. The four subskills – representing, structuring, measurement, and properties – identified by Pittalis and Christou (2010) may be required when thinking about either 2-D or 3-D objects and are represented by the columns under 2-D and 3-D respectively.

Note that each step of a problem may require more than one spatial skill. For example, step 7 requires 3 distinct spatial skills. First, the student must identify the radius by visualizing a particular property of a slice in a particular orientation. Then, the student needs to identify the range of appropriate values for that radius by accounting for the orientation of the stack of slices.

Using this framework to map other calculus problems that have spatial skills requirements (e.g., related rates, optimization, etc.) will allow us to identify which spatial skills are most used in the calculus curriculum and should get particular attention in remediation attempts. We now turn our attention to the spatial skills students possess when entering calculus.

Discussion

Although some programs have provided avenues for engineering students to improve their spatial thinking skills, there is a lack of attention to the development of spatial thinking skills for other majors. We need to promote students' understanding of geometric concepts and properties beyond using an algorithm or formula to get an answer. There are many places in the calculus curriculum where a lack of spatial skills hinders the understanding of concepts. One of the largest obstacles to college success is that students are arriving unprepared for the rigors of the college math curriculum; in particular, more than 40% and as many as 75% of students entering college place into a developmental math course (Twigg, 2013). While there are mechanisms in place for the development of other skills, such as arithmetic, algebra and geometry, not many colleges are addressing spatial thinking skills.

Further refinement of the framework for classifying calculus problems will allow us to analyze more of the curriculum to determine appropriate diagnostics and interventions. Considering the findings from the analyses of the curriculum, we will choose diagnostic spatial skills tests that align with the skills needed. Two examples of possible tests are the Purdue Spatial Visualization Tests: Visualization of Rotations (PSVT:R) (Guay, 1976;Yoon, 2011) and the Santa Barbara Solids Test (SBST) (Cohen and Hegarty, 2012).

The next step in this project is to observe the development of a group of 10-12 first semester calculus students' spatial thinking skills. We will obtain a baseline by using the PSVT:R and SBST tests and conducting semi-structured interviews with each student. The students will then be interviewed periodically over the course of several semesters as they encounter problems involving spatial thinking skills to identify how their spatial skills develop.

This work is the first step toward understanding what spatial thinking skills students have when entering calculus and determining how we can better understand where they need to improve. From the findings, we hope to pinpoint areas in the curriculum that rely on spatial thinking skills and determine if students have those necessary skills. We intend to develop interesting activities, both non-technology and technology-based, that could be used at critical points in the curriculum to assist students in developing and refining these critical spatial thinking skills. Much like the program developed by Sorby (1999) for engineering students, our goal is to determine which calculus students may have problems and with what concepts, and then provide additional activities to assist those students in the further development of their spatial skills.

We are interested in receiving feedback from the audience on the following questions:

1. What spatial skills are critical for success in mathematics that we have not captured?
2. What categories of problems should we examine next?
3. What ideas do you have for activities to build spatial skills?

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Student responses to instruction in rational trigonometry

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In this paper I discuss an investigation on students' responses to lessons in Wildberger's (2005a) rational trigonometry. First I detail background information on students' struggles with trigonometry and its roots in the history of trigonometry. After detailing what rational trigonometry is and what other mathematicians think of it I describe a pre-interview, intervention, post interview experiment. In this study two students go through clinical interviews pertaining to solving triangles before and after instruction in rational trigonometry. The findings of this study show potential benefits of students studying rational trigonometry but also highlight potential detriments to the material.

Key words: [Rational Trigonometry, Undergraduate Mathematics, Interviews]

Introduction

Students struggle with trigonometry. This struggle is a contributing factor to students not pursuing studies in the STEM fields. Students struggle with trigonometry at many points during their mathematical studies. While many pedagogical changes to trigonometry instruction have been tried (Bressoud, 2010; Kendal & Stacey, 1996; Weber, 2005) little has been done looking at replacing or augmenting trigonometry instruction with a mathematical alternative.

Rational trigonometry is a system for studying triangles using different units to measure length and the separation between two lines instead of using distance and angle (Barker, 2008; Campell 2007, Franklin, 2006; Henle, 2007; Wildberger, 2005a, 2005b). The use of a different unit necessitates different formulas than traditional trigonometry. Wildberger (2005a, 2005b) claims that rational trigonometry is simpler to learn, understand, and use than its traditional counterpart. He believes this based on the formulas for rational trigonometry lacking the sine, cosine, tangent or other transcendental functions. Little if any research has been conducted looking into educational benefits of rational trigonometry.

To investigate his claims I conducted task-based interviews before and after lessons in rational trigonometry to explore the following: How do mathematics majors approaches to solving problems pertaining to triangles change after studying rational trigonometry?

Traditional Trigonometry

Trigonometry as we know and teach it causes many difficulties for students. Previous research on students' difficulties with trigonometry include studies using quantitative methods (Brown, 2005), teaching experiments (Moore, 2009, 2013; Weber, 2005, 2008), and theoretical pieces (Bressoud, 2010; Gilsdorf, Moore, 2012; Wildberger, 2005a, 2005b, 2007).

What is trigonometry?

This is a question that is rarely answered explicitly in mathematics texts (Wildberger, 2005a). One method to defining words is the etymological approach. "Tri" being the prefix for three, "gon" referring to a polygon (e.g. pentagon, hexagon etc.) and "metry" referring to measure. Putting these together yields trigonometry as the study of the measure of three sided polygons.

A second way to define a word is to look at its use throughout history. The predecessor of the sine function was developed in the second century BCE (Bressoud, 2010). This was a relationship between central angles and chords of a circle (Bressoud, 2010; Gilsdorf, 2006). Using these techniques for triangles started in the 11th century CE and was formalized as sine and cosine in the 16th century (Bressoud, 2010). Introducing students to the trigonometric functions through the use of triangles began in the 19th century (Bressoud, 2010).

A third approach to defining trigonometry is to see how the word is currently used in the literature. Looking at texts yields the following list of topics: triangles, trigonometric functions, trigonometric identities, trigonometric equations, trigonometric graphs, imaginary numbers, polar coordinates, De Moivre's theorem, McClaurin Series, integral substitutions, waves, Fourier Analysis and more (Hirsch, Fey, Hart, Schoen, & Watkins, 2009a, 2009b; Larson & Edwards 2014, Liebeck, 2005). This would lead us to defining trigonometry as the study of anything pertaining to angles, triangles, or the functions sine, cosine, and tangent.

Based on these three perspectives, trigonometry is the study of everything pertaining to the functions, which resulted from applying the study of circles, to the study of triangles. For this study I am going to focus on the mathematics of triangles.

Student difficulties with trigonometry. Many difficulties pertaining to trigonometry are well documented (e.g., Akkoc, 2008; Blackett & Tall, 1991; Bressoud, 2010; Brown, 2005; Marchi, 2012; Moore, 2009, 2012, 2013; Weber, 2005, 2008, Wildberger, 2005b). Most of the documented difficulties can be sorted into two categories: 1) difficulties pertaining to the concept of angle (Akkoc, 2008; Bressoud, 2010; Moore, 2009, 2012, 2013; Wildberger, 2005b), and 2) difficulties pertaining to the sine, cosine, and tangent functions (Bressoud, 2010; Brown, 2005; Marchi, 2012; Moore, 2012; Weber, 2005, 2008; Wildberger, 2005b).

Student difficulties with angles. Moore (2012, 2013) and Akkoc (2008) claim that student difficulties with angles stem from gaps in their teachers' understanding of angles. Bressoud (2010) attributes difficulties with angles to incompatibilities between the ratio and the unit circle approaches to understanding trigonometry. These approaches are associated with degrees and radians respectively. Students are then taught that they are interchangeable yet certain problems are to be done in terms of one and other problems in terms of the other without any justification for the decisions made (Akkoc, 2008; Bressoud, 2010). Wildberger (2005b) takes these views to an extreme by claiming that the unit itself is overly complicated and that with the exception of a few values cannot be calculated without a background in calculus.

Student difficulties with trigonometric functions. Moore (2012) attributes flawed understandings of the trigonometric functions on the volume of inconsistent definitions used for them. Brown (2005) found that students compartmentalize two different definitions for sine and cosine. These two definitions for sine and cosine are as the ordinate and abscissa respectively of points on the unit circle and as ratios of side lengths of a right triangle. Some authors have found that the meanings of the trigonometric functions are obscured by the use of the unit circle instead of the use of ratios of side lengths of right triangles (Kendal & Stacey, 1996; Markel, 1982). Markel (1982) argues that the unit circle includes angles above 180° which are unnecessary and does nothing to help students differentiate sine and cosine. Kendal (1996) found that the unit circle approach gave students more opportunities to make mistakes. However, Weber (2005) states that the unit circle was a more effective pedagogical tool than right triangles. He found that students were more likely to recognize sine and cosine as functions if taught using a unit circle approach. Students have problems viewing sine, cosine, and tangent as functions due to their non-algebraic nature and as such are unsure about how to perform algebraic operations with them (Weber, 2005). This could be due to the pedagogy straying away from beginning with the study of circles and chords (Bressoud, 2010; Gilsdorf, 2006) or it may be due to the transcendental nature of the functions (Weber, 2005; Wildberger 2005b).

Need for trigonometry. One debated topic is the importance of studying traditional trigonometry. While the importance of many mathematical topics is debated in the K-16 curriculum the inclusion and exclusion of trigonometry can be seen in multiple scenarios. Multiple groups believe that high school students are not being taught enough trigonometry and that it should be the penultimate high school course instead of calculus (Bressoud, 2012; Markel, 1982). While many college calculus courses expect a prior knowledge of trigonometry many colleges now offer variants of their calculus courses that attend to the same topics with the exception of omitting trigonometry-based problems.

Rational Trigonometry

Rational trigonometry is a reformulation of trigonometry based on replacing the units of distance and angle, with the units of quadrance and spread (Wildberger, 2005a, 2007). Quadrance is distance squared. The spread between lines l_1 and l_2 is the quadrance of $\overline{BC}$ divided by the quadrance of $\overline{AB}$ shown in Figure 1.

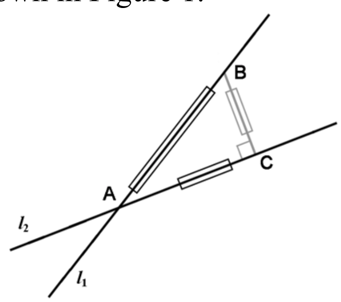

Fig. 1 The elements of the spread between two lines

Replacing the concept of angle with the concept of spread, results with the main formulas in trigonometry needing to be reformulated. The result is that the traditional trigonometry laws are replaced with the laws of rational trigonometry. They are analogous to the tradition trigonometric laws but the trigonometric functions are replaced with algebraic operations shown in Table 1 (Barker, 2008; Franklin, 2006; Henle, 2007; Wildberger, 2005a).

Table 1. Analogous Formulas in Traditional Trigonometry and Rational Trigonometry

Traditional	Rational
$c^2 = a^2 + b^2 - 2ab\cos C$	$(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2(1 - S_3)$
$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$	$\dfrac{S_1}{Q_1} = \dfrac{S_2}{Q_2} = \dfrac{S_3}{Q_3}$
$A + B + C = 2\pi$	$(S_1 + S_2 + S_3)^2 = 2(S_1^2 + S_2^2 + S_3^2) + 4S_1S_2S_3$

Curricular change

For something new to be adopted by the mathematics community it needs one of two things. It needs to either be able to do old tasks better than older approaches or it must be able to do new things.

Arguments in favor of rational trigonometry. Arguments in favor of rational trigonometry being simpler than traditional trigonometry are that it gets rid of the difficulties caused by the angle and the trigonometric functions by replacing them. With Rational Trigonometry, sine, cosine and tangent are no longer needed to study triangles (Barker, 2008; Franklin, 2006; Henle, 2007; Wildberger, 2005a, 2005b). Wildberger (2005a, 2005b) claims that the most complex operation needed for trigonometry becomes the square root function and that a student who has learned the quadratic formula has the prerequisite skills needed to study rational trigonometry.

Arguments against rational trigonometry. Three arguments have been made against rational trigonometry. One of these is that the units are less intuitive (Campell, 2007; Gilsdorf, 2006). Consecutive spreads of 1/4 and 1/4 combining to 3/4 is less intuitive than adding adjacent angles. Another is that many triangle problems would have irrational solutions when solved with rational trigonometry and that the irrational solutions from rational trigonometry are no more useful than the transcendental solutions from traditional trigonometry (Gilsdorf, 2006). A third is the inflexibility of the educational system (Campbell, 2007). Educational sequencing rarely changes and it pushes students to study traditional trigonometry before higher mathematics.

Questions from the arguments. The two sides of this argument bring up some interesting points in comparing the systems. Is the benefit of avoiding trigonometric functions worth a unit that is less visually intuitive? Should simpler be defined in how one uses the material or in how one learns the material? Is there any benefit to rational trigonometry when you have to study traditional trigonometry anyway?

While all of these are interesting my research question only addresses aspects of the first two. This study shows a glimpse at students working with quadrance and spread instead of the trigonometric functions. It also lets us see how two students use both trigonometric systems to address the same problems.

Mathematical research

As stated earlier there are two reasons for the mathematics community to adopt alternative mathematics. The second of these mentioned was that if it does something that has not been done before. There is a small yet existent body of literature in higher mathematics that makes use of rational trigonometry. Authors have applied the concepts of rational trigonometry to geometry (Alkhaldi, 2014; Le & Wildberger, 2013; Vinh, 2006, 2013; Wildberger, 2010), computer programming (Kosheleva, 2008), and robotics (Almeida, 2007).

Factors influencing students pursuing mathematics.

One of the factors that determines students' course taking patterns in college mathematics is their overall confidence with mathematics.

Students who expressed confidence in their mathematical abilities are more likely to take additional mathematics courses (Fennema & Sherman, 1977; Else-Quest, Hyde & Linn, 2010, Oakes, 1990). Those courses tend to be at a higher-level than the ones taken by their less confident peers (Fennema & Sherman, 1977; Else-Quest et al., 2010; Laursen, Hassi, Kogan, Hunter, & Weston, 2011; Stodolsky, Salk, & Glaessner, 1991). Typically, a loss in confidence is caused by performing lower than one's expectations (Ahmed, van der Werf, Kuyper & Minnaert, 2013). Improving students' performance in trigonometry would help their confidence and positively influence their future studies.

Students' problem solving strategies.

Students tend to use the strategies and techniques they are most recently familiar with when approaching problems (Chi, Bassok, Lewis, Reimann, & Glaser, 1989; Owen & Sweller, 1985). This explains why students might solve a quadratic by formula instead of factoring or use the law of sines when solving a right triangle. This phenomenon is stronger in weaker students who are less likely to stray from the patterns established in examples (Chi et al., 1989). Situation and context also influence how students attempt to solve problems (Moore, 2012). A student is most likely going to use the formulas they think an instructor or exam wants them to use.

As it pertains to trigonometric problems the strategies are the same in both rational and traditional trigonometry but the techniques are different. For example consider a problem where a student is given the measurements for two sides of a triangle and the vertex between them and asked for the third side. A strategy would be to use a formula that relates those four quantities. In traditional trigonometry the technique would be to use $c^2 = a^2 + b^2 - 2ab\cos C$ while in rational trigonometry the technique would be to use $(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2(1 - S_3)$.

Methodology

The comparative nature of this study influences many design decisions. Only distances are given and asked for in these tasks. To give or ask for spread or angle would inherently design the questions towards the use of a particular approach. A second outcome of this is the triangles presented in both interviews are geometrically similar. Without similarity is it possible that one interview task was inherently simpler due to the triangles used. A third result is the tasks asking for an altitude, median, and vertex bisector. These three concepts have been studied since antiquity (Heath, 1956) and as such are not dependent upon rational trigonometry for analysis.

Research Design

The inquiry approach for this study is case study design. Case study is the study of a case across a timespan (Hatch, 2002; Yin 2009). Case studies can be exploratory or explanatory in nature (Yin, 2009). For this study the cases are the two participants and the timespan is five days (the pre-interview, three days of lessons, and the post-interview).

Combining the need for a before and after and the exploratory affordances of task-based interviews (Confrey, 1981; Maher & Sigley, 2014; Schoenfeld, 2002) leads to the design of pre task based interview, lessons, post task based interview. The first interview is being used to look at strategies and techniques used by participants without a background in rational trigonometry. The lessons are used to create a background in rational trigonometry. The post interview is being used to see how a participant's behavior and/or reasoning when approaching the same task is altered after studying rational trigonometry.

Three video lessons on rational trigonometry were given to the participants. I designed these lessons to give familiarity with the units and formulas for rational trigonometry. The first lesson was focused on the units. The second lesson focused on the formulas $(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2(1 - S_3)$ (the cross law) and $\frac{S_1}{Q_1} = \frac{S_2}{Q_2} = \frac{S_3}{Q_3}$ (the spread law). The third lesson focused on $(S_1 + S_2 + S_3)^2 = 2(S_1^2 + S_2^2 + S_3^2) + 4S_1S_2S_3$ (the triple spread law). Each lesson was accompanied with a worksheet that acted as practice for the participant, additional data for myself, and verification that they watched the videos.

The first lesson focused on the units of quadrance and spread. Quadrance was described as distance squared and spread was first defined geometrically. After that I detailed arithmetical properties of spread and showed examples of how to calculate the spread for lines given in both slope-intercept and standard forms. Spread bisection was also shown. The second lesson focused on the spread and cross laws. The lesson was an example of solving a triangle knowing the quadrances of two sides and the spread between them. It started with the cross law being used to find the missing side and was followed by using the spread law to find the two remaining spreads. The last lesson focused on the triple spread law. The lesson was an example of using the triple spread formula to find the third spread in a triangle if only two spreads were known.

Participants

Due to the comparative nature of this study, participants with a strong background in mathematics in general, trigonometry in particular, and with no background in rational trigonometry were recruited. To ensure this, mathematics students with a 4.0 in their first year mathematics courses including Euclidian trigonometry were chosen.

Data Collection

Data was collected through a pre-interview, three worksheets and a post-interview.

Interviews. Task based interviews were used to gather information about the participants approaches to solving problems pertaining to triangles. The two interviews were audio recorded and occurred four days apart. Between the pre and post interviews the participants watched all three lessons and completed all three worksheets. Participants were supplied with pencil, paper, and a selection of traditional trigonometric formulas. During the second interview they were also given the rational trigonometry formulas from the lessons. From the interviews both their spoken word and written work were collected.

The three tasks chosen for the interviews were chosen to have no inherent bias towards traditional or rational trigonometry. The first task was to find the length of an altitude of a triangle. This task is commonly shown in the high school curriculum and is often done with and without the use of the sine, cosine, and tangent functions (Keenan & Gantert, 1989; Hirsch, et al., 2009b). The second task was to find the length of a median of a triangle. The last task was to find the length of a vertex bisector.

Worksheets. The primary purpose of the worksheets was to ensure that the participants watched the videos. The work was analyzed with respect to the findings from the interviews for triangulation purposes. All three worksheets were collected at the second interview.

Data Analysis

I began my data analysis by transcribing the interviews. At this point I was already making decisions about what data had the potential to show interesting findings. After this my next step was coding the data. That data was separated and regrouped for organizational purposes (Creswell, 2014; Maxwell, 2013; Saldana, 2009; Seidman, 2012). My coding efforts were focused on the written work and verbal statements given during the interviews. Once this was done I focused on the findings that were most abundant and different between both interviews. The strongest examples are highlighted here.

Findings

After my analysis three themes emerged. These themes were strategies, numerical properties of triangles, and confidence. The strategies used involved the Pythagorean theorem and the relationships represented by the laws of sine and cosine and the spread and cross laws. Numerical properties were that distances must be positive, the triangle inequality, and that the longest sides of a triangle are across from the largest angles / spreads.

Maureen

Maureen is a mathematics major with the goal of becoming a high school mathematics teacher. Her undergraduate course on trigonometry ended four months before the study.

Strategies. Maureen started the pre-interview using the Pythagorean theorem in an attempt to find the value of an altitude. After multiple iterations gave her more unknowns than equations or values that did not make sense to her she abandoned this strategy. Her next attempt was to use the Law of Cosines to find one of the angles. Her goal was to use that angle in the Law of sines to find the altitude. Once she found $\cos\theta = \frac{13}{14}$ she abandoned that approach as well.

During the second interview Maureen used the cross and spread laws in the manner she intended to use the Laws of cosines and sines in the first interview. In this attempt she successfully used both formulas. Though her use of the spread law gave her the quadrance of the altitude she did not turn that value into a length as the question was asking for. When questioned she said that the answer she gave was the length of the altitude.

N**umerical properties.** During the pre-interview Maureen made ample use of numerical properties of triangles. In particular she made use of the fact that side lengths cannot be negative and she made use of the triangle inequality. She used these to check her computational results.

The triangle inequality was also used to determine ranges for the answers to the interview tasks. Since she did not compute an angle there was no opportunity to observe if she would have used that the longest side is opposite the largest angle.

In the post interview there was no use of the triangle inequality. This could have been used to alert her to not having the right answer in the first task. She did however use the property that the largest spread has to be across from the largest side of a triangle.

Confidence. Maureen's confidence in approaching these tasks appeared to increase after the lessons in rational trigonometry. In the first interview she spent a lot of time staring at the tasks without performing any calculations. After a particularly long silence she said:

> As much as I hate to admit that I can not remember how to solve for altitude, I'm just going to spend 20 minutes staring at this, because I'm not liking what I'm getting. I feel very bad saying that and admitting that, but it's not gonna happen.

In the second interview the gaps in work and expressions of frustration lessened. After the interview she gave the following two statements: "That was really really cool the whole quadrance and [spread]" and "if I had more time to practice I think I could have gotten all 3." These statements point towards a higher confidence level using rational trigonometry.

Tom

Tom is a mathematics major aiming towards graduate studies in applied mathematics. He took his trigonometry course approximately three years before the study.

Strategies. In the first interview Tom's strategy was to solve for anything he could find in hopes that he would come up with pieces he needed to solve the tasks. When he found $\cos\theta = \frac{13}{14}$ he used that value in another Cosine Law equation in order to solve one of the tasks.

In the second interview his strategies were nearly identical. The biggest change between the two interviews was he was using the rational trigonometry formulas instead of the traditional trigonometry formulas.

Numerical relationships. There was no evidence in either interview that Tom used the numerical properties listed above. He submitted answers to all three tasks and he could have but did not find two of them to be impossible due to the triangle inequality. In both interviews he was confident in his strategies (which would have worked) and his computations (which contained errors).

Confidence. Tom showed no notable change in confidence.

Discussion

Based on the findings I believe it is safe to say there may be some benefits to students studying rational trigonometry. The strongest evidence for benefits come from Maureen's case. Maureen falls into the category of students who are weaker with their algebraic manipulation of functions, which hindered her mastery of trigonometry (Weber, 2005). She seemed to increase in confidence after studying rational trigonometry and appeared more capable of solving problems when using the rational trigonometry formulas. Tom showed a strong mastery of the algebra of functions and little change in performance using the rational formulas. This may point to potential benefits being more likely for students with a weaker skill set pertaining to functions.

Potential weaknesses also need to be mentioned. Maureen did not apply the numerical properties that she showed earlier evidence of using. She also at one point equated quadrance and distance. Quadrance being less intuitive than distance (Campbell, 2007; Gilsdorf 2006) is likely a contributing factor of this.

In conclusion there is more to research here. While benefits may exist it is possible that they are outweighed by the costs.

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Adaptations of learning glass solutions in undergraduate STEM education

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One of the main issues STEM faculty face is promoting student success in large-enrollment classes while simultaneously meeting students' and administrators' demands for the flexibility and economy of online and hybrid classes. The Learning Glass is an innovative new instructional technology that holds considerable promise for engaging students and improving learning outcomes. In this report we share the results of an efficacy study between an online calculus-based physics course using Learning Glass technology and a large auditorium-style lecture hall taught via document projector. Both courses were taught with the same instructor using identical content, including exams and homework. Our quasi-experimental design involved identical pre- and post-course assessments evaluating students' attitudes and behavior towards science and their conceptual learning gains. Results are promising, with equivalent learning gains for all students, including minority and economically disadvantaged students.

Keywords: Learning Glass, Effective Online Classroom, Online Inquiry Oriented Courses

Introduction

Against the economic backdrop of workforce demands forecasted by the report of the President's Council of Advisors on Science and Technology (PCAST, 2012), universities in the US are striving to meet the needs of their students with fewer resources. One way universities are trying to address these needs with shrinking resources is to offer large enrollment courses using large lecture halls or online or partially online (also called "hybrid") formats.

In this paper, we report the results of an investigation integrating an innovative new technology, the "Learning Glass", and how it can be leveraged to meet these challenges. One of the authors developed the Learning Glass as a low-cost, open technology that facilitates communication in STEM courses by allowing a lecturer to look at his or her audience while writing on a transparent surface. The Learning Glass screen acts as a transparent whiteboard. The instructor writes on a glass screen with LED illuminated edges. A camera on the opposite side of the glass records the video and horizontally flips the image (and hence the instructor is not required to write backwards as seen in the Figure 1). The Learning Glass allows the instructor to use a full range of communication modes and visual cues to clarify ambiguous or subtle concepts, and engage students.

Figure 1. Learning Glass enable the instructor to face the students while writing on the board

Schmid et al. (2014) defined an effective online classroom as one that provides learning outcomes equivalent to the same course offered face-to face. In order to assess efficacy of the Learning Glass in the undergraduate classrooms, we conducted a quasi-experimental study in two undergraduate calculus-based physics courses with the following research questions:

Research Questions

1. How does student success in the Learning Glass courses compare with student success in standard courses? In particular, does the Learning Glass environment mitigate differential student success by demographic subgroups?
2. What instructor behaviors (e.g., gestures, modeling real time struggle and errors) take advantage of the affordances of the Learning Glass and contribute to student engagement? In particular, how do students perceive instructor presence in standard and Learning Glass classes?
3. Is it effective to integrate the Learning Glass into online classrooms?

Theoretical Background

The PCAST report (2012) calls for the wide-scale adoption of empirically validated teaching and learning practices as one way to increase retention of STEM majors, and the key practice the report highlights is "engaged learning." In this study we examine how the Learning Glass might help transform traditionally passive courses into more engaged courses by increasing instructor immediacy and facilitating communication. In the following paragraphs we describe each of these potential affordances of the Learning Glass in more detail.

Media Richness

Media richness theory is used to analyze communication and media choices with the goal of reducing ambiguity of communications (and increasing immediacy) by selecting the most appropriate media type (Newberry, 2001). The theory postulates that for ambiguous tasks, understanding improves when communicators use "richer" media. "Richness" is determined by the capacity of the medium to facilitate instant feedback, transmit verbal and nonverbal cues, enable the use of natural language, and convey a more personal focus (Daft & Lengel, 1984). Rich media are characterized as having the capacity to convey the most information, while lean media have a lesser capacity. The Learning Glass is very rich media having the capacity to provide all of the affordances associated with rich media when used in a synchronous manner.

Immediacy, and Student Learning

In online and hybrid settings, communication technologies vary in their capabilities to convey messages and verbal or visual "immediacy" cues. Immediacy has been defined as perceived psychological or physical closeness (Christophel, 1990), created in part by nonverbal physical cues such as smiling, a relaxed body posture, making eye contact, and verbal cues that include the use of humor and personal examples (Hostetter & Busch, 2006). In recent experimental studies conducted by Schutt, Allen, and Laumakis (2009) and Bodie and Bober-Michel (2014) confirmed that instructor behaviors that have been shown to reduce psychological distance in face-to-face settings also positively influenced learning outcomes in online settings. Interestingly, results of the Bodie and Bober-Michel (2014) study of cognitive learning showed that participants exposed to higher-immediacy instructor behaviors using rich media performed significantly better at post-test, immediately following the teaching session.

Student engagement & Peer Instruction

In a meta-analysis of existing literatures on online learning (Means, Toyama, Murphy, Bakia, & Jones, 2009) and on using technology in the classrooms (Schmid et al., 2014) researchers have concluded that "learning is best supported when the student is engaged in active, meaningful exercises via technological tools that provide cognitive support" (Schmid, et al., 2009, p. 285). A study by Zhang, Zhou, Briggs, and Nunamaker (2006) found that the effect of video on learning depended on the learner's ability to control the video ("interactive video").

There is a wealth of research in mathematics and science education research demonstrating the effects of peer instruction in large classroom settings (Crouch, & Mazur, 2001). Peer instruction is a pedagogical strategy in which the lecture is interspersed with short conceptual tests designed to reveal typical difficulties and misunderstandings and to actively engage students during the lecture. Students' interactive-engagement within the class has been shown to be correlated to students' learning gain (Prince, 2004; Redish, Saul, & Steinberg, 1998).

We conjecture that using the Learning Glass has significant potential for increasing the extent to which learners feel connected and form relationships with their instructor, even in large-lecture and online/hybrid classes. Such connectedness and feeling of instructor presence (immediacy) has been shown to be a key indicator of persistence, especially for women and underrepresented students (Good, Rattan, & Dweck, 2012; Seymour, 2006).

Research Design and Methods

We used a quasi-experimental design to investigate the adaptation of Learning Glass technology during the spring 2015 semester. This study included two classes of Introductory Calculus-Based Physics, both taught by one of the authors, using identical pre- and post-course assessments to evaluate students' attitudes and behavior towards science and their conceptual understanding of physics concepts. At the post assessment, we also included a survey to assess students' measure of instructor immediacy. When signing up for the course students did not know that the two sections would be different. We also used the pre-assessments to create a baseline for our quasi-experiment study. We then used the results of the post-assessment to compare students' conceptual understanding between the two classes, their attitude towards physics, and their view of instructor immediacy.

Population and Setting

The first class (LG section), which met at 8:00 AM, used Learning Glass in front of a live studio audience of rotating sets of 20 students that were enrolled in the course (shown in the Figure 1). As seen in Figure 2, the lecture streams to the remote students via MediaSite and incorporates peer instruction techniques such as clicker questions via online breakout rooms and virtual whiteboard of Blackboard Collaborate technology to engage students at home. There were 215 students in this class.

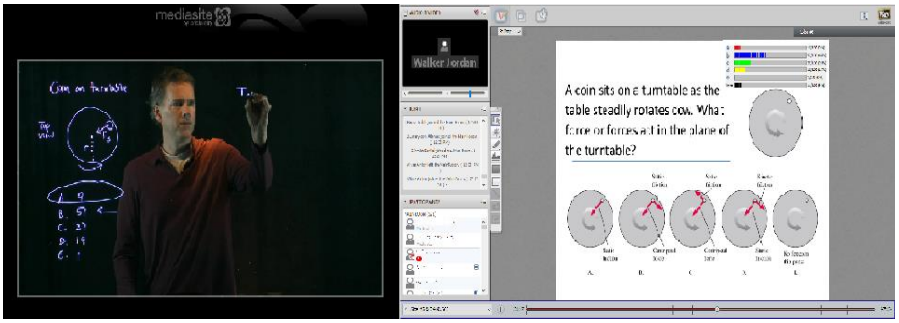

Figure 2. While instructor discussing students' difficulties with a clicker question, students can interact online and discuss the questions with their peers and their instructor.

The second class (standard face to face section) was delivered at 9:00 AM in a large auditorium-style lecture hall via document projector and used in-class clickers to engage the students. This class had 327 students. The content, homework, and exams were identical in both classes. Because of administrative difficulties, it was impossible to conduct a complete random-control study with these two classes, but as previously mentioned, students did not know that the sections would be different when registering for the course. During the first week of the semester there was very little movement of students to different sections, about 10% attrition from Learning Glass to face to face section.

In order to provide peer-instruction opportunities for the students in the Learning Glass section, we utilized an online conferencing system called "Blackboard Collaborate". Blackboard Collaborate is an online system that allows users to conduct and record synchronous virtual classes and meetings. Synchronous viewers at home were able to participate in "clicker" questions posed by the instructor, using the Blackboard Collaborate toolbox (essentially a high-tech virtual chat room with common whiteboard). These students were also able to watch the rich media lecture using MediaSite TM created by Sonic Foundry (left part of Figure 2). Belvins and Elton (2009) had shown that among three instructional hardware/software packages, MediaSite is the preferred format when video and audio of the instructor are shown simultaneously with the power point slides (clicker questions).

Data Collection

To answer the first research question, we needed to build a baseline for students' attitude and prior knowledge of physics concepts. Identical surveys were used as pre- and post- assessment so we could analyze how the Learning Glass intervention had impacted students' conceptual understanding of physics and their attitudes. Our pre- and post-course assessments included the Force Concept Inventory (FCI) (Hestenes, Wells & Swackhamer, 1992) and Colorado Learning Attitudes about Science Survey (CLASS) (Adams et al., 2006). In order to answer the second research question, in addition to the conceptual understanding and attitude surveys, we used a well-established instrument (Bodie & Bober-Michel, 2014) to measure the instructor immediacy at the end of the semester. Towards the end of the semester, we were also able to collect analytical data on the students' usage of Learning Glass videos from the MediaSite server where the class videos were streamed and recorded and recently we were able to obtain the demographic data for our students in both classes. These data included students' SAT scores, ACT scores, and their majors. Two indicators helped us distinguish minority and economically disadvantaged students: being part of the minority educational opportunity program and eligibility for financial aid.

Data Analysis

To answer the first research question, we used the students' responses to the pre- and post-assessment surveys from both sections. To answer the second research question, we compared students' responses to the immediacy survey. Based on the findings for first and second research questions, we addressed the third research question as well.

Students' responses to the force concept inventory were assessed based on a rubric provided by the designers of the FCI (Hestenes, Wells, & Swackhamer, 1992). Each student received a comprehensive score for their correct responses to the FCI survey (out of maximum 30). Students also received a comprehensive score for their responses to the attitude survey.

Most of the attitude survey's questions were quantified based on the psychometric Likert scale. So if a student had chosen C on the second question of attitude survey (question 37 on our survey), their answer would be quantified as 3. Students' responses on the immediacy survey were quantified similarly. Factor analysis of the CLASS survey had revealed several categories (Adams, et al., 2006) that we used in comparing the two sections: students' attitudes on personal interest, real world connection, problem solving confidence, sense making efforts, and conceptual understanding.

To compare the students' learning gains between the two classes, the learning gains were calculated using the following formula for each student:

$$Learning\ Gain = \frac{Post{-}Assessment\ FCI\ Score \quad - \quad Pre{-}Assessmnt\ FCI\ Score}{30 \quad - \quad Pre{-}Assessmnt\ FCI\ Score}$$

Results

The data was first cleaned to remove incomplete surveys; out of 215 students in the Learning Glass class we registered 125 valid surveys, for the face-to-face classroom we registered 205 valid surveys. First we looked at the students' performance on pre and post surveys between the two sections. Then we compared the learning gains between the two sections and the learning outcomes of minority and economically disadvantaged students. We then analyzed the results of immediacy surveys and the effects of Learning Glass on student learning.

Students' Performance on the Pre- Attitude Survey

To compare students' comprehensive scores on the attitude survey, we conducted an independent sample t-test. Levene's test for equality of variances came out to be 0.183 which is larger than 0.10, so it confirmed the assumption that variances of students' scores on the pre-attitude survey for both sections were the same. The independent-samples t-test confirmed that there was no significant difference in comprehensive attitude score between the face-to-face class (section 2) (*M*=151.09, *SD* = 13.561) and the Learning Glass class (section 1) (*M*=148.63, *SD*=14.954). The magnitude of the differences in the means was small (eta squared = 0.007).

Students in both sections had similar scores on the CLASS survey categories except in the conceptual understanding domain. Comparing to the students in the Learning Glass section (*M* = 20.31, *SD* = 3.704), students in the standard face-to-face class (*M* = 21.39, *SD* = 3.361) had shown larger mean scores ($p < 0.01$) for the questions related to their understanding that physics is coherent and is about making sense, drawing connections, and reasoning not memorizing.

Students' Performance on the Pre-FCI Survey

The one-way ANOVA test indicated a significant difference between the students' performance on the FCI ($p < 0.01$). Students in the standard face to face section performed higher on the pre- FCI (*M* = 12.75, *SD* = 5.694) than the students in the Learning-Glass section (*M* = 11.40, *SD* = 5.585).

It appears that the difference in the mean value of students' performance on the pre-assessment FCI could be due to differences in the students' majors (standard section had more engineering majors) or it could be due to students' limited prior knowledge and skills necessary to advance in the undergraduate classroom. Analyzing the demographic data between the two sections illustrated that statistically there was a significant difference in student scores on SAT Math ($p < 0.05$) with standard section scoring higher, while there was no significant differences on their scores in SAT Verbal exams or ACT scores.

In order to see whether there was a significant difference in the student performance on pre FCI while controlling for their majors and their prior preparation (quantified by their SAT Math scores), we conducted one-way between-groups analysis of covariance. After adjusting for their majors and their prior preparation, there was no significant difference between the two section performances on Pre- FCI ($F\ (1, 272) = 0.184$, $p = 0.668$).

Students' Performance on Post Attitude Survey

An independent-samples t-test was conducted to compare the students' scores on the attitude survey for face-to-face class and the Learning Glass class. Similar to the pre-attitude results, there was no significant difference between the students' comprehensive scores on the post-attitude survey for face-to-face students (*M* = 143.57, *SD* = 14.872) and Learning Glass students (*M* = 140.62, *SD* = 18.40). The magnitude of the differences in the means was very small (eta squared = 0.007). The difference we had seen in the conceptual understanding category had been eliminated and both sections had similar mean scores on all the CLASS survey categories.

Students' Performance on the Post FCI

Similar to the pre-assessment, the face-to-face class outperformed the Learning Glass students. An independent-samples t-test was conducted to compare students' performance on the post FCI in face-to-face and Learning Glass classes. There was a significant difference ($t\ (327) = -2.694$, $p = 0.007$) in the face-to-face students' performance on post FCI (*M* = 17.54, *SD* = 5.933) versus the Learning Glass students (*M* = 15.72, *SD* = 6.006). The magnitude of the differences in the means was very small (eta squared = 0.009). In order to normalize these differences, students' learning gains were compared.

Learning Gains Comparison

The independent-samples t-test was conducted to compare the students' learning gains between the two classes. There was no significant difference in learning gains for students in the face-to-face class (*M* = 0.27, *SD* = 0.31) and students in the Learning Glass class (*M* = 0.24, *SD* = 0.26). Students' learning gains were the same between the two classes.

Learning gains of minority and socially economically disadvantaged students

A two-way between-group analysis of variance was conducted to explore the learning gains for minority and economically disadvantaged students for each section. The effect of being a minority ($F\ (2, 323) = 0.006$, $p = 0.994$), being in either section ($F\ (1, 323) = 0.128$, $p = 0.226$) and the interaction effect ($F\ (2, 323) = 0.675$, $p = 0.24$) did not reach statistical significance. So there was no significant difference in the students' learning gains between the two sections. One could claim that despite the differences in students' prior knowledge, this instructor had created equal learning opportunities for the students from diverse backgrounds.

Level of immediacy between the two sections

The students in the Learning Glass section found the instructor more immediate. The independent samples t-test, a parametric test of significance was used to determine a significant difference between the Learning Glass students' scores on the immediacy survey ($M = 80.13$, $SD = 11.790$) and the traditional face-to-face section ($M = 75.25$, $SD = 12.260$; $t\ (327) = 3.554$, $p = 0.0001$, eta squared = 0.04). The relation between level of immediacy students feel towards the teacher and learning gains in both sections was investigated using a Pearson product-moment correlation coefficient. There was a strong, positive correlation between the immediacy and learning gains in the Learning Glass section ($r = +0.25$, $n = 125$, $p < 0.01$) however there was no correlation between these two variables in the face to face classroom ($r = 0.126$, $n = 204$, $p = 0.073$).

Immediacy in the Learning Glass classroom

Students in the Learning Glass section felt their teacher being more immediate towards them. This could be due to the rich media provided by the MediaSite delivery method. We therefore looked at the correlation of MediaSite analytical data and level of immediacy. Level of immediacy was significantly correlated to the total number of times students watched the videos ($r = 0.306$, $n = 123$, $p < 0.01$) and the number of times they watched the lecture live ($r = 0.247$, $n = 123$, $p < 0.01$). There was no correlation between the immediacy and total number of hours students spent watching the lecture videos ($r = 0.168$, $n = 123$, $p = 0.064$). The last piece of analysis is investigating the relationship between the learning gains and level of immediacy and the MediaSite analytical data.

Effect of the Learning Glass on student learning

The data obtained from the MediaSite hosting the Learning Glass videos showed three variables: For each student, it showed the number of times they watched the lecture live, the total number of times they watched the videos including asynchronous views, and finally the total number of hours they spent on watching the videos. In order to look at the predictability of these factors, we employed hierarchical multiple linear regression. Hierarchical multiple regression is a regression technique used to generate and compare the predictive models for a continuous dependent variable (learning gains in the Learning Glass section) using different sets of independent variables (level of immediacy, total number of live views, total number of times students watched the video and total number of hours spent watching the videos). The hierarchical multiple regression revealed that level of immediacy ($p < 0.05$) and total number of times students watched the lecture live ($p < 0.05$) contributed significantly to the learning gains ($F(2,122) = 8.828$, $p < 0.001$).

Conclusion

Based on this preliminary analysis we can claim that integrating Learning Glass technology into the online classroom can provide an effective learning opportunity where students reach the same level of learning outcomes as the students in a standard face-to-face classroom.

An ideal efficacy study would have had randomly distributed populations to create a baseline. As we saw in the results of the analysis, our two populations had a similar attitude towards physics to start with but performed differently on the pre-conceptual assessment. We saw the same differences occurred at the end of the semester, but the students in the Learning Glass class and the standard face to face class had similar learning gains.

Students in the Learning Glass section had found the instructor more immediate than the face to face section. This feeling of connectedness is a key predictor of persistence in STEM fields so one can claim that Learning Glass technology has the potential to increase retention rates in undergraduate STEM majors. Learning Glass is an effective technological replacement to the large auditorium style classroom and has the capability of providing equal learning opportunities for students from all walks of life.

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Student conceptions of definite integration and accumulation functions

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Prior research has shown several common student conceptualizations of integration among undergraduates. This report focuses on data from a written assessment of students' views on definite integration and accumulation functions to categorize student conceptualizations and report on their prevalence among the undergraduate population. Analysis of these results found four categorizations for student descriptions of definite integrals: antiderivative, area, an infinite sum of one dimensional pieces, and a limit of approximations. When asked about an accumulation function, student responses were grouped into three categorizations: those based on the process of calculating a single definite integral, those based on the result of calculating a definite integral, and those based on the relationship between changes in the input and output variables of the accumulation function. These results were collected as part of a larger study on student learning in multivariable calculus, and the implications of these results on multivariable calculus will be considered.

Key words: Calculus, Integration, Accumulation

Integration is a key concept in the undergraduate mathematics curriculum. Student conceptions and difficulties in that realm have been well-documented (Bezuidenhout, J. & Olivier, A., 2000; Orton, 1983; Rasslan, S. & Tall, D., 2002; Hirst, 2002). The central conceptions of a definite integral from the literature could be characterized as: antidifferentiation procedure, area, and accumulation. These categories are consistent with those used by Jones (2015) and Hall (2010).

Students with an antidifferentiation conception focus their interpretation on the function whose derivative is the function being integrated. Students have been found to be generally competent in executing the procedure of integration (Mahir, 2009; Orton, 1983; Grundmeier et al., 2006). However, only a small proportion of students are able to translate to the graphical representation and solve when the original problem contains an expression not elementary for integrating (e.g. $\lfloor x \rfloor$) (Rasslan, S. & Tall, D., 2002; Mahir, 2008).

In the area conception, students refer to the (signed) area bounded by the curve on a given domain on the graph of the function. The area can be conceived as an infinite collection of lines or limit of narrowing rectangles (or trapezoids) (Sealey 2006; Jones 2013; Czarnocha, B., Dubinsky, E., Loch, S., Prabhu, V., & Vidakovic, D., 2001).

Accumulation is an important but less widely understood interpretation (Thompson, 1994; Thompson & Silverman, 2008; Jones 2015). It represents the conception of the integral as a sum of amounts. Tall (1992) called it cumulative growth; Thompson (1994) called it accumulation; Jones (2013) called it adding up pieces; Jones (2015) called it multiplicative-based summation. Understanding of this form is directly related to the ability to writing the integral for an application problem (Jones, 2015; Sealey, 2006).

Whereas a definite integral has fixed boundaries, an accumulation function is an integral with at least one variable boundary, so the result is a function. Student conceptions on this particular aspect of integration have been less deeply explored. Its importance is clear from its appearance in the Fundamental Theorem of Calculus.

Our desire with this study is to understand the current landscape of students' conceptualizations of integration as they learn multivariable calculus. Our goal is to investigate the question, "What are the primary conceptualizations of integration present among students entering multivariable calculus who have completed single variable calculus, and how prevalent are the various conceptualizations in this population?"

Theoretical Perspective

In this study we chose to focus on students' descriptions of integration rather than examine their ability to give correct or incorrect responses to mathematical questions. This decision is based on our theoretical perspective which stems from Tall and Vinner's (1981) work on concept images and concept definitions. In short, our primary interest is in understanding the mental images, processes and connections that a student brings to mind when considering the topics of integration and accumulation, that is, their *concept image* of definite integration and accumulation functions. It is important to keep in mind that students involved in our study may possess elements of their concept image that were never uncovered by their responses, for this reason we say that we are studying their *evoked concept images* in response to the questions posed to them.

Methodology

The data for this report comes from a larger project studying student learning in multivariable calculus at 26 universities. The preliminary results presented here were collected from 57 students in two separate classrooms from two separate universities. At the beginning of their multivariable calculus course, students in the study were asked to complete a collection of open ended written responses on various topics in introductory calculus. The current report will focus on student responses to the following two questions featuring the concept of integration in single variable calculus.

> Question #1: Suppose the result of $\int_a^b q(x)\,dx$ is a real number, k. Explain what k means and how it was measured. Sketch any images you have in mind in the space below.
>
> Question #2: Suppose that a function G is defined: $G = \int_a^t f(x)\,dx$. Is G a function of t or a function of x? Justify your response.

Student responses were analyzed using an open coding scheme to sort them into distinct categories. Some student responses were unclassifiable due to misinterpreting the question, leaving the question blank, or responding in a manner that was uninterpretable. Among the 57 students involved in the study, 55 gave classifiable responses to question #1 and only 35 gave classifiable responses to question #2.

Preliminary Results

Following are the categories which emerged during our preliminary analysis of student responses to questions #1 and #2 above regarding the concept of integral in single variable calculus.

Categories of responses to question #1 on the definite integral

Integral as representing an antiderivative

Students within this group primarily respond in terms of symbolic representations of functions and describe k as the computed result from manipulating those representations. Student responses typically omit sketched images or present symbolic representations as the 'image' accompanying the description.

Example response: "You take the antiderivative of $q(x)$. Once you do that, you substitute b for x and a for x and subtract. The difference is k"

Integral as representing an area

Students within this group primarily describe the integral in terms of area without reference to how the area can be computed or interpreted. Accompanying sketches typically include the graph of a function with the area underneath the function shaded; however, the sketches contain no means of dividing the area into simpler shapes.

Example response: "k is the area under $q(x)$ between a and b"

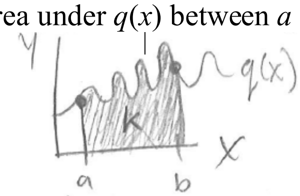

Figure 1: Sketch by a student interpreting a definite integral as area

Integral as representing an infinite sum of one-dimensional objects

Students within this group describe integration as a process of adding together an infinite number of infinitely small pieces, often referred to as 'lines' or 'slices.' Students within this group often describe this process as a means of measuring the area underneath the function. This category closely resembles the "collapse metaphor" of limit as described in Oehrtman (2009). Sketches accompanying these descriptions often contain either a single representative 'slice' of the function or an area composed of vertical lines.

Example response: "The area under the curve, $q(x)$, is equal to some real number, k. k was measured by taking an infinite number of slices of the area under the curve."

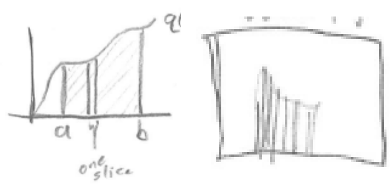

Figure 2: Sketches by students interpreting a definite integral as an infinite sum of one-dimensional objects

Integral as representing a limit of approximations

Students within this group describe integration as an approximation process, usually in terms of Riemann Sums. The students' descriptions of the limiting process of these approximations can vary widely including descriptions of repeating the approximation process indefinitely, doing a single approximation at a very high level of accuracy, or creating approximations to meet a desired accuracy as described in Sealey and Oehrtman (2007). Like the previous category, students within this group often describe the approximation process as a means of measuring the area underneath the function. Sketches accompanying these descriptions often reflect the traditional images associated with Riemann Sums.

Example response: "k means area under the curve. It was measured by taking segments of the curve and multiplied by the height of the function, thus creating rectangles. This process was then repeated by taking a limit and taking smaller and smaller segments each time."

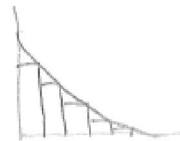

Figure 3: Sketch by a student interpreting a definite integral as a limit of approximations

For question #1 55 of 57 students gave a classifiable response. From those responses the majority of students (69%) describe the definite integral in terms of area and over half of the remaining students (16%) describe it in terms of an antiderivative. The remaining students were divided evenly among those who represented integration as an infinite sum of one-dimesional objects (7%) and those who represent the integral as a limit of approximations (7%).

Categories of responses to question #2 on the accumulation function

x and t are described by their role while performing integration

Students within this group focus on the process of computing a single value from integrating and the roles of x and t within that process. Students within this group tend to describe x as the "variable" involved in the process and t as a "parameter" or "boundary value." For this reason, students within this group respond that x is the variable present and argue based on the roles of either x or t rather than in terms of the function G. This may be due to a weak understanding of the covariational nature of functions (Carlson et al., 2002) or an unreified view of the process of integration (Sfard, 1991)

Example responses: "G is a function of x. It's not t because t is just a boundary." "G is a function of x because x is the input."

x and t are described by their role after performing symbolic integration

Students within this group tend to speak primarily in terms of the symbolic process of integrating; however, unlike the previous group of students, students within this category focus their attention on the result of integrating rather than the process of computing the integral. Students within this group respond that t is the variable because after using the fundamental theorem to integrate, t is substituted back for x to achieve the final answer.

Example response: "G is a function of t because t will replace the x from $f(x)$ when integrating."

x and t are described by how changes in each variable affect the value of the function, G

Students within this group emphasize in the input-output nature of the function G and respond in terms of how changes in either t or x will result in changes in G. Students within this group respond that G is a function of t because changing the value of t changes the resulting value of G.

Example responses: "G is a function of t. By modifying t, one can change the interval over which $f(x)$ is integrated." "A function of t, since any change in t would change the value of G."

For question #2 the preliminary analysis shows that among the students who gave a classifiable response, the majority (54%) described x and t in terms of their roles while performing integration with most of the remaining students (34%) described x and t in terms of their role after performing symbolic integration.

Table 1: Students with certain concept images for definite integral in Question #1

Evoked Concept Image for Definite Integral	*%*
antiderivative	16
area	69
infinite sum of one-dimensional objects	7
limit of approximations	7

Table 2: Students with certain concept images for accumulation function in Question #2

Evoked Concept Image for Accumulation Function	*%*
x and t are described by their role while performing integration	54
x and t are described by their role after performing symbolic integration	34
x and t are described by how changes in each variable affect the value of the function	12

Discussion

These preliminary results confirm the primary conceptions of a definite integral from the literature. They also offer a new result, categories for the conceptualization of the accumulation function. The data indicate that the majority of the students in our study evoked conceptualizations of integration that fail to emphasize the underlying structure of the definite integral as a limit, a summation, and a product. Such conceptualizations could be an indication of pseudo-structural thinking (Sfard & Linchevski, 1994) among the students in our study. It is interesting that, although a majority of the students responded to question #1 in terms of the area underneath the function, very few students used area as a means to reason about the roles of x and t in question #2, opting instead to reference the symbolic process of calculating a definite integral using the fundamental theorem of calculus. This is likely due to the increased complexity when moving from an integral with constant limits of integration to an integral with variable limits of integration. This requires the integral to be treated as an object within the function process defining the accumulation function, thus requiring a reified conceptualization of the definite integral. It is also likely that students' evoked images of the definite integral and accumulation functions are highly influenced by their experiences with each concept in the classroom. The impact of which interpretation is available for a student has been noted (e.g. Jones, 2015), but would benefit from further study.

The data for this report is situated within a larger project exploring student learning in multivariable calculus. In particular, the authors are interested in exploring the implications of these results for teaching and learning multivariable integration. For these reason, we have chosen to focus our audience questions on the implications of these results for teaching and learning in multivariable calculus.

Questions for discussion:

- How do these results impact instruction and learning in multivariable calculus?
- What effect would you expect multivariable calculus instruction to have on student responses to a post-test?
- What types of multivariable calculus experiences would most likely influence students' evoked images of integration?

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Support for mathematicians' teaching reform in an online working group for inquiry oriented differential equations

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There is more need for research on how mathematicians can alter their teaching style to a reform approach (Speer, Smith, & Horvath, 2010), especially if they have always been teaching the same way (Speer & Wagner, 2009; Wagner, Speer, & Rossa, 2007). One particular area that needs more work is investigations of support structures for mathematicians hoping to reform their teaching practice. This poster focuses on supports designed to aid in the reform of teaching practice and specifically discusses the Teaching Inquiry-oriented Mathematics: Establishing Supports (TIMES) project and one online working group (OWG) used as a mode of support in the project. Results indicate that facets of the OWG are successful support structures for mathematicians who desire to align their practice to an inquiry oriented (IO) approach to undergraduate differential equations (Rasmussen & Kwon, 2007; Rasmussen, 2003).

Key words: inquiry oriented, differential equations, support, instructional reform

There is clear evidence of the effectiveness of the inquiry oriented differential equations (IODE) curriculum and materials (Kwon, Rasmussen, & Allen, 2005; Rasmussen, Kwon, Allen, Marrongelle, & Burtch, 2006) and a plethora of research on the enactment of that curricula (e.g., Keene, Lee, & Lee, 2011; Rasmussen & Marrongelle, 2006; Rasmussen, Stephan, & Allen, 2004; Stephan & Rasmussen, 2002). However, just because reform curricula have been proven to be effective in certain contexts does not mean that mathematics instructors will be able to effectively implement the reform curricula unless they have support to do so (Speer et al., 2010; Speer & Wagner, 2009; Wagner et al., 2007).

The TIMES project is currently supporting the development and refinement of a set of instructional supports to aid university mathematics faculty in shifting towards an inquiry-oriented (IO) practice. Important to the project is not just that the mathematics instructors use the IO materials, but that they receive multiple forms of contact with the lead researchers to aid in that desired shift of practice. The form of contact which this poster focuses on, is a weekly online working group (OWG) which uses lesson study (Lewis, Perry, & Hurd, 2009) that are composed of four week-long segments. In two of those weeks, instructors seek help from their fellow instructors and from the facilitators to improve their teaching practices, and they use video clips from their own classrooms to facilitate that discussion. I observed one OWG in which the instructors were sharing videos of themselves, and particular attention was paid to the types of questions that the instructors asked their fellow instructors. Specifically, this report addresses the following research question: *What types of questions do mathematicians ask in an online working group for inquiry oriented differential equations and how do these questions relate to their perception of support from the online working group?*

Methodology

This report focuses on three of the participants currently in the IODE OWG. All three come from small liberal arts colleges/universities across the country and are all teaching IODE for the first time. One form of data came from interviews with the three participants (audio-recorded and self-transcribed). Additionally, I observed and took live field notes of one OWG where the instructors were sharing videos of their teaching of the systems of differential equations unit. To analyze the interview data, I began with an initial set of codes (i.e., content, pedagogical, logistical, and advice questions) that would be useful in answering the research question. Then I open coded transcripts to discover emergent themes. Lastly, I compared the field notes from the OWG observation to the interview transcripts to confirm instructors' claims of their dominant question type in the OWG.

Preliminary Findings and Conclusions

Results from the analysis highlighted several components of support afforded by the OWG. That support is directly correlated to the types of questions that instructors asked in the OWG. Because all three instructors are from small colleges/universities, being part of a collaboration with fellow instructors is unique for these participants. They can collaborate about innovate approaches to their teaching without being tied down by departmental regulations. Participant A remarked about the importance of honest feedback from her fellow instructors,

> I think that the fellow instructors tend to give honest feedback. And so since I am in such a small department here, it is really helpful to have honest feedback from people who are not involved in any departmental politics.

Furthermore, the instructors typically are not afforded opportunities to reflect on their teaching practice in their work environment simply because there are no support structures that exist within these participants' departments specifically focused on allowing them to reflect on their teaching practice. Participant B noted,

> And so where I think I can get the most benefit out of this of this kind of experience is when I can reflect with others on how that went so that I can see where I need to be anticipating or what I need to be thinking about more in that chaotic moment when I am processing the whole class.

Similarly, Participant C stated how he receives, "an on the spot, real time, type of feedback" from the OWG. Further, Participant B mentioned he does not observe other teachers teach in their classroom and has no one observe him in his classroom. This, however, does happen in the OWG when they all watch videos of their fellow instructors teaching in the lesson study.

Every instructor noted how the OWG allows them to ask questions to their fellow instructors about their teaching practice, past or future. Thus, questions that emerge in the OWG are used by the instructors to address the concerns that they have about teaching practice, which are not addressed by colleagues at their own colleges/universities. However, at the root of all instructors' questions type choices are notions of structures that are missing from their practice, yet are inherent components of the OWG. Ultimately in answer to the research question,

> *The OWG allows instructors to reflect on their teaching practice because they can ask for advice and feedback from the fellow instructors in a safe and collaborative environment to improve their implementation of the IODE material and their holistic teaching practice, which does not happen in their normal work atmospheres.*

These results are from an early investigation of support structures for mathematicians hoping to reform their teaching practice. They show promise for additional and exciting research on support structures for mathematician's instructional reform. The implications from this work add knowledge to the field of undergraduate mathematics education and instructor professional development and highlight the power of OWGs in mathematicians' instructional reform.

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Students' conceptualizations and representations of how two quantities change together

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In this article I discuss the nature of two university precalculus students' meanings for functions and graphs. I focus on the ways in which these meanings influence how these students reasoned about and represented how two quantities change together. My analysis revealed that a student who views a graph as a static shape and does not see a graph as a representation of how two quantities change together will not be successful in constructing meaningful graphs, even in instances when she is able to reason about two quantities changing together. Students made progress in seeing graphs as emergent representations of how two quantities change together when they conceptualized the point (*x,y*) *as a multiplicative object that represented the relationship between an x and y value.*

Key words: Function; Covariational Reasoning; Graphing

There is a growing body of research that documents the importance of covariational reasoning, imagining quantities' values varying together, when conceptualizing rates (Johnson, 2015; Thompson, 1994a; Thompson & Thompson, 1992), the behavior of exponential and trigonometric functions (Castillo-Garsow, 2010; Moore, 2010; Thompson, 1994c), and graphs (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Moore & Thompson, 2015). After high school, reasoning about variation is essential to understand calculus (Thompson, 1994b; Zandieh, 2000), differential equations (Rasmussen, 2001), and continuous functions (Roh & Lee, 2011). While the research community understands the importance of covariational reasoning, researchers do not yet understand how students come to reason covariationally.

Carlson et al. (2002) developed a framework to classify student's covariational thinking but they did not describe how students might develop these ways of thinking. Whitmire (2014) studied how undergraduate students solve a task whose solution necessitated covariational reasoning. He found that static shape thinking, reasoning about a graph based on one's perceptions of the shape of the graph (Moore & Thompson, 2015), stood as a distraction from covariational thinking. In contrast, he found that simultaneously attending to two quantities was associated with instances of covariational reasoning. In this report I extend Whitmire's findings by elaborating the ways static shape thinking inhibits reasoning covariationally and I provide evidence that simultaneously attending to two quantities' values is propitious for engaging in covariational reasoning and representing how two quantities' values change together.

Images of Variation

To engage in covariational reasoning, one must construct an image of how two quantities' values change together. As Saldanha and Thompson (1998) described, this requires the student construct a multiplicative object and conceptualize the two quantities' values at once. Then the student "tracks either quantity with the immediate, explicit, and persistent realization that, at every moment the other quantity also has a value" (Saldanha & Thompson, 1998, p. 2). For example, in the context of graphing, the student must construct the point (*x, y*) as a multiplicative object that simultaneously represents both the value of x and y. Then the student can track (or imagine tracking) the value of x with the awareness that as x varies, y varies as well.

Castillo-Garsow (2010) described two ways for students to imagine tracking the value of x: (1) the student imagines the tracking already happened and conceptualizes a completed change in the value of x – a chunky image of variation or (2) the student imagines change in progress and conceptualizes sweeping over a continuum of x-values – a smooth image of variation. Lakoff and Núñez (2000) argued that conceptualizing sweeping over a continuum involves *fictive motion* – using a motion verb when the subject is not actually moving. For example, in the phrase "the value of x goes from 1 to 4" the value of x is not moving but we talk as if it is. Fictive motion enables one to go between static and dynamic conceptualizations of the value of x.

Tracking a quantity's value is a nontrivial activity for students. This necessitates that the student conceptualize a quantity, the measure of that quantity, and that measure varying in a situation. If the student does not construct this image the student is said to have no image of variation.

Methodology

I conducted one-on-one task-based interviews with three university precalculus students, Sara, Carly, and Vince. The students were selected from three different sections of precalculus to account for differences in instruction. The interview consisted of two phases. The first phase was a clinical interview (Clement, 2000; Hunting, 1997). I engaged the students in tasks I anticipated would support me in understanding their meanings of functions, tabular representations, and graphical representations. The second phase of the interview was a task-based-teaching interview (Castillo-Garsow, 2010; Moore, 2010). My primary teaching goal was to support students in conceptualizing a graph as an emergent representation of how two quantities change together. In this part of the interview I used dynamic animations to support students in conceptualizing *change in progress*. I anticipated this would support students in imagining sweeping over a continuum of values thus constructing an image of smooth variation.

Results

After I completed the interview process I engaged in grounded coding (Strauss & Corbin, 1998). After each interview I engaged in open coding and gathered evidence of how students conceptualized functions, graphs, and variation. After reviewing videos and transcripts of each interview I found that while all three students were able to describe how two quantities changed together, not all students were able to represent their conceptualization of how two quantities changed together. Since Sara and Carly exhibited similar ways of thinking, I will focus on contrasting Sara and Vince's ways of thinking.

The Story of Sara

Over the course of two interviews, Sara consistently used shape thinking and memorized procedures to make sense of problem situations. For example, at the beginning of my interview with Sara I asked her to explain what it means for something to be a function. She responded by describing a procedure to compute the change in the value of y and divide it by the change in the value of x. She did not discuss the meaning for these calculations nor did she explain the meaning of the result of dividing.[1] Her responses throughout the interview suggest that she was unable to view graphs or other function representations as a means of representing how two varying quantities change in tandem. Data to support this claim follows in the next section.

Sara associated graphs with shapes she had previously seen in math class

I presented Sara with a graph that appeared to show two vertical lines, one at $x = 1$ and the other at $x = -1$ (Figure 1a). I asked her to determine whether the graph represented a function. She gestured that it was like an upside down 'u' shape and explained, "only you can't see the top". She concluded that the graph represented a function since the graph was like one she had seen in class. She matched the shape with one she had seen before and justified her response based on her perception of the graph's shape. This suggests that for Sara, graphs represent shapes as opposed representations of how quantities' values vary together.

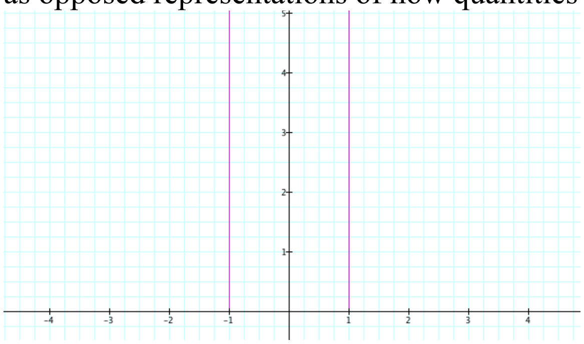

Figure 1a: Sara was asked to determine whether this relationship is a function.

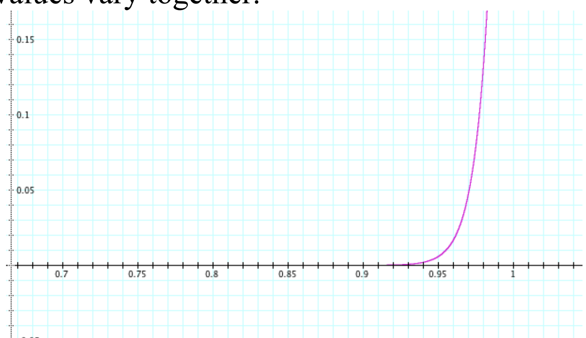

Figure 1b: Graph displayed to Sara after she anticipated the behavior around at $x = 1$.

I asked Sara to anticipate what we would see if we zoomed in on the graph around the point (1, 0). She described that the line would come down perpendicular to the horizontal axis but just stop and not go below the horizontal axis. Then I highlighted a region on the graph around $x = 1$ and zoomed in on this region (Figure 1b). I asked Sara if she believed what she saw and she said,

> "No. (Laughs) I mean I haven't seen it before and I just feel like a graph would look weird if it is like going down and then curving also. If like. Especially if there is a top to graph, which I don't know if there is or not now because I am second-guessing myself. But. (4 seconds of silence) Or, there is and there are asymptotes there. There wouldn't be a top. So that could work if there was like two asymptotes."

She proceeded to sketch of her new understanding of the graph (Figure 2)

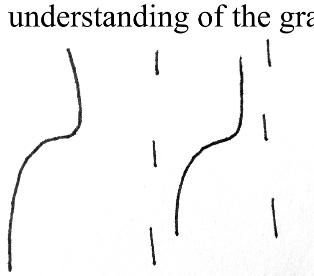

Figure 2. Sara's graph after she conceptualized graph as a shape with asymptotes.

[1] It is noteworthy that Sara had recently learned a method for determining the average rate of change of a function on an interval of a function's domain.

Sara approached this task by trying to think of a graph she had seen before that matched her conception of the situation. When her prediction did not match the graph I displayed, she used the new information displayed in the graph to develop a new possibility for the shape of the graph – a graph with asymptotes. Instead of conceptualizing what the zoomed in graph told her about how x and y varied together, Sara concluded that the graph would look weird "going down and curving". This provides further evidence that Sara engaged in static shape thinking when making sense of graphical representations.

Implications of Sara's tendency to engage in static shape thinking

The last task in the interview was based on an item from a diagnostic instrument for professional development (Thompson, 2011). I presented Sara with an animation where the value of Quantity A was represented with a red bar along the horizontal axis and the value of Quantity B was represented with a blue bar along the vertical axis. As the animation played the lengths of the red and blue bars changed together to describe how the two quantities change together. I presented Sara with three versions of this task.

While I intended this to be a novel task, Sara had done a similar task in her precalculus class. Sara said she was bad at these types of problems but her ability complete the first version of this task with ease suggests that Sara learned a strategy to complete these problems. Her way of thinking broke down in the second version of the task which represented the behavior of $y = \sin(x) : -2\pi < x < 2\pi$. As Sara watched the animation, she appropriately described, "As x was increasing at the beginning y was decreasing. But as x comes closer to 0 y also approaches 0 and they both increase for a little bit and as y keeps increasing or as x keeps increasing y starts to decrease." This suggests that Sara was imagining change in progress. Although Sara was able to describe the how the quantities' changed together, she struggled to represent this graphically. In the following excerpt Sara explained her approach to constructing a graph from the animation.

Sara: In my head I like know like as that one is increasing you have to like. I try and like think of the shape of the line or the point or whatever to get to the line. Or yeah.
Interviewer: What do you mean you think of the shape to get to the line?
Sara: So like for this like I have to see how like. Since that [value of x] is like increasing (gestures left to right) and that [value of y] is decreasing (gestures up and down) like what I am thinking in my head. Like I am like trying to figure out which way it needs to go.
Interviewer: Which way what needs to go? The graph?
Sara: Yeah. So. I don't know. That is why it takes me so long when I am just staring at the graphs.

Sara appeared to abandon her thinking about changing quantities when constructing a graph. Instead of conceptualizing the graph as a trace of how the value of x and y change together, she broke the graph up into chunks based on whether the value of y increased or decreased. Then she determined a shape that depicted the appropriate behavior of y as x increased. For example, if the value of y increased as the value of x increased then she knew the graph had to go up and to the right. While Sara was able to appropriately describe how the values of x and y changed together, her tendency to engage in static shape thinking prevented her from leveraging her reasoning about how the two quantities were changing together to construct a meaningful graph.

The Story of Vince

At the beginning of my interview with Vince I asked him to construct a graph from a table of values. He explained that since the table was a function the graph would be a smooth line and he could sketch an approximation but would need the function, “the y equals something”, to determine the exact graph. He elaborated that he needed the formula so he could plug in all of the x values, determine the associated y values, and plot all of the resulting (x, y) pairs. This understanding of graphs enabled Vince to draw smooth curves, “a bunch of dots put together that now looks like a line to me.” This suggests that Vince conceptualized graphs as collections of points where a point simultaneously represented an x and y value.

While there are limitations to this understanding of functions and graphs, Vince was able to complete all tasks in the interview with this way of thinking. For example, since Vince conceptualized graphs as a collection of points, whenever he attended to two points he also addressed the many points in-between. Specifically, when sketching a graph from a table of values Vince acknowledged that “anything could happen between the given points”. Thus, even though Vince often attended to individual points Vince was likely imagining the quantity’s value varying. This image of variation seemed to enable Vince to conceptualize sweeping over a continuum of x values with the awareness that he can construct and (x, y) pair at every value of x. This allowed Vince to imagine that “anything could happen” between two values of x.

Leveraging one’s image of a correspondence point to construct meaningful graphs

Vince’s point-wise meaning for functions and graphs broke down at the end of his interview. The last task Vince completed was the same task that Sara completed (described above). I presented Vince with and animation where the value of Quantity A was represented with a red bar along the horizontal axis and the value of Quantity B was represented with a blue bar along the vertical axis. As the animation played the lengths of the red and blue bars changed together to describe how the two quantities change together (Thompson, 2011). I presented Vince with three versions of this task and each time asked him to sketch a graph of how the two quantities changed together. The following exchange occurred at the beginning of this task. (The animation he was watching during this exchange represented the behavior of $y = 0.2x^3$: $-12 < x < 12$.)

> **Interviewer**: So a little bit different. I have a red line that represents that value of x and the blue line represents the value of y. These two values change together. (Animation plays). Well the question is how do they change together? What if I wanted a graph that showed how these two quantities change together?
> **Vince**: So are you asking for like. Like um. The intersection of the two?
> **Interviewer**: So x is changing and y is changing. Suppose you have a friend in Australia and you can’t send them videos. And for some reason you need to tell your friend how x and y are changing – you need to tell him what is going on in this video. So what are you going to do? Anything you could convey in snail mail.
> **Vince**: Um. I would probably do. I don’t know. (Gestures two vertical lines). How they are changing? You are no just looking for? I mean I would probably. Hm. If I want him to see the graph. What I am imagining what is happening is there are points. Like you intersect the two and there would be a line – a series of dots. (Gestures smooth curves)
> **Interviewer**: What would that line look like?
> **Vince**: (Gestures curved shape and then draws an appropriate graph).

After three minutes of puzzling about the task, Vince was able to successfully represent the behavior in the animation in the plane. He described a point that he imagined as the "intersection" of the red and the blue line segments and he described keeping track of this intersection as the animation played (Figure 3). Although Vince described this approach at the very beginning of the excerpt, it took more than three minutes of reasoning for him to believe that this would represent how the values of x and y changed together.

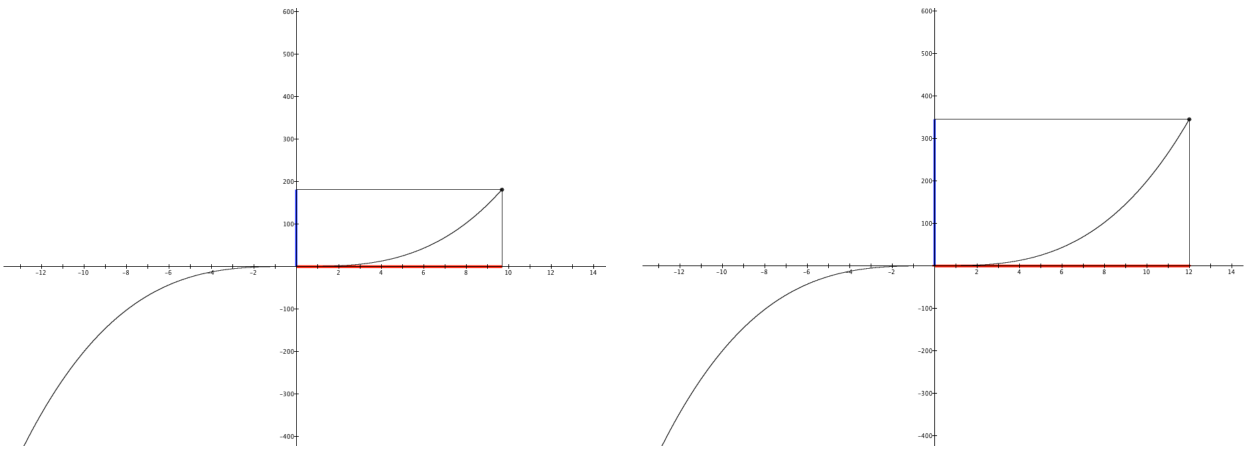

Figure 3: Researcher's representation of the imagery Vince described while constructing a graph to represent the behavior of two continuously varying quantities.

After Vince constructed his graph, I asked him to explain his approach. He began by describing the changes he was conceptualizing:

> "The red is changing at a pretty consistent speed. The blue when it is a lot further down moves a lot quicker then it goes slower and then it starts moving quicker again. So like as. So that is why I thought of this (points to graph in Quadrant 4) because we move up here, you get less bang for your buck. As x increases by one the increase of y becomes less and less."

Vince operationalized his image of the red line changing at a consistent speed by conceptualizing equal changes in the value of x. This gave him a means to think about how the blue line was changing: for a consistent change in x the corresponding change in the value of y was getting smaller (in magnitude). He was able to use this imagery to confirm his graph appropriately represented the behavior in the animation.

Throughout Vince's interview he exhibited two different tendencies. The first was to describe a continuum of values by attending to all the points in-between two given points and constructing graphs by tracking a correspondence point. His other tendency was to focus on points, numerical values, and calculated changes in the value of a quantity. One possible explanation for these different ways of thinking is to juxtapose Vince's daily experiences with his mathematical experiences. In his day-to-day life, Vince engages with and represents continuous motion. However, in his mathematical experiences he focuses on numerical values, points, and calculated changes in the value of x and y. As a result of his daily experiences he developed an ability to use fictive motion to construct dynamic conceptualizations of otherwise static objects. He engaged his understanding of fictive motion when he first completed this final task. However, he had little to no experience coordinating fictive motion with mathematics. Thus, he used his understandings from classroom experiences to justify the graph he created.

Vince was able to successfully complete all the tasks in the interview, including representing the relationship between two continuously varying quantities. He was successful because he consistently imagined a quantity's value varying over some continuum and he conceptualized a point as a multiplicative object that simultaneously represented an x value and associated y value.

Conclusion

Vince consistently described graphs as a collection of points really close together. Although Vince experienced some cognitive conflict when trying to graphically represent two continuously changing quantities, he was ultimately successful because (1) he imagined a continuum of x-values, and (2) he had conceptualized a point on the graph as a multiplicative object that simultaneously represented an x value and associated y value.

Sara, on the other hand, constructed meanings for function and graphical representations based on memorized procedures and shapes. While Sarah's conception of a graph as a shape led to many seemingly inconsistent answers throughout the interview, static shape thinking did not prevent Sara from conceptualizing a quantity's value varying continuously and describing how two quantities changed together. Static shape thinking did impact Sara's ability to construct a graph as an emergent representation of how two covarying quantities change together. This suggests that conceptualizing a graph as an emergent trace of two quantities' values requires more than imagining smooth variation and conceptualizing how two quantities change together.

This finding is consistent with Moore and Thompson's (2015) explanation of emergent shape thinking. They explain, "Emergent shape thinking involves understanding a graph simultaneously as what is made (trace) and how it is made (covariation)" (p. 4). This suggests that conceiving covariation is only one aspect of understanding a graph as an emergent representation of how two quantities change together. Vince's ability to conceptualize a graph as an emergent representation suggests that the other aspect of emergent shape thinking entails constructing multiplicative objects. First the student must construct the point (x,y) as a multiplicative object that unites the value of x and the value of y. Then the student must construct the graph as a multiplicative object that unites a way of representing the spatial movement of the multiplicative object with a conception of of covarying values of two quantities.

This study suggests that there are different ways to engage in covariational reasoning. The multiplicative object the student constructs determines the nature of his covariational reasoning. For example, one must construct the point (x,y) as a multiplicative object in order to conceptualize a graph as an emergent trace of how two quantities change together. However, Sara's interview provides evidence that this construction is not necessary in order to describe how two quantities change together. Future studies should explore how to support students in conceptualizing both the point (x, y) and the graph as multiplicative objects. Students have developed robust coping mechanisms that enable them to understand and complete novel tasks using their existing ways of thinking. Thus, educators will need to use unconventional representations, such as leveraging fictive motion, to support students in developing more robust ways of engaging in covariational reasoning and new meanings for graphical representations.

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Results from a national survey of abstract algebra instructors: Math ed is solving problems they don't have

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Temple University	Virginia Tech	Virginia Tech

There is significant interest from policy boards and funding agencies to change students' experiences in undergraduate mathematics classes. Abstract algebra specifically has been the subject of reform initiatives, including new curricula and pedagogies, since at least the 1960s; yet there is little evidence about whether these change initiatives have proven successful. Pursuant to answering this question, we conducted a survey of abstract algebra instructors to generally investigate typical practices, and more specifically, their knowledge, goals, and orientations towards teaching and learning. On average, moderate levels of satisfaction were reported with regard to the course itself or student outcomes; moreover, little interest in, or knowledge of, reform practices or curricula were identified. We found that 77% of respondents spend the majority of class time lecturing – not surprising when considering 82% reported the belief that lecture is the most effective way to teach.

Keywords: abstract algebra, instruction, reform, lecture,

Teaching matters. It is the single most important factor in terms of what students might be able to learn from a class and what they can't learn from a class. Teaching matters because it affects how students understand their roles in the class, what it means to learn and understand the material, and the ways that students come to understand the content, and almost certainly what kind and how much students put into mastering the material. Students know this. In a time when retention of STEM majors could not be more critical for our nation, fewer than 40% of students entering college in pursuit of a STEM degree complete that degree (PCAST, p. i) citing ineffective teaching methods and uninspiring atmospheres in introductory-level STEM courses as the primary reason for attrition (PCAST, p. 5).

Mathematics, like other STEM majors, is not immune to the retention issue: even as the number of entering freshman declaring mathematics as a major in increasing, the number completing the major is constant (Kirkland, 2013); however, unlike other STEM majors, mathematics must be acutely aware of the effects of poor teaching in introductory-level courses because these courses are required for a myriad of disciplines and often act as a gateway to STEM careers. Mathematics courses, without the siren song of labs and experiments beckoning, historically have resorted to the use of lecture-style presentation in disproportionate numbers relative to other STEM majors despite mounting evidence contradicting its effectiveness.

Background and Literature

Lecture-based pedagogy has been labeled problematic for undergraduate learning, persistence, and success; instead, researchers recommend pedagogical reforms that are more reflective of how people learn and better reflect the nature of doing mathematics (Kyle, 1997; National Academy of Science, 2007; National Research Council, 1996; National Science

Foundation, 1992, 1996). Critics who do not wish to see the lecture vilified will argue that it is the students who are to blame, for they do not understand the pedagogical contract, they can't comprehend the intellectual difficulty of the work, and they have the inability to even pay attention to the correct things in a lecture (Burgan, 2006; Wu, 1999). Although these are valid concerns, the research community is fairly resolute in the position that diversifying teaching methods enhances critical thinking skills, long-term retention of information, and subsidiarily, retention of STEM majors (PCAST, p.9 – multiple resources cited).

There is mounting evidence to believe that mathematicians are not only aware of reform practices and goals, but that they do, or at least would consider, using them. There have been numerous articles published in the journals of the AMS (American Mathematical Society) about reforming teaching (c.f., Leron & Dubinsky, 1995; Halmos, Moise, & Piranian, 1975; Jones, 1977). Thanks to outreach efforts at the Joint Mathematics Meetings by proponents of the Moore Method (Copping, Mahavier, May, & Parker, 2009), there is reason to believe that its basic precepts are well-known. Calculus reform specifically has been very extensive with reform activities being supported by commercial publishers, discussed in the *American Mathematical Monthly* (c.f., Kaput, 1997; Ostebee & Zorn, 1997), and examined in session at the Joint Mathematics meetings. What is certainly true is that the National Science Foundation has spent a large amount of money, and mathematicians and mathematics education researchers have spent a large amount of time, designing new curricula. On a smaller scale, many instructors have developed their own materials, some via participation in Project NExT, the Academy of Inquiry-Based Learning, or Moore-Method conferences.

In terms of mathematicians, national professional organizations (e.g. the MAA), and mathematics education researchers, it is quite possible that no other upper-division course has gotten anywhere near a comparable amount of attention in terms of reform initiatives as undergraduate abstract algebra (e.g., Dubinsky & Leron, 1994; BLINDED; Hibbard & Maycock, 2002). Almost exclusively, these initiatives have concentrated their efforts into changing the undergraduate abstract algebra experience; namely, with more doing of mathematics during class. Yet, we believe that despite this single-mindedness, these efforts have had little to no effect on most students' experience of the abstract algebra course. This suggests that the field might have misplaced beliefs about what change is possible, or more importantly, that we are missing or misunderstanding something fundamental about the class, instructors, or instructors' beliefs about the class, students, and learning.

Many theories have been posited about why new curricular practices have not been adopted. Coverage concerns seem to be paramount. There is evidence that faculty feel a significant tension between the breadth of required topics and the ability to focus on teaching and learning through problems, subsequently driving instructors to resort to more expeditious lecture approaches (e.g., Roth-McDuffie & Graeber, 2003, p.335). Other commonly cited barriers included: the demands of the position not allowing for innovation, lack of support from colleagues or supervisors, and a lack of common vision for reform (Roth-McDuffie & Graeber, 2003; Henderson & Dancy, 2007). While these studies do offer some reasons why mathematicians might not change their instructional practices, the results have limited applicability because the participants were neither mathematicians (Henderson & Dancy, 2007), nor instructors (Speer, 2008), or were not teaching abstract algebra (Roth-McDuffie & Graeber, 2003).

Theoretical Framework

The fact is that there is essentially no research that helps researchers and policy makers understand why some mathematicians adopt reform practices in their teaching and some do not (Speer et al., 2010). Maybe the goal of the funders and policy boards is inappropriate; alternatively, maybe the goal is good but there are no meaningful avenues for change. There has been little research attempting to explore these issues from the perspective of the instructors who are the ones being asked to change practice; consequently, we believe there is a considerable need for more investigation into university mathematicians' beliefs, knowledge, and goals about the teaching of abstract algebra. The present report is based upon a survey of abstract algebra instructors to examine typical practices in general, and more specifically, orientations towards teaching and learning. We investigate the following research questions: (1) What kinds of pedagogical practices do abstract algebra professors report using in their classrooms and why? (2) What affordances and constraints on their use of non-lecture practices do they perceive?

We designed our inquiry and analyzed our results through the lens of Schoenfeld's with Schoenfeld's (1999) framework of knowledge (resources), goals, and orientations. This framework, identified is useful for analyzing long-term decision making, supports the theory that mathematics instructors' "*thinking, judgments, and decision-making* as they prepare for and teach their class sessions" are important and shape their instruction (Speer, et al., 2010, p. 101).

Methods and Data Analysis

To create an instrument designed to measure the knowledge, goals, and teaching/learning orientations of mathematicians, we adapted questions from both Henderson and Dancy's physics-education survey (Henderson & Dancy, 2009) and Characteristics of Successful Programs in College Calculus survey (see surveys at www.maa.org/cspcc). In addition to basic demographic information, the survey questions asked the professors to rate the importance of various sources of information and to list factors that influenced their teaching decisions. In an attempt to elicit their beliefs about teaching and learning, we asked them to describe and characterize their classroom practices, including the motivation behind those choices. Finally, we asked questions to test claims from the literature about why undergraduate mathematics instructors were resistant to changing their pedagogical practices.

Requests for participation in our online survey were sent to departmental administrators at approximately 200 institutions, targeting instructors who teach undergraduate abstract algebra. We had 131 completed surveys (initial response rate of ~30%). In general, the respondents (92% tenure-stream faculty) had significant experience, both with teaching in general and abstract algebra specifically, and were most likely to be teaching an undergraduate groups-first course designed for a mixed audience. (See Figure 1.)

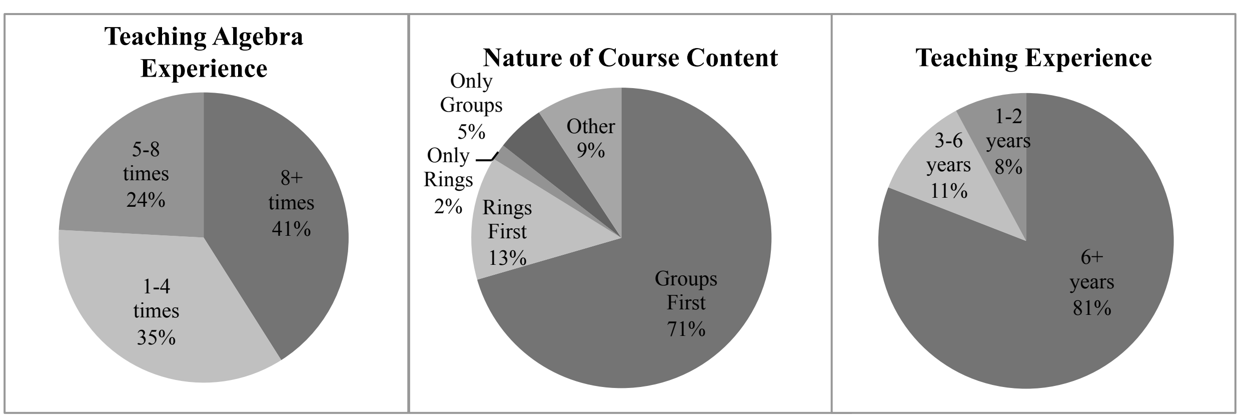

Figure 1. Information about Survey Respondents

To analyze the data, we first calculated basic descriptive statistics appropriate for each item. After compiling the demographic information, we focused our attention on instructor satisfaction in order to determine if any impetus for change existed. To address the first research question, we examined the self-reported teaching practices of the respondents and compared that to both level of satisfaction and extent of agreement with the Likert-scale belief statements designed to measure teaching/learning orientations. In our discussion, we highlight areas where the respondents appear to hold beliefs that should lead to certain pedagogical actions but who do not report engaging in those actions. To address the second research question, we categorized instructor reports on constraints and affordances to implementation of non-lecture reform practices, and we compared these with those cited in the literature. In each case, we have attempted to align these with Schoenfeld's (1999) framework of knowledge (resources), goals, and orientations.

Results

Satisfaction

When measuring satisfaction, several dimensions were considered. For this report, we choose to discuss two in particular: textbook and student learning outcomes. Of all the factors contributing to abstract algebra professor's overall levels of satisfaction, the aspect with the greatest percentage (87.6%) of satisfied or very satisfied respondents was the textbook. Instructor comments indicated that the satisfactory rating stemmed from the breadth, depth, and sequencing of content. It is important to note however, that even amongst the satisfied, complaints about pricing and frequency of new editions was rampant.

When reporting on satisfaction with student learning outcomes, approximately half of the classified responses (a number gave responses that we could not reliably categorize) reported being satisfied (44), with the remainder being evenly split between very satisfied (23) and dissatisfied (22). The responses were organized by domain and level of satisfaction, allowing us to look for common themes. Figure 2 shows a matrix illustrating typical comments.

	Very Satisfied	**Moderately Satisfied**	**Dissatisfied**
Student Engagement	▪ My students work hard. ▪ My students ask a lot of questions. ▪ My students put time in outside of class ▪ My students are excited to see how this course fits with past/future coursework	▪ The students who want to learn put in the time and do well ▪ My students generally work hard enough to get through the course but I wish they were more motivated to learn ▪ My students demonstrate infrequent or inconsistent participation in class	▪ My students don't appreciate the material ▪ My students don't do work outside of class ▪ My students are not interested in math ▪ My students view the course as irrelevant to their careers ▪ My students don't participate in class
Student Preparation	▪ My students are very well-prepared ▪ My students have a working understanding of prerequisite material and understand how to construct proofs ▪ My students' preparation is sufficient to be successful in my class	▪ My students' preparation varies by background and major ▪ Most of my students have weak proof backgrounds but develop this over the course ▪ Most of my students have insufficient prior knowledge relative to what I would like, but with the right work ethic can be successful in my class	▪ My students are unprepared to take this course ▪ My students lack proof skills ▪ My students have poor general math skills ▪ My students' insufficient preparation and ability hinders their ability to be successful in my class

Student Performance	▪ My students get good grades on exams ▪ My students produce high quality projects ▪ My students submit carefully considered homework assignments ▪ Very few of my students fail the course	▪ My students get decent grades on exams, but not as good as I would like ▪ My students produce mediocre projects ▪ My students submit homework that is often inadequate, incomplete, or rely on help to finish it satisfactorily ▪ I often have as many D/F/W grades as I do A/B/C	▪ My students do poorly on exams, without a curve, the majority would not pass ▪ My students produce poor projects or are incapable of completing them altogether ▪ My students don't/can't do homework or need extensive help to do so ▪ A large portion of my students fail or withdraw
Student Understanding	▪ My students are capable of coauthoring journal articles with faculty ▪ My students leave my class prepared for future advanced coursework and often get accepted to reputable grad school programs ▪ My students demonstrate algebraic reasoning and mathematical maturity	▪ My students leave my class adequately prepared for future coursework, but not necessarily grad school ready ▪ My students don't grasp all the subtleties, but come away with a level of understanding suitable for their backgrounds, abilities, and future plans ▪ My students have a working understanding of fundamental concepts and can usually make definitions, sort conjectures, and build useful examples	▪ My students master only a small fraction of the topics covered ▪ My students don't come away with a real understanding of the material ▪ My students leave without really getting the point ▪ My students are generally unprepared for future coursework
Curriculum Issues	▪ My curriculum covers lots of presently relevant examples from applications in diverse fields (physics, chemistry, math, etc) ▪ My curriculum requires that students work on finding proofs for themselves and this approach has been successful in generating student growth. ▪ Having the students work in small groups instead of traditional lectures has proven successful ▪ My curriculum gives the students the right taste of modern math and supplies them with the right language to be successful ▪ My curriculum has struck a successful balance between abstraction and computational topics to keep all students engaged	▪ My curriculum is ok but could benefit from extended motivation for topics and guided self-discovery ▪ My curriculum is ok for math majors but does not adequately serve the pre-service teacher population ▪ I am satisfied that they get a good introduction to group theory but would like to go deeper into the subject and have the students formulate and explore conjectures on their own ▪ I consider my course 'algebra appreciation' rather than a careful, complete introduction for those who should master the material	▪ My curriculum is out of date ▪ My curriculum is divorced from the true motivations and applications of algebra ▪ My curriculum materials are lacking and I often have to supplement with worksheets/handouts ▪ I spend too much time teaching how to write proofs and not enough time on algebra topics

Figure 2. Satisfaction Matrix

In summary, instructors who were moderately satisfied indicated (unsurprisingly) that students learned most of the important content and worked reasonably hard. The courses might be in need of a little reorganization or supplemental materials, but major pedagogical overhauls were not considered warranted or desired. The comments of the instructors who were dissatisfied were complaints about the unsatisfactory work ethic, motivation, and ability of the students. Instructors who reported high levels of satisfaction were the most likely to comment on the format and curriculum of their courses, with approximately half of them indicating belief that their course was different than most traditional abstract algebra courses due to the use of some form of inquiry-based learning (increased use of examples, student research, Modified Moore Method, etc.).

While the groups did vary widely in typical responses, it was interesting to note that there were two common themes that emerged across all levels of satisfaction. The first observation was a general frustration with students' lack of prerequisite proof skills and poor proof-writing ability. The other common opinion was that it was both difficult and inappropriate to design and teach a course for different constituencies (most often cited was the comingling of Math and Math Education majors). Due to different backgrounds, abilities, and occupational goals, the consensus was that neither population was being adequately served by teaching them simultaneously. However, even with this mixed sense of satisfaction with student learning outcome, we were surprised to find that, for the instructors completing the survey, the passing rates were quite high with the average grade break down being: A 33.37%, B 33.85%, C 20.55%, and D/F/W 12.18%.

Teaching methods

Lecture was the most common pedagogical practice with 77% of respondents claiming that they currently lecture to teach abstract algebra, 15% of respondents currently teach in some other way, and 8% used to do something different in the past but now lecture. Of the 23% who either now, or in the past, used non-lecture pedagogy and curricular materials, most (15 respondents) created it themselves without formal support (typically drawing on a mixture of texts and problem-sets). There were only two respondents who cited use of a particular established curriculum (Teaching Abstract Algebra for Understanding, Larsen, 2013; Learning Abstract Algebra with ISETL, Dubinsky & Leron, 1994). The others used their own experiences with Moore Method classes, collaboration with other Moore Method instructors, or participation in the Academy of Inquiry-Based Learning as a guide to develop their materials and shape their practice.

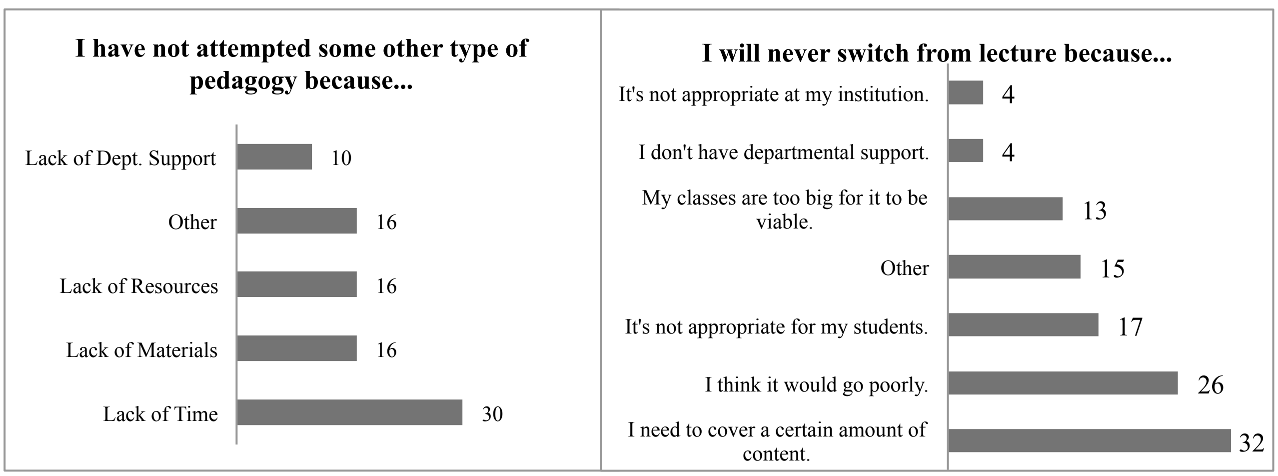

Figure 3. Perceived constraints on the use of non-lecture practices

Of the 85% who are currently teaching with lecture, 56% of them say that they would consider teaching with non-lecture practices (the remaining 44% say they would never do so). The reasons instructors provided for not yet attempting other pedagogy and the concerns mentioned explaining why they would never change their habits can be seen in Figure 3. In short, the two main themes in the comments related to the effort and support needed to revise and teach such a class and concerns about covering the appropriate amount of material. Of the 32 instructors who stated coverage as a reason to not adopt a non-lecture format, 23 of them answered “no” when asked “Do you feel pressure from your department to cover a fixed set of material in your abstract algebra course?” It appears therefore, that concerns about coverage may be more tied to an internalized goal or orientation, as opposed to an external pressure.

One of the most interesting findings was the apparent contradiction that emerged when comparing the responses to the following prompts. 82% of respondents agreed with the statement: *Lecture is the best way to teach.*; however, 56% agreed (and 26% more slightly agreed) with the statement: *I think students learn better when they do mathematical work (in addition to taking notes and attending to the lecture) in class*. This result was promising for the prospect of non-lecture class activities; yet when asked what students do in class besides take notes (given a list of options), the only things that instructors claimed that students did in class, even at a rate of once per month, was doing calculations, working with examples, or working with applications. Moreover, 63% reported that students never spent time working on mathematics problems in class. It appears that what instructors think is best for student learning (students doing mathematical work in class) is not happening with any frequency; thus, we argue that there exists a mismatch between beliefs about student learning and actual teaching practice.

Findings and Implications for Future Research

There are three primary findings that we highlight. First, that lecture is the predominant mode of instruction, and that even those who have tried other pedagogies appear to switch back to lecturing at very high rates. Moreover, given the significant amount of time, money, and energy spent developing, testing, promoting, and training mathematicians to use new curricula and pedagogies, there is almost no uptake. Those using non-traditional materials are far more likely to have developed their own materials than to have adopted NSF-supported curricula.

The second primary finding relates to the factors that influence pedagogical decisions. In decreasing order of significance, the participants reported that their experiences as a teacher and student were far and away the most significant (more than 90% agreement) influence; followed by talking to colleagues about how to teach specific content, and looking at other texts (70-90% agreement that it is a significant influence). Little importance was assigned to the normal means of learning about new teaching ideas; e.g., Project NExT, MathFest, MAA mini-courses or other workshops, or reading publications about teaching such as the MAA Notices series or PRIMUS (ranging from the single digits to about 15% indicating that it was significant). If mathematicians essentially give no weight to the traditional means of dissemination of new pedagogical ideas and techniques (and evidence of their effectiveness), reformers have little means of promoting change other than individual conversation. This alone suggests why reforming undergraduate abstract algebra instruction is difficult, especially with the currents modes of dissemination.

Finally, while faculty claim they have the ability to change their courses, the reported satisfaction levels indicate they do not have the desire to do so; furthermore, the majority of dissatisfaction stems from the students and not the course materials. Given the strong content focus and high belief in the efficacy of (and preference for) lecture, it appears that as a collective, the abstract algebra teaching faculty have little interest in adopting new pedagogical approaches at this time. Thus, we propose two concurrent research directions: first, we need to better explore the reasons that mathematicians appear to strongly believe in their current practice, the types of evidence that they hold as dispositive, and what means of dissemination of new approaches achieve meaningful penetration. Second, we argue that we need to further explore the types of changes to the practice of lecture that mathematicians would adopt. In other words, how can the RUME researchers meet the perceived needs of the abstract algebra community while taking into account what is understood as practical and feasible in the eyes of the faculty?

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An Insight from a Developmental Mathematics Workshop

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Abstract: In this report, we present data from 404 students in a developmental mathematics course at a large research university and try to better understand academic and non-academic factors that predict their success. This work is the first step in a larger project to understand when science, technology, engineering, and mathematics (STEM) intending students who begin in developmental mathematics courses are successful and continue to be successful in higher-level mathematics courses. To gain some preliminary insight, we analyze SAT and ACT mathematics scores for STEM and non-STEM majors who succeeded in our developmental mathematics course and also look at personality traits and anxiety levels in these students. Specifically, we sought to answer the following questions for STEM intending students: (i) what SAT and ACT mathematics scores correlate with success in developmental mathematics? and (ii) what other non-academic factors predict success in developmental mathematics?

Key words: Developmental Mathematics, STEM majors, Success.

Introduction and Theoretical Background

Many first-year college students are underprepared in the mathematics needed for their chosen majors and are in need of remedial education courses (Chambers, Ferlazzo, Ho, Pearson, & Radford, 2012). In addition, more than one third of all science, technology, engineering and mathematics (STEM) intending students in the U.S. enroll in mathematics remediation (Radford, Pearson, Ho, Chambers, Ferlazzo, 2012). In this study, we begin to analyze data collected from and about students planning to pursue a STEM degree who enter our university unprepared for college level mathematics. Our ultimate goal is to predict and model academic success patterns in order to intervene and support student success and promote opportunities for underprepared STEM students.

The psychology of learners in developmental mathematics classrooms is complex (Eden, Heine & Jacobs, 2013; Hembree, 1990). This research seeks to identify student characteristics, as indicated by demographic profiles, information collected through personality inventories (John, Naumann & Soto, 2008) and anxiety surveys (Alexander & Martray, 1989), that lead to success and persistence in STEM majors.

Methodology

The course that is the setting for this project is a mastery-based course requiring students to complete online modules at their own pace with specific levels of competency required before students can progress to the next chapter. Students are considered to have completed the course if they earn a 80% or better in each of seven in class exams and 70% on the final exam. Data were collected from 404 (almost 50% of total enrollment) developmental mathematics students who agreed to be part of this study. Surveys were administered to collect personality trait characteristics and to measure levels of exam anxiety (EA), course anxiety (CA), and numerical task anxiety (NA) at the beginning of the term and student progress was recorded at several points during the semester. The preliminary success data has been correlated to the personality traits and anxiety measures.

Data

Student performance data at various weeks during the semester are shown in Table 1 and SAT and ACT mathematics average scores for STEM and non-STEM intending populations who completed or did not complete are presented in Table 2.

Table 1
Student progress during weeks five, six, eleven, twelve and seventeen

Week	Number of Study Participants Completing Each of 7 Exams							
	Exam 1	Exam 2	Exam 3	Exam 4	Exam 5	Exam 6	Exam 7	Final
5	250	89	57	24	7	1	1	0
6	288	140	89	36	12	12	2	0
11	343	316	292	149	59	59	21	10
12	346	330	325	265	160	160	68	23
17	358	334	326	284	245	245	158	62

Table 2
ACT & SAT comparison between STEM and Non-STEM intending students

	Developmental Course	Average SAT Math	Average ACT Math
Non STEM (n = 320)	Completed	471.48	18.94
	Not Completed	455.14	17.94
STEM (n =84)	Completed	484.02	19.97
	Not Completed	466.9	18.95

Results

ACT mathematics score correlates significantly with stalling – defined as having passed only one exam by a given week - for almost all weeks ($p = .001$ to $p = .012$), but its low variance makes it difficult to use as a predictor. Two personality traits, extraversion and neuroticism, show signs during some weeks of having strong impact on student performance (week 6, $p = .059$ and week $11, p = .051$, respectively) but these traits are not consistent across all weeks. All three anxiety measures taken at the beginning of the semester were deemed to be significant indicators for a student stalling by week twelve ($p = .002$ for EA, $p = .039$ for CA, $p = .003$ for NA), with exam anxiety statistically significantly correlated to not completing the first exam by week eleven.

Conclusion

These developmental mathematics students will be tracked through subsequent mathematics courses and once more complete information about student performance is collected the data will be combined to determine factors that may affect performance and persistence. The ultimate goal is to develop a profile for a student that will be successful in mathematics courses and be able to persist in a STEM major. This will also allow identification of students who will struggle so that interventions can be developed and applied early in a student's academic career. The more we can understand who the students are and what makes them succeed or fail the closer we will be to devising programs and courses that will assist all students in achieving their desired goals.

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Integrating oral presentations in mathematics content courses for pre-service teachers

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In this paper we report on a study of assessment-based oral presentation tasks in a mathematics content course for pre-service teachers at a public university in the western United States. We used statistical inference to test for the significance of the observed improvement in pre-service teachers' attitudes towards using oral presentation tasks in their mathematical learning and towards teacher preparation. Our results suggest that use of oral presentation improves pre-service teachers' attitudes and beliefs towards mathematics learning. Moreover, responses to the post-presentation questionnaire provide insights on the benefits of using oral presentation tasks in mathematics courses for pre-service teachers.

Key words: Attitudes, Beliefs, Oral presentation, Pre-service teachers

Introduction

Since the last decade, there has been an increased desire for developing students' mathematical verbal skills and vocabulary (CCSSI, 2010; NCTM, 1989 & 2000). As a result, now the mathematics courses for pre-service teachers focus on strengthening pre-service teachers' mathematical content knowledge by improving verbal and writing skills. These verbal and writing skills are believed to foster conceptual understanding (Berry and Houston, 1995) and increase students' confidence (Butler and Stevens, 1997).

We noticed most of our pre-service teachers use incorrect terminologies and struggle in explaining concepts in a logical sequence. This made us consider how to help pre-service teachers in developing these skills as a large portion of their work involves oral mathematical communications. As the proverb says "No one learns as much about a subject as one who is forced to teach it" (Drucker, n.d.), oral presentations could be a valuable learning and teaching tool. This could help pre-service teachers learn to use correct mathematical vocabularies and to explain concepts. Fan & Yeo (2007) defined oral presentation as a classroom practice where students share ideas verbally and see and understand their own doubts. They noted that oral presentation gives students an opportunity to express their understanding in their own words.

Since pre-service teachers will be responsible for developing children's mathematical vocabulary and oral mathematical communication skills in the future, it is important to first develop their own understanding and disposition. Various studies have shown connection between mathematical disposition and mathematical learning (Maas & Schloeglmann, 2009; McLeod & Adams, 1989, Philipp, 2007). Disposition includes one's attitudes, beliefs and aptness to act in positive ways (NCTM, 1989). Attitude is a mental concept representing favorable or unfavorable feelings for objects, persons or other identifiable entities and a belief is a known or perceived information about an object (Koballa, 1998). For example, statements involving likes and dislikes reflect one's feelings toward an object and a statement such as, "*Math is hard*" represents one's beliefs. Fishbein & Ajzen (1975) noted that a person having a favorable attitude toward an object is more likely to perform or act with respect to the object. That is if pre-service teachers have favorable feelings and beliefs about oral presentations, they are more likely to use them in their own learning and teaching. Hence, it is worthwhile to investigate pre-service teachers' beliefs and attitudes toward oral presentations.

In this study, we investigate what are pre-service teachers' general beliefs and attitudes toward the use of oral presentation tasks in their mathematical learning and toward teacher preparation. We ask if oral presentation changes pre-service teachers' beliefs in their own ability to teach mathematics successfully to young children and in their own dispositions towards mathematics. In addition, we study pre-service teachers' self-reflection on the use of oral presentations on their own learning.

Theoretical Framework

Our theoretical framework is based on the theory of constructivism, social constructivism and multiple intelligence. From the point of view of constructivism, an individual develops meanings by organizing and reorganizing his own experiences and by constructing schemes ("a scheme is what can be repeated and generalized in an action" (Piaget & Garcia, 1991, p. 159)) (von Glasersfeld, 1995). We also consider individuals' meanings are constructed through social interaction (Steffe & Olive, 2010, Brooks & Brooks, 1993). In addition, from the point of view of multiple intelligence theory, each individual has a unique learning style. Hence, there must be different forms of learning opportunities (Fan & Yeo, 2007). In this study, we provide opportunities to pre-service teachers to use communication skills as one of the means to construct meanings besides writing assignments. Pre-service teachers were put into situations where they had to take responsibility for their own learning and continuously critique their own thought processes.

Methodology

Study participants and course description

This study took place in a geometry course for elementary pre-service teachers at one of the public universities in the western United States. The study participants were twenty four elementary pre-service teachers in the undergraduate program. One of the researchers was the instructor for this geometry course. The geometry course is the second course in the sequence of two-semester mathematics course for pre-service teachers preparing to teach children in pre-Kindergarten through grade six. This sequence of mathematics courses, which follows the CCSSI (2010) offered to strengthen pre-service teachers' oral and written explanations of mathematical concepts. The textbook for this course is Beckmann's text (2013), which focuses on the reasoning behind mathematical ideas pre-service teachers will teach in the future.

Study design

The course was taught using an inquiry-based approach (Bruner, 1961) allowing pre-service teachers to explore the mathematical concepts in a hands-on way. Throughout the course, pre-service teachers were encouraged to present their work in front of the class regularly. Additionally, thirty-five minutes were assigned per class to accommodate two pre-service teachers for the oral presentations. The instructor implemented two assessment-based oral presentation tasks: pre-structured oral presentation and impromptu presentation (Fan & Yeo, 2007).

Fan & Yeo (2007) described pre-structured oral presentations as tasks that are prepared in advance and impromptu presentations as tasks carried out without rehearsals. Below, we describe the two tasks in the context of our study.

Task 1: <u>Pre-structured oral presentation</u>: Each pre-service teacher presented once in the semester about any geometrical concept (K-6) relevant to the course for 10-15 minutes. This activity counted 5% towards the total grade. Pre-service teachers were asked to choose a geometrical concept from the textbook, either covered in class or a new topic and their presentation dates. They were responsible for reading the topic on their own, planning a draft of their presentation and informing the instructor about their topic two weeks prior to their chosen date for presentation. The instructor read their drafts and provided feedback before their oral presentation. Pre-service teachers also met with the instructor before their presentation to resolve doubts.

Task 2: Impromptu oral presentation: a) The same pre-service teacher had to answer instant questions related to the presentation from the audience (including the instructor) after the pre-structured oral presentation. This counted 3% towards the total grade.

b) Each pre-service teacher had to summarize a day's lesson twice throughout the semester. This counted 2% towards the total grade.

Data collection

We collected data in the form of pre-and post-tests (Appendix A), post-presentation questionnaire (Appendix B), and pre- and post-surveys (modified survey questionnaire from Fan & Yeo (2007) (Appendix C)). The survey consisted of nineteen questions on pre-service teachers' general beliefs and attitudes toward the use of oral presentation tasks, their beliefs in their own ability to teach mathematics successfully to young children and in their own dispositions towards mathematics. The pre-test/survey was conducted on the first day of the course, post-test/survey was conducted on the last day of the course. The post-presentation questionnaire was collected from pre-service teacher immediately after their oral presentation.

Results

Quantitative analysis

Below we discuss the pre- and post-surveys. Figures 1 and 2 show stacked column charts for the responses before and after the oral presentation tasks. Four pre-service teachers were not present on the first day and one pre-service teacher was absent on the last day of the course. Hence, n= 20 and 23 for pre- and post-surveys. Questions 4, 6, 8, 9, and 13-16 (see Appendix C) are worded such that a response with "agree" or "strongly agree" reflect negative feelings towards oral presentations. We observe a decrease in the green shaded area in the post-survey. In the remaining questions, we notice an increase in the green shaded area that represents the percent of the class who agree with the given statement. This shows an overall improvement in pre-service teachers' perception of using oral presentations in learning geometry.

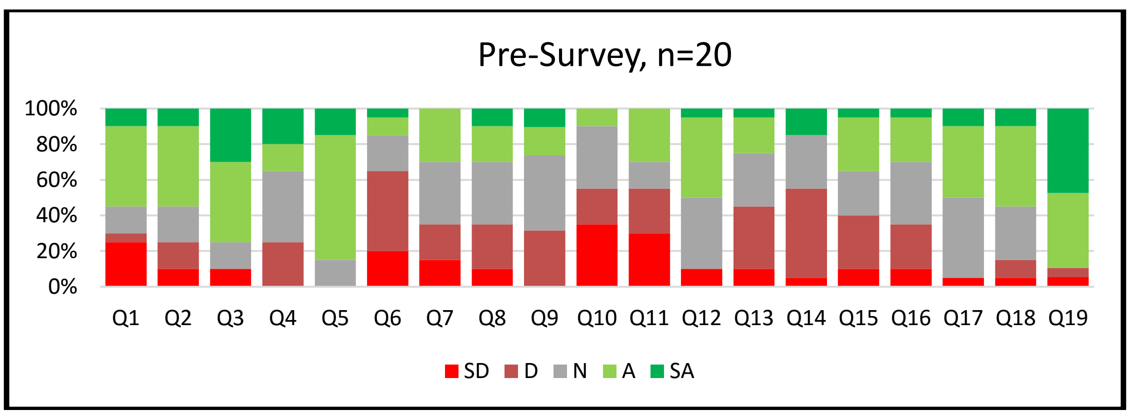

Figure 1: Stacked column chart for the survey responses before the oral presentation tasks.

To analyze the data from the two surveys, we first invert the responses from questions that highlight negative attitudes by treating a response of Strongly Agree as Strongly Disagree and so on. The responses are coded on a scale of 1-5 where 1=SD and 5=SA.

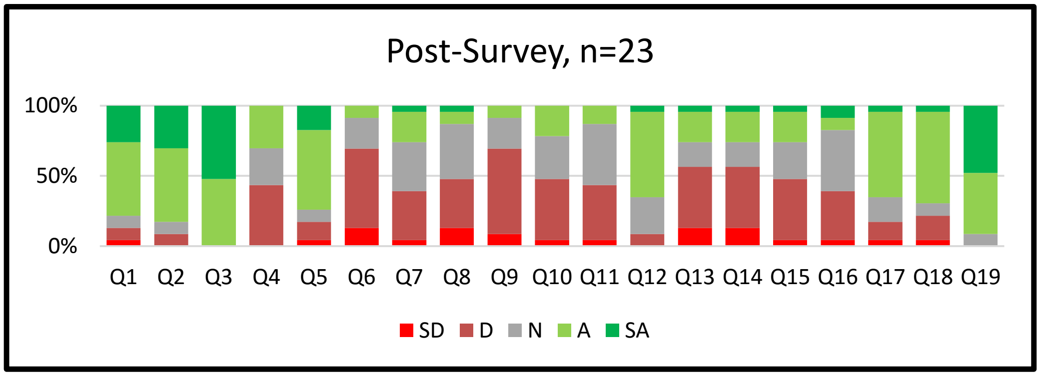

Figure 2: *Stacked column chart for the survey responses after the oral presentation tasks.*

Our study is threefold, we aim to investigate changes in attitudes of pre-service teachers toward oral presentations, in beliefs in their mathematics teaching ability and their disposition toward mathematics. Hence, we divide the survey into three categories. Questions 1-7 and 9-12 represent pre-service teachers' general attitudes and beliefs toward oral presentation. Questions 11, 12, 14 and 17 represent pre-service teachers' beliefs in their own ability to teach mathematics successfully. Questions 1, 5, 7, 8, 13, 15, 16, 18 and 19 represent pre-service teachers' disposition toward mathematics. We create the following score functions:

Oral presentation score = (Sum of responses to questions 1-7 and 9-12)/55*100

Teaching ability score = (Sum of responses to questions 11, 12, 14 and 17)/20*100

Disposition toward math score= (Sum of questions 1, 5, 7, 8, 13, 15, 16, 18 and19)/45*100[1]

We consider the responses of each pre-service teacher as paired data. Pre-service teachers who did not fill both the pre-and post-survey are not included in the paired t-tests but are counted in the summary of the responses. The paired differences in the three scores pass a normality test, thus we apply a paired t-test to see if the observed sample differences are significant. The improvement in the three scores is deemed significant at the 0.05 level of significance. Table 1 summarizes the results of our statistical inference:

Table 1

Summary of the statistical tests on the three score function. The data is considered normally distributed because the p-value of Anderson-Darling's test exceeds 0.05. The observed sample difference is significant because the p-value of the paired t-test is less than 0.05.

Score functions	P-value for Anderson-Darling's normality test	Mean sample of differences (After-Before)	P-value for the paired t-test
Oral presentation score	0.06682	6.84%	0.0062
Teaching ability score	0.5124	4.41%	0.04131
Disposition towards math score	0.07238	4.05%	0.02888

[1] 55, 20 and 45 are the maximum possible scores for each score and can be obtained by multiplying the number of questions by 5 which is the score of a *Strongly Agree* response.

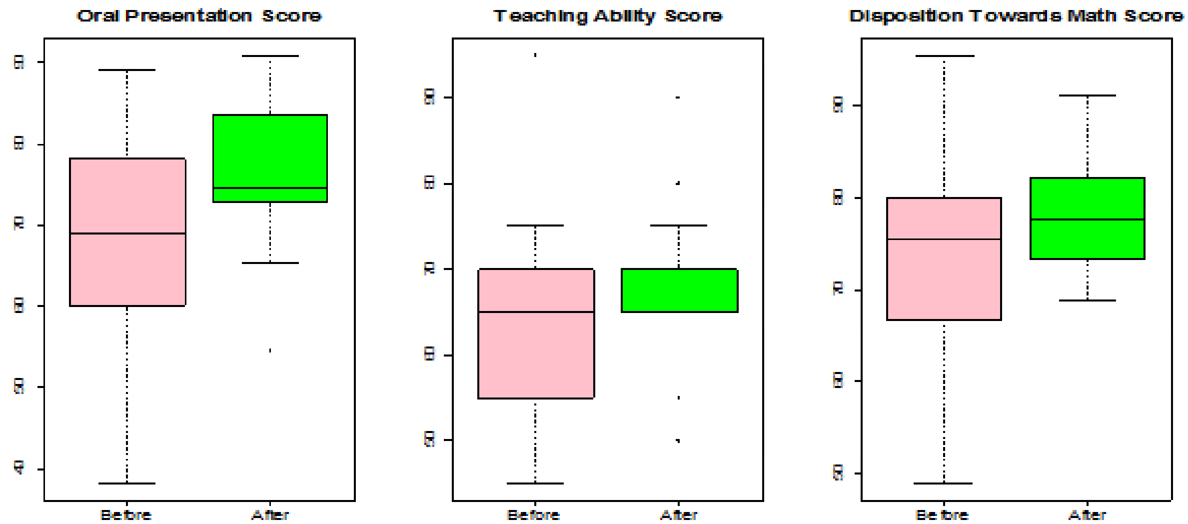

Figure 3: *Box-plots for the survey responses before and after the oral presentation tasks*.

Figure 3 displays side-by-side boxplots for each of the three scores we introduced. The box in each figure represents the middle 50% of the data. The solid line in each box represents the median (the 50th percentile) and the whiskers extend to show the minimum and the maximum. Outliers are displayed as dots in each panel. The graphs show improvement in the three scores after the oral presentation tasks.

Next, we analyze pre-service teachers' pre- and post-test scores. Each question is worth 10 points and is graded on a 2-5-10 scale. 2 points are awarded for identifying a correct answer with no explanation, 5 points for identifying a correct answer with explanation missing some logic and 10 points for a correct answer with correct explanation. The grades are recorded on a 30 point scale then converted to a percentage. Each pre-service teacher shows an improvement in their comprehension of the basic concepts that are tested. Our sample size of 21 shows an average improvement by 38.7% in the scores of the post-test compared to the pre-test. The data pass a normality test (Anderson-Darling test with a p-value of 0.0688), and the paired t-test results in a p-value of $8.39 * 10^{-8}$ indicating a statistically significant improvement in the post-test scores. However, we are not claiming this improvement is only due to the implementation of oral presentation tasks.

Qualitative analysis

In addition to quantitative analysis, we analyze the responses of the post-presentation questionnaire using a qualitative research method called the constant comparative method (Boeije, 2002). Here, we use an axial coding method (Strauss & Corbin, 1998). We compare and contrast the responses of the pre-service teachers on the questions in the post-presentation questionnaire. We identify the similarities and differences on each of the responses.

Out of 24, twenty three pre-service teachers report that they challenged themselves more to develop a deeper understanding of the topic for oral presentation. They report that the concepts they did not understand before or missed during exams, were cleared as they studied more for the oral presentation. However, one of the pre-service teachers mentioned, "maybe" she challenged herself more to get a deeper understanding of the topic, Graphs and Graphical representations, for oral presentation. She reports, "When I had to find graphs that could be confusing to students, I had to look at them from the perspective of the student." This suggests that even though this pre-service teacher was not sure whether she challenged herself more or not, she expended some efforts to explore graphs and to understand which graphs might be confusing for students. This is an important skill for pre-service teachers to develop understanding of the concepts that students might have difficulties with and how to address those issues. Hence, oral presentation might help pre-service teachers in recognizing the misconceptions of children through exploration of the topic.

Others mentioned, they explored many avenues, books, internet and videos to understand the concept better before explaining it to others. They ensured the concept made sense to them completely before presenting it to others. One pre-service teacher wrote, "I researched a lot and read the chapter like 10 times. I looked up YouTube videos on how teachers could teach this lesson." Another pre-service teacher reported, "I tried to really understand volume in case any questions were brought up. You have to have a full understanding of math before you teach it." This suggests oral presentation motivates pre-service teachers to study thoroughly the text books, to spend more time in studying and researching, and to pay more attention to the topic by teaching themselves before presenting it to the class.

These oral presentation tasks helped pre-service teachers reduce their own doubts on the concepts they had. For instance, one pre-service teacher wrote, "I did not completely understand the moving and additivity principles before, now I feel like I do." Another reported, "I selected a topic that I did not completely understand during lecture. I had to re-read the section in the chapter and I looked up videos online to further my reading." Pre-service teachers revealed that oral presentation had helped them to understand the concepts better by giving them the opportunity to investigate more about the concepts on their own. For example, one pre-service teacher whose topic was Platonic solids reported, "I read the information given in the chapter and I also did some research online. I was curious about the role Plato played and the Grecian history behind the platonic solids."

Conclusions

Our quantitative analysis provides evidence that implementing oral presentation tasks in the geometry classroom resulted in a significant improvement in preservice teachers' attitudes towards the oral presentations. It improved their confidence in their ability to teach mathematics and improved their disposition towards mathematics in general. Our qualitative analysis shows why our pre-service teachers feel oral presentation tasks are beneficial for their learning. For example, oral presentation tasks serve two purposes- developing mathematical meanings and assessing one's own understanding. Oral presentation tasks encourage pre-service teachers to take responsibility for their own learning and give them autonomy to take initiatives to make connections between ideas and a particular concept through self-arguing and validating their reasoning. In addition, oral presentation tasks give opportunities to pre-service teachers to present their ideas and to reflect on others' ideas. This facilitates the "meaning making process" (Brooks & Brooks, 1993).

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Appendix

A. Pre-/ Post-test Questionnaire:

1) Imagine floating in outer space above the North Pole. Looking down on the earth, which way is the earth rotating, clockwise or counterclockwise? Explain your answer.
2) What do you understand by area of a shape? Explain.
3) What is area of a circle? Explain why it makes sense.

B. Post-presentation questionnaire:

1) Did you challenge yourself more to get a deeper understanding of the topic for oral presentation?
2) Was the process (preparation for oral presentation and the oral presentation) beneficial for your learning? If yes, why? If not, why?
3) Were you able to demonstrate your oral presentation skills in front of the class?
4) Are you confident with your use of mathematical vocabulary while explaining math orally and in writing?

C. Pre-/ Post-Survey questions:

Q1. Oral presentations improve my understanding of mathematical concepts.
Q2. Oral presentation skill is important in mathematics learning.
Q3. Oral presentation skill is important in mathematics teaching.
Q4. Oral presentation makes me feel inadequate.
Q5. Listening to other classmates' presentation helps me understand other's perspectives.
Q6. Oral presentation is a waste of time.
Q7. I am not afraid of mathematics presentation in front of the class.
Q8. I don't know how to get started when I am doing mathematics.
Q9. I feel lost when I am doing mathematics oral presentation.
Q10. I like to do mathematics oral presentation.
Q11. I would like to have more mathematics oral presentations for my mathematics lesson.
Q12. I like to implement mathematics oral presentation while teaching math.
Q13. I have trouble understanding ideas that are based on mathematics.
Q14. If I taught in in a team or with a teaching partner, I would like to have another teacher teaching the mathematics.
Q15. I get frustrated when I do mathematics.
Q16. I do not do well on tests that require mathematical reasoning.
Q17. I feel confident in my ability to teach mathematics.
Q18. I feel confident in my mathematics ability.
Q19. I see mathematics as practical and useful.

Pre-service teachers' meanings of area

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Betsy McNeal — Ohio State University
David Bowers — Ohio State University

An exploratory study was conducted of pre-service teachers' understanding of area at a public university in the western United States. Forty-three pre-service teachers took part in the study. Their definitions of area and their responses to area-units tasks were recorded throughout the semester. We found a wide gap between pre-service teachers' meaning of area and their use of area-units. Initially, pre-service teachers had weak definitions of area. Over the semester, these definitions were refined, but misconceptions about area and area-units were illuminated in activities involving non-standard units and areas of irregular regions. We conclude that, despite detailed models of children's understanding of area, much work is needed to understand the learning trajectories of pre-service teachers, particularly when misconceptions exist.

Keywords: Area, Geometry, Pre-service teachers, Units

It has been demonstrated time and time again that many current and future elementary teachers have substantive weaknesses in their geometric content knowledge (e.g., Browning, Edson, Kimani, & Aslan-Tutak, 2014). It is notable that very few of the peer-reviewed studies on pre-service teachers' (PTs') geometric knowledge listed here deal specifically with area. Of the 112 studies published on elementary PTs' content knowledge reviewed for the special edition of the Mathematics Enthusiast in which Browning et al's article appeared, only 4 deal with the status of PT knowledge of area (Enochs and Gabel, 1984; Baturo and Nason, 1996; Reinke, 1997, and Menon, 1998). The findings of all four of these articles are similar, each indicating that the PTs under study demonstrated "incorrect, incomplete, and unconnected" knowledge that was very "rule driven" (Browning et al, 2014, p. 344). Perhaps as a byproduct of this issue, Enochs and Gabel (1984) found that a large percentage of PTs were unable to distinguish volume from surface area, a sentiment echoed by Baturo and Nason (1996) as well as Reinke (1997) which both found that PTs tended to conflate methods of finding perimeter with methods of finding area.

We were teaching a geometry course for elementary teachers when we observed that our PTs did not show a consistent understanding of area[1]. As in the literature cited above, our students confused the attribute of area with its measurement when they defined area as "length times width", as well as confusing area with perimeter and volume. At our two universities, PTs had completed a course in arithmetic before enrolling in the geometry course. This arithmetic course heavily emphasizes the meaning of the multiplication operation so our PTs often gave well-developed explanations for why we multiply to find the number of squares in an array. Our instruction, therefore, aimed to emphasize the meaning of area and to separate this from the process of measuring area and from the formula for the area of a rectangle. We did this by taking a more general approach using non-standard units and looking at the area of irregular shapes. In this context, we observed that most misconceptions that our PTs had about area showed up when they engaged in tasks involving non-standard units and conversion of units, tasks that did not involve computations of area using formulae. In class and in this paper, we take the following definitions. *Area* is the amount of two-dimensional space taken up by a 2D shape. An *area-unit*

[1] All three taught the same course with the same textbook and supplementary materials.

is any two-dimensional object used to measure a 2D shape. Finally, we describe *measurement* as a comparison of the area-unit with the 2D shape that is accomplished by covering the shape with an iteration of the area-units. We began to look for ways to trace the progressions of individual understanding of area over the course of the semester.

In this initial study, our goal was to examine changes in PTs' understandings of area over the one semester geometry course. More precisely, we asked the following questions: As seen in their written work, what area definitions did PTs bring to this course and how did those definitions change over the semester? What ideas about standard and non-standard area units did PTs demonstrate in their written work?

Theoretical Framework

The theoretical background for this study is drawn from the constructivist theory. From this point of view, one generates ideas by fitting new situations into existing ideas. If the situation does not fit, or if it cannot be explained, one modifies one's existing framework or generates new ideas (von Glasersfeld, 1995). From a social constructivist perspective (e.g., Cobb, Wood, Yackel, & McNeal, 1992), PTs' understandings of classroom conversations and actions are interpreted against a background of prior beliefs about the culture of school mathematics – about the norms and expectations for mathematical behavior and thinking in school – as well as against their prior understandings of mathematical ideas. In order to help them work through misconceptions, our instruction put PTs in situations where these prior understandings would be challenged. For example, we asked PTs to find the area of irregular shapes using non-standard units and we constantly required explanations for any answer given. We did this in part because asking about area of rectangular regions yielded responses that could appear correct, when a second look actually showed misconceptions. Our pedagogy is thus very similar to that described in Simon & Blume (1994), though our course used a textbook as an external resource.

Methodology

After many conversations together, one of the authors decided to collect data from her classes at a public university. Study participants consisted of 44 PTs from two sections of the mathematics course. One PT was absent for most of the tasks, hence we did not use that data. The geometry course is the second course in a two-semester mathematics course sequence for elementary PTs. All participants completed the first course prior to this study. The mathematics textbook for this course is Beckmann (2013), which aligns with the standards of the Common Core State Standards Initiative (2010). The class set-up and the textbook both used an inquiry-based approach (Bruner, 1961) towards learning, where students are encouraged to explore content on their own and discuss with their peers.

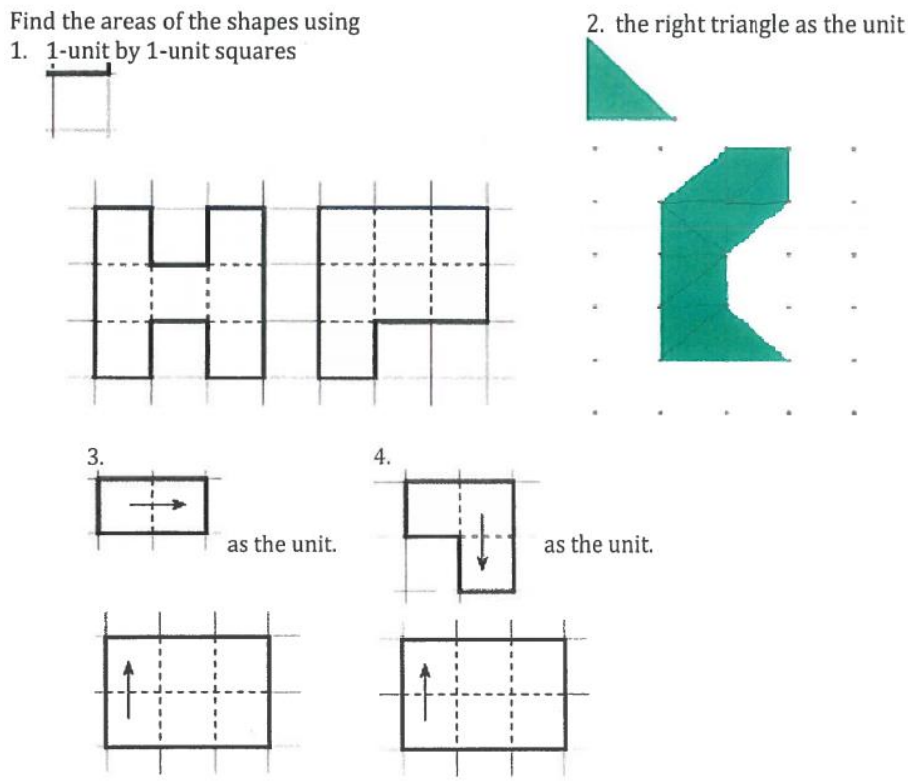
The study was conducted throughout the semester and area problems were collected from PTs' in-class writing assignments, quizzes, tests, and from the final exam. The in-class writing assignments (Figure 1) contained questions related to area and area-units and they were repeated multiple times throughout the semester. Each time the answers were discussed in class after PTs got their writing assignment back.

Our choice of tasks here represents a first pass at making a deeper examination of our PTs' area concepts. We intend to take a more precise look at their understandings of area in a future study by adding clinical interviews (Clement, 2000) including conversations about their written work.

Figure 1. Tasks used in the study.

Area Definition Task:
Discuss area of a shape. Give an example.

Units Task 1:



Units Task 2:
What do you understand by 12 m^2?

Data Analysis

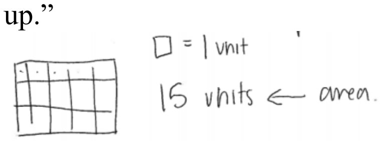
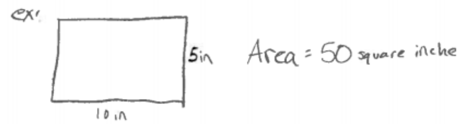
Area Definition Task: We analyzed the area definitions following an open and axial coding method (Strauss & Corbin, 1998). Each of the three authors read the PTs' written definitions of area and created a rubric to assign a score to each PT. Then we discussed our rubrics and created a common rubric (Table 1) for assigning scores to each PT's area definition.

Units Tasks: We analyzed Units Task 1 (non-standard units) and Units Task 2 (*What do you understand by 12*m^2*?*) by recording each PT's answers. We created a spreadsheet of the PTs' responses to each task so that we could trace an individual PT's progress across the tasks and simultaneously compare responses of all PTs to the same task at the same point in time. Responses to both units tasks were recorded as "correct" and "incorrect".

Results and Discussion

After comparing PTs' definitions of area, their use of non-standard units, and their responses to the question *What do you understand by 12*m^2*?*, we concluded that their understandings of area differed across these three contexts. Focusing first on responses to the Area Definition Task, we found about 86% of the 43 PTs started with a low understanding of area as measured by scores less than or equal to 3. By our rubric, this suggests that a majority of the study participants did not have a comprehensive understanding of area in the beginning of the semester because their definitions included only “measures space” or “the amount of space that an object takes up”, but did not specify two-dimensional space and made no reference to use of units.

Table 1
Rubric for assigning scores to PTs’ Area Definitions

Score	Description	Corresponding Examples
5	Used covering OR fitting concept, explicitly mentioned measuring a 2D shape AND clearly described a unit of area.	“The area of the shape is the 2 dimensional measurement of the amount of space it takes up.” 
4	Used covering OR fitting concept and indicated measuring a 2D shape (either expressed in words or pictures) OR a unit of measurement has been used specifying it as a length or area unit.	“Area of a shape is how much space it takes up in specified units in a 2-dimensional plane.” 
3	Used covering OR fitting concept OR mentioned measuring a shape/ space or outside of a shape (2D is not explicit through words or pictures) OR used *length times width* as an example beside their definition.	“If you were to put something inside it. The area is how much you could fit in.”
2	Discussed area with *length times width* as a requisite part of the definition (not just as an example). No indication of measuring 2D space.	“Length times width, because you want to find the area you have to multiply all of the sides together.”
1	Used volume formulae OR 3D figures as parts of definition OR unclear vocabulary OR did not write anything.	“The area of the shape are the dimensions inside of the shape or the volume of the shape.”

Those scoring 2 described area *only as* “length times width” and those scoring 1 wrote irrelevant or unclear statements with no reference to space at all. Throughout the semester, the same questions were asked and discussed multiple times. On the final exam, a similar question asked for a definition of area compared to perimeter or volume. About 81% of the 43 PTs scored at the level of 4 or 5 on this question. This suggests that PTs’ area definitions improved over time.

Results of Units Task 1 that required PTs to describe the area of a shape using standard and non-standard units showed that only 5 PTs initially identified correct units. Most PTs initially designated all non-square units as “units squared” or as “square units” (see examples in Table 2). Even after three repetitions of this task, each followed by discussions of the answers, only 22 PTs (about half of the total number of PTs) identified correct non-standard units. Analysis of the Area Definition Task suggested improvement in PTs’ understanding of area, but Units Task 1 suggested half of the class still had misconceptions about area. Combining our analyses of Units Tasks 1 and 2, we found that PTs at different levels of area definition answered the two tasks differently (see Table 3). Although 21 PTs reached a level 5-area definition, only 5 of them correctly responded to both units tasks at the end of the semester.

Table 2

PTs' initial responses to Units Task 1

Units in Units Task 1	PTs referring to the corresponding units
1. Square unit	Units^2, Units squared, Unit squares, 1 by 1 squares, Squares, Square units, squares^2, No units, Units, Unit squares^2, 1 by 1 unit squares, 1-unit by 1-unit squares
2. Right Triangle unit	Units^2, Units^2(of the triangles), Units squared, Right triangles, Right triangle units^2, Units^2 triangles, Square units, Triangle units, Units, Right triangles^2, Unit squares, No units
3. Two square unit	Units^2, Units squared, Two square units, Units, Square units, Units of two squares, No units
4. L-shape unit	Units^2, Units, Square units, Units of two squares, No units, L-shape units

This data suggests that, although the idea of "area as covering/ fitting" is mathematically linked to the units used to measure area, these concepts were conceptually distinct for many of our PTs. None of the PTs giving a level 3 definition of area were able to give a correct response to the non-standard units task (Units Task 1), but 3 out of 5 were able to correctly answer the question, "What do you understand by $12m^2$?" There were 11 students across all levels of definition who could give good or even excellent definitions of area and explanations of the meaning of $12m^2$, but could not apply these ideas correctly in situations involving non-standard area-units. It seems likely that these students had memorized these two ideas, but did not really understand the meaning of area when measured with a non-standard unit.

Table 3

Summary of results of area definition and the two units tasks at the end of the course

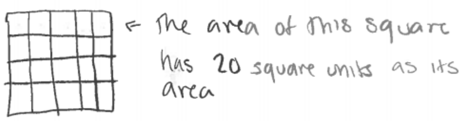
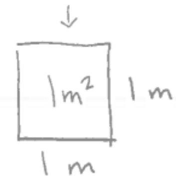
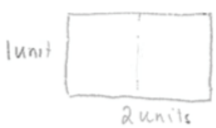
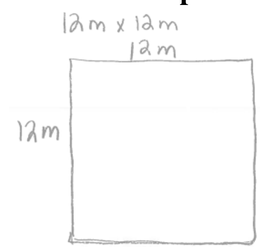
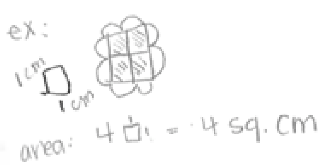
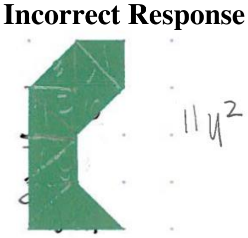
Area Definition Levels	Both Units Tasks Correct	Correct Units Task 1 + Incorrect Units Task 2	Incorrect Units Task 1+ Correct Units Task 2	Both Units Tasks Incorrect	Total
Level 5	5	6	6	4	21
Level 4	3	5	2	4	14
Level 3	0	0	3	2	5
Below 3	0	0	0	3	3
*Note: The numerical values above denote the number of PTs in each category.					

We can see that these ideas are distinct by isolating individual PT's work by level of area definition. Table 4 shows examples of four individuals' work across the three different tasks in their final attempt. Each row shows a different PT's work. For simplicity, we show only examples at definition level 5.

Conclusions and Implications

Our data clearly shows the ability to write a clear and complete definition of area, including reference to the units used to measure it, does not imply full understanding of area. Although area and area-units are tied together mathematically, these ideas were split in the minds of many of our PTs. This is consistent with a long-standing body of literature illustrating the psychological phenomenon of context-dependent understanding (e.g., Carraher, Carraher, & Schliemann, 1985). The only students who had both units tasks correct were those who had attained a level 4 or 5 definition by the end of the semester. In contrast, even after multiple repetitions, 5 of the 8 PTs (see Table 3) who ended the course still writing definitions at level 3 or below were still not able to correctly complete either of the units tasks. This suggests a well-articulated area definition is a necessary, but not sufficient, indicator of PTs' understanding of area.

Table 4
Four PTs' work on area and units tasks.

PTs	Area Definition Task	Units Task 1	Units Task 2
PT 1	**Level 5** The area of a shape is how many two dimensional units fit inside a flat shape. For example 	**Correct Response** 11 triangular units 	**Correct Response** That an area or space is filled up with 12 1m x 1m units 
PT 2	**Level 5** The space a plane shape takes up on a plane. The area, the space the shape takes up, can be defined as the units that makes up the shape. In this case, the shape is made up of 2 square units. 	**Correct Response** 	**Incorrect Response** 
PT 3	**Level 5** The area of a shape is a two dimensional measurement of space such as how many 1 cm by 1 cm squares can fit in a shape.	**Incorrect Response** 	**Correct Response** 12 1meter by 1 meter squares.
PT 4	**Level 5** Area of a shape is the amount of space the shape takes up. It is two dimensional. 	**Incorrect Response** 	**Incorrect Response** It is 12 square meters. So there are 12 square meters in the shape. It is NOT meters squared.

Just as we need caution when assuming that a correct definition implies understanding of area, we need to consider whether incorrect labeling of area-units (e.g., "12 triangle units^2")

necessarily implies deficient understanding of the units themselves. For example, PTs' experiences of units in science classes (where units are cancelled as if they were variables) might have helped to change their interpretation of "square units" from a correct understanding of this to the incorrect "squared units" or "units squared". The student errors we found suggest that PTs are multiplying words (*inches times inches)*, like variables, without regard to the units or the meaning of multiplication.

This study shows there is a need for development of progressions of PTs' understanding on geometric topics. Our results overlap with much of the work done with children by Battista (Battista, 2012; Battista et al., 1998). Using teaching experiments (Steffe, 1983), this research breaks the concepts of area and volume into "levels of sophistication" through which children must pass on their way to full understanding of area and volume. Battista (2012) classified reasoning about area into 8 broad levels, with the first four levels all explicitly about units. This suggests that a very deep understanding of units is required in order to attain a comprehensive understanding of area. At the lowest level described, the child "uses numbers in ways unconnected to appropriate area-unit iteration" (p. 112). At the next lowest level, the child "incorrectly iterates area-units" (p. 112). In contrast, our data from PTs show two different levels that indicated no understanding of the relationship between area and the area-units. Looking only at our PTs' definitions, they had two ways to be incorrect: 1) At our Level 1, PTs gave definitions having nothing to do with area or its measurement (e.g., they defined volume instead), and 2) At our Level 2, PTs gave incorrect definitions that relied on the formula for the area of a rectangle, a definition possibly derived from memorized school learning. Our PTs' responses were consistent with observations described by Simon and Blume (1994) who wrote that many of their PTs had a

> *rote procedure for finding area given two linear measures (expressed in common units of length). According to this scheme, one multiplies the two numbers and expresses the product in "square units", so that the second word in the area referent is the same as the referent for the linear measures. It is likely that for some of these [PTs], square units do not conjure up an image of a square. (p. 485).*

Looking at the other developmental levels found by Battista (2012), our PTs seem to be consistently at his sixth level – they "understand and use procedures and formulas for determining areas of rectangles" – but many have not reached the next level where they "generalize their understanding of area measurement to non-squares and to area-unit conversions" (p. 113). Where there is often overlap between the levels of understanding that we see in our PTs with the literature on children's developing understandings, many PTs have succeeded in school for years despite having misconceptions about area and area-units. This causes a PT's learning path to deviate from a child's.

It is clear that we must study the learning paths of PTs directly, not simply compare their understandings to the development of children. We suggest several improvements for future studies. Tasks should be designed specifically to focus on PTs' understanding of area-units, and written work should be paired with interviews asking them to explain their thinking. Study should focus on those PTs who have completed an arithmetic course for teachers to examine how they relate area formula for rectangles and two-dimensional area-units and the role played by multiplication. With enriched understanding of how our PTs learn geometric concepts, teacher educators will be better prepared to work with them to unravel misconceptions and strengthen and rebuild PTs' mathematics for teaching.

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How well prepared are preservice elementary teachers to teach early algebra?

Funda Gonulates, Leslie Nabors Oláh, Heejoo Suh, Xueying Ji, Heather Howell

As algebra has gained more attention in the K-12 curriculum, mathematics educators and policy makers have studied ways to support early algebraic thinking (e.g., Carraher, Martinez & Schliemann, 2008; McCallum, 2011). However, algebra in the early grades is sometimes misunderstood and is misrepresented as merely bringing algebra content down to the early grades (Kaput, 2008, p. 6). Instead of adding new content to an already packed curriculum, experts have suggested that elementary school teachers can support their students' algebraic thinking by being more selective and attentive to mathematical content as it is related to algebra during routine classroom discussions (e.g., noting that when you add two numbers, the order of numbers does not change the answer). Teachers can also support this thinking by considering ways to highlight algebraic connections and recognize patterns for generalization (Wu, 2001). This approach contrasts with a more traditional focus on computation and symbolic manipulation which Smith and Thompson (2008) consider a "fundamentally flawed" introduction of algebra, noting that "developing students' abilities to conceptualize and reason about situations in quantitative terms is no less important than developing their abilities to compute" (p.128). Therefore, even beginning elementary school teachers need to be knowledgeable about relevant algebraic content and what pedagogical choices will support their students in developing early algebraic thinking.

Although researchers theorize that early development of algebraic thinking is important for students' later understanding of algebra, the research base is not yet sufficient to identify what teachers know about their students' understanding of basic algebraic concepts (Asquith, Stephens, Knuth, & Alibali, 2007). This study builds knowledge by documenting the responses of a sample of preservice elementary teachers to a set of early algebra items designed to measure their mathematical knowledge for teaching (Ball & Bass, 2002). This will help us understand what knowledge such undergraduates have of the content needed to teach early algebra. We will use these findings to discuss whether and how teacher education programs across the nation are preparing undergraduates to teach early algebraic thinking. For this purpose we asked the following research questions:

- How do undergraduate preservice teachers interpret and respond to common patterns of student thinking in early algebra topics?
- What are the strengths and weaknesses among undergraduate preservice teachers in preparing appropriate materials to support students' early algebra development?

Conceptual Framework

We adopted the conceptualization of teacher knowledge as introduced by Shulman (1986) as pedagogical content knowledge, and later Ball, Thames, and Phelps (2008) elaborated on and operationalized as Mathematical Knowledge for Teaching (MKT). In this particular study we have attended to Knowledge of Content and Students (KCS) and Knowledge of Content and Teaching (KCT) from the Ball et al. framework. KCS refers to teachers' knowledge of their students with respect to mathematics (e.g., common misconceptions or students' level of understanding). In KCT the content knowledge is related to teachers' knowledge of teaching (e.g., choosing a mathematically valid representation to use in introducing a concept).

This study aims to investigate undergraduate preservice teachers' knowledge in the domain of early algebra. In building an understanding of early algebra many researchers mention the importance of "Equivalence Statements" and how students are challenged to see the equal sign as indicating equivalence. Rather, students tend to understand the equal sign as indicating an action to carry out (e.g., Nathan & Koellner, 2007). In addition, the transition to algebra is related to gradual "symbolization of computations" (Kaput, 2008). The literature refers to the importance of having students attend to and be able to "use structure in solving problems" (Kaput, 2008). This kind of work can enhance students' algebraic thinking skills. In addition, to develop functional thinking students need to be able to attend to and make sense of variables involved in a problem and try to explain relationship between variables in a problem situation, often referred to as "relational thinking"(Carraher, Martinez & Schliemann, 2008).

Methods

We conducted 90-minute clinical interviews with 15 preservice teachers (PST) in their fourth year of a five-year long teacher preparation program. At the time of the interview participants were enrolled in the Teaching Methods in Mathematics undergraduate course, and three of the 15 were math majors.

These interview sessions collected the PST's responses to a series of 17 assessment[1] items designed to measure their content knowledge for teaching early algebra, with follow up questions probing their content-based reasoning. We also collected self-reported information about their preparation in this content area. Our initial coding of items was designed to separate those that focused on student thinking from those that focused on preparation for instruction. The algebra focus of these items included equivalence statements, symbolization of computations, using mathematical structure in solving problems, and relational thinking. The distribution of items in terms of algebra focus is given in Table 1. Algebra focus categories were not mutually exclusive; therefore categorization of the items reflected primary content focus of the items.

Table 1
Distribution of Interview Items According to Early Algebra Content

Primary Content Focus	Definition	Number of Items
Equivalence Statements	Items included meaning of the equal sign and equivalency statements, understanding of equivalency, and properties of equivalence relations (reflexive, symmetric, transitive).	6
Symbolization of computations	Items included the use of variables in solving problems, moving from one representation to another, representing verbal information in symbols, and illustrating a story problem by using a graphical representation.	3
Using Structure in Solving Problem	Items included using the mathematical structure of the problem in finding a solution. These items also focused on ways students can make use of properties of operations and identifying flaws in student use of operations (e.g., incorrectly commuting over subtraction).	7
Relational Thinking	Items assessed the ability to move from recursive thinking to general, or to characterize the relationship among variables.	1

[1] These assessment items were developed by researchers at Educational Testing Service.

As demonstrated in Table 1, the majority of the items' main focus was either equivalence statements or using structure in solving problems. An Equivalence Statement item can assess PST's evaluation of student work or can require the PST to consider examples to support their students' view of equal sign as a balance. An example of an Equivalence Statement item (where the PST needs to understand that equivalence is *not* highlighted by the examples provide) is given in Figure 1.

Mr. Keller's sixth-grade class is learning about algebraic equations. In his teachers' edition of the textbook, Mr. Keller finds a page that suggests he ask students to critique the following two solutions to determine if they are valid.

$4x + 2 = 12$ $6x = 12$ $x = 12 \div 6$ $x = 2$	$5 = 2x + 3$ $5 = 5x$ $5 \div 5 = x$ $x = 1$

Which of the following is most clearly highlighted by asking students to critique the invalid strategies?

- Understanding the meaning of the equal sign
- Understanding the importance of combining like terms
- Understanding the use of properties of operations to simplify expressions
- Understanding the use of inverse operations to solve equations

Answer the question above by clicking on the correct response.

Figure 1. Example of a released Early Algebra item.

With respect to assessing PST's use of Structure, an item might ask a test-taker to consider responses to simplifying the expression 7 – 3 +2. A common misconception would lead a student to evaluate the expression as 2 by adding 3 and 2 before subtracting (Hewitt, 2012).We would code an item asking the test-taker to interpret this kind of work as *using structure in solving problems* and having a pedagogical focus on *student thinking*.

Data Analysis

Interviews were recorded and were transcribed to allow for data analysis in NVivo. A team of researchers used a grounded theory approach with open and axial coding techniques (Glaser & Strauss, 1967) followed by constant comparative analysis (Miles & Huberman, 1994). These methods allowed us to arrive at a set of themes describing PST interpretation of common patterns of student thinking, strengths and weaknesses in PST use of algebraic concepts, and refinement of those themes by going back to existing literature and to the data.

Coding Framework

The coding framework was developed in multiple steps. First an initial framework with broader themes was developed and later refined by reviewing the data more closely and by revisiting our research questions. An initial round of item-level coding documented whether items required consideration of the equal sign, presented or asked for a student misconception, and/or presented student thinking. In addition, we worked together as a group to code three items and revised the coding framework before starting the pair-coding process. Analysis was conducted at the item level because items differed in the content they targeted. A sample of our coding framework that distinguishes PST understanding of the equal sign is provided in Figure 2.

Area/ Focus	Detail	Code (Node)	Definition
Mathematical Focus			
Equal Sign	Equal sign as a balance	M_EQ_Balance	This code is used for evidence that the PST understands or uses the equal sign as a balance or considers that both sides of the equation need to be equivalent.
	Equal sign as an action	M_EQ_Action	This code is used for evidence that the PST understands or uses the equal sign as an indication of an action or computation. (i.e., the equal sign is considered as a signal to produce an answer).

Figure 2. Example of the coding framework.

Coding for these two views of equal sign, for example, will help us to characterize PSTs' common view of the equal sign and how they use these views in interpreting students' work or addressing student misconceptions related to the use of the equal sign. For example, a response like the following:

> *... maybe they don't understand the equal sign means that both sides of the equation are going to be the same value so this side of the equation is going to equal be the same value as this side of the equation*.

was coded as M_EQ_Balance because this PST clearly noted a balance view of the equal sign.

Preliminary Results

The presentation focuses on findings that we believe have direct interest for RUME participants: (1) the study participants were least likely to answer correctly on items targeting the meaning and use of operational properties, (2) they struggled in evaluating the appropriate use of the equal sign when presented with different uses in student work; and (3) they reported that they had had few opportunities to learn about early algebra as mathematical content and as a topic to teach.

When solving the assessment items by "thinking aloud," some participants shared their embarrassment at not knowing the definition of the commutative property or the associative property. In other assessment items, for example, they showed that they knew that the order of numbers does not matter when adding numbers. In other words, the participants had the mathematical knowledge of the properties but lacked the knowledge on their names. Some participants also did not recognize that the use of multiple equal signs was problematic. In terms of opportunities to learn about early algebra, a number of participants reported that they did not have many chances to discuss early algebra as content to learn and content to teach in the previous four years. Although it is possible that they studied early algebra topics in their mathematics content courses and in their mathematics teaching method courses, the participants' lack of recall in this area suggests the need for more emphasis on early algebra.

Evidence of Impact

While this study was conducted at one institution and, therefore, is not intended to be representative of all programs, the number and types of mathematics and mathematics methods courses these undergraduate PSTs take are similar among teacher preparation programs nationwide. Our results are likely typical of the types of challenges other undergraduate PSTs would be expected to have. We will detail these challenges in the presentation, such as an appropriate use of the equal sign and a flexible and appropriate use of properties. In addition, we will talk about how it is important for teachers who are teaching undergraduate PSTs to provide a broader view of algebra and help their students to move from a "fundamentally flawed" view of algebra with a focus on computations (Smith & Thompson, 2008).

Research provided evidence that students can learn so-called difficult algebraic concepts and overcome their misconceptions with appropriate pedagogical choices (Hewitt, 2012). Therefore it is important to know what PSTs are in need of the most in preparing to teach early algebra. Such information matters because it can inform the development of teacher-education curricula and support materials. Participants in this presentation will get a summarized list of findings and an opportunity to discuss implications for designing courses for undergraduates who will become teachers.

Organization of the Session

In this session we will present the findings of the study and use excerpts from the interviews to allow teachers of undergraduate preservice teachers to characterize the mathematical knowledge and reasoning revealed in the interviews. In addition, we will have participants discuss what kind of curriculum and course work is needed so that undergraduates who will become teachers will be well prepared to teach this content area. We will present the following questions for consideration:

1. If these findings were indicative of a broader need for increased undergraduate instruction in algebra, where should responsibility for this instruction sit within the undergraduate program?
2. What opportunities are currently given to undergraduate students to use algebraic reasoning in authentic problem solving contexts?

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Unraveling, synthesizing and reweaving: Approaches to constructing general statements.

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Abstract

Learning progressions for the development of the ability to look for and make use of mathematical structure would benefit from understanding how students in mathematics-focused majors might construct such structures in the form of general statements. The author recruited ten university students to interviews focused on tasks that asked for the reconstruction of a general statement to accommodate a broader domain. Through comparative analysis of responses, four major categories of approaches to such tasks were identified. This preliminary report describes in brief those four categories.

Keywords: Undergraduate mathematics, mathematical practices, structure, generality, general statements.

Rationale

One goal of mathematics education at all levels is to promote the development of proficiency in mathematical thinking. In recent years, the Common Core Standards (Council of Chief State School Officers [CCSSO] & National Governors Association Center for Best Practices [NGA], 2010) have become a well-known framework for describing the mathematical practices that students should develop during their K-12 education.

The Standards for Mathematical Practices are meant to “describe varieties of expertise that mathematics educators at all levels should seek to develop in their students” (p. 6). Included among these Standards is the practice to “look for and make use of structure” (p. 8). Mason, Stephens, and Watson (2009) define mathematical structure as “the identification of general properties which are instantiated in particular situations as relationships between elements” of some kind of collection. For the purpose of this preliminary report, a statement that describes such a structure will be referred to as a *general statement.* The construction of general statements is an essential component of mathematical activity, without which the knowledge of individuals and of the discipline cannot grow (cf. Mason, Drury, & Bills, 2007).

According to NGA and CCSSO (2010), the Standards were constructed on “research-based learning progressions detailing what is known today about how students’ mathematical knowledge, skill, and understanding develop over time” (p. 4). A learning progression for the ability to create general statements should include research-based descriptions of the various ways that individuals might construct general statements as their formal education in mathematics increases. Research efforts have led to insights into the ways that elementary, middle, and secondary students construct general statements through patterning and generalizing (e.g., Becker & Rivera, 2004, 2005, 2006, 2007; Ellis, 2007; English & Warren, 1995; Fuii & Stephens, 2001; Garcia-Cruz & Martinón, 1997; Jurow, 2004; Lannin, 2005; Lannin, Barker, & Townsend, 2006), yet much less is known about the approaches that students, engaged in formal postsecondary study of mathematics, use to construct general statements

This preliminary report focuses on findings from data collected as part of a research study designed to investigate the following question:

What are the characteristics of approaches that postsecondary students in math-focused majors use when constructing general statements?

Theoretical Framework

Examples and generality

Watson and Mason (2005) suggest that example construction is an important aspect of mathematical activity. Among other possible uses, examples may serve as “placeholders used instead of general definitions and theorems” (p. 3) or as “representatives of classes used as raw materials for inductive mathematical reasoning” (p. 3). The generality that one encodes in the examples that are produced may have an influence on the process of developing a general claim. For example, as Mason and Pimm (1984) noted, the numeral 6 can be used to represent a specific value, or as a representative example of an even number, or even as a generic representation of any element of the even numbers. The claims that one makes about a specific inscription may or may not be general claims about a class of objects, depending on the generality that the inscription is meant to represent.

In addition to the generality that one encodes in an example (or attributes to an example), the symbols used to represent an example can influence the process of developing a general claim about a collection. Lannin, Barker, & Townsend (2006) hypothesized that individuals are more likely to develop numerical patterns involving recursive relationships when elements of a collection are represented in such a way that one can perceive one figure as an intact subfigure of another, such as in the arrays shown in Figure 1, and that learners are more likely to work toward patterns that relate ordinal position and numerical values when presented with figures that are not so easily perceived as embedded one-within-another (see Figure 2).

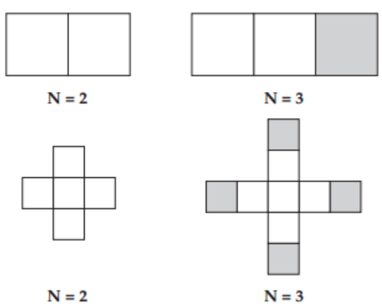

Figure 1. Recursively oriented patterns (Lannin et al., 2006, p. 22)

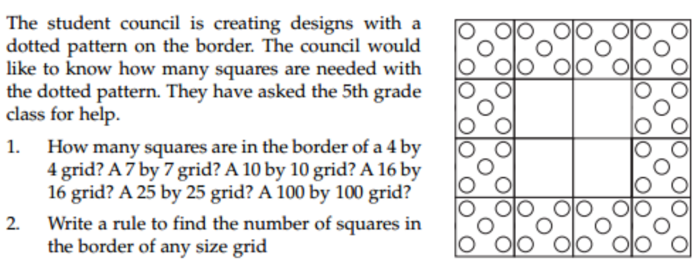

Figure 2. The Border Problem (Lannin et al., 2006, p. 18)

Relationships

A general statement is, in its presentation, nothing more than a claim that one is making about elements in a collection. Behind a general statement, however, are the structures and relationships that one understands and that undergird the statement itself. The relationships to which one attends when examining examples and building relationships can influence and even characterize the resulting general statement. Stacey’s (1989) illustration of students’ approaches to linear generalizing tasks indicated that some learners identify relationships between examples and use those relationships to transform one example into another. For example, a student who is given the images shown in Figure 1 and asked to predict the number of rectangles and squares in each set for N=4 might identify an additive relationship and predict that the number of rectangles will be one more than for N=3 and that the number of squares will be four more, thereby transforming the total of 3 rectangles and 9 squares for N=3 into totals of 4 rectangles and 13 squares for N=4. Alternatively, some respondents will focus on relationships between the index value and the number of elements, noting that, for example, the number of rectangles is 2 for N=2 and 3 for N=3 and hypothesizing that the number of rectangles will always equal the index value. In the case of patterning activities such as those used by Stacey (1989) and Lannin and colleagues (2006), the type of relationship that the participant finds salient can impact the development of either a recursive relationship or a functional relationship.

Methods

Ten students from a large mid-Atlantic university were recruited as participants. All were pursuing degrees in math-focused majors: Six were pursuing degrees in secondary mathematics education, and four were pursuing degrees in mathematics. Each participant was enrolled in mathematics coursework intended for students in their fourth year of study, and each had completed at least one mathematics course at that level prior to participating. Participation consisted of three task-based interviews, each lasting approximately one hour and consisting of one or more tasks designed to engage the participant in the construction of a general statement. Recordings were used to capture participants' statements and to provide a video record of the participants' written work and nonverbal gestural communication. Each interview was transcribed and each transcript was parsed into responses that began at the introduction of a task prompt and ended at the introduction of a subsequent task prompt or at the end of the recording.

This preliminary report is based on participants' responses to tasks that provided a general statement (we will refer to this as the *anchor statement*) and that asked the participant to reconstruct the claims made in the anchor statement as claims that would be true for a superset containing the original domain (we will refer to the superset as the *target domain* and to the requested set of claims as the *target claim*). Specifically, participants responded to one or more of the following task prompts:

Reconstructing products (RP). Consider the following statement: Any four consecutive whole numbers is divisible by 12. Can you rewrite the statement so that it is true for products of three or four consecutive whole numbers?

Reconstructing Unit Ball (RB). Every point (x ,0) on the interior of the interval [-1 , 1] has the property that $|x| < 1$. Can you rewrite the statement so that it is true for all points on the unit circle and its interior?

Reconstructing Sums (RS). Consider the statement that the sum of the first n counting numbers is $n(n + 1)/2$. Can you find a way to rewrite this statement so that it is true for any sequence of n consecutive integers?

Consistent with the theoretical framing presented here, participants' responses to these tasks were analyzed and categorized by comparing the ways that the participants exemplified the anchor domain and target domain, to the presence of evidence that illuminated the generality encoded in the examples that participants created, to the relationships (if any) that the participants analyzed while responding, and to the relationships that participants constructed while responding.

Findings

The comparative analysis of responses yielded five qualitatively distinct approaches to the tasks presented in the methods section of this preliminary report. Rough descriptions of each approach are presented in Table 1, and illustrative examples will be shared here, as space permits.

Characterizing Approach: Don, RP

In his response to the RP task, Don (a pseudonym) wrote examples of products of three consecutive whole numbers as shown in Figure 3. He noted that each 3-tuple contained an even number and a 3, and hypothesized that products of 3 consecutive whole numbers might always be divisible by 6. He then tested this for 4*5*6, 5*6*7, 6*7*8, and 7*8*9. This part of Don's response consists of characterizing the collection of examples without reference to the anchor statement.

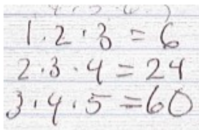

Figure 3. Don's examples of products of 3 consecutive whole numbers.

Oblique Approach with Specific Examples: Chris, RP

Chris created a set of specific examples similar to those used by Don (see Figure 4). However, instead of developing a claim inductively from examples, Chris searches for those 3-tuples that satisfy the anchor claim – in other words, those whose products are divisible by 12.

Figure 4. Chris' examples of 3-tuples in the RP task.

Unraveling and synthesizing: Jolene, RP

Jolene approached the RP task by analyzing the anchor claim. She determined that a 4-tuple would always have two even factors using a generic representation shown in Figure 5, and used the placeholder representation shown in Figure 6 to conclude that a 4-tuple would always include one number that was divisible by 3. She then used these understandings to synthesize the claim that a 3-tuple would always include one number divisible by 2 and one divisible by 3 and would, therefore, have a product that is divisible by 6.

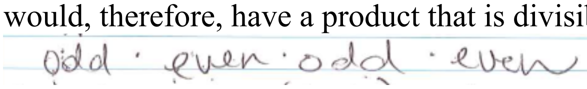

Figure 5. Jolene's general representation of 4-tuple.

Figure 6. Jolene's placeholder representation.

Unraveling and adapting: Edward, RS

Edward conceptualized an arbitrary sequence of positive consecutive integers as the difference between two sequences of counting numbers:

> Let's say we started just at 5 and . . . . I wanted to know the sum of the numbers from 5 to 10. I would do the first ten counting numbers and then I would take away the first four counting numbers.

Edward then used this relationship between the target domain and the anchor domain as a conceptual lens through which to adapt the anchor claim, writing a target claim that the sum of a sequence of integers from *k* to *n* would be computed through the expression in Figure 7.

$$\frac{n(n+1)}{2} - \frac{k(k+1)}{2}$$

Figure 7. Edward's formula for the sum from k to n.

Table 1
Approaches to Reconstruction Tasks

<table>
<tr><td></td><td></td><td colspan="4">Approaches</td></tr>
<tr><td rowspan="4">Representation of Domain</td><td></td><td colspan="2">Structural</td><td>Oblique</td><td>Empirical</td></tr>
<tr><td>Collection of Specific Examples</td><td rowspan="3">Unravel relationships between the anchor domain and anchor claim, then synthesize relationships from the target domain to a target claim.</td><td rowspan="3">Unravel relationships between the anchor domain and the target domain, then adapt the anchor claim.</td><td>Find examples that satisfy the anchor claim.</td><td>Reason inductively.</td></tr>
<tr><td>Generic example</td><td></td><td></td></tr>
<tr><td>General representation</td><td></td><td></td></tr>
</table>

The essential differences among these approaches lies in the generality with which the elements of the domain are represented and in the relationship that grounds the response. Structural approaches seem to be amenable to the greatest variation in ways of representing the anchor and target domains, and are more well-suited to general representations than are approaches in which the target claim is populated without calling on an analysis of relationships among the anchor statement and target domain.

Implications

As one advances in mathematics education, strategies for developing general statements do not necessarily become more sophisticated. However, some individuals develop the ability to analyze and utilize mathematical structure with respect to general representations to produce new general statements. Questions for those who attend this session will be:

1. Are there particular tasks that might provoke more structural approaches?
2. What teaching strategies might help students learn to use more structural approaches?

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Inquiry-based learning in mathematics:
Negotiating the definition of a growing pedagogy

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Abstract: Inquiry-based learning is one of the pedagogies that has emerged in mathematics as an alternative to traditional lecturing in the last two decades. There is a growing body of research and scholarship on inquiry-based learning in STEM courses, as well as a growing community of practitioners of IBL in mathematics. However, despite the growth of IBL research and practice in mathematics, wide uptake of IBL remains hamstrung in part by the lack of a sophisticated discussion of its definition. Using a diffusion and framing analytical framework, this qualitative research paper offers a first step toward addressing this problem by describing how a group of IBL practitioners define IBL and how they adopt IBL to fit their specific teaching needs. We argue that early diffusion of IBL, for the group we studied, was constrained by the initial framing of the pedagogy and ongoing conflict over the proper definition and application of it in the classroom. Over time, however, the conflict proved to be beneficial to the community and a general consensus is developing among practitioners of IBL focusing on two core beliefs: 1) that it is only IBL if the student takes ownership of course material on a regular basis, and, increasingly, 2) that students collaborate as a class or in small groups to produce the mathematics.

Introduction

Calls to evaluate and reform undergraduate teaching in higher education have been commonplace since the 1980s (Boyer, 1980; Brint, 2011; NGA, 1986), and in the discipline of mathematics these efforts have been particularly urgent (MAA, 1988; NSF, 1993; NRC, 1991; Tucker & Leitzel, 1995) because of the central role of mathematics in both general education and preparation for careers in technical and scientific fields. However, the literature suggests that reforms often result in minimal, lasting change. Research focusing on higher education writ large suggests that reforms are ineffective because of the challenges faculty experience in balancing their teaching and research roles (Cuban, 1999), because an emerging consumer-centric culture surrounding higher education has prevented reforms from influencing student achievement (Brint, 2011), and because faculty themselves have little reason to change given disciplinary cultures (Abbott, 2002) and institutional incentive structures. For reform in STEM fields specifically, research points toward poor implementation or the lack of quality collaboration and communication between curriculum researchers and instructors as additional reasons for sluggish adoption of innovations (Henderson & Dancy, 2008; Henderson, Finkelstein, & Beach, 2010).

Inquiry-based learning is one of the pedagogies that has emerged in mathematics as an alternative to traditional lecturing in the last two decades. There is a growing body of research and scholarship on inquiry-based learning in STEM courses, as well as a growing community of practitioners of IBL in mathematics. In the late 1990s, the Educational Advancement Foundation founded the Legacy of R. L. Moore annual conference, which began as an effort to spread the specific teaching method of the late topologist R. L. Moore but has become a focal point for a group of young instructors who identify themselves as IBL practitioners. In 2009, the Academy for Inquiry-based Learning was developed to foster further growth of the community through its blog, list serves, and workshops (www.inquirybasedlearning.org). However, despite the growth of IBL research and practice in mathematics, wide uptake of IBL remains hamstrung in part by the lack of a sophisticated discussion of its definition. This paper offers a first step toward addressing this problem by describing how a group of IBL practitioners define IBL and how they adopt IBL to fit their specific teaching needs.

Analytical Framework

This paper defines IBL as an educational innovation and thus our analytical framework utilizes perspectives and concepts from the literature on the diffusion of innovation. One of the central features in diffusion theory is the extent to which individuals can positively identify with an innovation. In this study, positive identification emphasizes the presence of shared understandings of IBL within the IBL community and this is important for both the diffusion of the innovation to potential members and the ongoing use of the innovation by existing members.

Rogers (2003) argues that an innovation is more likely to successfully diffuse if it meets several criteria: it has relative advantage over alternative courses of action, it is consistent with the values, experiences and needs of potential adopters (what Rogers refers to as “compatibility”, see also Givan, Roberts, & Soule, 2010 on “theorization”), it is not too complicated for the average potential adopter to understand and implement, it is something that can be adopted to specific circumstances, and it is relatively easy for potential adopters to observe in action. The criteria of compatibility with individual values, experiences, and needs and the perception of the fit of the innovation to individual circumstance are particularly relevant to our concern about identification with the change process. For all of these criteria, the perceptions of potential adopters about the innovation are crucial, and these can be particularly influential when potential adopters form opinions and make decisions that influence whether they will even consider trying an innovation.

Individuals, then, will be more likely to adopt an innovation if 1) they already identify with the innovation in some capacity, 2) they identify with the definition of the problem that is addressed by the innovation, and 3) they feel they will benefit from adopting the innovation because the proposed changes fit their needs and circumstances. This focuses our analysis on how different individuals define IBL in mathematics, how they identify the problem addressed by IBL, and their perceptions of how they perceive it to fit (or not) their specific needs.

Central to the ability of individuals to connect to innovations is the issue of discursive framing, or what Strange and Soule (1998) call "interpretive work." Discursive framing refers to the ways that various discourses are used by different groups to frame an innovation's purposes, values, and actions. In many cases, individuals who seek to diffuse innovations must negotiate their own multiple understandings and identities related to an innovation itself and the understandings of it projected by other innovators (see also Givan, Roberts, & Soule, 2010). This cultural work, establishing the boundaries of key concepts and what it means to be a member of the group of innovators, is a dynamic, ongoing process. Taken together, this analytical framework focuses our attention on how the faculty in our study define IBL, how they interpret and enact it in practice, and how they perceive their perspective to fit into their broader understanding of the IBL community.

Methods

Data for this paper were gathered primarily through semi-structured interviews that took place over the telephone or occasionally in person. We strove for a conversational style in the interviews rather than a simple question and answer approach (Burgess, 1984; Seidman, 2006). For the purpose of our project, we defined the IBL community as the group of practitioners who attend and are connected to the annual Legacy of R. L. Moore conference. We used knowledge of the community gained from past-research and evaluation projects to identify a preliminary list of the core members based on current and past involvement in putting together IBL workshops or participating in the organization of the annual conference. Additional names were added to the list through the use of snowball sampling as data gathering unfolded (Merriam, 1998; Mason, 2002). Potential interviewees were sent a solicitation email providing them with background information about the study and how their interview would contribute to it. The majority of interviews were 60-90 minutes in length, were digitally-recorded and transcribed verbatim. These interviews covered a variety of topics related to instructors' knowledge of the history of the IBL community, how they defined and applied IBL in their own courses, their perspectives on the values and shared behaviors of the IBL community, and their hopes for the future of IBL. This paper is based on 25 such interviews with mathematics instructors at institutions across the United States.

These interviews were transcribed and coded using the data analysis software *Nvivo 10*. The coding process was informed by Miles and Huberman (1994), particularly the strategies of pattern coding, clustering, identifying intervening variables, and making conceptual coherence. The first stage of coding utilized concepts from the diffusion literature. Initial codes were developed based on how speakers defined IBL (what it is and is not), and how they have reinvented or adapted it to fit their various teaching needs. Secondary coding schemes were developed through analysis and exploration of the patterns in the relationships between different clusters of the primary coding data.

Additionally, we made use of an open-ended survey item from a past study of this community that asked respondents to define IBL. As this survey sampled a broader group, the survey responses provided a necessary and insightful complementary data set that allowed us to compare and contrast the definitions of IBL for the core leadership group of the community and by members who are more peripheral.

Results

All of the early adopters we interviewed remembered the last years of the 1990s as a period when they all realized they needed a new generation to join the IBL community. The early adopters are all instructors who were taught by and learned IBL directly from R. L. Moore or one of his academic descendants. Beginning in 2003, these early adopters developed three different programs to reach out to potential members and educate them about IBL: 1) they began a series of annual workshops inviting new people to attend to learn about IBL and network with experienced IBL users, 2) they started four IBL centers in mathematics departments in universities across the United States that taught courses to future mathematicians using IBL and exposed graduate students and post-doctoral fellows to the teaching method, and 3) they connected with the Mathematics Association of America's Project NexT program, which focuses on the professional development of early career faculty in mathematics. Collectively, these three programs successfully exposed a new generation of instructors to IBL by 2010.

However, this exposure did not automatically translate into widespread diffusion of IBL because of two primary framing-identity problems. First, the pedagogy was intimately connected with the memory of the late R. L. Moore, which linked it to a personality and social perspective that many potential new users find troubling. Moore was a controversial figure because of his racist and sexist perspectives (Parker, 2005, pp. 287-290). Nearly all instructors interviewed acknowledged this fact but there were generational differences in how they negotiated its meaning. Older generations rationalized his social views by emphasizing the culture of the South that surrounded him or by attempting to focus only on his teaching method and ignoring the problematic aspects of his image. This enabled the older generation to maintain a positive identification with IBL. However, younger members of the community were not able to maintain such a positive identification with IBL so easily. Many of these younger instructors discussed the limiting role those social views have on getting some of their colleagues to attend community events. For example, one younger community member spoke about Moore's connection to IBL preventing many potential new members who already had teaching philosophies in line with the group from attending community events like the annual conference.

> [P]art of Moore’s legacy is racist, and...there are people who would be on board with [our] ideas and wouldn’t have any trouble being part of the community at the conference, but because his name is still associated with it, sort of on principle, wouldn’t consider it.

In attempting to spread IBL to a new generation of instructors, most early adopters failed to recognize the importance of Moore's racism and sexism to the population they were targeting.

Secondly, the original labeling of IBL as the “Moore Method” and later the “Modified Moore Method” was not ideal, as this framing using a name that is not broadly known in the discipline of mathematics prevented potential adopters from understanding or identifying with the teaching method. In fact, the majority of those interviewed who joined the community after its founding had never heard of the “Moore method” until they heard about the annual conference or encountered someone who was already in the community and realized they were teaching in a similar way. Together, these naming and framing issues made the processes of persuading new users to implement IBL and become active in the community challenging.

Many of the young faculty who joined the community as a result of the workshops, the IBL centers, or Project NExT expressed feeling out of place the first time they attended the annual conference. One instructor even referred to the culture of the group in the early years as a “cult of personality” surrounding Moore.

> There's something about the structure and the history of the group that, for one reason or another has not resonated with everyone, I think. Certainly, it was the case at my first Legacy meeting. It was sort of a cult [of] personality meeting. Everybody talked about R.L. Moore and what a great influence he had had on each of them personally, and what a transformative experience they'd had and how they felt empowered by him.

Another community member from the same generation agreed and offered a stark criticism of the group's culture when he first started.

> It was extraordinarily off-putting the way that the older members of the community would talk about their bloodline, their genealogy, their purity, if you will; with whom they studied and which descendant of whom they took courses from.

While the older generation of this community clearly championed the value of their experiences with Moore, or his descendants, and the type of IBL he taught, the ways in which they spoke about their experiences ostracized potential new members. Rather than stop going to the conference altogether or ceasing to use IBL, some of these younger generation members formed their own subgroup that is more welcoming and less Moore-centric in their thinking about IBL. One instructor offered a brief history of these group dynamics.

> One of the first few...conferences I went to, it felt like there was a divide. There was this group of people that was really devoutly following Moore and what Moore did and that what he did was very important and it should be preserved. Then there was another group of people who were trying to introduce new ideas, and this was, I feel like this group of people that were open to new ideas, seemed like the minority the first time I crashed the conference. Then, as the years went on, it started becoming the vocal majority, they were the ones organizing.

As the younger generation increased in number and had more influence in community events, their ability to challenge the dominant framing of IBL as connected to Moore increased as well. In contrast to the early group of instructors who identified with Moore-centric definitions of IBL, the growing sub-group of instructors increasingly embraced broader adaptations of the pedagogy.

Once enough new users identified with IBL and adopted it in their own practice, conflict increased between competing visions of how IBL should be labeled and defined. Several instructors, old and new, remembered arguments during paper sessions at the annual conference about the "proper" definition of IBL over the years. For example, one of the central pieces of conflict has been over whether collaboration among students should be considered part of IBL teaching methods. Older adopters stand by Moore's insistence that it is ultimately more empowering if students arrive at answers on their own while newer members argue that collaboration is more comfortable for students and ultimately more effective in today's increasingly diverse classrooms. Furthermore, the community has recently been publicly challenged to move away from an association with Moore altogether in favor of a strategy intended to recruit the faculty that were historically turned away by the problematic connection between Moore and IBL. The complete impact of this discursive move remains to be seen, but it is clear that it has energized the younger majority of the community. However, this relative, and perhaps even temporary victory for this group, took years to develop and was the result of conflict as original members and new members attempted to work through their initial differences in definition and understanding.

Today, though many in the IBL community embrace a much broader framing of the pedagogy, many practitioners are concerned that it is becoming too broad. Over time, as more instructors use IBL methods in their mathematics courses, the group has increasingly realized that it needs to be adapted to fit new circumstances, new groups of students, and new courses. As a result, many early adopters and change agents are concerned about how they police the boundaries of the pedagogy. For example, the annual conference has recently accepted papers on flipped-classrooms and not all members of the community agree that this is appropriate. Nonetheless, the persistence of instructors with a broader definition of IBL have therefore seemingly won the day by providing the discursive framing necessary for newer faculty to identify with the movement and the pedagogy. The emerging dominant frame is one that avoids ideological connection with Moore and his problematic legacy and instead highlights two core beliefs: 1) that it is only IBL if the student takes ownership of the material on a regular basis, and, increasingly, 2) that students collaborate as a class or in small groups to produce the mathematics. Thus far these core values have been enough to successfully recruit new members in the last few years while also preventing IBL from being too watered down in the eyes of the community.

Conclusions and Significance of the Research

Results of this study raise important implications for understanding reform efforts in higher education. Arguments for why reforms of teaching are slow to take hold—or die off altogether—focus on well-known contextual issues: that faculty focus on research more than teaching, that changing teaching requires more time than faculty have, or that instructors struggle to adequately implement new curriculum or pedagogy. This study reminds researchers, as well as practitioners, that it is also important how reform efforts are framed by those advocating for them and how potential adopters perceive new curriculum or pedagogy matters. While finding teaching methods that engage students or promote learning is important, researchers and practitioners must frame their innovative reforms in ways that connect with instructors' preexisting identity, values, perceived needs for their students.

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Developing an open-ended linear algebra assessment: Initial findings from clinical interviews

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The primary goal of this study was to design and validate a conceptual assessment in an undergraduate linear algebra course. We work toward this goal by conducting semi-structured clinical interviews with 8 undergraduate students who were currently enrolled or had previously taken linear algebra. We try to identify the variety of ways students reasoned about the items with the intent of identifying ways in which the assessment measured or failed to measure students' understanding of the intended topics. Students were interviewed while they completed the assessment and interview data was analyzed by using an analytical tool of concept image and concept definition of Tall and Vinner (1981). We identified two themes in students' reasoning: the first theme involves students reasoning about span in terms of linear combinations of vectors, and the second one involves students struggling to resolve the number of vectors given with the number of entries in each vector.

Key words: assessment, linear algebra, inquiry-oriented, student thinking

Students from a variety of science, technology, engineering, and mathematics (STEM) disciplines are required to take linear algebra as part of their undergraduate mathematics coursework. Students typically struggle with the theoretical nature of linear algebra as it is often their first time grappling with abstract mathematical concepts (Wawro, Sweeney, & Rabin 2011). Students' mathematical background up to this point is often primarily computational in nature; this often creates a barrier for students to overcome when they reach linear algebra (Carlson 1993).

Linear algebra is a pivotal course that includes mathematical underpinning of different STEM fields, but it is rife with challenges for students. According to Wawro (2011), "The content of linear algebra, however, can be highly abstract and formal, in stark contrast to students' previous computationally-oriented coursework. This shift in the nature of the mathematical content being taught can be rather difficult for students to handle smoothly." The abstract concepts of linear algebra are often taught in such a way that students do not find any connections between new linear algebra topics and their previous knowledge of computational mathematics (Carlson 1993). Researchers have worked to address this issue by developing inquiry-oriented instructional materials that help instructors and students bridge students' informal and intuitive ideas with more formal and conventional understandings (Wawro et al., 2013) This work aims to move toward documenting the effectiveness of these materials in supporting students' conceptual understanding of central topics in an introductory undergraduate linear algebra course.

In this study we have designed an assessment that aligns with four focal topics typically covered in an introductory linear algebra course: (1) linear independence and span, (2) linear systems, (3) linear transformations, and (4) eigenvalues and eigenvectors. We aimed to identify two questions for each of these four topics in order to develop an 8-item written assessment that could be completed by students in less than one hour. Based on findings from similar studies, we anticipate that we might see greater conceptual learning gains for students who learned in inquiry-oriented classrooms along with similar procedural learning gains (Rasmussen & Kwon, 2007). Research questions for this proposal are:

- What is the nature of student thinking elicited by the items on our assessment draft?
- To what extent do the items accurately measure student thinking?

Literature & Theoretical Framing

Difficulty in teaching and learning of linear algebra during students' first year of undergraduate study is well documented (Hillel, 2000; Sierpinski, 2000; Stewart & Thomas, 2009). Students often struggle with fundamental concepts like span, linear dependence, linear independence, and basis (Stewart and Thomas, 2009). Additionally, the need to learn and coordinate modes of the description and representation of abstract concepts of linear algebra can function as a source of difficulty for students (Hillel, 2000).

A theoretical construct that has been useful in many areas of mathematics education for making sense of students' struggles as they work to make sense of a new idea is the notion of concept image and concept definition (Tall & Vinner, 1981). The key distinction here is that the ways in which students reason with and about a mathematical construct is often different from (and often at odds with) the definition of that construct which is accepted by the broader mathematical community.

Researchers have been using the constructs of concept image and concept definition to analyze and understand students' thinking and understanding of concepts for more than three decades (Wawro et al. 2011). Britton and Henderson (2009) made use of concept image and concept definition to analyze the conceptual difficulties of students in linear algebra, especially about vector space and subspace. We draw on Tall and Vinner's (1981) notion of concept image and concept definition as an analytic tool for interpreting students' responses to assessment items.

According to Tall and Vinner (1981) concept image is the "total cognitive structure that is associated with the concepts, which include all the mental pictures and associated properties and process" (Tall & Vinner, 1981 p.152). For a given concept, every individual creates an image or structure in their mind that helps the individual understand and remember that concept. This concept image may or may not be similar to other individuals' images, and these images can be quite different from the formal definition of the concept. Moreover, Wawro et al. (2011) contend that concept image is not a static entity; it instead changes over the time and with new knowledge. Tall and Vinner (1981) use the term 'formal concept definition' to refer to the definition that is largely accepted by the mathematical community; they argue that this can be different from an individual's 'personal concept definition,' which may change over the time and with new knowledge as is the case with one's concept image. For our analysis, we look for alignment between a student's elicited concept image and the formal concept definition as evidence of understanding.

Data Sources

In this study, we conducted hour-long semi-structured clinical interviews (Bernard, 1988) with 8 university undergraduate students: 6 males and 2 females. One of the participants was taking linear algebra at the time of the interview, and the other participants had taken linear algebra within the last two years. The participants' majors covered fields that included mathematics, education and economics. Participants had taken an average of four math classes after linear algebra.

Every participant was asked to work through eleven assessment questions using a think-aloud interview protocol, in which the interviewer asked the student to read each item aloud and think aloud as he or she came to an answer. The interviewer then asked follow-up questions as needed to understand the student's reasoning in arriving at their answer. Each interview lasted for approximately one hour and was audio and video recorded. In this preliminary report, we consider participants' responses to the first interview question, shown below in Figure 1.

1. Answer the following questions regarding the set of vectors $V = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$.

a. Which of the following best describes the <u>span</u> of the set V?
   i. A point
   ii. Two points
   iii. A line
   iv. Two lines
   v. A plane
   vi. Two planes
   vii. A 3-dimensional space

b. Give an example of one vector in the span of V, and show or explain how you found that vector.

c. Give an example of one vector in R^3 that is not in the span of V and explain how you know it is not in the span of V, OR explain why there is no such vector in R^3.

Figure 1: Assessment item focused on span

We developed the assessment items used in this study by consulting past assessments prepared by 5 different mathematics faculty members at different universities, some of whom had been involved in the development of the IOLA materials, and others of whom had not. After identifying a set of questions related to each of our four focal topics, three mathematics faculty members from three different institutes were consulted to identify which items these experts felt focused on key ideas and had the potential to assess students' conceptual understanding of these ideas. We modified our assessment according to experts' initial feedback, and the assessment items to be used in interviews were selected after receiving a second round of feedback from these experts. We piloted the assessment with two students and made minor adjustments based on these students' responses. This modified assessment was used for the remaining interviews.

Methods of Analysis

In order to identify the kinds of student thinking elicited by our assessment items and the extent to which these accurately assessed student understanding, we conducted our analysis in four phases: (I) characterizing individual students' concept images, (II) identifying themes across students, (III) documenting students' written responses, and (IV) relating written responses to concept images elicited. Phases I and II will allow us to identify the kinds of student thinking elicited by our assessment items. Phases III and IV will offer insight to the extent to which the items accurately assessed students understanding. Specifically, we look to see whether the assessment item accurately documents alignment between student's concept image and the formal concept definition. The phases of analysis are described in greater detail below.

Phase I: Characterizing individual students' concept images. We developed a short description of each student's concept image of span by first watching the video and transcribing each student's interview response to question 1. We then developed a list of themes that characterized how he/she thought about span and collected quotes that exemplified characteristics of the student's thinking.
Phase II: Identifying themes across students in how students reason. In this phase, we grouped students according to the nature of their concept images. This helped us document themes in how students reason about the items. These groupings of students' concept images were organized in a table to make it easier to identify trends in thinking.
Phase III: Documenting written responses. In this phase, we identify what students stated their final answer would be (and other answers they offered if they changed their mind) as well as the justification they offered for their answer(s). This was identified by drawing on students' written work as well as using audio/video data as needed in cases when the response was given orally but not written on the student pages.
Phase IV: Aligning concept images with item responses. Students' responses to each item on the assessment were aligned with their corresponding concept image. Each response was color coded to indicate whether a correct response corresponded to correct or incorrect reasoning and whether an incorrect response corresponded to correct or incorrect reasoning. This will be used to assess the extent to which the item accurately measured what we intended.

Findings

After interviewing students and transcribing their interviews we analyzed the first assessment item to document students' concept image of span. In this preliminary report, we summarize themes we noted in students' concept images on this item and speculate on what this tells us about what our item is measuring, as well as what it needs to measure. In our presentation, we will provide a synopsis of all four phases of analysis for this item as well as other items on this assessment.

We identified two themes in students' reasoning as evidenced by their responses: the first theme involves students reasoning about span in terms of linear combinations of vectors, and the second theme involves students struggling to resolve the number of vectors given with the number of entries in each vector. (In addition, there was one student who didn't remember what span was, so he answered all parts of the question as if it was just referring to the set of vectors V rather than the span of that set of vectors; using this reasoning the student gave correct answers for 1b and 1c.)

Three of the students reasoned about all parts of the span assessment item in terms of the set of all possible linear combinations of the set of vectors given. Unsurprisingly, these three are the students whose concept image was consistently well-aligned with the formal concept definition. Interestingly, all three of these students offered rich geometric interpretations as part of their elicited concept image. This suggests to us that geometric intuition might be an important aspect of the concept image needed for students to successfully reason through this item (and potentially other items) regarding span, even though the formal concept definition of span does not necessarily entail a geometric interpretation. For example, one student Lewis explained his reasoning to question 1a: "A span is a linear combination or it would be any kind of linear combination of these two [pointing towards V]… because they are linearly independent, so any span of these two vectors will be linear combination of the two vectors so reproduce a plane."

Four students struggled to resolve the number of vectors given with the number of entries in each vector, but they resolved this issue in a variety of ways. For instance, one student noted the vectors were in three dimensions and concluded (incorrectly) that the span must be 3-dimensional, meaning that no vector in R^3 can be outside of the span. Another student, Beth, struggled with the same issue, but resolved it correctly by reasoning that "each vector has three entries in its column … that means that it is in the third dimension, I think there is only two vectors though, I think I need a third vector in order for this to actually span the third dimension and so since there are two it will span just the second dimension and the second dimension will be a plane so then it might actually just be a plane." One student resolved the issue by putting the vectors of the matrix V into a matrix, row reducing the matrix, and counting the number of pivot columns. Since there were two of something, he felt the span should be either two points or two lines, but he wasn't sure which because he didn't have a geometric interpretation. The final student concluded that the span of V would be two planes because each vector represents a plane. Interestingly, these students tended to give a linear combination of the vectors of V on part b of the question when asked for an example of a vector that was in the span (though some had interesting limitations on how those combinations should be formed, e.g. thinking the coefficients had to be integers). This suggests to us that a significant source of difficulty for students developing a rich concept image of span lies in coming to think simultaneously about *all* possible linear combinations of a set of vectors.

Implications/Future Work

Our findings suggest two things about the design of an assessment item focused on documenting students' understanding of span. First, is that we need to find a way to assess students' strategies for resolving differences between the number of vectors and the number of entries in each vector. Second, we likely want to include separate prompts that offer insight into students understanding of linear combinations and their understanding of span as the set of *all* possible linear combinations. We have endeavored to assess the latter using a geometric approach. Extending our analytic strategy, we intend for our analysis of other items to similarly inform aspects of student understanding that need to be measured by our assessment.

Questions for Audience

- When we conduct quantitative analysis, how do we account for the relatedness of subparts of questions (e.g. 1a, 1b, 1c)? We view this as a strength of the assessment, but are unsure of how to account for it methodologically.
- What is the contribution of this work? Is it methodological (is the method of refining assessment items new/novel/worth writing about)?
- How can we think about assessing the quality of the assessment as a whole rather than item by item?

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A qualitative study of the ways students and faculty in the biological sciences think about and use the definite integral

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In this poster, I share my methods and pilot interview results concerning a qualitative study of the ways undergraduate students and faculty from the biological sciences think about and use the definite integral. In this research, I utilize task-based interviews including five applied calculus tasks in order to explore how students and faculty think about area, accumulation, and the definite integral. Early results from pilot interviews helped me revise the interview protocols and indicate that student reasoning may be affected by experience and context. In presenting this poster, I hope to gain feedback from the community on my research methodology and potential analytical strategies.

Key words: calculus, biology, definite integral

The teaching and learning of calculus is currently a topic of great interest. The MAA recently supported a large-scale survey of calculus programs that has generated a great deal of literature surrounding calculus instruction at the undergraduate level (Bressoud, Mesa, & Rasmussen, 2015). Furthermore, there have been calls by researchers for investigations of how calculus is utilized by non-mathematics majors. Rasmussen, Marrongelle, & Borba (2014) call for "research that closely examines the ways in which calculus ideas are leveraged in the client disciplines, how these ideas are conceptualized and represented in the client disciplines, and what these insights might mean for calculus instruction" (p. 513). One of the most prominent client disciplines of calculus are the biological sciences. In their survey of over 10,000 undergraduate calculus students, Bressoud, Carlson, Mesa, & Rasmussen (2013) found that 30% of all Calculus I students intended on pursuing careers in the biological and life sciences (p. 691).

Mathematics is important for the biological sciences. In 2003, the National Research Council (NRC) published a report titled BIO2010, suggesting university biology programs develop stronger connections between the life sciences and the mathematical, physical, and computer sciences (NRC, 2003). As a result of that report, a number of undergraduate biology departments across the country incorporated changes in how the quantitative sciences are utilized; some revised the calculus sequence to focus more on mathematical techniques while others created a brand new degree program focusing on quantitative biology (e.g., Usher et al., 2010).

The definite integral is an important topic in introductory calculus that has been a focus of researchers since the 1980s (e.g., Jones, 2013; Orton, 1983; Sealey, 2014; Thompson & Silverman, 2008) and plays an important role in both mathematics and biology. Definite integrals are used when modeling population growth and cardiac output, as well as in chromatography (Horn, 1987). When asked what biology students need from calculus, biologists typically cite numerical approximation methods (e.g., the trapezoidal rule for approximating area under a curve) and a focus on modeling as opposed to the ability to manipulate complicated algebraic techniques (Horn, 1987; NRC, 2003). Additionally, researchers have shown that biology students tend to have lower self-efficacy when it comes to their mathematical ability when compared to physics and engineering students (Brent, 2004; Chiel, McManus, & Shaw, 2010). While calculus plays a vital role in the preparation of biology students and biological science programs have attended more to quantitative skills, there has not been a great deal of research on how students understand and use their calculus knowledge specifically in biological settings.

This research project serves as such an examination as I explore how undergraduate students and faculty members from the biological sciences think about and use the definite integral. My research questions in this study are: (1) What are the ways beginning and advanced undergraduate students majoring in the biological sciences think about and use the definite integral? (2) What are the ways professional biologists think about and use the definite integral? and (3) What are the similarities and differences in how beginning undergraduate students, advanced undergraduate students, and faculty members in the biological sciences think about and use the definite integral?

Methods

In order to investigate how students and faculty think about and use definite integrals, we need rich descriptions of the ways in which they attend to and use the definite integral while solving problems and working in their field. Therefore, I am using task-based interviews in which I ask participants to talk about their knowledge of definite integrals and calculus, as well as solve applied calculus problems as the data source for my study. I am interviewing 10 beginning and 10 advanced undergraduate students majoring in the biological sciences, and 5 faculty members from the biology department at a large southeastern public university. The calculus tasks span graphical, analytical, and tabular representations and are set in primarily biological contexts. Two of the tasks parallel each other in structure and form, using the same graph but with different axes labels. One task is biologically-based and the other utilizes a car's position and velocity. Interviews with the faculty members focus on how the participant uses calculus and the definite integral and how important they feel mathematics in general, and calculus in particular, is to their work and to their students.

Conceptual framework and data analysis

Researchers have previously investigated the ways in which students reason about applied integration problems (Jones, 2015; Sealey, 2014). My data analysis procedures begin by analyzing the students' responses for the overarching conceptualization of the definite integral they are attending to using the three primary conceptualizations illustrated by Jones (2015) and then drill down into how they are using those conceptualizations to solve the problem using aspects of Sealey's (2014) Riemann Integral Framework as applicable. While these frameworks serve as a foundation to my data analysis, I will employ a pseudo open-coding scheme in order to identify additional themes that may be unique to individuals from the biological sciences.

Pilot Study

This past semester, two undergraduate students volunteered to participate in informal interviews in order to help me revise my interview protocol and I was able to both edit my items, as well as determine that the items were sufficient for collecting appropriate data. For example, I found that each student reasoned differently on the parallel tasks; one student called on an area under the curve conception in the biologically-based task but not the velocity task. I hope to continue exploring the ways undergraduate students and faculty reason about the definite integral with a full run of interviews in January 2016.

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Measuring student conceptual understanding: The case of Euler's method

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This preliminary paper reports on early work for a differential equations concept inventory, which is being developed for an NSF-funded project to support mathematics instructors as they implement inquiry-oriented curricula. The goal is to assess student learning of differential equations. Preliminary results show that the iterative method of developing and field testing items, conducting student interviews, and modification may prove successful to complete a valid concept inventory. The field testing and piloting of questions concerning Euler's method show that students do respond as the research suggests but that Euler's method can be recreated by students and the correct response can be "figured out."

Key words: Differential Equations, Concept Inventory, Assessment, Euler's Method

One of the most challenging components of education research is measurement. How do you measure student understanding? How can you tell if your measurements are accurate? In the so-called "hard" sciences, measurement involves a physical instrument, rulers for lengths, scales for weights, etc. Alternatively, education researchers spend a great deal of time and effort both defining and describing their constructs as well as outlining how they can measure those constructs. Along those lines, we are developing an assessment to enable us to measure student understanding of the big ideas in differential equations (DE), as part of a larger study to assess how students taking an inquiry-oriented differential equations course understand the basic concepts of DE compared with students who have taken a traditional DE course. With this report, we discuss this process with a focus on one particular concept: Euler's method to solve a differential equation. Our research question is: How can a written assessment effectively measure students' conceptual understanding in differential equations, particularly the numerical technique called Euler's method?

Literature Review

Assessment

Researchers and instructors have developed a vast array of different types of assessment to aid in measuring learning. Concept inventories are research tools designed to measure learning with special attention paid to their validity and reliability in gauging how students think about the underlying concepts of a subject. There have been a number of concept inventories created in many different academic subjects. Two of the physics assessments, the Mechanics Diagnostic Test (Halloun & Hestenes, 1985) and the Force Concept Inventory (Hestenes, Wells, & Swackhammer, 1992) have been influential in the development of mathematics concept inventories. Common themes include the use of student interviews in the validation process and the iterative nature of writing and revising both the taxonomy and assessment items. Following from the two physics concept inventories, two foundational mathematical concept inventories are described below, the Precalculus Concept Assessment (Carlson, Oehrtman, & Engelke, 2010) and the Calculus Concept Inventory (Epstein, 2007). In our review of these assessments, we identified four primary steps in developing a concept inventory: (1) deciding what concepts to cover, i.e. the taxonomy of the assessment, (2) writing the assessment items, (3) validating the assessment items, and (4) validating the assessment as a whole.

Calculus Concept Inventory

The Calculus Concept Inventory (CCI) was designed and validated based on the Force Concept Inventory: the major concepts to be assessed were outlined, the items were written by a team with knowledge of the content, then items were reviewed using clinical interviews or what Epstein calls “cognitive laboratories” (2007, p. 167). Finally, a cyclic process of revision and analysis took place. In their first pilot of about 250 students, Epstein noted that scores were near the random guess level, which led to significant modifications of the items, specifically making them much easier. The CCI continues to be in use, but there has not been a great deal of studies published about the results.

Precalculus Concept Assessment Inventory

Carlson et al. (2010) provide great depth on the creation of the Precalculus Concept Assessment (PCA). First, the researchers developed a 34 item, open-ended assessment for the purposes of investigating students understanding of function. The results of the administration of this assessment came to form a Function Framework, which would serve as an initial draft of the PCA Taxonomy. The same process as the other concept inventories continued and the developers are validating the PCA currently.

Selected research on student understanding of differential equations

Analytical Solution Strategies

Researchers have found that students overwhelmingly elect analytical solution strategies when prompted to solve differential equations (Arslan, 2010; Camacho-Machin, Perdomo-Diaz, & Santos-Trigo, 2012a; Habre, 2003; Rasmussen, 2001). However, while students are relatively successful in finding solutions using various analytical solution techniques, they struggle to identify the appropriate strategy for various problem types (Camacho-Machin et al., 2012a) and that ability in solving DEs analytically does not necessarily imply deeper conceptual understanding (Arslan, 2010).

Graphical Solution Strategies and Representations

Students tend to devalue graphical representations of both the DE and their solutions instead relying heavily on analytical techniques and algebraic representations (Habre, 2003; Rasmussen, 2001; Trigueros, 2001). Specifically, researchers have shown that students have trouble both understanding graphical representations as well as constructing them (Camacho-Machin et al., 2012a; Camacho-Machin, Perdomo-Diaz, & Santos-Trigo, 2012b; Rasmussen, 2001). Fortunately, a few studies have provided evidence that students can use graphical representations productively, usually after being prompted (Habre, 2003), and that certain instructional strategies have been shown to help students retain more knowledge concerning graphical representations (Habre, 2003; Kwon, Rasmussen, & Keene, 2005).

Numerical Solution Strategies

Very little educational research exists on student understanding of Euler’s method. Rasmussen (2001) characterizes students’ conceptions of approximate solutions in three ways: (1) “A numerical approximation inscribes the exact solution,” (2) “A numerical approximation ‘tracks’ the exact solution by using the slope of the exact solution at each step in the approximation,” and (3) “A numerical approximation ‘tracks’ the exact solution via nearby solutions” (p. 76). Furthermore, Rasmussen describes how students’ ideas about other approximation methods in mathematics (e.g., Riemann sums and the definite integral) may play a role in how students think about numerical approximations in DEs. These potential mental images of the relationship between approximate and exact solutions informed the selection of multiple-choice items in our concept inventory (see Selected Findings).

Theoretical Framework

Some research and writing to develop a version of a concept inventory for differential equations exists. This earlier work uses the Relational Understanding of Procedures

Framework to address how knowledge in DE may be constructed (Keene, Glass, & Kim, 2011). The primary categories in this framework are: anticipate the outcome, identify the correct procedure, correctly use the procedure, understand the "whys" of the procedure, verify the solution graphically and symbolically, and make connections across the representations involved. An assessment was developed by mathematics educators and piloted in 2008. The assessment was comprised of 30 questions that all related to two analytic solution techniques and one numerical solution technique (Euler's method). Field testing was conducted and revisions were made. Our current work is framed by and directly utilizes this earlier research and framework.

Methods

In our review of the pre-existing concept inventories, we identified four primary steps in developing a concept inventory, (1) deciding what concepts to cover, i.e. the taxonomy of the assessment, (2) writing the assessment items, (3) validating the assessment items, and (4) validating the assessment as a whole. Therefore, in order to create the differential equations concept inventory (DECI), we first had to decide on what topics and concepts should be covered by the assessment and then write or compile the items we felt best assessed students' knowledge. To create the list of topics and concepts, also referred to as the taxonomy for the assessment, we completed a syllabus analysis of various DE courses, spoke with experts, and pulled from our own expertise informed by our experience as well as our work with the existing literature on differential equations assessments and student thinking.

Syllabus analysis

We investigated a sampling of ten DE courses from across the United States, including a wide range of universities. We selected a mix of public and private universities (six and four respectively), various sizes (smallest: approximately 3,200 undergraduates, largest: approximately 46,000), and a range of different primary textbooks (six different textbooks from the ten chosen courses). From these syllabi, we created a list of topics covered by each course and then aligned them to generate a list of any topic or concept that were listed frequently. Even though not complete, this did provide a useful reference in deciding what specific topics to include on the assessment and was a useful store of information when drafting the taxonomy of the assessment.

Taxonomy

We constructed the taxonomy for the DECI by beginning with the relational understanding framework and then referencing results from a syllabi analysis, discussions with experts on DEs, the cognitive research on DEs, the list of topics from the inquiry-oriented differential equations course, and the research teams' experience. We started by using the overarching themes discussed by DE experts as well as the topics covered on the relational understanding framework from Keene, Glass, & Kim (2011). From here, we looked at the major topics with considerable overlap from the syllabus analysis (including the inquiry-oriented differential equations course) to ensure we were hitting a number of the important topics from standard DE courses. This taxonomy underwent significant revision as we gathered evidence of student thinking on the various items through the task-based interviews. Space does not permit us to publish the taxonomy here.

Field testing data collection

In Spring 2015, we piloted a selection of possible items in two DE classrooms. Two differential equations teachers (referred to herein as classes A and B) who were participating in an online workgroup for an NSF-funded project to investigate instructional change agreed to use some of the potential assessment items in their introductory DE class. Class A was an introductory DE course of 15 students using inquiry-oriented materials at a small public liberal arts college in the eastern United States and they responded to the items on the final exam. Class B consisted of 20 students at a private university in the southern United States and responded to the items in an out of class assignment. This course focused on more traditional procedures and proof.

In Summer 2015, we conducted task-based interviews with five students in a summer section of DEs at a large research university in the southeastern United States. During the interviews, students were asked to work on assessment items in front of the researcher and to think aloud about their process. The interviewer asked probing questions while trying to minimize the introduction of any new mathematical concepts or vocabulary. Questions were selected for the interview protocol based on the preliminary findings of the field test described earlier and included both multiple choice and open-ended items. Interviews were audio and video recorded, and all written work was collected and scanned as PDFs for preservation and analysis.

Data analysis

Written Assessment

During the spring administration of the DECI, there were both open-ended and multiple choice formatted items. In this report we are only discussing multiple-choice items and so we will focus our discussion of analysis methods to those items. For the multiple-choice items, analysis consisted primarily of investigating the appropriateness of the answer choices and the difficulty of each item. The data were entered into a spreadsheet where each participant's answer choices were included for each question. Two separate analyses were carried out on these items, the first concerning how many students answered the question correctly and the second concerning how many students chose each of the provided distractors. In terms of difficulty, we looked to determine if any of the questions were either too challenging or too easy.

After the initial analysis for difficulty, analysis followed on the students who answered questions incorrectly. Primarily, the goal of such analysis was to ensure that the distractors were working effectively and that each was being chosen at least some of the time. In previous work, assessment authors have marked any distractors that were not chosen at least 5% of the time for potential revision (Carlson et al., 2010) and so this rule was our initial guide for throwing out distractors.

Task-Based Interviews

In the near future, we will be conducting an analysis of the task-based interviews in order to continue validating the assessment items. Identifying the ways in which students thought about the items will provide us with evidence that the items are actually measuring the concepts we assume they are measuring. On multiple-choice items, our primary goal will be to investigate what knowledge students attend to as they complete the items. To do this, we will employ an open-coding strategy on the video data in an attempt to outline the big ideas from DEs that students are attending to as they work through the problems.

Selected Findings: Euler's Method Question

For this preliminary report, we focus on one particular question that was administered to both pilot groups and used in the task-based interview (Figure 1).

5. Imagine you were to carry out the Euler method approximations with one step with $\Delta t = 1.0$ and two steps with $\Delta t = 0.5$ for the same differential equation. Assume that for both approximations you use the same initial condition $y(0) = 0.4$, that your rate of change is positive, and that the exact solution is concave up. If you sketched this on the same set of axes you would see:

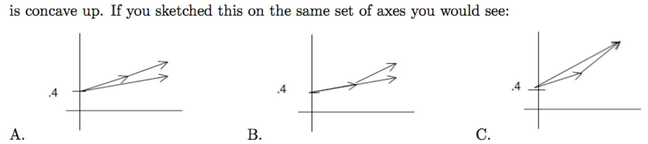

Figure 1. Euler's question.

The correct answer to this question is B. In using Euler's method, one uses the value of the derivative (rate of change) at one point to create a line segment for a defined constant change in the independent variable (in this case t). You then reevaluate the derivative (rate of change) and create a new line segment.

The results from the two classes and the interviews were particularly interesting and informative (see Figure 2). For the inquiry oriented class ($n=14$), the majority of the students answered correctly. We posit this is because this may have been explicitly discussed in the course, as the materials focus on the understanding of Euler's method. The traditional class ($n=14$) answered primarily C. This aligns with Rasmussen's (2001) findings on approximations, specifically that when doing the approximations, students want the lines to "track" the actual solution.

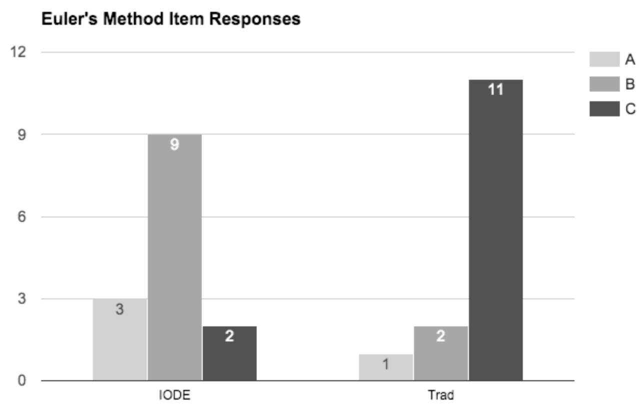

Figure 2. Results of choices for the two classes.

When the researcher asked this question of the five interview students, we found out more interesting information. The students in the interview had not ever seen Euler's method before so initially they did not know how to answer. However, two of the students were able to use information presented in the problem context with their knowledge of other approximation methods (e.g., Taylor series approximations of functions) to reason their way through to the correct answer, just as Rasmussen (2001) discussed. Afterwards, they were still not confident they were correct but had, in that moment, recreated Euler's method for approximating solutions to DEs.

Conclusion

The analysis and field testing of this assessment will continue during the next two years, but we have found that this method of developing a concept inventory seems to be useful. We know that the distractors on this question and others need to be revisited to make sure they are effective. We intend to continue with this same work, even though it is very time consuming. Thus far, we have found interest in the DECI to be high and we would like to include the RUME community in the continuation of this work. To this end, we will ask the following questions in the presentation:

1. What alternate conceptions do you see when teaching your students techniques to solve differential equations?
2. What do you consider the most important conceptions students need to develop in differential equations?
3. Do you know of alternatives to concept inventories to help assess student learning?

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Example construction in the transition-to-proof classrom

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Abstract. Accurately constructing examples and counterexamples is an important component of learning how to write proofs. This study investigates how one instructor of a transition-to-proof course taught students to construct examples, and how her students reacted to the instruction.

Keywords: transition-to-proof; undergraduate instruction; example construction

Introduction

Learning to write proofs is a complicated process, and students develop a variety of beliefs about how to construct a proof (Harel & Sowder, 2007). Using examples is one possible strategy in the proof writing process. Examples can be used for several purposes when developing and proving conjectures (Alcock & Inglis, 2008; Alcock & Weber, 2010; Lockwood, Ellis, & Knuth, 2013).

The term example can have many different meanings in mathematics (Watson & Mason, 2002). Within this study, the term example is limited to a mathematical object which satisfies specific characteristics and illustrates a definition, concept or statement (Moore, 1994). This definition excludes sample proofs, e.g., demonstrations of the direct proof technique or proving by induction. Alcock and Weber (2010) claim that this definition of example is "probably the most common intended meaning of the term when it is used by mathematicians and mathematics educators in the context of proof-oriented mathematics" (p. 2).

Research questions. In this study, the following questions are addressed: 1. In what ways do students construct examples effectively and ineffectively on tasks in their transition-to-proof course? 2. How did the instructor teach example construction? 3. What connections are found between the students' construction of examples and the instruction given?

Literature and Theory

Considerable literature is available on the proving abilities of students and mathematicians, and the use examples on such tasks. However, in the interest of space, much of this literature has been omitted. The review below focuses on the literature concerning the construction of examples, and the role of example in teaching advanced mathematics courses.

Example construction. Antonini (2006) sought to answer how examples are constructed by conducting clinical interviews with seven mathematicians. From these interviews three distinct techniques emerged: *trial and error*, *transformation*, and *analysis*. *Trial and error* is characterized by constructing objects, and then testing whether the object has the desired characteristics. *Transformation* is characterized by taking a known object which has some of the necessary characteristics, and then performing adjustments until the object has all the required characteristics. *Analysis* is characterized by deducing additional properties the object has to have. Eventually, this list of properties reaches a point that either a known example is evoked or an algorithm for constructing an example is determined.

Antonini (2006) observed that mathematicians often follow a process of starting with *trial and error* and then using *transformation* only when *trial and error* fails. The *analysis* technique was only used when after failing to construct an example with both the *trial and error* and *transformation* techniques. Antonini (2006) notes that the *analysis* technique is appropriate when there is a possibility that no example with the given properties exists, because the derivation of properties could lead to a proof by contradiction.

Behavior on one task can impact the conceptual knowledge gained from other topics. A particular instance of this occurred in a study by Iannone, Inglis, Mejia-Ramos, Simpson, and Weber (2011), where students were asked to generate examples of a particular type of function. The research team found that most students generated examples with a *trial and error* technique. Other students used a *transformation* technique where they modified known examples, or an analytic technique where the student deduced additional properties of an example. Iannone et al. (2011) theorized that the *trial and error* strategy resulted in weaker conceptual gains than the other strategies.

However, when it comes to the source of the examples used by students, Iannone et al. (2011) found that there was no significant differences between the proof productions of students who generated their own examples and those who were provided examples. This result is contrary to other literature that supports example generation as an important pedagogical tool (Dahlberg & Housman, 1997; Moore, 1994; Watson & Mason, 2002, 2005; Weber, Porter, & Housman, 2008). In fact, Iannone et al. (2011) found that the proof productions of the example reading group was slightly higher than the proof productions of the example generating group, although the difference was not significant.

The teaching and learning of mathematics. One of the primary goals of mathematics education is to develop and implement interventions that change mathematics teaching (Fukawa-Connelly, 2012a). At the undergraduate level, Speer, Smith, and Horvath (2010) criticized that "very little empirical research has yet described and analyzed the practices of teachers of mathematics" (p. 99), even though poor undergraduate mathematics teaching is often cited as a reason students change majors away from science, technology, engineering, and mathematics fields (Seymour & Hewitt, 1997). In fact, Mejia-Ramos and Inglis (2009) conducted a literature of 102 mathematics education research papers concerning undergraduate students' experience reading, writing and understanding proofs, yet none of these papers described the instruction the students received. Although some studies have investigated instruction in proof writing since the publication of these critiques (e.g. Fukawa-Connelly, 2012a, 2012b; Mills, 2014), there is still a need for additional studies in this area.

Instruction can influence the choices that students make and their preferences when solving problems, including proofs. Students need strategic knowledge in order to select appropriate strategies (Weber, 2001). It is known that heuristics are difficult to teach, but that students typically do not learn them unless an attempt was made to teach them (Lester, 1994). However, some instructors do try to design the courses they teach in order to explicitly teach students strategic knowledge (Weber, 2004, 2005).

Theoretical framework. This study is framed in the emergent framework developed by Cobb and Yackel (1996). This framework links the social perspectives of classroom social norms, sociomathematical norms and classroom mathematical practices to the psychological

Table 1
The characteristics of the sampled students.

Name	Year	Major	GPA	Course Attempt
Amy	Sr.	Mathematics for Secondary Teaching	2.50-2.99	3rd
Carl	Soph.	Mathematics for Secondary Teaching	2.50-2.99	1st
Raul	Jr.	Applied Mathematics and Biochemistry	3.50-4.00	1st
Mike	Sr.	Mathematics and Spanish	3.00-3.49	2nd

perspectives of beliefs about an individual's role in mathematical activity, mathematical beliefs and values, and mathematical conceptions and activity. This study was concerned with the links between the actions of the students, and the activity of the classroom community.

In addition, this study utilized grounded theory, a methodological technique developed by Glaser and Strauss (1967). Within this method, a researcher collects and organizes data by constantly organizing the data into categories or themes (Charmaz, 2006; Creswell, 2013; Glaser & Strauss, 1967; Merriam, 2009).

Method

The case for this study is a section of a transition-to-proof course at a large university. The participants in this study are the instructor, Dr. S, and the 27 students enrolled in her course during the semester of the study.

Due to the constraints on time and resources, four students were selected for more detailed data collection during the fourth week of the semester using maximal variation sampling (Creswell, 2013). By varying the students' levels of academic success (indicated by a self-reported grade point average), mathematical preparation (indicated by self-reported grades in mathematics coursework), and specialization (pure, applied, secondary teaching, mathematics minor), the findings have increased transferability (Merriam, 2009). The characteristics of the four students included in the sample are presented in Table 1. These students were purposefully selected because they frequently spoke during class, both by asking the professor questions and presenting their own work on the blackboards.

Data collection. Several sources of data were used to triangulate the results (Patton, 2002; Merriam, 2009). Interviews were conducted with the four selected students, in order to observe each student's process on proof-related tasks while working independently. These interviews occurred three times during the semester: around the seventh week of the semester, the twelfth week of the semester, and the last week of the semester.

Each interview with a student had three components: a semi-structured portion addressing proof strategies and goals for the course, a task-based portion where students attempted several proof-related tasks, and a reflection on the tasks. The semi-structured portion asked the students to talk about their impressions of the course, namely what they had learned and what they thought they should be learning. The tasks for the interviews were selected from the textbook, or other studies on undergraduate proof writing (Alcock & Weber, 2010; Dahlberg & Housman, 1997; Iannone et al., 2011). The mathematical content of the questions varied over the three interviews, matching the recent content from the course. After a student completed all tasks, then the students were asked to reflect on their work. Sometimes the final reflection was omitted due to poor time management.

The classroom was observed daily to observe the examples used by the instructor during lectures and student presentations. The observations are supplemented by three interviews with the instructor. These interviews focused on the choices made during class and how those choices influenced the desired instructional goals.

Results

Construction of examples. Knowing how to accurately construct examples is of crucial importance for using examples effectively. Two levels of analysis were done: 1) the accuracy of the example, and 2) the construction technique used.

Table 2
This table summarizes the construction abilities of the students.

Construction	Amy	Carl	Raul	Mike	Total
Accurate Construction	30	16	18	4	68
Inaccurate or Incomplete	6	3	3	2	14
Authoritarian Source	3	2	2	0	7
Trial and Error	17	15	11	4	47
Transformation	16	2	8	2	28

Three categories were used to describe the construction technique: *trial and error*, *transformation* and *authoritarian*. An *authoritarian* example is retrieved from a source, instead of being constructed by the prover. The terms *trial and error* and *transformation* were consistent with the definitions of Antonini (2006). Neither the students nor the professor discussed the *analysis* technique, so this category was not used.

The construction of examples was a difficult task for many of the students. During the first interview, Raul and Mike both made errors in constructing examples because they did not know which conditions a construction needed to satisfy to be classified as an example or a counterexample. In fact, they both identified constructions as counterexamples that did not satisfy the hypotheses of the statement.

The students generally constructed examples that were accurate, but their examples were frequently not useful. For instance, Mike was seeking a potential counterexample on a divisibility problem and chose $a = 1$ as the value for the divisor, stating that he chose this value because "1 divides everything." Mike did not realize that this choice for a meant that every possible example would be true. Although other students constructed examples that were not useful for their purpose, this was the only instance in which a student stated a fact that would directly indicate the lack of usefulness.

The students transitioned to more advanced construction techniques late in the semester. During the first interview, Mike was the only students to utilize the *transformation* construction technique, and he only did so once. By the final interview, the students were using the *transformation* technique more frequently than *trial and error*. This interpretation of the result may be conflated with the choices the students make due to the mathematical content. Specifically, the first interview consisted entirely of number theory tasks which the students may have limited previous experience, whereas the final interview concerned real-valued functions and the students should have significant experience with these from their calculus courses. Although the students likely used the *transformation* technique due to increased experience, they also knew more examples of real-valued functions to draw upon as the starting point for the transformation process.

In particular, when asked to construct an example of a fine function on question 3 of interview 3, the first example constructed by each student was a transformation of $y = \sin x$. These students recognized that the pattern of the zeros in $y = \sin x$ could be adjusted to satisfy the conditions of a fine function. It is unlikely that the students could have constructed an example of a fine function via *trial and error* because of how difficult it would be to verify. However, it is equally difficult to imagine a students utilizing a *transformation* technique on $a|(bc)$ implied $a|b$ or $a|c$, especially for an initial example of the statement. Most students will not have a sufficient background in the formal language of divisibility to have such examples in their personal example space.

The instruction. Dr. S modeled example construction very rarely during the lecture. Although she presented many examples throughout the semester, she seldom talked about how these examples were constructed. Dr. S did model how to determine which properties an example or counterexample of a statement needs to satisfy, and how to go about verifying that a construction satisfies those properties. Dr. S knew that *trial and error* is the first technique used in example construction, and that the most important aspect of that is knowing which properties need to be verified. Dr. S assigned student presentations that she intended to be opportunities for the students to learn how to construct examples. She knew that the students would often fail before they succeeded at example construction, and that the best way to help the students improve would be to review their constructions attempts during their presentations.

There were two episodes from the lecture where Dr. S emphasized example construction, and the care that must take place when constructing examples. The first instance occurred shortly after defining functions. Dr. S emphasized the importance of a function being well-defined, particularly when the domain is a partition. To do this, Dr. S presented three potential functions:

$$\begin{array}{ll} f : \mathbb{Z}_3 \to \mathbb{Z}_6 & f\left([x]_3\right) = [3x+2]_6 \\ g : \mathbb{Z}_4 \to \mathbb{Z}_2 & g\left([x]_4\right) = [3x]_2 \\ h : \mathbb{Q} \to \mathbb{Z} & h\left(\frac{a}{b}\right) = a + b \end{array}$$

The first example was generated using numbers suggested by the students, the last two were purposely chosen by Dr. S. Dr. S showed that f and h are not well-defined by producing counterexamples that show that two different representatives of the equivalence classes produce different outputs. For g, Dr. S provided the students with a proof that it was well-defined. Ultimately this episode was demonstrating what it means to be well-defined, but Dr. S knew that this would help the students when constructing their own functions especially in their Modern Algebra course.

Dr. S seldom lectured explicitly about constructing examples and counterexamples, because Dr. S had the expectation that the students would attempt and present many example construction questions on the board, and that these presentations would provide the opportunity to discuss example construction techniques.

Another reason that Dr. S did not lectured about example construction frequently is because she expected the students to utilize *trial and error* by randomly trying constructions and to test whether these are examples or not. Although this is not a sophisticated strategy for example construction, Dr. S believed that students at the earliest stages of proof writing “are not always ready yet” for other strategies. Dr. S wished that the students would move towards the *transformation* construction strategy by asking themselves questions such as “is the statement similar to one [I] know?” and then using that response to construct their example. During the final interview, Dr. S reiterated this by saying “I would like to move them toward more directed examples where they are intentionally trying to go certain places but I doubt that most of them are ready for that. Right now I’m happy if they try random examples to see what’s going on, as long as they don’t stop there.” Perseverance was a frequent theme when discussing proof and example constructions in the lecture.

When the students presented example construction tasks that we incorrect, Dr. S would usually ask the student who presented (or sometimes the whole class) to help her revise the construction. In one instance, Carl presented a relation on $A = 1, 2, 3$ that should have the properties of symmetry, transitivity and not reflexivity. Carl presented the relation $\{(1, 2)(2, 1)(1, 3)(2, 3)(3, 2)(3, 1)\}$, but this example is not transitive. Dr. S argued that if $(1, 2)$ and $(2, 1)$ are in the relation, then transitivity requires that $(1, 1)$ and $(2, 2)$ must also be included. As such, Dr. S changed the relation to $\{(1, 2), (2, 1), (1, 1), (2, 2)\}$, which is symmetric and transitive, but not reflexive because it is missing $(3, 3)$. Through this discussion, Dr. S walked the students through using the *transformation* technique for example construction, since she transformed an existing example to satisfy the given criteria.

Comparing the instruction and the students. The students used the *trial and error* construction technique for all of the examples constructed during the first interview, with one exception. However, as the semester progressed the students used the *transformation* technique with increasing frequency. Dr. S predicted this behavior of the students. The *analysis* technique was not demonstrated by the instructor or used by the students; however, during the member checking interview, Dr. S argued that the *analysis* technique was too advanced to be useful to the students at their current level of understanding. In the first interview, Dr. S said

> It depends on the problem, but to some extent, *trial and error* is the very first step. You just try stuff. I’ve seen this even with advanced REU students, where there is a good strategy. They’re not always ready yet. I’m okay with them randomly trying at first. Now, I want them to move toward more careful construction. As they go through this, they should be looking for things that are similar and using that to give them a hint.

Dr. S recognized that as beginning students, they would not have the mathematical experience to use the more advanced *transformation* and *analysis* techniques, but she hoped they would grow to that point. During the same interview, Dr. S elaborated that although she expects the students to have some familiarity with using examples from their calculus classes, “they just never had to construct [examples] themselves before.” As such, some of the difficulties the students had with example construction were expected.

Dr. S did not vocalize an expectation of the accuracy problems exhibited by some the students during the initial interviews. Both Raul and Mike had created examples that violated the statement hypotheses. Raul did not seem to realize that failing the hypotheses was a problem. During the member checking interview, Dr. S said students often make these types of construction errors at this point in their development. She furthered this by explaining that many students present counterexamples that are not actually counterexamples, especially on the first test of the course.

Dr. S usually did not talk about the construction technique when she presented examples to the class. She designed the course so that most of the example construction tasks were assigned as student presentations, and that she would talk about example construction as she reviewed and corrected the examples in the presentations. Unfortunately, the students did not present many problems and they tended to present problems asking for proofs rather than the problems asking for examples. Consequently, Dr. S did not have the opportunity to talk about construction techniques with the expected frequency.

Overall, Dr. S had the experience to know the capabilities of the students with respect to example construction. She recognized that *trial and error* would be the primary technique at the beginning of the semester, and that many of the students would not be able to move beyond that technique in this course. However, towards the end of the semester, she introduced the *transformation* construction technique for the benefit of the students who were ready for more advanced techniques. The students in the sample were able to apply the *transformation* technique in some circumstances, and likely will be able to utilize it more frequently in their subsequent courses.

Discussion

By the end of the semester, all of the students were selecting examples with more thought, and used the *transformation* construction technique with increased frequency. It is unclear exactly what caused this growth. Possible explanations include the students' individual development throughout the semester, the influence from the instruction, and the new content.

Previous research on undergraduate example construction showed that the students used *trial and error* techniques approximately 80% of the time (Iannone et al., 2011). This percentage is considerably higher than than the 57% *trial and error* observed in this study. It is unclear what accounts for this discrepancy, although the most likely causes are the sample and the task selection. Both studies also had small samples, this one had four participants and Iannone et al. (2011) had nine, so the individual characteristics of the participants strongly affected the percentages.

Implications for teaching transition-to-proof courses. One implication is that students should be explicitly taught strategies for constructing and verifying examples. One of the hardest parts of trial and error is picking the construction to test. However, by explaining how the examples in the course are constructed, it may be possible to guide the students beyond blinding picking parameters to test.

In this study, most of the students became convinced that a *prove or disprove* statement was true after constructing only one or two examples. However, when mathematicians obtain conviction from empirical evidence it is often from multiple examples or for unusual properties (Weber, 2013; Weber, Inglis, & Mejia-Ramos, 2014). Although it is unreasonable to assume that numerous examples should be constructed before trying to prove a statement, we need to teach students to consider the quality of the examples they construct, and to view the examples as a collection. For example, a statement that is true for a prime number, a perfect square, and another composite number is far more believable than a statement evaluated only with a prime number. But students need to be taught to consider examples collectively rather than individually.

Future research. Additional research concerns the instruction on example construction. How does instruction impact a provers ability to effectively use and construct examples? It is unclear whether or not such instruction will actually help the students learn how to construct examples effectively. Some studies suggest that instruction in problem solving frameworks alone does not help students become better problem solvers (Garofalo & Lester, 1985; Schoenfeld, 1980), so it is possible a similar phenomenon will occur here. This can only be established through additional testing and study.

Finally, it is unclear whether effective example construction will positively impact proof writing. Iannone et al. (2011) found that generating examples provided no benefits to the students as compared to receiving a list of examples. One interpretation of this is that it does not matter where the examples come from, what matters is how the examples are used and what conclusions are drawn from the examples. As such, it is possible that knowledge in using examples effectively can improve a persons ability to successfully write proofs, but additional study is needed on this topic.

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Probabilistic Thinking: An initial look at students' meanings for probability

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Probability is the central component that allows Statistics to provide a useful tool for many fields. Thus, the meanings that students develop for probability have the potential for lasting impacts. Using Thompson's (20015) theory of meanings, this report shares the results of examining 114 undergraduate students' conveyed meanings for probability after they received instruction.

Key words: probability, statistics, meanings, introductory statistics course

Probability is the engine that makes inferential Statistics run. This first statement is one that hardly any practitioner of Statistics will disagree with. Statisticians and Statistics Educators freely acknowledge that the central ideas of probability allow us to move beyond merely describing a data set to using the data set as evidence for supporting/refuting claims. This is even one area in which Frequentists, Bayesians, and Subjectivists all agree. However, how practitioners think is often vastly different from how students think before, during, and after instruction. In a recent discussion with a university instructor about introductory statistics courses, I was surprised to hear this individual say "I skip by probability because my students don't really need it and we need the time to talk about doing hypothesis tests." This statement caught me off-guard for two reasons: 1) this instructor had a Ph. D. in Statistics, and 2) the instructor continued to talk about how she wanted her students to develop "rich and productive meanings for hypothesis tests and *p*-values". While I believe that students *can* and *will* develop meanings for hypothesis tests in the absence of a way of thinking about probability, I challenge the claim that students can develop "rich and productive" meanings.

Cobb and Moore (1997) took the position that "first courses in statistics should contain essentially no formal probability theory" (p. 820). I agree with the spirit of their position. While this may seem like I am in the camp of the aforementioned instructor, there is a critical distinction. Cobb and Moore's position is *not* that first courses should avoid discussing probability, but rather emphasis on formal rules, such as the rules for $P\left(A \cup B\right)$, are of little consequence in these courses. Rather, they suggest that "informal probability" is sufficient, especially if the course focuses on the idea of sampling distributions. Thus, rather than skipping over probability entirely, an introductory course should skip over calculational rules of probability and focus on helping students construct *ways of thinking* about probability. I agree with Liu and Thompson (2002) that trying to debate the question of "What *is* probability?" is a fruitless endeavor in a first course. Rather, in a first course on statistics and probability, our focus should be on what we (us and our students) *mean* by the term "probability".

Introductory statistics texts that cover probability focus almost entirely on *how to calculate* rather than *how to think about*. The introductory text *Statistics for the Life Sciences,* 4th edition (Samuels, 2012) devotes 15 pages to probability. However, there are only two sentences related to how to think about probability. Out of the 29 exercises provided for the students to use for homework, 26 ask for students to calculate a value of the probability of some event, 3 ask students to make a claim about whether or not two events are independent, and 0 questions ask students to interpret/make use of a way of thinking about probability. Likewise, *Introduction to the Practice of Statistics*, 7th edition (Moore, McCabe, & Craig, 2012) devotes 18 pages to probability and randomness. Of these pages, only 3 sentences (all variations of each other) focus on how to think about probability. There are only two questions of the 45 that focus on something other than a calculation of probabilities or judgment of independence; one asks whether or not a probability value is applicable to a larger set of colleges, and the other asks students to explain what a probability value means. In both of these cases, students' major takeaway is that probability is a calculation.

The above issues created a backdrop for an informal, observational study that aimed to serve as a first step in looking at how undergraduate students think about probability after enrollment in an introductory course on Statistics. In the spring of 2014, students in the four sections of an introductory statistics course designed for life science majors at a large, public, Southwestern university responded to two questions related to their thinking about probability. A senior lecturer taught two sections of the course, and two graduate students (one Ph. D. Statistics and one Ph. D. Mathematics/Statistics Education) each taught one section. While the aforementioned *Statistics for the Life Sciences* served as the official text for the course, only three sections followed this text. The fourth section (taught by the Ph. D. Math/Stat Ed. graduate student) moved away from the text. This section followed an experimental design curriculum intended to support students developing ways of thinking about statistics beyond procedures and placed a heavy emphasis on meanings. In addition to the students answering the questions, the three instructors also answered the questions.

Methodology and Theoretical Background

I conducted this observational study at a large, public university located in the Southwestern region of the United States in the spring of 2014. Given that this study only serves as a first step to a more formal study, the selection of course and school was convenient to the researcher. The instructors of the selected introductory statistics course for life science majors asked their enrolled students to respond to three questions after the students had already received instruction on probability. Two questions related directly to the general purpose of this study: how the students think about probability.

Question 1: How do you think about probability? That is, how would you explain probability to another person?

Question 2: Consider the following statement:

The probability of observing a value of 4 when looking at the product of two dice is 3/36.

How should someone think about (interpret) 3/36 given the above statement?

Given the qualitative nature of the written responses to these questions, I made use of a coding methodology consistent with that of Strauss and Corbin (1990). I initially used open coding for the responses and then I made use of an axial coding system. I used the axial codes in my analysis.

In coding, I focused on the meaning conveyed by the students' responses. *Meaning* refers to the space of implications (including actions, images, and other meanings) that results from an individual assimilating some experience and thereby forming some understanding of that experience (Thompson, 2015). Using a student's responses, we may postulate the meaning that he/she has for probability. Just as responses may be viewed as more/less productive, so can meanings. I view *productive meanings* as those meanings that provide coherence to ideas that students have and those meanings which afford students a frame that supports the students in future learning (Thompson, 2015). *Productive meanings* are clear, widely applicable (within reason), and rely on explicated assumptions. To help demonstrate this, let us look at an example using Question 2. Suppose that we have two students, George and Sally. George's meaning for probability deals with notions of long-run relative frequency while Sally's meaning is a blend of circular and Classical (see Results for details on these meanings). George's meaning orients him to view 3/36 as a measure of the relative frequency of seeing a value of 4 when carrying out the dice experiment an indefinite, large number of times. George's thinking supports him in making statements such as "That $3/36^{th}$ of the time, we will observe a product of four" or perhaps "About 8% of the time, we'll observe a product of four". George's meaning is flexible enough so that if his teacher adds information that the two die have an unequal number of sides, or the dice are not fair, he won't feel a need to alter his initial response. For George, his initial interpretation works in light of this new information.

On the other hand, Sally's meaning supports her in thinking about 3/36 as a statement that there are 3 ways to get a product of 4 when rolling two dice out of 36 total ways of getting products. Implicit to Sally's thinking is that the two dice are standard, six-sided dice. This fact enables Sally to make sense of the 36. If the teacher were to reveal that one die was four-side while the other was a twenty-side die, Sally would struggle to make sense of the 36. Likewise, should the teacher state the two dice are unfair, but not state how they are unfair; Sally's meaning for probability does not necessarily enable her to give an interpretation in light of unfair dice. Additionally, the use of equivalent fractions could create issues for Sally's way of thinking. We know that 3/36 reduces to 1/12. However, the interpretation could change significantly for Sally; "there is only 1 way to get a 4 and 12 possible outcomes". The underlying process is no longer the same; a Classical meaning appears to allow individuals to ignore/forget the process all together. Additionally, 3/36 could be re-written as 4/48, 7/84, or even 30/360. The same issue with reducing still applies. George's meaning, particularly if he moves from the fraction to a percentage, will have no issue with using an equivalent fraction.

The productivity of George's and Sally's meanings for probability has broader implications. Consider the statement "The probability of selecting a random US man, 20+ years old, who is under six foot tall is 2/3." in place of the dice statement. George's meaning still supports him reasoning about the relative frequency of observing US men in the age group whose height is under 6ft. However, Sally either has to reason that there are 2 heights under six foot out of 3 total possible heights that US men can be or she needs to have a completely separate meaning for probability in continuous contexts. Having separate meanings for probability in different contexts does not lend itself to the student building a coherent way of thinking about probability.

Results

I characterized students' responses to Question 1 in five broad categories. The first category of responses deals with thinking about probability as being about the long-run relative frequency of some event (L.R.R.F.). For these students, they seem to think about probability as something that emerges after imagining carrying out some process a large number of times. While these students may speak about the probability of some event, from discussions they do not appear to think that the event is the next outcome of the process. Rather, they always reference needing to imagine the process carried out many, many times.

The second category is "Frequency" and contains all of the cases where the students appeared to focus on the frequency (or relative frequency) of some event occurring, but their responses do not clearly indicate that the student imagines the frequency stemming from repeating a process an indefinite number of times.

The third category covers those students' responses that dealt with prediction. The responses that fall into this category are reminiscent of the outcome-approach of probabilistic thinking (Konold, 1989). Often these students only spoke about the very next time you carry out some process.

The fourth category of responses I called "Circular". Typical responses that fall into this category are "Probability is the chance that something happens", or "the likelihood of some event". The descriptor of "circular" is highly indicative of how these students seem to think. During discussions, students who spoke of probability as being "chance" or the "likelihood" of some event, would often answer the follow up question of "What is chance/likelihood?" with the statements along the lines of "well, chance is, umm, just probability." The way students thought about probability appeared to be a near unending cycle of labels with little meaning behind those labels. The seemingly only way these student broke out of this cycle was when they had to deal with a concrete situation, a specific value for them to speak about, and, occasionally, restrictions on what words they could use (i.e. not use the words "chance", "likelihood", "probability"). The fifth category, "Other", covers those responses not captured by the other categories.

The following bar chart (Figure 1) shows the frequency of responses that fall into these categories. Overwhelmingly, 89 students (78.1%) gave a response that seems indicative of circular thinking. Nineteen students (16%) appear to think about probability in terms of frequency/relative frequency. Of these students, 15 think about probability as the long-run relative frequency of some process.

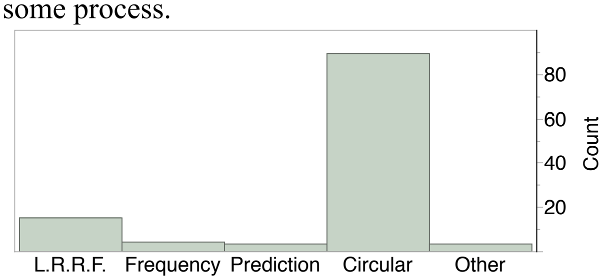

Figure 1. Students' responses for Question 1.

How do students appear to interpret a specific value of probability?

I used five codes to characterize students' responses for interpreting the probabilistic value 3/36. The first category consists of those students who seemed to think of 3/36 as one number rather than two numbers separated by a bar. These students spoke about 3/36 as representing the percent of the time you would see a product of 4 if you carried out the process of rolling two dice an indefinite number of times ("many, many times").

The second category, "Classical", covers those responses where students appeared to view the fraction 3/36 as two numbers. The upper number represented the number of ways to get the outcome of interest while the second number represented the total number of different outcomes. This way of thinking is exactly like that used in "classical" probability. In this school of thought, the sample space consists of a finite number of unique outcomes, which we assume as having the exact same probability of happening. In addition to the equi-probability assumption, the students also must make the assumption about details of the stochastic process. Namely, that there are 36 distinct outcomes.

Similar to the "Classical" category, another group of responses reflected thinking about the probability value 3/36 as telling us that either we already had observed 36 rolls of two dice and saw exactly 3 products of 4 or if we were to roll the dice 36 times, we would then see exactly 3 products of 4 ("fixed number of rolls"). These students also appear to view the fraction as two numbers. In both cases, students appear to think that the probability value tells us exactly how many outcomes of interest we saw for a set number of trials.

The forth category, "Chance", are those students whose response to the question was to essentially say that 3/36 was the chance getting a product of 4. The fifth category, serves as the catch-all for responses that did not fall into any of the other categories. This includes students who repeated the given statement (4 students), either simplified or wanted simplification of 3/36 (2 students), or expressed the need for dice (2 students) among the responses. I did not include the three students did not respond to this question in the final count.

As shown in the following bar chart (Figure 2), a majority of students interpreted the probability value as being about a fixed number of rolls of the dice and a fixed number of 4's (40.5%). Only 17.1% (19) of the students thought about 3/36 as representing the percent of the time we would see a product of 4. Fifteen students appeared to use a "classical" way of thinking, while 19 just substituted "chance" for "probability".

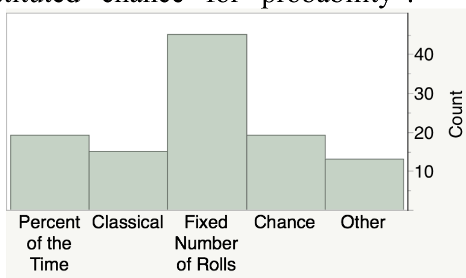

Figure 2. Students' responses for Question 2.

How does apparent student thinking about probability in general relate to how they interpreted a specific value of probability?

A natural question that follows from the previous two questions, is how do the students' responses to each question relate to one another? Table 1 shows the two-way contingency table for students' responses to both questions. The vast majority of individuals who appeared to think about probability as the long-run relative frequency of some event interpreted the given probability value as the percent of the time we would see some event happen in the long run. The majority of students who interpreted 3/36 as being two number separated by a bar (either Classical or Fixed Number) or as a "measure of chance", gave a circular meaning for probability. The wide range of interpretations given by students with a circular meaning is not surprising. Given that the students' meaning for probability appears related to a word-exchange, the students would need to draw upon some other meanings to help make sense of the value 3/36. All but one student who explained 3/36 as the "chance" of getting a product of 4, gave responses that indicated a circular meaning to Question 1.

Table 1. Students' responses to Question 1 by their responses to Question 2.

	Percent of the Time	Classical	Fixed Number of Rolls	Chance	Other	total
L.R.R.F.	12	1	0	1	0	14
Frequency	1	0	1	0	2	4
Prediction	2	1	0	0	0	3
Circular	4	12	42	18	10	87
Other	0	0	1	0	2	3
total	19	14	44	19	18	111

Is there a difference between how students appear to think about probability in general when accounting for the instructor?

To explore this question, Table 2 provides a good visualization of how students appeared to have thought about probability in regards to Question 1. A striking aspect to notice is that all of the students who appeared to think about probability as a long-run relative frequency all have Instructor A. Additionally, the vast majority of students for both Instructor B and Instructor C gave responses that appear indicative of a circular meaning for probability in general.

Table 2. Students' Responses to Question 1 by Students' Instructor

	L.R.R.F.	Frequency	Prediction	Circular	Other	total
Instructor A	15	2	2	8	0	27
Instructor B	0	2	1	58	1	62
Instructor C	0	0	0	23	2	25
total	15	4	3	84	8	114

To further explore this difference, I conducted a Kruskal-Wallis test with $\alpha = 0.05$. The test statistic has a value of 58.0382. Thus under a χ_2^2 distribution, the approximate probability of observing the differences we did or ones more extreme is $p \le 0.0001$. A post-hoc analysis using the Steel-Dwass method shows that Instructor A's students' responses are significantly different from the responses of Instructor B's students ($p \le 0.0001$) and significantly different from Instructor C's students ($p \le 0.0001$). However, the responses from Instructor B's and Instructor C's are not significantly different from each other ($p \approx 0.1752$).

Is there a difference between how students interpret a specific value of probability when accounting for the instructor?

Much like the prior question, a two-way contingency table provides insight into answering the question about the difference in how students interpret a given probability value in relation to the students' instructor. Notice in Table 3 that the vast majority of students who interpreted 3/36 as a percent of time have Instructor A and two-thirds of Instructor A's students gave this type of interpretation. None of Instructor C's students and only 1 of Instructor B's students gave a response that fell into this category. Given that the majority of Instructor B's and Instructor C's students appeared to have a meaning for probability that was circular (see Table 2), the spread of their students' interpretations is not surprising.

Table 3. Students' Responses to Question 2 by Students' Instructor

	Percent of the Time	Classical	Fixed Number of Rolls	Chance	Other	total
Instructor A	18	3	3	1	1	26
Instructor B	1	10	27	12	11	61
Instructor C	0	2	15	6	1	24
total	19	15	45	19	13	111

I conducted a second Kruskal-Wallis test (with $\alpha = 0.05$) to test the difference between the students' responses in relation to instructor. The test statistic has a value of 32.2145. Under a χ_2^2 distribution, the approximate probability that we observe the differences we did or one greater is $p \le 0.0001$. Post-hoc analysis using the Steel-Dwass method indicates that Instructor A's students' responses are significantly different from those of Instructor B's students ($p \le 0.0001$) and Instructor C's students ($p \le 0.0001$). The responses of Instructor B's students are not significantly different from Instructor C's students ($p \approx 0.9801$).

Discussion

The vast majority (78.1%) of students describe thinking about probability in circular ways, with roughly 13% (of 114) describing probability as being about the long-run relative frequency of some event (given a stochastic process). Similarly, a majority of students

(40.5%) interpreted the given probability value, 3/36, as being about a fixed number of rolls (of dice) and observing exactly 3 outcomes that were the event of interest. Even after students had received instruction on probability, there was still variation in the responses that students gave. This being said, there appeared to be clear distinctions between the majority response for each instructor's students. In the case of Instructor A, the majority response for probability (both in general and for interpreting) students gave is consistent with thinking about probability as the long-run relative frequency of an outcome of some repeatable process. For Instructor B's and Instructor C's students, the dominant responses for probability seemed to focus on a circular word-exchange and a fixed number of trials. The coded responses for the three instructors appear in Table 4. Like with the students, the teachers' responses offer insight into the meaning for probability that each teacher has. While further investigation into each teacher's actual meanings for probability is necessary, their responses appear to match up with the prevailing responses of their students. This seems logical, given that a teacher's mathematical meanings serve as one of the key components of how that teacher teaches (Thompson, 2013).

Table 4. Instructor's Responses to Question 1 and Question 2.

	Response to Question 1 (probability in general)	**Response to Question 2 (interpret 3/36)**
Instructor A	L.R.R.F.	Percent of the Time
Instructor B	Circular	Classical
Instructor C	Circular	Classical

A limitation to this study is that responses to two questions do not necessarily provide enough information to confidently describe an individual's meanings for a mathematical topic. While some informal discussions with students have taken place, interviews with more students will help to support the claims about the possible meanings students might operate with when they give particular responses. Additionally, given that this was an observational study, we cannot definitively say that Instructor A is the cause for stark differences between the three sets of students' responses. However, given that Instructor A made the decision to follow a curriculum centered on assisting students in developing productive ways of thinking, there is evidence of a strong causal relationship. Further research could substantiate this claim.

This study serves as but a first step in examining how undergraduate students think about probability after receiving instruction. While only drawing upon data from two questions, the inclusion of similar questions can help to refine items that serve as a means to measure a progress variable for probability. Progress variables represent "(a) the developmental structures underlying a metric for measuring student achievement and growth, (b) a criterion-reference context for diagnosing student needs, and (c) a common basis for interpretation of student responses to assessment tasks" (Kennedy & Wilson, 2007, pp. 3–4). Establishing a progress variable for probability along with items that measure such a variable has the potential to change how we teach probability at all levels. Additionally, a progress variable for probability is of use for other areas of statistics education research including students' notions of *p*-values, hypothesis testing, and distributions of random variables.

The present study into how a set of undergraduates thought about probability has shown that there are some stark differences between different sections of the same course. Sadly, the dominant meanings that these students appear to use for probability are circular and calculationally oriented. One section of the course, which used a "reformed" curriculum, does have a number of students who appear to have a highly productive meaning for probability. Further work needs to be done in order to help more students develop a rich and deep meaning for probability that is coherent and does work for the students in statistics.

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What Would the Research Look Like? Knowledge for Teaching Mathematics Capstone Courses for Future Secondary Teachers

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Abstract. *Mathematics Capstone Course Resources* is a 14-month proof-of-concept development project. Collaborators across three sites aim to: (1) develop and pilot two multi-media activities for advanced pre-service secondary mathematics teacher learning, (2) create guidance for college mathematics faculty for effective use of the materials with target audiences, and (3) gather information from instructors and students to inform future work to develop additional modules and to guide subsequent research on the implementation of the materials. The goal of this poster presentation is to provide information about capstone module development and brainstorm research design suggestions with the long term aim of developing a grant proposal to research the knowledge college mathematics faculty use to effectively teach mathematics to future teachers.

Research Questions

The preparation of the highest quality teachers of mathematics is a national imperative. There is a notable need for future math teacher "capstone" course materials, including guidance for math faculty, that deliberately and explicitly connect undergraduate mathematics content to the knowledge needed for teaching secondary mathematics. The poster includes details about the development processes and content of two capstone pilot modules. Our goal is that RUME attendees will discuss a preliminary research study design and outline the tools and analysis processes that will help us address the questions:

(1) How does incorporating the modules into instruction shape instructor mathematical knowledge for teaching future teachers (MKTFT; and how might we define MKTFT)?

(2) What are the relationships among varying conditions of implementation (differing degrees of fidelity of implementation) and the extent to which students are achieving the desired results?

Background & Conceptual Framework

Several NSF-funded efforts have been made in the past to create *courses* in mathematics departments that are secondary teacher candidate capstone courses. These courses appear to belong to three categories: connecting big ideas, connecting big ideas with deep content understanding, and connecting big ideas, deep content, *and* an applied understanding (where the application is to teaching). Some departments have a capstone course that emphasizes upper-division mathematical content with some connections to the topics found in secondary school curricula. Other departments' capstone courses also provide opportunities for pre-service teachers to enrich their knowledge of content *and* place more of an emphasis on developing mathematical knowledge for teaching, often in the form of knowledge of how students may think (productively and unproductively) about particular mathematical ideas. The range of needs such courses may fulfill for programs make it challenging for instructors to find and/or create an adequate supply of appropriate materials (Banilower, et al., 2013).

Results

We are aiming for a design that will allow reporting on results about the development of math knowledge for teaching future teachers. This includes articulating specifics for that type of knowledge/knowing. By definition, pedagogical content knowledge (PCK) is the collection of knowledge teachers and other instructional personnel need about the challenges learners encounter, strategies for helping students, ways to listen to identify not only learners' thoughts but also thinking processes, and skills for regulating teaching practices (Shulman, 1986). Teachers at all levels acquire PCK in many ways: grading, examining their own learning, observing and interacting with students, observing and interacting with colleagues, and reflecting on and discussing practice. Since Shulman's seminal statement on the blends of pedagogical and content knowledge needed for teaching, a rich collection of theories and models of it has grown in mathematics education (Depaepe, Verschaffel, & Kelchtermans, 2013).

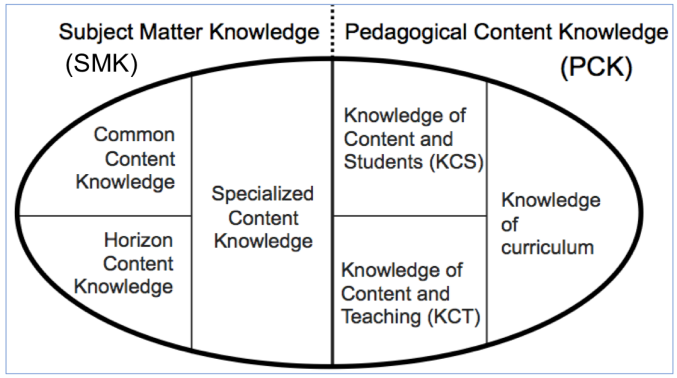

Figure 1. Dimensions of mathematical knowledge for teaching (MKT; Hill et al., 2008)

The framing of knowledge for teaching mathematics has centered on the question: What mathematical reasoning, insight, understanding, and skills are entailed for a person to teach mathematics effectively? We want to add to the end of that question: …to teach mathematics effectively *to future teachers*? Work on math knowledge for teaching in secondary and post-secondary settings (Hauk, Toney, Jackson, Nair, & Tsay, 2014; Speer, King, & Howell, 2015) builds on Ball's model of three types of subject matter knowledge (SMK) and three types of pedagogical content knowledge as categories in the domain of *mathematical knowledge for teaching* (MKT; Figure 1). Recent work on secondary and post-secondary models of MKT note that aspects of mathematical semantics, definitions, and discourse may be very important (Hauk et al., 2014).

Implications for Practice

We seek advice from RUME poster session attendees to identify and categorize the potential implications for practice (and future research). We also hope to gather ideas for communication of implications to college instructors, department chairs, and other stakeholders.

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Exploring the Factors that Support Learning with Digitally-Delivered Activities and Testing in Community College Algebra

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Abstract. A variety of computerized interactive learning platforms exist. Most include instructional supports in the form of problem sets. Feedback to users ranges from "Correct!" to offers of hints and partially to fully worked examples. Behind-the-scenes design of such systems varies as well – from static dictionaries of problems to "intelligent" and responsive programming that adapts assignments to users' demonstrated skills, timing, and an array of other learning theory-informed data collection within the computerized environment. This poster presents background on digital learning contexts and invites lively conversation with attendees on the research design of a study aimed at assessing the factors that influence teaching and learning with such systems in community college elementary algebra classes.

Research Questions. Funded by the U.S. Department of Education, we are conducting a large-scale mixed methods study in over 40 community colleges to address:

RQ1: What student, instructor, or community college factors are associated with more effective learning from the implemented digital learning platform?

RQ2: What challenges to use-as-intended (by developers) are faculty encountering and how are they responding to the challenges as they implement the learning tool?

Background and Conceptual Framing. First, there are distinctions among cognitive, dynamic, and static learning environments (see table).

Summary Table		Static	Dynamic
A particular model of learning is explicit in design and implementation (structure and processes)	No	Text and tasks with instructional adaptation external to the materials	Adaptive tutoring systems (Khan Academy, ALEKS, ActiveMath)
	Yes	Textbook design and use driven by fidelity to an explicit theory of learning	"Intelligent" tutoring systems (Cognitive Tutor)

Learning environments can vary along at least two dimensions: (1) the extent to which they adaptively respond to student behavior and (2) the extent to which they are based on a careful cognitive model. Static learning environments are those that are non-adaptive and devoid of a cognitive model – they deliver content in a fixed order and contain scaffolds/feedback that are identical for all users and have a design based on intuition, convenience, or aesthetic appeal. An example of this type of environment might be online problem sets from a textbook that give immediate feedback to students (e.g., "Correct" or "Incorrect"). Dynamic learning environments keep track of student behavior (e.g., error rates or time-on-problem) and use this information in a programmed decision tree that selects problem sets and/or feedback based on students' estimated mastery of specific skills. An example of a dynamic environment might be a system such as ALEKS or the "mastery challenge" approach now used at the online Khan Academy. For example, at khanacadmy.org a behind-the-scenes data analyzer captures student performance on a "mastery challenge" set of items. If a student gets all six items correct, the next level set of items in a programmed target learning trajectory is offered. Depending on the number and type of items the particular user answers incorrectly (on the path to six items in a row done correctly), the analyzer program identifies target content and assembles the next "mastery challenge" set of items. In addition to such responsive assignment generation, programming in a *cognitive* learning environment is informed by a theoretical model that asserts the cognitive processing necessary for acquiring skills (Anderson et al. 1995; Koedinger & Corbett, 2006). For example, instead of specifying only that graphing is an important skill necessary for mastery of elementary algebra, a cognitively-based environment will also specify the student thinking and skills needed to comprehend graphing (e.g., connecting spatial and verbal information), and provide feedback and scaffolds that support these cognitive processes (e.g., visuo-spatial feedback and graphics that are integrated with text). In cognitive environments, scaffolds themselves can also be adaptive (e.g., more scaffolding through examples can be provided early in learning and scaffolding can be faded as a student acquires expertise; Ritter et al. 2007). Systems can also provide summaries of student progress, which better enable teachers to support struggling students. Some studies have shown preliminary support of their promise in post-secondary mathematics (Koedinger & Sueker, 1996).

Method. The study is a multi-site cluster randomized trial. Half of instructors at each community college site are assigned to use an adaptive web-based system in their instruction, the other half teach as they usually would. The primary outcome measure for students' performance is an assessment from the Mathematics Diagnostic Testing Program (MDTP), which is a valid and reliable assessment of students' algebra knowledge (Gerachis & Manaster, 1995). In the stratified sampling approach we first did a cluster analysis on all community college sites eligible to participate in the study based on college-level characteristics that may be related to student learning (e.g., average age of students at the college, the proportion of adjunct faculty). This analysis led to five clusters of colleges. Our recruitment efforts then aim to include a proportionate number of colleges within each group. The primary value of this approach is that is allows more appropriate generalization of study findings to the target population (Tipton, 2014).

Quantitative Analysis. The primary aim of the quantitative analysis is to address RQ1, how and for whom the tools are effective. To this end, we employ Hierarchical Linear Modeling (HLM). Models include interaction terms between instructors' treatment assignment and covariates at different levels (e.g., students history of course-taking, self-concept of ability), to explore the moderating impact of tool use on student learning.

Qualitative Analysis. To address RQ2, a great deal of textual, observational, and interview data are being gathered. These data allow careful analysis of the intended and actual use of the learning environment and the classroom contexts in which it is enacted – an examination of implementation structures and processes. Indices of specific and generic fidelity derived from this work also play a role in HLM generation and interpretation.

Results. Fall 2015 is the first full semester of data gathering for the project. It is our "practice" semester in that researchers are refining instruments and participant communication processes while instructors are trying out the web-based learning tool with their classes for the first time. The "efficacy study" semester in Spring 2016. By the time of the conference we will have early results from the practice semester. We are eager to share these and to gather feedback from RUME attendees on (1) design and how to best explain it to stakeholder audiences and (2) strengthening connections between the cognitive science research community and the RUME community.

Acknowledgement

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Classroom Observation, Instructor Interview, and Instructor Self-Report as Tools in Determining Fidelity of Implementation for an Intervention

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Abstract. A web-based activity and testing system (WATS) has features such as adaptive problem sets, videos, and data-driven tools for instructors to use to monitor and scaffold student learning. Central to WATS adoption and use are questions about the implementation process: What constitutes "good" implementation and how far from "good" is good enough? Here we report on a study about implementation that is part of a state-wide randomized controlled trial examining student learning in community college algebra when a particular WATS suite of tools is used. Discussion questions for conference participants dig into the challenges and opportunities in researching fidelity of implementation in the community college context, particularly the role of instructional practice as a contextual component of the research.

Research Questions

(1) What is the nature of alignment between how the program is implemented and how the developer/publisher envisioned it (i.e., what is the fidelity of implementation)?
(2) What are the relationships among varying conditions of implementation (differing degrees of fidelity of implementation) and the extent to which students are achieving the desired results?

Background & Conceptual Framework

The theoretical basis for our approach lies in program theory, "the construction of a plausible and sensible model of how a program is supposed to work" (Bickman, 1987, p. 5). Having such a model in place allows researchers to conjecture and test causal connections between inputs and outputs, rather than relying on intuition or untested assumptions. As in many curricula projects, developers of the program in our study did attend to learning theory in determining the content in the web-based system, but the same was not true for implementation processes and structures. The pragmatic details of large scale classroom use were under-specified. Developers articulated their assumptions about what students learned as they completed activities, but the roles of specific components, including the instructor role in the mediation of learning, were not clearly defined. As Munter and colleagues (2014) have pointed out, there is no agreement on how to assess fidelity of implementation but there is a growing consensus on a component-based approach to measuring its structure and processes (Century & Cassata, 2014).

Fidelity of implementation is the degree to which an intervention or program is delivered as intended (Dusenbury, Brannigan, Faleo, & Hansen, 2003). This requires a careful articulation of what "as intended" means! Fidelity is rooted in the question: In what ways does the program-in-operation have to match the program-as-designed to be successful? For example, if a program calls for 15 hours of contact time, and only 10 are achieved, in what ways can the anticipated result still be reached? Do implementers understand the trade-offs in the daily decisions they must make "in the wild" and the short and long-term consequences on student learning as a result of compromises in fidelity? Century and Cassata's (2014) summary of the research offers five core components to consider in fidelity of implementation: Diagnostic, Procedural, Educative, Pedagogical, and Student Engagement. The poster will illustrate each (also see Table, next page).

Method

The project's research team has developed a rubric for fidelity of implementation, identifying measurable attributes for each component (for example, see the table, next page, for some detail on the "educative" component).

Educative: These components state the developers' expectations for what the user needs to know relative to the intervention.			
	High Level of Fidelity	**Moderate Fidelity**	**Low Level of Fidelity**
Users' proficiency in math content	Instructor is proficient to highly proficient in the subject matter.	Instructor has some gaps in proficiency in the subject matter.	Instructor does not have basic knowledge and/or skills in the subject area.
Users' proficiency in TPCK	Instructor regularly integrates content, pedagogical, and technological knowledge in classroom instruction. Communicates with students through WATS.	Instructor struggles to integrate CK, PK, and TK in instruction. Occasionally sends digital messages to students using WATS tools.	Instructor CK, PK, and/or TK sparse or applied in a haphazard manner in classroom instruction. Rarely uses WATS tools to communicate with students.
Users' knowledge of requirements of the intervention	Instructor understands philosophy of WATS resources (practice items, "mastery mechanics," analytics, and coaching tools),	Instructor understanding of the philosophy of WATS tool has some gaps. NOTE: Disagreeing is okay, this is about instructor knowledge of it.	Instructor does not understand philosophy of WATS resources. NOTE: Disagreeing is okay, this is about instructor knowledge of it.
Users' knowledge of requirements of the intervention	Instructor understands the purpose, procedures, and/or the desired outcomes of the project (i.e., "mastery")	Instructor understanding of project has some gaps (e.g., may know purpose, but not all procedures, or desired outcomes).	Instructor does not understand the purpose, procedures, and/or desired outcomes. Problems are typical.

Results

Our focus for the poster are the preliminary rubric results of data collected through observation, interview, and teacher self-report in weekly surveys (also known as "teaching logs"). From these, we may need to refine research tools (e.g., observation protocol, interview prompts, log items) as measures of fidelity. The purpose of a fidelity of implementation rubric is twofold: (1) to determine the degree of alignment between how the program is implemented and how the developer/publisher envisioned it and (2) identify conditions under which students are achieving the desired results. That is, what works, for whom, under what conditions? It provides the opportunity to discover where productive adaptations may be made by instructors, adaptations that boost student achievement beyond that associated with an implementation faithful to the developers' view. The factors included in this poster are meant as a starting point for conversation. They are not an assertion of a final collection of factors to be considered. The poster shares the theory behind the protocol and seeks to gather ideas from RUME attendees on revisions, additions, and deletions that might be productive as we move forward into the full study (2015 is a "practice" year for the study).

Implications for Practice

By definition, high fidelity implementation of an instructional tool is use that results in greater learning gains than non-use. Instructors and students are better equipped to implement with high fidelity when they have answers to questions like: What are the characteristics of good implementation? Among preferred actions in implementation, which are the highest priority? What are the trade-offs and consequences of making particular decisions about use of the tool? We seek advice form RUME-goers on effective ways to communicate implications to college instructors, department chairs, as well as stakeholders in the larger public arena.

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Helping instructors to adopt research-supported techniques: Lessons from IBL workshops

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Abstract

Inquiry-based learning (IBL) is a research-supported form of active learning in mathematics. While studies continually show benefits of active learning, it is difficult to get faculty to adopt these methods. We present results from a set of intensive, one-week workshops designed to teach university mathematics instructors to use IBL. We use survey and interview data to explore why these workshops successfully got many participants (at least 58%) to adopt IBL. Results are framed through a three-stage theory of instructor change developed by Paulsen and Feldman (1995). We focus specifically on the first stage, 'unfreezing.' In this stage, instructors gain the motivation to change, so these findings may provide the most useful lessons for helping more instructors to adopt research-supported instructional strategies. One of the key factors for the high adoption of IBL was portraying it broadly and inclusively in a variety of contexts, rather than as a highly prescriptive method.

Keywords: Inquiry Based Learning, Pedagogy, Professional Development

Background

Numerous studies have found benefits for the use of active learning methods in science, technology, engineering and mathematics (STEM) fields (Freeman et al., 2014). Freeman et al. (2014) stated that the benefits are so strong that, "If the experiments analyzed here had been conducted as randomized controlled trials of medical interventions, they may have been stopped for benefit—meaning that enrolling patients in the control condition might be discontinued because the treatment being tested was clearly more beneficial" (p. 4). While the evidence in support of the use of active learning strategies is strong, getting large numbers of faculty to adopt new methods is difficult (Fairweather, 2008; Henderson & Dancy, 2007; 2008; 2011). Professional development workshops are one strategy for helping instructors to adopt research-supported teaching methods. Workshops are the preferred method of National Science Foundation (NSF) program directors, particularly when they are multi-day, immersive workshops and include follow-up interaction between participants and organizers (Khatri, Henderson, Cole, & Froyd, 2013). There is some evidence to support this belief. In one study with engineering faculty, among six different types of professional development, the most strongly correlated form of professional development with instructors' use of student-centered pedagogies was workshop attendance (Lattuca, Bergom, & Knight, 2014).

In this report, we present findings from weeklong, intensive workshops designed to help mathematics faculty implement Inquiry-Based Learning (IBL) in their classes. IBL is a form of active, student-centered instruction in mathematics that helps students develop critical thinking through exploring loosely-structured problems and by constructing and evaluating mathematical arguments (Prince & Felder, 2007; Savin-Baden & Major, 2004). IBL has its roots in the teaching methods of mathematician R.L. Moore (1882-1974) (Mahavier, 1999), but the term IBL is used more broadly to include various practices which share the spirit of student inquiry through the core features of (1) deep engagement with rich mathematics and (2) collaboration with peers (Yoshinobu & Jones, 2013).

Like most studies of post-secondary professional development workshops, we involve only volunteer participants (Bobrowsky, Marx, & Fishman, 2001). However, learning from the experiences of motivated, volunteer participants may provide valuable lessons that can then be leveraged to meet the challenge of expanding the use of research-supported teaching methods among other instructors who are initially less familiar with or less motivated to use those methods. Therefore, we explore the questions:

(1) What lessons can we take from these faculty development workshops about ways to increase the use of research-supported, active learning techniques?

(2) How might we use those lessons to motivate non-volunteers to adopt these techniques?

Conceptual Framework

Surveys and interview scripts were designed to evaluate the workshops, to learn about participants beliefs about teaching, and participants' use of inquiry-based learning strategies. Rather than impose a conceptual framework from the start, we let one emerge from the analysis. To interpret findings, we used a three-stage model of instructor change developed by Paulsen and Feldman (1995), based on Lewin's (1947) theory of change in human systems. These authors described three stages of (1) unfreezing, (2) changing, and (3) refreezing. During *unfreezing* instructors gain motivation to change through experiencing incongruence between their goals and the outcomes of their teaching practices. Key to this stage is "psychological safety" through "envisioning ways to change that will produce results that reestablish his or her positive self-image without feeling any loss of integrity or identity" (Paulsen & Feldman, 1995, p. 12). In the next stage, *changing,* instructors learn, apply, and reflect on new teaching strategies to help align their behaviors with desired outcomes. While teaching strategies may be fluid and changing during this stage, in the final stage, *refreezing,* either these new strategies are confirmed through positive feedback and solidified, or the instructor returns to his or her original strategies. While all three stages are important, our main focus in this paper is on how these workshops supported instructors through the *unfreezing* stage, since gaining motivation may be particularly challenging with non-volunteer participants. Elsewhere, we have discussed features of these workshops that supported participants through the *changing* and *refreezing* stages (Hayward, Kogan, & Laursen, 2015, accepted).

We also draw on Rogers' *Diffusion of Innovations* (2003) to help explain results. Rogers's widely used model views adoption of innovations as a social process in which innovations spread among social networks. Various factors affect how innovations spread and the speed at which new users adopt them. We use some of these factors to help explain findings related to the adoption of IBL practices following these workshops.

Methods

Data were collected from three workshops held between 2010 and 2012. Each of the workshops was four or five days long and featured a mixture of various activities designed to help instructors learn about IBL and prepare for implementing IBL in their own classrooms. These included activities such as presentations from IBL 'experts,' panel discussions with IBL practitioners, and video observations of IBL classes. The first and third workshops were more hands-on and provided guided work time in the afternoons. During these times, participants were able to plan for their own IBL-based courses with help from the experienced staff members. The second workshop was more conference-style with formal presentations, and did not feature this guided work time. While the workshops were all part of a larger project, they were organized and run independently and therefore varied on the activities they used. All three workshops exemplified characteristics of effective research-based professional development that have been identified in previous literature on K-12 teacher development (Cormas & Barufaldi, 2011; Garet, Porter, Desimone, Birman, & Yoon, 2001). These included such features as active participation, engaging participants in discussions of their students' learning, and promoting participant self-reflection.

As evaluators of these workshops, we collected surveys pre-workshop, post-workshop, and one academic year later. Of the 139 participants at the workshops, we received 124 pre-workshop surveys (89%), 125 post-workshop surveys (90%), and 96 follow-up surveys (69%). Using anonymous identifiers, we were able to match individuals' surveys across the three time points. We successfully matched 100 (80%) post-surveys to pre-surveys and 69 (72%) follow-up surveys to pre-surveys. The high response rates indicate that the responses can be generalized to the workshop population, and are not strongly biased by subgroups such as adopters versus non-adopters.

The surveys included quantitative items and open-ended questions aimed at both evaluating workshop delivery and understanding the impact the workshops had on the participants' teaching methods. Items were developed to monitor participants' self-reported knowledge, skills, and beliefs about inquiry-based learning, as well as their motivation to use inquiry methods and their perceptions of the overall quality of the workshop. For example, on all three surveys, participants assessed their current knowledge of IBL on a scale of 1 to 4 (1=None, 2=A little, 3=Some, and 4=A lot).

To measure impact of the workshops on their subsequent teaching, we asked participants to report both *directly* and *indirectly* whether or not they had implemented IBL. We measured implementation *directly* through a multiple-choice question on the follow-up survey asking participants if they had implemented no IBL methods, some IBL methods, one full-IBL course, or more than one full-IBL course. We measured IBL implementation *indirectly* through comparing changes in participants' reported frequencies of use of eleven specific teaching practices that were probed on both pre-workshop and follow-up surveys. Available research indicates that self-report is most accurate when it is retrospective over a clearly defined time frame, when it is confidential, and when it is behavioral rather than evaluative (Desimone, 2009). Therefore, we designed these *indirect* measures of teaching practice to ask participants to anonymously report their frequency of use of eleven behaviors in a course that they had taught recently. The eleven behaviors included some that are consistent with inquiry-based learning as presented at the workshops, other behaviors that are characteristic of other forms of active learning but not necessarily IBL, and some that are characteristic of lecture-based instruction.

Open-ended questions addressed the perceived costs and benefits of using inquiry strategies and participants' impressions and learning from the workshop, which helped to provide more detail and deeper understanding of the factors that affected their use of IBL practices. Additionally, participants reported personal and professional demographic information such as career stage, institution type, gender, race, and ethnicity, so that we could test for possible differences in results among groups.

Survey data were analyzed using SPSS v. 21 (IBM Corp., 2012). Descriptive statistics were calculated for all variables, and inferential statistics were calculated as appropriate. Open-ended responses were entered into Microsoft Excel (Microsoft, 2011) and coded for common themes.

In addition, we conducted sixteen interviews. During these interviews, we asked questions to gain a deeper understanding of participants' development as instructors, their views on teaching and learning, and more detail about their classroom activities and the factors that affected whether or not they implemented IBL. Interviews were semi-structured so that participants could reveal their own perspectives instead of fitting their responses into categories introduced by researchers. As a result, we did not ask questions in the same order or with the same wording in every interview. Some topics arose spontaneously and thus were not represented in every interview. We used these interviews to help explain findings from the surveys.

Interviews were audio recorded, transcribed verbatim, and entered into *NVivo* v. 9 (QSR International Pty Ltd., 2010). We carefully read through all of the transcripts and identified recurring topics. Then, we identified segments of the transcripts that related to these topics and assigned them a code to identify that topic. If an individual passage covered multiple topics, we assigned multiple codes. Topics were coded each time they were discussed, so a code was sometimes used multiple times over the course of an interview. Groups of codes that shared similar themes were organized into domains (Spradley, 1980).

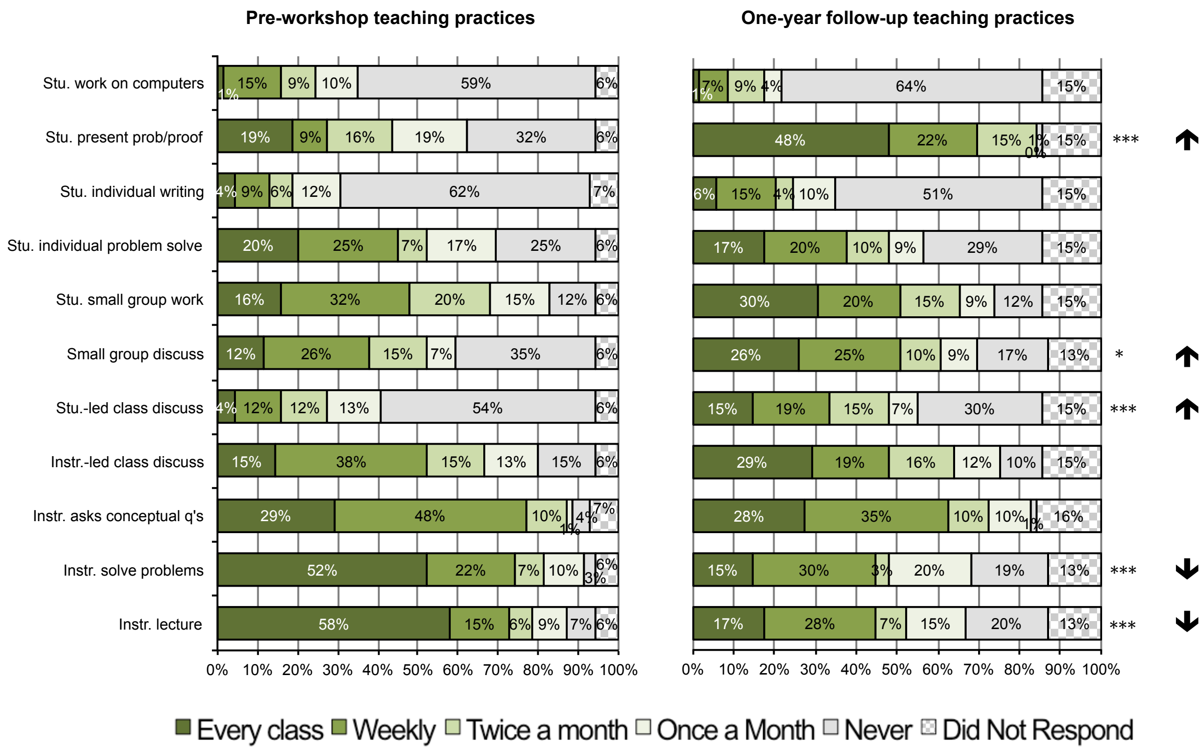
Results

Overall, the workshops were successful in helping participants to adopt IBL techniques. On a *direct* question on the follow-up surveys, 58% of all participants reported using IBL in their own classrooms, with 28% using "some IBL methods", 14% teaching "one full IBL course," and 15% teaching "more than one full IBL course." Only 8% did not report using any IBL methods, and the remaining 34% did not respond (to either the entire survey or just this question). To assess implementation of IBL *indirectly,* we compared respondents' reported teaching practices prior to the workshop with those reported on the one-year follow-up. We have presented at RUME before about the issues related to measurement and why we believe we can draw conclusions from these two types of self-reported data (Hayward & Laursen, 2014).

Significant changes in teaching practices were consistent with the use of the 'core' practices of inquiry-based learning: decreases in the reported frequencies of use of instructor lecturing and instructor solving problems or examples on the board; increases in the reported frequencies of use of student-led whole class discussions, small group discussions, and students presenting problems or proofs at the board. 'Preference IBL' practices, which instructors implement to varying degrees, showed non-significant changes. These included practices such as students working in small groups and instructor-led discussions. IBL instructors vary on whether or not they use group work and how active they are in leading discussions. Other forms of active learning that are not characteristic of IBL remained stable over time. These included students using computers or writing individually in class. Full results are presented in Figure 1 below.

These results corroborate participants' self-reported level of IBL implementation (none, some, full-IBL). The patterns related to core, preference, and non-IBL practices are also important in the context of Paulsen & Feldman's *unfreezing* stage. Key to *unfreezing* is the idea of 'psychological safety,' or being able to envision ways to change that fit with the individual's identity as a teacher. At the workshops, IBL was presented as a broad range of related practices that share common features, rather than prescriptive techniques or curricula. Having options for 'preference IBL' practices helped provide psychological safety as it allowed participants to implement a type of IBL that fit with their own teaching style. The *unfreezing* stage is especially relevant for spreading research-supported practices beyond volunteers, who are willing and able to commit to an intensive workshop.

Figure 1 Frequencies of pre-workshop and one-year follow-up teaching practices, matched survey responses



*p<0.05, **p<0.01, ***p<0.001

In the interviews, ten of the sixteen participants commented on the presentation of IBL as a range of related practices. They explained how this broad presentation of IBL provided psychological safety. For example, one participant was struck by "how enthusiastic everyone [at the workshop] was about teaching and helping other people learn what IBL is about and how to integrate it into your classroom," but "tuned out" one presenter that he found "aggressive" in communicating that "this is the only way to go, and that if you don't do this, then it somehow diminishes your classroom" (Male participant, cohort 2). Another participant explained that seeing IBL as a spectrum of related practices "was kind of a big moment for me because it made it seem less scary. …Feeling like I can pick and choose aspects of it, and find something on the spectrum that I feel comfortable with, was empowering" (Female participant, cohort 2).

This concept of a "spectrum" of IBL was particularly powerful for some participants who were familiar with other instructors who employ the teaching model of R.L. Moore "dogmatically." For example, one participant explained that,

> *the Moore method is on one extreme and I think when I was deciding whether or not to use IBL in my classes before the conference, I had always viewed it as all or none. I hadn't realized that some of the projects and stuff that I'd been doing in my classes were IBL in nature and that that's okay, that you don't have to do an entire class inquiry-based, or that you don't have to be as rigid as the Moore method prescribes (Female participant, cohort 2).*

Other participants also mentioned this idea of not doing 'full IBL' courses, but instead starting with smaller steps and then building over time. One participant explained that this was because "you saw people doing full inquiry classes, and it seems very intimidating, very time consuming. …You don't have to go to a full inquiry class … as the different semesters go on, I'm planning on turning more and more into inquiry" (Female participant, cohort 1).

In addition to portraying IBL as a broad, inclusive set of practices, the workshops also showed IBL being used in a variety of settings. For example, workshops featured sessions about how to tailor IBL to different groups of students (such as first-year students and pre-service teachers) and in different types of courses (such as calculus and proof-based). These differences shaped how interview participants structured their IBL courses. Interviewees described a number of situational factors that led them to vary the IBL strategies they used, depending on the level of the class (first-year, sophomore, etc.), the size of the class, or the audience (mathematics majors, pre-service teachers, etc.). All sixteen interviewees commented on these situational factors a total of 101 separate times throughout the interviews, meaning that they often considered multiple factors in designing their courses.

As one interview participant explained, seeing a diversity of IBL practices portrayed at the workshop, as well as a diversity of practitioners and situations, was important because it was "frustrating" when one presenter "had so many resources at their disposal that the rest of us didn't have. …how many graders and TAs they have and how they keep the class size small. These were things that just don't apply to most universities" (Female participant, cohort 2).

Other participants made positive comments about the diversity of opinions and viewpoints, such as one who identified the best aspect of the workshop as offering,

> *A good diversity of ideas and approaches, which I feel that I can adapt to my own teaching. As an inexperienced IBL user, I was very interested in learning from experts, but I was also interested in meeting people in my situation, who I can identify with, and hearing how they have worked through the same problems that I have (Male participant, cohort 1).*

Another participant felt that the workshop "gave me more ways and more tools to introduce IBL into [lower level and pre-service courses]" (Male participant, cohort 2). As a result, he was able to incorporate IBL methods into classes he previously thought could not be taught with IBL.

Application/Implications to Future Research or Teaching Practice

These findings suggest that the workshop leaders' choice to portray IBL as a broad, inclusive set of practices, rather than as a prescriptive, rigid method, may have been essential for helping new instructors during the *unfreezing* stage, as it helped them to envision a way to change their teaching that was consistent with their own self-image and thus felt safe. This also gave participants the freedom to use a "hybrid" style where they incorporated some IBL strategies into a more traditional class. This may have served as a more feasible and less daunting entry into IBL, but may then lead to "full IBL" as instructors experience success and observe positive student outcomes.

Using broader, more inclusive portrayals may also help increase the adoption of other research-supported strategies. Particularly, it may help instructors to start small and increase the use of these innovations over time. There is already evidence for this outside of mathematics. Biology education researchers call this process "phased inquiry" and suggest that it is "an important step toward expanding adoption of inquiry practices in college science courses" (Yarnall & Fusco, 2014, p. 56). "Phased inquiry" may be useful for overcoming time constraints, which physics instructors cite as one of the biggest barriers to implementing research-based instructional practices (Dancy & Henderson, 2010). Communicating broad definitions may help instructors to learn and phase in new strategies piece-by-piece over time, which may in turn make time seem like less of a barrier to implementation. In Rogers's (2003) model, *trialability* is positively related to the rate of adoption of innovations. However, further longitudinal research is needed to explore how teaching practices change after instructors take these initial steps to incorporate "hybrid" methods.

From their studies of physics education reform, Henderson and Dancy (2008) recommend providing instructors with easily modifiable curricular materials, so that individual instructors may use their expertise to adapt the materials to their own local environments. While their recommendation applies to reforms focused on curricular materials, our findings suggest that this feature of easy portability may also be important for sharing primarily pedagogical strategies such as IBL. Showing diverse examples of IBL helped participants to customize IBL for their individual context and may have made implementation more likely. Rogers (2003) calls this process *re-invention* and states that it may reduce mistakes, help to fit innovations to local contexts, and help to make those innovations more responsive to changing conditions.

The workshop leaders' choice to present IBL as a variety of related approaches may have been inviting for participants, but may cause others to doubt whether the fidelity of IBL was maintained. Studies in both physics (Dancy & Henderson, 2010) and biology (Yarnall & Fusco, 2014) have reported that instructors often adapt and modify research-based instructional strategies, usually in ways that align more with traditional methods and reduce the amount of student inquiry. Instructors often change materials to match their own individual style and preferences, because they do not expect materials created elsewhere to work without modification (Henderson & Dancy, 2008).

However, there are various ways to measure fidelity of implementation, which can be categorized into 'fidelity of structure' (i.e. adherence and duration of use) and 'fidelity of process' (i.e. quality of delivery, and program differentiation, or whether the "critical features that distinguish the program from the comparison condition are present") (O'Donnell, 2008, p. 34). For pedagogical innovations like IBL, fidelity to process may be more important whereas curricular reform may focus more on fidelity to structure. IBL, specifically, may be somewhat robust to variation in structure, as student outcomes are improved over traditional courses despite notable variations in how IBL is implemented (Laursen et al., 2014). It may be the case that portraying IBL as a spectrum of related practices helped participants by outlining ways in which they could modify the methods to fit their context while still maintaining the fidelity of the core features of IBL, including high levels of student inquiry. If research-supported active learning strategies are defined in a way that allows for and helps to outline appropriate modifications, this may be important to maintaining the fidelity of their core features (fidelity of process) and promote the same positive outcomes supported by the research.

Evidence from our study of this example of professional development of IBL that communicates broad, inclusive definitions seemed to help transform teaching practices in three ways: it (1) lowered the initial resistance and increased psychological safety by allowing for comfortable, personalized approaches to IBL teaching, (2) allowed for increasing adoption over time through "phased inquiry," and (3) helped to maintain fidelity to IBL's core features through outlining modifications that preserved the core principles of the approach. Our findings suggest that research-supported innovations that are inclusive and allow for context-appropriate modifications are likely to support broader adoption and greater user success than those that are restrictive and inflexible.

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A Collaborative Effort for Improving Calculus Through Better Assessment Practices

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Abstract

Like many institutions across the country, Utah State University's Department of Mathematics and Statistics has embarked on an effort to improve the calculus sequence with the following objectives: (1) improve our students' comprehension and application of key topics, (2) retain/recruit more students into STEM majors, and (3) provide more consistency across sections. After initial planning and preparation in the 2014-15 academic year, new practices were ready for implementation. In the fall of 2015, teams of instructors worked from common guided course notes, and met weekly to discuss instruction and develop common assessments. This poster displays the methodology of test design and item analysis we employed in the Calculus 2 course. While our team is only at the beginning stages of this work, the methods for creating reliable and relevant measures of student learning hold promise for achieving the goals of our reform.

1 Introduction

For many university students calculus courses serve as the gate-keeper that either attracts or repels them from pursuing STEM majors and professions, and is consequently a priority area for reform [1]. Here we share the background and motivations of our current work at Utah State University:

- The College of Engineering presented our department with data showing wildly inconsistent DWF rates in various sections of calculus by instructor.
- A Calculus Committee was formed and after a series of organizational meetings, we set about organizing the presentation, content, and evaluation in all courses and sections of Calculus.
- To provide cohesion between the sections of Calculus, we produced a set of guided lecture notes. These published notes act as a scaffold upon which instructors can build their lectures, as well as a learning tool for students during class sessions.
- To add consistency across all sections we scheduled common midterms and final exams. Exam questions are proposed and vetted by all instructors, and grading of exams was accomplished by assigning one problem to each faculty member thus ensuring consistent assessment of each test item.
- During weekly meetings instructors address questions about content and scope of material, propose exam questions, and share instructional techniques. These meetings also identified much-needed corrections to be included in the next iteration.
- In Calculus 1 and Calculus 2, we began the semester with a skills pre-test to help ascertain the level of preparedness of the students with varying backgrounds.

In this poster, we share our methods and findings as they pertain to the skills pre-test we offered at the beginning of the semester in Calculus 2. We also conducted item analysis after a midterm later in the semester and found improvement of test validity. Collaborative work throughout the semester was likely a quality professional development experience for all of the instructors involved.

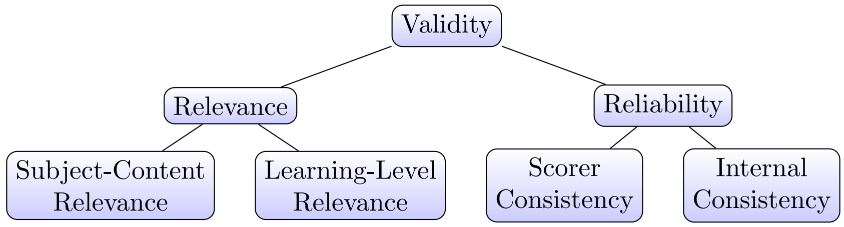

Figure 1: Cangelosi's hierarchy of test validity, dependent on relevance and reliability of test items.[3]

2 Methods

If we wish to maximize student achievement we then must pay greater attention to the improvement of classroom assessment. Instructors need training in creating better assessment tools. [2] To this end we established the following practices and methods.

- A common exam schedule was determined for all sections of Calculus 2, allowing for one exam to be administered to all six sections of the course.
- We agreed upon specific learning objectives to be tested. Instructors listed topics, but we also discussed the level of learning we were after and used Cangelosi's cognition scheme to guide our work. [3]
- Once we laid out a test blueprint of topics and their relative emphasis, committee members took ownership of test items for each topic. This entailed creating the test item, writing a scoring rubric, and then grading the item on all exams across all sections. This effort made scoring quick and consistent.
- Exam scores across all sections were collated for analysis (via boxplots, ANOVAs, and Tukey's HSD).
- Scoring of individual items as well as total scores permitted test item analysis. In addition to quantifying item difficulty and discrimination, we also calculuted Hoffman's Efficiency Coefficient for each item. Finally we compute the exam's Kuder–Richardson Formula 20 (KR-20) coefficient as a measure of internal reliability [3]. These provide guidelines for reliability of test questions, feeding back to future test design. Our poster will include the formulae and interpretations to illustrate this process for others.

We have not only seen our test reliability increase over the semester, but our level of collaboration overall has served as professional developing experience for both the graduate students and faculty. For example, as the semester progressed, faculty began observing one another's classes in an effort to better understand how material was presented by peers, and teachers shared supplemental materials to enhance lectures with computer animations and interactive learning activities.

3 Recommendations for the Next Cycle

As the world and calculus evolves, effective instruction will always require changes. This work is an iterative process, and should never be considered complete.

Extending beyond the traditional efforts to make common exams, we have made considerable strides in applying item analysis toward informing our exam design. While attention has been paid to instructional objectives and materials treatment of quantitative thinking skills, classroom assessments often fail to match these aspirations. Some students use tests to understand the teachers' expectations and prioritize topics. Thus poor quality assessments that fail to consider higher-order thinking skills will inhibit the development of those skills. [5] The item analysis provides a powerful tool for quantifying the validity of these assessments. Our work has has beneficial side effects such as motivating the empowering practice of observing each other's classes.

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Framework for Mathematical Understanding for Secondary Teaching: A Mathematical Activity perspective

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Abstract: *A framework for mathematical understanding for secondary teaching was developed from analysis of the mathematics in classroom events. The Mathematical Activity perspective describes the mathematical actions that characterize the nature of the mathematical understanding that secondary teachers could productively use.*

Mathematics teaching at the collegiate level focuses on enabling students to develop solid understanding of mathematics. Although collegiate mathematics students often describe mathematics as learning specific topics and strategies and applying this knowledge to their work, their instructors may have additional but less explicit goals such as valuing the structure of mathematics, being able to create a deductive argument, or exploring and comparing systems of mathematics. These latter goals are especially important for prospective teachers of secondary mathematics, and college mathematics instructors are attending in new ways to the mathematical preparation of those who will teach mathematics.

Over the past three decades, mathematics education researchers and theorists have increased their focus on the mathematical knowledge of teachers that helps teachers reach their goals of promoting a more robust understanding of mathematics in their students. During that time, researchers have refined the focus from Shulman's (1986) construct of pedagogical content knowledge to constructs such as mathematical knowledge for teaching (MKT) (Ball, 2003; Ball & Bass, 2003; Ball & Sleep, 2007a; Ball & Sleep, 2007b; Ball, Thames, & Phelps, 2008) and knowledge of algebra for teaching (KAT) (Ferrini-Mundy, Floden, McCrory, Burrill, & Sandow, 2005; McCrory, Floden, Ferrini-Mundy, Reckase, & Senk, 2012). Work on MKT is, perhaps, the best known of the research programs focused on teachers' mathematical knowledge. MKT originated with a reflection on the mathematical knowledge involved in the mathematical work of teaching at the elementary level. MKT partitions the territory of mathematical thinking into categories such as specialized content knowledge, common mathematical knowledge, and mathematics at the horizon. While the MKT categories can partition mathematical knowledge at the secondary level as well as at the elementary level, those categories do not characterize the nature of mathematical thinking that seems to distinguish mathematics at the secondary school level.

In their work in secondary mathematics, students expand their mathematical knowledge to include new ideas such as irrational numbers, complex numbers, static and rotating objects, sample spaces, and a variety of ways to represent these ideas. But the differences between mathematics at the elementary and secondary levels are not solely extensions of the topics involved, but also a change in the nature of mathematical thinking involved. Whereas both elementary and secondary mathematics honor deductive reasoning, secondary mathematics places a much stronger emphasis on deductive thinking within a closed mathematical system. It is in the context of secondary mathematics that curricula focus on reasoning on the basis of a well-defined system of given properties and relationships. For example, the work of secondary students in the study of geometry is more likely to occur at the third or fourth van Hiele level (making deductive connections and constructing proofs) rather than the first or second levels (focused on visualizing or recognizing properties of geometric objects) that are more prominent at the elementary level. At the elementary level, students develop ways to represent mathematical relationships. As students progress through school mathematics, their repertoires of ways to represent mathematical relationships expands so that, as they engage in secondary mathematics, they can be expected to link representations of the same mathematical entities and to reason about a mathematical entity in one representation making conclusions about that entity in another representation.

Secondary teachers need to be able to reason flexibly enough to recognize and act on opportunities for their students to build capacities for reasoning in a closed system and for capitalizing appropriately on a range of representations. They need mathematical understanding that enables them to perform such activities as creating examples, nonexamples, and counterexamples of entities encountered in secondary mathematics, to identify special cases of broad categories of mathematical objects, and to explain when a general statement can or cannot be extended to a larger or different domain or set of mathematical objects. Secondary teachers need to make connections between mathematical systems. In order to facilitate learning secondary mathematics, the work or context of teaching requires a depth of specific mathematical understanding that incorporates the more subtle but important goals of mathematics teaching. Mathematics teachers must not only understand mathematics but they must enable others to understand mathematics in the fullest sense. They need to pose interesting questions and tasks that bring the structure of mathematical systems alive. They need to understand the mathematical thinking of students in order to correct or challenge their thinking. They need to be able to reflect on the curriculum and organization of mathematical ideas. The context of learning mathematics requires specific mathematical understanding beyond pedagogical knowledge.

The six faculty involved (G. Blume, J. Kilpatrick, J. Wilson, and R. M. Zbiek, in addition to the authors) wanted to build a framework that would account for the proficiencies, actions, and work of secondary mathematics teachers. We committed to developing a framework that accounted for the mathematical opportunities secondary teachers actually encounter, and so we began in the classroom. As we began to study the mathematical opportunities unfolding in the classroom, we recognized many of the ideas expressed by others who have attended to secondary mathematics (e.g., Adler & Davis, 2006; Cuoco, 2001; Cuoco, Goldenberg, & Mark, 1996; Even, 1990; McEwen & Bull, 1991; Peressini, Borko, Romagnano, Knuth, & Willis-Yorker, 2004; Tattoo et al., 2008). While the framework incorporates previous ideas, it attends directly to the secondary mathematics built on data from mathematics classes.

Our source of data was a set of what we came to call Situations. A Situation is a mathematical description, based on an actual event that occurred in the practice of teaching, of the mathematics that teachers could productively use in the work of teaching mathematics. Teams of mathematics education faculty at Penn State and at University of Georgia worked with dozens of doctoral students in mathematics education to develop more than 50 Situations. Although any one Situation is too large to report in this paper, we provide a brief outline of one of the Situations (from Heid & Wilson, in press) here. Each Situation includes a Prompt (a description of a mathematical opportunity–an event that one of the authors observed happening in the course of teachers planning or implementing a secondary mathematics lesson) and several Mathematical Foci (development of mathematics that a teacher could productively use in the context of that mathematical opportunity). A short statement about the nature of the mathematical understanding being targeted precedes each Mathematical Focus. Other parts of each Situation are Commentaries (a description of how the Mathematical Foci for the Situation fit together) and PostCommentaries. One of the Situations is outlined in Figure 1.

CHAPTER 22. INVERSE TRIGONOMETRIC FUNCTIONS

Prompt
Three prospective teachers planned a unit of trigonometry as part of their work in a methods course on the teaching and learning of secondary mathematics. They developed a plan in which high school students would first encounter what the prospective teachers called *the three basic trig functions*: sine, cosine, and tangent. The prospective teachers indicated in their plan that students next would work with "the inverse functions," which they identified as secant, cosecant, and cotangent.
Commentary
The Foci draw on the general concept of inverse and its multiple uses in school mathematics. Key ideas related to the inverse are the operation involved, the set of elements on which the operation is defined, and the identity element given this operation and set of elements. The crux of the issue raised by the Prompt lies in the use of the term *inverse* with both functions and operations.
Mathematical Focus 1
An inverse requires three entities: a set, a binary operation on that set, and an identity element given that operation and set of elements.
Secondary mathematics involves work with many different contexts for inverses. For example, opposites are additive inverses defined for real numbers and with additive identity of 0, and reciprocals are multiplicative inverses defined for nonzero real numbers and with multiplicative identity of 1. [Discussion follows about the nature of inverses, the role of an identity in inverses, and the importance of domain and range in consideration of inverses.]
…
Mathematical Focus 2
Although the inverse under multiplication is not the same as the inverse under function composition, the same notation, the superscript -1, is used for both.
[Discussion follows about notation used in different inverse relationships, and the specific use of that notation in consideration of trigonometric functions.]
…
Mathematical Focus 3
When functions are graphed in an xy*-coordinate system with* y *as a function of* x*, these graphs are reflections) in the line* y = x *of their inverses' graphs (under composition).*
The graph of a function reflected in the line $y = x$ is the graph of its inverse, although without restricting to principal values, the inverse may not be a function. Justifying this claim requires establishing that the reflection of an arbitrary point (a, b) in the line $y = x$ is the point (b, a). [A geometric proof follows, using a coordinate plane representation of the reflection of a point (a,b) over the line $y = x$.]
…

Figure 1. Outline describing a Situation appearing in (Zbiek et al., in press).

The Situations we (the cross-university teams) developed suggested a range of mathematical abilities, actions, and settings that could underlie potentially productive mathematical thinking on the part of the teacher. It was on the basis of these abilities, actions, and settings that we embarked on the challenging task of developing our Framework for Mathematical Understanding for Secondary Teaching. As we examined the Situations we had created, we recognized that we needed several different perspectives to explain the mathematics we had identified. Akin to Plato's allegory of the cave, the framework on which we settled consisted of three perspectives, each of which cast a different shadow representing a student's mathematical understanding (See Figure 2).

From one perspective, Mathematical Proficiency, we could use the strands of proficiency to describe the nature of the mathematical understanding, but this perspective did not account for the mathematical actions that secondary teachers could productively take. The second perspective addressed this as Mathematical Activity. However, neither the first nor second perspective accounted for the settings in which teachers needed to call on their mathematical knowledge. The third perspective, Mathematical Context of Teaching, addressed the mathematical context in which teachers could productively call upon their mathematical knowledge.

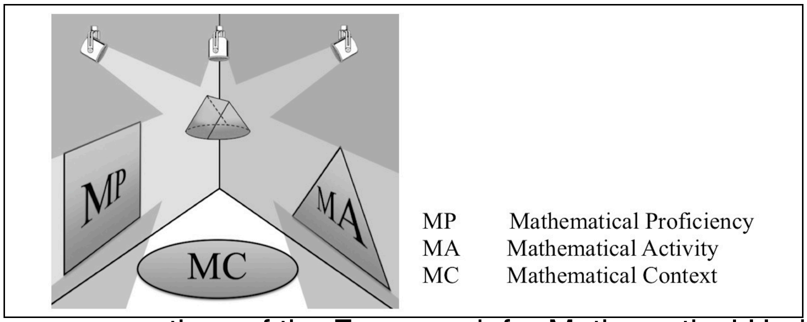

Figure 2. Three perspectives of the Framework for Mathematical Understanding for Secondary Teaching (Heid & Wilson, in press).

The first perspective, Mathematical Proficiency, is likely to be familiar as a way to think about students' mathematical capability. The third perspective, Mathematical Context, provides a description of the mathematical understanding that is particularly relevant to teaching. This perspective was more implicit than explicit in our data, but we realized that the Mathematical Context of teaching indicates why it is critical to recognize and attend to the importance of Mathematical Activity. In this paper, we confine our discussion to the development of the second perspective, Mathematical Activity.

Mathematical Activity

We used the final set of Mathematical Foci as data from which to generate our Framework for Mathematical Understanding for Secondary Teaching. First we identified mathematical actions implicit or explicit in each of the Foci. We then categorized those actions, including categories such as creating mathematical entities and interpreting mathematical representations and orchestrating movement among them.

For example, one set of mathematical actions that we grouped into a single category included the following actions:

- Creating a counterexample for a given structure, constraint, or property
- Creating an example or non-example for a given structure, constraint, or property
- Creating equivalent equations to reveal information
- Creating problems to foreground a concept
- Creating a file (a computer application) whose creation requires mathematics beyond what the file is used to teach
- Constructing an object given a set of mathematical constraints
- Generating specific examples from an abstract idea
- Creating a representation for a mathematical object with known structure, constraints, or properties

Having grouped these actions into a single category, we developed a description of a mathematical action that encompassed these actions. In this case our description was “Creating a mathematical entity or setting from known (to the one creating) structure, constraints, or properties.” An example of a specific mathematical action that might fit this category is the task of constructing a quadrilateral with specific characteristics. Other mathematical actions were developed in a similar fashion. A few of the final set of mathematical actions at this juncture, along with specific examples drawn from the Situations, are shown in Figure 3.

Category	Example
Create: Creating a mathematical entity or setting from known (to the one creating) structure, constraints, or properties	Sketch quadrilateral ABCD with $m\angle D = m\angle A = 90$ and $\overline{AB} \parallel \overline{DC}$ such that ABCD is not a parallelogram.
Recognize: Recognizing mathematical properties, constraints, or structure in a given mathematical entity or setting, or across instances of a mathematical entity	Recognizing that strategic choices for pairwise groupings of numbers are critical to one way of developing the general formula for summing the first *n* natural numbers
Choose: Considering and selecting from among known (to the one choosing) mathematical entities or settings based on known (to the one choosing) mathematical criteria	The mathematical meaning of a/b (with $b\neq0$)) arises in different mathematical settings, including: slope of a line, direct proportion, Cartesian product, factor pairs, and area of rectangles. One might choose slope of a line as a setting to illustrate the need for $b\neq0$.
Use representations: For given representations, interpret them in the context of the signified, orchestrate movements between them, and craft analogies to describe the representations, objects, and relationships	Using tabular and graphical representations to estimate the value of $2^{2.5}$
Assess (interpret and adapt) the mathematics of the situation: Interpret and/or change certain mathematical conditions/ constraints relevant to a mathematical activity	Assess and use a modulus definition of absolute value in evaluating $f(x) = \left\lvert\sqrt{x-10}\right\rvert$
Extend: Extend the domain, argument, or class or objects for which a mathematical statement is/remains valid.	Extending: the absolute value function from the real to the complex domain; "triangle" from Euclidean to spherical geometry
Connect: By recognizing structural similarity, make connections between: representations of the same mathematical object; different methods for solving a problem; mathematical objects of different classes; objects in different systems; or properties of an object in a different system.	Identifying structural similarities of the Euclidean algorithm and the long division algorithm

Figure 3. A few of the set of mathematical actions that comprised the Mathematical Activity perspective of the Framework for Mathematical Understanding for Secondary Teaching, along with specific examples drawn from the Situations.

Reason: Reason about a mathematical entity in more than one way, including, but not limited to: from mathematical definitions, from given conditionals, from and toward abstractions, by continuity, by analogy, and by using structurally equivalent statements.	Reasoning about the sum of the first n natural numbers by appealing to cases, by making strategic choices for pair-wise grouping of numbers, and by appealing to arithmetic sequences and properties of such sequences.

Figure 3, continued.

Finally, we organized the set of mathematical actions to account for the actions arising in the Situations as well as reasonable mathematical actions that were not captured in the categories that were derived from the Situations. The final set of categories is displayed in Figure 4.

I. Mathematical noticing: *Recognize and choose from among known mathematical entities or settings based on known mathematical criteria such as:*
   A. Structure of mathematical systems
   B. Symbolic form
   C. Form of an argument
   D. Connections within and outside mathematics

II. Mathematical reasoning**:** *Reason about a mathematical entity in one or more than one way, including, but not limited to: from mathematical definitions, from given conditionals, from and toward abstractions, by continuity, by analogy, and by using structurally equivalent statements.*
   A. Justifying/proving
   B. Reasoning when conjecturing and generalizing
   C. Constraining and extending

III. Mathematical creating*. Create (Creating a mathematical entity or setting from known (to the one creating) structure, constraints, or properties)*
   A. Representing
   B. Defining
   C. Modifying/transforming/manipulating

IV. Integrating strands of mathematical activity. *Coordinate (Coordinate mathematical knowledge, student mathematical thinking, school curricula, and knowledge development); Reflect (self-reflect) (Reflect on mathematical aspects of one's practice or on one's own doing math); and Apply (Employ algorithms, definitions, and technology in mathematical settings and/or real world quantitative settings when applicable.)*

Figure 4. Mathematical Activity Perspective of the Framework for Mathematical Understanding for Secondary Teaching (Heid & Wilson, in press).

The final categories differed from existing frameworks in their mathematical nature. The mathematical actions we described derived from the mathematical decisions that teachers confront. Their work in mathematics classrooms would benefit from their ability to notice similar mathematical structures. Being comfortable enough with mathematical entities, properties, and structures to create and modify new representations would allow them the freedom to pursue their students thinking. They could productively use a flexible and robust repertoire of techniques for justifying their mathematical work.

The framework is intended to be a work in progress. It can serve as a research tool to study the mathematical understanding of secondary teachers. Researchers might investigate, for example, what collegiate mathematics courses contribute to the development of the capabilities suggested in each of the perspectives. They might also investigate how the aspects of secondary mathematics teachers' own mathematical understandings as described in the Framework influence the mathematics to which they expose their students.

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Developing mathematical knowledge for teaching in content courses for preservice elementary teachers

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In recent years, much attention in the teacher education literature has been given to ways in which inservice teachers develop facility with the construct known as mathematical knowledge for teaching (MKT). Much less is known about the ability of preservice teachers to construct MKT. To address this, the current preliminary report adds to the research base by investigating two primary questions: (1) Can teachers build MKT in their content courses?, and (2) Can teachers engage in meaningful mathematical discourse as a result of their content courses? The report examines the effects of a semester long course on number and operations designed to allow preservice elementary teachers opportunities to build different aspects of MKT. Very preliminary analysis shows that many students lack this knowledge upon entering the course, but most are able to begin to build a degree of facility in it by course completion.

Key Words: Mathematical Knowledge for Teaching, Mathematical Discourse, Reasoning, Justification

Background and Research Questions

A central tenet of teacher education research has long been identifying the types of knowledge that teachers need to know in order to teach mathematics. Such attempts date back to Shulman's (1986) original proposal of a new type of knowledge that he called *pedagogical content knowledge* (PCK), defined as "the particular form of content knowledge that embodies the aspects most germane to its teachability" (p. 9). Since then, research teams such as Ball and company (2008) and Hauk and her colleagues (2014) have worked to conceptualize PCK. Ball and company have developed typologies for the much broader realm of *mathematical knowledge for teaching* (MKT), shown in figure 1, for which PCK is a subconstruct. Note that the left half of the oval consists of subject matter knowledge (SMK) which they claim requires no knowledge of students, which distinguishes it from the right half which is PCK. It is worth noting that the Ball model is specifically designed for the K-8 setting; this is important because as Speer et al (2014) note, generalizability comes into question when trying to apply the model outside the K-8 context.

Within the Hill, Ball, and Schilling (2008) model, *common content knowledge* (CCK) is defined as "knowledge that is used in the work of teaching in ways in common with how it is used in many other professions or occupations that also use mathematics"(p.6). In contrast, *specialized content knowledge* (SCK) is specialized in the sense that it is specific to the task of teaching. SCK includes various ways to represent mathematical ideas, provide mathematical explanations for rules and procedures, and examine and understand innovative solution strategies(Hill et al, 2008, p.377). As an example, consider fraction division. Most middle school graduates can readily use the invert-and-multiply algorithm to divide fractions. Thus, this piece of knowledge is CCK. Yet, few can explain to a novice learner *why* the algorithm exists in school mathematics nor why it is justified, thereby making this particular piece of knowledge SCK. Within the realm of PCK are *knowledge of content and students* (KCS) and *knowledge of content and teaching* (KCT). KCS is

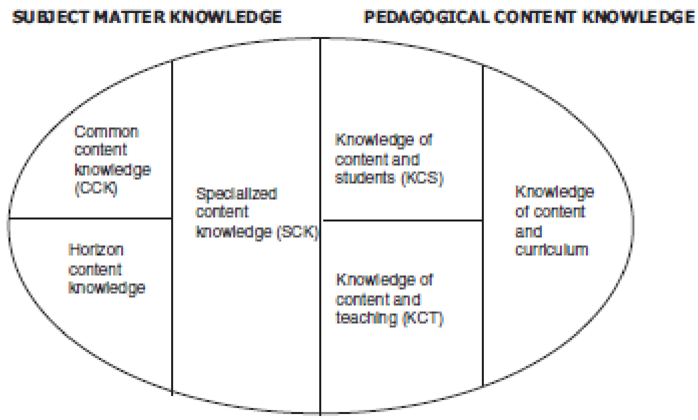


Figure 1. Domain Map for MKT (Hill et al, 2008)

"content knowledge intertwined with knowledge about how students think about, learn, or know this particular content" (p. 375), while they define KCT as a knowledge of teaching moves. So, using our division of fractions example again, a teacher who is aware that students often invert the first fraction instead of the second fraction is demonstrating KCS, and might use fraction diagrams as a way of scaffolding student understanding of division of fractions by using her KCT.

Implicit in the use of KCS and KCT is an awareness of the words, grammar, syntax, and forms of standard mathematical language in use – what Gee (1996) would call the "little d" discourse of mathematics. Also at work in the teaching of mathematics are nuances about what is valued in mathematical discourse in a mathematics class (as opposed to mathematics in a physics or biology class), the socio-mathematical norms for questions and answering, and myriad other interactions that make a mathematics lesson recognizable in an instant (e.g., by someone listening in or looking through the window of a classroom for just a few seconds). This kind of *situated* "little d" discourse is what Gee called "big D" Discourse. Hauk and colleagues (2014) have brought these ideas into a further unpacking of the components of PCK. The extended model, shown in Figure 2, adds a fourth dimension to PCK called *Knowledge of Discourse* (KD). Hauk et alia argue that effective teaching of mathematics includes facilitating student learning of *mathematical discourse* (along with other discourses). Such Discourse is enacted in the classroom when students and teacher engage in mathematically appropriate, accurate, and effective communication situated in the context of reasoning and justification of mathematical ideas. Clearly, a rich and textured Knowledge of Discourse is required for teachers to use and promote the valued mathematical skill of justification: engaging in reasoning about and explaining how one knows something is true (Cioe et al., 2015).

To measure PCK and MKT more generally, both research teams developed multiple choice assessments designed for administration to inservice teachers receiving professional development for Ball's team and completing a master's degree for mathematics teachers in the case of Hauk's team. It should be noted that in the case of Hauk's team, the focus was on the PCK development of teachers at 7-12 level unlike Ball's team. While items in the instrument developed by Hauk's team measured in large extent the syntactic structure of KD and did attempt to measure the teachers' ability to engage in proof validation, neither their instrument nor the items developed by Ball's team measure the larger components of discourse required to engage in reasoning and justification: i.e. neither team tried to specifically measure mathematical discourse more generally. Hence, the current project is designed to address two key missing

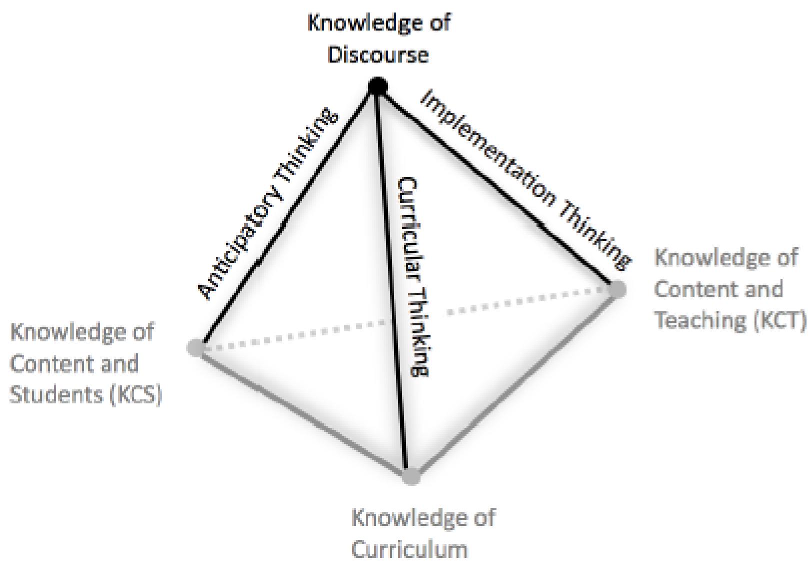


Figure 2. Tetrahedron Model of PCK (Hauk et al., 2014)

ideas in the existing literature: (1) Can *preservice elementary teachers* (PSETs) develop facility in MKT as a result of their learning in the content courses for PSETs?, and (2) Can PSETs learn to engage in meaningful mathematical discourse as a result of their experiences in these courses?

Research Methods

Beginning in the summer of 2011, an instrument was developed to begin measuring different aspects of MKT of preservice elementary teachers, with particular emphasis on items which require some combination of SCK, KCS, and KD. The items requiring a combination of SCK and KD to answer generally require the teachers to engage in mathematical discourse to justify certain mathematical facts or procedures. Two examples of items from the most recent administration of the instrument are given below:

7. John asks you in math class one day why $4^0 = 1$. Give John an explanation that he can understand for why this is true.

8. Nancy, a student in your 5th grade math class, asks you day why she cannot divide 5 by 0. That is, why she cannot do 5 ÷ 0. Give Nancy an explanation that she can understand for why she cannot do this.

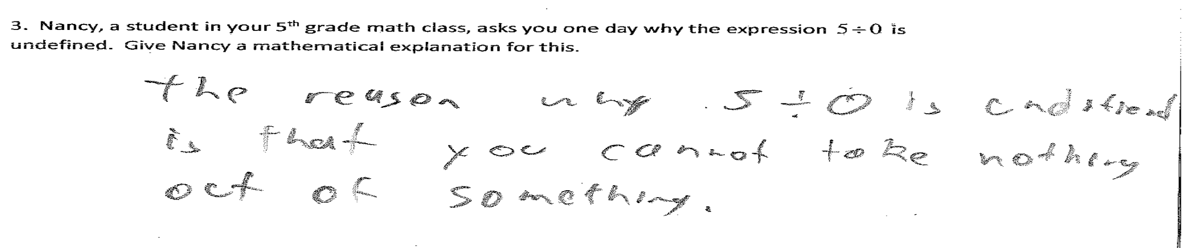
PSETs enrolled in a course on number and operations at a large public state university in the northeastern US were given the instrument upon entering the course as well as upon exiting in a standard pre-post format. During the course, PSETs are expected to engage in mathematical discourse through reasoning and justification consistently as a socio-mathematical norm in class and group discussions, online homework exercises, and on exams. The instrument contains 13 items, and the current report focuses on data collected from 4 sections of the course in the Fall 2014 and Spring 2015 semesters, with N=78 teachers. In addition to administering the instrument, five teachers were interviewed in the spring semester concerning their answers to 3 of the items to discern their ability to communicate effectively orally in addition to written formats. Participants were also presented with novel tasks for them during the interviews that gauged their abilities to engage in reasoning and justification more generally through validation. For instance, one of the items in the instrument asks for a justification of the invert-and-multiply algorithm. During the interviews, the teachers discussed their own justifications for the algorithm and then were presented with justifications that had not been discussed during the course and were asked to discuss the appropriateness of the justification for an elementary classroom.

Preliminary Data Analysis and Results

Data analysis is ongoing and will continue through the end of the Fall 2015 semester. Pre and post responses to the free response items in the MKT instrument are to be scored by researchers with emphasis on interrater reliability based upon predetermined criteria involving mathematical accuracy of the responses, the effectiveness of the responses in reaching the intended audience of elementary students, and the appropriateness of the responses based upon the grade level of student whom the teacher is communicating with. The five interviews are to be transcribed and coded based upon similar criteria. However, in addition, the interviews will also be analyzed to look for evidence of surface validity in instrument items in constructing various components of MKT.

Early analysis shows that a significant proportion of teachers did build some facility in different aspects of MKT, although some teachers were more successful than others. To highlight some these successes or lack thereof, a few sample corresponding pre-post response pairs are given below:

Pre Responses


3. Nancy, a student in your 5th grade math class, asks you one day why the expression $5 \div 0$ is undefined. Give Nancy a mathematical explanation for this.

the reason why $5 \div 0$ is undefined is that you cannot take nothing out of something.

6. John asks you in math class one day why $4^0 = 1$. Give John a mathematical explanation for this.

Don't know

Corresponding Post Responses:

3. Nancy, a student in your 5th grade math class, asks you one day why the expression $5 \div 0$ is undefined. Give Nancy a mathematical explanation for this.

$5 \div 0$ is undefined since $5 \div 0$ is really asking what times 0 is going to give you five. As we know any number times 0 is 0 so there is no number times 0 that gives us five. Therefore 0 is undefined

6. John asks you in math class one day why $4^0 = 1$. Give John a mathematical explanation for this.

$4^4 = 256 \quad 4^3 = 64 \quad 4^2 = 16 \quad 4^1 = 4 \quad 4^0 = 1$

As the exponent goes down by 1 the answers are divided by 4 each time, so if $4^1 = 4$ then to figure out 4^0 you would do $4 \div 4$ which equals 1.

As is readily seen, there are dramatic shifts in ability to engage in mathematical discourse in these two PSETs. Upon entering the course, the first teacher gave a very common mathematically inaccurate response among PSETs which makes effectiveness and appropriateness moot, while the second teacher was unable to justify the given statement. Upon leaving the course, both teachers effectively engaged in mathematical discourse that is commonly found in elementary curricula. These outcomes are not unique of course, but are shared throughout the data. However, there is another interesting aspect of the project design feature that the current report does not deal with: the attitudes and beliefs of the PSETs. The data shows a clear shift for some teachers in how they think about mathematics: many participants entered the course with responses that included some mention of a type of rule as a justification for a given statement, whereas their post responses seldom if ever talk about rules in mathematics. Beliefs and attitudes about mathematics and teaching it also surfaced in the interviews as some participants talked about why they felt it was important for teachers to know certain things based upon those beliefs. Again, this is not a focus of the current report, but it is indeed an avenue of exploration for future study.

Questions for the Audience

(1) What kinds of things would you like to see as teacher educators/researchers in the responses to the items? Why?
(2) Are there other ways of measuring the ability to engage in justification which in and of itself requires measuring the ability to engage in mathematical discourse? What are they? What are the advantages and disadvantages to each method?
(3) What are some of the most important topics in the elementary curriculum that you as teacher educators/researchers believe that no PSET should exit their content courses without being able to have a somewhat stable mathematical discourse in those topics?

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The case of an undergraduate mathematics cohort of African American males striving for mathematical excellence

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Historically Black Colleges and Universities (HBCUs) provide a different milieu as it pertains to supporting students academically in all disciplines, and this study champions an HBCU effort within the context of undergraduate mathematics. Specifically, it highlights the case of a cohort of 16 African American male mathematics majors at an all-male HBCU. The overarching research question sought to delve deeper into these participants' educational experiences to ascertain factors that influenced their mathematical persistence. Using qualitative research methods grounded in critical race theory, preliminary data show that these African American male mathematics majors were affirmed racially and mathematically in their undergraduate mathematics space.

Keywords: Undergraduate mathematics education, African American males, Equity

Introduction

This preliminary research report analyzes the mathematics experiences of a cohort of 16 African American male mathematics majors at an all-male Historically Black College/University (HBCU) in the southeastern region in the United States. More specifically, this research work seeks to ascertain intrinsic and extrinsic factors that led to their persistence in undergraduate mathematics. This study adds to the body of scholarship on the schooling experiences of African American male students (see, e.g., Berry, 2008; Duncan, 2002; Jett, Stinson, & Williams, 2015; Noguera, 2008). With respect to African American male students' college experiences, some studies focus on how African American men experience and grapple with racism in college settings (see, e.g., Bonner & Bailey, 2006; Cuyjet, 2006; Davis, 1994; Harper, 2015; Seymour & Hewitt, 1997). Research specifically on African American men's college mathematics experiences highlights the fact that many of them often experience difficulties with mathematics (see, e.g., Stage & Kloosterman, 1995).

Despite some of these reported findings regarding African American students in mathematics, there are, however, African American students who achieve in undergraduate mathematics. Thus, it is important to gain insights from studying African American male students who are persisting in college mathematics. African American male students' stories of mathematical persistence are largely absent from the research literature. This research project is designed to fill this void in the research literature and shift the discourse concerning the mathematics experiences of African American male mathematics collegians.

Review of the Literature

There have been fruitful efforts designed to improve the mathematics achievement outcomes of African American students. One effort that has been successful in promoting high levels of undergraduate mathematics performance among African American (and other underrepresented) students is the Mathematics Workshop Program (MWP) at the University of California, Berkeley (Fullilove & Treisman, 1990; Treisman, 1992). The MWP is cited as being successful for the following reasons: the workshops create environments that promote mathematics academic excellence among peers; the students spend more time on learning activities and learning tasks as opposed to just solving mathematics problems; and the students who participate in MWP are believed to continue in college longer than those students who do not participate in the workshop because they obtain social and study skills that can be used throughout their college matriculation.

A research team at the University of Maryland Baltimore County studied high-achieving African American men (Hrabowski, Maton, & Greif, 1998). At this institution, researchers became concerned about the status of African American male students in college science, mathematics, and engineering (SME; SME is synonymous with STEM) majors and decided to learn more about this group by studying the habits of the highest-achieving students who were enrolled in the Meyerhoff Program. Although the program now serves students from all racial and ethnic backgrounds who desire to pursue a doctorate in the sciences or engineering, the first year consisted of African American male students only. Hrabowski et al. (1998) hoped to identify attitudes, behaviors, habits, perspectives, and strategies of the highest-achieving African American male students in the program. According to Hrabowski et al. (1998), the following factors are critical for success in college among African Americans in mathematics and science: an adequate high school academic preparation, analytical skills, strong study skills, time management skills, advising, academic as well as social integration, and motivation and support.

McGee's (2005) work studied 14 high-achieving African American mathematics and engineering majors in their junior and senior years of college. She found that these students exhibited positive racial identities and continued in the African American spiritual tradition. She also found that parents were important factors in the students' success. Furthermore, she found that most students embraced a "succeeding against the odds" ideological paradigm. In sum, Hrabowski et al.'s (1998) findings support other findings of mathematical persistence and success factors among African American students. Early exposure to and access to rigorous and culturally specific mathematics provides the crux for which later mathematics success is attained. Moreover, the MWP (Fullilove & Treisman, 1990) and the work of McGee (2005) buttress the claim that efforts must be made to transform the undergraduate mathematics education discourse.

All in all, this study builds on scholarship from scholars who believe in the gifted mathematical abilities of African American students (see, e.g. Cooper, 2004; Ellington & Frederick, 2010; Jett, 2010; Leonard & Martin, 2013; Noble, 2011; Stinson, 2006; Thompson & Lewis, 2005; Walker, 2014). This work extends my own scholarly efforts concerning the importance of HBCUs as it relates to producing African American male mathematics majors (Jett, 2013). Moreover, this study reveals how complexities about the constructs of race and/or gender may influence the undergraduate mathematics education of African American male students. This study, too, complements and expands existing research efforts in the field.

Theoretical Framework

Critical Race Theory (CRT) was employed as the theoretical framework for this research project. Historically, African Americans in the United States have experienced this hierarchical race system that places Europeans at the top and people of color at the bottom since slavery (Bell, 1992; DuBois, 1903/2003). Racism is an institutionalized force that has been used both historically and currently to dismiss and oppress people of African descent and other people of color. Solórzano and Yosso (2002) argue that "substantive discussions of racism are missing from critical discourse in education" (p. 37). As it stands, issues of race and racism have been underexplored in mathematics education research (Martin, 2009). In an attempt to utilize CRT in undergraduate mathematics education research, CRT was used to analyze the experiences of a cohort of 16 African American male mathematics majors.

There are five foundational tenets of CRT, and these tenets are the hallmarks driving this theoretical perspective. These philosophical underpinnings include the following:

1) CRT asserts that "racism is normal, not aberrant, in American society" (Delgado & Stefancic, 2000, p. xvi).
2) CRT adheres to interest convergence, which advances that the dominant culture advances racial justice and other race based initiatives when it serves their interest (Delgado & Stefancic, 2001).
3) CRT asserts that race is orchestrated as a social construction (Ladson-Billings, 2013).
4) CRT explores the intersectionality of various constructs such as race, sex, class, gender, and sexual orientation to explore how these intersections make for broader understandings of these constructs (Delgado & Stefancic, 2001).
5) CRT utilizes voice to serve as a counter-narrative to the dominant discourse surrounding racial groups (Dixson & Rousseau, 2005).

These tenets of CRT were be used to frame the interview questions and to analyze the data.

Research Question

The overarching research question for this study was as follows:
How do African American male mathematics majors describe their educational experiences?

Methodology

The qualitative research data collection methods included the following: 1) a pre-survey, 2) a semi-structured interview, and 3) a member checking prompt (Bogdan & Biklen, 2007).

1) The pre-survey was given to the participants prior to the first interview. This pre-survey solicited information from the participants pertaining to their demographics, family dynamics, and education. The information obtained from the pre-survey was used to inform the first interview as well as to substantiate the data for coding and analysis. 2) The interview allowed the participants' voices to be heard using their own words. The utilization of "voice" as well as narratives aligns with qualitative research methods and CRT. Additionally, the semi-structured interview caused the participants to reflect upon and (re)construct their mathematics experiences in their own words. Each interview lasted anywhere between one and two hours chronicling their mathematics schooling experiences as African American male students. 3) The member checking aspect allowed the participants to verify whether I reported their words, findings, and interpretations accurately.

Preliminary Findings

Data have been collected and coded for this research project, and the data are in the early stages of analysis (Saldaña, 2013). However, preliminary data indicate that these 16 African American male mathematics majors were affirmed at their HBCU. Preliminary findings from the study also indicate these 16 African American men had a sense of mathematical brotherhood in college, benefited from an affirming Mathematics Laboratory at the college, solidified mathematics passions during their early childhood experiences, and largely expressed that their high school preparation did not fully prepare them for collegiate mathematics, which is in stark contrast to the research literature.

Discussion Questions

The following discussion questions will allow participants in this session to engage in dialogue, offer feedback for strengthening the work, and recommend suggestions for future areas of scholarly exploration:

• Please share any stories of your experiences working with African American male mathematics majors.

• What are your thoughts and feedback concerning the aforementioned study of this cohort of 16 African American male mathematics majors at this HBCU?

• What are the implications of this work for mathematics instructors as it pertains to making the undergraduate mathematics space a more inclusive one at different institution types?

• What are the implications of this work as it pertains to future research regarding African American male undergraduate mathematics students' experiences?

Goals

A goal of this session is to highlight the robust and longstanding history of mathematical excellence at this all-male HBCU. This particular institution has a legacy of producing many African American male mathematics majors. Another goal is to disseminate more stories of mathematical persistence to influence and filter more African American male students into the mathematics pipeline who have a desire to explore mathematical pursuits. Finally, a goal is to generate more conversations concerning the participation and underrepresentation of African American male students in undergraduate mathematics.

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Using the effect sizes of subtasks to compare instructional methods: A network model

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Abstract

Networks have become increasingly important in studying air pollution, energy use, genetics and psychology. These directed graphs also have features that may be useful in modeling student learning by answering questions such as the following: How can we determine if one teaching approach has better outcomes than a second method? In this paper we present a framework for dividing an approach into subtasks, assigning a numerical value (such as an effect size) to each subtask and then combining these values to determine an overall effectiveness rating for the original approach. This process allows researchers to investigate potential causes for student achievement rather than simple correlations, and can compare the effectiveness of a method for various types of students or instructors.

Key words: effect size, teaching methods, subtasks, network

Introduction and context

As university faculty face the challenge of teaching a new generation of students, some have adopted alternative assessments or a variety of teaching methods to enhance the learning experience (Cohen, 1977, Hastings, 2006, Hattie, 2009). In general, this transition away from lecturing has been slow as indicated by a recent survey of 700 calculus instructors in which a majority still believes that students learn best from clear and well-prepared lectures (Bressoud, 2011). In a related survey of over 700 faculty members who teach introductory physics, an impressive 72% had used at least one research-based instructional strategy; however, nearly one-third of this 72% no longer use any of the strategies (Henderson et al., 2012). Why do so many instructors think and react this way? Three major reasons are the following: (1) some may not be aware of the existing research supporting new approaches, (2) many are skeptical about the effectiveness of newer methods (often based on their own observations), and (3) if a faculty member is willing to try a different approach, which choice among the alternatives should she choose to produce the greatest impact?

The focus of this paper is to address reason (3) from above on how to select the best approaches. Issue (1), increasing faculty awareness of current research, is already being addressed by several professional organizations. The National Council of Teachers of Mathematics (NCTM) has recently published a survey of over fifty studies related to seven principles behind motivational strategies (Middleton & Jansen, 2011) and its 73rd yearbook, *Motivation and Disposition: Pathways to Learning Mathematics*. The Mathematical Association of America (MAA) sponsors the SIGMAA on RUME and strands on teaching and learning theory at its national meetings each January and August. Similarly, the American Mathematical Association of Two-Year Colleges (AMATYC) highlights research-based topics in several sessions at its annual meeting. As for issue (2), if faculty members have a way of choosing more effective methods, then their skepticism may be diminished due to better classroom results and replaced by a lasting commitment to incorporating classroom change.

Educator Spencer Kagan (Kagan & Kagan, 1998, p. xxii) claims that “the greatest sustained change results from the smallest changes in instruction;” however, a challenge related to selecting methods is that one general “method” may be accomplished in several ways – each with varying levels of success. For example, suppose two instructors wish to motivate students more. Instructor A chooses to focus on the *relevance* of mathematics since there seems to be general agreement that mathematics, particularly for low-achieving adolescent students, must be made more relevant in order to increase student performance (Haylock, 1999). However, Haylock cautions that not all applications are equally motivating to students. For instance, when the teacher states that certain material will be used in the next chapter or in the next course, students do not value the content as much as if they see the topic addressing a perceived need. However, even though many students see the task of finding the cost to carpet a 12 foot by 18 foot room when carpet costs $6.75 per square yard as a real problem, it is seen as someone else’s problem and not the students’. Thus, it is not as motivating as if the students see an immediate, personal need for the content.

Similarly, Instructor B desires to motivate students; however, she chooses *social interaction* rather than relevance. At many institutions one of the major areas addressed in student evaluations of faculty has been “Student-Instructor Interaction.” Contact between students and faculty, as well as student-student interactions such as reciprocity and cooperation among students are two of the seven research-based principles in undergraduate education advocated by Chickering and Gamson (1987). She considers the following passages from (Middleton & Jansen, 2011):

> Mathematics classrooms’ social dimension can support students’ learning of mathematics, particularly when teachers purposefully structure opportunities for social needs to converge with academic needs. All students’ needs for relatedness – among them avoiding disapproval, achieving social affiliations, demonstrating competence, acquiring social concern, and building shared meaning – can become channeled into opportunities to engage in mathematics. Teachers’ efforts to support students’ mathematics learning – how they choose mathematical tasks, treat students’ errors, evaluate students, reduce competition, raise status, and build positive relationships with students – can help students meet their needs for relatedness as well.[2]
>
> Integrating all of these practices into your instructional repertoire at once is not realistic . . . cycling them gradually into your teaching can help scaffold student learning.[3]

Observe in these two excerpts that many classroom changes are endorsed; however, are all of the revisions listed in the first passage important to the progress of adult learners, and if so, which changes should be prioritized (rather than merely cycled through) to achieve the greatest impact quickly?

A current approach for comparing teaching methods is to compute Cohen’s effect size, *d*. This statistic was popularized in the 1960’s by Cohen (Cohen, 1977) and has been used extensively to evaluate the effectiveness of many educational approaches on student achievement. This number is computed by finding the difference between two means and then dividing this difference by the paired standard deviation (if the data is matched) or by the pooled standard deviation (if the sets of data are independent). Two common scenarios where the effect size arises in educational studies are the following where the difference of means in the numerator is either (i) the average score on a pre-test subtracted from the average score on a post-test, or (ii) the mean score from a test for a control group subtracted from the mean score of a treatment group.

The effect size is easy to compute – even in meta-analyses of several studies with varying populations and sample sizes, and it is considered reliable. For instance, John Hattie has compiled the value of d from large meta-analyses for almost 150 educational interventions including the following (Hattie, 2009):

1) Staying in college residence halls	$d = 0.05$
2) Cooperative learning	$d = 0.41$
3) Teacher-student relationships	$d = 0.72$
4) Providing formative evaluation of programs to teachers	$d = 0.90$

These results can be interpreted as follows: there seems to be little – if any – change in achievement scores for students simply staying in college residence halls, while students who participate in cooperative learning experiences with other students see a greater improvement in achievement scores. However, developing relationships between the teacher and students, or providing feedback to instructors, appear to be associated with even more growth in student achievement.

Hattie's work focuses mainly on the effect sizes between various *approaches* or *tasks* and the final *outcome* of student scores – without considering which intermediate *tasks* or student *responses* may be potentially high-impact revisions.

A network model

The theoretical model described in this paper to address the questions of selection and priority divides a particular instructional method into a sequence of "subtasks." For the rest of this paper we will think of a method as a path with the following six steps:

1) instructor
2) motivational *principle*
3) *approach*
4) *task*
5) student *response*
6) *outcome*

For instance, if an instructor tries to motivate students using the *principle* of social interaction, then one path that might produce higher student scores would include the student-student interaction *approach* (rather than the student-instructor interaction approach, for example), followed by a classroom *task* of having pairs of students solve problems where the two students take turns explaining how to solve a problem to each other. This task results in the student's *response* (or attitude) such as valuing the mathematics or feeling more confident, and ultimately leads to an *outcome* such as student achievement as measured by test scores. This progression is shown by the path in Figure 1.

Instructor → Social interaction → Student-student interaction→ Explain in pairs → Value math → Test score

Figure 1. A "subtask" path linking the instructor and the student

In reality, a single motivational *principle* can be addressed with various approaches, one *approach* can be accomplished with several tasks, some *tasks* may invoke multiple responses (or two tasks may produce the same response), and several *responses* may result in the same outcome. Thus, a network models a more complete picture of the interactions between various subtasks. In Figure 2 a network is shown which includes several (but not all) subtask paths from the instructor to the measure of student achievement – the phrases will be referred to as *nodes* and the arrows called *arcs*. The paths begin with the instructor choosing one motivational *principle* from three (social interaction, technology, and immediate feedback). Moving in the direction of the existing arcs, the teacher next chooses an *approach* in the second column that aligns with the motivational principle, followed by one of several possible *tasks* in the third column that support that approach. Each task correlates with at least one student *response* in the fourth column which ultimately correlates with the desired *outcome* (student achievement). As already mentioned, the network shown is not complete, since there may be other motivational principles, approaches, tasks and responses not shown (as illustrated by the node labeled "In-class Technology" and the arc emanating from it).

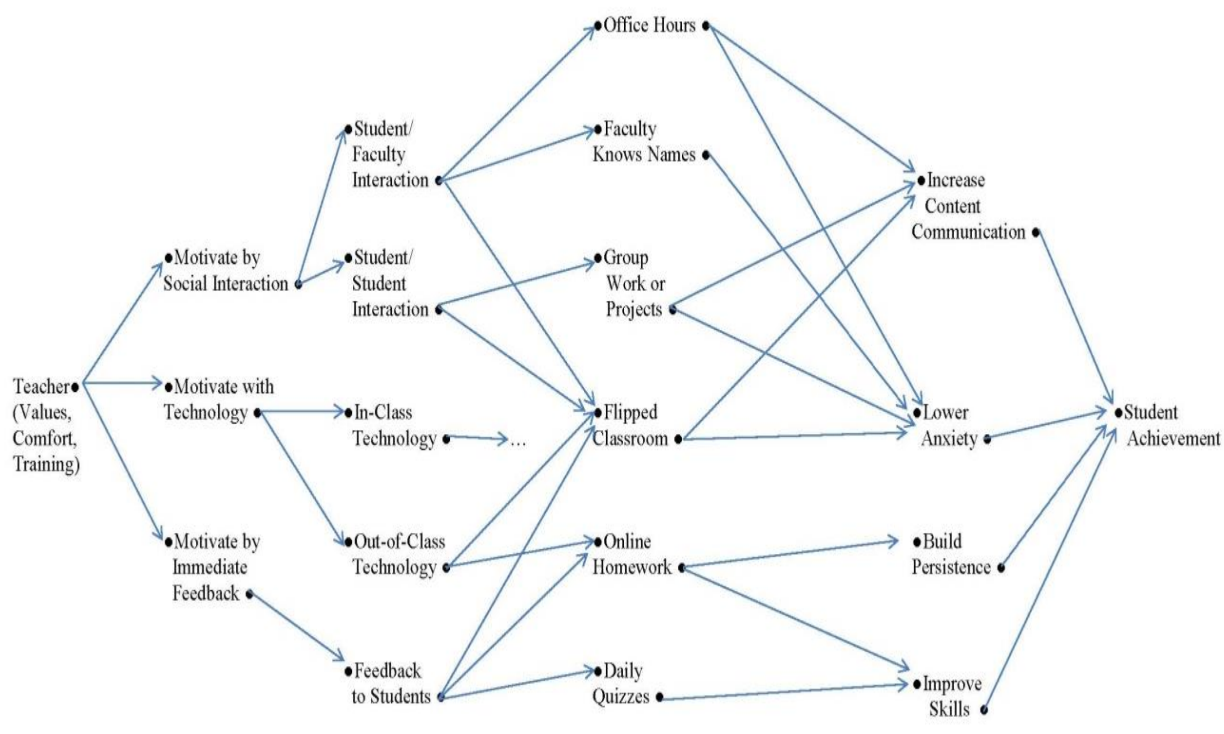

Figure 2. A network model linking the instructor and the student

Networks have become increasingly important in studying fields such as air pollution, energy use, genetics, psychology, economics, ecosystems, voting behavior, and traffic flow (Roberts, 1976). These directed graphs also have features that may be useful in modeling student learning. As is, the network in Figure 2 shows potential links between tasks and responses or between responses and outcomes. However, how can we determine if Task A results in a larger response than Task B, or similarly, when does one response lead to a better outcome than a second response? For instance, in Figure 2, if an instructor wants to provide relatively immediate feedback to her students, she could incorporate daily quizzes, online homework or a flipped classroom format. The quizzes and online homework result in just one student response each (improvement of skills and persistence, respectively) while the flipped classroom leads to both lower anxiety and increased content communication between the students. One might think that addressing two responses is better than focusing on just one in improving student achievement, but (Dweck, 2008) and (Reason, 2009) claim that persistence is seen as a necessary pre-requisite response to the outcome of student success and in (Perkins-Gough, 2013), Angela Lee Duckworth is said to argue that persistence – or "grit" – is a better indicator of success than talent or intelligence. Thus, tasks that develop the single response of persistence may be more productive than tasks that result in multiple responses.

Arc weights

By assigning a numerical value (or *weight*) to each arc, the intent is that possible comparisons could be made between various principles, approaches, tasks, or responses by combining the numbers in some way to determine an overall effectiveness. One value that has been used in similar problems in the area of path analysis is the Pearson correlation coefficient, r (Simpkins et al., 2006). Thus, the weights in our network could be based on the correlation determined by known statistical studies. For instance, if a strong positive association existed between the task of "having students explain problems to each other in pairs" and the response where "students value mathematics," then the correlation coefficient, r, would be close to 1. Unfortunately, the correlation coefficient has some limitations. First it indicates only correlation – not causation. Second, r measures only *linear* correlation. Third, there is no natural way to combine the correlation coefficients for the arcs on a path to determine a cumulative correlation for the entire path.

A second value that could act as the weight of an arc is the *effect size, d,* discussed earlier with Hattie's work. This parameter seems more viable because statistical studies can be done to find d for each of the arcs in the network, some overall values of d are already known (for example, $d = 0.43$ for the motivational principle of immediate feedback (Hattie, 2009)), and formulas can be developed for combining the d-values along a path to determine an overall effect size.

One last note about the arc weights relates to the values assigned to the arcs between the teacher and the motivational principles in the first step in the network. That is, how does one measure why an instructor chooses one principle over another? A possible method would be to use an attitudinal survey to rank the instructor's value of, comfort with, and training in that principle. It is natural to believe that if one instructor values the use of technology more than a second instructor, then the achievement for the students of the first instructor will probably be greater than that of the second instructor if both use technology in their classes; however, a second question worth studying is the following: If an instructor does not value (or is uncomfortable with) a motivational principle, could the student achievement of her students still be higher than if she focused on another principle she valued more?

Conclusion

In an effort to select more effective instructional methods, Hattie and others have used effect sizes to relate educational interventions with student achievement. This process may be able to be refined by dividing a teaching approach into subtasks and creating a network model of the interactions between these subtasks. By studying the effect of each subtask and assigning a corresponding value to each arc of the network, we may be able to determine if certain classroom tasks should be implemented, or if specific student responses should be targeted to attain more growth in student achievement. This theoretical model provides a framework for designing statistical studies that determine the arc weights and the opportunity for using methods from graph theory to combine these weights into cumulative values that may help in evaluating questions in mathematical instruction at the college level.

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Prototype images of the definite integral

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Research on student understanding of definite integrals has revealed an apparent preference among students for graphical representations of the definite integral. Since graphical representations can potentially be both beneficial and problematic, it is important to understand the kinds of graphical images students use in thinking about definite integrals. This report uses the construct of "prototype" to investigate how a large sample of students depicted definite integrals through the graphical representation. A clear "prototype" group of images appeared in the data, as well as related "almost prototype" image groups.

Key words: calculus, definite integral, graphical representation, prototype

Mathematical representations are an essential part of doing mathematics and have been studied extensively by mathematics education researchers (e.g., Cuoco & Curcio, 2001). In particular, much attention in the mathematics education literature has been given to the *graphical* representation (e.g., Romberg, Fennema, & Carpenter, 1993). While some have advocated for increased visualization in the teaching and learning of mathematics (Cunningham, 1991; Eisenberg & Dreyfus, 1991), others have also warned about an overreliance on the graphical representation, which may lead to "uncontrollable imagery" (Aspinwall, Shaw, & Presmeg, 1997). Since graphical images can potentially be either beneficial or problematic, I believe it is crucial to understand how students make use of such representations for mathematical concepts.

This paper is intended to examine the graphical representation in the context of the calculus concept of the definite integral. Recently, several researchers have begun analyzing how students understand and conceptualize the definite integral (e.g., Jones, 2013; Kouropatov & Dreyfus, 2013; Sealey, 2014). From this research has emerged the conclusion that students tend to rely heavily on the graphical "area under a curve" interpretation of the definite integral over other potential interpretations (Jones, 2015). While there is certainly nothing wrong with graphical interpretation of the definite integral, it is important that calculus educators have an understanding of the types of graphical images that prevail in student thinking, and that we examine possibly inadvertent predilections instructors may have in presenting graphical images of definite integrals. In particular, this paper is meant to address the questions: (a) What graphical images do students and instructors tend to use to depict definite integrals? (b) Are there certain features that are common to these graphical images? Answers to these questions may help us begin to understand how certain prevalent images might help or hinder student thinking regarding definite integrals.

Prototype images and social construction

This paper uses the notion of "prototypes" (Rosch, 1973), which is built on the idea that certain categories seem to have a hierarchical nature to their membership. For example, in the category "bird," people often think of robins as better examples of "bird" than chickens, which are themselves better examples than penguins, even though the people understand that all three meet the standard scientific definition for "bird" (Lakoff, 1973). Rosch later clarified that there is not necessarily a cognitive object that *is* the prototype (Rosch, 1978), but that "prototype" represents a sort of judgment of "best fit" for possible members of a category.

Prototypes are a useful lens for this study, since its purpose is to identify commonly-used graphical depictions of the definite integral among many possible depictions. That is, one could think of a range of images that could portray the concept of "definite integral," and I am interested in documenting certain types of graphical images that students seem to use as "best depicters" or "default depicters" for the concept of the definite integral. Inherent in this is the notion that students would not necessarily believe that other images are *not* included in the idea of "definite integral," but that certain images may represent it more naturally.

While prototypes have been studied for individuals, it is clear that there is an across-individual theme to the research as well. That is, prototypes seem to extend to larger groups of people beyond individuals. In certain cases, such as prototypes for focal colors (Kay & McDaniel, 1978), there is a biological justification (the photo-receptors in the human eye), but for others, such as prototypes of birds, there does not seem to be a biological basis. In these cases, I claim that prototypes are social constructions (Ernest, 1994) in that within a community of people a certain sense of an "ideal" may emerge for a given category of objects, which becomes the dominant *shared* ideal for that particular category. Again, this is not to say that the ideal actually *exists*, but rather that judgments on prototypicality become uniform and homogenized among the community.

The socially constructed aspect of prototype is central to this paper. As such, this investigation is not focused on individual students per se, but rather on socially shared prototypical graphical representations of the definite integral. What individual students think about definite integrals is, of course, important to this study, since individuals of students, instructors, and others make up the school mathematics community. Yet my analysis is centered more on similarities that range *across* students regarding graphical representations of definite integrals that are perpetuated through the community.

Origins of the data

This paper is an outgrowth of a series of studies regarding definite integrals (Jones, 2013, 2015, *under review;* Jones & Dorko, 2015). It is important to note that none of the studies was originally intended to produce this particular report as an outcome, and each was rather centered on trying to explore how students understand and make sense of definite integrals, or how instructors teach integration. Through the process of conducting these other studies, however, a clear and unmistakable trend began to take shape in the data. In this way, this paper admittedly represents an *a posteriori* investigation into how students from this series of studies graphically represented the definite integral.

The set of data initially used for this paper consists of interview sessions with 23 students and surveys administered to 205 students at two higher education institutions with a wide range of backgrounds and classrooms experiences. However, 67 of the surveyed students did not provide a graphical image in their responses (despite many stating "area under the curve" in words), and these 67 students were consequently removed from the data set since the study was only focused on the types of graphical images produced by the students. This left 23 interviewed and 138 surveyed students. The data set also included videotaped classroom observations from seven different instructors at these same two institutions. Since the interviews and surveys were not all done with the same purpose in mind, and therefore do not consist of exactly the same set of prompts and questions, I focused only on the parts of the overall data set that were generated from open-ended prompts in which students were asked to explain what definite integrals meant, how they understood definite integrals, or how they would describe definite integrals to others. Placing such constraints on the data is an attempt to capture in this paper how students naturally depicted definite integrals through the graphical representation. The following list provides examples of the types of prompts from the interviews and surveys that were used in the analysis for the present paper.

- Consider the expression $\int_a^b f(x)\,dx$. What does it mean? What does it represent?
- Let's say you had a friend in your calculus class who had been sick for the last week or so and missed *everything* your class learned about integrals. How would you explain integrals to them? What would you say an integral means?
- Explain in detail what $\int_a^b f(x)\ dx$ means. If you think of more than one way to describe it, please describe it in multiple ways. Please use words, or draw pictures, or write formulas, or anything else you want to explain what it means.

Analysis of the student and instructor data

The first step in analyzing the data for this study was to simply recognize many similarities between graphical representations of the definite integral used among the 161 students, as I analyzed the data for other purposes. Once these similarities were recognized (details are provided in the results section), the images were organized according to similarity. That is, images that were very similar to each other were grouped together. This organization resulted in a "web" of image clusters, since, for example, one group might differ from another in one characteristic, but then also differ from a third group through a separate characteristic. Once this organizational web had been created, the frequency of the different groups was tabulated, which led to the uncovering of one particular image group as clearly the most common. This image group was also positioned in something approximating the "center" of the web. Accordingly, I labelled this group the "prototype" group and identified what I considered to be seven key characteristics shared by the images in the group. These characteristics were then compared to the images in surrounding "similar" groups. Since there were so many other groups of images that were so close in nature to the prototype group, I decided to create a secondary label, "almost prototype." I defined an "almost prototype" as an image group that contained all but one of the seven characteristics.

Once the interview and survey data had been analyzed, I turned my attention to the videotaped classroom observations I had from seven instructors at two higher education institutions. All of these instructors had had their first two hour-long introductory lesson on integration observed, with some having had additional observations as well. With the "prototype" characteristics and "almost prototype" characteristics defined through the student data, I watched the lesson videos to identify any images that the instructors created in the classroom that matched either definition.

Results: Student data

In this section, I first display images from the student data to show examples that were included in the "prototype" group (see Figure 1). I then use these example images to highlight the seven characteristics I identified regarding the prototype image. Note that, as discussed previously, I wish to avoid the false conclusion that there exists one, single "prototype image," in the same way that Rosch (1978) clarified that "prototype" does not mean that an actual cognitive object exists that *is* the prototype. This can be seen in Figure 1, since the images are certainly not identical to each other. However, the shared features of the graphical images produced by a significant portion of the students indicates that there is clearly some sense of “best fit” features for graphically depicting the definite integral, which represents “the prototype.”

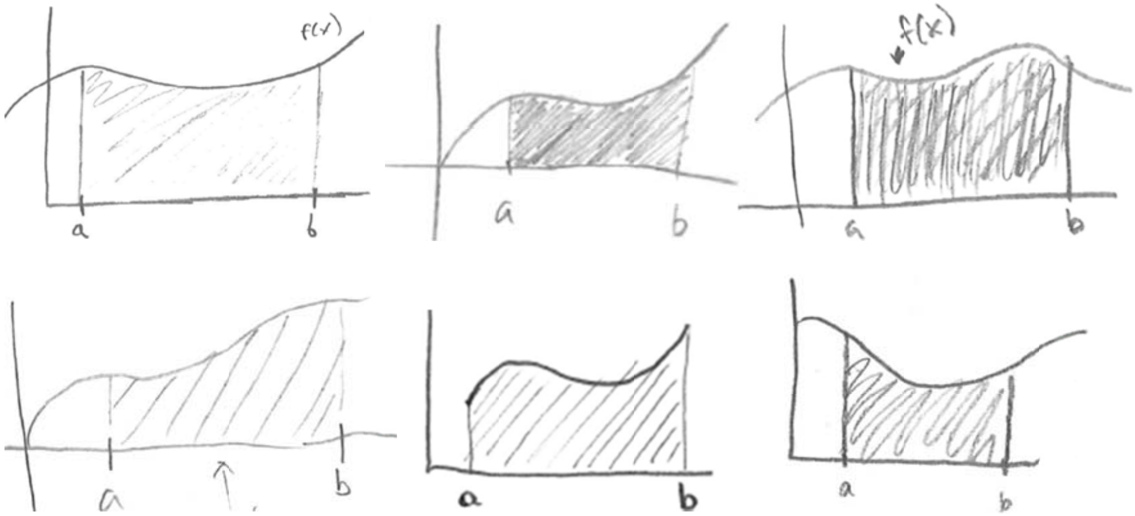

Figure 1. Typical examples of images in the “prototype” group

While artistically distinct, the images shown in Figure 1 have strikingly similar features, especially given the range of students involved in the various studies. In particular, I identified seven major characteristics the images in this group shared, as follows:

1. Neither x nor $f(x)$ are negative over the interval $[a,b]$
2. The graph does not touch the x-axis nor y-axis over the interval $[a,b]$
3. Vertical lines (either solid or dashed) extend vertically from a and b to the graph to form the left and right sides of a region
4. The graph is “wavy” in that it contains at least one change in concavity over $[a.b]$
5. There are no apparent “steep” slopes for the graph over $[a,b]$
6. The graph seems smooth in that it visually appears differential at all points in $[a,b]$
7. The graph over $[a,b]$ does not deviate significantly from the average function value over $[a,b]$

A couple of these characteristics merit clarification. First, I initially had lumped the first and second features together as “x and $f(x)$ are non-negative.” However, in light of certain related image groups (compare Figure 2b and Figure 3, for example), it seemed that having a graph simply *touch* the x- or y-axis had a notably different qualitative feel than having an interval that extended to the left of the y-axis or a graph that went below the x-axis. As a result, it seemed necessary to split these into separate characteristics. Second, the feature of having no “steep” slopes is subject to visual interpretation, since it is not possible to accurately calculate the slope of a hand-drawn graph at every point. To make this judgment, I approximated what the greatest slope might be if an evenly-scaled coordinate system was placed directly on top of the drawing. For all the images in the prototype image group, there appeared to be no slopes (positive or negative) having magnitude greater than two. That said, the vast majority of these images even appeared to have slopes of magnitude no greater than one. Third, it is also difficult to precisely identify what counts as “deviating significantly” from the average function value. In this paper, however, I only attempt to highlight this particular characteristic and do not attempt to define a precise quantitative measure to provide a cut-off for "significant deviation." For an example of an image that violates this feature, see Figure 2d. Finally, I note that for convenience I have defined these characteristics in terms of the symbols x, $f(x)$, a, and b to match the expression $\int_a^b f(x)dx$, but that these symbols could easily be switched for different symbol, such as t, $g(t)$, t_1, and t_2, for example.

Having discussed the features present in the images in the prototype group, I return to the idea of "almost prototype," which was defined in the previous section as an image having all but one of the seven characteristics listed for the prototype. Figure 2 shows examples of images from four different "almost prototype" image groups. I note that by far the two most common almost prototype groups were those in which the graph contained no inflection point (Figure 2a) and those in which the graph touched one of the axes, but still contained no negative values for x or $f(x)$ over $[a,b]$ (Figure 2b).

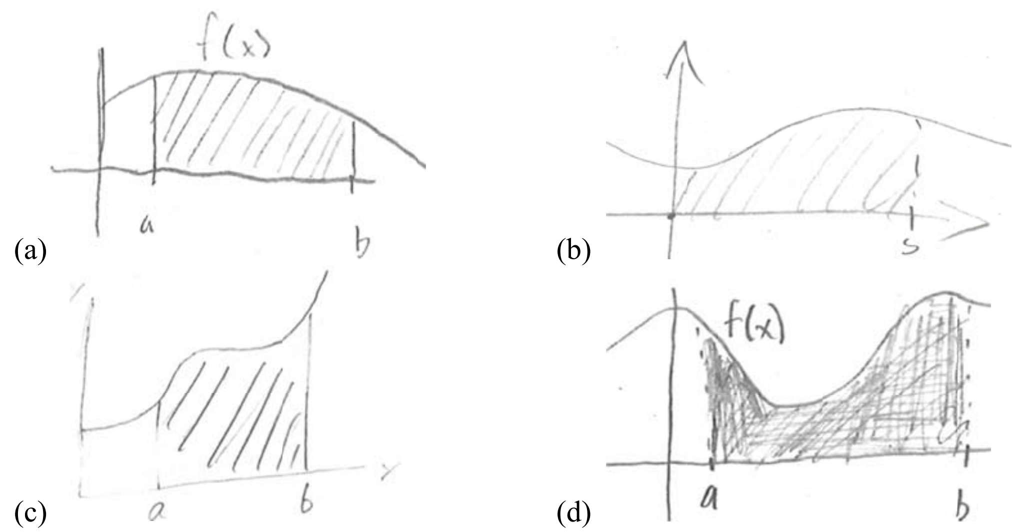

Figure 2. Example "almost prototype" images, including (a) no inflection point, (b) touches axis, (c) steep slopes, and (d) significantly deviates from average

As defined, any image that violated two or more of seven characteristics were excluded from being an "almost prototype." For example, a continuous graph over an interval $a < 0 < b$ (in violation of #1 and #2) was excluded from being an "almost prototype." However, many of these types of images were still quite *related* to the prototype group, but were not counted as being in either the "prototype" or "almost" groups. Figure 3 shows examples of these kinds of images.

Figure 3. Example images NOT counted as "prototype" nor "almost prototype" for violating at least two characteristics, but that are still related to the prototype group

Having discussed the prototype group and the almost prototype image group, I now provide overall summary percentages from the student interview and survey data. Note that in

Table 1, each student is counted *only once*, regardless of how many images they produced in the interview or survey. A student is counted in "produced prototype" if they created an image satisfying all seven characteristics, regardless of whether they also drew "almost" images or other images. Conversely, a student is *only* counted in "almost prototype" if they did not create an image that fit in the prototype group. The "neither" group consists of students who produced at least one image, but none of which fit into the "prototype" or "almost prototype" groups.

Table 1
Frequencies from the student interview and survey data

	Interviewed students (n=23)	Surveyed students (n=138)
Produced prototype image	18 (78.3%)	69 (50.0%)
No prototype, but produced "almost prototype" image	4 (17.4%)	42 (30.4%)
Produced images, but neither "prototype" nor "almost"	1 (4.3%)	27 (19.6%)

Results: Instructor data

We can see in the previous section that a large percentage of students produced prototype or almost prototype images. All but one of the interviewed students produced either a prototype or an almost prototype image and about 80% of the surveyed students produced one of these kinds of images. As such, the characteristics that define the prototypicality of definite integral images seem to be shared in the community of calculus students. The next obvious question is: from where do students adopt this shared sense of prototypicality?

To provide a partial answer, I relate the results from the seven observed calculus instructors at two institutions of higher education. Of the seven instructors, five of them produced exact matches for the prototype group that emerged from the student data. Examples of these images are shown in Figure 4.

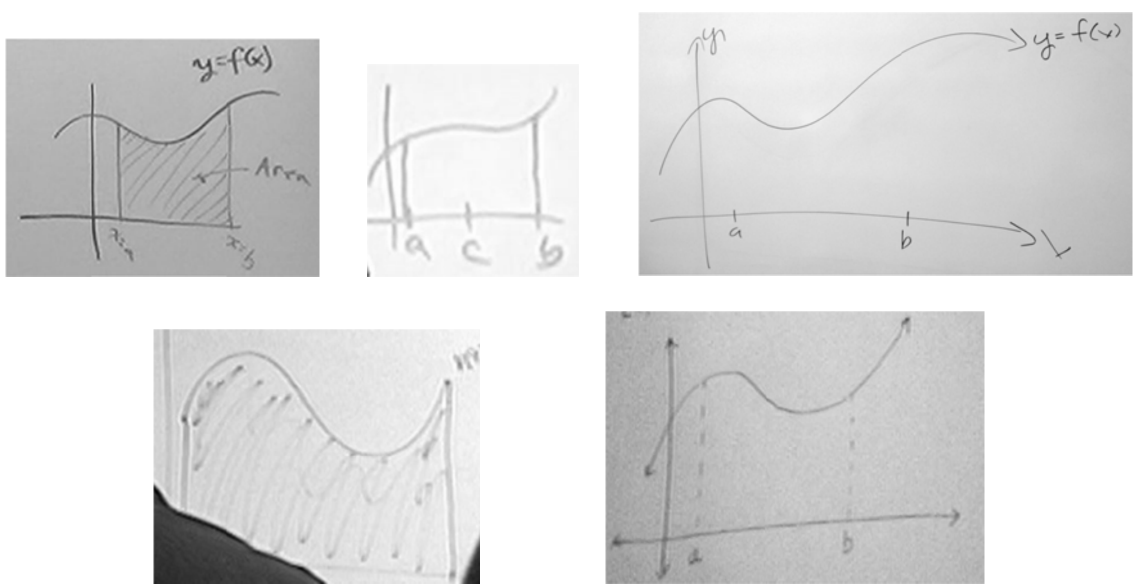

Figure 4. Examples of images produced by five of the seven instructors

These images are directly related to the student-produced images seen in Figure 1. Note that one of the images does not contain vertical lines at *a* and *b*, but the instructor did use his hands to indicate vertical lines at these two points. These results suggest that students may be inducted into the usage of these kinds of "best representational fit" images from their calculus instructors. While the underlying idea of teachers inducting their students into a shared practice is obviously neither new nor revelatory, this portion of the data does reveal, though, that it is more than students who share a sense of "prototypicality" of graphical representations of definite integrals. It seems to be shared by instructors as well. As such, it appears deeply embedded in the calculus education culture.

Discussion

As discussed in the beginning of this paper, graphical representations play an important role in mathematics education, and it is consequently important to understand how graphical images are used by students. In this study I have presented a set of seven characteristics that define a measure of "prototypicality" for graphical representations of the definite integral that seems pervasive in calculus education. Given that this paper, together with past studies (Jones, 2015), suggest this type of image may be a "default" image for students (and instructors), and since graphical images can potentially override other forms of representation (Aspinwall et al., 1997), it is important to understand the ways in which this particular type of image may benefit or hinder student thinking in relation to definite integrals. The results of this study provide some initial insight into possible benefits and hindrances.

On the positive side, this type of graphical image is simple, free of visual clutter, and contains a function that both increases and decreases and whose slope continuously changes. These characteristics may provide individuals with a quick image in which to check the plausibility of certain integral properties or to imagine the quantities involved in a real-world-based integral. Yet, on the negative side, it seems problematic that neither the "inputs" nor "outputs" (i.e. x and $f(x)$) attain negative values, which may have important ramifications for both integral properties and real-world quantities. Also, the fact that the graph has no dramatic rises or drops and is always continuous and smooth may oversimplify the nature of definite integrals if this kind of image is too dominant in a student's thinking.

In stating these possible benefits and hindrances, I wish to be clear that I am not taking the position that this default image is bad. However, I am advocating that we, as calculus educators, should take a careful look at the types of graphical images we use in connection with the definite integral in order to develop a more robust catalogue of images that could serve more flexibly in a wider range of situations. Having a single graphical image that is so prominently culturally embedded may be problematic for thinking about definite integrals. By contrast, if this image were included as just one in a set of easily-accessible graphical images, students may possibly develop a more robust understanding of definite integrals. Since no alternative images came up with nearly as much frequency in this study, it may be that many students might not have such a catalogue of useful images, and may be overlying on this one particular type of image. If this is the case, we, as calculus instructors, might wish to emphasize a greater variety of graphical images in connection with integrals.

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Ways of understanding and ways of thinking in using the derivative concept in applied (non-kinematic) contexts

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Much research on students' understanding of derivatives in applied contexts has been done in kinematics-based contexts (i.e. position, velocity, acceleration). However, given the wide range of applied derivatives in other fields of study that are not based on kinematics, this study focuses on how students interpret and reason about applied derivatives in non-kinematics contexts. Three main ways of understanding or ways of thinking are described in this paper, including (1) invoking time, (2) overgeneralization of implicit differentiation, and (3) confusion between derivative expression and original formula.

Key words: calculus, derivative, applications, ways of understanding and thinking

The calculus concept of the derivative is important both within mathematics and in other disciplines like physics, engineering, economics, biology, and statistics. As such, researchers have been interested in how students use the derivative in a range of contexts (e.g., Bucy, Thompson, & Mountcastle, 2007; Christensen & Thompson, 2012; Zandieh, 2000). However, much of the mathematics education research dealing with applications of the derivative has been centered on the kinematics applications of position, velocity, and acceleration (e.g., Berry & Nyman, 2003; Marrongelle, 2004; Petersen, Enoch, & Noll, 2014; Schwalbach & Dosemagen, 2000). While velocity and acceleration are certainly common and useful applications of the derivative, there are myriad other uses of this concept in fields of study outside of mathematics. Given this deficit in exploring student understanding of the derivative in a wider variety of applications, I focus this paper on how students interpreted and reasoned about the derivative concept in applied, *non-kinematics* contexts. Specifically, I relate certain *ways of understanding* and *ways of thinking* exhibited by students that seemed particular to working with applied, non-kinematics derivatives.

Ways of Understanding, Ways of Thinking

For this paper, I draw on the constructs of *ways of understanding* and *ways of thinking* (Harel, 2008; Harel & Sowder, 2005) to explore certain aspects of how students might think and reason about derivatives in applied, non-kinematics contexts. First, Harel and Sowder (2005) use the term "mental act" to denote any internal mental action, such as interpreting, inferring, explaining, or searching. One single cognitive product of a mental act (or acts) in one given situation is termed a *way of understanding*. For example, if a student sees "*dy/dx*" and thinks "that's the slope," then the student has produced a single way of understanding *dy/dx* through the mental act *interpret derivative symbol*. If, in observing a student, a particular characteristic is found to be repeatedly associated with a given mental act, then Harel and Sowder term that a *way of thinking*. In the example, if the same student indicates in many situations or problem contexts that a derivative is a "slope," then that student is considered to have a *way of thinking* associated with interpreting the derivative.

The constructs of ways of understanding and ways of thinking are used in this paper to explore idiosyncrasies and difficulties evidenced by students in thinking about derivatives in applied, non-kinematics contexts. Some of the idiosyncrasies, which are discussed as possible ways of thinking, seemed specific to the applied context of the derivatives. Some of the difficulties appeared to be particular ways of understanding produced by the students as they performed mental acts related to reasoning about derivatives in applied contexts.

Interview and Survey Data

The data used for this paper consists of hour-long, task-based interviews with six first-semester calculus students, and surveys conducted with 38 first-semester calculus students. The six interviewed students were recruited at the end of the same first-semester calculus class, which was taught in a fairly "traditional" manner by a mathematics department faculty member at a large university in the United States. Here I use "traditional" simply to indicate that nothing seemed unusual in the presentation of the material in this course. In this paper, the interviewed students are given the pseudonyms: Jack, Lily, Noah, Zoe, Oliver, and Toby.

The 38 surveyed students came from two different calculus classes at the same university (25 students in one class, 13 in the other), with these two classes being different from the one in which the interviewed students were recruited. Thus, students were recruited from a total of three different classes with three different instructors at the same university. The classes were all taught in what could be described as a fairly typical manner.

The six interviewed students were given a range of contexts in which they were asked to discuss the derivative concept. The interview consisted of five prompts (see below), though for the purposes of this paper I focus only on the three "applied" prompts, which asked the students to calculate and discuss derivatives in applied, non-kinematics contexts. Note that all "fractional" expressions, like "df/dx," are formatted this way for the purposes of the paper, but were given to the students as "$\frac{df}{dx}$."

1. Let $f(x) = x^4$. Calculate df/dx and explain what it means.
2. Given the formula $z = rt + st^2 + rs/t$, calculate dz/dt and explain how you did it.
3. Suppose we have a cylinder with radius r and height h [an image of an unlabeled cylinder is provided]. The volume formula for a cylinder is $V = \pi r^2 h$. (a) Calculate dV/dr. What does this answer tell you? (b) Calculate dV/dh. What does this answer tell you?
4. The force of gravity (F) is dependent on how far an object is from the Earth's center (r), given by the formula $F = GmM/r^2$. (M and m are the mass of the earth and the object and G is the "gravitational constant.") (a) Calculate dF/dr. What does that tell you? (b) Calculate dF/dm. What does this answer tell you?
5. What would these following derivatives tell you? Should they each be positive or negative? (a) dS/dp, if p = price of a book, and S = number of books sold; (b) dV/dr, if V = volume, and r = radius of a sphere; (c) dM/dt, if M = memory, and t = time.

Many follow-up questions were used during the interviews based on the students' responses, such as "Why does your answer tell you that?," "What does it mean that the answer has a negative sign?," or "For every increase in __, will I get the same change in __?"

The interviews were fully completed before the administration of the surveys. This was done intentionally to allow a preliminary analysis of the interview data to occur prior to the survey creation. In this way, I identified individual students' potential ways of understanding and ways of thinking from the interview data and then tailored the survey questions to see if some of those same ways of understanding/thinking might be replicated by a larger sample of students. In order to create a brief survey protocol, the surveys only contained two applied, non-kinematics questions, which corresponded to the third and fourth interview prompts:

1. The volume of a cylinder with radius r and height h is given by $V = \pi r^2 h$. (a) Calculate dV/dr and then state the meaning of dV/dr. (b) Suppose that r increases. Describe as much as you can what your answer to part (a) tells you about the cylinder's volume. *How* does your answer tell you that?
2. The force of gravity between an object and the Earth is given by $F = GmM/r^2$ (r is the object's distance from Earth's center, M and m are the masses of the earth and the object, and G is a constant). (a) Calculate dF/dr and then state the meaning of dF/dr. (b) Suppose that r increases. Describe as much as you can what your answer to part (a) tells you about the force of gravity. *How* does your answer tell you that?

Data Analysis

Since much work has already been done in examining how students understand the fundamental ideas contained in the derivative concept (e.g., Habre & Abboud, 2006; Orton, 1983; Zandieh, 2000), this paper is not meant to repeat the results of these prior studies. Consequently, I did not focus the analysis for this study on students' overall understandings and meanings assigned to the generic derivative concept. Rather, I focused on exploring aspects of students' thinking and reasoning that seemed pedagogically important regarding working with applied, non-kinematics derivatives.

The preliminary analysis of the interview data, mentioned in the previous section, consisted of using open coding (Strauss & Corbin, 1998) to identify plausible ways of understanding/thinking specific to applied, non-kinematics derivatives exhibited by the individual students. This led to the creation of three main categories, which are described in the results section: (1) invoking time, (2) overgeneralization of implicit differentiation, and (3) confusion between the derivative expression and the original formula. Once this preliminary analysis had been conducted, the survey protocol was created and administered to identify whether these categories would be observed in a larger sample.

Following the survey administration, a more systematic coding of the data occurred by going through the interviews and surveys to code for all instances of the three categories. Throughout the process, I remained open to the possibility of new categories emerging. While no "top-level" categories were introduced at this stage, a distinct subset of the third category took shape that centered on confusion around applied derivative expressions that were constant. The data was re-coded a final time looking for instances of this subcategory. Unfortunately, this subcategory was explicitly identified *after* the survey administration, meaning no question had been included on the survey to target it in the larger survey sample.

Results

In this section, the three categories listed in the previous section are discussed through the lens of ways of understanding and ways of thinking. That is, there appeared to be certain idiosyncratic tendencies from many of the students, which provided evidence of ways of thinking related to applied derivatives. In addition, a common difficulty became evident in terms of how students interpreted applied derivatives. While perhaps not a way of *thinking*, it appears to be a common way of *understanding*.

Invoking time

To preface this subsection, I wish to draw attention to the fact that none of the "applied" interview prompts (with the exception of 5c) and none of the survey prompts explicitly required *time* as a factor in the derivative. For example, the derivative dV/dr does not require

r nor V to change quickly or slowly in time, nor even at a steady rate with respect to time. It is therefore interesting that four of the interviewed students and 14 of the surveyed students interjected time explicitly into the contexts as they calculated and explained the applied derivatives, as demonstrated by these examples:

Lily: [Explaining *dV/dr*] Like say [*r*] is changing at a rate of one meter per second, that's really fast, but if it's getting bigger constantly, this is going to, the volume itself… if it's one meter per second… it changes smaller at first, but then bigger.

Noah: [Explaining *dV/dh*] If we're increasing the height by one every time, assuming that it happens in one-second or, like, the next time interval, the next time, then that would just be the same relationship. So, it would increase at the same, at a constant rate.

Survey: [Explaining *dF/dr*] It's talking about the force in relation to distance. It's related to time and mass.

Survey: [Explaining *dV/dr*] Volume is changing in relation to *r* in time.

Thus, for many students the mental act "describe" or "explain" applied derivatives produced a way of understanding that explicitly attended to time. I hasten to add that involving time is *not incorrect*, since changes in real-world quantities can essentially only be envisioned over time. Furthermore, for many of the students, it seemed that interjecting time was a useful way to explain the meaning of these derivatives. For example, in Lily's excerpt, she used the context of a radius increasing at a steady rate in time to help explain that the volume would always grow, but by a smaller rate at first and then by a larger rate later.

While these could be characterized as ways of understanding, since they are stand-alone explanations, some interviewed students had a strong tendency to insert time into most of the problem contexts, as exemplified by Zoe:

Zoe: [Explaining *dV/dr*] If it was normal, let's say it's normal, it would be *dV/dt*, which would mean we would have meters cubed divided by time, in seconds. [Attempts to use analogous reasoning to interpret *dV/dr*, but unsuccessfully.]

Zoe: [Explaining *dS/dp*] As the price gets cheaper, the number of books sold would decrease. That doesn't, well, it depends [trails off]. But I suppose over time, if it's a cheaper price for a longer amount of time, it would increase [*S*].

Interviewer: [Regarding prompt 5b] Why would the values of [*V*] be getting bigger?
Zoe: I don't know [pause]. Alright, [*dV/dr*] means change in volume over change in radius, so [long pause]. The rate at which—there's no time involved!… So, as we're changing the radius, imagine the radius is time, because as you're affecting the radius, you can't do it without time, because you can't do things outside of time… If we negate the middle-man and negate the change in radius, then we'd just have the change in volume as the time changes.

Zoe's repeated inclusion of time shows that these were more than ways of understanding, but together demonstrate a strong way of thinking. Whereas some students, like Lily, could use time effectively to imagine a non-time-based derivative as needed, Zoe's way of thinking seemed to hinder her reasoning at times, becoming more of a crutch than an aid. She often desired to alter the nature of the derivative from one that is time-less to one that is based on time. In the last excerpt, she even cut the radius from the context altogether in order to bring the derivative in line with her strong time-dependent thinking.

Overgeneralization of implicit differentiation

The second category I discuss in this paper is less conceptual in nature and represents more of an overgeneralization of a specific class of derivative problems. In typical first-semester calculus courses students study applications of the derivative including optimization problems and related rates. Related rates deal with implicitly defining variables in terms of another "latent" variable, requiring implicit differentiation to solve the problems. For example, in $V = {}^{4}/_{3}\pi r^{3}$, the volume and radius could be thought of as functions of temperature (say, if the sphere is metallic), leading to $V(T) = {}^{4}/_{3}\pi[r(T)]^{3}$. Then derivatives such as dV/dT or dr/dT could be calculated through implicit differentiation. In this study, four of the interviewed students and 27 of the surveyed students assumed some of the variables in the formulas to be implicitly defined in terms of either the variable of differentiation or some other variable. For example, many students seemed to think that a derivative such as dV/dr required all or some variables to become implicitly defined in r—sometimes even the variable r itself! Time was also often invoked as a latent variable, making this category connected, in part, to the previous category. The following are examples from the students' work (note that not all calculations would represent correctly calculated derivatives):

- $dV/dr = 2\pi r \, {}^{dh}/_{dr}$
- $dV/dr = 2\pi rh \, {}^{dr}/_{dr}$
- $dV/dr = 2\pi rh \, {}^{dr}/_{dt}$
- $dV/dr = 2\pi rh + \pi r^{2} \, {}^{dh}/_{dt}$
- $dV/dt = \pi r^{2} \, {}^{dh}/_{dt} + 2\pi rh \, {}^{dr}/_{dt}$
- $dF/dr = G \, {}^{dm}/_{dr} M + Gm \, {}^{dM}/_{dr}$
- $dF/dr = [r^{2} G \, {}^{dm}/_{dr} \, {}^{dM}/_{dr} - GmM(2r)] / r^{4}$
- $dF/dr = [r^{2}(Gm \, {}^{dM}/_{dt} + MG \, {}^{dm}/_{dt}) - (GmM)2r \, {}^{dr}/_{dt}] / r^{4}$

I once again note that implicitly defining some variables in terms of others is not necessarily incorrect, though many of the ways in which students did so in this study could be considered incorrect. For example, in $V = \pi r^{2}h$, unless an extra condition is placed on the relationship between the radius and the height, height is not a function of the radius at all.

Most of the students who forced some variables to be implicitly defined in terms of others did so for more than one problem. For example, most surveyed students who did this did so on both problems. As such, I consider this category to represent a way of thinking for many students. While more procedural in nature, it is important for educators to be aware of this tendency, given that over two-thirds of the students in this study forced implicit differentiation onto the non-implicitly-based applied derivatives.

Confusion between the derivative expression and the original formula

Perhaps the most important category to discuss in this paper is the confusion many of the students exhibited at times between the derivative formula and the original formula. Five of the interviewed students and 16 of the surveyed students gave evidence of this type of confusion, which is nearly half of all the students in this study. When interpreting and making sense of the derivative formula they calculated, many students began to explain the derivative formula as though it provided a value for the *original* quantity of interest. The following excerpts are examples of this type of confusion.

Oliver: [After correctly calculating $dF/dm = GM/r^2$] It's the change in force as we change the mass of the object. And what this is telling us is, because there is no mass [little m] in the equation that the force isn't subject to the mass of the object.
Interviewer: OK, so would that mean that I could make the object more massive or less massive and that has no effect on force?
Oliver: Right.

Lily: [Explaining whether dS/dp would be positive or negative] I guess that would be positive, because there's no such thing as selling a negative number of books.
Lily: [Discussing what it would mean if $dS/dp = 0$] Zero would mean that no books are being sold at that specific price.

Zoe: [Explaining $dV/dh = \pi r^2$] This would imply that there is no change!
…
Interviewer: Whether you feel like it makes intuitive sense or not, what do you feel like that [points to the derivative formula] should be telling you?
Zoe: That no matter how h changes, V remains the same. That's doesn't make sense!

Survey: [Explaining $dV/dr = 2\pi rh$] The rate at which the volume of the cylinder is increasing is 2x the rate at which the radius is increasing (π and h are constants). Answer tells me that, because the derivative of the function is $2r$ (π and h being constants). Thus I know that the rate at which the volume is increasing is double the rate the radius is increasing.

In these examples, it is clear that the students had essentially read the derivative expression as directly providing the value of the original quantity whose derivative was being calculated. In other words, the students explanations would make sense if the following substitutions were made:

- $dF/dm = GM/r^2 \rightarrow F = GM/r^2$
- $dS/dp \rightarrow S$
- $dV/dh = \pi r^2 \rightarrow V = \pi r^2$
- $dV/dr = 2\pi rh \rightarrow V = 2\pi rh$

Essentially, these students seemed to have a particular way of understanding equations, in these moments, in which an equation takes on the meaning [quantity] = [expression]. That is, regardless of what is on the left of the equation, whether F or dF/dr, it seems to mean "quantity" instead of other possibilities, like "rate of change." I wish to point out that many of the students were *not* consistent in doing this, but that they only occasionally made this error. As such, I am careful not to call it a way of thinking, which would assume a greater regularity than was visible in the data. Rather, for most students, I see it as a way of understanding, since it was the result of a particular mental act at one point in time. Even so, this conceptual mistake happened so often with both the interviewed and the surveyed students that this way of understanding seems to be an issue educators should be aware of.

Confusion with applied derivatives that are constant

In discussing the confusion between the derivative expression and the original expression, I note that the most frequent context for this confusion during the interviews was when the applied derivative yielded an expression that was *constant*. (Note that since I did not provide a constant derivative on the survey, I cannot comment about this issue for the surveyed students.) In expressions like $dV/dh = \pi r^2$ and $dF/dm = GM/r^2$, the variable of differentiation is not present on the right side of the equation. These cases would indicate a constant *rate of change*, which many of the students overlooked, believing it to mean a constant *quantity* instead. This led to much frustration in the students as they struggled to identify *why* the variable, such as h or m, would not have an impact on the quantity V or F. Some students, like Oliver and Zoe, were never able to reconcile the discrepancy between what the derivative expression seemed to be saying and what they intuitively believed to be true. It is important to note that these same students, during the pure-mathematics prompts, showed no difficulty whatsoever in making sense of a constant derivative in pure mathematics contexts (the interviewer asked about this as a follow up to prompt 1). As such, it seems that there was something fundamental about the applied nature of the derivatives used in this study that prevented the students from accessing resources they certainly had about the meaning of constant derivatives in pure mathematics contexts.

Discussion and Implications

In this paper I have highlighted three pedagogically important ways of understanding or ways of thinking exhibited by many of the interviewed and surveyed students in the study. *Invoking time* seemed to be a useful way of understanding for some students, though problematic for others when it became an almost uncontrollable way of thinking. This suggests that it would be important for calculus instructors to have explicit discussions regarding *time* and how it comes into play with applied, non-kinematics derivatives. Since typical applications of the derivative, including velocity and acceleration, *are* time-based, it may be important to explore other, non-time-based derivatives during instruction as well.

Overgeneralization of implicit differentiation was the most commonly observed of the three categories in this study. This suggests that many students, when learning certain types of applications, such as related rates, may overgeneralize the implicit differentiation procedure into a belief that non-kinematics-based applied derivatives require variables to be defined implicitly with respect to either the variable of differentiation or time. While in some cases this may be fine, in many cases it may be incorrect, or at the least very burdensome. Thus, calculus instructors may need to have meta-discussions on the types of applications studied in class, so that students do not mistakenly believe that those procedures must be used for *all* applied problems.

Perhaps the most important conceptual difficulty students had was in *confusing the derivative expression with the original expression*. This seems similar to what Musgrave and Thompson (2014) call "function notation as idiom," wherein the symbol on the left of the equation just represents a "name" for the equation, leading to potentially problematic expressions like "$f(x) = n(n\text{-}1)/2$" (p. 283). In other words, students might not pay careful attention to *what* exactly is on the left side of the equation, but may rather simply view it as a label, usually for a quantity's value (as opposed to other possibilities, like a rate of change). The right side of the equation is "where the math happens" (Musgrave & Thompson, 2014, p. 286), and the expression's value tends to represent the *magnitude* of the quantity of interest.

Overall, this study shows that there are additional conceptual and procedural layers to working with applied derivatives in *non-kinematics* contexts. As such, perhaps the extensive emphasis placed on kinematics examples in calculus (Berry & Nyman, 2003; Marrongelle, 2004; Schwalbach & Dosemagen, 2000) may not be adequately developing the resources needed to work with and reason about non-kinematics derivatives. Since there is a significant range of applied derivatives in other fields of study that are *not* based on kinematics, or even on time, it may be important for calculus educators to bring in these types of examples more regularly during calculus instruction. Doing so may help students develop the conceptual and procedural resources to effectively use and reason about these types of derivatives.

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Reasoning about changes: a frame of reference approach

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In a RUME 18 Theoretical Report my co-authors and I presented our cognitive description of a conceptualized frame of reference, consisting of mental commitments to units, reference points, and directionality of comparison when thinking about measures. Here I present a pilot study on how a focus on conceptualizing a frame of reference impacts students' ability to reason quantitatively about changes. The two-part empirical study consisted of clinical interviews with several students followed by teaching interviews with three students chosen because of their varying abilities to conceptualize a frame of reference. My initial evidence shows that the ability to conceptualize a frame of reference greatly benefits students as they attempt to reason with changes.

Keywords: Frames of Reference, Quantitative Reasoning, Quantities Versus Changes, Reasoning About Changes

In a RUME 18 Theoretical Report (Joshua, Musgrave et al. 2015) my co-authors and I presented our cognitive description of a conceptualized frame of reference, consisting of mental commitments to units, reference points, and directionality of comparison when thinking about measures. At the same time, my experiences working with reform curricula for pre-calculus (Carlson, Oehrtman et al. 2013) and Calculus 1 (Thompson, Byerley et al. 2013) led me to be surprised at how much students struggle with thinking about and reasoning about changes. In mathematics, rate of change is known to be a main idea in calculus (Carlson, Jacobs et al. 2002), and important to introduce as early as Algebra 1 with the idea of slope, but in order to reason about rate of change, a student must be able to conceptualize and reason about changes themselves.

I hypothesized that student struggles were due at least in part to the fact that they were taught that measures of changes in quantities had reference points and directionality yet did not conceptualize measures of quantities themselves with reference points and directionality. Therefore, they did not have parallel attributes with which to compare and contrast the ideas of quantities versus changes, and to distinguish the two in their minds.

The pilot study I propose to share via a poster presentation is an empirical study that I conducted on the connections between a student's ability to conceptualize a frame of reference, and his or her ability to reason about changes. There are several issues surrounding changes that I explored. Among them are:

a) Changes in Quantity vs. Values of Quantity
   - How do students conceptualize a change in a quantity versus the value of a quantity?
   - Does a focus on frames of reference affect students' ability to reason about changes in quantity and values of quantity, by drawing explicit attention to reference points?

b) Changes in Changes

- How do students think about changes in changes, in tasks such as being asked to identify whether a function is increasing/decreasing at an increasing/decreasing rate?
- Does a focus on frames of reference affect students' ability to reason about changes in changes, by drawing explicit attention to a directionality of comparison?

c) Changes in the Context of Velocity & Accelerations
   - How do students conceptualize velocity as it relates to both displacement and acceleration?
   - Does a focus on frames of reference affect the common misconception that "positive acceleration mean the object is speeding up?"
   - How could inconsistent use of a frame of reference (as described in the anecdote) affect student's thinking and consistency, and/or cause future problems, if at all?

The interview processes were carried out with students who have taken at least one algebra class and one physics class. The first part of the study consisted of clinical interviews on eight tasks with seven students to gather data to help me form models of each student's ability to conceptualize a frame of reference. I then picked three students that I found demonstrated varying abilities to conceptualize a frame of reference (roughly described as high, medium, and low) and conducted teaching interviews on eight new tasks with them. All interviews were videotaped and analyzed to form models of how the student thought about measures and measure comparisons before, during, and at the end of the teaching experiment, as well as hypotheses about how these ways of thinking about measures (within a frame of reference or not) affected the student's ability to reason about changes.

Through this pilot study, I found strong initial evidence that a student's ability to conceptualize a frame of reference and reason about measures within a frame of reference had a large positive effect on their ability to reason about changes. The students' abilities to reason through tasks about changes in the teaching interviews frequently reflected the initial positions of 'high' 'medium' and 'low' that I had placed them simply on their abilities to reason about a frame of reference in the clinical interviews. More significantly, the language that the students used to explain their reasoning about tasks involving changes was often about aspects of a frame of reference (units, reference points and directionality of comparison) when the students were successful, and almost never about aspects of a frame of reference when the students gave up or were unsuccessful. Finally, there were many fascinating details in my teaching interviews about how students might begin to conceptualize a frame of reference and apply such an ability to dealing with changes, that have provided starting places for my next expanded project on how a focus on conceptualizing a frame of reference impacts students' ability to reason quantitatively about changes. I believe that not only will the results of my pilot study be of interest to the RUME community, but that discussions with and advice from colleagues during and after my poster presentation would be greatly beneficial to me as I continue to design the next stage of this project.

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TEACHING AND LEARNING LINEAR ALGEBRA IN TERMS OF COMMUNITY OF PRACTICE

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Abstract

Communities of practice (CoP) are defined as groups of people who share a concern, a set of problems, or a passion about a topic, and who interact in an ongoing basis to deepen their knowledge and expertise. The purpose of this study is to examine the process of teaching and learning linear algebra within this theoretical framework. In this research, we used an ethnographic case study design to study three linear algebra instructors and their students at a large public university. The instructors have different educational and cultural backgrounds. Data included observations, a Linear Algebra Questionnaire, and semi-structured interviews. We observed significant differences in teaching methods between the instructors.

Keywords: Linear Algebra, Community of Practice, Ethnographic Case Study.

Communities of practice are based on social learning theory and situated learning theory. CoP are defined as groups of people who share a concern, a set of problems, or a passion about a topic, and who interact in an ongoing basis to deepen their knowledge and expertise (Wenger, McDermott and Snyder, 2002). A CoP includes three crucial components: the domain, the community and the practice (Wenger-Trayner and Wenger-Trayner, 2015). Education is one of the most important applications of the CoP framework. According to Wenger (1998), education is a mutual developmental process between communities and individuals. In this study, the process of teaching linear algebra will be examined within this framework.

Many studies have documented the challenges of teaching and learning linear algebra, which is a main subfield of mathematics (Dorier and Sierpinska, 2001; Hillel and Sierpinska, 1993). To this point, the CoP framework has not been applied to studying the teaching and learning of linear algebra.

The research questions addressed by this study are

1. How do instructors' cultural and educational experiences affect their teaching?
2. How do institutional policies and culture affect instructors' teaching?
3. How do the students experience the teaching?
4. How do instructors' cultural and educational experiences affect the students' experiences and learning?

Methodology

This ongoing study uses an ethnographic case study design. The participants consist of three linear algebra instructors, who have different educational and cultural backgrounds, and their students. Data consists of observations, a questionnaire and semi-structured interviews with both teachers and students. Each teacher was observed for ten weeks by the lead researcher. Responses to a Linear Algebra Questionnaire were collected from students at the end of the course. The questionnaire included questions about students' experiences in the course, and also asked them to solve several linear algebra problems from the Magic Carpet Ride Problem Sequence (Wawro et al., 2012). Data were analyzed and interpreted using standard tools of content analysis.

Preliminary Findings

Institutional Policies and Culture

The same general course outline, the same syllabus and the same textbook were used by all three instructors of MATH 33A (Linear Algebra and Applications).

Instructors' backgrounds

All of the instructors have different cultural and educational backgrounds.

Table 1. Instructors' Educations

	College	PhD	Post-Doc	Visiting Scholar
Lara	Netherlands	U.K.	U.S.A	-
Andres	Spain	Canada	U.S.A.	-
George	U.S.A.	U.S.A.	Canada	Germany, France, Japan, India

Teaching Methods

Categories were generated inductively from the classroom observation data. Results from some of the categories are presented in Table 2.

Table 2. Classrooms Observations

		Lara	Andres	George
Teaching Methods		Teacher-centered approach		
Order of topics		Different from the course syllabus	Similar to the course syllabus	Similar to the course syllabus
Resources used		Lecture notes, online resources, textbook	Textbook	Textbook
Interacting in class		Medium Level	Low Level	High Level
Interacting out of class		Piazza	-	-
Classroom Activities	**Reminding**	-	Recall	Last time-Today
	Proofs	Informal Proof on Examples	Formal Proof	Formal Proof
	Giving Example	Linear Algebra Questions + Daily life example (rarely)	Linear Algebra Questions	Linear Algebra Questions + Daily life example (often)
	Big Picture	The relations between the concepts	-	The processes of problem solutions
	Using multiple representations	The representations of matrix, algebraic, graphic (often R^2, rarely R^3)		
Assessment		Exams, Homeworks	Exams, Homeworks	Exams, Homeworks, Quizzes

In this ongoing study, it has been observed that there are significant differences in teaching methods between the instructors. More results will be added after data analysis is completed.

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A new perspective to help analyze argumentation in an inquiry oriented classroom

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Using argumentation to help understand how learning in a classroom occurs is a compelling and complex task. We show how education researchers can use an argumentation knowledge construction framework (Weinberger & Fischer, 2006) from research in online instruction to make sense of the learning in an inquiry oriented differential equations classroom. The long term goal is see if there are relationships among classroom participation and student outcomes. The research reported here is the first step: analyzing the discourse in terms of epistemic, social, and argumentative dimensions. The results show that the epistemic dimension can be better understood by identifying how students verbalize understanding about a problem, the conceptual space around the problem, the connections between the two and the connections to prior research. In the social dimension, we can identify if students are building on their learning partners' ideas, or using their own ideas, and or both.

Key words: inquiry, argumentation, knowledge, differential equations discourse

Discourse, argumentation, and how to codify and analyze them in collegiate mathematics is in ongoing study in Research on Undergraduate Mathematics Education. Much of the research has been about discourse in the classroom (for example: Lee et al., 2009; Mesa, 2010; Stephan & Rasmussen, 2002), teacher questioning, and other pedagogical moves in the classroom (Nicol, 1998; Moyer & Milewicz, 2002). This work has moved the field forward and provided ways for mathematics instructors to reflect on their teaching and classrooms in productive ways. However, much of this work has been more at the level of identifying the kinds of language that students and teachers use. There is still a need for research about the complex relationships between how students participate in a classroom and achievement. This is important for the growing call for improvement of undergraduate STEM Education (see [Termos, 2011] for one call).

In this report, we offer results about classroom argumentation from a different point of view. The results are part of an ongoing research project where we are hoping to relate classroom argumentation to student achievement in an active learning environment. The call for more work on connecting classrooms and student outcomes has come from several areas, and some are already reporting on it (Cazden & Beck, 2003; Singh, Granville, & Dika, 2002). Additional work has been more generally focused on "active learning" STEM classrooms (e.g., Freeman et al., 2014) and shows that there is building evidence that students that are more active in classrooms perform better on tests. For example, a recent meta-analysis about active learning in undergraduate STEM classes found that students in active learning classrooms earned higher grades (Haak, HilleRisLambers, Pitre, & Freeman, 2011).

Our larger research question is: How does students' class participation relate to student achievement? First, we are trying to more carefully define discourse and argumentation, particularly by evaluating the actual content of the verbal exchanges among students and teachers. The research question that we answer is: How can we use a framework on argumentative knowledge construction to characterize students' contributions to an inquiry-oriented (IO) undergraduate mathematics class?

Argumentation Knowledge Construction Framework

We have adopted a framework used to analyze online scripts (when students use discussion boards, etc. in an asynchronous setting). In their 2006 paper, Weinberger and Fischer offer the following theory about constructing knowledge by argumentation:

> Argumentative knowledge construction (AKC) is based on the assumption that learners engage in discourse activities and that *the frequency of these discourse activities is related to knowledge acquisition*. Learners construct arguments in interaction with their learning partners in order to acquire knowledge about argumentation as well as knowledge of the content under consideration (Andriessen, Baker, & Suthers, 2003). This *definition of argumentative knowledge construction includes that discourse activities on multiple process dimensions may facilitate knowledge acquisition*. Analyzing and facilitating argumentative knowledge construction on multiple process dimensions may extend and refine our understanding of what kind of student discourse contributes to individual knowledge acquisition (van Boxtel & Roelofs, 2001). (p. 73, italics added)

This conception of knowledge acquisition and how it is part of discourse activities resonates with the notion that learning is a social activity (Wenger & Lave, 1991) and that classrooms are where learning may take place (Yackel & Cobb, 1996). It shows a different perspective, in that it discusses knowledge acquisition, something that is very difficult to identify and measure.

The original framework consists of four dimensions: participation, social mode, epistemic, and argument. The participation dimension is two-fold; quantity of participation describes whether learners participate at all, while heterogeneity of participation describes whether they participate equally. For the social modes of co-construction, highly related to knowledge acquisition, characterizes to what extent learners make reference to contributions of other learners in class. The epistemic dimension goes beyond the participation dimension which confirms quantity; it examines the content of learners' contributions by considering how learners work on the task at hand. Lastly, the argument dimension holds the notion that learners encounter difficult problems, and must balance arguments and counterarguments to ultimately find solutions to problems. For the purpose of this analysis, we excluded the first dimension as we felt that we can include that as we inspect the coding of the other three; participation will be evident and the number of talk turns was the most important aspect of our work.

A few additional modifications arose during analysis (See Table 1). The term 'learning partner' is used to describe anyone in the classroom participating in the development of the mathematics, including the instructor.

Table 1

AKC Framework, adapted from Weinberger and Fischer (2006)

Dimension	Categories of Discourse by Learning Partners	Brief Description
Social Mode	Externalization (EXT)	Articulating thoughts to the group

	Elicitation (ELI)	Questioning the learning partner or provoking a reaction from the learning partner
	Quick consensus building (QCB)	Accepting the contributions of the learning partners in order to move on with the task
	Integration-oriented consensus building (IOCB)	Taking over, integrating and applying the perspectives of the learning partner
	Conflict-oriented consensus building (COCB)	Disagreeing, modifying or replacing the perspectives of the learning partners
Argumentation	Argument (ARG)	Statement put forward in favor of a specific proposition
	Counterargument (COU)	An argument opposing a preceding argument, favoring an opposite proposition
	Integration (reply [RPY])	Statement that aims to balance and to advance a preceding argument and counterargument
	Non-argumentative moves (NAR)	Questions, coordinating moves, and meta-statements on Argumentation
Epistemic	Construction of problem space (CPS)	Learners relate case information to case information within the problem space with the aim to foster understanding of the problem
	Construction of conceptual space (CCS)	Learners relate theoretical concepts with each other and explain theoretical principles to foster understanding of a theory
	Construction of adequate relations between conceptual and problem space (CAR+)	Applying the relevant theoretical concepts adequately to solve a problem. Learners relate theoretical concepts to case information.
	Construction of inadequate relations between conceptual and problem space (CAR-)	Applying theoretical concepts inadequately to the case problem. Learners may select the wrong concepts or may not apply the concepts according to the principles of the given theory
	Construction of adequate relations between prior knowledge and problem space (CRP+)	Applying concepts adequately that stem from prior knowledge rather than the new theoretical concepts that are to be learned
	Construction of inadequate relations between prior knowledge and problem space (CRP-)	Applying concepts inadequately that stem from prior knowledge rather than the new theoretical concepts that are to be learned
	Non-epistemic activities (NEA)	Digressing off-topic

Methods

Setting and Participants

This study took place during an IO differential equations course for teachers working to earn a master’s degree in Mathematics Education. The course was held in the summer at a large southeastern university. Twenty-one students participated in the course, which was taught by a professor experienced in teaching inquiry mathematics courses. The student population was comprised of students seeking a master’s degree in Mathematics and Mathematics Education and doctoral students in Mathematics Education. Some of the students had previously taken undergraduate differential equations; however, such coursework was not a prerequisite for the course and the majority of the students indicated that they were starting with minimal or no knowledge of differential equations.

The course met three times a week for two and a half hours in a classroom designed for group work. The classroom had tables where the students sat in assigned groups which were changed at least once a week. The class was taught using the tenets of (IO) instruction (Rasmussen & Kwon, 2007). This meant that students worked on research-based tasks to reinvent the mathematics of the course. The students inquired into the mathematics, working in cycles of small group and whole group discussion spaces. The mathematics involved using differential equations to model real world situations, and understanding the analytical, qualitative, and numerical methods to solve. The students took a pretest and posttest to assess conceptual understanding of the material. Additionally, there were weekly conceptual and procedural homework assignments and two exams.

Data Collection

Each class session was video recorded by a researcher with two cameras, one in the back and another on the side near the front of the room. For this report, we chose one hour of whole-class discussion from four class sessions that occurred before the midterm exam, three in the early part of the course and one the day before the midterm. In order to capture a representative glimpse of the contributions made by the learning partners, we randomly selected one of the 15-minute time moments in the class to use as our beginning time. From that time period, the next full hour of whole-class discussion was transcribed for coding. The transcriptions were divided into talk turns; we define *talk turns* as a single utterance made by any of the learning partners. The instructor's talk was not transcribed verbatim, but all other talk was.

Data Analysis

Two members of the research team coded each talk turn using the transcripts of whole-class discussions by using the descriptors from the framework. With the exception of the professor under the epistemic dimension (as the instructor was assumed to not be constructing new mathematical conceptions), each learning partner's talk turns were coded for epistemic, social, and argument dimensions according to the framework identified above. If the two researchers' codes were not in agreement, the third researcher gave the talk turn its final code, breaking the tie or providing a new code. In order to establish reasonable agreement between the two coders a trial coding was conducted for one of the class sessions. Originally, the two coders had poor agreement (see Table 2 for July 5, Argumentative). As a result, the three researchers spent 10 hours discussing coding discrepancies, clarifying language that was not used the same in another research field and modifying when it did not seem to be appropriate for whole class in-person discussions.

Table 2

Inter-rater reliability by class date and dimension

Day	Epistemic	Social	Argumentative
July 2	80.58%	77.70%	68.35%
July 3	76.22%	78.32%	60.14%
July 5	73.65%	67.70%	48.60%
July 16	81.63%	89.12%	80.95%

As shown above, the interrater reliability increased dramatically after the hours of discussion. The days were coded in the following order: July 5, July 3, July 2, July 16. After the coding, all the ties were broken and analysis began.

Results

We present the results of our coding, relationships among the codes that can be identified statistically, and a discussion of our experience with the framework and its utility for codifying argumentation in a classroom setting. Table 3 depicts the compilation of all codes given to all talk turns from students only. The 8 in the top section of the table indicates that 8 student talk turns were coded with the chain, CPS-ELI-NAR, indicating that students were asking authentic questions about the problem space 8 times during the 4 hours of whole-class discussion. Notable values include the 26 talk turns coded with CAR+ and IOC in the epistemic and social dimensions. These codes indicate that students made adequate connections between the problem and concept spaces while building from previous students' thoughts. These talk turns made up roughly 46% of the turns coded CAR+. Another notable value is the 102 talk turns coded as blank in all three dimensions. These talk turns make up nearly a quarter of student contributions, and illustrate the times in which students displayed Quick Consensus Building. Comments such as, "Right," or, "I agree," were exemplars of quick consensus building.

Table 3
Number of each code sequence for student contributions

	CPS	CCS	CAR-	CAR+	CRP-	CRP+	NEA	(blank)	Grand Total
ELI									
ARG	2	1	1					1	5
COU	1	1			1	2			5
NAR	8	5	1	1	1			18	34
RPY	2	4		1				2	9
(blank)								1	1
ELI Total	**13**	**11**	**2**	**2**	**2**	**2**		**22**	**54**
IOC									
ARG	4	3		3		2			12
COU	5			3	1				9
NAR								1	1
RPY	14	7	1	20		2		3	47
IOC Total	**23**	**10**	**1**	**26**	**1**	**4**		**4**	**69**
COC									
ARG								1	1
COU	7	6	1	4				1	19
NAR								1	1
RPY	5	2	1			1		1	10
COC Total	**12**	**8**	**2**	**4**		**1**		**4**	**31**
(blank)									
ARG	34	10	2	9		1		2	58
COU	11	6	2	4		1		1	25
NAR	6				1		1	9	17
RPY	33	13	2	12		4		12	76
(blank)	7						6	78	91
(blank) Total	**91**	**29**	**6**	**25**	**1**	**6**	**7**	**102**	**267**
Grand Total	**139**	**58**	**11**	**57**	**4**	**13**	**7**	**132**	**421**

To further display student contributions to discourse, we provide Figures 1 and 2. These figures display the number of talk turns by each student (pseudonyms) that were assigned each code from the Epistemic and Social dimensions (respectively). In Figure 1, it can be seen that many of the contributions by students were focused on constructing the problem space. However, for students with more than 20 talk turns, a bulk of their contributions made adequate connections from concepts to problems. Excitedly, inaccurate connections from prior knowledge or between concept and problem spaces were student contributions that occurred least often.

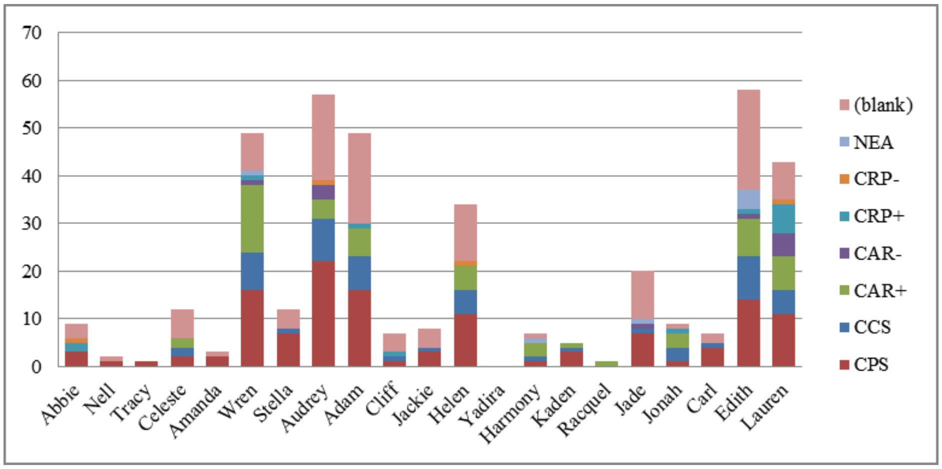

Figure 1. Epistemic codes by students.

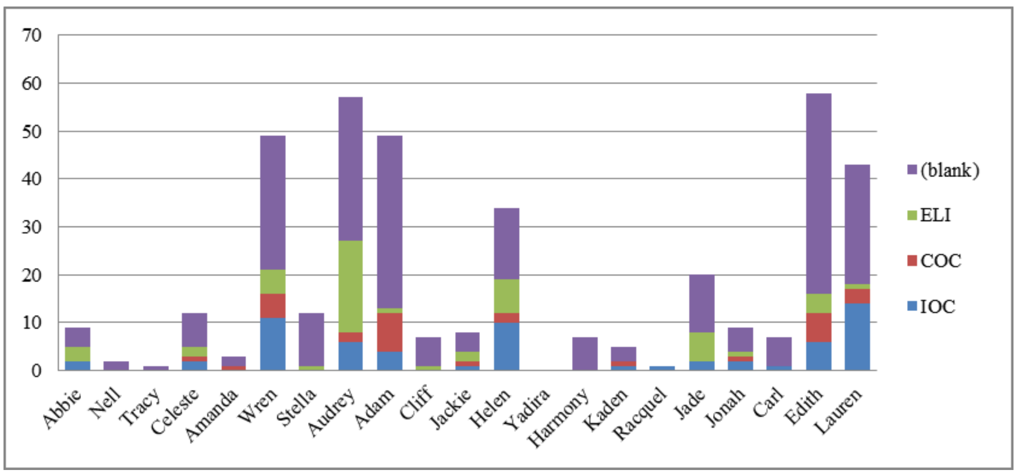

Figure 2. Social codes by students.

Together, Figure 2 and Table 3 respectively illustrate the frequency and source of the blanks from the social dimension. From a social standpoint, we noticed that the more vocal students had an abundance of IOC and COC. In contrast, the less vocal students had most of the blanks for these categories with the social dimension. All students but one (Racquel) had blanks for their talk turn codes, which may indicate quick consensus building, a code we decided to not consider in analysis. Students were either constructing the problem space, quick consensus building, or externalizing a thought – without providing much to the argument. The framework for argumentative knowledge construction used here was originally used on scripts from asynchronous online discussions. However, we found the framework very useful as the dimensions of epistemic, argument, and social were highly evident in active and more traditional classrooms (Weinberger & Fischer, 2006). Following are some issues we found.

First, the coding is exceptionally time consuming. After transcription, for the 6 hours we coded, the two coders spent approximately 42 hours coding and another 10 hours breaking the ties. We also spent 12 hours meeting to refine, modify, and agree on descriptions of the codes. We believe the results are accurate and useful, but considering just the coding, not the preparation work, this means that each hour of class took about 16 hours to complete the coding only, a significant time commitment.

Secondly, we had to make modifications to the knowledge framework in some key (good) ways. We were able to watch the class videos and see nuances that were not present during the original framework development. There were many places where graphs and tables were being discussed, and it was sometimes hard to interpret the words, but we felt that the videos were an important resource for this. We also had samples of student work to help analyze their knowledge construction on the research-based tasks implemented in the course.

Third, we found that thinking in terms of problem space, conceptual space, and prior knowledge was a very helpful way to determine how students were developing their ideas in terms of the epistemic dimension. We have seen other work in this area (Sfard, 1998), but this seems particularly effective with understanding the actual construction of knowledge. Connected to this, thinking about the way students either express their own thinking and/or build on others in a social dimension helped us see how this actually occurs in an (IO) environment. The argument dimension was the least useful at this point; it appeared to be the most difficult to code and find agreement. Ultimately, we saw few connections. Future work might involve only considering the epistemic and social dimension.

Conclusion

In summary, this framework is a new valuable tool that can help us understand knowledge acquisition of students in all mathematics classrooms, not just at the undergraduate level. The educational community outside of mathematics education can provide new and effective ways to do research. The framework allows us to identify when students are building on each other's knowledge and bring in their own ideas. It also helps show when students are thinking about a given problem, the conceptual space behind it, and previous knowledge used to solve the problem. Although time consuming, we propose this as an effective tool to analyze knowledge development in an active learning classroom. The next step is to take this information and see how it connects to student outcomes in terms of homework, projects, tests, etc. By thinking about how our students participate in a more focused way, we can provide instructors with ways to think about implementing and improving inquiry and other active learning situations.

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How should you participate? Let me count the ways

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Retention of students in STEM majors is an issue of national stability because government projections indicate our nation to need one million additional STEM majors by 2022 (PCAST, 2012); thusly, the current trends in attrition are alarming. Students leave STEM for various reasons, but poor experiences in Calculus I seem to be a significant contributing factor for many switchers, especially female students. Using data situated within a larger study (Characteristics of Successful Programs in College Calculus), the present report looks specifically at student participation and its influence on Calculus I success. Results indicate that while participation is significantly correlated with success, this effect is not uniformly distributed across types of participation or gender groups. Interestingly, overall success rates were equal, but gender differences were noted in frequency of participatory behaviors and distribution of grades; specifically, males (who reported more A grades) preferred in-class participation and females preferred out-of-class participatory activities.

Key words: Calculus, Student Success, Participation, Gender

Calculus serves as an introductory course for college freshmen everywhere, but especially for those intending to enter into science, technology, engineering, and mathematics (STEM) majors; therefore, of critical importance is student success in Calculus I – without which continuation in a STEM major is impossible. The retention of students in STEM majors has been identified by the President's Council of Advisors on Science and Technology (PCAST) as a key contributor to the ability of the United States to remain a leader in the STEM fields (2012); PCAST specifically advises that over the next decade, in order to retain our dominance, the nation will require an additional one million STEM majors beyond those currently projected. With calculus acting as a gatekeeper to a student's ability to successfully complete an undergraduate STEM degree, post-secondary educators and students alike must develop a better understanding of what factors may contribute to success in calculus. This work aims to serve that goal by exploring the relationship between student participation and success in Calculus I. In this report, we investigate the following research questions: (1) Does there exist a positive correlation between student engagement in participatory behaviors and student success in Calculus I? (2) If so, can particular behaviors be identified as critically influential and is this association consistent for both genders?

Theoretical Framework and Literature

Research (e.g., Rasmussen & Ellis, 2013; Seymour & Hewitt, 1997) indicates that many students are in fact leaving STEM majors as a result of poor experiences in calculus, and that instructional factors within the calculus classroom contribute to this departure. The historically predominant reliance upon lecture as the conduit of calculus material appears to be a contributing factor in students' discontent with their experiences in STEM majors. Instructors of calculus need to carefully reconsider their pedagogical decisions if they wish to combat the disengagement that leads to attrition; however, it is important to note that each student's achievement is ultimately the result of her own actions within the course; i.e. participation, and thus students must share in the responsibility for their success. While we concur with previous research (e.g., Rasmussen & Ellis, 2013; Johnson, Ellis, & Rasmussen, in press) that indicates student retention and success is influenced by instructor actions, we choose to neglect that variable for the purposes of this study and opt instead to focus our analysis on students' investments in their learning

We believe that a student's interactions – with the material, with the instructor, with her classmates – are of critical importance in determining success and therefore we frame our research within the theory of social constructivism. Social constructivism emphasizes "the claim that higher mental functions in the individual have their origins in social life" (Wertsch, 1990). Thus, in order for students to learn and achieve academically, they must engage socially with others. This engagement can occur in many forms. Engagement with the instructor, for example, occurs when the student contributes to class discussion, corresponds with the instructor regarding course content, or completes assignments designed by the instructor.

The theory of social constructivism is well represented in the educational literature. An example of such research in support of social constructivism is Tinto's (1997) work, which indicates that "the more students invest in learning activities, that is, the higher their level of effort, the more students learn" (p. 600). The implication is that in order to be academically successful, students must first engage in the learning process. This finding is neither unique nor modern; put bluntly, participation increases student learning (e.g., Johnson, Johnson, & Smith, 1991; 1998; Lyman, 1981).

Student participation manifests itself in a variety of ways, both inside the classroom and outside of it, but there has been research (Lucas 2009, Rasmussen & Ellis, 2013) indicating that homework completion and participation in classroom discussion are of critical importance. In addition to considering different types of participation, it is important to consider that participation patterns do not indiscriminately influence student success across demographic groupings. Tinto (1997) determined that students from various minority groups necessarily seek inclusion in the learning community as their main goal prior to seeking academic success. This is consistent with cognitive evaluation theory, which indicates people must feel competent, related, and autonomous prior to engaging academically (Deci & Ryan, 2000). Tinto's contribution to this theory is that different individuals require different levels of satisfaction of competence, relatedness, and autonomy. Tinto does not further elaborate on which minority groups are more likely to seek out relatedness prior to competence, or any other combination of factors, but his findings are important in understanding that not all students will participate in, gain interest from, or learn from the same activities in an equal manner.

One particular minority group of interest in the STEM community is women. Karp and Yoels (1975) identified differences in the participation of male and female students in the college classroom, and moreover, that these differences are influenced by the instructor's gender. Specifically, female students participate more in classes led by female instructors (42.4% of interactions compared to 24.6% in male led classes). Conversely, male students are responsible for 75.4% of interactions in classes led by males as compared to 57.8% of interactions under female instructors. These differences must be seen as a function of both the student's choice to participate and the instructor's choice to prompt participation. Karp and Yoels' (1975) findings are mirrored in more recent literature. Sadker and Sadker (1995; Sadker, Sadker, & Zittleman, 2009) have similarly determined that both the quantity and quality of teacher-student interactions with male and female students are different. Teachers tend to ask more questions of male students, allow more wait time for male students, and ask more follow-up questions. Female students, on the other hand, are asked lower level questions and provided less constructive feedback and encouragement than male students in the mathematics classroom (Sadker & Sadker, 1995). The persistence of these gender differences in the participation and inclusion of female students over time in the mathematics classroom are troubling – if the circumstances are such that female students are not provided an equal opportunity to participate in classroom discussion, then their learning is being affected before they can even make the choice whether or not to participate.

Perhaps not coincidentally, gender differences are also being noted in STEM retention in addition to participation. PCAST (2012) specifically notes that the retention and success of women in STEM majors is critical, as they represent a majority of college students but a minority of STEM graduates. Despite this need for female STEM graduates, significantly more women switch out of STEM majors (20%) than do males (11%) (Rasmussen & Ellis, 2013).

Data Sources and Methods of Analysis

The present study is situated within the larger research project entitled Characteristics of Successful Programs in College Calculus (CSPCC) that was designed to gain a nationwide overview of the college calculus programs as well as to identify more successful programs based on a combination of factors including: grades, affective variables (e.g., interest, enjoyment, and confidence), and intention to continue on to Calculus II. The CSPCC project[1] used a stratified random sample of colleges and universities in the U.S. based on the highest degree granted at each university (Associate's, Bachelor's, Master's, or Ph.D.). The first phase was comprised of a total of six surveys—three for the students (one at the beginning of Calculus I, one at the end of Calculus I, and one a year later to the students that gave their email addresses), two for the instructors (one at the beginning of Calculus I and one at the end of Calculus I), and one survey given to the Calculus course coordinator. For the purposes of this study, we limited our dataset to those student respondents who had completed the end of semester survey.

In order to answer our research questions, it was necessary to operationally define both success and participation. Previous research had suggested the use of the rates of *persisters* and *switchers* as a proxy for success; however, we feel that measure is more appropriate as an indication of the success of a university's academic courses and STEM programs overall and not the best measure of individual student success. We chose instead to define success in terms of reported/expected[2] course grade (A-F). As educators, we acknowledge that success cannot and should not be measured only in terms of final grades; however, we were both limited by our use of a pre-existing data set and also constrained by our desire not to duplicate research already performed in this area. We recorded the reported/expected letter grade for each student and also coded each student as 'successful' (A, B, C) or 'not successful' (D, F).

For consideration in the initial regression analysis, we selected eight items from the Student End survey that we felt captured what we considered to be instances of participatory behavior: talking in class, preparing for class, reinforcing content, seeking help. These questions (see Figure 1) reflect activities for which the ability to participate was provided, placing the choice to participate in the hands of the student. From the perspective of social constructivism, participation is the vehicle of student learning; thus, these items were selected to demonstrate the student's perception of her engagement.

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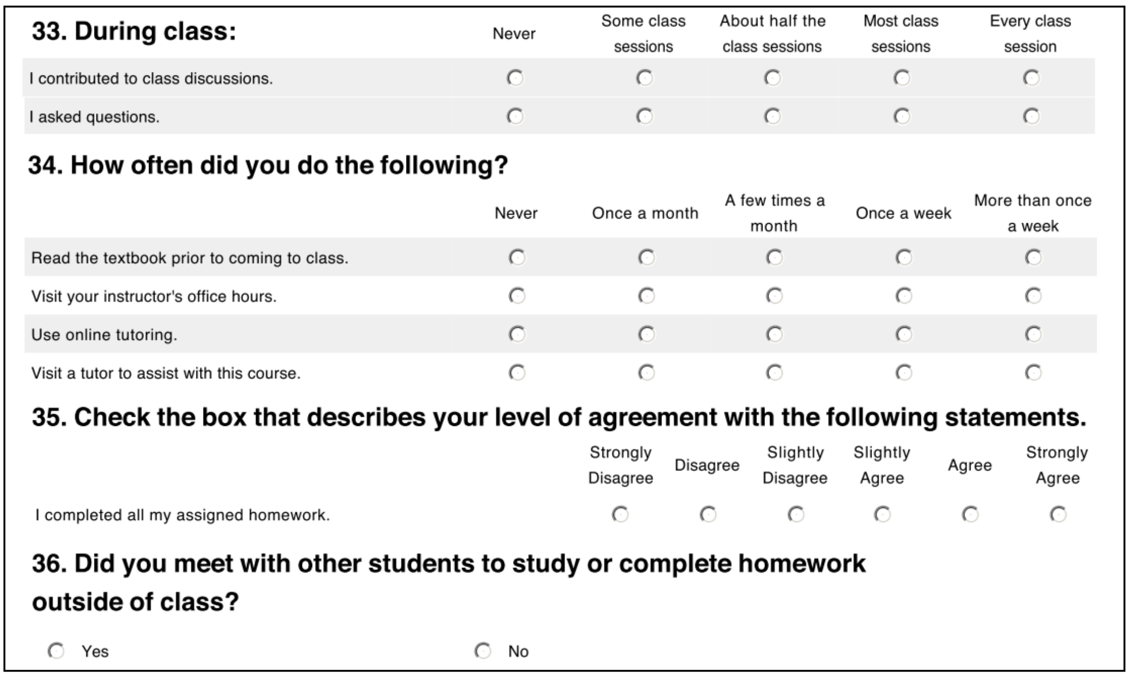
[1] For further details, see the *MAA Notes* volume *Insights and Recommendations from the MAA National Study of College Calculus* (Bressoud, Mesa, & Rasmussen, 2015) or visit the website at www.maa.org/cspcc).

[2] On the post-survey students were asked: *What grade do you expect (or did you receive) in this course?* We are unable to determine if this question was asked before or after students received their final grades.

From the eight questions, we collapsed this into seven independent variables: *ContributedtoDiscussion, AskedQuestions, ReadText, OfficeHours, UsedTutor (*composite variable computed by summing frequency of tutor and online tutoring), *CompletedHW,* and *MetToStudy*. Depending on the analysis being conducted, the dependent variable was measured either by *Grade* (recoded from 0.0 to 4.0 to reflect the usual grading scale) or by *Success* (coded 1 for A, B, C grades and 0 for D, F grades). A combination of ordinary least-squares and binary logistic regression models were run in order to address the first research question; i.e. to determine the ability of our selected participatory behaviors to predict academic success. Subsequent analyses involved comparing the behavior of specific groups (successful versus unsuccessful, males versus females) on those behaviors deemed statistically significant.



33. During class:	Never	Some class sessions	About half the class sessions	Most class sessions	Every class session
I contributed to class discussions.	○	○	○	○	○
I asked questions.	○	○	○	○	○

34. How often did you do the following?

	Never	Once a month	A few times a month	Once a week	More than once a week
Read the textbook prior to coming to class.	○	○	○	○	○
Visit your instructor's office hours.	○	○	○	○	○
Use online tutoring.	○	○	○	○	○
Visit a tutor to assist with this course.	○	○	○	○	○

35. Check the box that describes your level of agreement with the following statements.

	Strongly Disagree	Disagree	Slightly Disagree	Slightly Agree	Agree	Strongly Agree
I completed all my assigned homework.	○	○	○	○	○	○

36. Did you meet with other students to study or complete homework outside of class?

○ Yes ○ No

Figure 1. Items Selected from the CSPCC Student-End Survey

Results and Discussion

Primary Analysis

The logistic regression model based upon the students' self-reported levels of participation, despite having a pseudo R-square value (Nagelkerke = .146) lower than what would have been preferred, had predictive accuracy of 95.59% in projecting success. Using a forward stepwise Wald procedure, the resulting model identified four of the seven initial independent variables as being significant: *ContributedtoDiscussion* (β = .482; $p < .001$), *CompletedHW* (β = .448; $p < .001$), *UsedTutor* (β = -.290; $p < .001$), and *ReadText* (β = -.168; $p = .002$). Interpreting this in terms of odds ratios, all other factors being equal, for a one-unit increase in frequency of homework completion (or contributions to discussion), a student would be 1.565 (or 1.619 respectively) times more likely to be categorized as successful by the model.

Interesting to note is that not all of these variables were positively associated with success as one might have assumed. Both *UsedTutor* and *ReadText* were negatively correlated with success; i.e. increasing the frequency of these behaviors decreases a student's odds of being labeled successful. This must be interpreted with caution because it would be tempting to think that reading the text or working with a tutor decreases one's odds of being successful; however, this is almost certainly not the case. All the model is telling us is that of those students who were unsuccessful, they were reading the text and using tutors at higher frequencies than those who were successful. Further analysis would be needed to identify the other factors at play that contributed to these variables having a negative correlation with success. Conceivably, the students who are successful do not read the text because they feel it unnecessary as they already have a firm grasp of the material. With regard to the use of a tutor, perhaps it is the timing of the help-seeking behavior that is confounding the situation. It is possible that failing students are waiting until they have already established themselves as unsuccessful before seeking tutoring help. This is perhaps less a reflection of the tutor's effectiveness and more a proxy for traits of unsuccessful students.

According to social constructivism, learning occurs through engagement in social activity; therefore, since all of the previously identified participatory behaviors are social activities, we would expect that they all positively influence success. That being said, because mathematics is constructed individually and understood uniquely, all participants in the social activity affect the quality of the mathematics being constructed. In this report, the two most significant and positively correlated variables contributing to student success are representations of high quality interactions with the course instructor. As Tinto (1997) indicated, high quality engagement is paramount to student learning. Therefore, we hypothesize that *ContributedtoDiscussion* and *CompletedHW* are both the most significant and positively correlated with success because they represent high quality, structured social engagement with the instructor who designs and assesses their learning. Class discussions are likely the result of the instructor's lesson plans, and homework directs the students toward the instructor's learning goals. As students participate in planned discussions led by the instructor and complete homework assignments designed to help the students review or learn new material, the student is actively engaging with the instructor and constructing mathematics in a manner which is consistent with that which will be assessed.

Previously in this article, we discussed two variables which were not found to positively correlate with success in Calculus I: *ReadText* and *UsedTutor.* Through the lens of social constructivism, we offer an additional theory as to why these are negatively associated with success in the current model. While each of these variables does represent a social interaction on the student's part, it is one in which the instructor is absent. Therefore, we hypothesize that the quality of these interactions is not likely to be as high as those student-teacher interactions previously described. When a student reads a textbook or engages with a tutor, they are interacting with an expert; however, the mathematics being constructed is not necessarily in alignment with that intended by the instructor.

The three variables remaining – *AskedQuestions, OfficeHours,* and *MetToStudy* – were either insignificantly correlated (logistic model) or negatively correlated (OLS model) with success in Calculus I. This is interesting because these certainly represent social interactions, and moreover, two of the three involve both the students and the instructor, so by our previous explanation it seems as though they should correlate to student success. The distinction is that asking questions and attending office hours are unplanned, unstructured interactions and therefore are likely lack the careful consideration and depth of a high quality social interaction. We hypothesize that the quality of these interactions is not as high as the pre-planned classroom discussions and carefully constructed homework assignments. Furthermore, the survey questions as written do not capture the level of sophistication and purpose of the questions being asked. If the nature of the questions is that of high-level cognitive demand (i.e. beyond-the-scope) and helps to advance the mathematical agenda, then we argue that this should be positively correlated with successful students; however, if the questions being asked are low-level clarification questions (e.g. What does that symbol mean? Why are we using that formula? etc.) or worse yet, logistical questions (e.g. Will this be on the exam? Do we have to memorize that? Does my calculator have a button for that?), then it seems plausible that these are being asked by students more likely to be unsuccessful and therefore would rightly be negatively correlated.

In the case of students meeting to study with other students, the quality of the social engagement is even more questionable, as the group of students working together may or may not have mastered the mathematics which they intend to learn. These variables certainly represent social activities. The learning which takes place during these participatory activities, however, is not necessarily high quality, as it was not designed by the same person who will assess the students' learning.

Secondary Analysis

In addition to determining a correlation between participation and success in Calculus I, we also sought to determine whether the distributions across categories of these positively correlated participatory behaviors were similar when comparing successful versus unsuccessful students and when comparing males to females (see Figure 2). Unsurprisingly, results from independent-samples Kruskal-Wallis tests reveal that the distribution across categories of *CompletedHW* was not the same for successful and unsuccessful students ($H(1)= 79.278$, $p < .001$) and the distribution across categories of *ContributedtoDiscussion* was not the same for successful and unsuccessful students ($H(1) = 30.941$, $p < .001$) either. Successful students, on average, contribute to discussion *more* frequently (2.65 as compared to 2.16) and complete homework *more* frequently (4.70 as compared to 3.73) than unsuccessful students.

Looking at gender differences, results from independent-samples Kruskal-Wallis tests reveal that the distribution across categories of *CompletedHW* was not the same for males and females ($H(1) = 58.6$, $p < .001$) and the distribution across categories of *ContributedtoDiscussion* was not the same for males and females ($H(1) = 42.94$, $p < .001$) either. Male students, on average, contribute to discussion *more* frequently (2.74 as compared to 2.48) and complete homework *less* frequently (4.51 as compared to 4.90) than do female students.

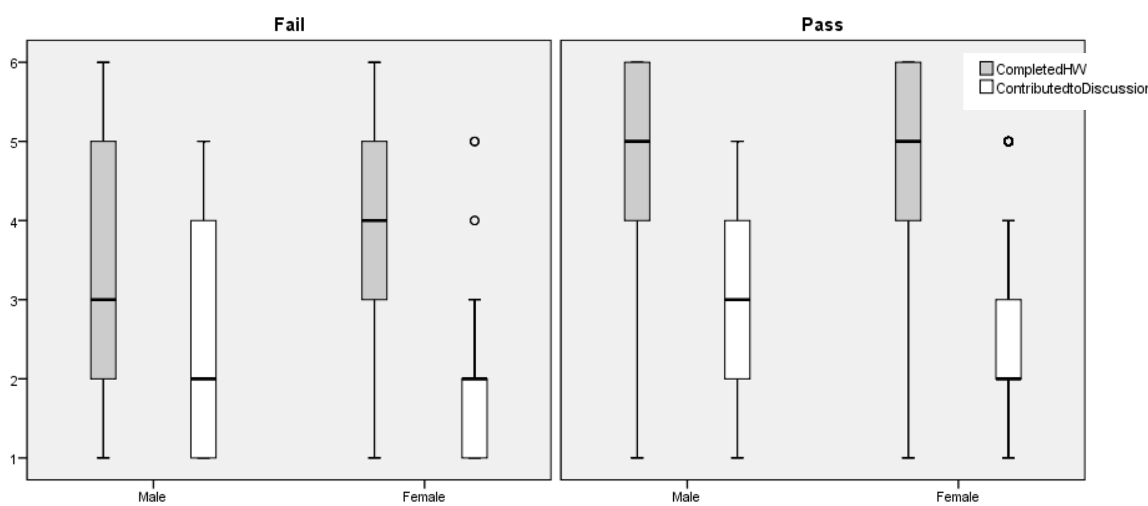


Figure 2. Patterns of Participatory Behavior by Gender and Success Category

These results are consistent with previous research. Researchers (Karp & Yoels, 1975; Sadker & Sadker, 1995) have previously determined that male students are more likely to be called on during class and to have higher quality in-class interactions with their instructors. This finding also coincides with the determination that different groups of students benefit from and engage in social activities in varying capacities as they are ready to do so (Tinto, 1997); it moreover extends Tinto's results in specifying the participation of males and females to be significantly different.

Implications and Future Directions

The results of this report demonstrate that successful students' participatory behavior is both qualitatively and quantitatively different than unsuccessful students. Coupled with the fact that the same can be said about the differences between male and female students, does that not raise the logical follow-up question: Do males and females succeed at different rates? An independent-samples t-test establishes no significant difference ($t(3094.252) = 1.583$, $p = .114$) between the percentage of successful students by gender; however, an independent-samples Kruskal-Wallis test does provide evidence ($H(1) = 5.773$, $p = .016$) that distribution across reported/expected grades is not the same for males and females. In other words, males and females are equally likely to have passed or failed the course, but for those who passed, the males are disproportionately likely to have reported earning (or expected to earn), an A ($z = 2.46$, $p = .014$).

It is important to note that while students are likely to accurately predict success or failure in a course, it is unlikely that they are equally adept at predicting final grades. Since the data in this study was based on *reported* grades that may or may not have actually matched the final grade received, the interpretation of the distributional analysis and subsequent conclusions must be interpreted with caution. Future research warrants attempting to replicate these findings in the cases for which final grades can be verified.

When choosing survey items for consideration in this research, the decision was based on participatory behaviors that we felt the student had the ability to self-select; however, based on the research of Sadker and Sadker (1995; Sadker, Sadker, & Zittleman, 2009), it seems that female students are not given equal opportunity to ask questions or contribute to class discussions and thus these participatory behaviors cease to be ones of personal choice. We conjecture that each student requires a certain minimal level of attention for social constructivism and since females don't receive attention, approval, or reinforcement during class time at levels comparable to their male counterparts, they seek to make up for it on their own time. This would explain the fact that female students complete homework and enlist the use of a tutor more frequently than male students. While this out-of-class participation leads to/contributes to success rates for females equivalent to those of males, it does not appear to translate into comparable levels of high performance (i.e. A grades), suggesting that in-class participation is somehow superior to out-of-class participation in terms of measuring success by academic achievement. This hypothesis, along with the implications for STEM attrition, warrants further research. Although both an A-student and a C-student might be equally likely to continue from Calculus I to Calculus II, can the same be said about the ability to complete a STEM degree or even persist in the major beyond Calculus II? It is our opinion that students who earn borderline grades in Calculus I are disproportionately likely to ultimately depart from their current major and possibly the STEM field altogether. This argument might explain why females represent approximately 57.5% of all college students, but only 29.7% of STEM graduates – a dangerous imbalance that carries societal and economic implications.

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Preliminary genetic decomposition for implicit differentiation and its connections to multivariable calculus

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Derivatives are an important concept in undergraduate mathematics and across the STEM fields. There have been many studies on student understanding of derivatives, from graphing derivatives to applying them in different scientific areas. However, there is little research on how students construct an understanding of multivariable calculus from their understanding of single variable calculus. This poster uses APOS theory to hypothesize the mental reflections and constructions students need to make in order to solve and interpret an implicit differentiation problem and examine the connections to multivariable calculus. Implicit differentiation is often the first time students are introduced to the notion of a function defined by two dependent variables, a concept vital in multivariable calculus. Investigating how students initially reconcile this new idea of two variable functions can provide knowledge of how students think about multivariable calculus.

Key Words: Implicit differentiation, student understanding, genetic decomposition, multivariable, APOS theory

Introduction and Relation to Literature

Student understanding of single variable calculus has been well researched (e.g. Bardini, Pierce, & Stacey, 2004; Habre & Abboud, 2006; Lauten, Graham, & Ferrini-Mundy, 1994; Simonsen, 1995; Tall, 1985; Thompson & Silverman, 2008; White & Mitchelmore, 1996; Williams, 1991). Comparatively, there are relatively few investigations of student understanding of multivariable calculus (e.g. Dorko & Weber, 2014; Fisher, 2008; Kerrigan, 2015; Martínez-Planell & Gaisman, 2012; McGee & Moore-Russo, 2014). In particular, while there is an abundance of research on the topic of single variable differentiation (García, Llinares, & Sánchez-Matamoros, 2011; Habre & Abboud, 2006; Haciomeroglu, Aspinwall, & Presmeg, 2010; Orhun, 2012; Santos & Thomas, 2001), there is very little knowledge of student understanding of multivariable differentiation (Martínez-Planell, Gaisman & McGee, 2015; McGee & Moore-Russo, 2014; Tall, 1992). Similarly, many researchers have explicitly investigated the mental generalizations and reflections students need to construct the concept of differentiation (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Clark et al., 1997; García et al., 2011). However, there has been little work done on what reflections and mental constructions students need to make in order to understand implicit differentiation. This poster is theoretical in nature and focuses, through APOS theory, on the connections between the mental reflections and constructions need for implicit differentiation and those needed for multivariable calculus.

APOS Theory

APOS theory emerged from Piaget's notion reflection abstration and is a theoretical framework for investigating mental construction of mathematical objects (Dubinsky & McDonald, 2001). There have been several addititions the the original *action*, *process*, *object*, *schema* stages in the theory, including *procept* and *procedural process*. The term *procept* refers to a duality of having both a process and object understanding and *procedural process* refers to when a student can mentally run through an action and has interiorized it but may not yet have a deeper conceptual understanding of the process. In APOS theory, a genetic decomposition is a hypothetical model of mental constructions needed to learn a specific mathematical concept (Arnon, 2014). This poster will exhibit of a genetic decomposition for implicit differentiation and examine the connections between the mental actions/reflections students make in implicit differentiation which may be useful later in multivariable calculus. This linking broadens the current ways researchers have been looking at the connections between single- and multivariable calculus.

Discussion

There are several key reflections and connections that students must make in solving an implicit differentiation problem that similarities to those made in multivariable settings. Due to space limitations, I provide two examples; the poster will contain a complete genetic decomposition..

The first reflection students must make in solving implicit differentiation problems is to identify an implicitly defined function. To identify that a function cannot be explicitly expressed in terms of a single variable requires students to reorganize their notion for function. Students often think of functions as a variable set equal to an expression, such as $y=2x+7$ (Thompson, 2013). However, an implicitly defined function it is dependent on both variables as opposed to one. This is an essential concept in the multivariable setting because the functions are explicitly defined in terms of two variables. A student who has already seen implicit differentiation should have at least a process level of understanding of function dependent on two variables. Thus when introduced more formally to two variable functions in multivariable calculus, students already have a process to reflect on in order to build the concept of multivariable functions.

Another key construction students must make in implicit differentiation is taking the derivative of y with respect to x, rather than taking the derivative of x with respect to x. The methodology for finding the derivative with respect to a single variable when multiple are present is different between implicit differentiation and multivariable differentiation, however, constructing the concept of looking at the change in one dependent variable with respect to another dependent variable is common to both. For instance, to find the derivative with implicit differentiation, students must take the derivative of each variable with respect to a single variable. This requires several new mental constructions including an encapsulate the process of the chain rule to be able to apply as an object in implicit differentiation. However, finding a derivative of a multivariable function requires the student to reflection on what variable the derivative is being taken with respect to and to treat the other variables as fixed. This does not require the same mental structure as implicit differentiation but the main reflection of taking a derivative with respect to a single variable when more that one is present is vital to both concepts.

These are just two examples of mental constructions that students make first in implicit differentiation that are vital to those in multivariable calculus. Understanding how student think about implicit differentiation and the underlying mental actions needed to construct the concept, can not only help implement better instructional methods but also lend insight into how students think about multivariable calculus.

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An interconnected framework for characterizing symbol sense

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Algebraic notation can be a powerful mathematical tool, but not all seem to develop "symbol sense," the ability to use that tool effectively across situations. Analysis of interview data with both novice and expert users of notation identified three interconnected viewpoints: looking at, with, and through the notation. The framework has implications for instruction and potential development of symbol sense.

Key Words: symbol sense, algebraic notation

Algebraic notation can be a powerful mathematical tool, yet not all students seem to develop "symbol sense." The ever-changing state of technology contributes to the motivation for mathematics educators to define symbol sense and design instruction to encourage its development. In particular, the *Common Core State Standards for Mathematics* (2010) calls for students to be able to both *decontextualize*—work with abstract symbols while allowing the referents to shift to the background—and *contextualize*—to reconnect with those referents as needed in order to appropriately interpret the relationships within the situation.

Background and Methods

Mathematical symbols, and algebraic notation in particular, can be the *focus of* one's attention, or the *means through which* one's attention on quantitative relationships is mediated. The ability to make and attend to such shifts can be considered *symbol sense*, or a coherent approach to algebraic notation that supports and extends mathematical reasoning. Symbol sense thus goes beyond efficient manipulation of symbols to being able to select, construct, manipulate, and interpret notational forms in service of mathematical work (Author). While the importance of developing such symbol sense is widely accepted, the process by which this happens is not yet well understood. Arcavi (2005) identifies three open issues related to the development of symbol sense. First, we do not have a full characterization of symbol sense, in terms of having a comprehensive set of categories to inform research and instruction. The second issue may be cast as *nature versus nurture*: can symbol sense be taught and learned, or are there "symbol experts" that have an inherent sense of symbols? The final issue addresses the interplay between technical practice and symbolic reasoning; that is, how does technical fluency interact with the development of symbol sense? These three issues are not trivial and will not be answered easily. Recent studies (cf. Banarjee & Subramaniam, 2012; Hewitt, 2012; Bokhove & Drijvers, 2012) have explored options for designing algebra instruction; reasoning about structural aspects of the notation is a common theme and resonates with the idea of developing symbol sense.

Views from previous work (Kinzel, 2000) were combined with Arcavi's characteristics of symbol sense (1994, 2005) in the design of the current study. An interconnected framework emerged that coordinates three viewpoints: *looking at*, *with,* or *through* the notation. *Looking at* the notation can involve noticing particular aspects and considering appropriate actions; more fluent users are able to consider a wider range of possible actions. An ability to *look with* the notation connects with Arcavi's notion of "friendliness" with symbols, that the individual recognizes the power of a symbolic representation and that symbols are readily available as a means of representation. Coordination of *looking at* and *with* the notation may involve conscious choices; for example, having chosen one representation, the individual is capable of considering the implications—either for representation or for manipulation—and perhaps changing or altering the initial choice. Arcavi also includes the ability to "read through" the symbols to assess affordances of manipulations, or to check for meanings within the implementation of procedures. Interview data supported the characterization of a third view, *looking through* the notation to explore underlying relationships. Effective work with symbolic forms on a single task can involve conscious coordination of these viewpoints.

Task-based interviews were conducted with 11 students enrolled in a proof-based mathematics course and with 9 practicing mathematicians. Data were analyzed in terms of the introduction of symbols, construction of expressions and/or equations, manipulation of symbolic forms, and interpretation of intermediate or final results, as well as any shifts in focus of attention or use of notation (e.g., choosing to change the nature of a representation). Explicit articulations related to interpretation or use of notation were noted and categorized. Narratives were created for each interview, capturing the individual's use of and articulations about the notation. The analysis of these narratives lead to the proposed framework, which characterizes three viewpoints (*looking at, with,* and *through* the notation) and interactions between them.

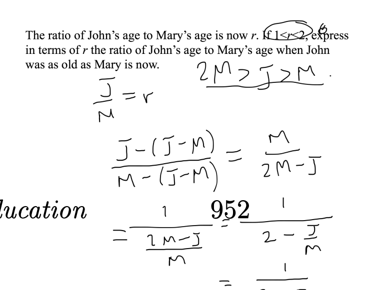
Sample Task Selection and Analysis

Tasks were carefully selected for the interviews, in order to be accessible to a range of participants but to also provide enough complexity so that a participant's approach to notation becomes apparent and an explicit focus of the interview. The Age Ratio task was used for all participants; a brief analysis of the task provides an illustration of overall task selection: *The ratio of John's age to Mary's age is now r. If 1<r<2, express in terms of r the ratio of John's age to Mary's age when John was as old as Mary is now.*

This task presents a concise yet complex set of information. The relationship between the ages now and at a point in the past are defined, but in an abstract manner rather than through specific numeric values. The relevance of the given restriction on the value of *r* is not necessarily immediately apparent. A successful response to the task involves constructing expressions for the ages *then* in terms of the ages *now* so that the new ratio can be expressed in terms of the current ratio. This requires noticing (1) that the point in the past is when John's age was equal to Mary's current age and (2) that the number of years between *now* and *then* is equal to the difference in their current ages. Once an appropriate expression for the ratio is constructed, the choice of manipulations to express this in terms of *r* is not necessarily immediately apparent. Thus, the participant may need to choose between options, and their thinking about such choices can be explored. The resulting expression for the ratio of ages *then* $\left(\frac{1}{2-r}\right)$ can be interpreted in terms of the relationship between the two ratios; the relevance of the restriction on *r* may also now be apparent. A potential difficulty with this task is the assumption that the ratio between ages will remain constant; such an assumption makes expressing the ratio of ages *then* nonsensical.

A mathematician's work

The following data excerpt illustrates the framework through the work of a fluent symbol user on the Age Ratio Task (see Figure 1). M6, a



practicing mathematician, read the task statement and wrote $\frac{J}{M} = r$ as he read. He interpreted the interval $1<r<2$ to mean that John is older than Mary and rewrote this as $2M<J<M$. Without much articulation, he then determined that the difference in their ages is a relevant quantity and can be represented as *J-M*. The ratio of ages "then" is expressed as $\frac{J-(J-M)}{M-(J-M)}$ and simplified to $\frac{M}{2M-J}$; as he simplified, he was pleased that (1) the denominator will be positive within the given restriction on *J* and *M* and (2) John's age then (the numerator) simplified to Mary's age now (*M*). At this point, he evaluated his work in relation to the goal of expressing this ratio in terms of *r*: "I need to manipulate that to be in terms of r, or solve for one of them in terms of the other." He chose to divide through by *M* and obtained $\frac{1}{2-r}$. He was pleased with this result "especially as it agrees nicely" with the given interval for *r*. "Barring typos," he is confident that he has an appropriate solution to the task.

Applying the framework

In terms of the viewpoints, M6 introduced the symbols *J* and *M* to *look with* and record given information. In interpreting the interval in terms of the ages, he demonstrates the ability to *look through* the notation to see a relationship. Determining that the desired ratio can be expressed by subtracting the difference in ages (*J-M*) from each age allows him to once again *look with* notation. Part of this *looking with* includes noting alignment with the context. His statement about needing to manipulate this ratio indicates a shift to *looking at* the notation and making choices related to the goal. Noting that his final expression "agrees nicely" with the interval indicates a tendency to continue to *look with* the notation and check meaning within the task context. It would be possible to consider the relationship between this ratio and the given ratio (*r*), which would be an instance of *looking through* the notation. M6 did not do this, but it was not explicitly required by the task.

In contrast, a less fluent user may be distracted from one viewpoint by another. For example, one student participant (S1) was asked to solve this system of equations for *x* and *y*: $xy=100$ and $(x-5)(y-1)=100$. Within his work, he produced the linear relationship: $x-5y=5$. This was unexpected (he had anticipated being able to solve for either *x* or *y* directly) and prompted him to find the *x*- and *y*-intercepts for this line. When asked if he had solved the system, he expressed surprise that these intercepts are not solutions to the original equations. In this case, the participant's fluency with a known procedure seemed to interfere with his ability to interpret the notation within the context, although he did attempt to *look through* the notation. The intercepts are not solutions to the system, but he did believe that the solutions will lie somewhere on this line. However, he incorrectly connected this to the first equation, stating "I keep coming back to [$xy=100$]. Somewhere along that line, the solution's going to be 100. But I don't know how to find it." When asked how he knows this, he states: "Because I was able to come up with an equation [referring to $x-5y=5$]. There you go." This comment indicates that his observations from *looking at* the notation take precedence over any attempts at *looking through*. Such instances seem related to Arcavi's third issue, the interaction between technical fluency and appropriate interpretation; the triggered known procedure was applied in spite of not having a clear connection to the context.

Implications

Constructing narratives of individuals' work served to refine the viewpoints within the framework. Including both novice (undergraduate students) and expert (mathematician) participants expanded the range of actions described by the framework. Linking the views through the framework emphasizes seeing notation as a tool to support mathematical reasoning. This may sit in contrast to instructional approaches in which manipulations are the primary focus of attention. Time spent developing fluency with specific procedures can strengthen one's ability to recognize and evaluate the potential of particular forms. Without the complementary views of *looking with* and *through* the notation, however, these manipulations can be empty processes. The framework can inform instructional design, in that the views can be incorporated into task selection. As with the interviews in this study, tasks can be evaluated in terms of their potential to provide opportunity for or even require explicit attention to shifts between and coordination of views, thus potentially contributing to the development of symbol sense.

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Classroom Culture, Technology, & Modeling: A Case Study of Students' Engagement with Statistical Ideas

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Advances in technologies have changed the way statisticians do their work, as well as how people receive and process information. The case study presented here follows two groups of two students who participated in a reform-oriented curriculum that utilized technology to engage students with modeling and simulation activities to develop their statistical literacy, thinking, and reasoning. Our analysis applies a social theory of learning and a framework for student engagement as a means for studying students' development of statistical reasoning. In addition, we investigate the impact of a curriculum focused on modeling and simulation on the development of students' statistical reasoning skills.

Key words: Statistics, Engagement, Statistical Reasoning, Technology, TinkerPlots™

Introduction

Today there appears to be a consensus among the statistics education community that approaches to teaching introductory statistics should utilize technology and place emphasis on data and the core concepts of inference rather than on the dissemination of statistical theory (ASA, 2005; Cobb, 1992; 2007, Garfield, delMas, & Zeiffler, 2012). Statistics educators argue that many of the components of our introductory statistics courses (e.g., using a z-score to calculate a 95% confidence interval; computing a standard deviation) are relics dating back to the 1900's historical roots of statistics and need to be reconceived in light of our data-driven, technologically based world (Cobb, 2007; Gould, 2010; Nolan & Lang, 2010). In the hope of aligning curriculum more with the practice of statistics, educators are developing new curriculum (e.g. Garfield et al., 2012, Lock et al., 2013; Tintle et al., 2011) that utilize technology and engage students with modeling and simulation activities to develop students' statistical literacy, thinking, and reasoning. We argue that a better understanding of the ways in which technology and curriculum work together to impact students' development of key statistical ideas is an important next step in statistics education research.

Theoretical Perspective

The authors take the perspective that learning is a result of participation in a classroom community (see for example, Bowers, Cobb & MacClain, 1999; Gresalfi, 2013). Our theory of learning leads us to the view that the curriculum materials, the classroom culture, and the technology all work together to compel students to engage at a critical level. Specifically we focus on a framework of *affordances* for students to *engage* with statistical ideas (see Greeno & Gresalfi, 2008; Gresalfi, 2013). Gresalfi defines affordances as "the set of actions that are made possible by a particular object"; effectivities as "an individual's ability to realize those affordances"; and, "the extent to which an affordance is realized depends on the dynamic *intention* that emerges among elements of the system" (p. 17). She suggests that this framework (affordances, effectivities, and intention) provides a way to document learning (where learning is seen as tied to context and situation). Gresalfi and Barab (2011) use four types of engagement in their work: procedural, conceptual, consequential, and critical. They define procedural engagement as "using procedures accurately" and conceptual engagement as "understanding why an equation works the way it does" (p. 302). Consequential engagement "involves recognizing the usefulness and impact of disciplinary content" and critical engagement "involves questioning the appropriateness of using particular disciplinary procedures for attaining desired ends" (p. 302). They argue that the goal of curricular design and implementation of curricula is to foster consequential and critical engagement so that students use procedures and concepts as tools for investigating problems in meaningful ways.

The research presented here investigates students' reasoning as they engage in modeling and simulation activities while using the Change Agents for Teaching and Learning Statistics (CATALST; see Garfield et al., 2012) curriculum, coupled with TinkerPlots^{TM} technology (Konold & Miller, 2015). In particular, in this report our intention is to study *if* and *how* the classroom culture impacts students' reasoning as small groups are presented with a statistical problem, asked to reason about the context to make conjectures, and then model and simulate the research question using TinkerPlots^{TM} technology. Our overarching research questions are: 1) How do students who receive the CATALST curriculum and use TinkerPlots^{TM} software develop and reason about the viability of their conjectures while engaging in the modeling process? and 2) What aspects of the classroom culture impact students' reasoning about the viability of their initial conjectures?

Methods

Data was collected in an introductory statistics course at a large urban university in the Northwest region of the United States. Students enrolled in this course as a prerequisite for the traditional introductory sequence (descriptive statistics, probability, inferential statistics) or to satisfy the required math elective needed to graduate. A total of 21 students enrolled in the course and all students consented to be participants in the study. Data collection consisted of all student work on in-class activities, video, audio, and screen capture recordings (with a subset of students from the class), and student assessment items.

The third author implemented the CATALST curriculum (Garfield et al., 2012) and TinkerPlots^{TM} technology during the 10-week course. The philosophical stances underlying the development of the CATALST curriculum harmonize with Gresalfi's (2013) affordances for engagement framework. Like Gresalfi's framework, modeling activities are fundamentally designed to provide *strong affordances* for students to *critically engage* with statistical ideas by providing opportunities for modeling, generalizing, and reflecting. To illustrate the alignment of the course materials with Gresalfi's framework we present excerpts from both the video data collected as two groups of students (containing 2 students in each group) reason through the Cereal Box Activity as well as the group work turned in upon completion of the activity.

In the Cereal Box Activity students are asked to model and investigate the number of boxes of Munchy Crunch cereal a person would expect to buy in order to collect all six possible prizes, assuming that the manufacturer placed one of the six possible prizes in each box at random during manufacturing. Once the students are presented with the problem, students are then asked to work in groups and make conjectures about the number of boxes of Munchy Crunch cereal they think would need to be purchased (on average) in order to collect all six prizes. Students were then asked to create a model (both a conceptual model and in Tinkerplots^{TM}) and generate data that would help them answer the statistical research question. Other then this initial information, students were not given any additional directions. The context and open-ended nature of the activity afforded students opportunities for engaging both consequentially and critically with statistical ideas.

Initial Results

Initial analysis of the transcripts of the video data from the two groups identified four primary instances during the activity where students reasoned about their conjectures: 1) when groups formulated their initial conjectures; 2) when the instructor asked them to discuss and explain their original conjectures; 3) after examining the results of a single trial; and 4) after constructing (a plot of) their empirical sampling distribution. To gain a better understanding of how students reasoned when making and evaluating their conjectures and to illustrate the alignment of the course materials with the engagement framework, we present excerpts from two groups' written and recorded video data. The remainder of the results section will be organized according to the instances presented above.

Formulating Initial Conjectures

During their initial conjectures, Group 1 and Group 2 reasoned that a person would need to buy 36 boxes of Munchy Crunch cereal in order to collect all six prizes. An excerpt from Group 1 is presented below to illustrate how they generated and reasoned about their initial conjecture.

Student A: MmHm.
Student B: Not, I think - well obviously you have to buy more than six...
Student A: Yeah.
Student B: To get all six of them. So at least six.
Student A: Maybe it'd be like, since there are six prizes and you'd probably have to get six boxes at least for each prize to rule out one of them. No.
Student B: So thirty-six boxes.
Student A: So like thirty-six. Yeah. That's a lot of cereal. Let's say thirty.
Student B: Thirty?
Student A: Yeah. Or...we can come back to it.
Student B: Okay.
Student A: Let's just say thirty-six question mark.
Student B: Okay.

In the above transcript, we see that the students engaged with one another to begin to reason through and formulate a initial conjecture of 36 boxes, however it is clear that both students are unsure of the initial conjecture and are struggling to articulate their reasoning behind it. We would characterize this response to the task as procedural engagement.

Reasoning About Conjectures While Interacting with the Instructor

After making initial conjectures, both groups had the opportunity to discuss them with the instructor. We believe that interaction with the instructor assisted students in articulating their reasoning behind the original conjecture and even resulted in Group 1 evaluating the validity of their conjecture. When the instructor first joins the students she inquires about their initial conjecture (36 boxes). When prompted to explain their reasoning behind the conjecture Student A offers the following reasoning, "Well I think it's because you have six possible prizes and let's say you have a one out of six chance of getting each prize. So kind of multiplying it on itself makes sense because if you get six boxes of cereal you have the chance of getting at least one different one then the rest of them". Student A's response demonstrates conceptual engagement as she is offering justification to support their original conjecture.

While the students in Group 1 are able to explain that one of the sixes (in their multiplication of six and six to obtain 36 boxes) comes from the number of possible prizes they admit that they are confused as to why they chose to multiply the number of possible prizes by another six. As the instructor and the students continue to discuss the multiplication by six, the students realize that rather than considering how many boxes of Munchy Crunch cereal a *single* person would need to buy to collect all six prizes they were considering how many boxes total *six different* people would need to buy for each of them to collect one prize that was unique from the other five people. In response to this realization, Student B states, "But this is only one person though, right? It's not six people…So it shouldn't have been the six". The discussion between the instructor and Group 1 and the assertion from Student B that the multiplication by six was wrong prompted student to reevaluate their initial conjecture. This provides evidence that the interaction between the instructor and group members assisted students in increasing their level of engagement (from conceptual to critical), as the students are now questioning the appropriateness of the reasoning behind their original conjecture (and therefore the appropriateness of their original conjecture all together).

Reasoning About Initial Conjectures After Examining the Results of a Single Trial

Group 1 showed further evidence of reasoning about their initial conjecture after running and examining the results of a single trial using their sampler in TinkerPlots™. Their single trial produced all six prizes in 11 cereal boxes. Given the results of the single trial, the students revisited their initial conjecture.

Student B: I would say twelve. Between six and twelve.
Student A: Yeah
Student B: Because...
Student A: I mean you need at least six, right?
Student B: Yeah.
Student A: And then, I feel like if you got more...
Student B: Cause the chances of you getting it on the first try are not...
Student A: I don't know. Yeah. The chances of getting like all six different prizes of your first six boxes doesn't make sense. But…
Student B: But two. I feel like if you get two and two and two and two. And then the last one.
Student A: Is like a fifty fifty chance.
Student B: Yeah.
…
Student B: So twelve is a more reasonable number.

The students reason together about the results in a way that leads them to conclude that a more reasonable conjecture would be "between six to twelve" boxes (an interval) because "you need at least six" and "twelve is a more reasonable number" than 36 based on the results they obtained and the reasoning that there is a 50/50 chance of getting a unique prize for each subsequent draw. While this reasoning in not entirely sound the students seemed to believe that the results of the single trial supported their conjecture. Therefore, we assert these students are demonstrating critical engagement because they recognize the need to re-evaluate their initial conjecture in light of the new information obtained and they question the appropriateness of using a single number (rather than an interval) to capture their conjecture.

Reasoning About Conjectures After Constructing an Empirical Sampling Distribution

After creating an empirical sampling distribution of the number of boxes of Munchy Crunch cereal needed to obtain all six prizes, both groups of students showed evidence of evaluating the validity of their conjectures. In the below excerpt from Group 2, we see Student C discussing with the instructor the likelihood of his group's original conjecture.

Instructor: What was your conjecture?
Student C: Um. Thirty six.
Instructor: Would your conjecture be shocking to you?
Student C: Yes, it would actually.

In light of the new information gleaned by constructing and examining the empirical sampling distribution (See Figure 1), Student C recognized that the group's original conjecture would be unusual. While he does not provide explicit reasoning in the above excerpt, further evidence of his reasoning can be seen in their written work. Student C said, "Based on the results of my simulation, I would give a point of estimate of 15. I arrived at this number by using the average function in tinkerplots. This gave me an average of 14.61, which I then rounded up to 15."

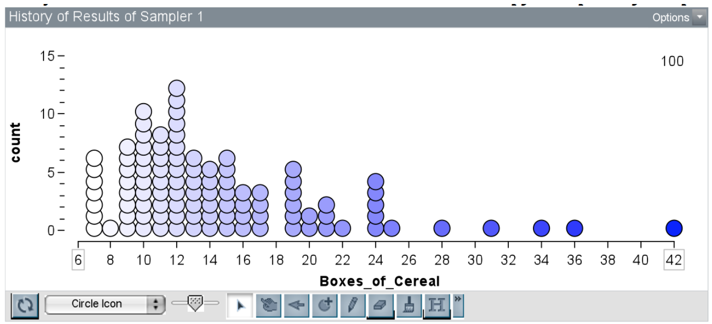

Figure 1. Student C's empirical sampling distribution

In the excerpt presented above and in Student C's written work we see evidence of Student C engaging with the technology, task, and his instructor to determine their original conjecture was unusual and a better estimate would be 15 boxes of cereal, which he based on the mean of the empirical sampling distribution. While selecting the mean may not be most desirable point estimate (since the sampling distribution is right-skewed) we believe that Student C's work demonstrates consequential engagement. Had Student C reasoned about the appropriateness of using the mean (versus another statistical measure) to determine the point estimate then he would have demonstrated critical engagement.

Conclusion

The question posed in the Cereal Box Activity is not a simple one. To arrive at the theoretical answer (14.7 boxes) students would need to have knowledge of formal statistical concepts such as expected value as well as knowledge of the geometric distribution. Although this problem is very complex, the curriculum, the technology, and the classroom culture afforded students the opportunity to generate conjectures about the number of boxes one would need to buy as well as opportunities to evaluate their conjectures and their reasoning. Our analysis provides insights into how the classroom culture impacted these students' reasoning while participating in modeling and simulating the Cereal Box Activity using TinkerPlots™. In particular, analysis of both groups' transcripts provided evidence that while each group was able to initially decide that 36 boxes seemed like a reasonable estimate, neither group was able to fully explain their reasoning behind the selection initially. This result is not entirely surprising given the difficult nature of the task. However, interactions between the group members and instructor, and using technology to explore single-trial results and empirical sampling distributions afforded students the opportunity to reconsider their initial conjectures and engage at a higher level with the statistical concepts. While our analysis is still in the preliminary phases we believe that the above work suggests that the curricular approach focused on modeling, the technology, and the classroom culture appeared to work together in a way that supported these students' engagement with statistical ideas at a consequential and critical level. We assert that this deeper level of engagement resulted in gains in students' statistical reasoning skills.

Questions for the Audience

1) Was the engagement framework useful in analyzing the impact of the classroom culture on the development of students' statistical reasoning skills?
2) Do you foresee any limitations in utilizing this framework as we continue to investigate our research questions?

3) Are there other frameworks that may be more useful in teasing apart the impact of the classroom culture on the development of students' statistical reasoning skills?

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Online calculus homework: The student experience

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The MAA advertises that the online homework system WeBWorK is used successfully at over 700 colleges and universities, and the institution selected for my study has implemented WeBWorK universally across all calculus courses. I used a mixed method approach to examine how students experience online calculus homework in order to provide insights as to how online homework might be improved. In particular, I examined the behaviors, perceptions, and resources associated with online homework. A survey was administered to all students in the mainstream calculus course that provides quantitative information about general trends and informs further questioning. For example, more than half of students reported that they never study calculus with classmates nor in office hours. In tandem with the large survey, I also closely studied the online homework experience of 4 students through screen recordings and interviews.

Key words: Online Homework, Calculus, Study Habits, Student Perceptions, Resources

In the following report, I will describe the importance of understanding how students perceive and experience online calculus homework, based on the fact that calculus is a gateway course for students seeking careers in STEM fields. Then, I will make an argument for the importance of understanding how online homework interacts with students' experiences and perceptions, noting that the literature is sparse in this regard. Finally, I will describe my investigation of students' experience doing online calculus homework. My research describes students' perceptions about online homework and how it interacts with their learning, portrays students' study habits, and identifies resources that are commonly utilized by students while working on online homework.

My motivation for studying this topic arises from witnessing struggles that highly motivated students have in succeeding in their calculus classes. Both as an instructor and as a tutor, I have watched as highly motivated students have spent hours trying to learn calculus by completing their online homework, only to be rewarded with failure on uniform midterm and final exams. It is likely that a large component of learning calculus is tied to doing homework, as is the case for other content areas (Cooper, Robinson, & Patall, 2006), so understanding the mechanisms that relate learning and homework is crucial to understanding how students learn calculus. The literature is fairly sparse in this regard, however, and this study is an effort towards filling some of the gaps. In particular, this study included a qualitative component that attempted to capture data about the student experience, as opposed to student outcomes, because inquiries focused on student outcomes constitute the majority of the research in this area. My purpose is to contribute to an understanding about how doing online calculus homework and learning calculus are connected in order to equip instructors and those who support instructors with knowledge that can be used to improve teaching practices.

Background of the Problem

An enduring challenge of educators across the Unites States is addressing the problem of persistence of students working toward a degree in the science, technology, engineering, and mathematics (STEM) fields. The educational system has struggled to produce graduates in STEM fields, in part, because students who enter postsecondary education intending to pursue an education and career in the STEM fields commonly move away from those career paths during their undergraduate education. It is estimated that between 40 and 60 percent of students who enter postsecondary education with the intention of pursuing a degree in a STEM field will switch their study to a non-STEM field (Bressoud et al., 2014).

It is often assumed that students drop out of degree-granting programs in STEM fields because they are either unable to afford the expense of higher education or because they are unable to succeed academically in the programs that feed STEM fields. To the contrary, poor instruction in mathematics and science courses, especially calculus, is often cited as a primary reason for students' discontinued STEM course taking (Seymour, 2006). Although poor instruction is related to poor academic performance, students who are academically successful still sometimes leave STEM fields because of negative reactions to the instruction and pedagogical styles that they experience in mathematics and science courses. In fact, it has been determined that students—particularly students who have done well in Calculus I—enter Calculus I with the intention of taking Calculus II, but finish their first-semester calculus class with a change of heart. In a study (Ellis, Kelton, & Rasmussen, 2014) that was part of the MAA's Characteristics of Successful Programs in College Calculus (i. e. Bressoud, 2011, 2013), it was reported 15% of students who started the term with the intention of taking Calculus II changed their mind by the end of the term. While it is a possibility that the taking of Calculus I correlates with other course-taking patterns that may play a role in this this behavior, the effect of Calculus I on students' intentions should be more fully understood as we work to solve the issue of student persistence.

Calculus as a Gatekeeper

Of the 69% of students enrolled in Calculus I that expect to continue and take Calculus II, 58% of students are required to do so based on their intended major (Bressoud, 2011). At the university selected for the study, both Calculus I and Calculus II are required to have been completed prior to admission into the College of Engineering. The only other courses that are also required are a first-semester chemistry course, a first-semester physics course, and a first-semester engineering course. Even among the courses from the other three disciplines, Calculus I is an especially important course because it is a both a requirement in itself and is a prerequisite for both Calculus II and the required physics class.

Online Homework

The literature reviewed for this research suggests that the implementation of online homework is unlikely to damage student outcomes (exam scores and course grades), but there is not strong evidence that online homework substantially improves student outcomes (Bonham, Beichner, & Deardorff, 2001; Cheng, Thacker, Cardenas, & Crouch, 2004; Cole & Todd, 2003; Hirsch & Weibel, 2003; Richards-Babb, Drelick, Henry, & Robertson-Honecher, 2011). These findings provide some justification for the implementation of online homework, because online homework appears to "cause no harm" in terms of exam scores and course grades, while still offering substantial affordances in other areas, such as freeing up department resources for other support programs, providing immediate feedback and grading, allowing for individualized homework sets, and promoting the acceptability of mistake-making (Bonham et al., 2001; Burger, 2012; Carpenter & Camp, 2008; Demirci, 2007; Epstein, Epstein, & Brosvic, 2001; Kortemeyer, 2014; Richards-Babb et al., 2011; Zerr, 2007). While it is still important to investigate the effects of online homework in terms of exam scores and grades to further justify its use, the continued propagation of online homework makes it necessary to explore forms of knowledge other than statistical comparisons of student outcomes on traditional measures such as grades and exams. It is important that teaching practices align with the use of online homework and are informed by knowledge about how students experience online homework. For example, my survey data suggests that students are more likely to work alone than in collaborative settings, which may influence how instructors choose to structure class time, perhaps choosing to facilitate more in-class collaboration.

Methods

My research combined a quantitative survey with qualitative observational inquiry to determine general trends in student perceptions, behaviors, and resource uses in tandem with providing a portrayal of the individual student experience with online homework.

Research Questions

1. How do students experience online calculus homework?
    a. What are student perceptions about how online calculus homework supports learning?
    b. What are student behaviors associated with online calculus homework?
    c. What resources do students employ while completing online calculus homework?

Sample

The survey was administered to all students the mainstream calculus course at a large public university and was completed with a 23% response rate. Further analysis will be completed to determine the representativeness of the sample.

Participants for the observational study were solicited from two sections that were deemed typical, based on their being taught by a graduate teaching assistants who had at least one year of teaching experience and had previously participated in a teaching mentoring program. Four participants were selected from a pool of 9 volunteers, with attention given to selecting participants that represented several different stories. All of the participants were freshman with varying backgrounds in terms of experience with AP Calculus, with other calculus classes at the institution, and as repeat students for the mainstream calculus course. Each of the four observational research participants also completed the survey that was administered to the rest of the student population.

The Survey

The survey was designed to gather data about students' demographic information (multiple choice), mathematical backgrounds (multiple choice), perceptions about how online homework supports learning (Likert-type questions), study habits (time estimates), and resource use (frequency estimates).

To analyze the survey data, I have examined the basic distributions of the responses in order to identify general trends in student responses. I have also attempted to uncover relationships between the variables by searching for correlations among survey items. For example, I compared perception of online homework as a useful learning tool to perceptions of the clarity of online homework questions as shown in Figure 1.

Observational Study

For the observational component of my study, I gathered two main forms of data: (a) video recordings of student homework sessions and (b) transcribed, audio-recorded interviews. As a secondary data source, I may draw on notes from informal conversations with the participants, most of which occurred while meeting with students to gather the video files they produced.

The video recordings of homework sessions include two data streams. Screen-recording software (Screencast-O-Matic) was installed on the participants' computers to capture the details of their computer work. The screen-recordings provide details about the exact input that students provide the online homework system, and also captured students online activity outside of the online homework system, such as browsing the internet for support and using online calculators like WolframAlpha. A webcam was used to simultaneously to capture students' real-world activity, but did not capture the same level of detail that was captured by the screen recordings. From the webcam recordings it is possible to identify when students are working with paper-and-pencil during their online homework session, but it is not possible to determine exactly what students are writing from the recordings.

I have thus far developed a coding scheme to analyze basic student behavior elements of the screen-recordings, such as the amount of time spent navigating the online homework interface, working on paper-and-pencil, submitting answers, and working with various learning resources. Deeper layers of analysis can be conducted to identify student strategies following an incorrect submission or student resource usage trends throughout homework sessions, for example. I will seek input from the audience in this regard.

Preliminary Results

Both the survey data and the observational data have proven to be informative through my preliminary analyses. The survey data suggest that students find online homework useful, but believe that written homework, in addition to online homework would help support their learning. Students indicated that several traditional resources (the textbook, the department tutoring center, office hours, and study groups) are largely under-utilized while some newer resources (YouTube, instructional websites, and online calculators) are heavily utilized. The observation and interview data suggests that while students find online homework useful, there are changes in the way that the online homework is administered that may better support student learning, including considering how newer resources may be integrated within the experience.

Survey Results

I found that student responses had responses that were skewed either towards agree/strongly agree or disagree/strongly disagree for questions about the usefulness of online homework, the prospect of adding written homework, the clarity of online homework questions, overall study habits, and resource use. The survey items related to student perceptions of online homework (Table 1) suggest that while students find online homework useful in learning calculus, they may also think that assigning written homework would help them learn calculus more effectively. The results also suggest that the clarity of homework questions may be an issue. The survey items related to study habits (Table 2) suggest that students work primarily alone, and few students substantially utilize classmates, private tutors, the department tutoring center, or office hours. The survey items related to resource use (Table 3) suggest that traditional resources are under-utilized, while some newer resources are heavily utilized. I plan to disaggregate the data to examine response trends and relationships between response items. For example, Figure 1 shows that there appears to be a relationship between the perceived usefulness of online homework and the perceived clarity of online homework questions. After a cursory analysis, the data appears to be less clear for other response items.

Statement	Agree	Disagree
WebWork is a useful tool for learning calculus.	53.2%	29.8%
Assigning written homework, in addition to WebWork, would help you learn calculus more effectively.	51.3%	25.9%
It is easy to determine what WebWork questions are asking.	31.8%	51.9%

Table 1: Responses to Selected Survey Items Related to Perceptions

Study Activity	0 minutes	0-60 minutes	>60 minutes
Alone	3.9%	22.7%	**73.3%**
With classmates	**52.6%**	25.3%	21.4%
Tutoring Center	**42.2%**	27.2%	30.5%
Office Hours	**75.3%**	22.1%	2.5%

Table 2: Time Spent Studying in Various Ways (per week)

Resource	Never	Rarely	Sometimes	Often	Always
Textbook	**45.5%**	**24%**	20.1%	8.4%	1.9%
Tutoring Center	**47.4%**	**18.2%**	13%	13.6%	7.8%
Office hours	**63.6%**	**22.7%**	8.4%	3.9%	1.3%
YouTube Videos	28.6%	17.5%	**29.2%**	**17.5%**	**7.1%**
Online Calculators	14.3%	9.7%	**27.3%**	**28.6%**	**20.1%**

Table 3:Reported Resource Use While Completing Online Homework

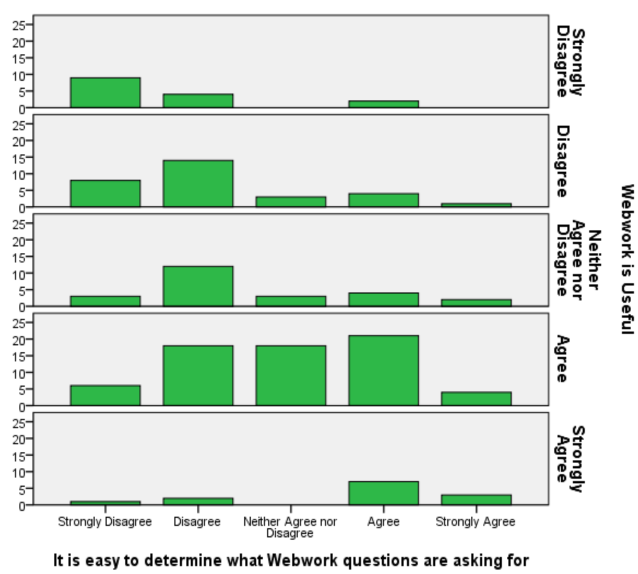

Figure 1:Multi-variate analysis of survey items

Questions for the Audience

1. What are suggestions for making sense of my video and interview data?
2. What other statistical techniques might reveal interesting relationships in the survey data?
3. How can this research be expanded into a more robust research program?

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Student resources pertaining to function and rate of change in differential equations

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While the importance of student understanding of function and rate of change are themes across the research literature in differential equations, few studies have explicitly focused on how student understanding of these two topics grow and interface with each other while students learn differential equations. Extending the perspective of Knowledge in Pieces (diSessa, 1993) to student learning in differential equations, this research explores the resources relating to function and rate of change that students use to solve differential equations tasks. The findings reported herein are part of a larger study in which multiple students enrolled in differential equations were interviewed periodically throughout the semester. The results culminate with two sets of resources a student used relating to function and rate of change and implications for how these concepts may come together to afford an understanding of differential equations.

Key words: Differential Equations, Function, Rate of change, Resources

Differential equations form the foundation of many topics in mathematics, science, and engineering, such as biological modeling, thermodynamics, electromagnetism and fluid dynamics. Due to its central role, students in these majors are often required to successfully complete a differential equations course before enrolling in more advanced topics. The subjects within differential equations, however, present challenges to students by invoking their understanding of familiar concepts and then building on them in unique and conceptually demanding ways. For instance, Rasmussen (2001) noted that understanding certain aspects of differential equations requires a "fundamental leap" (p. 67) in one's thinking. Considering the importance of differential equations and the recent calls for increasing the number of students majoring in STEM fields (Engage to Excel Report, PCAST, 2012), research focusing on student understanding of differential equations is necessary and valuable.

The concepts of function and rate of change are necessary and important for understanding differential equations (Donovan, 2007; Habre, 2000; Keene, 2007; Rasmussen & Blumenfeld, 2007; Rasmussen & King, 2000). These concepts also transcend the subject, present in topics such as existence and uniqueness theorems (Raychaudhuri, 2007), phase planes (Keene, 2007), slope fields, and fundamental sets of solutions (Stephan & Rasmussen, 2002). In this way, function and rate of change are important for understanding what a differential equation is, and have a large impact on student understanding of many mathematical ideas embedded within a differential equations course. Though the importance of these concepts is a theme throughout the research literature, few studies explicitly focus on the role of function and rate of change with regard to student understanding in differential equations. In their review of mathematics education literature, Rasmussen and Wawro (2014) call for research that examines how the ideas of function and rate of change grow and change across a differential equations course. The goal of the research presented in this paper is to characterize how students utilize their notions of function and rate of change in differential equations, how these notions interact with each other, and how these ideas might support students in developing an understanding of differential equations.

Literature Review

The importance of the concept of function is evident in a large portion of the research literature on student learning in differential equations. For instance, Rasmussen (2001) noted students have difficulty conceptualizing solutions as functions in ways that have been documented concerning functions in general. Namely he found that students have difficulties interpreting solutions as functions with graphical representations, interpreting equilibrium solutions as functions, and interpreting the quantities represented by solutions. In her discussion of students' use of time as a parameter, Keene (2007) found that students often reason about solutions in ways that are commensurate with how they reason about functions. Concerning the development a framework for student understanding of the existence and uniqueness theorems, Raychaudhuri (2007) discussed that students' notions of continuity, function, and integration play a significant role in how they interpret and apply the theorems. For example, often times the students believed the coefficients needed to be continuous functions in order for a differential equation to have continuous solutions. A significant challenge for students is developing an understanding of the terms within the differential equation as both variables and functions. Difficult as it may be, however, it has been shown to be immensely important for students' understanding of differential equations (Donovan, 2007; Stephan & Rasmussen, 2002; Whitehead & Rasmussen, 2003), and research has suggested that constructing such an understanding requires putting together images from both concepts to create new ways of reasoning (Whitehead & Rasmussen, 2003).

Student understanding of rate of change has been shown to be connected to the ways in which students reason about various representations of differential equations. For instance, Whitehead and Rasmussen (2003) discuss student use of rate to build images of population, prediction, and function. They noted that students often used rate as a quantity that determined the behavior of a function. With regard to student reasoning with slopes, Stephan and Rasmussen (2002) documented student reasoning with regard to how slopes change over time, slopes of autonomous differential equations being horizontally invariant, and the existence of infinitely many slopes in a slope field. Additionally, it has been suggested that students can reason with rate in ways that promote the construction of new mathematical objects such as straight line solutions (Rasmussen & Blumenfeld, 2007).

Theoretical Perspective

Considering the goals of the research, the interconnected nature of rate of change and function in differential equations, and the way they are utilized to build new mathematical understandings as discussed in the literature, a theoretical perspective that is sensitive to the nature of these concepts is required. The analysis presented here makes use of the epistemological perspective, Knowledge in Pieces (diSessa, 1993; Smith, diSessa & Roschlle, 1993). Within this perspective, knowledge is characterized as a dynamic system of elements and their connections, which is shaped by the learner's interactions with their environment. These elements of knowledge are context specific, in that certain knowledge is associated (to varying degrees) with being useful in certain situations. As such, from the Knowledge in Pieces (KiP) perspective, knowledge elements are either productive or unproductive for accomplishing a certain task within a certain situation. This means the knowledge elements themselves are not evaluated as correct or incorrect (Smith, diSessa & Roschlle, 1993); the evaluation of correctness is only relevant to the application of the elements from an observer's point of view. From a KiP standpoint, learning is characterized as the reorganization, contextualization, and systematization of knowledge elements (diSessa & Sherin, 1998; Wagner, 2006). Studies in which KiP is utilized are often designed for the identification of these elements and the mechanisms that afford their systematization (Adiredja, 2014; Kapon, Ron, Hershkowitz, & Dreyfus, 2015). I contribute to this body of literature by identifying knowledge resources students utilize while completing tasks in differential equations.

There are a multitude of "pieces" used to model a learner's system of knowledge within the KiP perspective. To complete the analysis, I make use of only a few of the various elements existing in the KiP perspective, namely, *knowledge elements*, *knowledge resources*, and *concept projections*. Knowledge elements refer to any one of the various structures within the larger system of knowledge and, as such, vary in size. While generally consistent in nature, the characterizations of knowledge resources found within the literature have been somewhat varied. Broadly speaking, knowledge resources are sets of one or more small-scale pieces and can take on many forms. In general, however, and consistent with the definition posed by Adiredja (2014) and Hammer (2000), here I take knowledge resources to be ideas consisting of small sets of small-scale knowledge elements with a specific use in a particular context. Given the large number of resources students may utilize in service to a single concept, and the context specificity of those resources, the term concept projection (diSessa, 2004; diSessa & Wagner, 2005; Wagner, 2010) is useful when discussing students' understanding of a certain concept. Wagner (2010) defines a concept projection as "a set of particular knowledge resources that enables the knower to attend to and interpret the available information necessary to 'perceive' or 'implement' a concept within a given situation" (p. 450). In general, concept projections provide a way of illuminating the specific ideas about certain concepts that students use when encountering certain tasks. For the purposes of the research presented in this paper, concept projections serve as a way to build on previous findings by identifying how students utilize and build on their ideas about function and rate of change while completing differential equations tasks.

Methods

The research presented here is part of a larger study with the following research questions: What resources concerning function and rate of change do students utilize to complete various differential equations tasks; how do these resources change as the students progress through a differential equations course; and how do students' resources concerning rate of change and function influence one another during the development of their understanding of differential equations? A total of 8 students participated in five one-on-one, task-based, semi-structured interviews (Clement, 2000), each spaced two to three weeks apart. The primary goal of each interview was to engage the participants in tasks centered on topics relevant to differential equations so as to elicit the students' knowledge resources concerning function, rate of change, and differential equations. This paper focuses on a single student's (Dominick) response to one of the tasks (see Figure 1) posed during the second interview and as such briefly addresses the first research question. Dominick was chosen because his responses explicate the interconnected nature of function and rate of change when reasoning about differential equations tasks. The task discussed here was adapted from the inquiry-oriented differential equations curriculum (Rasmussen & Kwon, 2007). The interviews were audio and video recorded and then transcribed. Student generated work was collected as a secondary data source.

Using the transcription, recordings, and student work, resources related to function and rate of change were identified. Recall that resources are small pieces of knowledge that served a productive role in the student's problem solving activity. In general, evidence of the productivity of a piece of knowledge is that the student indeed used that knowledge to attain a (not necessarily correct) solution. The first step in identifying a knowledge resource is determining the meaning behind various parts of the student's arguments (Adiredja, 2014). To accomplish this I inquired into each of the actions and statements the student made while completing the task. More specifically, I tried to determine why these actions and statements were important to the student, and what it was from these actions and statements that he utilized to complete the task. For instance, Dominick noted that the $-2xy$ term "describes the interaction between the two [species] and has a negative effect on the initial state of one of them [species x].” At first glance this statement may seem rather straightforward; however, the various phrases, such as “negative effect,” have complex meanings for Dominick. While inquiring into the meaning of the various parts of the argument, I was also analyzing the attributes Dominick was attending to and ideas Dominick coordinated with them. The above statement includes ideas about the sign of the rate of change, how changes in y affect $\frac{dx}{dt}$, two species with a finite food source, and functions. By trying to uncover what Dominick did, the meaning behind his statements and actions, the ideas he employed to draw his conclusions, and why these ideas were important, the resources he utilized could be identified. The last step is to determine which resources Dominick associated with rate of change and function. This entire process is briefly elaborated on in the analysis section that follows.

In this task, we look at systems of rate of change equations designed to inform us about the future populations for two species that are either competitive (that is, both species are harmed by interaction), or cooperative (that is, both species benefit from interaction, for example bees and flowers). Which system of rate of change equations describes competing species and which system describes cooperative species? Explain your reasoning.

A	**B**
$\frac{dx}{dt} = -5x + 2xy$	$\frac{dx}{dt} = 3x - 2xy$
$\frac{dy}{dt} = -4y + 3xy$	$\frac{dy}{dt} = y - 4xy$

Figure 1: Competing/Cooperative Species Task

Analysis

This was the second time Dominick encountered the task (the first being during his first interview), and he quickly determined System B described competing species and System A described cooperative species. The analysis starts with inquiring into the meaning behind Dominick’s statements and actions. Dominick began to explain how he made the determinations by stating the $-2xy$ term “describes the interaction between the two [species] and has a negative effect on the initial state of one of them [species x].” Specifically he noted that without the interaction, the rate of change of species x by itself would be positive, “but with the competing reaction it takes away from species x.” When asked what it was that indicated a change in the initial state, he replied “so this here [points to $-2xy$], the interaction, is basically the constraint that there is a finite food source and if one species gets so much, then the other species can’t.” He then said that $\frac{dx}{dt}$ is “the rate of change of the population of species x” and that it describes *how* the population changes, while the right hand side of the equation tells you *why* the population is changing. When prompted to elaborate on what he meant, he stated “well, so there is a finite food source and there are only two species pulling from that food source…Species y gets more food than species x, species x will, population will decrease. So, it [the population of species x] will have a negative rate of change.” For Dominick this meant the population of species y is increasing (because it is getting more food) which in turn causes species x to decrease, as indicated by the negative rate of change. This was determined after considering one of his final responses to a question about the relationship between $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in the systems of differential equations. Dominick replied “…if there is a relationship between x and y, then the change in one is definitely going to affect the change in the other, so if the population of x gets exponentially bigger…

that means that x is getting more food… which would cause y to get even smaller and x to get even bigger." Here, for Dominick, "getting more food" means an increase in population.

Identifying the resources Dominick used requires determining what attributes he attended to and how he made use of them. Much of Dominick's argument revolved around the $-2xy$ term. From this we can see that he attended to the $-2xy$ term, interpreting it as "the interaction" between species x and species y. Additionally he attended to $\frac{dx}{dt}$, referring to it as "the rate of change of the population of species x." He concluded that $-2xy$ has a negative effect on the population of species x by coordinating "the change in the initial state" (a decrease in x) with ideas relating negative rate of change to decreases in population. In other words, Dominick considered how "the interaction" affected the value of the rate of change of x and how this in turn affected the value of x. Concluding that the interaction caused the population of species x to decrease, these ideas came together to signify two competing species with a finite food source. It seems that for Dominick the population of species x is decreasing because of the existence of a finite food source from which species y is getting a higher proportion of food. This is the "why it's changing" he made reference to concerning the right hand side of the equation. With these attributes identified, the focus of the discussion shifts toward identifying the resources utilized in the construction of his argument.

To identify the small-scale knowledge elements he coordinated with these attributes to construct his argument, consider his responses to a follow up question that explicitly inquired into how he thought about x, y, $\frac{dx}{dt}$ and $\frac{dy}{dt}$. Dominick noted, "Each one of these variables [x and y] could be functions of time, so how much, like the current population at that time. So how these [x and y] change affects the total rate of change." Dominick was then asked to elaborate on what he meant by x and y being functions and variables. He replied "they are functions so they have a graph, but if you plug in a certain, you plug in their independent variable that is going to give you a number, which you would then plug in for the variable." Describing this procedure, he went on to say " if you plug in t_0 into each y and x, you get x_0 and y_0 and you plug that into this differential equation, you are gonna get the rate of change of x at time t_0." Dominick was then asked what he meant by rate of change, which he explained as the "slope of the line tangent to the curve at the point t_0, x_0" and that it describes how the function is behaving: a positive slope means the function is increasing, a negative slope means the function is decreasing and a zero slope means the function "is transitioning."

His descriptions of x, y, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are connected to the attributes he attended to, as well as various parts of his argument and their meanings. For instance, the combination of ideas indicating that x and y are variables whose values can be substituted into the differential equation to find the rate of change at certain population values, provide insight into how he was able to determine how increases in y affect x. Namely, based on his description of x and y as variables, he was coordinating the effect of larger and larger y values on $\frac{dx}{dt}$. Additionally, Dominick utilized these ideas as he reasoned about the "competing interaction," the "changes in the initial state," and when making conclusions about the finite food source. For example, when he noted that the competing interaction takes away from species x, he considered how different values of y impact the value of $\frac{dx}{dt}$. Here he was treating y as a variable. In other words, he was treating $\frac{dx}{dt}$ as if it was dependent on y. Both of these are small-scale ideas Dominick utilized to complete the task, in other words, resources.

Results

Dominick utilized many resources associated with function and rate of change as he completed the task. These resources were used in coordination with certain attributes from the task, which served as affordances for him. In other words, the attributes of the task that Dominick perceived and attended to both influenced and were influenced by the knowledge resources Dominick had available to him at that particular time. Some of these attributes were terms within the task such as "interaction," terms within the differential equation such as $-2xy$, and the problem situation (two competing or cooperating species). Additionally he was attending to and interpreting these attributes in ways that promoted the coordination of multiple resources related to function and rate of change with these various attributes. By analyzing his argument I was able to construct his concept projections for these constructs (see Figure 2 and Figure 3, respectively). It should be noted that there might be additional resources Dominick utilized while performing the task which are not captured within the concept projections. Therefore they should not be thought of as a representation of the totality of Dominick's thinking during the task.

- The terms x and y are variables that could be functions of time.
- x and y represent the current populations of species x and y respectively, at some time, *t*.
- As time changes, so does the population/function value.
- Functions have graphs.
- Plug in an independent variable (in this case t) and that gives a number (in this case population).

Figure 2: Dominick's concept projection of function

- $\frac{dx}{dt}$ is a change, the rate of change of the population of species x.
- The value of $\frac{dx}{dt}$ is dependent on the value of y.
- Population decreases when the rate of change is negative.
- Rate of change is the slope of the line tangent to the curve at a point on the curve.
- Rate of change tells you how the function behaves. (+) implies increasing (-) implies decreasing.
- The function/variable y affects the rate of change of x.

Figure 3: Dominick's concept projection of rate of change

All of these resources where utilized to enable Dominick to determine which system of equations represented a competing relationship. Namely, these are ideas Dominick saw as useful and productive for completing the task and provided different affordances for him at different times. For instance, the idea that x and y are variables allowed Dominick to consider multiple values for y and what happens when y "gets more food." However, attending to x and y as functions allowed him to reason about $\frac{dx}{dt}$ and $\frac{dy}{dt}$. In short, depending on how he was attending to x and y at the different times he used resources that allowed him to treat x and y as static quantities in some cases and dynamic quantities in others. This is evident for example in his statements about y increasing or "getting more food," and y representing the population value at a certain time. These resources, among others, allowed Dominick to treat $\frac{dx}{dt}$ as a quantity that depended on the values of x and y, to treat x and y as quantities that depended on t (and hence make sense of $\frac{dx}{dt}$), to coordinate changes in y with changes in $\frac{dx}{dt}$, and to utilize the sign of $\frac{dx}{dt}$ to determine the behavior of x.

The concept projections found in Figure 2 and Figure 3, directly address the first research goal. To address the interaction between the resources related to function and rate of change, consider the resources in Figure 4. Each of these resources could be categorized as both resources relating to function or resources relating to rate of change. Additionally, each of these resources seem reasonably related to Dominick's understanding of differential equations. Take for example the resource, "after evaluating $x(t)$ and $y(t)$ at some value t_0, you get x_0 and y_0, this then gets plugged into the DE in place of x and y respectively." This is indicative of an understanding that x and y in the differential equation are both functions and variables, an idea researchers have noted as being key to understanding differential equations (Donovan, 2007; Stephan & Rasmussen, 2002). In other words, Dominick was able to simultaneously coordinate resources relating to function and resources relating to rate of change in ways that afforded him the ability to interpret and implement ideas associated with differential equations to complete the task.

- Changes in the values of the variables x and y affect the rate of change of x.
- After evaluating $x(t)$ and $y(t)$ at some value t_0, you get x_0 and y_0, this then gets plugged into the DE in place of x and y respectively.
- The DE gives you the rate of change at a certain time on $x(t)$.
- The function/variable y affects the rate of change of x.

Figure 4: Resources relating function and rate of change

Conclusions and Implications

The results generated from the analysis of Dominick's reasoning during the task indicate that he utilized many ideas about function and rate of change. Particularly, it is important to note that he was not solely using knowledge strictly pertaining to what a function or rate of change is, he also utilized much knowledge about these concepts as a tool for recognizing and implementing them. For example, the ability to recognize x and y as functions and variables and attend this as a useful piece of information afforded him a productive line of reasoning about "the interaction term." This highlights the importance of knowledge for implementing and utilizing certain concepts with regard to student thinking, and points to the value of including this type of knowledge in the analyses of student learning.

More importantly, the results enlighten why understanding x and y as both variables and functions, and recognizing the differential equation as a function are so important. Dominick's ability to attend to the dependence of $\frac{dx}{dt}$ on y and to coordinate this with the resources related to y being both a variable and a function provided him with powerful ways of reasoning about the systems of equations. Specifically he was able to coordinate changes in the value of x and y with respective changes in the value of $\frac{dx}{dt}$ and $\frac{dy}{dt}$. This formed the basis on which he was able to draw his conclusions.

In light of the importance the research literature places on function and rate of change for understanding differential equations, looking at resources that overlap both concept projections may provide insight into how students construct an understanding of differential equations. The results suggest that supporting the development of students' understanding of differential equations also requires supporting their abilities to attend to relevant attributes and implement mathematical ideas from which differential equations are built.

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Supporting undergraduate teachers' instructional change

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Teaching Inquiry-oriented Mathematics: Establishing Supports (TIMES) is an NSF-funded project designed to study how we can support undergraduate instructors as they implement changes in their instruction. One factor in the disconnect between the development and dissemination of student-centered curricula are the challenges that instructors face as they work to implement these curricular innovations. For instance, researchers investigating mathematicians' efforts to teach in student-centered ways have identified a number of challenges including: developing an understanding of student thinking, planning for and leading whole class discussions, and building on students' solution strategies and contributions. This research suggests a critical component needed to take curricular innovations to scale: supports for instructional change. In this poster we address our current research efforts to support undergraduate teachers' instructional change.

***Keywords**: Instructional support, Inquiry-oriented, instructional change*

Purpose

The main goal of this NSF funded project, Teaching Inquiry-oriented Mathematics: Establishing Supports (TIMES), is to study how to support undergraduate instructors as they implement changes in their instruction. Additional goals of this project are to: 1) understand how best to support undergraduate mathematics instructors in effectively implementing inquiry-oriented instruction, 2) understand the relationships and interactions between instructional supports, instructors, and instructional practices, 3) characterize and measure inquiry-oriented instruction, and 4) assess student learning in inquiry-oriented instructional settings.

Inquiry-Oriented Instruction

We adopt Rasmussen and Kwon's (2007) characterization of inquiry, which applies to the activity of both the students and the instructor. Here, students learn new mathematics by: engaging in cognitively demanding tasks prompting exploration of important mathematical ideas, engaging in mathematical discussions, developing and testing conjectures, and justifying their thinking. Instructor inquiry seeks to reveal students' intuitive and informal ways of reasoning, especially those that can serve as building blocks for more formal ways of reasoning. Instructors inquire into students' emerging ideas to facilitate and support the growth of students' self-generated mathematical ideas. The instructor's role is to guide and direct the mathematical activity of the students by using their reasoning to support the development of new conceptions. The instructor's role is to guide and direct the mathematical activity of the students as they work on tasks by listening to students and using their reasoning to support the development of new conceptions. With an inquiry-oriented instructional approach, instructors: elicit student thinking, build on student thinking, develop a shared understanding, and connect student ideas to standard language and notation.

The Three Curricula

TIMES is centered on three sets of research-based, inquiry-oriented curricular materials being scaled-up for post-calculus undergraduate mathematics courses: linear algebra (e.g. Wawro, Rasmussen, Zandieh, and Larson, 2013), differential equations (Rasmussen, Kwon, Marrongelle, Allen, & Burtch, 2006), and abstract algebra (Larsen, Johnson, Weber, 2013). The instructional design heuristic of Realistic Mathematics Education formed the foundation on which these curricula were developed and, as such, each aims to foster student reinvention of important mathematical concepts (Freudenthal, 1973). To support this reinvention, the curricular materials contain task sequences developed to utilize and build on student reasoning. Such task sequences form the basis of inquiry-oriented instruction.

Instructional support

We currently have a three-pronged instructional support model consisting of curricular support materials, summer workshops, and online instructor work groups. Each of the curricular documents include a set of support materials created by the researchers responsible for developing the respective curricular innovations. These include: student materials (e.g., task sequences, handouts) and instructor support materials (e.g., learning goals and rationales for the tasks, examples of student work, implementation notes). The summer workshops span 2-3 days and have two main goals, 1) building familiarity with the materials, and 2) developing an understanding of the intent of the curricula, and inquiry-oriented instruction. Lastly, during the semester the participants meet in small groups for one hour a week to discuss selected lessons from the curricular materials. For each of the focal lessons, the groups discuss the mathematics embedded in the lesson and plan for implementation. The goal is to help instructors develop their ability to interpret and respond to student thinking in ways that support student learning.

Data

There are currently 18 instructors participating in this project in various universities across the united states. In order to address the research goals, data is being collected from a multitude of sources including: an instructor background survey, video recordings of the summer workshops, post summer workshop surveys, one-on-one interviews with instructors participating in the online working groups, video recordings of the online working group meetings, clips of instruction from the online working groups, video recordings of the participants' instruction, and student content assessments.

Research Progress

Currently TIMES is refining the instructor support materials utilizing data collected from the first group of participants, and leveraging findings from prior work - that indicates early implementers tend to be successful at eliciting but not building on student thinking, whereas whole class discussions facilitated by repeat implementers are statistically more likely to entail both eliciting and building on student thinking. Preliminary analysis suggests that the instructors felt the weekly online working group meetings had the highest impact on their successful implementation of the curricula, but that they desired more experiences discussing examples of how to utilize student thinking in their instruction. Additionally, a preliminary version of the inquiry-oriented instructional measure has been created and will be piloted on the data collected as part of reliability and validity testing. The next steps include continuing the investigation of how to support teachers' instructional change, analyzing the student assessment data (from both inquiry-oriented and more traditional non-IO classrooms), further refining the inquiry-oriented measure and assessing student learning in IO settings.

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Inquiry-oriented instruction: A conceptualization of the instructional the components and practices

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In this paper we provide a characterization of inquiry-oriented instruction. We begin with a description of the roles of the tasks, the students, and the teacher in advancing the mathematical agenda. We then shift our focus to four main instructional components that are central to carrying out these roles: Generating student ways of reasoning, Building on student contributions, Developing a shared understanding, and Connecting to standard mathematical language and notation. Each of these four components is further delineated into a total of eight practices. These practices are defined and exemplified by drawing on the K-16 research literature. As a result, this conceptualization of inquiry-oriented instruction makes connections across research communities and provides a characterization that is not limited to undergraduate, secondary, or elementary mathematics education. The ultimate goal for this work is to serve as a theoretical foundation for a measure of inquiry-oriented instruction.

Key words: Inquiry-oriented, instructional practices, K-16

Rasmussen and Kwon (2007) refer to an inquiry-oriented approach to instruction as one in which "important mathematical ideas and methods emerged from students' problem-solving activities and discussions about their mathematical thinking" (p. 190). Importantly, they state that the students are not the only ones that engage in inquiry. Instead, in inquiry-oriented instruction students inquire into the mathematics and the instructor inquires into student mathematical thinking and reasoning. In this type of instruction the tasks, the students and the teacher work to support the classroom participants in advancing the mathematical agenda. The carefully designed tasks engage students in meaningful mathematical activity that generates student thinking which is then leveraged by the instructor to support the development of more sophisticated mathematics.

In the following section we provide a description of inquiry-oriented instruction by explicating the role of the tasks, the students, and the teacher. We then shift our focus to the components of inquiry-oriented instruction that support the tasks, the students, and the teacher in carrying out their roles. These components will be discussed in relation to relevant K-16 literature, allowing us to draw connections between the RUME and K-12 research communities.

Roles in Inquiry-Oriented Instruction

In inquiry-oriented instruction, the students, task sequence, and the teacher each have an important and interactive role for advancing the mathematical agenda. Here we discuss each of these roles.

Role of the Tasks

Meaningfully designed instructional tasks, regardless the form of instruction, provide a medium through which student mathematical ideas and reasoning can be generated. In inquiry-oriented instruction, tasks are specifically designed to *evoke* informal student strategies and ways of reasoning that can then be leveraged (in subsequent tasks or whole class discussion) to support the development of more formal mathematics (Gravemeijer, 1999; Larsen, 2013; Speer & Wagner, 2009; Rasmussen & Marrongelle, 2006). Instructional activities provide students with an opportunity to re-invent important mathematical ideas by supporting the students in mathematizing both the problem context and their own mathematical activity.

Role of the Student

In an inquiry-oriented classroom students, "learn new mathematics through *inquiry* by engaging in mathematical discussions, posing and following up on conjectures, explaining and justifying their thinking, and solving novel problems" (Rasmussen and Kwon, p. 190). These activities promote the emergence of many important student generated ideas and solution methods which one can think of as providing the mathematical "fodder" available to the teacher for the progression of the mathematical agenda (Speer and Wagner, 2009; Stein et. al., 2008). This fodder is generated through engaging with the mathematical activities that comprise the instructional sequence and by participating in argumentation and justification as students explain their own ways of reasoning and make sense of the reasoning of others. Importantly, by engaging in inquiry and supplying the mathematical fodder for the mathematical agenda, the students assume responsibility for the classroom's mathematics. Indeed, an important goal of inquiry-oriented instruction is for the "learners to come to regard the knowledge they acquire as their own private knowledge, knowledge for which they themselves are responsible" (Gravemeijer & Doorman, 1999, p. 116).

Role of the Teacher

Inquiry serves as an important aspect of the teacher's role. Inquiry-oriented teachers regularly inquire into their students' mathematical thinking and reasoning (Rasmussen & Kwon, 2007). Inquiring into student thinking helps the instructor promote the students' development of a more sophisticated mathematical understanding. In this way, the teacher is a co-participant in the development of the mathematics, in terms of both the mathematics of the moment, and the long and short-term mathematical trajectory intended by the curricula materials (Yackel, Stephan, Rasmussen & Underwood, 2003). Much of the focus on the role of the teacher has emphasized whole class discussions (e.g., Stein et al., 2008, Speer & Wagner, 2009). During these whole class discussions the teacher's aim is to bridge the gap between where the students are and the mathematical goals of the lesson. Specifically, the instructor leverages student ideas to move the students to a more sophisticated mathematical understanding. As noted by Stein et al. (2008) "the role of the teacher during whole-class discussions is to develop and then build on the personal and collective sense-making of students rather than to simply sanction particular approaches as being correct or demonstrate procedures for solving predictable tasks" (p. 315). Given this explication of the roles in inquiry-oriented instruction, we now turn our attention to specific components of instruction that allow these roles to be fulfilled.

Four Instructional Components of Inquiry-Oriented Instruction

Informed by our experiences with inquiry-oriented instruction, a starter-list of components was initially outlined. This list was then refined and explicated through a review of the K-16 research literature. This resulted in the following four components: Generating student ways of reasoning, Building on student contributions, Developing a shared understanding, and Connecting to standard mathematical language and notation. It should be noted that the four components are somewhat artificially separated for the purposes of explication. In actuality, these components are quite intertwined and work together throughout the lesson to support student development of more sophisticated mathematics. To better characterize inquiry-oriented instruction, we delineate each of the components into sets of practices. These practices are grounded in relevant research literature and each set works together to support its respective component. The practices exist at a smaller grain size and provide a high level of detail in terms of how each of the components supports the progression of the mathematical agenda. Some of the practices transcend individual components as they may serve different purposes at different times depending on nature of the component.

Generating Student Ways of Reasoning

In order to utilize student ideas and thinking to move forward the mathematical agenda, the teacher must first have student generated ideas and thinking to work with. Research indicates that eliciting meaningful student contributions requires the teacher to support the production of such contributions (Stein et. al., 2008; Hufferd-Ackles, Fuson & Sherin, 2004). One characteristic of instructors that promote meaningful student contributions is asking questions which drive student investigation of mathematics, support students in explaining their solution strategies, and help the instructor understand students' thinking (Munter, 2014). Questions of this nature require that the students engage in problem solving activity that affords the instructor with opportunities to inquire into student thinking and reasoning.

In inquiry-oriented instruction, purposefully designed tasks are utilized to engage the students in such authentic mathematical activity and lead the students to discover key mathematical ideas (Larsen, 2013; Rasmussen & Marrongelle, 2006; Rasmussen & Kwon, 2007; Speer & Wagner, 2009). The tasks provide a context in which the students engage in mathematical activity, which in turn provides opportunities for the instructor to inquire into student thinking and reasoning. This reasoning can then be used to promote a more sophisticated mathematical understanding. The interaction between the teacher, student and tasks affects the quality of the contributions that can be elicited (Jackson et al., 2013). Jackson et al. (2013) note that the cognitive demand of a task can be lowered depending on how the students are expected to engage with the task or if solution methods are posed before the students begin the task. Their research suggests that, when the cognitive demand of high quality tasks is maintained and when the students are supported in describing the contextual and mathematical features of the task, students are provided with higher quality opportunities to learn.

With this characterization of the practice of *Generating Student Ways of Reasoning,* we have identified three critical components in the literature:

1) *Students are engaged in meaningful tasks and mathematical activity that support the development of important mathematical ideas.* This practice is characterized by student engagement with cognitively demanding tasks, that support students in mathematical activity and are designed to promote ways of thinking about the mathematics that can be leveraged to advance the students' mathematical understanding (Jackson et. al, 2013; Hiebert, 1997; Speer & Wagner, 2009).
2) *Teachers actively inquire into student thinking.* This practice means that instructors purposely and intently inquire into student thinking for the purposes of determining if and how student generated ideas can be utilized to promote a more sophisticated understanding of the mathematics. The questions asked by teachers not only direct student investigations and provide the teacher with insight into student thinking, they also help students refine and reflect on their own thought process (Borko, 2004; Hiebert & Wearne, 1993; Rasmussen & Kwon, 2007). In this way, by inquiring into

student thinking, teachers are able to support students in generating more sophisticated ways of reasoning.

3) *Teachers elicit student thinking and contributions.* Teacher prompt students to explain their reasoning and justify their solution strategies, with the focus on the reasoning the students utilized during the task as opposed to solely focusing on the procedures used. Research on instructional quality indicates that the type of contributions teachers elicit is directly related to the students' opportunities to learn. Thus it is important that teachers elicit thinking and reasoning that "uncover the mathematical thinking behind the answers" (Hufferd-Ackels, Fuson & Sherin, 2004, p.92).

Building on Student Contributions

Researchers have noted that the practice of building on student thinking is quite complex and difficult to implement (e.g., Ball & Cohen, 1999; Sherin, 2002). Leatham, Peterson, Stockero, and Van Zoest (2015) characterize building on student contributions as engaging the class in student-generated contributions in ways that result in developing students' more sophisticated understanding of important mathematical ideas and relationships. To facilitate such building, teachers must elicit and inquire into student contributions to determine which ideas (correct or incorrect) are important and relevant to the development of the mathematics, which ideas can be leveraged to move the understanding of the class toward the goals of the lesson, and then engage the students in each other's contributions in ways that forward the mathematical agenda (Johnson & Larsen, 2012; Leatham et al., 2015; Speer & Wagner, 2009). Building on student thinking in this way requires that the classroom participants create the "mathematical path as they go," (Yackel et, al, 2003, pg. 117), because student contributions form the trajectory along which the mathematics develops (Johnson, 2013). In this way, teachers need to be sensitive to the ideas students contribute and use them to inform the lesson.

Orchestrating class discussions that build to certain educational goals while allowing the students to retain ownership of the mathematics requires that the instructor "slide between being noninterventionist and assuming greater responsibility" (Rasmussen & Marrongelle, 2006, p. 399). In other words, while the students' own ideas form the basis for the mathematics being developed, it is the instructor's responsibility to guide the development of the mathematics toward the mathematical agenda. Inquiry also plays an important part in how teachers carry out this role during the practice of building. By inquiring into student thinking with an eye towards important mathematical ideas, teachers must determine where to position themselves on the continuum between noninterventionist and interventionist. In either case, "'You are still the teacher. The students might not see your teaching. But you are still in control.' However, the nature and degree of control is different in this setting. Instead of controlling the exact content that gets stated in a lecture, the teacher's responsibility is to monitor, select, and sequence student ideas." (Johnson et al., 2013, p. 13).

With this characterization the practice of *Building on Student Contributions,* we have identified five critical components in the literature:

1) *Teachers elicit student thinking and contributions.* Leatham et al. (2015) note that student contributions can provide opportunities for the class to make sense of each other's thinking as well as opportunities for the teacher to build on student thinking. Hufferd-Ackles, Fuson, and Sherin (2004) echo this idea, stating that the "questioning of students allows their responses to enter the classroom's discourse space to be assessed and built on by others" (p. 92).

2) *Teachers actively inquire into student thinking.* Teacher inquiry serves many functions and roles throughout a lesson (see Hufferd-Ackles, Fuson, & Sherin, 2004; Johnson, 2013; Rasmussen & Kwon, 2007). With regard to building on student contributions, teacher inquiry allows teachers to form models of student thinking and understanding, reconsider important mathematical ideas in light of those models, and formulate questions and tasks which enable the students to build on those ideas (Rasmussen & Kwon, 2007).
3) *Teachers are responsive to student thinking and use student contributions to inform the lesson.* Rasmussen and Marrongelle (2006) state that, "an important part of mathematics teaching is responding to student activity, listening to student activity, notating student activity, learning from student activity, and so on" (p. 414). By doing so, the teacher can generate instructional space where "the nature of student mathematical thinking might compel one to take a particular path because of the opportunity it provides at that moment to build on that thinking to further student mathematical understanding" (Leathan et al., 2015, p. 118).
4) *Teachers guide and manage the development of the mathematical agenda.* Teacher need to actively guide and manage the mathematical agenda and can do so by: identifying and sequencing student solutions to "ensure that the discussion advances his or her instructional agenda" (Jackson et al., 2013, p. 648); utilizing *Pedagogical Content Tools* "to connect to student thinking while moving the mathematical agenda forward" (Rasmussen & Marrongelle, 2006, p. 389); or by refocusing the class towards the use of certain student generated ideas, marking important student contributions, and assigning tasks meant to clarify and build on students' ideas/questions. In these ways, teachers can guide and manage the development of the lesson while building on student contributions, developing mathematical ideas in directions commensurate with the mathematical agenda, and maintaining the student ownership of the mathematics.
5) *Students engage in one another's thinking.* Stein and colleagues (2008) provide several examples of how teachers can support students in making mathematical connections between differing student contributions and important mathematical ideas. Some of these examples include asking students to reflect on the contributions of other students, assisting students in drawing connections between the mathematics present in solution strategies and the various representations that may be utilized, and facilitate mathematical discussions about different student approaches for solving a particular problem. Doing so can prompt students to reflect on other students' ideas while evaluating and revising their own (Brendehur & Frykholm, 2000; Engle & Conant, 2002).

Developing a Shared Understanding

As discussed by Stein et al. (2008), "a key challenge that mathematics teachers face in enacting current reforms is to orchestrate whole-class discussions that use students' responses to instructional tasks in ways that advance the *mathematical learning of the whole class*" (p. 312, emphasis added). Within the inquiry-oriented instruction literature base, many articles make use of and highlight the importance of developing a shared understanding (e.g. Stephan & Rasmussen, 2002; Rasmussen, Kwon & Marrongelle, 2008; Rasmussen, Zandieh & Wawro, 2009). For instance, Stephan and Rasmussen (2002) discuss ways in which important mathematical ideas and ways of reasoning, emerging from ideas originating with individual students or small groups of students, become *taken-as-shared* within a classroom. Elaborating on how this occurs, Tabach, Hershkowitz, Rasmussen and Dreyfus (2014) discuss the reflexive relationship between ideas formulated by individuals or small groups and the normative ways of reasoning evident in whole class discussion. Their research suggests that the development of shared understandings supports student construction of mathematics by allowing ideas to be formulated by individuals or small groups and become normative ways of reasoning during whole class discussions. Further, McClain and Cobb (1998) note that supporting the development of taken-as-shared understandings help students with less sophisticated understandings participate in and benefit from whole-class discussions.

The important distinction between *Building on Student Contributions* and *Developing a Shared Understanding,* is characterized by who is making sense of the evolving mathematical agenda: the teacher and a select group of students who have provided the bulk of the contributions, or the classroom community as they develop and co-construct a taken-as-shared understanding. As described by Fredericks (in Johnson et. al., 2013):

> There is this risk that you can pose the problem and then you can have five groups share how they did it and then you can go to the next problem [without any additional discussion of the groups' ideas]. And you can assume that the students will make the connections, and some of them will and some of them won't. I think to really be effective you have to push yourself further than that. That you have to think about what those connections are and you have to make sure that they explicitly come out. Otherwise you don't know who got it and who didn't. You are right back to where you were when you taught the old way. (p. 13-14)

With this characterization the practice of *Developing a Shared Understanding,* we have identified three critical components in the literature. It should be noted that, while these three practices are also important for building on student thinking, their use and purpose is slightly different for developing a shared understanding.

1) *Teachers are responsive to student thinking and use student contributions to inform the lesson.* When teachers are responsive to student contributions they can create new instructional space (Johnson and Larsen, 2012). In regards to this component, the instructional space is created for the purpose of developing a shared understanding within the classroom community.
2) *Students are engaged in one another's thinking.* By engaging with one another's thinking, students are able to deepen their thinking, generate new ideas, and make mathematical connections. As discussed by Jackson et al. (2013), "the teacher plays a crucial role in mediating the communication between students to help them understand each other's explanations" (p. 648).
3) *Teachers guide and manage the development of the mathematical agenda.* Here the focus in on guiding and managing the development of the mathematical agenda *for the whole class.* This involves monitoring and assessing what is taken-as-shared.

Connecting to Standard Mathematical Language and Notation

One of the major tenants of inquiry-oriented instruction is the idea that formal mathematics emerges from students' informal understandings (Gravemeijer, 1999). This is contrasted with more traditional forms of instructions where formal definitions or standards algorithms serve as the starting place for students' mathematical work. However, this does not mean that mathematically standard language and notation have no place in inquiry-oriented instruction. As Stein at el. (2008) discuss, there is an "increasingly recognized dilemma associated with inquiry- and discovery-based approaches to teaching: the challenge of aligning students' developing ideas and methods with the disciplinary ideas that they ultimately are accountable for knowing" (p. 319). One way for a teacher to approach this challenge is to act as a broker "between the entire classroom community and the boarder mathematical community by the insertion of formal convention and terminology" (Rasmussen, Zandieh, Wawro, 2009, p. 201).

With this characterization the practice of *Connecting to Standard Mathematical Language and Notation,* we have identified two critical components in the literature.

1) *Teachers introduce a minimal amount of language and notation prior to students' engagement with a task*. Formal notation is introduced after the students have generated an understanding of what is being notated and a need for it has been established. "In contrast to more traditional teaching in which formal or conventional terminology is often the starting place for students' mathematical work, this teacher [one implementing an inquiry-oriented curriculum] chose to introduce the formal mathematical language only after the underlying idea had essentially been reinvented by the students" (Rasmussen, Zandieh, Wawro, 2009, p. 203)
2) *Teachers support formalizing of student ideas/contributions*. In inquiry-oriented instruction, as the students reinvent the mathematics, their reinventions build to be commensurate with formal mathematical ideas. The instructor must be able to promote the students' ability to connect the their mathematical ideas to more formal mathematics. "The teacher plays a crucial role … in supporting students to link student-generated solution methods to disciplinary methods and important mathematical ideas" (Jackson et al., 2013, p. 648).

Implications

Within the undergraduate mathematics community, the last decade has seen a sharp rise in inquiry-oriented, research based, instructional innovations. Inquiry-oriented instruction is being used in mathematics classes from calculus through abstract algebra. The limited research that does exist on mathematicians teaching practices has shown that these inquiry-oriented curricular materials present a number of challenges for implementation. Such challenges include: developing an understanding of student thinking, planning for and leading whole class discussions, and building on students' solution strategies and contributions (Johnson & Larsen, 2012; Rasmussen & Marrongelle, 2006; Speer & Wagner, 2009; Wagner, Speer, & Rossa, 2007). Given these challenges with the implementation of inquiry-oriented instructional materials, the need for a measure of instructional quality becomes an important way to understand differences in these classrooms.

Before such an instrument can be developed, "inquiry-oriented instruction" first needs to be operationalized in a way that can be observed, measured, and analyzed. The work here contributes to this in two ways: it represents a conceptualization of inquiry-oriented teaching, including the identification of the components and the specification of small-grain practices that support those components, and it can be used to as a theoretical foundation for a measure of inquiry-oriented instruction. Importantly this conceptualization draws on a wide spectrum of literature from the K-16 research base, allowing us to make connections across research communities and provide a characterization that is not limited to undergraduate, secondary, or elementary mathematics education.

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Design research on inquiry-based multivariable calculus:
Focusing on students' argumentation and instructional design

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In this study, researchers design and implement an inquiry-based multivariable calculus course as well as derive the characteristic of instructional interventions for enhancing students' argumentation in proof construction activities. Over the course of 14 weeks, 18 freshmen mathematics education majors participated in this study. Multiple sources of data were collected, students' reasoning in the classroom discussions were analyzed within the Toulmin's argumentation structure, and the instructional interventions were gradually revised according to the iterative cyclic process of the design research. The students' argumentation structures presented in the classroom gradually developed into more complicated forms as the study progressed, and the researchers conclude that the interventions were effective at improving students' arguments.

Key words: Design research, Multivariable calculus, Inquiry based learning, Argumentation, Flipped classroom

One of challenges in undergraduate mathematics classrooms is the shift from traditional teacher-centered and textbook-dominated approaches to new instructional approaches that are student-centered and inquiry-based (Holton, 2001). However, there is a shortage of studies that go beyond basic topics of calculus into areas such as multivariable calculus and differential equations (Rasmussen, 2014). Also, there is a lack of instructional tasks developed for inquiry-based learning (IBL) and a lack of research dealing with classroom interactions and the instructor's role in multivariable calculus teaching/learning. This study attempts to develop an inquiry-based multivariable calculus course and derive the characteristic of instructional interventions for enhancing students' argumentation.

In the fall semester of 2013, a multivariable calculus course for first year students majoring in mathematics education was organized as a flipped classroom at a university in Seoul, Korea. In the flipped classroom, instructors' explanatory lectures can be replaced by online video clips in order to assign more time to student inquiry during the face-to-face in-class sessions. The researchers applied the flipped classroom model to development of inquiry-based multivariable calculus course in order to provide students with opportunities for mathematical inquiry in the classroom as well as instructors' lecture in the online courseware.

In this paper, we focus on the design research methodology based on systematic qualitative analysis that the researchers applied to the development of the course in order to 1) understand the characteristics of students' argumentations in the proof construction activities in the inquiry-based multivariable calculus course, and 2) derive the characteristics of three sites of intervention for enhancing students' arguments: instructional design, classroom interaction, and the instructor's role.

Theoretical Background

Analysis on argumentation structure

Toulmin (1958, 2003) describes argumentation structures using six components for discourse analysis: claim, data, warrant, backing, qualifier, and rebuttal. In the meantime, van

Eemeren and Grootendorst (1992) suggest four types of patterns for argumentation structures: single argumentation, multiple argumentation, coordinate compound argumentation and subordinated argumentation, and Kwon et al. (2013) use these patterns to analyze the argumentation structure of the mathematics classroom, but they combine coordinate compound and subordinated argumentation into compound argumentation. A single argumentation structure includes only one claim and warrant, and a multiple argumentation structure contains a claim supported by more than one warrant. The compound argumentation structure includes a variety of warrants for supporting a claim that induces a new claim. In this study, the researchers adopt the framework consisting of these three argumentation structures to analyze the complexity of students' arguments.

Argumentation in general is understood as a process in which one's opinions are justified, or a discourse in which one convinces others of his/her opinion (Krummheuer, 2007; Wood, 1999). Argumentation can become more complicated when the antagonist reveals an unconvincing part of the given arguments, and the protagonist brings forward more arguments to meet this criticism. Consequently, some arguments may have a single argumentation structure while others have a multiple or compound argumentation structure (van Eemeren et al., 2007). Therefore, a more complicated argumentation structure shows that the students participate in more diverse discursive activities such as suggesting arguments, providing counterarguments, giving additional arguments or refuting counterarguments than a less complicated argumentation structure. In this study, the researchers consider the change in students' argumentation structures from single to compound as an evidence of the improvement of students' argumentation and justification.

Argumentation in mathematical inquiry

Inquiry-Based Learning (IBL) has been implemented in mathematics education in the form of problem-solving, the theory of didactical situations, realistic mathematics education, modeling perspectives, anthropological theory of didactics, and dialogical and critical approaches (Artigue, & Blomhøj, 2013). Since justification or persuasion in argumentation is recognized as being similar to theoretical demonstration in mathematics or mathematical proof, argumentation is considered to be an important part of mathematical learning (Krummheuer, 2007; Staples et al., 2012). According to Richards (1991), inquiry in mathematics is characterized by learning to speak and act mathematically through engaging in mathematical discussions, suggesting reasons, and following the process of solving new and unfamiliar problems. In this sense, Goos (2004) consider learning from an IBL perspective as participation in communities of mathematical inquiry, and Rasmussen et al. (2008) argue that inquiry enables students to learn new mathematics through taking part in genuine argumentation.

The complexity of an argumentation structure depends on the reactions between the arguments of the protagonist and the critical responses of the antagonists. The complexity of the argumentation structure grows as the discussion becomes more active (van Eemeren et al., 2007). Thus, argumentation structure analyses can serve as a quality criterion for mathematical inquiry through proof construction activities in IBL. Considering that learning in IBL is to learn to act and think like a mathematician, students' change of argumentation structure is a proper criterion for the students' learning in IBL. For this purpose, the researchers adopt an empirical approach to study students' arguments in the classroom, and use Toulmin's argumentation structure (1958, 2003) and the classification of argumentation structures suggested by van Eemeren and Grrotendorst (1992) as the frameworks of analysis.

Design research

Design research is appropriate when researchers develop innovative and complex teaching methods to implement unknown principles and guidelines, to quickly test an early model on site, and to refine such a model (Borgman et al., 2008; Kelly, 2009; Nieveen, & Folmer, 2009). Therefore, we adopted the designed research methodology to move them toward the goal of this study: to discover the characteristics of an instructional design that supports students' argumentation through in-class session discussions.

In design research, intervention involves the use of the curriculum, students' learning and teaching strategies, educational materials, and learning environments to improve students' ability to solve complex problems in a real educational context through repeated experiments. After researchers design and implement the interventions, they examine the educational products (e.g. student achievements) to determine whether they are able to answer their research questions. If the research questions are not answered with the current interventions, the researchers reflect on the educational products and improve the interventions (Plomp, 2007). The researchers consider instructional tasks, classroom interactions, and instructor's role as three elements of interventions and derive the features of these interventions from the literature reviews, which serve as a starting point to the iterative design process in this research.

The study aims to derive characteristics of the interventions for the multivariable calculus IBL classroom that induce the development of argumentation structure. This aim is addressed in the following research questions: 1) how do students present their argumentation in proof construction activities in the inquiry-based multivariable calculus course? 2) what are the characteristics of an intervention that improves students' argumentation?

Methodology

Research process

At the preliminary stage, results from previous studies on teaching/learning of mathematical content related to the subject of multivariable calculus, analysis of existing textbooks, and collegiate math education were analyzed in light of the current educational setting. At the design stage, tasks were developed based on each session's objective and content, and the researchers planned interventions for the in-class sessions based on anticipated characteristics on interventions. At the implementation stage, the researchers, consisting of one instructor and three research assistants, played the role of field participants for these in-class sessions. At the reflection stage, the researchers met to debrief on the implementation of the approach and the observation immediately after each in-class session. This approach enabled the gradual improvement of the interventions, and the cyclical process of the design research contributed to the final proposal of the characteristics of instructional interventions for inquiry-based multivariable calculus.

Settings

Over a total period of 14 weeks, the students observed two or three online video lectures (20–30 minutes each) and participated in one face-to-face in-class session (75 min) every week. The class was composed of 18 freshmen majoring in mathematics education majors who had taken the course "Calculus I" as a prerequisite, and a total of five small groups of three or four students each were set up for learner-centered discussions during the in-class sessions. Depending on the task at hand, laptops or tablet computers were provided for the students to use for discussion or problem-solving purposes.

Data collection and analysis

All in-class sessions were video-recorded and the reflection journals written by students after the session were collected. Additionally, a focus group interview with selected students was conducted at the end of the semester in order to complete the triangulation on the analysis. In this paper, the researchers analyze three in-class sessions that focus on mathematical proof construction activities in order to present a detailed account of students' argumentation structures. Two coders transcribed all the utterances of the students and the instructor, and coded the elements of students' discussions according to the components of Toulmin's argumentation structure. Afterwards, they cross-checked the argumentation structures of these components and reviewed the work sheets and the reflection journals in order to validate the results of the analysis. In order to validate the assumptions of the above questions, the researchers compared the Hypothetical Argumentation Structure (HAS) with the actual implemented argumentation structure and derived the characteristics of interventions by refining them in each cycle.

Result

The students' argumentation structures presented in the in-class sessions gradually developed into more complicated forms as the study progressed, and the researchers conclude that the interventions were effective at improving students' arguments.

Phase 1

The aim of the week's in-class session was to provide students with the opportunity to observe whether the symmetry of partial derivatives holds for two functions *f* and *g* and to examine several aspects of the functions, such as graphs, limits, and continuity, in order to inquire about the conditions that would satisfy the property. In the in-class session, however, the students could not reach the final step, in which they were to suggest their own conjectures about the symmetry of partial derivatives. In some steps, students had difficulties constructing their arguments as the researchers had intended, and the instructor had to directly convey certain mathematical knowledge to students that they were expected to be able to derive themselves. Finally, students could not performe well in the last two steps of the task, and the argumentation structure was also different from what the researchers had expected (Figure 1).

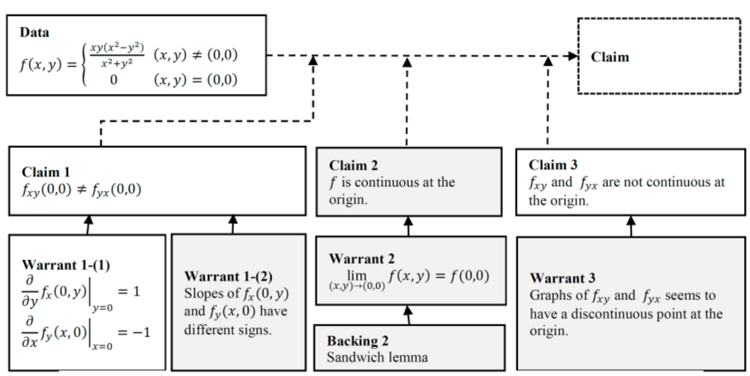

Figure 1: Student's argumentation structure in Phase 1

In the Figure 1, a solid line is used to represent stages of argumentation that students performed well and a dotted line is used to link parts of the students' argumentation that did not occur in the in-class session; shaded regions indicate parts that the researchers did not anticipate in the design stage or had to change spontaneously during the in-class sessions.

Phase 2

At the end of the in-class session in phase 1, the instructor had explicitly presented Young's theorem and the above lemma and asked students to suggest how it could be proved and to complete the proof of Young's theorem in their reflection journals using the MVT. Student S2 proposed an argument using the MVT twice, and the researchers decided to begin the discussion of how to prove Young's theorem in the fourth in-class session by sharing her idea with her peers. The researchers anticipated that during the session, students would point out some of the problems with S2's proof.

Students proposed three different ways, including S2's proof mentioned above. All proposals were based on the same idea, namely exhibiting the difference in terms of the function *D*(*x*,*y*) and to determine when the concept of limit should be used in the proof. During the whole-class discussion, a multiple argumentation structure focusing on showing the validity of each proof and on comparison between them was observed (Figure 2).

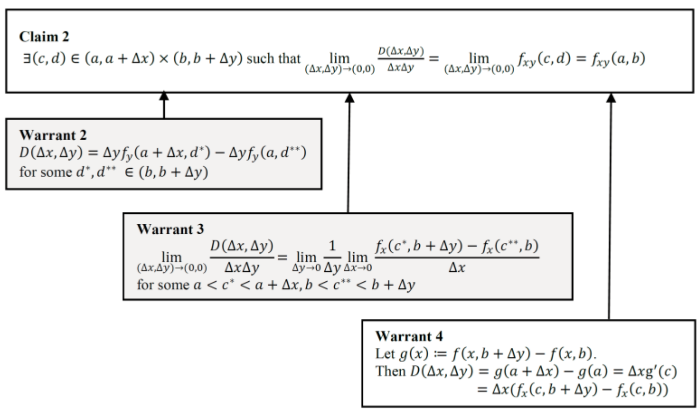

Figure 2: Student's argumentation structure in Phase 2

In this session, the more complicated task of proving Young's theorem was proposed, and a task sequence was implemented beginning with an incomplete solution. It seems that this approach — posing a relatively difficult question by incorporating a suggested idea — was more effective than simply providing students with the idea on its own without a specific starting point. By explicitly revealing the controversial point in the proof, the tasks enabled students to suggest multiple warrants for one claim in each small-group discussion, causing the whole-class discussion to result in a multiple argumentation.

Phase 3

In vector calculus, conservative vector fields can be defined in different ways, and most textbooks introduce the definition with several equivalent statements. The task asked students to prove that a potential function exists if the value of line integration is independent of the curve when the starting point and the terminal points are fixed. Researchers design the sequence of the task to construct a new function and examine the function to ensure that it satisfies the definition of potential functions. Although the instructor showed part of the proof to students in the online session to reduce their burden with this unfamiliar and complex task and to improve their concentration, she didn't provide students with individual steps to the proof. In other words, students need to find strategies to develop proofs by themselves.

In this session, the students' proof construction activity was implemented as expected in the HAS, but the instructor had to provide students with scaffolds to help them reach certain sub-claims. Therefore, the students' argumentation structure appeared in the form of the compound argumentation, but showed a slight difference in the shaded regions of the HAS. The shaded regions indicate the instructor's active engagement in the discussion (Figure 3).

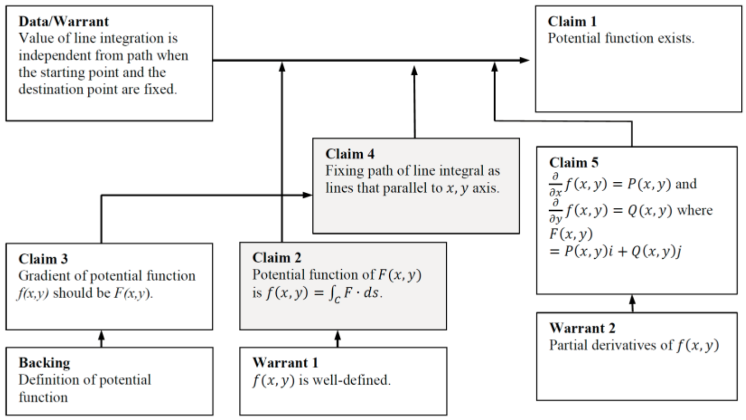

Figure 3: Student's argumentation structure in Phase 3

The main goal of the task in this session was to find and specify new ideas to accurately advance and complete the proof. While the task was described relatively clearly, it was difficult for students as it demanded several complex sub-claims and warrants, and promoted more elaborated arguments. Also, it led to active small-group discussions and required the instructor's engagements and discussions between small-groups. Therefore, the task contributed to the appropriate environment for IBL so that the students can construct the desired compound argumentation.

Conclusion

The students' argumentation structures presented in the sessions gradually developed into more complicated forms as the study progressed. That is, the structures transformed from single argumentation to multiple argumentation and compound argumentation structures as the interventions changed. The revised interventions employed in Phase 3 can exemplify the characteristics of interventions that are effective at changing argumentation structures.

Instructional tasks consist of sub-claim-based questions that can be used to provide students with room for inquiry to solve each question and to motivate them to take ownership of the entire proof construction process. Each question should be set at an appropriate level of difficulty in order to promote students' mathematical inquiry with discussion, and it should also provide the necessary prior knowledge, skills, and crucial idea required to help them find a valid orientation to their inquiry. Incomplete, but improvable solutions suggested by students can induce active student participation.

Classroom interaction should have a flexible structure consisting of within-small-group discussions, between-small-group discussions, and whole-class discussions. Students are encouraged to participate in whole-class discussions after sharing opinions with each other in small groups and reaching a similar degree of understanding.

The instructor should encourage students to argue for their ideas even when they could not definitively convince their peers of the validity of those ideas. The instructor should consistently monitor the discussions and take appropriate actions to indirectly guide students in the right direction in constructing their argumentations. In the environment with flexible interaction structures and open-ended questions, the instructor should re-organize classroom interactions and the tasks according to the students' progress observed in the discussions.

The iterative application and improvement of the interventions acquired in this study provided students with a structure in which they could participate more actively in whole-class discussions, while the instructor, who directed them to productively construct knowledge, played the role of facilitator of discourse. In addition, given a lack of explanatory lectures, the students were able to solve inquiry-based tasks in small groups and draw conclusions regarding the solutions in whole-class discussions during in-class sessions. Overall, the students participated responsibly and productively in knowledge construction and learning, as confirmed by the gradual development of their argumentation into more complex structures.

Discussion

In this study, the researchers focus on the design products and the design principles in the inquiry based multivariable calculus course, which are derived from the systematic qualitative analysis on students' reasoning in argumentation. The complexity of the students' argumentation structure serves as the quality criterion for optimizing the interventions in the IBL multivariable calculus course. The systematic qualitative analysis based on the well-established theoretical framework contributes to the methodology of this research, which assures the effectiveness of the design products from the empirical data.

The design research methodology thus made a clear contribution to the development of a multivariable calculus course based on the flipped classroom model and the pursuit of IBL. Using a cyclic process of design research, researchers design, implement, and reflect on the curriculum and instruction in order to validate their assumptions about three instructional interventions based on evidence from practice. This implies that design research can be beneficial to the many instructors who have troubles designing effective instructions without sacrificing the quality of education due to a lack of well-established design principles or practical guidelines.

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A case study of a mathematic teacher educator's use of technology

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The use of technology in mathematics classrooms remains an important focus in mathematics education due to the proliferation of technology in society and a lag in the implementation of technology in classrooms. In this paper, I present data from clinical interviews with a mathematics teacher educator (MTE) and observations from that MTE's class in order to discuss his use of technology. Specifically, I describe three themes that emerged from the MTE's technology use and how they relate to his epistemological stance. These themes are: (a) his developing a classroom environment around the use of technology, (b) technology providing a precise and dynamic environment, and (c) his using technology to help engender students' mental imagery. Finally, I discuss how the ideas emerging from this paper can be helpful for the mathematics education community.

Key words: Technology, Mathematics Teacher Educators, Epistemology

"The use of technology to study mathematics has changed the very nature of the mathematics we are studying … [T]eaching and learning methods will need to be regularly reconceptualized to take advantage of the power of modern technology to improve mathematics education" (Leung, 2013, p. 523). In other words, technology provides us a way to transform the content we teach and the ways of thinking students construct. Instructional methods must be updated to include the tools we have available to us in ways that help students construct meaningful and sophisticated mathematics. In this paper, I expand upon Leung's notion: teaching and learning methods need to be regularly reconceptualized to take advantage of powerful technological tools. Specifically, I describe one mathematics teacher educator's (MTE) conceptualization and use of technology in a pre- and in-service education course focused on middle and high school mathematics content. I first provide relevant background information to this study and describe the methods used to collect and analyze data. I then provide background on the epistemological stance of the MTE and describe some of the main themes emerging from the MTE's use of technology. Finally, I close the proposal by discussing some productive takeaways from investigating the MTE.

Background

Over the past two decades, the proliferation of technology has been expansive and the use of technology in the classroom has been increasing as well (Kaput, 1992; Zbiek et al., 2007). This societal change raises several questions about how emerging technology can be implemented to support student learning. Pea (1985) argued that technology should be used as more than just an amplifier (i.e., making things faster and/or easier); technology should be used as a tool for reorganizing thinking. He posited, "I take as axiomatic that intelligence is not a quality of the mind alone, but a product of the relation between mental structures and the tools of the intellect provided by the culture" (p. 168). He also pointed out that much could be missed if we confine ourselves to defining technology as just a cognitive amplifier. Echoed by Leung's quote above, Pea argued that not only does the use of technology allow us to do things "faster" and more efficiently, it also changes the way that we engage in tasks.

Speaking to specific ways in which technology can influence the way we engage in tasks, Kaput (1992) differentiated extensively between traditional and dynamic computer media. Dynamic media offer a situation in which variation is easy to achieve. Geometer's Sketchpad

[GSP] (Jackiw, 2006) provides an example of a software program where variation is easy to achieve. Using GSP, a user can create sketches and then drag/modify parts of the sketch around the screen to see a variety of possibilities. This is much more tedious in a pen-and-paper drawing of the situation, as that medium requires the user to draw each instantiation. A program like GSP provides students opportunities to “see” such concepts as variation because instead of being constrained to static media and instantiations of variation, they can “experience” variation in experiential time. Repeated experiences with such activities can help students construct variation cognitively (Thompson, Byerley, & Hatfield, 2013).

Several researchers have conducted studies on how technology can help engender *student* learning in various content areas (e.g., Tall, 1986; Sinclair & Robutti, 2013; Geiger et al., 2012; Heid, Thomas, & Zbiek, 2013). However, there is still a need for studies that focus on how *teachers* implement technology and how their epistemological stance effects that implementation. Some researchers (e.g., Grossman, 1991; Margerum-Leys & Marx, 2002; Mishra & Koehler, 2006) have studied different variations of this issue. Specifically, Niess (2005) discussed the construct of technology pedagogical content knowledge (TPCK), in which teachers must develop an overarching conception of their subject matter with respect to technology and what it means to teach with technology. There is still much work to be done with respect understanding how teachers, including mathematics teacher educators, are using technology in their classrooms. This paper is an attempt to fill part of this void.

Subject, Setting, and Methods

The main participant in this study is a mathematics teacher educator (hereafter referred to as the MTE) employed as a professor in the mathematics education program of a large university in the southeastern U.S. The participant is an experienced member of the faculty at that university, working there for over 40 years. I chose the MTE strategically as someone who would provide me an opportunity to gain insights into his epistemology and how it shapes his teaching. I also wanted to choose an MTE who was teaching a class in which student learning and secondary mathematics content topics would be a main focus.

The study took place around the MTE’s class, which met weekly for 2 hours and 45 minutes. There were 15 people in this class and they consisted of both Master and Doctoral students who had a variety of teaching experience (from only student teaching to currently teaching). These students normally formed 4 groups and worked together on many of the tasks during the semester. The majority of these tasks were completed on GSP. The class, which is labeled as a class on curriculum, focuses on student learning of many different mathematical content areas in middle and high school. The syllabus of the class includes the following topics: multiplication, counting/combinatorics, measurement and oriented quantities, rate of change, Pythagorean theorem and square roots, quadratics, parabolas. The observation period for this study consisted of the classes covering the latter four topics.

The study included a pre-interview, lasting about 90 minutes, in which I probed the MTE’s views on teaching, the use of technology, and student learning. The crux of the data in the study originates from an eight-week observation (eight class sessions) of the MTE’s class. During each of those eight weeks, I: (a) interviewed the MTE before class, (b) observed the MTE during the class, and (c) interviewed the MTE after class. The interview before class, which lasted between 30 and 60 minutes, provided a lens into the MTE’s intentions for that class. The interview after class, also lasting between 30 and 60 minutes, allowed the MTE to provide his thoughts on events in the previous class. In some instances, I showed the MTE video clips from my observation to frame my questions. After the observation period completed, I conducted a 90-minute post-interview with the MTE during which I probed the

MTE's use of technology throughout the semester as well as asked some questions on his epistemological stance. All interviews conducted were semi-structured (Roulston, 2010).

I video- and audio-recorded all interviews and observations and digitized them after each event. Field notes were also taken during the observation sessions and screen-recording software was used to capture work done on the computer during those sessions. I analyzed the data following an open and axial coding approach (Corbin & Strauss, 2008). First, I watched the interview and observation videos for any utterances or actions by the MTE that stood out. Such utterances included specific mentions about how and why he would (or did) use technology, or instances in which he described his stance on knowing and learning. Actions targeted included the MTE's implementation of technology and interactions with students when using the technology. Then, I identified any themes emerging from the data to provide insights into the MTE's technology use. As I developed these themes, I repeated the analysis to revise or further clarify a theme if needed.

Results

I focus on three themes that emerged from the interviews and observations with the MTE. These themes are: *MTE developing a classroom environment around the use of technology, technology providing a precise and dynamic environment, and MTE using technology to help engender students' mental imagery*. Although these themes are listed separately, they are connected and each tied to the MTE's epistemological stance. Therefore, before providing an explanation of the themes, I describe aspects of the MTE's theoretical perspective that provide cohesiveness to these themes. First, the MTE identifies as a radical constructivist. As a radical constructivist, he follows von Glasersfeld's (1989) tenets that (a) "knowledge is not passively received, but actively built up by the cognizing subject" (p. 164) and (b) "The function of cognition is adaptive and serves in the organization of the experiential world, not the discovery of ontological reality" (p. 164). Taking this perspective means the MTE believes students cannot be passive receivers of mathematics as he understands it. Instead, they must construct a personally viable mathematics that is idiosyncratic and fundamentally unknowable to another. This stance will be evident throughout the description of the themes.

The MTE expressed his epistemological stance many times during the study. One example arose during a class post-interview. I showed him a clip from the class observation in which he said, "you can't take anything for granted. When you're teaching math you can't take your mathematics for granted." He clarified this statement in the interview (Excerpt 1).

Excerpt 1. The MTE's describes not taking your mathematics for granted as a teacher

MTE: What I mean by that is, so many teachers take their math- the way their thinking mathematically as that what should be learned … They take as a given what they believe is valued in their culture and sometimes that's- you know- what people say is valued in society. Their representation of that, that's what's taken for granted and I'm saying you can't do that, you shouldn't do that because the students have their own culture. Students have their own ways of thinking and you don't know what those ways of thinking are until you start working with them … Being a teacher, the primary challenge of being a teacher is to learn students' mathematics. To learn how they think and take those into consideration.

The MTE's words provide insights into the importance he places on (a) working closely with students and (b) helping teachers understand the significance of building models of students' mathematics. Although later in the interaction he explained that building models of students' mathematics is something that is very difficult to accomplish, he maintained that all teachers must try to accomplish this goal in order to productively support students' learning. Paying attention to how students learn shapes his teaching and consequently the way in which he implements technology. Dialogue in the post-course interview provides an example of his attention to student learning when planning to use GSP (see Excerpt 2).

Excerpt 2. Using GSP to help students learn.

MTE: Forcing yourself to work in GSP, although it's extremely detailed, I think brings out … the ways that you think and you almost- you have to start being more explicit in how you put things together mathematically in GSP. That's one of the basic reasons I want to use that. That's one of the basic reasons. Another basic reason is, is that when you think about the construction of a mathematical reality … This notion of a mathematical reality with the way you implement your thinking in GSP- that- I think it changes how you think mathematically and it changes your ways of thinking about the nature of mathematics.

His belief that GSP can be used to change how students think mathematically and how they perceive mathematics was pervasive throughout the observations of his teaching and the interviews. I discuss some examples in the description of the three themes below.

MTE Developing a classroom environment around the use of technology

An important part of the MTE's teaching when using technology was the emergence of a welcoming and mathematically productive environment shared by him and the students. To solidify this point during one of the interviews, the MTE said, "I want the students to develop a sense of ownership over the class so it's not [The MTE's] class, it's our class." Through this co-ownership of the class, the MTE hoped students would be actively engaged in tasks and discussing the mathematics with other students and the instructor. In this way, he hoped students would not only engage in their own mathematics, but they would also have to think outside of their own mathematics, considering those around them and the instructor as well.

Throughout the course of the observation period, students were grouped together and asked to work on tasks requiring the use of GSP. In class, the MTE stated, "I believe in my students working independently side-by-side. Does that make sense? So it's interdependent." I asked him to clarify what he meant by having his students working interdependently in a post-class interview. His explanation is found in Excerpt 3.

Excerpt 3. The MTE explains working interdependently.

MTE: Working together in groups of 2 or 3 provides them with uh, ways of- one student has an insight they explain it to the other students, the other students interpret and they may be able to assimilate what the other student is doing but yet by those assimilations be able to modify what- how they're operating and understand how they're operating but it also could provide this person that's there making the explanation- it gives them a way to uh, learn to communicate with other people mathematically and start carrying on mathematical conversations.

Such conversations were the norm, not the exception during my observation of the class. To understand why he feels such communication is important, we can go back to his epistemological stance. In the process of assimilating what another person has done, a student needs to consider another's mathematics and how it might relate to one's own mathematics.

Another aspect of the MTE developing a classroom was the idea of having students present and discuss their work. There were very few instances in which the MTE lectured to the class for an extended period of time. Instead, he let the students take the lead in explaining the mathematics, allowing students to examine their (and other students') thinking more explicitly. An interesting discussion resulted from this strategy. After the MTE mimicked a student's directions on a construction, one student discovered an error in the construction and said, "So the way it's written there shouldn't it be *x* and *y* instead of *x*-prime and *y*-prime?" The MTE replied to the student by saying, "well I'm just doing what you told me. I'm your robot." This exchange provides evidence that he intended the students to take ownership of the class, both in its discourse and in its mathematical focus. He mimicked the actions suggested by the students and allowed them to discover the error on their own.

As the MTE continued to call on students to show their work, I asked him to explain how he decided which students to call on. One reason for his decision can be found in Excerpt 4.

Excerpt 4. The MTE explains his decisions on call students to the front.

MTE: One of my goals for the course is that they become explicitly aware of how they think … I want them to become more explicit with the way they are thinking and start explaining how they are thinking. But it's also important that I put people up there who have solutions so- so the people- so the class- oh my classmate's solving this problem. And that's really important that I understand that the classmate's solving the problem that makes it more possible for me and I'll work harder. It's a matter of energizing the students and setting goals so they become goal directed in their mathematical activity.

Like in the other examples in this section, the MTE described the importance of considering others' mathematics when working on a task. In the same interaction, the MTE discussed other ways he decided who would present solutions. One notable way was selecting someone who has not taught but will teach in the future in order to get them comfortable (in his words "building a social confidence") with communicating mathematics in front of a class.

Technology providing a dynamic and precise environment

Over the duration of the observation period, it became apparent that the MTE used GSP for both its dynamic and its precise nature. By dynamic, I mean that one can create a sketch, click on a point, and drag it around in ways determined by how the sketch was constructed. Similarly, aspects of the sketch can be animated. This allows students to see the changes in experiential time rather than requiring students to redraw every possible shape, as is the case in pen-and-paper constructions. Precise refers to the ability to, for example, create a circle and know that the shape created is representative of a circle (and not subject to the error of human hands). Although it is true the limitations of computers (e.g., the screen is made up of pixels) do create some error, such error is visually minimal. One cannot easily, by hand, draw a line with length of the square root of a non-square number. But, GSP allows students to do provide a near exact representation through a construction in which the students are hands-on.

Specifically, the MTE exploited the dynamism of GSP by developing constructions with the intention of helping students develop proving skills. One example occurred when the class was tasked with showing something is true for all cases. For instance, the class was asked to justify that the inscribed shape in figure 1 is a square. The students were then asked to drag the points of the inscribed square along the sides of the outer square. Is the shape still a square? Will it always be a square? The MTE explained the affordances technology provided in proof building, "In GSP, you can move this thing. Now you know that you can see all cases … it seemed to be visually palatable that it was true for all cases." Although he says later that a GSP construction does not represent a complete proof, it can help students generalize. A teacher only showing the base case and the inductive step would be taking his own mathematics for granted and perhaps ignoring the mathematics of students who have not yet constructed a scheme for generalization.

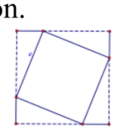

Figure 1. Square inscribed within another square

For similar reasons, the MTE felt the preciseness of the software was important in teaching his class. Specifically he felt that precision was pivotal in the unit covering square roots. I asked him to expand on the reasons GSP was important for that unit and how it compared to previous units. His response can be found in Excerpt 5.

Excerpt 5. The importance of preciseness

MTE: It's more essential to do these constructions because GSP offers you the power of being extremely precise. OK, and it does in the context of the other problem sets as well. … I can't do this easily without GSP. I really can't. Using GSP is much more essential here with the precision it offers you and the opportunities it offers you. So how can you construct a segment whose length is precisely the square root of two long, unless you go to GSP? You can't do it. I can't do it!

The preciseness was important for the MTE in other areas of the course as well. For example, the students were tasked to prove two lines were perpendicular (they were constructed as perpendicular in GSP). To do so, they needed to find the slope of both lines. The MTE said, "That was a difficult problem so GSP was a beautiful medium- I could have written it on the whiteboard but it wouldn't have been quite so dramatic because it wouldn't have been as precise. GSP provides you something that's precise." Through this precision, the MTE hoped that the students would be forced to become more explicit in their thinking and the use of GSP would perhaps change the very nature of the way they think of perpendicular lines.

MTE using technology to help engender students' mental imagery

The third theme emerging from the data, which relates to the dynamic affordances of technology, is the MTE's use of technology to help students develop mental imagery. This was an idea that permeated through the observation period. The MTE hoped to have students construct an imagery of the situation through use of the technology. This does not mean just remembering pictures. Instead the MTE wants the student to be able to reconstruct the image of the situation, including its dynamic nature, in his mind. In doing so, the MTE hopes this re-presentation adds constructive building blocks to the students' mathematical reality so the student can operate mathematically on similar situations in the future.

A specific example where helping students create and operate on imagery was an explicit goal occurred during the unit on rate of change. The unit relied heavily on reasoning about quantities and proportions (many tasks involved conversions between monetary units). The MTE discusses engendering students' mental imagery for rate in Excerpt 6.

Excerpt 6. Rates and imagery

MTE: One of the aspects of the rate scheme is the dynamic imagery. That you can regenerate the operations. So when this when-uh- this point moves [*referring to a point on a sketch*]. Well if you think about that as a motion, okay. Which we do. Well if we think about it conceptually if I can regenerate these two, the pounds and the dollars in relationship, then I- we're talking about mental operations. So what the motion becomes if I can regenerate the motion and produce the motion mentally, those are conceptual operations and no longer just a physical motion. And so that transformation is something I can't do for the students but that's what I'm trying to engender is this dynamic imagery.

The MTE's statement fits with the epistemological stance that students cannot passively receive knowledge. The teacher must be cognizant of students' mathematics in order to provide tasks and experiences that engender mathematical reasoning.

There are several other examples of attempts by the MTE to engender dynamic imagery that occur during the class periods (quadratic functions using area model, oriented quantities, etc.). One of the more dramatic examples in the class was the construction of the parabola.

The MTE showed them an animation in GSP that, when put in motion, created the envelope of a parabola. This animation not only provided a stimulating image but also showed the step-by-step construction. The class sounded a collective "oh!" during the first run of this animation. Although the "oh!" is nice, the MTE's goal was for the students to operate on this imagery and productively re-present in a situation without the technology. In the post-course interview, the MTE expounded upon the importance of mental imagery (see Excerpt 7).

Excerpt 7. The importance of mental imagery.

MTE: By using GSP, you can develop this concept of mental imagery. So, in which I really think is important, you know, in all aspects of mathematics. This is something quantitative reasoning really has going for it because imagery is really-is really foundational. But it's just not imagery. It's how you operate on your imagery okay. And so you gotta spend a lot of time developing the imagery and how you operate on imagery.

Later in the interaction, the MTE relates this development of a mental imagery to his epistemological idea that a student creates his own mathematical reality. The construction of various imagery and the operations on that imagery help construct the students' reality.

Discussion

At the beginning of this paper, I quoted Leung (2013) and agreed with the idea that using technology can transform the very content that is being taught. I contend that the MTE discussed in this paper provides an example case of where his conceptualization of and the students' engagement with the technology transformed the content being taught and also the way in which it was taught including the environment of the classroom. The MTE made several considerations in his decision to use technology. Like Kaput (1992) described, technology can be used in dynamic ways. The MTE used the dynamic nature of GSP to help students generalize situations with infinite cases. Similarly, in conjunction with Pea (1985), the MTE attempted to use technology as a cognitive reorganizer by using it to help engender students' mental imagery. The MTE intended for students to construct imagery in their minds and then operate on it in a way they can re-present later in a situation absent the technology.

Another important takeaway from this study is the importance of a teacher's epistemological stance on the way in which he or she implements technology. If the teacher does not hold that student knowledge is actively built, that person is unlikely to see the importance in using technology as a way to engender dynamic imagery (for example). Nor are they likely to consider student interactions essential for student learning, as they may not have considered the importance of others' mathematics. Although this may seem like a fairly obvious contention, I believe that one must consider a teacher's stance before making any pedagogical recommendations. The writers of standards and recommended practices may tell teachers that using technology is important but if the teachers have not considered how students learn at any depth, then it would be hard to expect them to implement the practices in the way the writers intend. Thus, attention to how students learn should be of utmost importance in teacher preparation classes (as it was in this MTE's classroom).

Finally, the study has implications on the community of MTEs, a group of people that has been studied minimally compared to other populations in the mathematics education field. MTEs are responsible for preparing PSTs and thus determining how to provide them with experiences that focus on ways of thinking important to the teaching and learning of mathematics is a pressing area of need. An MTE's use of technology is a model for the PSTs he teaches. This paper provides examples of the different considerations MTEs must make when using technology.

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Lacking confidence and resources despite having value: A potential explanation for learning goals and instructional tasks used in undergraduate mathematics courses for prospective secondary teachers

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In this paper, I report on an interview-based study of 9 mathematicians to investigate the process of choosing tasks for undergraduate mathematics courses for prospective secondary teachers. Participants were asked to prioritize complementary learning goals and tasks for an undergraduate mathematics course for prospective secondary teachers and to rate their confidence in their ability to teach with those tasks and goals. While the mathematicians largely valued task types and goals that mathematics education researchers have proposed to be beneficial for such courses, the mathematicians also largely expressed lack of confidence in their ability to teach with these task types and goals. Expectancy-value theory, in combination with these findings, is proposed as one account of why, despite consensus about broad aims of mathematical preparation for secondary teaching, these aims may be inconsistent with learning opportunities afforded by actual tasks and goals used.

Key words: secondary teacher education, mathematicians' instructional dispositions

Each year, many prospective secondary teachers are enrolled in undergraduate programs intended to prepare them to apply mathematical knowledge to their future teaching practice. The field has called for improving these programs, including teachers' mathematical preparation. In many programs, mathematicians teach the mathematics courses for these programs. However, there are few studies of how mathematicians teach (Speer, Smith, & Horvath, 2010), including how mathematicians make instructional decisions.

Scholars in teacher education have argued that mathematical knowledge for teaching develops through reasoning mathematically in ways that interact with pedagogical considerations, and that such reasoning should play a prominent role in teachers' preparation and continual development (e.g., Ball, 2000; Gallimore & Stigler, 2003; Mason & Davis, 2013; Shulman, 1986). Prospective teachers, as those who have not yet accrued experience teaching their own class, are unlikely to be able to contextualize mathematical knowledge into how it would apply to teaching. Thus from prospective teachers' viewpoint, even ostensibly useful knowledge may seem irrelevant to future practice and they therefore may not be invested in learning—a viewpoint that has been shown in multiple studies of prospective secondary teachers (e.g., Goulding, Rodd, & Hatch, 2003) Tasks that are "practice-based" (Ball & Bass, 2003)—those that engage the doer in mathematical reasoning situated in a pedagogical context provided—can potentially bridge this disconnect. By engaging in such tasks, pre-service teachers could apply mathematics in ways that are authentic to the demands of teaching (Stylianides & Stylianides, 2014; Ball, 2000). Moreover, practicing teachers' achievement on assessments using these tasks correlates positively with their student outcomes and teaching quality (Baumert et al., 2010; Hill, Rowan, & Ball, 2005; Hill et al., 2008; Rockoff, Jacob, Kane, & Staiger, 2011). Practice-based tasks, then, could play a potentially powerful role in the mathematical preparation of teachers by giving prospective teachers a window into teaching that engages them in mathematics with which they may otherwise not engage.

As the first part of the full paper will elaborate, such tasks do appear to be common to specialized courses[1] for elementary but not secondary level teaching. Given prospective secondary teachers' perception of their mathematics coursework as irrelevant, the reported study investigated: *Which task types used in specialized courses for secondary teaching are prioritized by mathematicians who may teach them, why, and for what purposes?*

The goal of the study is to identify reasons why practice-based tasks may have had a slower adoption in specialized courses for secondary level teaching as compared to elementary. This small scale study was designed to elicit potential reasons by conducting think-aloud interviews with mathematicians (n = 9), in which the mathematicians were asked to prioritize tasks and goals for use in a specialized course for secondary level teaching. The results of the study are four hypotheses that to be examined in a future, larger scale study:

1. Mathematicians generally value practice-based tasks but lack confidence in using practice-based tasks for specialized courses for secondary level teaching.
2. Mathematicians are generally more confident about teaching tasks from a secondary from an advanced perspective than practice-based tasks, even if they may value it less than practice-based tasks.
3. The confidence of a mathematician for using practice-based tasks is mediated by perceived access to resources where practice-based tasks are paired with pedagogical guidance about questions or prompts to use with prospective teachers.
4. Mathematicians frame programmatic goals in terms of assessment and lesson-level or task-level goals in terms of instruction.

These hypotheses suggest a potential reason why practice-based tasks are not commonly integrated into specialized courses at the secondary level, and this reason runs contrary to the idea that mathematicians, due to their training in the discipline of mathematicians, may simply value discipline-based more than practice-based problems. Instead, practice-based tasks may not be common because mathematicians may not feel that they can adequately teach or design such tasks, even though they would like to be able to. Additionally, the hypotheses are significant in that investigating them may explain why, despite the appearance of consensus about the programmatic aims of mathematics teacher education as evidenced by policy documents co-written by leaders of mathematics and mathematics education (CBMS, 2001; CBMS, 2012), the aims may not be coherent with the learning opportunities afforded by tasks and goals used in practice. If broad aims, tasks, and lesson-level goals are not consistent, it will be hard to improve mathematical preparation for secondary teaching in any substantive way. I take up this issue in the conclusion.

Rationale for Interview Design and Relation to Literature

I took the perspective that instructors use tasks to accomplish particular goals. Because specialized course goals are likely to be based on ideas about mathematics and teaching, and goal attainment in general is influenced by a number of cognitive and affective factors, the study design drew from literature in mathematics teacher education and cognitive science.

The role of practice-based tasks in mathematics teacher education

Practice-based tasks. I use the phrase "practice-based" in reference to Ball and Bass's (2003) description of mathematical knowledge for teaching as a "practice-based" theory. Practice-based mathematics tasks, of which examples include those used in the Learning

[1] In this paper, I use the term *specialized courses* to refer to courses designed primarily for prospective teachers, which are intended to address mathematics broadly useful for teaching a grade band within K-12 mathematics.

Mathematics for Teaching (LMT, 2008) and COACTIV (Baumert et al., 2010) instruments, are those for which successful performance on the tasks require mathematical reasoning based on inferences about the pedagogical context (Hill, Schilling, & Ball, 2004; Lai, Jacobson, & Thames, 2013). Pedagogical context refers to elements of teaching and learning provided by the task, including the purpose of an embedded teacher or information about students. Task (d) in Table 1 (d) is an example of a practice-based task.

Tasks potentially used in specialized courses for secondary teaching include tasks addressing secondary mathematics from an advanced standpoint, secondary mathematics with connections to tertiary mathematics, practice-based contexts, and common content knowledge. These are the four task types used in the interviews. The first three represent goals for specialized courses for secondary teaching as described in the guiding document *The Mathematical Education of Teachers II* (CBMS, 2012). The last type, common content knowledge (Ball, Thames, & Phelps, 2008), represents proficiency (NRC, 2001) at secondary level content. Examples of each task type used in the study are provided in Table 1.

Table 1. Task types and examples

(a) *Secondary mathematics from an advanced standpoint* Suppose $x \neq 0$. Prove that $x^0 = 1$. You may use the additive law of exponents ($a^{b+c} = a^b a^c$ for all $a \in \mathbb{R}$, $b, c \geq 0$, and $b, c \in \mathbb{Z}$) and the definition that $a^1 = a$ for all $a \in \mathbb{R}$	(b) *Secondary mathematics with connections to tertiary mathematics*[2] During a lesson on exponentiation, Ms. Waller's students came across the expression $\left((-4)^{\frac{1}{2}}\right)^2$. Two students obtained different answers when they tried to evaluate this expression. **Anna**: *I got* -4. *I started with* $(-4)^{\frac{1}{2}} = \sqrt{-4}$. *And* $\sqrt{-4} = 2i$. *So* $(2i)^2 = 4i^2 = -4$, *and so* $\left((-4)^{\frac{1}{2}}\right)^2 = -4$. **Brenda**: *My answer was* 4. *I did* $\left((-4)^{\frac{1}{2}}\right)^2 = (-4)^{\frac{1}{2}\cdot 2} = (-4)^{2\cdot\frac{1}{2}} = \left((-4)^2\right)^{\frac{1}{2}} = 16^{\frac{1}{2}} = \sqrt{16} = 4$. Explain the apparent contradiction between Anna's and Brenda's answers in terms of a multi-valued exponential function.
(c)*Practice-based*[3] Ms. Madison wants to pick one example from the previous day's homework on simplifying radicals to review at the beginning of today's class. Which of the following radicals is best for setting up a discussion about different solution paths for simplifying radical expressions? 1. $\sqrt{54}$ 2. $\sqrt{72}$ 3. $\sqrt{120}$ 4. $\sqrt{124}$ 5. Each of them would work equally well. Explain your reasoning.	(d) *Common content knowledge* Find three different pairs of functions g and h such that $g \circ h = (x+3)^2.$

Common content knowledge was included because it is necessary for teaching, and also to represent the viewpoint that teachers only need to be able to do the mathematics their

[2] Adapted from an instrument developed by the Educational Testing Service © 2013, with permission

[3] Source: Thames (2006), p. 6.

students need to learn to do. Although this viewpoint is not often expressed in the teacher education literature, it nonetheless implicitly or explicitly presents itself in our culture. I see this viewpoint as underlying the design of studies on teacher effectiveness using teachers' SAT or ACT as a proxy for knowledge (e.g., Rockoff, Jacob, Kane, & Staiger, 2011).

Note that Task (b) in Table 1 might be thought of as practice-based, but it is classified first as a connections to tertiary task because of its reference to complex analysis which is not a secondary topic. Moreover, the level of inference about pedagogical context needed for the task is arguably less than that of the example practice-based task.

Practice-based tasks in prospective teachers' development of mathematical knowledge for teaching. Practice-based tasks may play an especially critical role in teacher preparation. Multiple researchers have commented on the potential of *practicing* secondary teachers to learn and apply mathematics from mathematics courses and tasks that do not provide pedagogical context, and to see these mathematical experiences as relevant to their teaching (e.g., Watson, 2008; Thompson, Carlson, & Silverman, 2008; Kleickmann et al., 2013). Yet *prospective* secondary teachers' documented perception of the irrelevance of their undergraduate mathematical experiences (Goulding, Rodd, & Hatch, 2003; Ticknor, 2012; Wasserman, Villaneuva, Mejia-Ramos, & Weber, 2015) suggests that even if the tasks they worked on drew on relevant mathematics, a different approach or at least supplement to teaching and learning is needed in order for the tasks to influence thinking during and outside of class (Doyle, 1988). Practice-based tasks, by situating mathematics in teaching, could play such a role (Stylianides & Stylianides, 2014; Ball, 2000). I am not arguing that all tasks in specialized courses should be practice-based but rather than some tasks should be, and that the tasks should be tightly connected to the mathematical theory developed, whether the theory is from an advanced standpoint, or with connections to tertiary mathematics, or an alternative that somehow connects to secondary mathematics teaching.

The availability of practice-based tasks integrated into the mathematical goals of a specialized course is greater at the elementary and middle levels than secondary level. As elaborated in the full paper, evidence for this assertion includes an analysis of the content and tasks of textbooks commonly used in and policy documents guiding the curriculum for specialized courses (e.g., Bassarear, 2011; Beckmann, 2003; Bremigan, Bremigan, & Lorch, 2011; CBMS 2012; Parker & Baldridge, 2004; Sultan & Artzt, 2011; Usiskin, Peressini, Marchisotto, & Stanley, 2003). Tasks provided were identified as practice-based or not, and the key mathematical knowledge needed for the tasks were compared to the knowledge from theorems or explicitly stated mathematical goals of the section of the chapter they were contained in. Textbooks and policy documents for specialized courses for elementary and middle grades teaching incorporate practice-based tasks. On the other hand, for specialized courses for secondary teaching, the most commonly used textbooks do so less centrally. When these textbooks do incorporate practice-based tasks, the tasks do not tightly connect to a mathematical theory being developed, and so can be treated as asides rather than a central part of the course.

Cognitive and affective factors influencing goal attainment

Broadly speaking, many studies have shown that a person's success in attaining a goal is strongly shaped by how much the person values the goal intrinsically, the person's confidence that they could attain the goal, and the quality of the person's ability to conceive of implementation intentions (statements of the form "If X happens, then I will do goal-attaining behavior Y"). (See Eccles and Wigfield (2002)'s review of research on the effects of motivational beliefs and values on goal attainment, and Gollwitzer and Sheeran (2006)'s review on the effect of implementation intention on goal attainment).

To represent "confidence", I used the notion of expectancy, that is, a person's belief about how well they will do at a task (Atkinson, 1964), as used in Eccles and colleagues' extensively validated expectancy-value theory that relative value and perceived probability of success influence achievement-related choices (e.g., Eccles, 1983; Eccles, Wigfield, Harold, & Blumenfeld, 1993). The phrasing of this study's interview questions on expectancy and value were adapted from those described in Eccles, Wigfield, Harold, and Blumenfeld (1993). To represent capacity for implementation intention, the interview design included asking mathematicians to articulate anticipating prospective teacher thinking on a task and how the mathematicians would respond in order to move the teachers toward a particular learning goal. While a separate study is planned for examining these implementation intentions, statements regarding expectancy and value expressed during this interview portion were used in the analysis for the present study.

Data, Interview Design, and Method

Mathematicians who self-reported as "having taught a course designed primarily for prospective secondary teachers or would be interested in teaching such a course were the opportunity made available" were recruited for the study via a national network of US mathematicians interested in mathematics education. Interviews of 9 mathematicians were conducted, each approximately 90 minutes in length. The mathematicians were located in 6 different states, had between 0-12 years of teaching specialized courses for secondary teaching, and 0-10 years of teaching specialized courses for elementary teaching. All mathematicians had previously taught or were teaching prospective or practicing teachers.

Each interview included these five parts: (a) Task Goal Sort (b) Goal Sort (c) Task Sort (e) Overarching Goal Sort (e) Wish List. In Task Goal Sort, mathematicians were asked to prioritize learning goals for prospective teachers in the context of using a particular task, and to describe what specifically they would anticipate prospective teachers thinking, how they would know, and how they would respond so as to move the class toward the intended goal. In Goal Sort, mathematicians were then asked to prioritize the goals for "how important are these goals for mathematical preparation for secondary teaching", independent of the task. Table 2 describes these goals and task. In Task Sort, mathematicians were presented with a set of 6 tasks and asked to prioritize them for "how well each task represents what secondary teachers should learn in their mathematical preparation". Table 1 provides a sample of 4 of the tasks used. The task types represented the set presented to each mathematician were: practice-based (Table 1c), tertiary connections situated in a pedagogical context (Table 1b), secondary from advanced standpoint (Table 1a), a variant whose mathematics matched the advanced standpoint task but situated in a pedagogical context, another secondary from advanced standpoint task addressing different mathematics and also situated in a pedagogical context, and common content knowledge (Table 1d). The Overarching Goal Sort used the same prompt as the Goal Sort with generic goals that paralleled those in the Goal Sort, also shown in Table 2. In Wish List, mathematicians were asked to describe the resources they felt they would need to "get better at teaching courses designed primarily for prospective teachers.

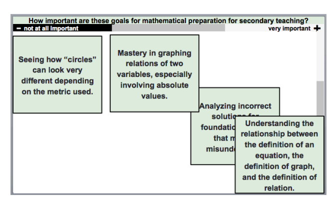


Fig 1. Card sort interface

In Goal Sort, Task Sort, and Overarching Goal Sort, mathematicians expressed their prioritizations by "sorting" the cards containing the goals and tasks horizontally, where more to the left/right meant lower/higher priority. Figure 1 shows a picture of this interface. They were then asked to sort the cards vertically by expectancy, where lower/higher meant "less/more confident that, if asked, that you could create or learn to create opportunities for teachers to do well at [these kinds of tasks/this goal].” Cards could overlap. Mathematicians trained on the interface by placing the cards “do math while drinking coffee” and “make mathematical puns” horizontally and vertically where more to the left/right meant “enjoy less/more” and lower/higher meant “less/more confident that, if asked, you could create or learn to create opportunities for fellow mathematicians to [do math while drinking coffee/make mathematical puns]”. Most mathematicians placed the coffee card at the very top, and the puns card on the very bottom. This activity was to ensure that study participants understood the notion of expectancy and that leftmost/rightmost and upmost/downmost represented extremes.

For each card for each participant, cards were assigned horizontal and vertical coordinates with values between 1 and 5 based on the approximate location of the center of the card as placed by the participant. Horizontal coordinates represented value and vertical represented expectancy. Interview transcripts were chunked into statements of beliefs, reasons, goals, and resources. The collection of statements were analyzed for themes (Glaser & Strauss, 1967). Patterns noted in card placements were triangulated with interview statements.

Table 2. Task and goals sorted by participants

Task Goal Sort/Goal Sort	*Overarching Goal Sort*
Understanding the relationship between the definition of an equation, the definition of graph, and the definition of relation.	Connecting ideas from higher mathematics to secondary mathematics
Seeing how “circles” can look very different depending on the metric used.	Experiencing secondary mathematics as a rigorous, challenging, coherent body of mathematics.
Analyzing incorrect solutions for foundational ideas that may be misunderstood.	Analyzing mathematical teaching situations
Mastery in graphing relations of two variables, especially involving absolute values.	Ensuring that teachers themselves would be able to do the problems that they are responsible for teaching K-12 students how to do.
Description of Task used in Task Goal Sort [4]	
Which of the following best shows the graph of $|x| + |y| = 6$? (a) [picture of a circle] (b) [diamond] (c) [shaded triangle in quadrant I] (d) [square] (e)[shape similar to a four pointed hypocycloid]	

Results

I summarize the logic of how the results generated the hypotheses described in the beginning of this proposal, with more elaboration in the full paper.

Hypotheses 1 and 2: *Mathematicians generally value practice-based tasks and goals but lack confidence in using practice-based task sand goals for specialized courses for secondary level teaching. Mathematicians are generally more confident about teaching tasks from tertiary connections and advanced viewpoint than practice-based tasks, even if they may value them less than practice-based tasks.* These hypotheses are supported by the general trend that practice-based goals and tasks were generally placed more right than down (below the 45^{o} line), representing higher value and lower expectancy than other types of tasks; and advanced standpoint and tertiary connections tasks were generally more left than up (above the 45^{o} line). Table 3 shows scatterplots of card sort placements.

Hypothesis 3. The confidence of a mathematician for using practice-based tasks is mediated by perceived access to resources where practice-based tasks are paired with pedagogical guidance about questions or prompts to use with prospective teachers. This hypothesis emerged from themes in statements about resources made by mathematicians, and

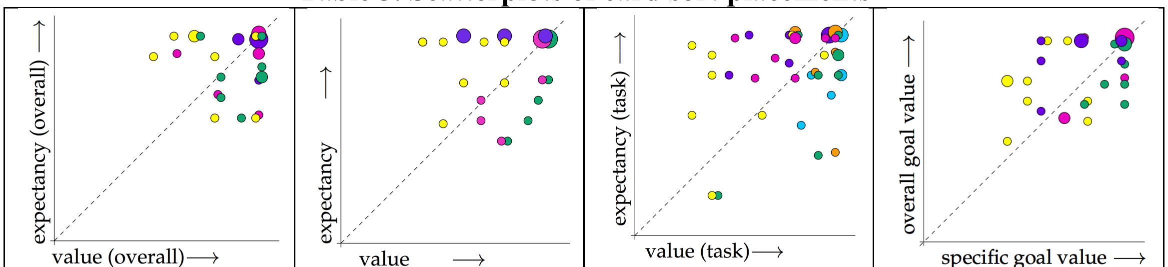
[4] Source: Begle (1972), p. 42

the statements about expectancy made by the mathematicians who expressed relatively higher expectancy about practice-based tasks.

Hypothesis 4. Mathematicians frame programmatic goals in terms of assessment and lesson-level or task-level goals in terms of instruction. This hypothesis was generated by comparing values ascribed to parallel goals in Goal Sort and Overarching Goal Sort. When looking over the interview chunks for reasons and beliefs concerning the placement of overarching goals, participants who placed types in Overarching Goal Sort differently than in the Goal Sort, tended to, in the Overarching Goal Sort, bring up the theme of assessment. That is, they appeared to frame programmatic goals in terms of certifying knowledge and more specific goals in terms of instruction; not one participant who placed them differently brought up the difference explicitly suggesting that the differing frames of assessment and instruction may not have been adopted deliberately.

Limitations of the study. As the findings of this study are based on a small scale study with limited examples from each type, the findings at most suggest hypotheses that bear examination in larger scale studies. Alternative explanations may account for the findings. For instance that participants happened to prefer or not prefer the particular examples of specific goals and tasks, but had other examples or variants of the goals and tasks been used, then the results may have been different. However, there are ways in which the findings are consistent with other literature. For example, if Hypothesis 2 is true, expectancy-value theory would predict that many specialized courses would be characterized by tertiary connections and an advanced standpoint, corroborating Murray and Star (2013).

Table 3. Scatterplots of card sort placements



Overarching Goal Sort Value vs. Expectancy	Goal Sort Value vs. Expectancy	Task Sort Value vs. Expectancy	Values in Goal Sort vs. Overarching Goal Sort
Key: Green = Practice Based, Yellow = Tertiary Connections, Pink = Advanced Standpoint, Purple = Common Content Knowledge. Larger circles represent more participants placing the card in that approximate location.			

Implications

The main aim of specialized courses is to prepare teachers to learn and apply mathematical knowledge to their future teaching. The CBMS (2012) policy document takes this position, signifying broad agreement in this aim. Practice-based tasks could play an important role in carrying this aim to fruition, but are not being used. The findings of this study suggest the uncommonness of practice-based tasks in specialized courses for secondary teaching is not explained by the idea that mathematicians do not value practice-based tasks. In fact, almost all participants remarked unprompted on the value of "tasks like the ones on the colored cards" that had practice-based elements, and almost all participants mentioned a wish for a repository of such tasks. The lack of practice-based tasks may be better explained by mathematicians' lack of confidence in using, accessing, and designing practice-based tasks. However, it is an open question as to what such a resource would look like and how it would be indexed.

Another implication of this work has to do with how discussion of goals actually drives programmatic and instructional decisions. If there is a difference in how parallel goals are prioritized when thinking about them on the program level and on the course level, then actions taken are likely to be incoherent. Increased awareness may be needed for the frames used in discussion and be clear when we are discussing overall certification or moment-to-moment instructional decisions.

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The graphical representation of an optimizing function

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Optimization problems in first semester calculus present many challenges for students. In particular, students are required to draw on previously learned content and integrate it with new calculus concepts and techniques. While this can be done correctly without considering the graphical representation of such an optimizing function, we argue that consistently considering the graphical representation provides the students with tools for better understanding and developing their optimization problem-solving process. We examine seven students' concept images of the optimizing function, specifically focusing on the graphical representation, and consider how this influences their problem-solving activities.

Key words: Calculus, Optimization, Function, Graphs

Word problems are notoriously challenging for students - not a surprise to anyone who has ever been a student or an instructor of mathematics. Slightly obscuring the mathematics involved by describing a situation using everyday "nonmathematical" language adds an extra level of difficulty. In first semester calculus, students have many opportunities to solve word problems. Here we examine optimization problems, which require students to read a short description of a scenario in which a quantity needs to be maximized or minimized. This quantity may be an area, a volume, a cost, a distance, an amount of material, or a production output. To solve the problem, the student must construct a function for this quantity (we call this the *optimizing function*) and then use calculus techniques to find the absolute maximum or absolute minimum of the function in the appropriate domain. White and Mitchelmore (1996) found that students are much more likely to be able to find a desired maximum or minimum when the function is given explicitly in the problem than they are when the problem is stated in the form of a word problem and the students must first construct the appropriate function. For this reason, we are interested in studying how students construct the optimizing function and how this influences their understanding of the rest of the problem.

Our research is guided by the following two research questions: 1) What facets of learners' concept images influence their construction of the optimizing function when solving calculus optimization problems? and 2) How can this shed light on teaching interventions that could support conceptual development of optimization?

LaRue and Engelke Infante (2015) identified six key mathematical concepts that play a role in students' understanding of optimization – specifically their construction of the optimizing function. These six mathematical concepts are: variables, function notation, function composition, properties of rectangles and the relationships between them, the role of the optimizing function, and the graphical representation of the optimizing function. Here, we examine in greater depth the students' concept images of the optimizing function, focusing on those aspects related to the ability to transition between the graphical and algebraic representations of the optimizing function throughout the problem-solving process. We observed that students' inclination to consider the graphical representation of the optimizing function was directly related to their ability to explain the reasoning for their work and to solve challenging optimization problems.

Literature Review

Graphical interpretations of functions can convey information about the functions in a single image. The graph of a function can benefit students because it allows them to consider the overall behavior of the function, rather than focusing on individual elements. Students, however, are often reluctant to consider the graphical representation of a function, and when they do, they frequently have trouble interpreting the information correctly (Eisenberg, 1992). Sfard (1992) noted, "Graphs provide another way of thinking about functions, but there is almost no connection between a graph and the underlying formula" (p. 75). Knuth (2000) reported, "three fourths of the students chose an algebraic approach as their primary solution method, even in situations in which a graphical approach seemed easier and more efficient than the algebraic approach" (p. 504). Even (1998) found that students have trouble solving problems that require them to move seamlessly between different representations of functions.

Arcavi (2008) noted that analytic techniques are frequently "devoid of meaning" for students and suggests having students examine the graph of a function *prior* to using analytic techniques to determine information about the function (p. 9). He suggested using a dynamic graphing tool that allows the students to watch the function being drawn and to see the relevant information about the function at various points on the function. He argued, "a dynamical graphical model highlights aspects of the situation that were not as salient had we investigated it alone or even by modeling it symbolically" and gave the example of using dynamic graphs to examine the relationship between the perimeter of a rectangle and the length of one its sides and the relationship between the perimeter of a rectangle and the length of its diagonals (p. 5).

Theoretical Perspective

Tall and Vinner (1981) define a learner's concept image as 'all the cognitive structure in the individual's mind that is associated with the given concept' (p. 1). Because the concept image exists in the mind of the student, and we know that students do not always have correct understandings of mathematical concepts, this cognitive structure may be incomplete, incorrect, or logically inconsistent. When a need arises, the parts of the concept image that are directly related to the need are called upon, but the rest of the concept image remains dormant, ready to be accessed if needed, but not until then. This means students may have conflicting information in their concept images without realizing it, and unless the two logically inconsistent parts of the concept image are evoked simultaneously, the student may never realize something is wrong.

Carlson and Bloom's (2005) problem-solving framework allows us to describe students' activity as they solve optimization problems. The framework is divided into four main phases: orienting, planning, executing, and checking. In the orienting phase, the student deciphers the problem and assembles the tools he or she thinks may be required. In the planning phase, the student uses conceptual knowledge to determine an appropriate course of action, which is then implemented during the executing phase. Finally, during the checking phase, the problem solver goes back to the original problem to see if the answer makes sense.

During the orienting phase, students will likely focus on algebraic aspects of the problem as they assemble useful formulas and equations. In the planning phase, we would expect students to work to construct an appropriate optimizing function. It is during this phase that we would like to see them consider the graphical representation of the function as a tool for quickly recognizing the algebraic techniques they need to employ to solve the problem. Considering the graphical representation should help them determine that the key next steps are to differentiate, find critical points, and use either concavity or the increasing and decreasing nature of the graph to verify that they have solved the problem. Much of the executing phase is computational in nature and will focus on the algebraic representation of the function. Finally, during the checking phase, the graphical representation of the optimizing function affords students the opportunity to verify that their answer makes sense.

Methods

Data was collected through a series of semi-structured interviews with first semester calculus students at a large state university in the United States. Interviews were conducted during the summer and fall semesters of 2014. Four students (Franz, Sam, Tracy and Lars) were interviewed during the summer of 2014 and three students (Ashod, Brandi, and Cy) were interviewed during the fall of 2014. In both cases, the students were interviewed after their exam covering optimization and just before their final exam for the class. The students were selected on a volunteer basis and self-reported average to strong mathematical backgrounds. All interviews were video recorded and transcribed for analysis. We used open and axial coding to isolate and further analyze portions of the interviews associated with the graphical interpretation of the optimizing function.

The students were asked to solve the following optimization problem, which is standard in most first semester calculus classes: *A rectangular garden of area 200 ft^2 is to be fenced off against rabbits. Find the dimensions that will require the least amount of fencing if a barn already protects one side of the garden.* We refer to this problem as the **garden problem**. After solving this problem, the students were asked questions about the connection between the area and the perimeter of rectangles and then were asked to solve an additional optimization problem involving the volume and surface area of a 3D object. The students interviewed in the summer of 2014 were asked to examine a graph of the optimizing function associated with the second optimization problem, while the students in the fall of 2014 were asked to examine a graph of the optimizing function associated with the garden problem and with the second problem. In both cases, the graphs consisted of two unlabeled axes and a rough sketch of the function. The students were asked to mark and label important information on the graphs.

Results

The seven students interviewed had varying levels of success when asked to transition from the algebraic expressions of the optimization problem to the graphical representation. We have grouped them based on the strength of their graphical connections, and we discuss these below.

No Graphical Connections: Franz and Ashod

Franz and Ashod did not have well-developed concept images of the optimizing function. Ashod constructed his optimizing function with the motivation of finding something to differentiate and set the derivative equal to zero because, "when it equals zero, that's when you know you have the least amount." His only rationale for deciding how to construct such a function, however, was, "whatever they give you, use the other equation." When he was given a graph of the function he had constructed and was asked to explain how the answer to the problem related to the graph, he said they were not related, and he was unable to correctly mark the place where his answer belonged on the graph (see Figure 1). He knew he had been using the derivative to solve for the answer, and since "the graph of the derivative looks completely different from the graph of the original equation," he did not think his answer related to the original graph at all. He was able to explain that the first derivative always tells you whether the graph is "positive or negative or increases or decreases," but even though this was part of his concept image for the first derivative of a function, he did not relate this information back to the graph of the optimizing function. Thus, Ashod's concept image for the optimizing function was developed just enough to allow him to be able to solve the problem and get an answer, but not enough for him to be able to move from the algebraic interpretation to the graphical interpretation.

Franz was unable to make any connections to the graphical representation of a function when solving the garden problem. Initially, he tried to set the optimizing function equal to zero and solve for *x*, but when he got a negative answer, he realized that wouldn't work. His next attempt was correct (setting the derivative equal to zero), but when he was asked why he was doing that, he said, "it's a standard thing that we do," and, "I have no idea. I just know that is the minimum." The interviewer repeatedly encouraged him to make connections to the graph, but with no success. When he was asked to label the graph of the optimizing function, he decided where to put his answer based on where he thought that number would generally be located on a number line (see Figure 1), completely disregarding the shape of the graph. His responses indicate that his concept image for the optimizing function contained little more than some basic facts about what to do with it, without any links to the graphical representation of the function or the context of the problem.

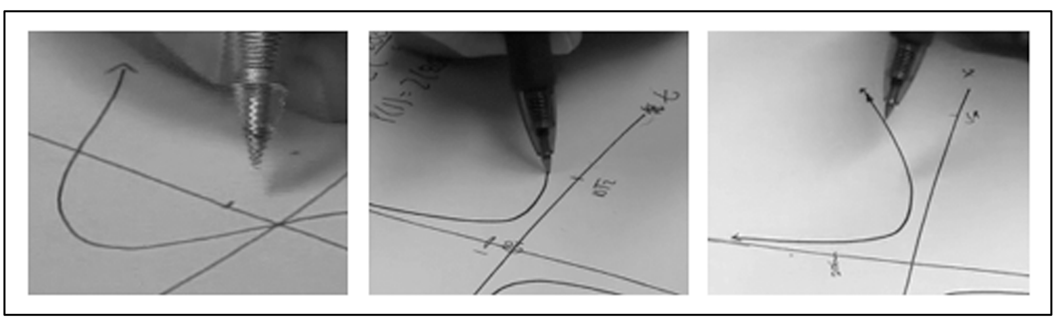

Figure 1. Franz, Brandi, and Ashod mark the location on the graph where they believe the critical number belongs on the axis. Note that all three students placed the mark somewhere other than below the obvious maximum or minimum.

Limited Graphical Connections: Cy and Lars

Cy and Lars were able to use the language of graphs to discuss the algebraic work they had done, but they had trouble making some connections initially. Cy knew there was a connection between the graph of the optimizing function and the algebraic expressions he was working with. Early in his explanation, he stated that the first derivate indicates where "the slope equals zero" and the second derivative indicates the concavity, signifying whether there is a maximum or minimum. However, Cy had a lot of difficulty interpreting the graph when it was first presented to him. Like Ashod, he was confused because he had used the derivative to solve for his answer, and was thus did not understand how his answer could have something to do with the graph of the original function.

Cy recognized that making the transition from the algebraic representation to the graphical representation made sense and was surprised that he was having trouble. He said, “I don’t know why this is so hard. This seems like something very easy.” Later in the interview, he said, “It’s weird to like make the jump from numbers into a graph sometimes. In some situations, the numbers are really just a stand in for the work I’m doing in my head with graphs, but in this situation, it’s more, the numbers are all I really ever thought about with this.” This is especially interesting, because he used the language of graphs when he was explaining his work, but was still unable to translate that to an actual graph. Eventually, with some prompting from the interviewer, he was able to make sense of the graph and connect it to the problem.

Lars brought up graphs without being prompted. He had trouble constructing his optimizing function, but explained that all of the information could be put on a graph. Unfortunately, even though he knew that there was a graphical component to the optimization problem, he did not know what function corresponded to the graph. He could not label the axes correctly or give any sensible information about what information could be obtained from the graph. At one point he thought the two axes corresponded to the two sides of the rectangular garden, and at another point he thought the two axes corresponded to one side of the garden and the area of the garden. Eventually, after a lot of intervention from the interviewer, Lars was able to construct an appropriate optimizing function and correctly relate it to the graphical representation. So even though he began with the recognition that he could use a graph to figure out how to solve the problem, he did not know what the graph should represent. Once he realized that the graph should represent the amount of material needed for the fence, he was able to complete the problem.

Strong Graphical Connections: Tracy, Sam and Brandi

Tracy began the problem by trying to recall how she had done similar problems, but quickly became confused and unsure how to proceed. She knew that she was trying to find the minimum of a function and that she could find that by taking the derivative, saying, “the derivative would just be the rate of change of the graph, so the best I can figure out of that is finding the critical values in the derivative would help you find the minimums because you look at areas of increase and decreases.” When she was asked why she wanted to set the derivative equal to zero, she said, “When the derivative is zero? Doesn’t it just have a horizontal tangent line which means it has to have that shape, the parabola shape?”

Unfortunately, even though she knew this, she did not know what function she should be differentiating. After a lot of intervention from the interviewer, she was able to recognize that for the garden problem, she was trying to construct a function representing the amount of material used to construct the fence. Once she figured that out, she was able to solve the rest of the problem easily, because she had such a strong understanding of the connections to the graphical representation of the optimizing function.

Sam very quickly set up and solved the garden problem with little difficulty. He explained, “we take the derivative of that function in order to find the points, uh, where slope equals zero in order to tell us where it stops increasing, decreasing, and that’ll tell us where the minimum or maximum values are.” However, as the interview progressed, he realized that he did not understand how the optimizing function, and particularly the *graph* of the optimizing function, related to the problem. He said, “Like how this fence has a minimum value that relates on a graph that’s a function of a different function.” He recognized that his initial equation represented the amount of fence of the garden, but once he eliminated one of the variables and constructed a single-variable function, he could not relate this "new" function to the amount of fencing. Thus, he knew graphical properties about functions in general, but he did not understand how the graph was related to his original algebraic expression.

When Brandi was asked to explain her work, one of her first responses was, "Cause like the amount of feet that could be used, if you think about it on the graph." She immediately made the connection to the graphical representation of the optimizing function, indicating that it is a well-developed part of her concept image. She was able to talk about this clearly and comfortably about the relationship between the graphical representation and the algebraic representation as she explained her thought processes, yet when she was presented with the actual graph, she had trouble marking the correct place for her answer, $10\sqrt{2}$. She placed it where she thought $10\sqrt{2}$ would fall on the axis (see Figure 1), not directly below what was clearly the minimum of the function. On a theoretical level, she appeared to understand the connection, but when she had to apply what she knew to an actual graph, she still had some difficulty.

Tracy, Sam, and Brandi had more developed understandings of the connections than the other students, and they moved more flexibly between the algebraic and graphical representations. All three, however, still had some difficulties with the two representations.

Discussion

The students in this study were not naturally inclined to consider the graphical representation of the optimizing function when solving optimization problems. When they did consider the graphical representation, most did so incorrectly or with an incomplete understanding of how it was related to the algebraic representation and the work associated with it.

In the planning phase of Carlson and Bloom's (2005) problem-solving framework, the student determines an appropriate course of action for solving the problem. Ideally, for the garden problem, students would consider that they need to construct a function representing the amount of fencing required for an area of $200ft^2$. They would then consider the graphical representation of this function and recognize that since they need to find the minimum, they should expect their function to be concave up at the value they find.

During this phase, some students recalled similar problems and simply attempted to duplicate a familiar solution path. Franz, Ashod, Lars, and Tracy all began this way. When this didn't work for Lars and Tracy, they were (with some encouragement) able to fall back on their knowledge about the graphical representation of the optimizing function to figure out how to move forward. Franz and Ashod were able to solve the first problem without intervention, but both had trouble solving the second more difficult optimization problem. They did not have any understanding of the graphical representation of the function to fall back on and were unable to determine how to move forward. Tracy did have this understanding, but she had so much trouble with the 3-D aspect of the second problem that she was unable to solve the problem without a lot of help. However, once she reached an answer, she was able to clearly explain what her answer meant in the context of the problem, suggesting that if she had a stronger concept image of surface area and volume, she would have been able to solve the problem on her own. Lars was able to solve the second optimization problem with ease, because once he had reasoned through the connection to the graphical representation once, he was able to draw upon this to make sense of the next problem.

The other three students, Brandi, Cy, and Sam, had some difficulties, but all began the first problem with well-developed concept images for the optimizing function that included at least some understanding of the graphical representation of the function. They began solving the problem by referencing this graphical connection and were able to give explanations other than "I don't know," or "because this is how my teacher did it" when they were asked why they were beginning the problem that way. Additionally, when they attempted to solve the more difficult second optimization problem, they were all successful. For these students, a well-developed concept image for the optimizing function, including at least some understanding of the connection to the graphical representation, led to more success in solving and understanding optimization problems.

Thus, we see that the graphical representation of the optimizing function has an important role to play in helping students develop their understanding of optimization in general. Because existing literature and our current research tell us that most students generally are not likely to move fluidly between the algebraic and graphical representations, we must work to find ways to encourage students to make these connections.

Conclusion

We found that even when students were able to accurately describe the connection between the algebraic and graphical representation of the optimizing function, they often had more difficulty when they were asked to put this information to use when dealing with an actual sketch of the graph. We suggest asking the following questions when teaching and/or assessing students on optimization problems.

1. Identify your optimizing function. What does it represent? How do you know it is a function?
2. What is the realistic domain of your optimizing function? What is the realistic range?
3. Draw a rough sketch of your optimizing function. Label the axes appropriately.
4. Consider an ordered pair *(a,b)* on the function. In the context of the problem, what does *a* represent? What does *b* represent? In the context of the problem, what is the relationship between *a* and *b*?
5. Mark the point in the domain of the function that corresponds to the answer you hope to find (or have already found) using algebraic techniques.

These questions are designed to encourage the students to think about and make the connections between the different representations of the optimizing function and to help them further develop their concept image of the optimizing function. Making these connections will help the students set up and solve these problems, particularly during the planning and checking phases of the problem-solving process.

In our study, the graphical representation of the optimizing function was only a portion of the interview protocol, but it has emerged as a significant theme in our research. We believe that there is room for a more targeted, small-scale research project focused on examining the role that the graphical representation of the optimizing function plays in students' work with optimization problems. The above questions could be a good starting point for such a project.

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Organizational features that influence departments' uptake of student-centered instruction: Case studies from inquiry-based learning in college mathematics

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Active learning approaches to teaching mathematics and science are known to increase student learning and persistence in STEM disciplines, but do not yet reach most undergraduates. To broadly engage college instructors in using these research-supported methods will require not only professional development and support for individuals, but the engagement of departments and institutions as organizations. This study examines four departments that implemented inquiry-based learning (IBL) in college mathematics, focusing on the question, "What explicit strategies and implicit departmental contexts help or hinder the uptake of IBL?" Based on interview data and documents, the four departmental case studies reveal strategies used to support IBL instructors and engage colleagues not actively involved. Comparative analysis highlights how contextual features supported (or not) the spread and sustainability of these teaching reforms. We use Bolman and Deal's (1991) framework to analyze the structural, political, human resource and symbolic elements of these organizational strategies and contexts.

Keywords:
Inquiry-based Learning, Departments, Case Studies, Reform, Teaching Assistants (TA)

Research evidence supports the use of student-centered teaching approaches to improve student educational outcomes in science, technology, engineering and mathematics (STEM) disciplines (Freeman et al., 2014; Singer, Nielsen & Schweingruber, 2012; Ruiz-Primo et al., 2011). The bottleneck in achieving these improvements on a national scale is not a lack of well-developed classroom approaches from which to choose, but rather slow faculty uptake of these proven teaching methods (Fairweather, 2008). This paper focuses on the important but under-studied organizational context for uptake, by examining the implementation of inquiry-based learning (IBL) at four research university mathematics departments. We address the question,

> *What organizational factors, including both explicit action strategies and inherent contexts, influence the spread and sustainability of inquiry-based learning in mathematics departments at US research institutions?*

Conceptual Framework

On the whole, prior studies of the uptake of student-centered teaching approaches have focused on individual STEM instructors, examining their knowledge and skills around instruction, and the internal and external barriers to pedagogical change that they face (e.g. Henderson & Dancy, 2007; Walczyk, Ramsey & Zha, 2007). Early disciplinary socialization inculcates a values hierarchy that privileges research over teaching and portrays teaching as an art or innate talent rather than a craft that can be studied, learned and mastered. Structural issues such as classroom seating arrangements complicate the logistics of methods such as small group work, and instructors fear real or perceived resistance from their students, colleagues, or chairs. In general, this focus treats teacher decision-making as individualized within a static setting.

However, STEM instructors are also embedded in dynamic social systems that influence their thinking in positive and negative ways (Austin, 2011). Thus it is also important to understand instructors' working contexts in higher education. Contextual influences at the *institutional level* include features such as overall teaching loads, the relative weight of teaching duties relative to research and service in faculty job descriptions, and the role and measurement of teaching outcomes within faculty reward systems. These may vary widely, for example, between a research-oriented institution and a liberal arts college.

These institutional influences are commonly manifested at the *department level*, where curricular structures are set and teaching assignments are made. Here colleagues communicate formal traditions and informal norms about teaching, shaped by their understanding of their student clientele and in turn reinforced by students' expectations. For example, the type of teaching seen as appropriate in "service" courses may be different from that in "majors" courses. Collegial agreements—stated or tacit—may inform expectations about the nature and amount of work that can be assigned, availability and use of office hours, and specific topics that must be covered as preparation for the next course in a sequence. Department chairs and committees control access to resources and rewards, and oversee graduate students' preparation as teachers.

Finally, *disciplinary contexts* shape instructors' understanding of the aims of education in their field, their notions of intellectual development and rigor, and their professional identities as researchers and educators (e.g., Brownell & Tanner, 2012). In mathematics, phrases such as "mathematical maturity" encode and signal the value of generalized skills in analyzing problems and developing solution approaches, creativity, flexibility, and recognition of mathematical concepts in varied contexts. Epistemological beliefs about the nature of knowledge and "truth" shape instructors' interest in and ability to make sense of education research findings about teaching and learning that rely on different disciplinary standards for what counts as knowledge.

In addition to considering the level at which these organizational influences are felt, we apply Bolman and Deal's (1991) multi-frame model to analyze their nature. In this model, four main perspectives serve as viewpoints for examining organizational issues: structural, human resource, political, and symbolic perspectives. Each of these perspectives functions as a frame or "lens" that can "bring the world into focus" (p. 11) in order to understand organizational issues—in this case, processes of change to support faculty use of research-based instruction.

- The *structural* frame emphasizes policy and procedure as tools for shaping instructor practice. This lens recognizes the importance of formal rules, policies, management hierarchies, and relationships within organizations.
- The *human resource* frame emphasizes the importance of the demographics, experiences, needs, and feelings of the people involved in an organization. Here the key human resource is the instructors who carry out the department's teaching mission, including both those who engage with the IBL Center's teaching reforms and those who do not.
- The *political* frame attends to issues of resource allocation and the sources and seats of power, whether tied to formal institutional roles or as informal thought leaders of high status.
- Finally the *symbolic* frame focuses attention on issues of meaning and culture within an organization, including rituals, stories, and celebrated individuals, and the process through which sense-making takes place within the organization (Eckel, Green, & Hill, 2001).
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Study Methods

Context of the Study

This report draws upon a large, mixed-methods study examining inquiry-based learning (IBL) in four mathematics departments where privately funded "IBL Centers" had been established to promote the use of IBL in teaching. These highly ranked research departments were assumed to have high visibility and influence in their discipline. All four were "very high" research schools by Carnegie classification and had full-time, four-year, selective undergraduate programs with low transfer rates. Each Center was led by an eminent mathematician with some track record of involvement in K-12 or undergraduate education.

The full study encompassed a wide range of issues, including student outcomes of IBL instruction and IBL teaching and learning processes, but in this report we focus on instructors' experiences in implementing IBL. Our analysis treats each department as a case, but also identifies common issues across the four cases that help to highlight challenges and opportunities for establishing and sustaining student-centered approaches to teaching in college mathematics.

Study Samples and Analysis Methods

This report draws primarily on qualitative analysis of 42 semi-structured interviews with 43 IBL instructors (one focus group had two teaching assistants from the same course). We use the general term 'instructors' to refer to all interviewees; when it is important to distinguish specific classroom roles, we specify "faculty" (anyone in the lead instructor role, regardless of appointment type) or "TA" (graduate student in a course assistant role).

The interview sample was drawn from institutionally provided lists of active or previous instructors of IBL courses for the period 2006-2009. We invited all instructors we could reach and scheduled in-person interviews during campus site visits in 2009, or telephone interviews if needed. The overall response rate was 77%, varying from 50% to 88% by campus. Nearly all interviewees were white; about 15% were born outside the US. Most taught "math-track" courses for math or STEM majors; seven taught IBL courses for pre-service teachers.

The faculty interview subset included 23 interviewees (3 women, 20 men) who held faculty appointments, including tenured, pre-tenure and non-tenure-track instructors in both long-term lectureships and short-term postdoctoral or visiting positions. Of these, 13 had prior IBL teaching experience of one year to decades, and 10 were new to IBL.

The TA interview subset included 20 teaching assistants (9 women, 11 men). Most were second- to seventh-year graduate students; at the time of the interview a few had graduated and moved on to postdoctoral or tenure-track faculty positions. Their IBL teaching experience ranged from one term to several years.

The interviews covered a range of topics and questions about IBL instruction, student outcomes, and the relationship of IBL teaching to the interviewee's career path. All protocols were approved by the Institutional Review Board, and interviewees provided informed consent.

Formal content analysis methods were used to analyze the interviews (Babbie, 2001; Berg, 1989). Digitally recorded interviews were transcribed verbatim and coded using *NVivo 8.0* (QSR International, 2009). The coded passages reflect a set of analytical themes that together describe the nature and range of issues in participants' collective report. Counting the frequencies with which different themes appear helped to characterize the relative weighting of these issues.

In all, about 2400 text passages were coded into five broad themes, each with 8-14 sub-themes, divided roughly as follows (counted as a percentage of all coded passages):

- observations of cognitive, affective and other outcomes of IBL for students – 16%
- students' learning processes that instructors observed or hypothesized – 16%
- instructors' reflections on the processes of teaching – 40%
- personal and professional outcomes of IBL teaching for instructors – 16%
- instructors' personal, departmental, disciplinary and institutional contexts – 10%

The present analysis focuses on these latter, context-focused codes, with some use of respondents' comments on teaching and learning processes and their own professional outcomes. Documents about the IBL Centers (e.g., annual reports to their funder) were used to elucidate or confirm some of the contextual features identified from interviews.

In addition, data from classroom observation (over 300 hours) in 42 course sections, as well as data from student surveys (1105 respondents) and interviews (68 respondents), provide context on campus-level patterns and variations in how IBL was conceived and executed in the classroom. These data primarily address the learning outcomes and learning processes observed in class and reported by students; insights from these data enrich this report but are not its focus.

Study Sites: Commonalities and Variation in IBL Instruction

As context for this analysis, we describe common features of IBL instruction and identify some sources of variation among them. We have described IBL instructional practices as applied in the IBL Centers elsewhere (Laursen, 2013; Laursen et al., 2014); the practices we observed at the Centers are consistent with those reported by practitioners in the broader IBL community (e.g., Yoshinobu & Jones, 2013; Coppin, Mahavier, May & Parker, 2009). Key features of these IBL classes were identified from classroom observation, course documents, and instructors' and students reports in surveys and interviews:

- Students solve challenging problems alone or in groups; they share their solutions, then analyze, critique and refine their solutions.
- Class time is used for these student-centered activities. Students often lead the activities (e.g. by presenting their work) and these activities change several times a class period.
- The course is driven by an instructor-built sequence of problems or proofs, rather than by a textbook; the pace of the course is set by students' progress through this sequence.
- Course goals tend to emphasize mathematical thinking skills and communication practices; "coverage" of specific content is less central in the syllabus.
- The instructor's role shifts notably from "sage on the stage" to "guide on the side" (King, 1993), playing stage manager, monitor and summarizer of key mathematical benchmarks, and cheerleader for students.

The details of practice varied somewhat from course to course (Laursen, 2013) but these features were consistently noted. Class time was predominantly used for student-centered work, which accounted for over 60% of class time observed.

While instructional practice was relatively consistent, other features of the IBL courses varied widely among Centers. Each IBL Center selected the courses where it would apply IBL methods. These ranged from first-year honors courses to upper-level courses for mathematics majors and students in fields such as physics, engineering or economics. Two Centers developed IBL courses for pre-service elementary and/or secondary teachers. These selections were guided both by theoretical considerations, such as courses thought well suited to IBL approaches or student audiences thought to benefit, but also by practical considerations, such as where class sizes were already amenable to IBL techniques.

As a result, the overall range of courses and student audiences in the study was large. This reflects real-world conditions of instructional reform in higher education, where individual faculty have high autonomy in how to teach their courses and where curricular sequences and student characteristics may vary widely among departments. Indeed, this high variability within and among departments is one reason for the slower pace and different path of educational reforms in higher education as compared to K-12 education.

Study Findings

Our data reveal both explicit action strategies and implicit contextual features that shaped departments' implementation of IBL. Explicit strategies were more easily noted by interviewees, as each Center developed ways to support IBL instructors and engage colleagues, including:

- Processes to engage faculty in IBL instruction: recruiting new participants, preparing and supporting them for IBL teaching
- Processes to engage the support of colleagues not involved in IBL teaching, especially those whose approval was politically important
- Engagement of graduate TAs in IBL instruction: recruitment, professional development, and context within the department's TA teaching preparation program (if any).

Contextual features were less often recognized by interviewees themselves and instead were embedded in interviewees' statements as taken for granted. Some of these were well-established aspects of the institution, such as its size and reputation. Other features of the IBL program were sometimes the result of explicit and strategic decisions made when the Center was established, but had come to be seen as pre-existing features of the Center's IBL program that shaped how things were done. These contextual features included:

- Nature of the institution and department: size, prestige, public or private status
- Characteristics of the IBL Center's leader: status and seniority in mathematics and education, leadership style, relationships to other STEM reform efforts
- Characteristics of faculty connected to the Center's work: seniority, status, nature and extent of involvement or resistance
- Nature of the IBL undergraduate program: targeted undergraduate audiences, selection processes for entry to IBL courses, predominant styles of IBL teaching
- Other components of the IBL program, if any: pre-service and in-service K12 teacher education, mathematics enrichment for K12 students, linkages to other STEM programs.

We argue that both the explicit strategies and contextual features help to account for the spread and sustainability of IBL as a teaching reform in undergraduate mathematics in these departments. In general, IBL teaching practices spread with adequate local fidelity within these departments, despite local variation in style. We use the contrasting cases to illustrate how different strategies and cultures helped or hindered the spread and sustainability of IBL.

Understandably, many of the explicit strategies fall primarily under Bolman and Deal's (1991) *human resource* frame, especially efforts to interest colleagues in IBL and to develop their skills as IBL teachers. These included collegial and informal mentoring (more rarely, structured mentoring); participation in formal workshops (helpful albeit not widespread); and extra-departmental support from a national meeting on IBL in mathematics and the broader network of practitioners who participated in this meeting and related events. Some human resource strategies sought to build an active IBL community, for example through lunches and talks that focused on IBL and other teaching topics, to which all interested persons were invited. TAs' professional growth as teachers was greatest when they too participated in this community and when they were generally treated as full instructional partners, for example by meeting regularly with lead instructors to share observations about students and troubleshoot daily problems in the class. TAs also discussed and shared practices within their own peer group, leading to rapid uptake of certain TA-initiated innovations in grading and student motivation.

Strategies that helped to recruit and engage non-involved instructors sought to develop awareness and positive impressions of IBL among those in formal and informal leadership roles. Primarily *political* strategies included inviting colleagues to observe an IBL course or to evaluate a graduate TA's teaching in such a course. One department made a point to share emerging research findings on student outcomes of IBL with key committee chairs, deans and provosts. One leader's style of "managing by walking around" was also a political strategy that fostered high buy-in and program coherence and alerted him to impending challenges. *Structural* strategies used existing policies and procedures to enhance visibility and acceptance of IBL, such as asking the standing undergraduate curriculum committee to review IBL courses, or engaging the department's educational thought leaders to serve as a steering committee for the Center. One department took advantage of the *symbolic* frame by successfully nominating an IBL leader for a major institutional teaching award, and by publicizing its IBL work in the department's annual newsletter sent to supporters and alumni. *Human resource* strategies that centered on building community also doubled as a means to engage non-involved instructors when departmental leaders participated or simply were informed of colleagues' participation in these events.

The culture and structure of TA preparation varied widely across the four departments, which in turn shaped opportunities for TAs to join the IBL effort. Departments differed philosophically as to whether opportunities to TA for IBL courses should be concentrated among a few to hone their IBL teaching skills, or be offered broadly to give more TAs exposure to IBL. In one case the chance to TA an IBL course was offered to all TAs moving through that department's formal, multi-part TA preparation process; elsewhere TAs were informally recruited by IBL faculty leaders. In both scenarios, TAs were carefully screened for aptitude and interest.

In practice, what commonly resulted from IBL participation by TAs and other early-career instructors (including postdocs) was a strong commitment to student-centered teaching, based on having seen it work for undergraduate students—sometimes despite their initial skepticism. Indeed, most said they would teach this way again. TAs in particular viewed IBL as a broadly applicable pedagogy, describing their enhanced skills as a nuanced "toolkit" that enabled them to apply IBL to varied student audiences. Interestingly, they often articulated a broader view of where IBL could be used than that expressed by their senior faculty colleagues. Overall, these IBL teaching experiences proved to be a powerful form of experiential professional development for early-career instructors. As they moved on to teaching roles at other institutions, they took along re-shaped teaching philosophies and expertise; many have remained active and taken on leadership roles in the larger IBL mathematics community.

Departmental cultures strongly shaped the predominant local style of IBL, as our classroom observation data make clear (Laursen et al., 2014; Laursen, Hassi & Hough, 2015). Courses at two Centers often featured formal, in-class group work, while courses at the other two emphasized student presentations at the board. These patterns occurred and persisted in part because of how instructors learned about IBL: what colleagues said about IBL, what they saw in colleagues' classrooms, and how they adopted or adapted prior versions of the same course as they prepared to teach an IBL course new to them. Because few had independent exposure to active learning approaches (e.g, via formal professional development, reading, or peers outside the Center), such informal transmission of IBL norms led to substantial homogeneity of IBL approaches in use within any one campus. However, variation *among* the Centers played a significant role in broadening understandings among the larger IBL community of "what is IBL" (Author, 2015).

Local culture also shaped views of what courses and students were seen as good fits for IBL. In considering which students would benefit from IBL, IBL was variously seen as

- a special experience to recruit talented (honors) students into mathematics

- a good way to help students make the transition from lower-division, computational courses to upper-division, proof-based courses
- a crucial experience for non-math majors (especially pre-service elementary and middle school teachers), to learn to think like mathematicians and to value IBL teaching.

These views reflect disciplinary beliefs about "who can do IBL," shaped in turn by prevailing views of "what is IBL." Some departments held multiple views and tested these various hypotheses in their choice of where to implement IBL. The choice to work with pre-service teachers was in part driven by a need to assert the department's primacy over mathematical preparation of teachers, but had a side benefit of requiring good cooperation with the School of Education, which in turn made them advocates for the IBL program.

As to mathematical content, there was little consensus as to what course content was best suited for IBL treatment; indeed, nearly every assertion about how a particular topic (e.g., linear algebra) "could not be taught" with IBL was countervailed in our data by a contrasting assertion of why that course worked very well in IBL form. Such lack of consensus in the evidence suggests that these beliefs, while informed by and often couched in disciplinary terms, were in fact department-based. It is thus noteworthy that TAs in particular could articulate broader uses of IBL, even in courses where IBL was not formally practiced in their department.

Implications for Practice

Over time, these departments have seen IBL spread and succeed within the department and beyond, as early-career trainees took their newfound IBL skills to new venues. But long-term sustainability of the IBL reforms in the home departments is less certain. A distinguished leader and a few senior faculty champion IBL courses, but overall senior faculty participation is low, and the programs rest on transient or low-status instructors such as postdocs and non-tenure-track instructors. There is some evidence of risk to programs' ongoing health when a senior leader steps down, and there has been little visible effort to absorb the costs of the IBL program.

We do not yet know whether and how these departments will sustain their IBL programs if funding is withdrawn. But in our judgment, the departments with the best prospects for sustaining their IBL programs are those which have consciously attended to the *political* and *symbolic* landscapes—by keeping key leaders in the loop, by making strong alliances with external constituencies for general education or teacher preparation, and by publicizing and promoting their work to a variety of stakeholders within and outside the department. These strategies help to broaden ownership of IBL so that the effort does not depend on a single leader.

Explicit strategies for *human resource* development have helped to strengthen student outcomes at the Centers and to enhance the spread of IBL beyond these Centers, but have had less effect on the sustainability of IBL within the department. Staffing the IBL courses with temporary instructors who develop skill and enthusiasm but then move on to other positions is a double-edged sword: it enhances the Centers' role as national leaders in IBL but fails to bolster their own long-term sustainability. Finally, long reliance on external funds—and the concomitant need to preserve the argument that these funds are essential to continued activity—seems to have limited departments' attention to *structural* aspects of sustainability. We note little sense of urgency to find other ways to cover the costs of team-teaching, maintaining small class sizes, or deploying extra TAs in IBL courses, and indeed there may be some risk to doing so. We acknowledge, however, that our data set is most limited on this point.

Overall, we propose that the set of action strategies must be well-rounded to enhance the growth, success and sustainability of an education reform within a department. Human resource strategies are necessary but not sufficient; program sustainability requires explicit attention as well to political, symbolic and structural elements of the organization. These strategies must also be designed to fit the department's unique context. Higher education is rife with stories of once-promising reforms that failed to take hold; analyzing these organizational features may be important in understanding why.

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Colloquial mathematics in mathematics lectures

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In this poster, we focus on mathematics professors' use of colloquial mathematics where they express mathematical ideas using informal English. We analyzed 80-minute lectures in advanced mathematics from 11 different mathematics professors. We identified each instance where mathematicians expressed a mathematical idea using informal language. In the poster, we use this as a basis to present categories of the metaphorical images that professors use to help students comprehend the mathematics that they are teaching.

Key words: Advanced mathematics; Language; Lectures

It is widely accepted that mathematics majors learn less from their mathematics lectures than we would like (e.g., Alcock et al., 2015; Leron & Dubinsky, 1995; Thurston, 1994). To account for this, our research team conducted a case study in which we compared the different meanings that a mathematics professor and his students attributed to the same lecture in real analysis (Lew, Fukawa-Connelly, Mejia-Ramos, & Weber, in press). We proposed the following account, among others, for why students had difficulty comprehending the lecture. The professor used what we called *colloquial mathematics* where he phrased technical mathematical ideas using informal English such that students' intuitions about informal English might help them understand the technical ideas. For instance, the professor framed the process of constructing a real analysis proof as asking how one can make a term small knowing that other quantities are small. His students, however, did not know what he meant by small and consequently did not understand the high-level summary of the proof that the professor attempted to convey. In our current project, we attempt to further investigate how colloquial mathematics is used in advanced mathematics courses.

Methods

At the beginning of the semester in three institutions, we sent an e-mail to every mathematics professor teaching a proof-oriented advanced undergraduate mathematics course, asking him or her to participate in our study. Eleven mathematics participants agreed to participate. For each participant, a member of our research team attended a randomly chosen lecture. We used a LiveScribe pen to audio-record the lecture and record what the professor wrote on the blackboard in real time. The 11 lectures were the corpus of data for our study.

We transcribed each of the lectures. Next, two members of our research team read each transcript, flagging every instance in which the professor used colloquial mathematics. More specifically, we coded a portion of transcript as being an instance of colloquial mathematics if one of the two following conditions held: (i) the professor represented a technical mathematical idea using ordinary English that was not equivalent to a formal description of that idea. An example of this is referring to an ideal that "sucks elements in from both sides", by which the professor meant that left and right multiplication by a ring element and ideal will be contained in the ideal. (ii) the professor discussed a meta-mathematical idea without a formal mathematical correlate, such as a particular structure as being "nice", "well-behaved", "interesting", or "boring". Analysis is ongoing, but we are currently sorting each instance of colloquial mathematics into categories using an open coding scheme in the style of Strauss and Corbin (1990).

Significance

Professors try to make mathematical concepts and mathematical practices accessible to undergraduates by using colloquial mathematics. Our prior research suggested that such natural and well-intentioned actions may not have their desired effect as students may be unable to interpret the professor's intentions when they hear this colloquial mathematics (Lew et al., in press). We view identifying commonalities between and categories of colloquial mathematics as a first step to a larger research agenda. We will use the categories generated in this study to see: (i) if there is a shared understanding amongst mathematicians as to what terms in colloquial mathematics means; (ii) how mathematics majors understand terms in colloquial mathematics and the ways in which such understandings align or do not align with mathematicians' understanding; and (iii) how mathematicians think students will understand colloquial mathematics and the accuracy of mathematicians' predictions.

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‘It’s not an English class’: Is correct grammar an important part of mathematical proof writing at the undergraduate level?

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We studied the genre of mathematical proof writing at the undergraduate level by asking mathematicians and undergraduate students to read seven partial proofs based on student-generated work and to identify and discuss uses of mathematical language that were out of the ordinary with respect to what they considered standard mathematical proof writing. Preliminary results indicate the use of correct grammar is necessary in proof writing, but not always addressed in transition-to-proof courses.

Key words: Mathematical language, Proof, Mathematicians, Undergraduate students

Introduction

Mathematicians and mathematics educators have found undergraduate mathematics students to have difficulties when constructing (Weber, 2001), reading (Conradie & Frith, 2000), and validating (Selden & Selden, 2003) mathematical proofs. One suggested reason for these difficulties is the students’ unfamiliarity with the language of mathematical proof writing (Moore 1994). However, mathematical language at the level of advanced undergraduate proof writing is a scarcely studied topic. As a result, little is known of how mathematicians and students understand and use this technical language.

Related Literature

Halliday’s (1978) introduction of the notion of register (and mathematical register in particular) was groundbreaking in the study of mathematical language:

> A set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings. We can refer to a ‘mathematics register’, in the sense of the meanings that belong to the language of mathematics […], and that a language must express if it is being used for mathematical purposes. (p.195)

Thus, the mathematical register contains not only technical vocabulary and symbols, but also phrases and the associated syntax structures. Various mathematics educators have considered how the mathematical register plays a role in mathematics learning and classrooms. For instance, Pimm (1987) discussed how students develop the mathematical register and Schleppegrell (2007) noted students’ difficulties with differentiating between the mathematically precise and colloquial uses of words like ‘if’, ‘when’, and ‘then’.

However, much of the existing work on mathematical language focuses on K-12 mathematics and little empirical research exists on how professional mathematicians view or use the language of mathematics. Konior’s (1993) analysis of over 700 mathematical proofs revealed a common style of construction of mathematical proofs that signals the organization of the proof’s arguments. Burton and Morgan (2000) identified the roles that the author’s identity and focus played in mathematical writing in research papers. Meanwhile, a number of manuals (AMS, 1962; Halmos, 1970; Gillman, 1987; Krantz, 1997; Higham, 1998; Houston, 2009; Alcock, 2013; Vivaldi, 2014) have been written describing how mathematicians and students should effectively use mathematical language. As these texts are based on the authors’ experiences rather than empirical research, the texts were used to guide the materials used for the study.

Theoretical Perspective

This study builds upon Scarcella's (2003) conceptual framework of academic English, which was designed to study the learning of academic English. Scarcella (2003) defined academic English as a "register of English used in professional books and characterized by the specific linguistic features associated with academic disciplines" (p. 9). Scarcella argued that academic disciplines have their own sub-registers of academic English and, as such, the mathematical register can be seen as a sub-register of academic English. Thus we consider the mathematical sub-register in this study with a focus on undergraduate proof writing.

This study is also informed by Herbst and Chazan's (2003) body of work on practical rationality. Intending to study norms by evoking repairing reactions from their participants, Herbst and Chazan adapted the ethnomethodological concept of breaching experiments (Mehan & Wood, 1975). The hypothesis of the design is that when a participant of a practice is presented with a situation in which a norm of such practice is breached, he or she will attempt to repair the breach highlighting not only what the norm is, but also the role that the norm has in the practice (Herbst, 2010). Adapting this methodology, this study investigates how mathematicians view and describe conventional uses of the language of mathematical proof writing at the undergraduate level and how students understand these conventions.

Research Questions

In this study, we am to investigate the following questions: 1) How do mathematicians view and describe common unconventional uses of mathematical language in undergraduate mathematical proof writing? 2) How do these unconventional uses affect how mathematicians evaluate student-constructed proofs? 3) How do students understand the conventions of mathematical proof writing at the undergraduate level?

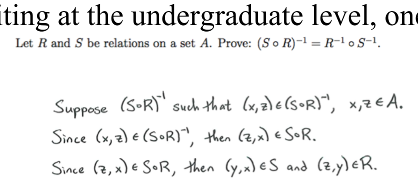
Methods

We investigate the linguistic dimension of undergraduate proof writing by presenting participants with student-generated proofs and asking the participants to identify and describe uses of mathematical language that are out of the ordinary with respect to undergraduate proof writing. By identifying non-standard uses of mathematical language, the participants discussed their understanding of the conventions of proof writing in this context.

The study was conducted at a large research university in the United States. Eight mathematicians and sixteen undergraduate students were interviewed (eight of the undergraduates were mathematics majors who had completed the proof-based courses required for graduation and eight were undergraduates enrolled in an introduction to proof course.) The mathematicians had 1-38 years of experience teaching undergraduate proof mathematics courses, with 1-15 years of experience teaching introduction to proof courses.

Materials

The materials for this study include seven partial proofs that are based on student-generated work. Each of the proofs was truncated to help participants focus on the use of mathematical language and not the attempted proof's logical validity. One of the partial proofs used in the study is provided below in Figure 1a. The partial proofs were chosen from student exams given in introduction to proof classes at the same university of the study. For each one of these partial proofs, a copy was created and marked for each of the instances of what we believed to be breaches of conventional uses of the language of mathematical proof writing at the undergraduate level, one example is shown in Figure 1b.



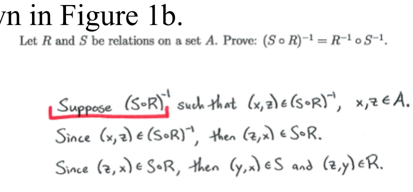


Figure 1a. Example of the partial proof. **Figure 1b.** Example of marked partial proof.

Procedures

The interview procedures for mathematicians and for undergraduate students were nearly identical. The semi-structured interviews were videotaped and lasted one to two hours. Participants were presented with the student-constructed partial proofs one at a time. They were asked to mark the partial proofs for anything that was out of the ordinary with respect to the use of the language in undergraduate mathematical proof writing. The interviews made two passes through the materials. In the first pass, participants were asked to explain why they had made each mark. Then for each mark, the participant was asked if the breach at hand was a logical issue, if it affected the validity of the proof, if it was an issue of mathematical writing, if it was definitely unconventional or a matter of personal preference, if it lowered the quality of the proof significantly, and if they (or in the students' case, if they thought a mathematician) would have deducted points based on this issue when grading the proof in an introductory proof course. These prompts were designed to elicit the participants' views on what they thought were conventional uses of mathematical language in proof writing. In particular, the prompts addressed the severity of each breach and enabled a differentiation between issues of logic and issues of mathematical writing in the analysis of the data.

In the second pass, for each of the predicted instances of unconventional use of mathematical language that had not been identified by the participant in the first pass of the data, participants were asked if they would agree that this was an issue of mathematical language. Specifically, mathematicians were asked whether or not they would agree with a colleague of theirs who had suggested these were unconventional uses of mathematical language and the undergraduate students were asked if they would agree with a classmate of theirs who believed a mathematician would think these were unconventional uses of mathematical language. If they agreed, they would be prompted to discuss the breach as in the first pass.

Analysis

Interview videos were transcribed and materials generated in the interviews were scanned for analysis. The interview protocol created clear episodes of discussion, each concerning a single breach of mathematical language. Thus the data is organized by these episodes and was then analyzed using open ended thematic analysis in the style of Braun and Clarke (2006). That is, we first familiarized ourselves with the data by marking for ideas and transcribing videos, generated initial codes by organizing the data into meaningful groups, searched for themes by focusing analysis at a broad level, and reviewed the themes to verify that the themes reflect he data set as a whole.

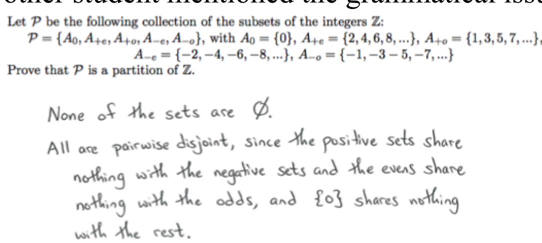
Results

One theme that has emerged from the data is that mathematicians believe that mathematical language is a subset of the English language whereas some students believe the two are independent. This theme was brought forth by the mathematicians' attention to the need for correct grammar and complete sentences as well as some of the undergraduate students' responses indicating the rules of English do not apply in mathematical settings. In particular, this theme emerged from three categories of responses from participants discussing what they considered was non-standard mathematical language use, which are described below using interview data three different proofs: Proof A, Proof B, and Proof C.

Mixing mathematical notation and English prose

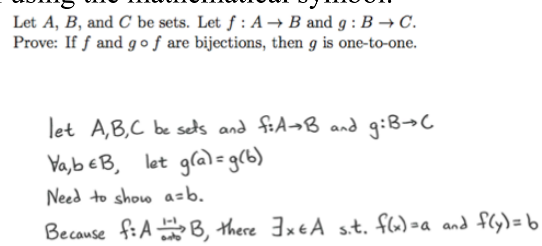
As shown in Figure 2, the first line of Proof A reads, "None of the sets are $\emptyset$." We expected that participants would indicate this sentence as an unconventional use of language because the mathematical symbol for the empty set, $\emptyset$, was used in a sentence that was otherwise written in English words. This was generally the case, however, two mathematicians explained further that the issue of using the symbol ∅, was an issue of grammar. For example, M8 indicated in Pass 1 through the proof that there is a problem with the part of speech of the symbol "because *empty* is an adjective and *the empty set* is a noun". So M8's comment highlights that when read, the statement says, "none of the sets are empty set" rather than the possible intended meaning, "none of the sets are empty". M5 gave a similar explanation for why the use of the symbol ∅ was inappropriate in this statement. Both M5 and M8 said that they would make a note to the student suggesting that they avoid this use of language in the future.

One undergraduate participant S2 made a statement arguing that words and symbols can be used interchangeably since "the symbol for the empty set is just as rigid as saying empty". From this quote, it appears that S2 is (at least implicitly) aware of the difference between the noun and the adjective forms, however, disregards the issue. With the exception of S2, no other student mentioned the grammatical issue of using the mathematical symbol.


Let $\mathcal{P}$ be the following collection of the subsets of the integers $\mathbb{Z}$:
$\mathcal{P} = \{A_0, A_{+e}, A_{+o}, A_{-e}, A_{-o}\}$, with $A_0 = \{0\}$, $A_{+e} = \{2, 4, 6, 8, ...\}$, $A_{+o} = \{1, 3, 5, 7, ...\}$, $A_{-e} = \{-2, -4, -6, -8, ...\}$, $A_{-o} = \{-1, -3-5, -7, ...\}$
Prove that $\mathcal{P}$ is a partition of $\mathbb{Z}$.

None of the sets are ∅.
All are pairwise disjoint, since the positive sets share nothing with the negative sets and the evens share nothing with the odds, and {0} shares nothing with the rest.

Figure 2. Proof A.


Let A, B, and C be sets. Let $f : A \to B$ and $g : B \to C$.
Prove: If f and $g \circ f$ are bijections, then g is one-to-one.

let A,B,C be sets and $f:A\to B$ and $g:B\to C$
$\forall a,b \in B$, let $g(a)=g(b)$
Need to show $a=b$.
Because $f:A \xrightarrow[\text{onto}]{1\text{-}1} B$, there $\exists x \in A$ s.t. $f(x)=a$ and $f(y)=b$

Figure 3. Proof B.

Punctuation and capitalization

In Proof B (as shown in Figure 3), there is a lack of punctuation and capitalized letters to indicate the ending and beginning of sentences. Mathematician M7 pointed this out during Pass 1 through the proof, saying: "the expression and the punctuation are not good" and "we can't allow writing like that". In Pass 2, the remaining mathematicians agreed that lacking punctuation and capitalization is definitely unconventional of mathematical proof writing. However, mathematicians M3, M4, and M5 each also agreed that they would not address this issue in their introduction to proof classes. For example, M4 explained:

> I look for understanding of the construction of the mathematical arguments. So I'm not sure you can require that deep understanding at the same time pushing them to be correct with punctuation and so on. […] And I consider that my task is to teach them reasoning, rather than to use punctuation.

Although all eight mathematicians in the study did agree that proofs should be presented in complete sentences, including appropriate punctuation and capitalization of letters, not all believed they should discuss this in class. Only M7 indicated that he would deduct points from his students' work for missing punctuation and lacking capitalized letters. Meanwhile, M4, M6, and M8 indicated that they would mark the punctuation and capitalized letters when grading, without deducting points, to illustrate to their students that one should use complete sentences in proofs.

None of the 15 undergraduate participants discussed the lack of punctuation or capitalization in Pass 1 through the proof. In fact, during Pass 2, 12 of the 15 undergraduate participants disagreed with the suggestion that this is an issue of mathematical proof writing. When asked why not, S2 explained "well, in my experience in my classes, some of my proofs were not full sentences with punctuation and capitalization and there was never really an issue about it." This suggests that students may not learn the conventions of mathematical writing by simply observing mathematicians write proofs in class and that students are not made aware of issues with their proof writing until points are deducted.

Others were even surprised that issues of English would be important in a math class, for example, S4 exclaimed "Oh my god, this is a mathematics major, not a linguistic major right? I think it's fine!" and S8 noted this is not an issue because "it's not an English class." This is not to say, however, that none of the undergraduates believed that mathematical language is a subset of academic English; three undergraduate participants did believe that capitalization and punctuation belonged in mathematical proof writing; for instance, S3 explained, "a proof is like a math essay of sorts and it should still be like grammatically correct".

The above suggests that the mathematicians in this study agreed that full sentences should be used when writing proofs. On the other hand, some of the responses from mathematicians and students indicated their beliefs that proper English does not play a role in proof writing at the introduction to proof level. With only one mathematician deducting points for lacking capitalization and punctuation, it is unsurprising that students do not see the necessity of proper grammar in proof writing.

Non-statement

Proof C (shown in Figure 1a) included the following phrase that was ungrammatical and meaningless: "Suppose $(R \circ S)^{-1}$ s.t. $(x, z) \in (R \circ S)^{-1}$". As an imperative phrase with a transitive verb, English grammar dictates the need for both a direct object and an object complement to be a complete sentence. That is, the sentence must suppose the direct object in relation to another object or a property about the direct object. While the mathematicians did not give this exact grammatical explanation, they did note the incompleteness of the sentence.

In Pass 1, seven of eight mathematicians discussed that the proof's first line is not a complete sentence and has no meaning. M8 explained, "the way that I would parse this sentence is, suppose $(S \circ R)^{-1}$. That's in itself a part and again it has no verb. Suppose $(S \circ R)^{-1}$?" M5 similarly noted "Students sometimes say 'let a set' which doesn't mean anything. This is just a nonsense thing to say, suppose this set." Thus, the statement does not suppose a property of the relation, is not a complete sentence, and conveys no meaning. Moreover, seven of eight mathematicians indicated they would deduct points for a nonsensical and incomplete statement. The eighth indicated they would make a note to the student to show the student that the statement was incomplete, but would not deduct points.

Meanwhile some undergraduate participants saw an issue with the statement and attempted to rectify it by completing the sentence, but were unable to articulate what was wrong in the first place. For instance, N3 explained, "I would say 'Suppose $(S \circ R)^{-1}$ is a relation such that (x, z)' is in this relation". On the contrary, some of the undergraduates found no problem with the incomplete sentence; for example, N4 saw no difference between saying 'Suppose $(S \circ R)^{-1}$' and 'Suppose there is a relation $(S \circ R)^{-1}$'. This suggests that some students do not view mathematical language as a sub-register of academic English and do not see the importance of using compete sentences in mathematical proof writing.

Discussion

As this qualitative study considers only a small sample of mathematicians and undergraduate students, the findings are simply suggestive of how mathematicians and undergraduates view the need for proper grammar in undergraduate proof writing. Based on the above, we see for the most part that mathematicians in the study believed that grammar and the parts of speech of mathematical words should be attended to when writing mathematical proofs. This need for complete sentences and attention to grammar is supported by the mathematical writing guides written by mathematicians (Gilman, 1987; Krantz, 1997; Higham, 1998; Houston, 2009; Vivaldi, 2014), who indicate that correct grammar and complete sentences should be used in proof writing. Meanwhile, the results suggest that mathematicians may not be attending to these issues in introductory proof courses.

Questions for the audience

- How might one instruct mathematical grammar to undergraduate mathematics students?
- How can we motivate students to use correct grammar in proof writing if they believe it is unnecessary?

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A critical look at undergraduate mathematics classrooms: Detailing mathematics success as a gendered and racialized experience for Latin@ college engineering students

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Latin@s demonstrated an increase of nearly 75% in engineering degree completion over the last 15 years (National Science Foundation, 2015). However, Latin@s remain largely underrepresented across STEM disciplines with scholars calling for analyses of their undergraduate education experiences to improve retention (Cole & Espinoza, 2008; Crisp, Nora, & Taggart, 2009). With calculus as a gatekeeper into advanced STEM courses, undergraduate mathematics must be examined as a social experience for underrepresented populations including Latin@s. This report presents findings from a phenomenological study on mathematics success as a gendered and racialized experience among five undergraduate Latin@ engineering students at a large, predominantly white institution. In light of recent calls for equity considerations in undergraduate mathematics education (Adiredja, Alexander, & Andrews-Larson, 2015; Rasmussen & Wawro, under review), this report focuses on the Latin@ students' classroom experiences with implications for broadening Latin@s' and other underrepresented groups' access to high-quality, supportive learning in undergraduate mathematics.

Key words: Gender, Intersectionality, Latin@s, Race, Teaching

Introduction and Related Literature

Mathematics has been well documented as a gendered and racialized space for marginalized populations including women (Boaler, 2002; Mendick, 2006), African Americans (McGee & Martin, 2011; Stinson, 2008), and Latin@s[1] (Oppland-Cordell, 2014; Téllez, Moschkovich, & Civil, 2011). Issues of gender and race, however, have largely been studied separately in extant mathematics education research with minimal insight on how their intersections lead to varying forms of mathematics experience. For example, while such intersections have informed sampling of participants such as African American males across studies using a critical race theory lens, race was the primary focus of their analyses with considerations of how gender shaped participants' racialized mathematics experiences left implicit (Leyva, accepted).

Scholars, therefore, are calling for intersectional analyses that highlight variation of mathematics experience among historically marginalized groups at different intersections of their identities (Esmonde, Brodie, Dookie, & Takeuchi, 2009; Martin, 2009; Oppland-Cordell, 2014). Latin@s, in particular, have "seldom been asked for their perspectives on their classroom mathematics experiences" (Varley Gutiérrez, Willey, & Khisty, 2011, p. 27) to shed light on how they negotiate their multiple identities with their mathematics success.

In undergraduate mathematics education, Rasmussen & Wawro (under review) argued that considerations of such equity issues are the "next steps" in understanding how mathematics instruction can be more responsive to the cultural and linguistic diversity in undergraduate classrooms. Last year's Research in Undergraduate Mathematics Education

[1] Drawing on Gutiérrez (2013), the term Latin@ decenters the patriarchal nature of the Spanish language that traditionally groups Latin American women and men into a single descriptor (Latino) denoting only men. The @ symbol allows for gender inclusivity among Latin Americans compared to the either-or form (Latina/o) implying a gender binary.

(RUME) conference proceedings echoed the calls for intersectional considerations of mathematics experiences and identities. Namely, Adiredja, Alexander, and Andrews-Larson's (2015) theoretical report offered a conceptualization of equity for undergraduate mathematics education challenging researchers to pursue data analyses and reporting of findings with a critical awareness of the "intersectionality of identity."

Study Overview and Research Question

In response to this need of equity scholarship to inform more critical approaches in undergraduate mathematics education, this preliminary research report presents findings from a phenomenological study that used intersectionality from critical race theory (Solórzano & Yosso, 2002) and Latin@ critical race theory, or LatCrit, (Bernal, 2002) to characterize mathematics success among five undergraduate Latin@ engineering students at a large, predominantly white four-year institution. The study used a three-tiered analytical framework from prior work (Leyva, under review) to detail the institutional, interpersonal, and ideological dimensions of the Latin@ students' mathematics success.

For this report, these three dimensions are considered in relation to instruction and student learning in the undergraduate mathematics classroom context. This analysis particularly focuses on the extent to which these classroom situations were gendered and racialized experiences as well as how this shaped the Latin@ students' academic pursuits at the university. More explicitly, this report addresses the question, "In what ways did undergraduate mathematics classroom experiences afford or limit opportunities for mathematics success as Latin@ engineering students at the university?"

Theoretical Perspectives

Critical race theory (CRT) in education is a perspective that "foreground[s] and account[s] for the role of race and racism" (Solórzano & Yosso, 2002, p. 25) in efforts to disrupt racism and other intersecting systems of societal oppression (e.g., sexism, classism) in schools and classrooms. One of the CRT tenets in educational research is recognizing what Kimberle Crenshaw (1991) coined as *intersectionality* referring to the mutual constitution of oppression at intersections of race, class, gender, and other identities (Solórzano, 1998). As a "close cousin" to CRT, LatCrit examines intersectionality among Latin@s in relation to issues such as culture, immigration, and language that often go unaddressed under CRT (Bernal, 2002).

Phenomenology informed the study's methodology of collecting and critically examining multiple "texts of life" (Creswell, 2013) to detail the phenomenon of mathematics success among the five Latin@ engineering students at a large, predominantly white institution. Under the CRT perspective, these "texts of life" inform the analytical construction of Latin@ participants' *counter-stories* (Solórzano & Yosso, 2002), or personal narratives challenging racial discourses of mathematics ability among people of color including Latin@s. The study's coupling of CRT with LatCrit guided the analysis of intersectionality across Latin@ participants' counter-stories and classroom experiences in undergraduate mathematics.

Data Sources and Research Methodology

Study Participants

This study took place at a large state university in the northeastern United States during the 2014-2015 academic year. Less than 15% of the 2011-2012 graduating class was

Latin@. These Latin@ graduates earned only 10% of the university's conferred degrees in STEM areas.

The Latin@ participants were purposefully recruited based on criteria informed by scholarship on "successful" underrepresented students in STEM (Cole & Espinoza, 2008; McGee & Martin, 2011; Stinson, 2008). Five Latin@ students (2 women: Diana and Zoila; 3 men: Benito, Cristian, and Daniel) were recruited from the university chapter of the Society of Hispanic Professional Engineers (SHPE), a national organization aimed at empowering the Hispanic community in realizing its potential in engineering through STEM outreach and professional networking.

Data Collection

Four types of data were used: (i) mathematics autobiographies, (ii) field observations, (iii) semi-structured interviews, and (iv) a focus group. Informed by critical race methodology (Solórzano & Yosso, 2002), the autobiographies, interviews, and focus group were used for the analytical construction of Latin@ participants' counter-stories as students in undergraduate mathematics classrooms. Field observations in their mathematics classrooms as well as the engineering and mathematics departments provided situated insights to complement participants' reflections of their experiences. Insights from the study's interviews and focus group will be the focus of the analysis presented in this report.

Throughout the academic year, participants completed four 60-minute, semi-structured individual interviews. All interviews were audiotaped and transcribed verbatim. The interviews were opportunities for participants to share and explore what being Latin@ and mathematically successful meant to them across different contexts (e.g., classroom, home, SHPE meetings). Interview questions were structured in an open-ended manner allowing participants to describe varying levels of consciousness of their different identities across these contexts including the mathematics classroom (Bowleg, 2008).

In addition, participants completed a focus group structured around three stimulus narratives based on observations in their lectures and recitation/workshop sessions. These narratives related to ideas of students taking up classroom space, stereotypes of mathematics ability, and faculty-student relationships. Participants were probed on the extent to which they observed such dynamics in mathematics classrooms and whether or not they saw themselves in similar situations. The focus group was audiotaped and transcribed verbatim.

Data Analysis

Phenomenology guided data analysis by focusing on patterns across participants' mathematics experiences to detail the phenomenon of mathematics success among these undergraduate Latin@ students (Creswell, 2013). Open codes were used to identify the institutional, interpersonal, and ideological influences on mathematics success while axial codes examined the intersectionality across participants' mathematics experiences (Bowleg, 2008; Creswell, 2013). While some axial codes were specific to individual identities (e.g., race, gender), other axial codes corresponded to different intersections of these identities such as gender-race (Bowleg, 2008). Implicit instances of intersectionality were made explicit by constructing analytical narratives for each participant (Angelillo, Rogoff, & Chavajay, 2007).

Validity was reinforced through triangulation of collected data, memoing, and member checking. I brought awareness of my positionality to pursue data analysis with strong subjectivity and build nuanced understandings of the Latin@ engineering students' mathematics success. In addition, I developed positive rapport and mutual trust with participants supported by our mutual identifications as Latin@ STEM majors.

Findings

The following section presents findings from the analysis of interview and focus group data organized by institutional, interpersonal, and ideological influences on the undergraduate Latin@ engineering students' mathematics success. In alignment with the constructive goals of RUME's preliminary research report presentations, discussion about this subset of the study's data analysis will guide my next research step in triangulating participants' classroom reflections with their observed behaviors in mathematics lectures and recitations/workshops. This will allow for consideration of confirming and disconfirming evidence across data sources to characterize the Latin@ engineering students' strategies in successfully navigating the gendered and racialized spaces of their undergraduate mathematics classrooms.

At the institutional level, participants described how instruction in their mathematics lectures limited their participation in terms of asking questions, volunteering answers, and correcting the instructor. Cristian described how although he "captures little concepts" during lectures, he "do[es] not learn in class" and instead does much of his mathematics learning at home when reviewing his notes. Diana also reflected on how lectures' fast instructional pace caused her to have to write notes without "taking them in or processing them." This resulted in what participants described as quiet mathematics lectures with only a few other students participating – namely, whites and Asian Americans. While Daniel characterized these more active class participants as the "same people who go above and beyond," Cristian asserted that they were the "nerdy kids" from high school who "want to know everything and get the highest grade" which typically did not include Latin@s.

Interpersonally, participants viewed strong relationships with their mathematics teachers as motivation to not let them down and be successful in their classes. They, however, described minimally connecting this way with their undergraduate mathematics instructors. Benito posited that in order to establish strong teacher-student relationships, professors and graduate teaching assistants (not just students) have to "make an effort to build that relationship" which he did not readily observe at the university. As an example of this limited teacher approachability, he commented on how the chemical engineering department attempted to "humanize" its faculty members by mandating them to make small self-introductions including personal interests on the first day of their courses.

Both Cristian and Daniel commented on how having shared racial and gender identifications with their instructors would positively impact their participation and performance in undergraduate mathematics. Cristian, for instance, saw himself being more comfortable and "willing to correct the professor… [if] he is a male professor" considering the underrepresentation of women faculty in mathematics. Daniel looked back on how his former Latin@ college calculus professor's use of the Spanish language and sharing of childhood stories in Honduras separated him from other university mathematics faculty who "felt like robots." It was this professor connection that Daniel raised as an explanation for his "metamorphosis" as a college mathematics student characterized by sitting toward the front in lecture, attending office hours, and ultimately passing first-semester calculus after failing it the first time and being placed on academic probation.

From an ideological standpoint, participants raised the discourse of a racial hierarchy of mathematics ability (Martin, 2009) with whites and Asian Americans being better at mathematics than African Americans and Latin@s. This discourse allowed them to make meaning of undergraduate mathematics classrooms' "competitive" feel and minimal opportunities for peer connection. Diana reflected on how most of her mathematics professors curved course grades so one's performance is contingent on how the entire class performed. As a result, students were aware of those "at the top" scoring near-perfect exam scores who Diana described as commonly being Asian American classmates "mak[ing] themselves known" by publicizing their high grades. Diana's racialized views of who was successful in undergraduate mathematics brought her to feel as though her classroom participation as a Latin@ was subject to closer scrutiny particularly from white and Asian American peers who were possibly thinking, "Why are you talking?" Cristian asserted that such peer judgments coupled with limited opportunities to connect with other students one-on-one in large lectures often resulted in "closed off" opportunities to study with classmates who he saw as more mathematically capable than him.

Implications for Teaching Practice

Although findings from this study are not generalizable to all Latin@ student populations and higher education institutions, they raise key implications to advance undergraduate mathematics teaching informed by critical awareness of mathematics as a variably gendered and racialized experience for Latin@s and other underrepresented groups. Questions for audience discussion during the RUME presentation are raised throughout this section.

First, the Latin@ participants' reflections on their mathematics lectures resonate with Rasmussen and Wawro's (under review) argument for equity considerations in structuring undergraduate mathematics instruction. I argue that such pedagogical mindfulness, however, should not be limited only to post-calculus courses considering how the Latin@ students expressed limited opportunities to meaningfully engage with instruction in calculus. Furthermore, it is well documented that entry-level mathematics courses like calculus serve as a gatekeeper for underrepresented groups' access to advanced STEM coursework (Chen, 2013). What literature would be useful to further explore values of classroom instruction and student learning across P-16 mathematics? How can university faculty better support Latin@s and other marginalized students whose pre-college mathematics learning approaches may differ from those used to inform undergraduate instruction?

Secondly, participants' discussions of feeling disconnected from their university instructors capture the importance of teacher-student relationships in their mathematics success. This aligns with scholarship that highlights how students of color's academic success is characterized by high-quality instruction coupled with caring, supportive teacher relationships allowing for increased access to mathematics and STEM at large (Battey, 2013; Brown, 2002). To what extent has culturally responsive pedagogy been examined in undergraduate mathematics education? What approaches to undergraduate mathematics instruction establish relational spaces with underrepresented students such that their ability and cultural backgrounds are acknowledged and valued throughout the learning experience?

Lastly, participants invoked gendered and racial discourses of mathematics ability to make meaning of their positioning along a hierarchy of success across undergraduate mathematics classrooms (Boaler & Greeno, 2000; Leyva, under review; Shah, under review). Zoila, for example, shared how she held ideas that "whites and Asian Americans are smarter" making her often feel intimidated by these peers. Such discourses, however, often hindered the Latin@ students from connecting with classmates even though such networking played an important role in their pre-college mathematics success. This challenges Treisman's (1992) claim of students of color as inherently not able to form peer networks and thus raises the question of what practices in undergraduate mathematics teaching can disrupt existing forms of gendered and racialized status of who are seen as "doers of mathematics" in classrooms.

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Students' conceptions of factorials prior to and within combinatorial contexts

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Counting problems offer rich opportunities for students to engage in mathematical thinking, but they can be difficult for students to solve. In this paper, we present a study that examines student thinking about one concept within counting, factorials, which are a key aspect of many combinatorial ideas. In an effort to better understand students' conceptions of factorials, we conducted interviews with 20 undergraduate students. We present a key distinction between computational versus combinatorial conceptions, and we explore three aspects of data that shed light on students' conceptions (their initial characterizations, their definitions of 0!, and their responses to Likert-response questions). We present implications this may have for mathematics educators both within and separate from combinatorics, and we discuss possible directions for future research.

Keywords: Combinatorics, Discrete mathematics, Factorials, Counting

Introduction and Motivation

Counting problems provide opportunities for interacting with problems that are easy to state and understand, but that require deep and non-algorithmic mathematical thinking. Brualdi says the following about counting problems: "The solutions of combinatorial problems often require *ad hoc* arguments sometimes coupled with use of general theory. One cannot always fall back onto application of formulas or known results" (2004, p. 3). In this paper, we explore one specific concept within counting that we feel plays an important role in combinatorial enumeration: factorials. The factorial of a natural number n is defined as the product of the first n natural numbers (for instance, Epp (2010) defines n factorial as follows: "For each positive integer n, the quantity n factorial denoted $n!$, is defined to be the product of all the integers from 1 to n" (p. 181)). Factorials themselves are defined and exist in contexts outside of a combinatorial setting, and yet they play a significant role in the solving of counting problems – both due to the fact that they can be interpreted as having inherent combinatorial meaning and because they are a basic component of many fundamental counting formulas. Because of this, we are interested in learning more about what conceptions students have of factorials within and without the counting context, and we seek to better understand how students' pre-existing conceptions of factorials (even prior to reasoning about them in a combinatorial context) might interact with their combinatorial thinking about factorials. We seek to answer the following research question: *How might we characterize students' initial conceptions about factorials, and how do such conceptions interact with students' solving of counting problems?*

Literature Review and Theoretical Perspectives

Counting problems have been shown to be difficult for students at a variety of levels. This is seen both in low success rates (e.g., Eizenberg & Zaslavsky, 2004) and in qualitative evidence of student struggles (e.g., Batanero, Navarro-Pelayo, & Godino, 1997; Hadar & Hadass, 1981). Although difficulties persist, there has also been growing evidence of students' success with counting problems. This has included identifying beneficial ways of thinking about counting problems (e.g., Lockwood, 2014; Halani, 2012; Maher, Powell, & Uptegrove, 2011), identifying and testing the value of particular instructional interventions (Lockwood, Swinyard, & Caughman, 2015; Mamona-Downs & Downs, 2004), and developing models and schemes of students' combinatorial thinking (Lockwood, 2013; Tillema, 2013). One possibility toward continued understanding of the state of students' combinatorial reasoning is to look at a very specific yet fundamental concept related to counting.

In this paper, we focus on the concept of factorials because it is itself a foundational combinatorial idea with a widely applied formula, and it is a key aspect of many other combinatorial concepts (and their formulas) that students encounter as they solve counting problems. Though we did not identify any research explicitly about students' reasoning about factorials, there has been a tradition, beginning with Piaget and Inhelder (1975), of closely examining students' mental processes involved in reasoning about particular concepts. We are also motived by a recent study (Lockwood, et al., 2015) in which students seemed to have preconceived notions of factorials that were affecting their reasoning about counting problems. We seek to better understand if other students hold similar preconceptions, and also to consider if and how certain conceptions come into play as students reason about factorials in counting contexts. We take conceptions to mean a students' mathematical understanding about a particular idea.

The study is framed within Lockwood's (2013) model of students' combinatorial thinking. This model proposes three components of students' combinatorial thinking and elaborates the relationships between these components. *Formulas/Expressions* are mathematical expressions that yield a numerical value, while *Counting Processes* refer to the processes in which a counter engages (either mentally or physically) as they solve a counting problem. Finally, *Sets of Outcomes* consist of the collection of objects being counted – those sets of elements that are generated or enumerated by a counting process (p. 252 – 253). In this paper, we use the model as a way of framing students' combinatorial reasoning about a particular combinatorial construct.

Methods

Participants. The participants in this study were 20 undergraduate students at a large university in the western United States. Fourteen of the students were taking a calculus course at the time of the interviews, and six were taking 300-level mathematics courses that came after discrete mathematics, including advanced calculus, topology, linear algebra, probability, and numerical analysis. We targeted these two groups of students because we wanted to include in our sample both some students who had taken discrete mathematics at the university level and some who had not. There were fourteen male students (ten in calculus, four in the advanced courses) and six female students (four in calculus, two in advanced courses).

Data collection. The design of our study was to conduct individual, semi-structured, task-based interviews (Clement, 2000; Goldin, 1997), and we used Livescribe pens to record the interviews, which allowed us to capture what students wrote and said in real time. We created a list of symbols commonly seen in college-level mathematics classes, and we asked the students to indicate which symbols they recognized. If the students recognized the factorial symbol, we proceeded to ask them a line of questioning to probe their understanding of factorial. We then asked them a handful of initial questions about factorials, including statements that required a response on a Likert scale. We concluded the interview by giving the students counting problems one at a time, asking them to verbally explain their reasoning and write down their thought processes as they went. If students did not recognize the factorial symbol, or recognized it and gave an incorrect interpretation of the symbol, we immediately gave them the counting problems, returning to the reflective questions and Likert response statements if they showed any subsequent sign of having remembered what the factorial symbol meant. Because students' responses and background varied considerably, and because the interviews were semi-structured, there is a variety among the specific tasks and counting problems each student completed.

Data Analysis. After conducting and recording the interviews, we re-listened to the interviews and watched the files outputted by the Livescribe pens. We created content logs of the interviews, which provided a detailed, time-stamped description of what happened and included transcriptions of particularly noteworthy episodes. The research team discussed and coded the students' responses to the initial questions, and the data revealed emergent dimensions of students' conceptions of factorial, which we discussed in relationship to Lockwood's (2013) model of combinatorial thinking. Next, we studied the Likert responses by first calculating the average and standard deviation of the responses for each question. We also discussed responses that were particularly surprising and attempted to determine and articulate why students might have responded in unexpected ways. Finally, each author individually coded student responses to the counting problems, looking for particular factors of interest for each problem. Any discrepancies in coding were addressed via discussion among the research team.

Results

In presenting the results, we focus on sharing empirical evidence of students' initial conceptions of factorials, as seen in their responses to initial questions and to the Likert-response questions. Due to space we do not report on their work on the counting problems.

Computational and Combinatorial Conceptions of Factorials

A major finding about students' reasoning about factorials is that there is a fundamental distinction between two conceptions of factorials: computational and combinatorial. We do not claim that is it mathematically a new insight. However, this distinction arose in our data in several ways, and we share it as a finding because it seemed to reflect an important difference in conceptions for our students. By a *computational* conception, we mean that students think of n factorial in terms of its numerical definition as the product of the first n positive whole numbers. A student with a computational conception might be able to use and manipulate factorials in expressions or equations, as strictly a numerical calculation. A *combinatorial* conception involves an understanding that factorials have some intrinsically combinatorial meaning – specifically as being related to the number of ways of arranging n distinct objects. Someone with a combinatorial conception may think of $n!$ as the number of arrangements of n distinct objects, and they additionally may be able to conceive of a process by which the product $n!$ can generate all of the arrangements of n things. With a combinatorial conception, there is a natural way to relate factorial to some combinatorial meaning and not only as a computable expression. We provide evidence of these two conceptions in students' initial characterizations and in their definition of 0!.

Students' Initial Characterizations of Factorial. For each student who had recognized the factorial symbol, we asked them what they thought it meant and how they would explain it to someone else. Of the 20 students, 15 had seen the factorial symbol before and could provide a statement of it. Of those 15, 12 of the students gave a correct definition, with all of these students expressing factorial in terms of its computational definition. Eight of them defined $n!$ as a decreasing product, such as Student 6 who replied, "You take whatever number n is and you multiply it by all whole numbers fewer than it, stopping at 1." The remaining four students who provided a correct definition described $n!$ as an increasing product. For example, Student 17 said, "You're multiplying all the integers together in order until the n."

In addition to providing correct computational definitions, there were two students (1 and 8) that additionally defined factorial combinatorially. Student 1 defined factorials as follows: "It's called n factorial, and it gives us the number of possibilities we can arrange things in order. Like, if we have n distinct objects, and we'd like to put them in a certain order, we'll have n factorial which is n times n-1 times dot dot dot times 1." Similarly, Student 8 gave his initial computational definition by saying "I would just say it's n times $n - 1$ times $n - 2$ all the way down to 1. I would, you know, explain it recursively." When asked if he could think of more than one way to explain a factorial to someone else, he responded,

Student 8: "Hmm. I guess if you're talking about, like permutations of, yeah, if you're talking about permutations of, like, 8 objects or something, you'd say, okay so for the first one I have 8 choices, then seven choices, then six choices, then five choices, and explain that and say 'Oh, and this is really annoying to write it out, so we'll call it the factorial function.'"

The three students who incorrectly characterized factorial did so as involving addition instead of multiplication, which suggests perhaps that these students were familiar with the notion of factorial but did not have a solid understanding of it. As an example, Student 20 said, "Um, I think it was just, um, to notate the fact that, um, uh, it repeat, well, the symbol is, like, like, 5 exclamation point is like 5+4+3+2+1, isn't it? And I forget why we needed to use that." Although only two students defined factorial combinatorially here, a number of students also brought up combinatorial interpretations of $n!$ in other parts of the interviews, such as when they were asked to solve counting problems or when justifying why 0! is 1. This suggests that while most of the students had existing computational conceptions of factorial, some had a sense that factorials could apply in a combinatorial setting.

Student definitions of 0!. We also saw evidence of computational versus combinatorial conceptions in students' discussions of how 0! should be defined. We asked 12 students about how they understood 0!, and nine students responded (the other students did not have a guess). We note that the convention of having 0! defined as 1 is easily justified by the combinatorial characterization of factorial, because there is only one way to arrange no objects. The computational justification for why $0! = 1$ is related to the convention of empty products (the product of no factors) equaling the multiplicative identity, which is 1.

Although there exists a computational, non-combinatorial justification for why 0! might be 1 (the empty product convention) the students who did not provide a combinatorial justification were not able to articulate this argument – indeed, many of the reasons the students provided are not correct or convincing. For example, Student 9 said 0! would be 0 and explained, "I'm not sure if this is the technical definition, but n factorial is all the numbers from n to 1, and so if n was zero, then it'd be like zero times nothing." In fact, the students who did not reason combinatorially about 0! did not provide reasonable justifications about why 0! is 1. These results suggests that students' computationally knew how 0! was defined but had not thought deeply about why that might conceptually make sense. The majority of students seemed strictly to have a computational conception of factorials, while a handful of students recognized that factorials may have some combinatorial meaning. Even fewer (just two students, 1 and 8) demonstrated a robust combinatorial conception of factorials. We are not saying that the calculus students *should have* been able to give a combinatorial justification (as they might not have been exposed to a combinatorial context), but these findings suggest that students may have existing conceptions of factorials that they may bring to combinatorial situations.

Responses to the Likert-response Questions

We also get a sense of students' conceptions of factorials by evaluating their Likert-response questions. We do not have space to share all of the responses to these questions, but we highlight a couple of points of discussion. First, the answers to some of the Likert-response questions suggested that many students do associate factorials with counting, if only vaguely, even if this was not demonstrated in their initial characterizations of factorials. To elaborate this point, we consider Statement #5 (*Factorials could be used to solve a counting problem like, "How many possible outcomes are possible if you flip a coin ten times?"*), which speaks to the combinatorial nature of factorial. We would expect the answer to this problem to be a 1 (strongly disagree) because factorials are not appropriate for solving this kind of counting problem (the answer should be 2^{10}, as the number of options does not decrease for each successive stage in the counting process). The average of all responses to this question was 2.77, and the standard deviation was 1.64. Looking more closely at the responses, 8 of the 13 students seemed to recognize that factorials were not appropriate for this counting problem. For example, Student 8 responded with a 2 and said, "Factorials are useful when you have problems that involve, like, uh, situations where your choices, diminish? Like, where you do something, and then the next thing you do you have fewer possible outcomes, um, and that's why they're—yeah, that's why that form is useful." However, 5 of the 13 students agreed or strongly agreed with this statement. In discussing this problem, those who responded with 4's or 5's suggested that they associated factorials with any kind of counting problem. For example, Student 7 responded to the statement with a 5 and said, "I think this has to do with probability, and we would always use factorials in probability. So, uh, I think there is definitely a way to use factorials to solve that." This suggests to us that for these students, factorials are vaguely associated with counting in their minds, but that its combinatorial meaning may not be precisely defined.

Similarly, Statement #7 (*n! is the number of ways to rearrange n objects, even if some of them are identical*) provides further evidence of this phenomenon. We would expect the answer to this statement to be a 1 as well, but the average of the student answers was 2.54 with standard deviation 1.51. More closely examining the students' answers, the majority of students understood a factorial to mean counting arrangements of distinct objects, but there were still four students who agreed or strongly agreed with this statement, again suggesting a vague association with factorials and counting.

Given our previous section that highlights the fact that most students had computational and not combinatorial definitions of factorials, it is not surprising that most students likely did not have a robust understanding of how factorials fit in with solving counting problems. However, the student responses to #5 and #7 suggest to us that students are bringing with them pre-conceived ideas about factorials as they relate to combinatorics that are perhaps not clearly or well defined. These findings suggest that instructors of discrete math and combinatorics should be aware of the kinds of pre-existing notions students might have about factorials.

We also saw some evidence of students' ability to see the multiplication in factorials as a counting process, which suggests perhaps a connection between computational and combinatorial conceptions of factorials. A final item to discuss is Statement #13 ($2\times4\times3\times1$ *is the same as 4!*). Every student responded to this statement with a 5, except for Student 17 who gave it a 4 and Student 1 who gave it a 3. The students overwhelmingly justified their responses by appealing to the commutativity of multiplication of real numbers. This underscores this idea that the multiplication in factorials is just the operation of multiplication, which is commutative. So, we want to emphasize that in some sense, the students are correct that, as a numerical result, the product $2\times4\times3\times1$ is the same as the product $4\times3\times2\times1$ – it must be, because the operation of multiplication of real numbers is commutative. However, we would argue that combinatorially, these two products are not "the same." To see this, we must consider the counting processes and how those processes might be structuring the set of outcomes. In terms of Lockwood's (2013) model, the expression $4\times3\times2\times1$ suggests a counting process with four stages, in which the first stage has 4 options, the second stage 3 options, the third stage 2 options, and the last stage 1 option. The expression $2\times4\times3\times1$ suggests something else entirely – that in a four-stage process the first stage has 2 options, the second 4, the third 3, and the fourth 1. There are plenty of counting processes that reflect this idea (for instance, forming outfits from 2 shirts, 4 pants, 3 hats, and a belt), but they are different than the processes that underlie 4!. In addition, while the multiplication $4\times3\times2\times1$ corresponds naturally to a particular organization of the arrangements of four distinct objects, it is less natural to find an organization of those arrangements corresponding to the multiplication $2\times4\times3\times1$. In this way, the orders of multiplication suggest different relationships between the Formulas/Expressions and Set of Outcomes components of the model.

Student 1 is the only student to have addressed this issue, and not surprisingly he is the student who seemed to have the most robust understanding of factorial in the entire study. Student 1 said about Statement #13, "I know they have the same value, but I don't think this one contains information that 4 factorial has." When asked about the information that 4 factorial has, he said, "4 factorial tells us I'm counting something. But, just, $2\times4\times3\times1$also tells us we're counting something, but, um, I don't know." To us, this suggested that 4 factorial had some different meaning for him in terms of counting objects than $2\times4\times3\times1$. When asked if the two had different meanings to him, he said, "Yeah, I feel differently. Like, if you have 2 options for the first place, then 4 options for second, then 3 and 1, you'll have this number of possibilities. But, for 4 factorial, it means you're doing a specific kind of counting, like, hmm, like ordering—yeah, ordering things. Not just counting the number of [possibilities]. I think there's something more." In saying this, he demonstrated a strong understanding of the multiplication principle, and in particular the way in which the order of multiplication corresponds to distinct, temporal stages in the counting process.

In contrast, Student 17, who gave Statement #13 a 4, only justified his response by saying, "It's the, it's commutative, so, it—they mean the same thing." When asked why he put a 4 and not a 5, he answered "Uh, kinda because it's, like, different to put it that way. It's not, it's not what you would normally put as 4 factorial. I mean, I put a 4 because it's not wrong." This suggests that even Student 17 did not think the expressions were different because they have inherently different combinatorial meaning, merely that $2\times4\times3\times1$ is an unconventional but equivalent way of writing 4 factorial. The responses to Statement #13 provide for us an interesting insight about how students might view factorials, and it suggests that there is more to be investigated about how students understand the multiplication within factorials, especially as it relates to counting processes (and how those processes might generate and structure sets of outcomes).

To summarize our results, the students' initial conceptions revealed a couple of salient points of discussion. First, 15 of the students had seen factorials before, even though only 6 of them had taken coursework beyond discrete mathematics. Their initial definitions revealed that students predominantly conceptualize factorials computationally, although, while only two initially gave a combinatorial definition of factorial, four additional students suggested that they understood that factorials were related to counting (even if imprecisely). The students' responses to the Likert-scale questions gave evidence that many of them thought that factorials were related to counting in general (even if imprecisely), and in addition, the Likert responses revealed the variety of ways in which students conceptualize aspects of factorials. Getting a better sense of students' pre-existing conceptions of factorial (particularly the distinction between computational and combinatorial conceptions) is important as we consider what it might mean for students to have a robust understanding of factorial (and how we might help them adopt such an understanding).

Conclusions, Implications, and Directions for Future Research

We have evidence that students without combinatorial experience (or without a clear combinatorial conception of factorial) are coming into counting situations with some knowledge of factorial. It is important for researchers and teachers of combinatorics to have a sense of what kinds of preconceived ideas their students might have about factorials, and highlighting the combinatorial/computational distinction illuminates different conceptions that student may have as they are introduced to factorials in a discrete mathematics or probability class.

Teachers who might introduce the computational definition of factorial (such as in a calculus class or an induction proof) should be aware of the fact that students might eventually need to understand factorials in a combinatorial context. Factorial should be framed not as some meaningless computational fact or rule, but rather as a flexible and efficient way of writing a product. Teachers who teach discrete mathematics should bear in mind that students might have been introduced previously and might have a purely computational understanding of factorial. It may also be beneficial to realize that students' combinatorial understanding of factorial may be reflected in the three components of Lockwood's (2013) model. The formula should relate to counting processes, which should also relate to the sets of outcomes, and computational facility should be explicitly tied to what that might afford combinatorially. Framing factorials in this way can help to develop in students robust and flexible understandings of factorial. Because factorials appears so frequently and are a key aspect of so many topics and formulas in counting problems, it is important for students to understand them well.

There are a couple of potential directions for future research based on this study. Given that our findings are based on a relatively small number of students, it would be possible to investigate conceptions about factorials among a larger set of students. In so doing, we could investigate whether the distinction between computational and combinatorial conceptions of factorials exists more broadly for other students. In addition, the multiplication principle is a key underpinning idea in factorials, more work can and should be done to examine this fundamental idea and even the role it may play in students' reasoning about factorials.

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Students' meanings of a (potentially) powerful tool for generalizing in combinatorics

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In this paper we provide two contrasting cases of student work on a series of combinatorial tasks that were designed to facilitate generalizing activity. These contrasting cases offer two different meanings (Thompson, 2013) that students had about what might externally appear to be the same tool – a general outcome structure that both students spontaneously developed. By examining the students' meanings, we see what made the tool more powerful for one student than the other and what aspects of his combinatorial reasoning and his ability to generalize prior work were efficacious. We conclude with implications and directions for further research.

Key Words: Combinatorics, Generalization, Counting problems, Mathematical meanings

Introduction and Motivation

Generalization is a fundamental mathematical activity with which students at all levels engage (Amit & Klass-Tsirulnikov, 2005; Lannin, 2005; Peirce, 1902), and yet there is still much to learn about ways to foster productive generalizing activity. In particular, most of the work on generalization has been with younger children, commonly in algebraic settings (Amit & Neria, 2008; Becker & Rivera, 2006; Cooper & Warren, 2008; Ellis, 2007b; Mulligan & Mitchelmore, 2009; Radford, 2006; Rivera, 2010; Steele, 2008). In the context of a larger study, we sought to better understand students' generalization in the domain of combinatorics which involves the solving of counting problems and provides students with opportunities to engage with accessible yet challenging tasks (e.g., Kapur, 1970; Tucker, 2002). In this paper, we compare and contrast two students' work on a series of combinatorial tasks, during which they each spontaneously introduced a potentially powerful tool for generalization in the combinatorial setting. Each of these students used this new tool, but they varied in the meaning they seemed to make of the tool. As a result, they differed in how effective they were able to be in using the tool generally and solving combinatorial tasks. We seek to answer the following research question: *What meaning do students make of the same spontaneously generated tool (which we refer to as the 11xx structure), and what do these meanings suggest about students' generalization in combinatorial contexts?* The results in this paper help to inform research on students' meanings in the context of both their generalizing activity and their combinatorial thinking.

Literature Review and Theoretical Perspective

The act of generalizing is a key aspect of students' mathematical development, and both mathematics education researchers (e.g., Amit & Klass-Tsirulnikov, 2005; Davydov, 1972/1990; Ellis, 2007b; Vygotsky, 1986) and policy makers emphasize its importance (the Common Core State Standards highlight generalization in both the content and the practice standards; Council of Chief state School Officers, 2010). We seek to extend the current work on generalization by focusing on undergraduate students in the context of combinatorics. The tasks we designed were designed with the overall aim of facilitating students' generalizing activity, and for this purpose we follow Ellis (2007a) (who drew on Kaput, 1999) in defining generalization as "engaging in at least one of three activities: a) identifying commonality across cases, b) extending one's reasoning beyond the range in which it originated, or c) deriving broader results about new relationships from particular cases" (p. 444). We chose the context of combinatorics in part to examine generalization in a novel context, but we were also motivated to contribute to previous work on students' combinatorial thinking. There is evidence that students struggle with solving counting problems correctly (e.g., Batanero, Navarro-Pelayo, & Godino, 1997; Hadar & Hadass, 1981), and we hope to contribute to the existing literature by providing instances of meaningful combinatorial reasoning that might ultimately inform instruction.

We draw on Thompson's (2013) notions of meanings in this paper. He argues for the importance of developing meaning of the idea meaning (p. 57) and that a greater emphasis on mathematical meaning could contribute to a more coherent educational experience for students overall. Thompson surveys different meanings of "meaning," and we adopt his alignment with a Piagetian view of meaning as assimilating a scheme (p. 60). Thompson notes that, "From a Piagetian viewpoint, to construct a meaning is to construct an understanding – a scheme, and to construct a scheme requires applying the same operation of thought repeatedly to understand situations being made meaningful by that scheme" (p. 61). Also, importantly, Thompson emphasizes meaning from the students' perspective:

> "The meanings that matter at the moment of interacting with the students are the meaning that students have, for it is their current meanings that constitute the framework within they operate and it is their personal meanings that we hope students will transform" (p. 62).

For this reason, in this paper we seek to understand and interpret students' meanings in order to gain insight about what made their work particularly productive (or unproductive) in the contrasting cases. We use this notion of meanings in this paper because we have a situation in which two students introduce and use a tool that externally seems very similar, but their different meanings of that tool cause them to use it differently. We thus find it useful to discuss the variety of meanings students had about what appears to be a very similar mathematical phenomenon.

Methods

In this study we conducted a set of single, individual, hour-long interviews with ten calculus students as they worked through what we call the Passwords tasks, and in this paper we report on two contrasting cases of students' work. We chose students who had not been taught combinatorics formally at the college level. The main goal of the tasks was to put students in a situation in which we could evaluate their generalizing activity as well as gain insight into their combinatorial reasoning. The progression of Passwords tasks is as follows:

First, we had students solve the problem, *How many 3-character passwords can be made using the letters A and B?*, and we explicitly directed them to organize their work by completing tables according to the number of As in the password. We had them begin with 3-character passwords, and then also make tables for 4-character then 5-character passwords (Figure 1 shows Tyler's table for the 4-character AB passwords). There were some opportunities to observe generalization in building up these cases, as students could observe relationships and similarities among the tables or could make combinatorial observations that held across cases.[1] We wanted the students to build (typically through partial or complete listing) the tables to see how they would use them as we progressed to the next part of the tasks.

[1] The reader may note that the emerging tables involve binomial coefficients, or the ways of selecting positions for the A's to go, as the remainder of the positions must consist of B's (these entries also correspond to rows in Pascal's triangle). We did not expect students to recognize binomial coefficients or even to conceive of these entries as involving choosing in any way, and indeed most students did not. Eventually, we could increase the *number* of numbers in the passwords, and ultimately move toward developing a formal statement of the binomial theorem (which we accomplished with one student, not discussed here).

Then, we moved onto passwords involving the number 1, and the letters A and B. We had students make tables for 3-character and 4-character passwords, organized according to the number of 1s in the passwords (Figure 6 shows Tyler's table for the 4-character 1AB passwords). Note that we can use the previous tables in the following way: we can think first of determining positions for the 1s (which the previous AB table provides), and then the problem is reduced to counting passwords involving only A's and B's. For example, in making a table for the number of 4-character passwords with 1, A, and B, for each respective row of the table (0, 1, 2, 3, or 4 1's), we can first think of counting the number of ways of placing the ones. There are 1, 4, 6, 4, 1 respective ways of doing this, which is reflected in the previous 4-character AB table. Once this is established, note that for each row in the table, once the ones are placed it is just a matter of counting passwords of length 3 using only A's and B's, reducing the problem to a previous problem (specifically, there are 2^3 such passwords). The point is that it is possible, with some combinatorial insight and understanding of the outcomes' structure, to leverage the previous work from the AB passwords in the more complicated 1AB passwords case. The interviews were videotaped and transcribed, and overall the videos and transcripts were analyzed so as to construct a narrative about the teaching experiment (Auerbach & Silverstein, 2003). We discussed the two contrasting cases with the entire research team and together formulated hypotheses about the students' meaning in each case via repeated viewings of video and reading of enhanced transcripts.

Results

In presenting our results, we describe different students' meanings of the same phenomenon. We highlight these results both to show an interesting phenomenon that emphasizes a potentially powerful tool toward meaningful combinatorial generalization and to suggest that students might need to ascribe certain meanings to such tools in order to leverage them in an impactful way.

Example 1 – Tyler. We begin with Tyler, who demonstrated an ability to reason comfortably and easily with outcomes. His method of solving the tasks typically involved some organized listing. For example, in trying to determine the number of 4-character AB passwords with exactly two A's, Tyler made the list in Figure 2 and gave the following explanation:

Tyler: Um. Yeah I guess I started with the first one being A um, and then I did like 2 A's consecutively and then B's, and then moved the B over one, and then, um, moved the next B over one… And then, after that I just start with the B and kind of did the same thing.

He ultimately correctly created the table for 3, 4 (Figure 1), and 5-character AB passwords.

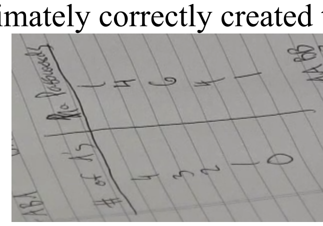

Figure 1

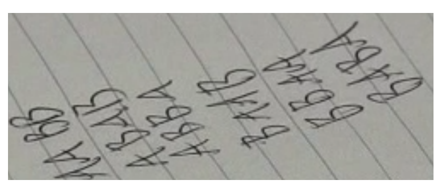

Figure 2

Early in the interview, Tyler had established that there were a total of 2^n n-length passwords using only A's and B's. He established this primarily through noticing a numerical correlation after giving the totals for 3, 4, and 5-length passwords, read from his empirical tables (noticing the 3-character AB passwords table had 1+3+3+1 = 8 total passwords). He went on to write the relationship "*n length = 2^n combos*," but we suspect that he did not meaningfully understand the multiplication principle as a combinatorial way of explaining the expression 2^n.

We then moved on to counting passwords that consist of characters 1, A, or B. Tyler felt that there was more to keep track of, but he persisted with listing outcomes and filling out the table as he had in the previous situations. He managed to list the entire table for the 3-character 1AB passwords, and again he used systematic listing used to do so, and he seemed to maintain a combinatorial understanding of the entries in his tables.

Next, we asked him with to fill out a table with the four-character 1AB passwords (organized according to number of 1s). He got started but paused and said, "I can't really think of any pattern," and he seemed to realize that this table would be more difficult to work out than the previous case. We directed him to perhaps start thinking about the rows for zero and one 1's (starting from that end of the table). Tyler then did something unexpected – he introduced a way of describing a general outcome involving 1's and x's, as seen in the following exchange.

Tyler: *Yeah. Ok so the 0 [row] was gonna be, what did I come up with here [refers back to the 4-character AB table] 10, 15, 16 if uses, um. And then the 1, so what I was thinking, what I was saying earlier. How there is only a certain amount of spots for it, like it has to be, like I'm just gonna use x cause, um, has to be in one of these spots... [draws Figure 3]*

Int.: *Great.*

Tyler: *So there's, now there's just 3 x's um, and I know that for…3 spots with 2 different letters there's going to be 8 different ways to do it [points back to the previous 3-character AB table, see Figure 4]…Um so I guess 8…there's 8 different of each of those just using this same table umm, there's just 32 so I want to say there's gonna be, um, 32 for just the 1.*

Int.: *Okay and you got, you're thinking of that as kind of the 4 times 8?*

Tyler: *Yeah I, just adding them all up.*

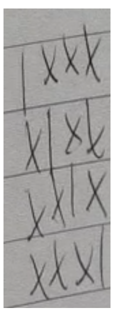

Figure 3

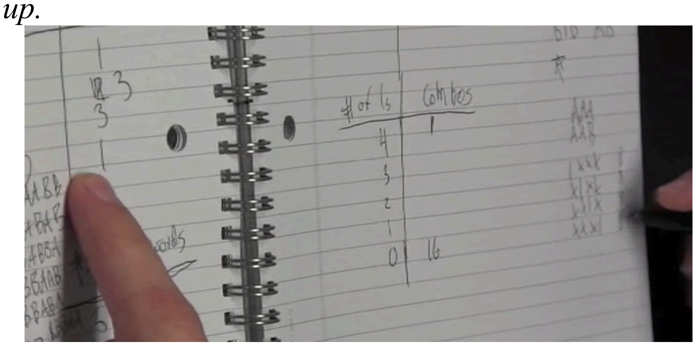
Figure 4

This was a key moment in Tyler's work. He spontaneously introduced a very powerful tool for how to count desirable passwords in the form of a general structure consisting of x's and 1's. (For ease of communication, we hereafter refer to the tool as "the *11xx* structure," which is meant to suggest the introduction of the variable of x as a means of representing a more general outcome.) We contend that this was a general representation of an outcome (a password), perhaps a product of his rich facility with listing. He realized that in each case where there was a 1 with three x's, there would be 8 such possibilities (because there were 8 total 3-character AB passwords), and his total would be 32. He was thus able to recognize that he could use his previous case as a part of the more complicated new situation. We can further explore this moment of insight as he continued to use the *11xx* structure in filling out the rest of the 40-character 1AB table. Figure 5 shows his listing of x's and 1's in the four-character 1, A, B case, with exactly two 1's. There are exactly 6 of them, and the following exchange demonstrates Tyler's meaning of those six general outcomes as they relate back to his previous work. Specifically, note that he understands why 6 such outcomes would make sense, because he can understand that he is in a situation of arranging two distinct objects, which is what his previous work involving AB passwords also entailed.

Tyler: *Yeah there you go. Is that all of them? Yeah so 6, 'cause that would make sense...*

Int.: *Does that 6 make sense?*
Tyler: *Does it? Uh, well that would that's um, 2 variables like instead of doing 3 things there's 2 umm, with the 4 combo, so 2, was 6 over here [points back to the 6 in the correct entry of the 4 character AB password table], so that's why I thought it made sense.*

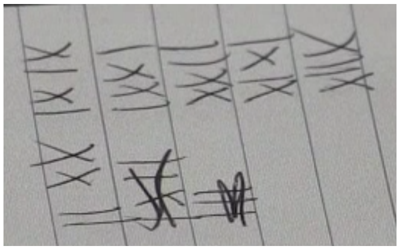

Figure 5 – Tyler's

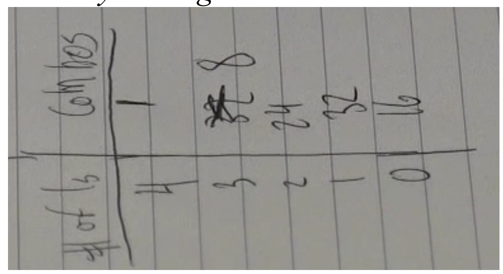

Figure 6

He continued to work in a similar fashion for the case of three 1's, and he ultimately arrived at the correct table for 4-character 1AB passwords (Figure 6). Although we do not have space to outline his subsequent work, Tyler did go on to use the same tool in subsequent cases involving 5-character 1AB passwords. He seemed to have a robust understanding of how he could use the tool to solve password problems involving more characters and more letters.

We point out a couple of important features of Tyler's work. First, it is noteworthy that Tyler spontaneously introduced a new, general structure that appropriately represented the situation and the outcomes he was trying to count. This is in and of itself impressive, and his work demonstrates an existence proof of the kind of thinking the Password tasks fostered in terms of combinatorial generalization. Second, Tyler was able to relate that new structure to his combinatorial activity to that point, and this relationship to prior work played a key role in him ultimately being able to solve more problems correctly. Importantly, he seemed to preserve the combinatorial meaning of the tasks and the situations as he related the *11xx* structure with his previous work. In terms of Tyler's meanings, we interpret that he understood the *11xx* structure as a general structure of the outcomes he had been working with, allowing him to relate back to a previous combinatorial situation involving just two objects (specifically, A's and B's). Although he did not demonstrate a deep combinatorial, multiplicative meaning of 2^n, he could recognize the 2^n as being numerically equivalent to a previous case, which he could use effectively.

Example 2 – Richie. We now contrast Tyler's work with another student, Richie. Richie, too, spontaneously introduced the *11xx* structure, but we highlight a key difference in that Richie was less successful in leveraging the new tool by relating it to previous circumstances. When making the tables for the AB passwords tasks, Richie correctly filled out the tables, often using some listing, but it seemed as though he was more attuned to the numerical patterns he observed than in finding a combinatorial explanation that made sense. For example, when making a table for the 5-characer AB passwords, we had the following exchange. Notice that his justification for why certain entries were in the table was based on the patterns he'd observed. This is not in and of itself problematic, but it shows perhaps that he was not establishing a robust combinatorial meaning but that his meaning was based on observed numerical regularity.

Richie: *So when you get to like the -- the second one, or it's not even like the second one, it's more like the one in the middle of 0 and 5 is going to be the most possibilities. And in previous problems it's been like 2 more than the preceding one.*
Int.: *Okay. Sure.*
Richie: *And I'm just assuming that this is 5, because the previous pattern's like increasing by 1.*

When we moved on to the 1AB passwords case, Richie, like Tyler, spontaneously moved toward a new structure involving 1's and x's. Figure 7 shows his drawing of the six ways to arrange two 1's and two x's.

Richie: *I'm just trying to think of all the different configurations – that 1s can (inaudible) – so like they can be starting with X, or just like that. Or like this or like this. Or like this. And then it could be 1 inside.*

Int.: *Perfect.*

Richie: *So, 1, 2, 3, 4, 5, 6. There's 6 different possibilities for that. And then each of these can have 4 different configurations.*

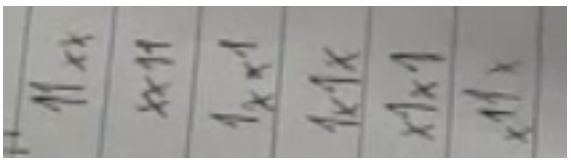

Figure 7

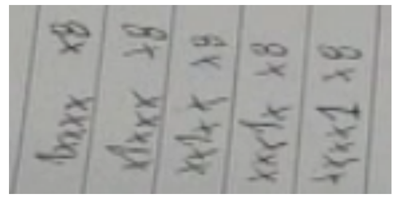

Figure 8

Richie then checked his work and reasoned that for each of those possibilities there would be four possibilities, running through BB, AA, AB, and BA. He concluded, "So 6 times 4 would be – it would be like 6 times all of these really [referring to the six configurations]…Yeah, okay. So I guess they would be 24. So it would be 24 possibilities." Richie then continued to work, and as he progressed to other rows in the table he made more diagrams involving 1's and x's.

We then had him move to the 5-character 1, A, B password, and I asked him to start making the table. Here again he made a similar diagram with 1's and x's, but here his work departed from Tyler's. Richie was able to think about there being a certain number of options for each case (each arrangement of the x's), and he knew there were two options for each x (A and B), but he added instead of multiplied the number of options, yielding 8 rather than 16 possibilities.

Richie: *So for 5 it would be 32. Same thing. And then for 1 there would be – (writes Figure 8 without the *8's) and so those would each have – this could be A or B, so that would be 2 for that, 2 for that, 2 for that, so these would each have 8 different possibilities [writes the *8's in Figure 8]. So it would be 5 – 5 times – it would be 40 for 1.*

In asking Richie to explain this work, we gain insight into his meaning of the diagram. He made no explicit connection to the previous tables or situations as Tyler had.

Richie: *This, like I made I want to say like a diagram basically of a position so 1 can be. And then I put Xs in for the – where the As and Bs could be, because those are variables that can be either A or B. And then I noticed that for every X it has 2 possibilities, either being A or B, and there's 4 Xs…So then I just multiply that by 2 to get 8. So each – for each 1 position there's 8 different possibilities for the password. And that's how I got 40.*

Richie continued his work and listed out all 10 of the configurations of two ones in a 5-character password. He demonstrated a consistent meaning by again adding the options – saying there were 6 passwords for each configuration, which is 2*3 rather than $2^3 = 8$.

At first blush it seems that perhaps Richie simply made a mistake, adding instead of multiplying the options, but we do not feel that he simply made a numerical error. Instead, the evidence seems to point to the fact that he did not make meaning of the new structure as being related to the previous case, at least not directly to the previous tables. Unlike Tyler, he did not recognize that the power of the *11xx* structure is that it can be very clearly related to the previous situation. There are two potential points of connection to the prior AB tables (relating the placement of the 1's to the rows of the AB tables, such as 1, 4, 6, 4, 1, and realizing the totals in the previous tables represent the possible number of passwords of a given length), both of which Tyler recognized, and neither of which Richie recognized.

This is not to say that Richie's meanings were unreasonable or that they did not make sense to him – indeed they did. His introduction of the *11xx* structure seemed to serve as a way of simplifying and organizing the problem so he could better break it down, but not in a way that facilitated rich connections to the previous problem. We asked him how and why he came up with the structure, and he suggested that he was motivated by efficiency: "I started to write here different configurations for where the 1s could be and where the Bs and As could be, and I noticed that basically the As and Bs were just switching places for wherever the amount of 1s were. So I started putting Xs there just so I wouldn't have to write as much."

Conclusion and Implications

By examining two students' meanings of the same tool that they each spontaneously developed, we gain insight both into students' generalizing activity and their combinatorial reasoning. Ultimately, we want to help students be more effective in creating productive generalizations, and we want to learn more about how students might effectively solve counting problems. We feel that Tyler's work – not only his production of the *11xx* structure but his ability to make meaning of it in light of prior activity – is a powerful example of a student-generated general structure that led to inroads in challenging combinatorial tasks. Set in contrast to Richie's work (which was also impressive in that he generated the *11xx* structure, but was limited in its lack of combinatorial meaning and connection to the previous situations), we can examine what aspects of Tyler's work and meanings were so efficacious. One aspect of his work that was powerful was that he remained grounded in his prior activity, and he had a rich combinatorial meaning of those prior situations. The AB tables Tyler made were combinatorially meaningful for him, in the sense that he reasoned about outcomes and did not lose sight of the combinatorial context. This is in line with previous work that emphasizes the importance of outcomes (e.g., Lockwood, 2013; 2014). We suspect that because Tyler had such a strong sense of outcomes (as seen through his listing activity in his creation of AB tables), the *xx11* structure really did represent to him a more general form of an outcome. It resembled a password (still a sequence of characters on the page), and we posit that this enabled him to maintain his reasoning about the structure of his outcomes and thus a connection to the previous combinatorial situation.

In terms of implications, our findings suggest that students can, on their own, produce potentially powerful tools involving general structures. However, this alone is not sufficient for productive generalizing or counting activity, and these contrasting cases show some of the other reasoning necessary to make full use of such tools. A pedagogical implication is that teachers may need to be vigilant in helping students maintain contact their with their prior activity. Specifically, in combinatorics this might mean that even as students notice patterns, teachers should help them to connect those patterns to the combinatorial context and not simply to numerical regularity. Combinatorially, another implication of the work is that this sequence of tasks does seem to be potentially useful in helping students to reason about the binomial theorem (or at least its initial stages). Tasks like these could be leveraged to introduce and teach combinatorial identities, which is a building block toward the learning of combinatorial proof.

Our findings show an example of rich generalizing activity in a combinatorial context. These findings emerged in a single interview, but we hope to extend this work through teaching experiments in which students' meanings can be developed and examined over time. Next research steps also include an investigation into more specific instructional interventions that might foster the kind of meanings that proved beneficial for students like Tyler.

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Reinventing the multiplication principle

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Counting problems offer opportunities for rich mathematical thinking, yet students struggle to solve such problems correctly. In an effort to better understand students' understanding of a fundamental aspect of combinatorial enumeration, we had two undergraduate students reinvent a statement of the multiplication principle during an eight-session teaching experiment. In this presentation, we report on the students' progression from a nascent to a sophisticated statement of the multiplication principle, and we highlight two key mathematical issues that emerged for the students through this process. We additionally present potential implications and directions for further research.

Key Words: Combinatorics, Reinvention, Counting problems, Teaching experiment

Introduction and Motivation

The multiplication principle (MP), called by some "The Fundamental Principle of Counting" (e.g., Richmond & Richmond, 2009), is a fundamental aspect of combinatorial enumeration. Broadly, it is the idea that for independent stages in a counting process, the number of options at each stage can be multiplied together to yield the total number of outcomes of the entire process. It is generally considered to be foundational to many of the counting formulas students learn and is also a much-needed source of justification for why these counting formulas work as they do. In spite of its importance, little has been studied about the MP in and of itself. In order to better understand student thinking about the MP, we had two undergraduate students reinvent a statement of the MP over the course of eight interviews. In this paper, we describe their overall reinvention process, discussing and presenting some of their statements. We also introduce and discuss two mathematical issues that are entailed in the MP and that arose for the students (the independence of stages in a counting process and the need to count distinct composite outcomes). We seek to address the following research goals: *1) Describe a pair of students' trajectory as they reinvent a statement of the MP, and, in so doing, 2) Present the mathematical issues in the MP to which the students attended as they reinvented the statement.*

Literature Review and Theoretical Perspective

Research about the MP in Combinatorics Education Literature

Previous work has demonstrated the importance of the MP in counting, and the lack of a well-developed understanding of the MP appears to be a significant problem and hurdle for the students, particularly in terms of their ability to justify or explain formulas. We have found that students can easily assume that they completely understand the MP in counting because multiplication is a familiar operation for them. As a result, they use the operation frequently but without careful analysis, and they tend not to realize when simple applications of the operation are problematic. While some researchers have discussed multiplication within combinatorial contexts (Tillema, 2013), there have not yet been studies that explicitly target student understanding of the MP.

Lockwood, Swinyard, and Caughman (2015) had students reinvent basic counting formulas, and the students in that study did not appear to have a solid understanding of the MP. They worked with outcomes empirically but lacked the understanding of how those outcomes related to the underlying counting process involved with the MP. This work suggested the need for more research that targets students' understanding of the MP as a fundamental counting process.

In addition, Lockwood, Reed, and Caughman, (2015) recently conducted a textbook analysis that examined statements of the MP in university combinatorics and discrete mathematics textbooks. This revealed a wide variety of statements of the MP (Figures 1, 2, and 3 reveal three very different formulations).

Product Rule: *If something can happen in n_1 ways,* ***and*** *no matter how the first thing happens, a second thing can happen in n_2 ways,* ***and*** *no matter how the first two things happen, a third thing can happen in n_1 ways,* ***and*** *..., then all the things together can happen in $n_1 \times n_2 \times n_3 \times \ldots$ ways.*

Figure 1 – Roberts & Tesman's (2003) statement of the MP

The Multiplication Principle*: Suppose a procedure can be broken down into m successive (ordered) stages, with r_1 different outcomes in the first stage, r_2 different outcomes in the second stage, ..., and r_m different outcomes in the mth stage. If the number of outcomes at each stage is independent of the choices in the previous stages, and if the composite outcomes are all distinct, then the total procedure has $r_1 \times r_2 \times \ldots \times r_m$ different composite outcomes.*

Figure 2 – Tucker's (2002) statement of the MP

Generalized Product Principle: *Let $X_1, X_2, \ldots, X_k$ be finite sets. Then the number of k-tuples (x_1, $x_2, \ldots, x_k$) satisfying $x_i \in X_i$ is $|X_1| \times |X_2| \times \ldots \times |X_k|$.*

Figure 3 – Bona's (2007) statement of the MP

Part of the motivation for the current study, then, is to build upon the textbook analysis by actually studying how students think about mathematical issues that arose in the textbook statements of the MP. The findings from Lockwood, Reed, et al. (2015) framed and informed the mathematical issues we pursued with the students, and in the following section we briefly discuss these key mathematical issues in the MP.

Key Mathematical Issues

Here we describe two mathematical issues in the MP, both of which are seen in Tucker's (2002) statement (Figure 2). In the Results section we will describe the students' reasoning about these key ideas, and so we briefly introduce them here to facilitate subsequent discussion. First, there is the notion of *independence* of stages in the counting process, which captures the idea that a choice of options at a given stage does not affect the number of outcomes in any subsequent stage. This is a necessary condition in order to apply the MP, or else overcounting may occur. Second, the MP must yield *distinct composite outcomes*, which means that when applying the MP we want to ensure that there are no duplicate outcomes. This qualification, too, prevents instances of overcounting. In the Results section we highlight two counting problems that demonstrate the need for each of these mathematical issues in statements of the MP.

Reinvention

Gravemeijer, Cobb, Bowers, and Whitenack (2000) describe the heuristic of *guided reinvention* as "a process by which students formalize their informal understandings and intuitions" (p. 237). From this perspective, students can formalize ideas through generalization of their previous mathematical activity. We had students reinvent statements of the MP because we felt this would allow students to meaningfully understand and articulate a statement, giving us insight into how students come to understand the MP. This is in line with other researchers who have used principles of reinvention to gain insight into students' reasoning about a particular concept or definition (e.g., Oehrtman, Swinyard, & Martin, 2014; Swinyard, 2011).

Methods

Data Collection

We conducted a teaching experiment (as described by Steffe & Thompson, 2000) in which a pair of undergraduate students solved counting problems over eight hour-long sessions. The students were enrolled in vector calculus in a large university in the western United States, and they were chosen because they had not been explicitly taught about the MP in their university coursework (and thus would not simply try to recall it). The interviews took place outside of class time over a period of four weeks. Broadly, the students solved a series of counting problems, and they were asked periodically to write down and characterize when they were using multiplication as they solved these problems. They wrote down several iterations of statements of the MP. Throughout the study the interviewer selected tasks to highlight various aspects of the MP and regularly asked clarifying questions.

For the sake of space we provide only a sampling of tasks from the teaching experiment. Broadly, the students engaged in three kinds of activities: solving counting problems that involve multiplication, articulating a statement of the MP, refining their statements of the MP, and evaluating given textbook statements of the MP. Although there was some overlap of activities in each session, Table 1 gives the overall structure of the teaching experiment by outlining the session number (and total number of tasks in each session), a sample task given in that session, and the predominant activity that occurred in each session.

Session	Sample Tasks for Each Session	Emphasis of Session
1 (6 tasks)	You have 4 different Russian books, 5 different French books, and 6 different Spanish books on your desk. In how many ways can you take two of those books with you, if the two books are not in the same language?	Solving counting problems that involve multiplication
2 (5 tasks)	How many ways are there to form a three-letter sequence using the letters a, b, c, d, e, f: (a) with repetition of letters allowed? (b) without repetition of any letter? (c) without repetition and containing the letter e? (d) with repetition and containing e?	
3 (5 tasks)	In a standard 52-card deck there are 4 suits (hearts, diamonds, spades, and clubs), with 13 cards per suit. There are 3 face cards in each suit (Jack, Queen, and King). How many ways are there to pick two different cards from a standard 52-card deck such that the first card is a face card and the second card is a heart?	Articulating a statement of the MP
4 (2 tasks)	How many ways are there to flip a coin, roll a die, and select a card from a standard deck?	
5 (2 tasks)	There are 7 professors and 5 grad students. In how many different ways could an advisor and a grad student be paired up?	
6 (3 total tasks)	How many 6-character license plates consisting of letters or numbers have no repeated character?	Refining their statement of the MP

7 (6 tasks)	How many rearrangements of the letters in the word DYNAMIC start with a vowel?	
8 (7 statements)	Please read the following statement [such as Tucker's (2002) in Figure 1]. How is it similar to or different from your own statement?	Evaluating given textbook statements

Table 1 – Overall structure of the teaching experiment

Data Analysis

The interviews were videotaped and transcribed, and overall the videos and transcripts were analyzed so as to construct a narrative about the teaching experiment (Auerbach & Silverstein, 2003). We used prior understanding of the MP that had emerged from the textbook analysis to guide our focus of particular mathematical issues. Key episodes involving mathematical issues were flagged and reviewed, and we scrutinized the students' statements of the MP and their explanations for insights about their reasoning.

Results

We organize the results into two sections. First we provide an overview of their progress and offer several of the statements that they developed as they reinvented the MP. This should demonstrate their overall progress and provide a broad narrative of what transpired during the teaching experiment. Then, we present their handling of two key mathematical issues (independence and distinct composite outcomes), highlighting student thinking about important aspects of the MP.

Overall Progression of Statements

The students went through between 20-25 statements (depending on how one defines a statement, as some were verbally articulated, and some involved minor adjustments from previous statements). Here, due to space, we provide seven statements (exactly as the students had written them on the board), which emphasize development from a nascent to sophisticated statement of the MP. Statements 2a and 2b, 3a and 3b, and 4a and 4b each represent minor changes that reflect the students' realization about a key mathematical issue.

Session	Statement
2	#1 – Use multiplication in counting problems when… there is a certain statement shown to exist and what follows has to be true as well.
4	#2a – For each possible pathway to an outcome there is an equal number of options leading to that path.
	#2b – For each possible pathway to an outcome there is an equal number of options leading to that path but without repeating the same pathway more than once.
6	#3a – For every selection towards a specific outcome, if one selection does not affect any subsequent selection, then you multiply the number of all the options in each selection together to get the total number of possible outcomes.
	#3b – For every selection towards a specific outcome, if one selection, no matter the previous selections, is no difference in the number of options, then you multiply the number of all the options in each selection together to get the total number of possible outcomes.

6	#4a – If for every selection towards a specific outcome there is no difference in the number of outcomes, regardless of previous selections, then you multiply the number of all the options in each selection together to get the total number of possible outcomes.
7	#4b – If for every selection towards a specific outcome, if there is no difference in the number of options, regardless of the previous selections, then you multiply the number of all the options in each selection together to get the total number of possible unique outcomes.

Table 2 – The students' progression toward a statement of the MP

We began the interviews by simply having students solve counting problems that involved multiplication, the motivation being to give them some experience using multiplication so they might extrapolate some key principles of when to use multiplication in counting. By the end of Session 2 we gave them the following prompt: *Can you take a stab at characterizing when you use multiplication when you're solving these problems?*, and they produced Statement #1. In Sessions 3 through 7 we gave them more counting problems and asked them to refine their statements. As they developed more sophisticated language in talking about the statements, we targeted key mathematical ideas by giving counting problems that would elicit certain ideas.

We note a couple of important observations about their progression in Table 2. First, we highlight the lack of sophistication in Statement #1 compared to Statement #4b (their final statement). Statement #1 is not well formed, and this suggests that the task of characterizing when to use multiplication is not a trivial one. By the end of the teaching experiment, however, they had developed a rigorous statement. We also point out the shift in language and emphasis from pathways and paths in Statements #2a and #2b to language of outcomes, selections, and options in Statements #3a, #3b, #4a, and #4b. This shift reflects an intentional pedagogical move that occurred after Session 5. In Session 4 the students had come up with Statements #2a and #2b. Session 5 saw no progress or refinement of their statement, and so we decided to redirect the students' attention away from the pathway language and toward more general language. This led them to introduce language of options, selections, and language.

Key Mathematical Issues

We now offer data examples that exhibit how the students' statements developed over time, and in particular how they adjusted their statements to address some key mathematical issues. These are meant to demonstrate the nature of the students' interactions with each other and with the interviewer and also to show how students' reasoning developed as they interacted with particular tasks.

Independence. In Session 6, the students had come up with Statement #3a from Table 2. We draw attention to the phrase "if one selection does not affect any subsequent selection then you multiply…" This is a valuable insight in and of itself, because it ensures that one only multiplies when the stages of a counting process are independent of one another. However, as stated, Statement #3a is not quite accurate, because multiplication is still appropriate if the actual options change from stage to stage. Instead, it is the *number* of options that cannot change (as the following episode demonstrates).

To draw the students' attention toward this issue, we presented them with the following problem: *How many 6-character license plates consisting of letters or numbers have no repeated character?* They immediately recognized that they could use multiplication on this problem –

they knew the answer would be 36*35*34*33*32*31, and they had the following exchange as they tried to justify the answer.

Pat: *It's still multiplication, but it's not the same as multiplication that we were thinking of. So as we, it has to change things now doesn't it?*

Caleb: *That one definitely put a damper on our – [cut off]*

They then thought about the problem a bit more, and Pat had the following realization.

Pat: *I'm just concerned about the idea that we're saying, that the selection is affecting the next selection, because technically in this case, the selections…affect subsequent selections, but it still is multiplication…*

Pat saw that their wording would not allow for the multiplication in a problem like the License Plates problem, because their statement #3a was too restrictive about how subsequent selections might be affected. The two students then had the following exchange (the A and 9 refer to choices for a license plate character):

Caleb: *Maybe you pick like A and then 9, and you can't do either one of those again.*

Pat: *Yeah but it's like A or 9 won't affect the number of next selections. 'Cause no matter what it's gonna be 36, 35, 34, 33…See what I'm saying? So like how do we incorporate that? Because like, so as long as it – as long as the next, the subsequent selections still have the same number of selections, it's okay.*

The students then proceeded to write down language that might help address this issue, now emphasizing the *number* of options to which they had not previously attended, ultimately writing statement #3b. This exchange provides evidence of how the students reasoned about a subtle aspect of independence, and the License Plate task was carefully chosen to refine an already-existing idea – that the *number* of options (not the options themselves) must be independent.

Distinct composite outcomes. The progression from statements #4a to #4b reveal an evolution in the students' thinking about distinct composite outcomes. This came about through the Three *e*'s problem: *How many ways are there to form a three-letter sequence using the letters a, b, c, d, e, f with repetition and containing e?*. In this problem, a common, tempting incorrect answer is to argue that there are 3 places in which to place an *e*, and then once that *e* is placed, since repetition is allowed there are 6 options for the next spot and then 6 for the remaining spot, yielding an answer of 3*6*6. Indeed, this is what the students first answered. However, in this answer an outcomes like *eee* gets counted more than once, because the process described above could generate *eee* both when *e* is placed in the first spot in the first stage (and then *ee* are in the last two spots in the second and third stages), and also when *e* is placed in the third spot in the first stage (and then *ee* are in the first two spots in the second and third stages). The students had written statement #4a, and then they revisited this problem. As they worked through the problem, they realized that someone could use their statement #4a to solve the Three E's problem but would end up overcounting. Caleb talks about wanting to disallow this kind of overcounting in their statement of the MP, but he acknowledges the difficulty of how to articulate that.

Caleb: *Well you have to make it seem like you can't over count something without saying don't over count it. Because the reason is, we're not wanting to over count it.*

Int.: *Okay, okay. So what do you mean? Yeah, say more about that.*

Caleb: *So our problem here is over counting, and you can't just like put in a clause of like don't over count. […] 'Cause right now we sort of have the difference in the number of options but that doesn't, it's not necessarily specific to over counting. We were sort of thinking of less.*

We encouraged them to think more about the problem, and they had the following exchange:

Int.: *Okay, cool. I might just let you guys think about this for a couple minutes.*
Caleb: *We could say without any repeated outcomes.*
Int.: *Okay. Say more about that.*
Caleb: *So our problem here is where we're getting like a repeated outcome. If we say, um.*
Pat: *Oh hey, we already have specific outcome in there [in statement #4a].*
Caleb: *Yeah.*
Pat: *So how about we say specific unique outcome?*

They then added the word "unique" to their statement, resulting in their final statement, #4b. This addition of the word "unique" is their way of addressing the possibility of overcounting, and ensuring that the outcomes must be unique, is equivalent to Tucker's (2002) clause of "distinct composite outcomes." By specifying that they only want to count unique outcomes, their statement technically does not allow for this kind of overcounting to occur. Thus, again we see an instance in which the students refined their statement after considering a particular mathematical task. This sheds light on how their thinking developed about the MP and the mathematical aspects of the statement on which they focused.

Conclusion and Implications

By having students reinvent a statement of the MP, and by closely analyzing aspects of multiplication to which they attend, we gain insight both into how students reason about the MP, and also how productive reasoning about the MP might be developed. In particular, by engaging with particular tasks, the students we worked with were able to come to reason about key mathematical aspects of the MP (such as independence and unique outcomes) that they wanted to include in their statement of the MP. In addition to insights about how they come to understand particular mathematical ideas, we can draw a couple of key conclusions from their overall progression from to a final statement. First of all, we have an existence proof that it is possible for students to develop, on their own, a mathematically rigorous statement of the MP. It is not trivial to characterize many of the subtle mathematical details of the MP, and it is impressive that the students were able to do so. Second, we see that although they were able to accomplish this task, it was not a trivial activity to characterize when to use multiplication in solving counting problems. This is demonstrated most clearly in their first statement, which shows that even after they had successfully used multiplication in counting problems for two sessions, they still struggled with articulating a statement about it.

Our findings suggest a couple of implications. First, as a fundamental aspect of counting, the MP is invaluable, yet potentially challenging, for students to understand well. Although it deals with a familiar operation, there are subtle mathematical features that it involves, which might take time and effort for students to learn. More work is needed to more carefully evaluate how best to teach the MP to students in a classroom setting, but our work suggests that it may be worthwhile to unpack some key mathematical issues with students. Instructors should appreciate the mathematical details in the MP and should help students think carefully about when multiplication properly applies in counting situations. In terms of research, we plan to continue to explore what might be entailed in a robust understanding of the MP, which includes interviews with more students and also with mathematicians. Based on our findings from this study, especially insights about understanding independence and distinct composite outcomes, we can look to design instructional interventions that might draw students' attention toward such ideas.

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Classifying combinations: Do students distinguish between different types of combination problems?

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In this paper we report on a survey designed to test whether or not students differentiated between two different types of problems involving combinations – problems in which combinations are used to count unordered sets of distinct objects (a natural, common way to use combinations), and problems in which combinations are used to count ordered sequences of two (or more) indistinguishable objects (a less obvious application of combinations). We hypothesized that novice students may recognize combinations as appropriate for the first type but not for the second type, and our results support this hypothesis. We briefly discuss the mathematics, share the results, and offer implications and directions for future research.

Key words: Combinatorics, Discrete mathematics, Counting

Discrete mathematics, with its relevance to modern day applications, is an increasingly important part of students' mathematical education, and prominent organizations have called for increased teaching of discrete mathematics topics in K-16 mathematics education (e.g., NCTM, 2000). Combinatorics, and the solving of counting problems, is one component of discrete mathematics that fosters deep mathematical thinking but that is the source of much difficulty for students at a variety of levels (e.g., Batanero, Navarro-Pelayo, & Godino, 1997; Eizenberg & Zaslavsky, 2004). The fact that counting problems can be easy to state but difficult to solve indicates that there is a need for more research about students' thinking about combinatorics.

One fundamental building block for understanding and solving combinatorial problems are combinations (i.e., C(n,k), also called binomial coefficients due to their role in the binomial theorem). Combinations are prominent in much of the counting and combinatorial activity with which students engage, and yet little has been explicitly studied with regard to student reasoning about combinations. This study contributes to our understanding of students' reasoning about combinations, and in particular to study beginning students' inclination to differentiate between typical combinatorics problems. This study addresses the following research question: *Do early undergraduate students recognize two different types of combination problems as involving binomial coefficients, and do they use binomial coefficients to solve both types of problems?*

Literature and Theoretical Perspective

Combinations in Mathematics Education Literature. As we have noted, there is much documented evidence for the fact that students struggle with solving counting problems correctly. Some reasons for such difficulty are that counting problems are difficult to verify (Eizenberg & Zaslavsky, 2004) and that it can be difficult to effectively encode outcomes[1] in terms of objects one knows how to count (e.g., Lockwood, Swinyard, & Caughman, 2015b). We seek to address potential difficulties by focusing on better understanding students' application and use of combinations – which are fundamental in enumeration. Piaget & Inhelder (1957) studied students' mental processes as they solved arrangement and selection problems, and they

[1] We take *encoding outcomes* to be the combinatorial activity of articulating the nature of what is being counted by associating each outcome with a basic mathematical entity (such as a set or sequence).

took a special interest in determining whether permutations, arrangements, or combinations would be most difficult for students. Dubois (1984), Fischbein and Gazit (1988), and Batanero, et al. (1997) have also investigated the effects of both implicit combinatorial models and particular combinatorial operations on students' counting, again considering differences in reasoning about particular problem types such as permutations and combinations. We extend existing work that focuses on students' mental processes of foundational combinatorial ideas, seeking specifically to explore the extent to which undergraduate students distinguish between two types of combination problems (which we develop in the following section). Our work also builds on a recent study by Lockwood, Swinyard, & Caughman (2015a) in which two undergraduate students reinvented basic counting formulas, including the formula for combinations. Based on the students' work on combination problems, Lockwood, et al., (2015b) suggested the importance of being able to correctly encode outcomes combinatorially (by which they mean the act of articulating the nature of what is being counted by associating each outcome with a mathematical entity such as a set or a sequence).

Mathematical Discussion. A combination is a set of distinct objects (as opposed to a permutation, which is an arrangement of distinct objects). Combinations can also be described as the solution to counting problems that count "distinguishable objects" (i.e., without repetition), where "order *does not* matter." The total number of combinations of size *k* from a set of *n* distinct objects is denoted C(*n*,k) and is verbalized as "*n* choose *k*."[2] As an example, combinations can be used if you want to select from eight (distinguishable) books three books to take on a trip with you (order does not matter) – the solution is C(8,3), or 56 possible combinations. By contrast, other combinatorial problems and solution methods, such as permutations, are organized in relation to some different possible constraints – see Table 1.

Table 1: Selecting *k* objects from *n* distinct objects

	Ordered	**Unordered**
Distinguishable Objects (without repetition)	Permutations $\frac{n!}{(n-k)!} = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-k+1)$	Combinations $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
Indistinguishable Objects (with repetition)	Sequences $n^k = \underbrace{n\cdot n\cdot n\cdot\ldots\cdot n}_{k}$	Multicombinations $\left(\binom{n}{k}\right) = \binom{k+n-1}{n-1}$

In this paper we refer to *combination problems* as problems that can be solved using binomial coefficients, in the sense that parts of their outcomes can be appropriately encoded as sets of distinct objects (i.e., Lockwood, et al., 2015b). Sometimes this encoding is fairly straightforward, as the outcomes are very apparently sets of distinct objects. For instance, in the Basketball problem (stated in Table 2), the athletes could be numbered 1 through 12 (because they are different people), and the outcomes are fairly naturally modeled as 7-element sets taken from the set of 12 distinct athletes. Any such set is in direct correspondence with a desired outcome; there are C(12,7) of these sets. We call such problems Type I. In other situations, a problem may still appropriately be solved using a combination, but recognizing how to encode the outcomes as sets of distinct objects is less clear. For example, consider the Coin Flips problem (stated in Table 2). Here, the problem also can be solved using combinations – that is, if the outcomes are encoded appropriately. The most natural way to model an outcome is as an ordered sequence of Hs and Ts; however, we can encode a desirable outcome as a set of distinct

[2] The derivation of the formula for $C(n,k)$ as $n!/(n-k)!k!$ is not pertinent to the study; Tucker (2002) provides a useful explanation.

positions in which the Hs are placed. Given the five possible positions (i.e., the set: {1, 2, 3, 4, 5}), the outcome HHTHT would be encoded as the set {1, 2, 4}. This sufficiently establishes a bijection between outcomes and sets because every outcome has a unique placement for the Hs (the Ts must go in the remaining positions). In this way, the answer to the counting problem is simply the number of 3-element subsets from 5 distinct objects (i.e., positions 1-5), which is C(5,3). We call these Type II problems (See Table 2).

Table 2: Characterizing two different "types" of fairly standard combination problems

	Description	Example problem	Natural Model for Outcomes
Type I	*An unordered selection of distinguishable objects*	*Basketball Problem.* There are 12 athletes who try out for the basketball team – which can take exactly 7 players. How many different basketball team rosters could there be?	{(1,2,3,4,5,6,7), (1,3,5,7,9,11,12), …}
Type II	*An ordered sequence of two (or more) indistinguishable objects*	*Coin Flips Problem.* Fred flipped a coin 5 times, recording the result (Head or Tail) each time. In how many different ways could Fred get a sequence of 5 flips with exactly 3 Heads?	{(HHTTH), (HTHHT), (TTHHH), …}

In light of various ways of encoding outcomes that facilitates the use of combinations, we point out that it may seem that combinations are actually being used to solve two very different kinds of problems. The outcomes in the Basketball problem are clearly unordered sets of distinct objects, but the outcomes in the Coin Flips problem are actually *ordered* sequences (not unordered) of two kinds of *indistinguishable* (not distinct) objects (Hs and Ts). Combinations are applicable in both situations, but we argue that there could be a difference for students in identifying both problems as counting combinations. Indeed, using combinations to solve Type II problems involves an additional step of properly encoding the outcomes with a corresponding set of distinct objects, and we thus posit that Type I problems would be more natural for novice students, more clearly representative of combination problems than Type II problems.

In spite of the widespread applicability of combinations, we posit that students may not recognize all fairly standard combination problems as involving combinations. This may be due in part to the fact that students tend not to reason carefully about outcomes (e.g., Lockwood, et al., 2015b), and because "distinguishable" and "unordered" are not always natural or clear descriptions of the situation or outcomes. We are thus motivated us to investigate whether or not students actually respond differently to the two different problem types.

Methodology

We designed two versions of a survey, and although the survey contained a number of elements, we focus in particular on features of the survey that serve to answer the research question stated above. Each survey consisted of 11 combinatorics problems, and each problem was designed with categories in mind that included problem type (I or II) and complexity (Simple, Multistep, or Dummy).[3] *Simple* combination problems refer to those that can be solved using a single binomial coefficient, in the sense that their outcomes can be appropriately encoded as sets of distinct objects; *multistep* combination problems would require multiple binomial coefficients in the solution. The authors coded the problems independently before finalizing the coding for each problem. Each version of the survey contained the same number of problems of

[3] We also coded the tasks according to other criteria that we do not report on in this paper, such as: sense of choosing (Active or Passive), and whether an object or process is to be counted (Structural or Operational).

each type and complexity, as well as two "dummy" problems to discourage students from assuming that every problem could be solved with a combination. Each problem was selected for one version of the survey with a companion problem in mind for the other version in order to compare responses with respect to the various coding categories.

We targeted Calculus students because they were believed to have been likely to have seen combinations at some point in their mathematical careers without having studied them in detail. In order to gain insight into students' familiarity with combination problems, each survey included a section on demographic information to identify previous mathematics courses taken in high school or in college, current mathematics courses being taken, and whether students recognized different representations of combinations and permutations. We have yet to incorporate the demographic data in our analysis, but this is a further avenue we plan to pursue.

In order to investigate the research question, we wanted to see whether or not students who would solve Type I problems using combinations would also solve Type II problems using combinations. That is, we wanted to see whether students would recognize a difference between Type I and Type II problems in terms of the applicability of combinations as a solution. In order to do this, we needed to focus on only those participants who had demonstrated some understanding of combinations, by using them in their solution to Type I problems. On each survey there were four *simple* Type I problems and three *simple* Type II problems. These seven problems were of particular interest. Two other problems, one *multistep* Type I and another *multistep* Type II, were also included, for which the answer was a product of combinations. We treated these two kinds of problems separately in the analysis.

The prompts for the combination problems asked for the students to use notation that suggests their approaches rather than numerical values,[4] but many students did not, on the whole, follow this prompt. Although there were 69 complete responses (we removed participants who did not finish the survey), 38 gave strictly numerical responses rather than expressions that indicated their solution method, which limited our ability to analyze their responses. Further, of the 31 remaining, only 12 correctly responded to at least half of the Type I problems using the appropriate combination notation. For the purpose of our preliminary report, we focus our analysis on these 12 participants, because, as noted, we sought to examine students who had used combinations to solve Type I problems. It is perhaps noteworthy that so few students followed the survey directions and also that so few correctly solve the Type I problems using combinations, but these are not points of discussion we are able to discuss in detail.

Findings

In this section we present two aspects of our data analysis that support a singular finding in regard to our research question.

Simple Type I and Type II problems. Out of the 12 people who correctly answered at least half of the Type I problems using the appropriate combination notation, 6 of these participants demonstrated very different responses between Type I and Type II problems. This is a considerable proportion (50%) of participants that displayed a dichotomous way of responding to these two problem types. For example, one participant gave correct responses of C(20,4), 9!/(7!2!), C(15,2), and C(250,6) for the Type I problems, and similarly correct but different

[4] Specifically, the prompt was: Read each problem and input your solution in the text box. **Please write a solution to the problem that indicates your approach.** If you're not sure, input your best guess. NOTE: Appropriate notation includes: 9+20, C(5,2), 5C2, 21*9*3, 5*5*5*5=5^4, 8!, 8!/5! = 8*7*6 = P(8,3), C(10,2)*3, Sum(i,i,1,10), 12!/(5!*7!), etc. *Only* if you individually count *all* of the outcomes should you input a numerical answer, such as 35.

responses of 8!/(5!3!), 55!/(20!35!), and 40!/(17!23!) for the Type II problems. In other words, even though this participant identified 3 of the 4 Type I problems as easily solvable by a combination, s/he did not recognize any of the Type II problems as a combination. Another participant gave correct responses of C(20,4), C(9,2), C(15,2), and C(250,6) for the Type I problems, but incorrectly gave permutation responses of P(13,2), P(55,2), and 40P(2,1) to the Type II problems. Again, this participant's responses indicate a difference between how s/he understood and solved these problems. Only two participants used combinations (correctly) to solve all of the Type I and Type II problems. Some other participants were mixed. In these cases, the Type I/II distinction does not seem to explain their responses as clearly and we hope further analysis of the data will lend additional insight into their responses.

Multistep Type I and Type II problems. As further support of the potentially differential responses between the two hypothesized problem types, we look at two multistep problems. Such problems may themselves be different given that participants have to view combinations as part of a process for arriving at the solution instead of the solution itself (like in simple combination problems). Regardless, for the multistep Type I problem, 10 of the 12 participants used some sort of combination in their solution. Although about half still arrived at an incorrect solution (for example, adding two combinations), the majority of these participants viewed combinations as vital to their solution approach. In contrast, for the multistep Type II problem, only 1 of the 12 participants used a combination in their solution at all (this participant, however, still arrived at an incorrect solution). The only participants with correct solutions – there were five of them – were of the form: 17!/(3!4!2!8!). Thus, we see further evidence in these multi-step combination problems to suggest that students seem to differentiate between these two problem types – viewing Type I problems as suitably involving combinations in the solutions whereas Type II problems require some other solution approach.

Conclusions and Implications

Despite the fact that all of these problems would be considered fairly standard combination problems, our findings suggest that the participants did not view the problems in this way. In particular, the 12 participants were mostly successful in using combinations to solve Type I problems, but often relied on other (at times incorrect) methods to solve Type II problems. These findings, while preliminary, suggest potential implications for the teaching and learning of combinations. They seem to indicate that students may not necessarily view the two problem types as the same, and perhaps with good reason: the descriptions of "unordered" and "distinct" do not seem to apply – at least in the most natural way to model the outcomes. Thus, students may need additional exposure to combinations and may benefit from explicit instruction about how Type II problems can be encoded in a way that is consistent with Type I problems. Generally, this point underscores a need for students to become more adept at combinatorial encoding (Lockwood, et al., 2015b). Encoding outcomes as sets is an inherent part of the field of combinatorics, but students may need particular help in making this connection explicit. Combinations are a powerful tool for enumeration problems, but without a robust understanding – including how and why Type II problems can be solved using them – students may possess a tool they do not really understand how to use. In addition, there are natural next steps and avenues for further research. We plan to investigate more questions and hypotheses with the data we have, such as analyzing effects of demographic data and investigating other relationships and potentially contributing factors in students' responses. Our findings also indicate that further investigating students' reasoning about encoding with combinations through in-depth interviews may give insight into the development of more robust understandings.

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Beyond procedures: Quantitative reasoning in upper-division Math Methods in Physics

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Abstract. Many upper-division physics courses have as goals that students should 'think like a physicist.' While this is not well-defined, most would agree that thinking like a physicist includes quantitative reasoning skills: considering limiting cases, dimensional analysis, and using approximations. However, there is often relatively little curricular support for these practices and many instructors do not assess them explicitly. As part of a project to investigate student mathematical learning in upper-division physics, we have developed a number of written questions testing the extent to which students in an upper-division course in Mathematical Methods in Physics can employ these skills. Although there are limitations to assessing these skills with written questions, they can provide insight to the extent to which students can apply a given skill when prompted.

Key words: Physics, Mathematics, Upper-division, Quantitative reasoning

Introduction

This work is part of a collaboration to investigate student learning and application of mathematics in the context of upper-division physics courses. Our project seeks to study student conceptual understanding in upper-division physics courses, investigate models of transfer, and to develop instructional interventions to assist student learning.

While physics education research (PER) has primarily focused on introductory-level courses, there are increasing efforts to expand into the upper division [1]. The core sophomore- and junior-level theory and laboratory courses taken by most physics majors have begun to receive the attention of researchers and curriculum developers, including electricity and magnetism [2], thermal physics [3], classical mechanics [4], quantum mechanics [5], and advanced laboratories [6]. One key course that remains under-researched (with a few exceptions [7]) is a course taught by most departments that is commonly known as "mathematical methods." Unlike similarly-named courses for prospective mathematics teachers, this highly theoretical course focuses on the mathematical techniques that students will encounter in upper-division physics courses. Such a course is typically intended to serve as a bridge between introductory level courses and the more challenging physics and mathematics students encounter in the core upper-level theory courses in the physics major (particularly electricity and magnetism, classical mechanics, and quantum mechanics).

Typically the learning goals for courses of this nature (called MM courses for this paper) focus on content goals, with a syllabus that covers a list of topics including differential and integral calculus, series, complex numbers, vectors and vector calculus, differential equations, and linear algebra. As if this daunting list of topics were not sufficient, often the MM course also has stated or implicit goals that go beyond specific physics and mathematics context. For example, students in these courses are expected to 'think like a physicist' when solving quantitative problems. However, despite its seeming importance, this phrase is not always operationally defined. Examples of skills that might be included in this term include connecting physical intuition with mathematics, checking units and performing dimensional analysis, considering limiting cases, and using approximations. While instructors value these skills, and there has been some previous discussion of them [8], their value is often left implicit and they are not often explicitly taught or assessed.

In this portion of the project, we have sought to investigate the development of mathematical understanding that goes beyond procedural skill and calculation, to probe the quantitative reasoning skills whose development is often left implicit. A key goal of our larger project is to develop a series of tasks that would be suitable for use by instructors in the MM course. For this report, we describe preliminary efforts to develop written tasks that ask students to apply such skills and to document student responses to them. For the purpose of this paper, we will refer to questions designed to assess several quantitative reasoning skills. The set of these skills is not intended to be complete, but we have identified several that appear to be relevant as starting points:

- Using dimensional / unit analysis
- Testing expressions with limiting cases
- Using approximations, e.g., with Taylor series
- Identifying errors in solutions
- Predicting the effects of problem changes on the resulting solution

During the current study we have examined several of these skills. Examples of using approximations and of predicting the effects of changes to a problem on the resulting solution have been described previously [9, 10]. For this short paper we restrict the discussion to the second: limiting cases.

This work has taken place in the context of a MM course taught at a large public comprehensive university serving a diverse student population. The course is required for physics majors and is a prerequisite for upper-division theory courses; for most students it is one of the first upper-division courses taken. The course uses the text by Boas [11] and covers a fairly standard list of topics. It meets for two 75-minute blocks per week. The course has as prerequisites three semesters of calculus, and most students have completed at least two semesters of introductory physics. The author has taught the course six times, with enrollments between 12 and 19. Data shown is from written responses to the *Evaluate the Expressions* task shown below.

Theoretical Perspective

This portion of our project is driven by practice; we are seeking to learn what is difficult for students and develop instructional interventions. The theoretical framework guiding our analysis for this portion is "identifying student difficulties" [12].

A growing body of work in PER has examined student use of mathematics in physics. In particular, several models have been proposed to describe student use of mathematics. In each of these models, successfully executing the mathematical procedure in question is only one element of success. Redish has proposed a framework to describe student usage of mathematics in science course [13], describing stages of modeling, processing, interpreting, and evaluating. For the specific case of upper-division physics courses, Wilcox et al. have proposed the ACER framework 'to guide and structure investigations of students' difficulties with the sophisticated mathematical tools used in their physics classes.' [14] In this framework, students must activate the appropriate mathematical tool in addition to constructing a model, executing the mathematics, and then reflecting on results.

In traditional physics courses, instruction and assessment tend to focus disproportionately on what Redish describes as the processing phase, or what the ACER framework frames as executing mathematics. In a recent paper, Redish states that 'our traditional way of thinking about using math in physics classes may not give enough emphasis to the critical elements of modeling, interpreting, and evaluating' [12]. The tasks in this study reflect our attempt to investigate other aspects of mathematical thinking. For example, to probe reflection or evaluation of results, students might be asked to evaluate expressions for correctness or identify errors in solution, rather than performing procedures to generate expressions.

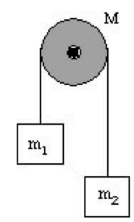
We have examined the RUME literature for corresponding studies but there appears to be relatively little focus on these sorts of skills, at least at this level of instruction. The work by Sherin describing how physics students read mathematical expressions bears upon the task described below [xx]. Thompson has examined 'quantitative reasoning' and its relationship to modeling of phenomena [1x5]. In addition, this project as a whole considers transfer broadly, as we examine how students apply their mathematical knowledge in the context of physics courses, and we have been guided in part by the work of collaborator Wagner [1x].

It is important to note there are limitations of the current study, which artificially separates these tasks from a problem solution. Qualitative studies of students in the process of solving problems can help to give insights into when and how students activate such resources and at what phase of problem solving they are employed. This project has a more modest goal, asking whether students can apply skills when their use is explicitly cued. Additional qualitative work, including interviews, is ongoing.

Methodology

For the purpose of this paper, we will focus on a sample task that is illustrative of the quantitative reasoning skills that we are describing. This task, which we refer to as the *Evaluate the Expressions* task, involves the evaluation of mathematical expressions for correctness. The problem is posed on the first day of the MM course on an ungraded quiz subsequently explored in a large group discussion while the tasks projected on a screen.

Consider the motion of two blocks connected to form an Atwood's machine. The masses of the two blocks are m1 and m2 and the mass of the pulley is M. The following expressions are proposed for the acceleration of block 1. For each, evaluate whether the expression could be correct an explain briefly:



$$a = \frac{m_2 - m_1}{m_2 + m_1} g \qquad a = \frac{m_2}{m_2 + m_1 - M/2} g \qquad a = \frac{m_2 - M/2}{m_2 + m_1 + M/2} g$$

FIGURE 1. A written task in which students are asked to evaluate multiple expressions for possible correctness given a physical situation.

The task describes a simple physical system (known in physics as an 'Atwood's machine') in which two massive blocks are connected by a string across a pulley (see Figure 1). Students are shown three expressions for the acceleration of one of the two blocks and asked to determine whether the expressions could be correct. (All three expressions are incorrect.) The Atwood's machine is a common instructional task and widely used in physics courses as an example of the application of Newton's laws. Most students would have solved a similar problem in their introductory mechanics course, encountering the problem first in the case of a massless pulley, solving the problem by drawing free body diagrams for the two blocks and applying Newton's second law with appropriate constraints to generate equations of motion. Later, in the section of the course focusing on rotational dynamics, the effect of torques on the pulley mass are treated explicitly.

The problem is different from many tasks that students have encountered to this point in that it asks for evaluation (per Redish) or reflection (per Wilcox) rather than a procedural computation. Students are not asked to solve the problem, which has no numerical values. The expectation is that students will use quantitative reasoning to arrive at an answer, by checking limiting cases (e.g., considering extreme values of variables; a very large pulley mass means small acceleration) or by identifying cases in which the expression becomes unphysical (e.g., in the second expression, if $M/2 = m_1+m_2$ acceleration would be infinite).

The problem has been administered in three sections of MM ($N = 47$) before any instruction. Student written responses were examined and coded. Answers were coded for answer and explanation. Student responses are tentatively assigned to categories that arise iteratively according to the major themes identified in the data. For the explanations, we use open coding (drawn from grounded theory), in which the entire data corpus is examined for common trends, and all data are reexamined and grouped into the defined categories.

Results

Student responses were coded iteratively based on both the correctness of their assessment and the explanation used in support. In the final version of the rubric, twelve distinct but not exclusive codes were used, with an 'other/blank' category used when students provided no intelligible explanation. Of the twelve codes, only a few were commonly used. We provide brief examples of several of the most common codes and criteria in Table I.

While some of the codes that emerged were expected, many of them were not. For example, a few students evaluated the correctness based on whether the resulting expression would have the correct algebraic sign, although no coordinate axis is specified. Of course the written explanation may not indicate fully the underlying reasoning that students are using. For example, a few students gave explanations in which they stated that the rotational inertia of the massive pulley mass would decrease the acceleration. Others simply noted the presence or absence of the pulley mass in the expression. We cannot be sure whether the students in two groups were using similar reasoning.

TABLE 1. Sample codes for *evaluate the expression* task. The codes were not exclusive, so a student response might include both mechanism and mass difference, for example.

Code	*Description*
Solution	Attempted to solve problem directly
Variables	Noted presence or absence of variables in expression
Mechanism	Described physical mechanism for motion (forces, energy)
Mass difference	Commented on presence or absence of term describing difference in masses $\pm(m_2\text{-}m_1)$

The data indicate that this task is challenging task for students. Only one student offered a completely correct solution. Ten others identified all three solutions as incorrect but with incomplete or incorrect explanations. Many students gave no response. About 10% of students were coded as 'blank' for the first expression, and 20% for the second. Written explanations often indicated a lack of confidence in responses: "This is not correct because the mass of the pulley needs to be incorporated (although to tell the truth I am not sure how)."

The approaches used by students varied considerably, and while many did give explanations that call upon physical intuition or an attempt to parse the mathematical expression, others seemed to respond as though this task were a more typical end-of-chapter problem. About 10% of the students solved the problem directly, and a few others performed algebraic manipulations of the given expressions. A few responses appeared as though they were to a multiple-choice question, with a circle or check mark next to one expression. One student wrote that the first response "Needs the pulley!" and circled the second response, writing "This one!!!" A few students mentioned partially remembered results: "my very rusty memory only recalls subtracting from the bottom." These responses suggest an epistemological stance that is quite different that the problem intends, and students may need to have the purpose of this activity framed very explicitly.

Limiting cases: Only a handful of students explained using quantitative reasoning from the categories described above to evaluate this expression. As an example of a response that we coded as using limiting cases, one student wrote, "No, if m_2= 0 kg then this formula makes a = 0 m/s^2 which is clearly not true."

Presence / absence of variables: A very common code reflected responses in which explanations referred to the presence or absence of variables: "This could be correct because all relevant variables are used." For the first expression, most students gave explanations that referred to the absence of the mass of the pulley. A few (~5%) stated that the expression was incorrect because of the absence of the pulley mass *M,* but the more common response was to state (correctly) that the expression would be correct if the mass of the pulley is negligible.

Physical mechanism: Several student responses described a physical mechanism for the motion. A smaller group reflected an attempt to reconcile the mathematical form of the expression with this sense of physical mechanism. The most obvious examples of this were students who referred explicitly to the presence or absence of a term with the difference in masses. These responses included some in which the presence of this term was noted: "Correct [first expression]; m_2 is countering m_1 so m_1 is accelerating at a portion of g." A few students gave explanations that reflected similar reasoning but with respect to other quantities: "this [second] expression raises the value of acceleration as the mass of the pulley increases leading me to believe this is incorrect."

Discussion

This portion of the project is in initial stages, and further research is needed. Interviews focusing on these skills are in progress, including the *Evaluate the Expressions* task. We offer two tentative observations.

First, many students entering the MM course do not successfully reason quantitatively even when explicitly prompted to do so. The responses given by some students suggest that they do not recognize that the tasks shown require them to step away from solving the problem directly or remembering its answer in order to reason whether a solution might be correct. Relatively few students spontaneously examined the expressions for special cases of the variables in the problem or related to a sense of physical mechanism. Traditional instruction, focused almost entirely on procedures, does not necessarily lead students to develop other quantitative reasoning skills.

Second, given that physicists value the quantitative skills described, that there is a need for tasks that can be used in instruction and assessment. Redish and Kuo [12] have recently written that students "need to learn a component of physics expertise not present in math class—tying those formal mathematical tools to physical meaning….We as physics instructors must explicitly foster these components of expert physics practice to help students succeed in using math in physics." Yet the majority of problems in the course text are merely mathematical exercises that do not explicitly address these reasoning skills.

Questions for audience

Is there existing theoretical or empirical work in RUME that would complement or inform this study?

PER-influenced physics instruction has led to increased emphasis on conceptual understanding and sense-making rather than procedural tasks and routine computations. We clearly have a lot to learn from the RUME community regarding the ways that students think about mathematical sense-making; what have we overlooked due to our disciplinary filters?

What are the implications for research and practice in mathematics education?

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Student Experiences in a Problem-Centered Developmental Mathematics Class

Community colleges serve an important role in providing access to college for students who may otherwise be unable to pursue post-secondary education. However, required pre-college level (or developmental) coursework often serves as a barrier to the college-level classes. Approximately 60% of community college students take at least one developmental class (Attewell, Lavin, Domina, & Levey, 2006; Bailey, Jeong, & Cho, 2010), with only around 30% of those who take developmental math actually completing their required developmental math classes (Bailey, 2009). The reasons for student attrition from developmental classes are complex (Cohen & Brawer, 2008), but unlike many of the challenges community college students face, the college teachers control both the curriculum and instruction of developmental classes. Thus, creating developmental classes that promote student success and empowerment has become an important goal in developmental math education.

A recent curriculum movement, often called *Mathematical Literacy*, uses group work and problem solving to 1) make the mathematical content more relevant, and 2) highlight the utility of mathematics. However, when classes like *Mathematical Literacy* were introduced in K-12 classrooms, some students resisted (Lubienski, 2000) which could limit the impact of the new curriculum on student outcomes.

Fields Community College (FCC; all names are pseudonyms) has offered *Mathematical Literacy* for about four years. For this study I investigate the *Mathematical Literacy* classroom of a course designer. The instructor's familiarity with the curriculum offers a window into the best case scenario of the *Mathematical Literacy* movement. In this context, I focus on student experiences because of the belief that mathematics instruction should be empowering: an important, but often overlooked outcome at the college level. Towards this end, I ask:

1) How do students in *Mathematical Literacy* experience the class, as taught by one of the course designers?
2) How does the student experience for students who did not successfully complete the course differ from those who did?

Population & Sample

The main population under study consists of the students enrolled in *Mathematical Literacy* at FCC in the spring 2015 semester. FCC students were advised to take *Mathematical Literacy* if they needed developmental algebra but were not pursuing a degree in science or math. The class of one of *Mathematical Literacy's* designers was observed. Eight of the sections' 22 students elected to be interviewed about their experiences in the class.

Methods

Data Sources

Data in this study comes from the eight student interviews and audio recordings of these students in their groups during class. I observed and audio recorded 12 two-hour class periods over the course of the semester. Interviews with the eight students took place outside of class and lasted between 30 and 45 minutes. Individuals in the observed section took a pre- and post-survey that included an attitudes towards math inventory and open ended questions.

Methods of Analysis

Survey data. Both the pre- and post-survey contained data from the four attitude scales. Scores on each scale were computed so that the lowest value (1) corresponded to "Strongly disagree" and the highest value (5) corresponded to "Strongly agree." I report these scale scores without further analysis of the survey results.

Student interviews. The audio recordings of the interviews were transcribed and coded using the first phase of grounded theory (Corbin & Strauss, 2007; Strauss & Corbin, 1990), which uses several deliberate steps in developing codes rooted in the themes of the data. The themes ultimately used for this study revolve around doing group work, problem solving, experiences with the teacher, and emotions and feelings about mathematics. Given the research questions and the importance of the individual in those questions, it seemed important to explore individual's experiences within each theme. As such, rather than performing a second round of grounded theory coding, for each interviewee I created a one- to two-page profile summarizing their data on the four identified themes.

Preliminary Results

Six of the eight interviewees completed the course and two did not. Table 1 presents the pre- and post-survey scores for each interviewee on each of the four measured mathematical attitudes. Of note, Ross and Emelia, the two students who did not complete the course, had lower than average confidence scores. Review of the audio of them in their group demonstrates that they were behind from the first week of class. On the other attitude sub-scales Emelia tended to be in the top half and Ross tended to be around the median of the group. Carrie decreased her scores all around, while Craig increased them.

Table 1. Interviewee's pre- and post-survey attitude scores by scale

	Motivation		Enjoyment		Value		Confidence	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Dave	3.78	3.56	3.50	3.38	4.38	4.63	3.40	3.40
Emelia[a]	3.44	-	3.25	-	3.63	-	2.60	-
Carrie	3.00	2.89	3.38	2.88	4.38	4.00	3.07	2.53
Vince	2.89	3.00	2.88	3.75	4.50	4.50	3.00	3.87
Ross[a]	2.78	-	3.00	-	3.75	-	1.60	-
Craig	2.67	3.00	2.25	3.13	3.00	3.63	2.13	3.20
Bea	2.56	2.00	2.63	2.75	3.13	4.25	1.80	1.60
Carley[b]	2.56	2.56	2.50	2.50	3.88	-	2.87	-

Note: The reported scales represent the scaled score on the pre- and post-survey, where a score of 1 corresponds to "Strongly disagree" and a 5 corresponds to "Strongly agree."
[a] Student was not present for the post-survey.
[b] Some of student's sub-scales were not complete.

The interview data highlight the fact that students' group work experiences varied widely, but student temperament seemed to play a role in their feelings: Emelia and Dave preferred working alone, which partially informed their dislike of group work. Dave's dislike was tempered when he thought his group mates were on the same level as him. Emelia, however, perhaps because she was dependent on her group to teacher her the content, found little about the group work enjoyable. Many of the students believed that individuals had some responsibility to ask for help if they were struggling, but only one, Carley, explicitly noted that individuals within the group had a responsibility to others in the group. Ultimately, how the individuals thought about their groups seemed to play the largest role in how students experienced the class.

Significance

By answering these questions future iterations of this class can better structure the group work environment to facilitate learning for the students community colleges math classrooms most need most to help—those who struggle early in the class.

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Supporting students in seeing sequence convergence in Taylor series convergence

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Virtual manipulatives designed to increase student understanding of the concepts of approximation by Taylor polynomials and convergence of Taylor series were used in calculus courses at multiple institutions. 225 students responded to tasks requiring graphing Taylor polynomials, graphing Taylor series, and describing relationships between different notions of convergence. We detail significant differences observed between students who used virtual manipulatives and those that did not. We propose that the use of these virtual manipulatives promotes an understanding of Taylor series supporting an understanding consistent with the formal definition of pointwise convergence.

Keywords: Taylor Series, Virtual Manipulatives, Calculus, Cooperative Learning, Approximation

As one of the more challenging concepts in calculus, Taylor series coordinates ideas of approximations of functions with the evaluation of limits of sequences and series. Previous studies of student understanding of Taylor series indicate that many students lack a developed, working model of this concept that is sufficient to afford meaningful progress on Taylor series tasks (Kung & Speer, 2010; Martin, 2013). Tasks typically proposed to students and the ways in which they engage in such tasks may not be adequate to promote the conceiving of and relating relevant quantities so as to coordinate notions of Taylor series convergence with sequence convergence. One approach for helping students coordinate these ideas is through the use of computer software. Specifically, we aim to understand if a Virtual Manipulative (VM) might aid students in their understanding of Taylor series convergence. In particular, we ask:

1. Do students in classrooms implementing VMs respond differently to prompts asking for the production of graphical representations of Taylor polynomials and Taylor series?
2. Can differences in conceptions be observed between students from VM and non-VM classes?
3. How do students in classrooms utilizing VMs respond to open-ended questions about convergence? What do they consider most relevant to the concept of convergence?

Background

For an analytic function *f*, a Taylor series is a power series of the form $\sum_{k=0}^{\infty} c_k (x-a)^k$ where $c_k = f^{(k)}(a)/k!$ for each *k*. Almost all studies of student understanding of Taylor series document student struggles to comprehend and interpret the complicated structure inherent in Taylor series (e.g. Champney & Kuo, 2012; Kidron & Zehava, 2002; Kung & Speer, 2010; Martin, 2013; Martin & Oehrtman, 2010). Martin (2013) noted that students rarely moved beyond algebraic reasoning to offer graphical interpretations of convergence, and often failed to coordinate their notions with sequence convergence by fixing values of *x*. When looking at Taylor series graphs, Oehrtman (2009) observed that students inaccurately concluded that a Taylor polynomial and the approximated function are identical over an interval after observing a polynomial “touching” the approximated function, overlooking nonzero differences (or error or remainder) between polynomials and the function for particular values of *x*. In contrast, Martin and Oehrtman (2010) noted that students attending to structures involving “approximations,” “error,” “accuracy,” etc., where for each approximation there is an associated error and a bound on that error, can lend itself to students describing the existence of nonzero differences between approximations (Taylor polynomials) and the approximated function for particular values of *x*.

What is a suitable understanding of Taylor series convergence for the Calculus student?

Our focus has not been to bring calculus students to a formal definition of pointwise convergence using Taylor series, but to support students in algebraic and graphical reasoning consistent with the formal definitions of sequence convergence and the recognition of sequence convergence as embedded within pointwise convergence. For graphical explanations of Taylor series convergence, our students were expected to elaborate on notions of sequence convergence using vertical number lines (Figure 1) for different values of *x* while coordinating notions of “estimates,” “error,” “accuracy,” etc. which involve an unknown quantity and a known approximation. This paper investigates differences in responses to prompts between students engaging in such activity compared to students from calculus classes that tend to focus mainly on completing convergence tests using algebraic approaches.

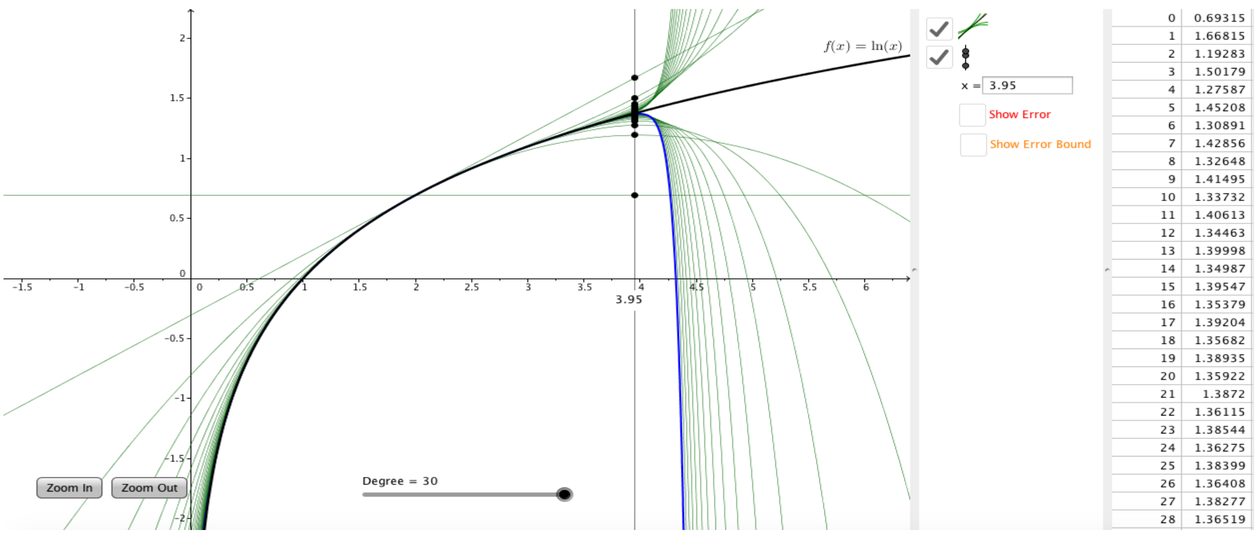

Figure 1: Screenshot of Taylor series VM

Virtual Manipulatives (VMs)

By a VM we mean an interactive computer representation of a mathematical concept. Moyer-Packingham and Westenskow (2013) note that VMs have been useful to develop certain understandings but that education research on VMs is lacking beyond 6^{th} grade. For notions related to sequence convergence, Cory and Garofalo (2011) observed that VMs can reinforce students’ understandings of the quantitative and logical relationships captured by dynamic imagery and that these relationships can be recalled months later. Yet, when it comes to Taylor series, Kidron and Zehavi (2002) found that students can fail to correctly interpret what they are seeing in the VM if working with the VM preceded interpretation of the algebraic representation coordinated with the graphical depiction. Taylor series VMs have been helpful in supporting students with noticing general graphical trends (Habre, 2009; Kidron & Zehavi, 2002), but these VMs have also unintentionally reinforced students seeing a Taylor polynomial and the approximated function as identical over an interval after viewing a Taylor polynomial literally touching the approximated function in the VM. Even with these potential setbacks, we hypothesize that well designed graphical images (including VMs) coordinated with approximation tasks can help students come to an understanding of Taylor series convergence consistent with formal theory.

Methods

Students were recruited from calculus classes to participate in this study after receiving instruction concerning sequences, series, and Taylor series. Students were from two types of classes: calculus classes that focused on algebraic approaches to common Taylor series tasks (referred to as non-VM students), and classes that included VMs and incorporated approximation tasks (Martin & Oehrtman, 2015; Oehrtman, 2008) using laboratory-style group exercises (referred to as VM students). Instructors utilized VMs to supplement classroom instruction and group activities. Students individually interacted with the VMs to complete homework tasks. To address notions of polynomials being the same as the approximated function on an interval, zoom features were included in most VMs.

Data from this study was taken from quizzes, classwork, exams, and questionnaires that students completed after the conclusion of relevant classroom activities concerning Taylor series. In total, 139 non-VM students and 86 VM students from four institutions participated in this study. For this analysis we focused on student responses to three tasks:

<table>
<tr><th colspan="3">Tasks</th><th>non-VM Students</th><th>VM Students</th></tr>
<tr><td colspan="3">1) “Using the graph of sin(x) below, on the same axes sketch three different Taylor polynomials for sine”</td><td>✓</td><td>✓</td></tr>
<tr><td colspan="3">2) “Using the graph of sin(x) below, on the same axes sketch the Taylor series for sine”</td><td>✓</td><td>✓</td></tr>
<tr><td colspan="3">3) Explain how sequence convergence is related to Taylor series convergence.</td><td></td><td></td></tr>
<tr><td rowspan="2"></td><td><u>Two Question Version</u></td><td>3a) “Briefly explain how sequence convergence is related to series convergence. Be as precise as you can.”
3b) “Briefly explain how series convergence is related to Taylor series convergence. Be as precise as you can.”</td><td></td><td>✓</td></tr>
<tr><td><u>One Question Version</u></td><td>“List all of the ways in which Taylor series convergence is related to sequence convergence and series convergence. Make sure your explanations reference
i. formulas when appropriate and
ii. includes a graphical explanation that highlights sequences and/or series on your graph above. (That is, add to the graph above to appropriately highlight sequences and/or series convergence as it relates to Taylor series convergence.)”</td><td></td><td>✓</td></tr>
</table>

Figure 2: Tasks given to students

Responses were collected and coded (Strauss & Corbin, 1990) by a team of two undergraduate and two faculty researchers. One faculty researcher and the two undergraduates developed the coding protocol for the three tasks. The other faculty researcher coded a random sample of 20 students for each of the first two tasks, achieving over 80% reliability for each coding decision. Task 3 was coded independently by the second researcher.

Initial Results

Task 1

Task 1 was intentionally ambiguous so that students could choose to have different Taylor polynomials be centered at different points, have different degrees, or both. Responses were coded for correctness twice: once using a strict rubric and once using a more relaxed rubric. The relaxed rubric was developed since students drew even degree Taylor polynomials, while the Taylor series expansion for sin(x) centered at x=0 does not have even degree terms. These answers were counted as correct under the relaxed rubric.

Using the strict rubric, 13% of the 86 VM students answered correctly, compared to 4% of the 139 non-VM students. Using the relaxed rubric, 28% of the VM students answered correctly while 6% of the non-VM students did. This suggests that students in the VM classrooms outperformed non-VM calculus students measured by either the strict rubric, $X^2(1, N = 225) = 4.3164$, $p = 0.038$, or the relaxed rubric, $X^2(1, N = 225) = 19.592$, $p < 0.001$.

Taylor polynomials beyond 0^{th} order must be tangent to the function being approximated. Of the VM students, 33 (38%) drew Taylor polynomials that were both tangent to the function being approximated, while 11 (8%) of the non-VM students did. Of the VM students, 28 (33%) drew Taylor polynomials not only tangent to the function being approximated, but also to each other, while 9 (6%) of the non-VM students did. This indicates a larger proportion of VM students are drawing Taylor polynomials tangent to the function being approximated than non-VM students, $X^2(1, N = 225) = 31.331$, $p < 0.001$.

Since students could answer the question using either Taylor polynomials of different degree or with a different center, we considered how students approached the problem. The number of students who answered correctly using different centers was quite small: 6 students, 5 in VM sections, 1 in non-VM sections. 62 students in VM sections (72%) answered the question by drawing polynomials of different degree, while 26 students in non-VM sections (19%) answered the question by drawing polynomials of different degree, indicating students in VM classrooms were more likely to solve the problem correctly using different degree Taylor polynomials than non-VM students, $X^2(1, N = 225) = 63.589$, $p < 0.001$.

Task 2

The ideal answer for Task 2 would be to sketch over the graph of sine. Possible misconceptions can introducing error either throughout the entire graph or part of the graph, or considering the Taylor series to be a collection of polynomials, either finite or infinite. As before, grading was done using a strict rubric and a relaxed rubric. Students were considered correct under the strict rubric if they traced over the graph. A correct answer under the relaxed rubric allowed for careless tracing.

Using the strict rubric, 41% of the 86 VM students answered correctly, while 11% of the 139 non-VM students did. Using the relaxed rubric, 44% of the VM students answered correctly, while 16% of the non-VM students did. VM students therefore outperformed non-VM students using both the strict rubric, $X^2(1, N = 226) = 23.595$, $p < 0.001$, and the relaxed rubric, $X^2(1, N = 226) = 18.29$, $p < 0.001$.

Another misconception of interest was drawing a collection of functions as a Taylor series. Students in the VM sections were more likely to report a collection of polynomials as a Taylor series than those in non-VM sections, whether including all students, $X^2(1, N = 226) = 12.34$, $p < 0.001$, or removing students who left the problem blank, $X^2(1, N = 148) = 5.23$, $p = 0.02$.

Also of interest are the students who introduce intentional non-zero error in the graph of the Taylor series on either the entire domain (negative infinity to infinity) or on some part of the domain. There was no statistically significant difference between the proportion of VM and non-VM students who drew graphs with error over the entire domain, $X^2(1, N = 226) = 1.877$, $p = 0.17$, including when students with blank responses were removed, $X^2(1, N = 148) = 0.35126$, $p = 0.553$. Similarly, there was no difference between the groups when comparing students who introduced error for only part of the domain, $X^2(1, N = 226) = 0$, $p = 1$, and still no significant difference when blank answers were removed, $X^2(1, N = 148) = 2.1018$, $p = 0.147$.

Task 3

The numbers of students describing key ideas related to Taylor series in task 3 are reported below.

Table 1: Number of students referencing concepts is task 3

	Highlight particular *x*	Mention error/error bound for fixed *x*	Mention vertical number line	Describe partial sums	Approximation language
Two question version (N=60)	4 (7%)	0 (0%)	1 (2%)	3 (5%)	3 (5%)
One question version (N=15)	10 (67%)	0 (0%)	1 (7%)	3 (20%)	0 (0%)
	Concept of error	Describe error as decreasing	Describe error bound	Describe sequence of terms	Describe interval of convergence
Two question version (N=60)	6 (10%)	2 (3%)	1 (2%)	6 (10%)	17 (28%)
One question version (N=15)	2 (13%)	2 (13%)	0 (0%)	1 (7%)	4 (27%)

Conclusion & Questions

Evidence suggests that these VMs encouraged students to have an understanding of Taylor series that may eventually support the formalization of a pointwise convergence definition. Students in classes using VMs were more likely to draw Taylor polynomials correctly, as well as to draw the Taylor polynomials tangent to the function being approximated. Students in VM classrooms, while more likely to correctly draw a Taylor series graph of sin(*x*), were also more likely to draw a collection of Taylor polynomials than students in non-VM classrooms. This may be an artifact of the VMs themselves, in which new Taylor polynomials are introduced while previous Taylor polynomials remain. Responses to the third task suggest that connecting many of the key concepts of approximation, error, and error bound to issues of convergence may not be in the fore of the VM students' minds. Despite the relatively low numbers and losses in some categories, we saw gains in highlighting a particular *x* to be especially promising as it is one of the key concepts of pointwise convergence of Taylor series. Combined with results from Task 1, VM students had improved understanding of general shapes and trends of Taylor series convergence, but more support may be necessary to promote further unpacking of the relevant quantities and move closer to a notion of pointwise convergence.

Currently, interviews are being conducted with students who have completed classes featuring VMs to further describe what students are observing when viewing a Taylor series VM.

We invite discussion about the following questions:

1) Analyzing the ways in which students interact with VMs compared to static images.
2) Is there some other way of bringing out the pointwise conception naturally?

Acknowledgment

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Student performance on proof comprehension tests in transition-to-proof courses

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As part of a project aimed at designing and validating three proof comprehension tests for theorems presented in a transition-to-proof course, we asked between 150 and 200 undergraduate students in several sections of one of these courses to take long versions (20 to 21 multiple-choice questions) of these tests. While analysis of these data is ongoing, we discuss preliminary findings about psychometric properties of these tests and student performance on these proof comprehension measures.

Key words: Proof reading, Proof comprehension assessment, Transition to proof

Most advanced mathematics courses are taught in a "definition-theorem-proof" format, where textbooks and lecturers present the definitions of new concepts and then prove theorems about those concepts (Weber, 2004; see also Dreyfus, 1991; Mills, 2011). Underlying this widely used pedagogical format is the assumption that mathematics majors can learn a great deal by studying the proofs that their professors present. Yet as mathematicians and mathematics educators observe, students' understandings of the proofs that they read are rarely assessed in a meaningful way (e.g., Conradie & Frith, 2000; Cowen, 1991). This is due, in great part, to the dearth of valid assessments that measure proof comprehension (cf., Cowen, 1991; Weber, 2012).

In a recent project we are developing proof comprehension tests for three theorems that are commonly presented in undergraduate transition-to-proof courses. Our aim is to validate these multiple-choice tests and make them available for others to use in their own courses and research projects.

Literature Review

Prior to our work in the area, we are not aware of the existence of systematic ways of assessing students' comprehensions of proof in undergraduate mathematics courses. However, there have been three important contributions in this area in the research literature. Conradie and Frith (2000) directly addressed the issue of proof comprehension tests in undergraduate mathematics. In addition to stressing their importance, these researchers provided comprehension tests for two different proofs. While their items were intriguing and called mathematics educators' attention to an underrepresented area of research, we note that these tests seemed to be created in a somewhat *ad hoc* manner, and it was unclear how these items were generated or what specific skills or understanding each item was designed to assess. Yang and Lin (2008) developed a model of reading comprehension for geometry proofs (RCGP) that consisted of four levels of understanding: *surface* (i.e., understanding the meaning of terms and statements), *recognizing the elements* (i.e., knowing whether a statement was an axiom, assumption, definition, or deduction), *chaining elements* (i.e., seeing how new statements are deduced from previous ones), and *encapsulation* (i.e., viewing the proof as a whole to comprehend the higher level ideas in the proof). Yang and Lin developed specific assessment items to assess the first three levels of understanding of a given proof, but notably did not attempt to assess how well students encapsulated the proof.

While some of the ideas in Yang and Lin's model are pertinent to proofs at the undergraduate level, we argue that the model by itself is not sufficient to probe students' understanding of a proof in an advanced mathematics class. For instance, Yang and Lin did not attempt to assess if students achieved the highest level of understanding in their model, which consisted of viewing the proof as a whole to comprehend its higher-level ideas. While this type of understanding might not be a central concern for a high school geometry teacher, we contend that skills such as being able to summarize a proof or being able to flexibly apply the methods of a proof to prove a new theorem are crucial skills for students in advanced mathematics classes. Further, there are also logical nuances that are present in some undergraduate proofs that are not accounted for in Yang and Lin's assessment model, such as how should proof by contradiction or proof by cases be understood. From Yang and Lin's perspective, this is not important as such proofs are rare in high school geometry classes, but they are common in undergraduate mathematics classes.

To address this limitation, Mejia-Ramos et al. (2012) built upon Yang and Lin's model to develop an assessment model for proof comprehension that is more suitable to the context of undergraduate mathematics. The components of their assessment model can be separated into two groups. The first group concerns *local* understandings of the proof, meaning that these questions can be answered by focusing on a small number of statements within the proof. In general, these questions would be concerned with describing the logical structure or evaluating the validity of the proof, and are adaptations of the first three components of Yang and Lin's (2008) model for geometry proofs. These local types of assessment items are:

- *Meaning of terms and statements*: items of this type measure students' understanding of key terms and statements in the proof.
- *Logical status of statements and proof framework*: these questions assess students' knowledge of the logical status of statements in the proof and the logical relationship between these statements and the statement being proven.
- *Justification of claims*: these items address students' comprehension of how each assertion in the proof follows from previous statements in the proof and other proven or assumed statements.

The second group concerns *holistic* understandings of the proof. In contrast with local understandings, one would not be able to answer questions about holistic understandings of a proof by focusing on a small number of statements in a proof, but would have to be addressed by inferring the ideas or methods that motivated the proof in its entirety. The holistic understandings relate to the "encapsulation" level in Yang and Lin's (2008) model, and include four types of assessment items that address students' understanding of the proof as a whole:

- *Summarizing via high-level ideas*: these items measure students' grasp of the main idea of the proof and its overarching approach.
- *Identifying the modular structure*: items of this type address students' comprehension of the proof in terms of its main components/modules and the logical relationship between them.
- *Transferring the general ideas or methods to another context*: these questions assess students' ability to adapt the ideas and procedures of the proof to solve other proving tasks.
- *Illustrating with examples*: items of this type measure students' understanding of the proof in terms of its relationship to specific examples.

Lecturers, textbook writers and researchers can use Mejia-Ramos et al.'s (2012) model to generate open-ended items that measure students proof comprehension along dimensions that are valued by both mathematicians and mathematics educators. However, this way of using the model has two shortcomings. First, these open-ended questions can be time consuming to generate and grade, which may limit their utility in teaching and research situations involving a large number of test takers. Second, the validity and reliability of these questions has yet to be verified. The purpose of our current project is to address both shortcomings by designing and validating multiple-choice proof comprehension tests for three proofs from a transition-to-proof course.

Methods

Materials

We have developed comprehension tests for proofs of the following three theorems:

- *Theorem 1:* The set of prime numbers is infinite.
- *Theorem 2:* Every third Fibonacci number is even. That is, if we define the n^{th} Fibonacci number (denoted by f_n) in the usual way, then f_{3k} is even for every $k \in N$.
- *Theorem 3:* The open interval $(0,1)$ is uncountable.

Theorem 1 is a central theorem in transition-to-proof courses, being one of the first indirect proofs that students encounter. The proof is also often misunderstood by students, as there is frequently confusion as to whether the constant generated in this proof is a prime number (e.g., Hazzan & Zazkis, 2003). The proof of Theorem 2 is a typical example of the kind of proofs by induction presented in transition-to-proof courses. Proofs by induction are a central concept in these courses and is notoriously difficult for students (e.g., Dubinsky, 1987, 1989; Harel, 2001). Theorem 3 is a more advanced theorem with a more sophisticated proof method that is usually covered in transition-to-proof courses.

In order to generate the current version of the multiple-choice, proof comprehension tests, we followed the following procedure:

1. For each one of the three proofs, we first generated open-ended questions of each type of assessment item in Mejia-Ramos et al.'s (2012) model.
2. We then conducted task-based interviews with 12 mathematics majors who had recently completed a transition-to-proof course. These participants were asked to read the three proofs and answer the open-ended questions.
3. We observed the correct answers that the participants provided as well as common incorrect answers. These data were used as a basis to generate a larger set of multiple-choice questions, with at least one question for every dimension of understanding in Mejia-Ramos et al.'s (2012) proof assessment model.
4. We then sought feedback from mathematicians at our institution and an advisory board (which included a prominent mathematician and a leading researcher on proof comprehension a the undergraduate level) regarding the accuracy and appropriateness of our items.
5. Finally, we piloted these multiple-choice items with 12 mathematics majors to make sure our items had appropriate wording and choices.

To illustrate the type of items in our proof comprehension tests consider the proof used for Theorem 1:

> Suppose the set of primes is finite. Let $p_1, p_2, p_3, \ldots, p_k$ be all those primes with $p_1 < p_2 < p_3 < \cdots < p_k$. Let *n* be one more than the product of all of them. That is, $n = p_1 \cdot p_2 \cdot p_3 \cdots p_k + 1$. Then *n* is a natural number greater than 1, so *n* has a prime divisor *q*. Since *q* is prime, $q > 1$. Since *q* is prime and $p_1, p_2, p_3, \ldots, p_k$ are all the primes, *q* is one of the p_i in the list. Thus, *q* divides the product $p_1 \cdot p_2 \cdot p_3 \cdots p_k$. Since *q* divides *n*, *q* divides the difference $n - p_1 \cdot p_2 \cdot p_3 \cdots p_k$. But this difference is 1, so $q = 1$. From the contradiction $q > 1$ and $q = 1$, we conclude that the assumption that the set of primes is finite is false. Therefore, the set of primes is infinite.

Item type	Open-ended items	Multiple-choice items
Meaning of terms and statements	1. Please give an example of a finite set and explain why it is finite. 2. Please give an example of a set that is infinite and explain why it is infinite.	Which of the following are examples of finite sets? Please select **all** that apply. a) The set with the following elements: 1, 2, and 3. b) The set of real numbers between -2 an 2 c) The set of all fractions $\frac{1}{r}$ where r is a natural number. d) The set of integers greater than 4.5 and smaller than 9999.
Justification of claims	1. Why is it valid to conclude that n is a natural number? 2. Why does n have to have a prime divisor? 3. Why exactly can one conclude that if q is prime, then $q > 1$?	In the proof, why is it valid to conclude that n is a natural number? Please select **the best** option. a) Because the product and sum of natural numbers is a natural number. b) Because n is greater than 0. c) Because $1, p_1, p_2, \ldots, p_k$ are all integers. d) Because it is a given in the proof that n is a natural number.
Summarizing via high-level ideas	1. Summarize in your own words the main idea of this proof. 2. What do you think are the key steps of the proof? 3. Give a three-sentence description of how the proof established the theorem.	Which of the following options **best** summarizes the main idea of this proof?[1] a) The main idea of the proof is to show that if the set of primes were finite, one could find a formula for a new prime number that is not in that finite set, contradicting the assumption. b) The main idea of the proof is to assume that the set of prime numbers is finite and to construct a natural number that has a prime divisor equal to 1, which is impossible.
Transferring the general ideas or methods to another context	1. In the proof, we define $n = p_1 \cdot p_2 \cdots p_k + 1$. Would the proof still work if we instead defined $n = p_1 \cdot p_2 \cdots p_k + 31$? Why? 2. Define the set $S_k = \{2,3,4,\ldots,k\}$ for any $k > 2$. Using the method of this proof, show that for any $k > 2$, there exists a natural number greater than 1 that is not divisible by any element in S_k.	In the proof, we define $n = p_1 \cdot p_2 \cdots p_k + 1$ Would the proof still work if we instead defined $n = p_1 \cdot p_2 \cdots p_k + 31$? Please select **the best** option. a) Yes, because n will still be a prime number, so the contradiction will still hold. b) Yes, because 31 is a prime number, which means that q must still be 1. c) No, because this definition of n would not be necessarily prime. d) No, because in this case q could be 31, which does not lead to a contradiction.

Table 1. Examples of items used in the proof comprehension tests for Theorem 1.

[1] This item has two other foils that did not fit in the table/proposal.

Table 1 contains examples of open-ended and multiple-choice versions of some of the different types of items used in the test for this proof. The multiple-choice tests generated for theorems 1 and 2 contained 20 items each, while the test for Theorem 3 contained 21 questions.

Participants and procedure

The proof comprehension tests were distributed to students in several sections of an undergraduate transition-to-proof course in a large state university. Each of the five participating instructors allocated 40 minutes of class to distribute each test. On the day each test was distributed, students in the course received a packet that contained the theorem and its proof, instructions on the different types of items in the test, and all the multiple-choice questions (the order of the items in each section of the test was randomized). The test for Theorem 1 was distributed after instructors had introduced proofs by contradiction in class (approximately a third of the way into the term), the test for Theorem 2 was distributed once instructors had discussed the principle of mathematical induction (usually by the middle of the term), and the test for Theorem 3 was distributed by the end of term, after instructors had discussed the notion of the cardinality of sets.

A total of 201 students took the proof comprehension test for Theorem 1, 192 students took the test for Theorem 2, and 152 students took the test for Theorem 3. [2]

Preliminary Results

Analysis of this data set is on going. However, preliminary analyses suggest several interesting trends:

1. There is a strong correlation between students' performance on any two of the three proof comprehension tests,
2. The tests, even before excluding poor or uninformative items, show a high internal consistency.

Taken together, these results suggest that proof comprehension can be a meaningful single-dimensional construct. Ongoing analyses will explore the extent that this is the case. We will also discuss items that the large majority of students answered correctly and the items that most students answered incorrectly, which can provide some much needed baseline data on how well mathematics majors understand proof in a transition-to-proof course.

Questions for the audience

1. Do you have any suggestions for further analysis of the data?
2. How would you recommend that we disseminate these tests to mathematicians?
3. How might we improve the test design process for future iterations of these types of studies?

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[2] The decreasing number of participating students was not only due to the regular reduction of class size as the term progresses. One of the participating instructors did not reach the topic of cardinality in class, which meant that we could not distribute the test for Theorem 3 in the two sections led by this instructor.

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Results from the Group Concept Inventory: Exploring the Role of Binary Operation in Introductory Group Theory Task Performance

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Binary operations are an essential, but often overlooked topic in advanced mathematics. We present results related to student understanding of operation from the Group Concept Inventory, a conceptually focused, group theory multiple-choice test. We pair results from over 400 student responses with 30 follow-up interviews to illustrate the role binary operation understanding played in tasks related to a multitude of group theory concepts. We conclude by hypothesizing potential directions for the creation of a holistic binary operation understanding framework.

Key words: Binary Operation, Abstract Algebra, Student Conceptions

Binary operations are at the heart of school mathematics from early arithmetic, to high school algebra, and their generalization: abstract algebra. The prominence and familiarity of operations can lead to the belief that they are a simple concept for university-level students. We validated this conjecture through surveying a panel of introductory abstract algebra instructors. All 13 felt that the difficulty of the binary operation concept was 5 or below on a 0 to 10 scale with an average value of 2.63.

However, while students may have a strong understanding of binary operation in straight-forward contexts such as determining if a given relation is in fact a binary operation, a robust understanding is required to leverage binary operations in the contexts of building groups, differentiating between binary operations, appropriately checking properties, and dealing with unfamiliar structures. While the majority of students we surveyed could correctly determine that division is not a binary operation, understanding of binary operation seemed to contribute to incorrect responses on questions targeting understanding of group, subgroup, associative property, identities, and inverses.

In this proposal, we present results from a large-scale implementation of the Group Concept Inventory (GCI). The inventory was designed to probe conceptual understanding around fundamental topics in introductory group theory. Over 400 students, representing a multitude of institution types across the United States, responded to each question. We pair these responses with interview data to hypothesize how binary operation understanding underlies conceptions around fundamental group theory topics.

Literature Review

All group theory relies on the concept of group: a set paired with a binary operation. A binary operation is a function that maps the cartesian product of a set of elements to that set of elements. For example, addition over the integers would be a binary operation as it inputs any two integers and returns one integer. In order to understand the generalized binary operation, not only would one need to make sense of operations and their properties, but also understand binary operation as a special case of function.

The majority of research on binary operations exists within the specialized case of arithmetic operations. Slavit (1998) discussed *operation sense* in a series of stages built around familiarity with standard arithmetic operations and their relationships to other operations, properties they may possess, and their understanding independent of concrete inputs. However, this framework is built in terms of operations that are not arbitrarily defined but rather represent a standard process, such as combining groups in the case of addition. In addition to operation sense, operations have been discussed in terms of their duality as both a process and object. Gray and Tall (1994) deem the symbol associated with an operation a *procept.* An expression such as "3+2" represents both the process of adding 3 and 2, as well as the resulting sum. Similarly a function defined as f(x)=3x+4 is both a direction for how to compute an output for any input, and also an object- the function for all x-values.

As a binary operation can be any relation that is a function between a cartesian product of a set and the set itself, the generalized notion incorporates many of the complexities studied in the contexts of function. Understanding functions is challenging across grade spans (Oehrtman, Carlson, & Thompson, 2008), with their role as both processes and objects in addition to numerous representations. Notably, understanding functions (or binary operations) involves seeing function as an *action* (mapping individual inputs to outputs), *process* (a general process for mapping inputs to outputs), and *object* (that can itself be operated on such as comparing if two binary operations are the same) (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Brown DeVries, Dubinsky, and Thomas, 1997). Students have frequently proceduralized functions such as evaluating *f(x+a)* as being equal to *f(x) + a* (Carlson, 1998). Rather than coordinating what the input and output are, the students are superficially altering the function. Students have also been shown to have limitations in terms of representations, desiring an explicit rule written symbolically rather than just a correspondence of ordered pairs (Breidenbach, et al., 1992; Vinner & Dreyfus, 1989).

Two additional frameworks have been contributed in terms of undergraduate understanding of binary operation. Novotná, Stehlíková, and Hoch (2006) approached binary operation from a structure sense view dividing understanding of binary operations into four levels: *Recognise a binary operation in familiar structures; Recognise a binary operation in non-familiar structures; See elements of the set as objects to be manipulated, and understand the closure property; and See similarities and differences of the forms of defining the operations (formula, table, other).* Rather than considering stages of mental constructions in terms of process/object reification, structure sense captures abstracting from familiar objects to unfamiliar. Ehmke, Pesonen, and Haapasalo (2005) contributed an analysis in terms of procedural and conceptual understanding. They identified students as having *procedure-based* understanding of binary operation if they could match binary operations if presented in different representations. The next level is *procedure-oriented* where students could also create different representations when prompted. The highest level is *conceptual* where students could not only move between representations, but also determine if a given relation was a binary operation.

A number of studies have shown that binary operations are not a trivial topic, illustrating struggles with varying undergraduate populations including linear algebra students (Ehmke, Pesonen, & Haapasalo, 2005), abstract algebra students (Brown, et al., 1997; Dubinsky, Dautermann, Leron, & Zazkis, 1994; Hazzan, 1999), in-service and pre-service secondary teachers (Zaslavsky and Peled, 1996), and statistics students (Mevarech, 1983). Mevarech (1983) found introductory statistics students assumed that unfamiliar binary operations such as mean and variance had properties found in groups including the associative property. Zaslavsky and Peled found secondary in-service and pre-service teachers struggled to produce a binary operation that was associative, but not commutative. Binary operation related issues include defining a unary operation, and incorrectly considering repeated binary operations such as *wrongly translating* the associative property on the operation |a+b| as |a|+|b+c|=|a+b|+|c| rather than ||a+b|+|c||=|a+|b+c|| or *overgeneralizing* such as considering the equality (a*b)+c=a*(b+c) to determine if (a*b)+c is a binary operation.

Each of these studies explored some of the complexities associated with the binary operation concept. The group theory context is often the first time that students are asked to reason about binary operations that may be unfamiliar. Furthermore, until group theory, they have likely not reasoned about the binary operation as a general concept.

Methods

These results stem from a larger project developing a concept inventory targeting conceptual understanding in introductory group theory. A 17-item instrument was developed, field-tested and refined through several rounds of validation studies (AUTHOR).The results reported here come from the final round of field-testing across the United States. Students from 33 institutions took this survey after finishing an introductory group theory portion of an undergraduate abstract algebra course. The survey was administered online. Institutions participating were geographically diverse and representative of varying levels of selectivity including 14 institutions with acceptance rates greater than 75%, 10 institutions with acceptance rates between 50-75%, and 7 institutions with acceptance rates less than 50%.

Throughout the field-testing, follow-up interviews were conducted to validate the interpretation of student responses. A total of thirty interviews were conducted including 15 with students during an open-ended round, and 15 with students completing the closed-form multiple-choice version. The students were prompted to explain their answer selection and their understanding of the relevant underlying concept.

Preliminary Results

The following results include examples of three GCI questions where understanding of binary operation appeared to influence student performance. In the first question, students are asked to define a binary operation on a set to form a group. In the second question, students determine if a given subset is a subgroup. In the third question, students evaluate if an unfamiliar operation is associative.

Students were asked to consider the set: $\{1,2,4\}$. This set was selected because it does not correlate nicely to any group students likely studied. Instead, to correctly address the prompt, students would need to recognize that a binary operation can be defined on any set with or without a symbolic rule. As can be found in Table 1, only 23% of students selected the correct response. Thirty-six percent of students responded with a familiar operation that would not meet group requirements, while the remaining students wanted the set to have additional elements in order to define a closed binary operation. This latter group represents a potential limitation in ability to construct an abstract binary operation. In follow-up interviews, a number of students explained that they tested various known operations declaring the sentiment that, "there's no operation I could think of" that would meet the requirements. In contrast, students selecting the correct response appeared to have a more sophisticated understanding of binary operation. In follow-up interviews, they explained that a binary operation can be made to meet group requirements by building an unfamiliar binary operation through leveraging alternate representations such as building a Cayley Table or defining the operation element-wise.

Table 1

Percentage of Students Selecting each Response for Defining a Group Question

Consider the set: $S = \{1, 2, 4\}$. Can an operation be defined such that S forms a group?

Response	Percentage (*n*=468)
Yes, because an operation can be defined on any three element set to form a group.	23%
Yes, multiplication mod 6.	36%
No, the set will not be closed under any operation.	18%
No, the identity element 0 would be needed.	10%
None of the above reasoning is valid	14%

Table 2 includes student response selections for the question on subgroup. This question (or a variant of it) has been used in several prior studies to illustrate student conceptions around subgroup and Lagrange's Theorem (Dubinsky, et al., 1994; Hazzan & Leron, 1996). Dubinsky et al. posited that students who identified Z3 as a subgroup of Z6 where not coordinating binary operation and set correctly - failing to see that the operation of a subgroup must be inherited from the supergroup. However, during many of the follow-up interviews conducted with students who selected the first and second option, the students articulated a notion that the subgroup's operation was "inherited." Several students explained that "it's the same operation" in Z3 and Z6, seeming to rely on a generalized version of modular addition. These students did not seem unable to recognize the need for the same binary operation, but rather did not appropriately address what it means to have the different operations. Instead of evaluating if the products of elements were the same, they instead relied on the general rule which appears to be the same type of operation.

Table 2

Percentage of Students Selecting each Response for Subgroup Question

Does the set $\{\bar{0}, \bar{1}, \bar{2}\}$ form a subgroup in $\mathbb{Z}_6$ (under modular addition)?

Response	Percentage (*n*=429)
Yes, because $\{\bar{0}, \bar{1}, \bar{2}\}$ is a subset of $\mathbb{Z}_6$.	13%
Yes, because $\mathbb{Z}_3$ is a group itself contained in $\mathbb{Z}_6$.	36%
Yes, because 3 divides 6.	6%
No, because the subset $\{\bar{0}, \bar{1}, \bar{2}\}$ is not closed.	44%

In this third question, students had to address an operation that was not associative, averaging. In relation to binary operation, there are two notable responses found in Table 3: the first where students did not feel the need to address a new operation because of its component parts being familiar associative operations, and the third option where parentheses are moved artificially. Students selecting the first choice may be superficially applying the idea of associativity being "inherited" in a new situation. Students selecting the third response fall into Zaslavsky and Peled (1996) overgeneralization category. We conjecture these students may have more fundamental issues with the binary operation procept. These students were not repeatedly operating on two elements to determine if (a ◊ b) ◊ c=a ◊ (b ◊ c), but rather treating subcomponents of the binary operation as if they were three different inputs. This mimics function issues where students struggle to appropriately evaluate expressions such as $f(x+a)$. A robust understanding of binary operation requires making sense of what constitutes the input and how repeated binary operations are calculated.

Table 3

Percentage of Students Selecting each Response for Associativity Question

Consider the binary operation of averaging $\diamond$, on the set of real numbers defined below. Is this operation associative?

$$a \diamond b = \frac{1}{2}(a+b).$$

Response	Percentage (n=432)
Yes, addition (and multiplication) are associative, so $\diamond$ is also.	29%
Yes, because $\frac{1}{2}(a+b)=\frac{1}{2}(b+a)$.	22%
No, because $(\frac{1}{2}a)+b \neq \frac{1}{2}(a+b)$.	17%
No, because $\frac{a}{2}+\frac{b}{4}+\frac{c}{4} \neq \frac{a}{4}+\frac{b}{4}+\frac{c}{2}$.	31%

Discussion

The three results above illustrate some of the additional complexities associated with binary operation as found in field-testing of the GCI. Binary operation conceptions can underlie performance in a number of essential group theory tasks. Furthermore, the student responses serve as a starting ground for expansion of previous work on student conceptions of binary operation. Ehmke, Pesonen, and Haapasalo's (2005) conceptual levels might need to be expanded where creating an unfamiliar binary operation on a given set may represent an even higher level of conceptual understanding. Novotná and Hoch (2008) identified determining if two binary operations are the same or different as the top level of binary operation understanding. This ability seemed crucial to appropriately addressing the question related to subgroup. Finally, in the associativity question, students may be more than just overgeneralizing (Zaslavsky and Peled, 1996), but have fundamental issues correctly operating. As binary operation is a special case of function, these complexities mimic many of the issues found in understanding functions for various level students. Exploring the role of binary operation can help provide insight into why students may struggle with various aspects of these algebra courses. Additional analysis of these results can hopefully build a more holistic framework of binary operation understanding,

Questions for the Audience

1. What might a comprehensive framework for student understanding of binary operation look like?
2. How might student conceptions around binary operations influence their understanding in other advanced mathematics courses?

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Using Adjacency Matrices to Analyze a Proposed Linear Algebra Assessment

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An assessment of student learning of major topics in linear algebra is currently being created as part of a larger study on inquiry-oriented linear algebra. This includes both the assessment instrument and a way to understand the results. The assessment instrument is modeled off of the Colorado Upper-division Electrostatics (CUE) diagnostic (Wilcox & Pollock, 2013). There are two parts to each question: a multiple-choice part and an explanation part. In the explanation part, the student is given a list of possible explanations and is asked to select all that could justify their original choice. This type of assessment provides information on the connections made by students. However, analyzing the results is not straightforward. We propose the use of adjacency matrices, as developed by Selinski, Rasmussen, Wawro, & Zandieh (2014), to analyze the connections that students demonstrate.

Key words: Linear Algebra, Assessment, Adjacency Matrices

As part of a larger study on developing materials for inquiry-oriented approach to Linear Algebra we created an assessment instrument to measure student understandings of major topics in linear algebra, including span, linear independence/dependence, invertibility, solutions to systems of linear equations, and transformations. In an attempt to gain a deeper understanding of student thinking on these topics without the use of a free-response assessment, we modeled our assessment off of the Colorado Upper-division Electrostatics (CUE) diagnostic (Wilcox & Pollock, 2013). In this type of an assessment, students are asked a standard multiple choice-question and then they are prompted to select all of the "because" choices that could justify their choice. An example is given in Figure 1.

1) The set of vectors $\left\{\begin{bmatrix}4\\5\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right\}$ is:

A) linearly independent
B) linearly dependent

Because … (select ALL that could justify your choice)

i) the set includes the vector $\begin{bmatrix}0\\0\end{bmatrix}$.
ii) the vector $\begin{bmatrix}0\\0\end{bmatrix}$ can be excluded from the set.
iii) the set has 3 vectors in $\mathbb{R}^2$.
iv) the **only** solution to $c_1\begin{bmatrix}4\\5\end{bmatrix} + c_2\begin{bmatrix}2\\3\end{bmatrix} + c_3\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ is the trivial solution (i.e., $c_1 = c_2 = c_3 = 0$).
v) the vectors span all of $\mathbb{R}^2$.
vi) $\begin{bmatrix}4 & 2\\5 & 3\end{bmatrix}$ is row equivalent to the identity matrix.

Figure 1: Assessment format

Interpreting the results of such questions, however, is complicated and prior work gives little insight into how to compile and make sense of the results. Moreover, in light of the fact that linear algebra is rich in connections, we were interested in measuring the nature of connections students made between topics in linear algebra. One possibility is to analyze student responses through the use of adjacency matrices.

Adjacency Matrices

Recent studies in linear algebra have used adjacency matrices as a way to analyze the connections students make between topics (e.g., Selinski et al., 2014). The use of adjacency matrixes starts by creating a list of codes that are topics and sub-topics in linear algebra. These codes make up the sides of the matrix and whenever a student makes a connection between two topics or subtopics, the matrix is marked in the corresponding cell of the row and column. From there, a quantitative measure can be placed on the types of connections made between topics by the students as well as preserving a connection to the qualitative data that created the matrix.

For the current study, we explored how adjacency matrices may be leveraged to analyze results from the assessment. The goal is to obtain a portrait of the connections students made between topics in linear algebra through the use of their assessment answers. However, before the assessment answers could be analyzed separate from the students, we needed to have knowledge of the ways in which the students were reading and understanding the assessment as well as how they were choosing their answers. Such information requires individual interviews.

Assessment Interviews

In the fall of 2014, 11 interviews were conducted using the current version of the linear algebra assessment at a large public university. Students were asked to explain their choices so as to gain an understanding of how they understood the problems and answers. The interviews were transcribed and coded independently by two researchers. The researchers discussed each code until consensus was reached. The codes were generated through a combination of open coding and a priori coding, with the a priori codes coming from previous studies done on student understanding in linear algebra (Selinski et al., 2014). The codes were used to create ordered pairs, much like what one may use to describe placement on a matrix, so as to create a basis for the adjacency matrix for each student. It is through these codes that an adjacency matrix was created. In the table below is an example of the coding done throughout the interviews. In this example, the student had just finished answering the multiple choice section of the problem shown in Figure 1 and had chosen the answer of linearly dependent. He is going through a couple of the explanation answers in the quotes below and chosing the ones he believes supports his answer choice.

Student	The set includes the vector [0, 0]. I think that's why.	(B1, B)
	The set has 3 vectors in R2. I didn't even think about that but that's why, right? Yeah. Pretty sure. [H: okay]	(B2, B)

The third column gives an example of how the coding was done and is read as B1 implies B2 and B2 implies B, where B1 stands for "the set includes the zero vector," B2 stands for "the set has more vectors than dimensions," and B stands for "set of vectors is linearly dependent." Much of the coding was done based on what the student had said previously as well as what was specifically stated in the utterance.

Conclusions

Adjacency matrices have provided both a quantitative and qualitative way of looking at student understanding in linear algebra. Through the use of interviews, for example, prior work provided a detailed explanation of the depth and types of connections students made between topics in linear algebra. In the current study, there is an additional level of complexity as we are attempting to use the adjacency matrix to help analyze whether or not the assessment shows the types of connections we expect. Our analysis suggests the following questions to discuss: If the adjacency matrix shows different or additional connections made by the students than what the assessment captures, how might the adjacency matrix inform future versions of the assessment? The number of explanations we can provide students is limited so how do we determine the optimal number that captures most of the possible explanations? What other possible frameworks could be useful in analyzing the assessment?

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A Proposed Framework for Tracking Professional Development Through GTA's

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There are several different models of graduate teaching assistant programs in mathematics departments across the nation (Ellis, 2015). One particular public university has recently reformatted their calculus program toward a peer-mentor model and this is the first year of implementation. In the peer-mentor model, there is a lead TA who serves as a support for the other TAs in the program. Because of this, the professional development in which the TAs are engaged is formally directed by both faculty and a peer. We are interested in discussing a framework known as the Vygotsky Space as a methodology for tracking the appropriation and sharing of pedagogical practices among those responsible for calculus instruction.

Key words: Vygotsky Space, professional development, Graduate teaching assistants

Across the nation, mathematics departments have begun to change the way they structure the teaching and support in their calculus sequence. Several different models that utilize graduate student mathematics teaching assistants have been identified, including the peer-mentor model, the apprenticeship model, and the coordinated-innovation model (Ellis, 2015). Of particular interest to us is the peer-mentor model in which, in addition to faculty support, there is a lead TA who observes and provides additional support to the other TAs. However, there has been little research into how the professional development in which the TAs are engaged is appropriated and shared among those responsible for instruction in the Calculus sequence. More specifically, how does the instructional practice of the TAs shift and what might provide support for those shifts, such as pressing for justification and providing multiple solutions to problems? What role does the lead TA play as both a learner and as one who provides support for the changes? This poster will report on a promising framework for understanding the ways in which the practices are appropriated and transformed by the graduate teaching assistants in a peer-mentor model.

The Setting

The 2015-16 academic year is the first year the peer-mentor model has been implemented at the large public southwestern university under investigation. In addition to a restructuring of the graduate student TA model, the breakout sections for the first two semesters of college calculus were restructured to include both an active learning problem-solving section, as well as a more traditional homework section. So as to support the TAs in leading more student-centered learning sessions, the TAs engaged in professional development for nearly two full days before the semester began and two half days during the first semester. Additionally, the TAs met weekly with the coordinator of their calculus course to discuss the past week and the week to come. Finally, the lead TA observed the teaching of their fellow TAs three times throughout the semester and gave each one of them feedback. Figure 1 provides a diagram of the various meetings throughout the first semester. The blue represents the weekly meetings, the yellow represents the professional development, and the green circles represent the observations conducted by the lead TA.

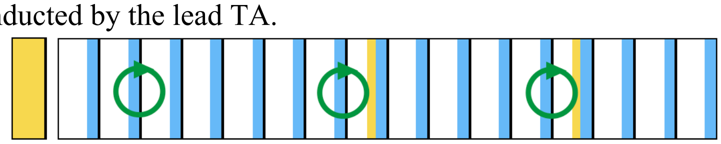

Figure 1: Diagram of Meetings

Thus there are formal opportunities for support in formal faculty-led professional development and informal opportunities in the debriefing meetings the lead TAs have with the other TAs after observing their classes and weekly meetings with the faculty coordinators. Through the various meetings, and through the classroom observations, there are opportunities for a researcher to observe the various ways in which the instructional practices are appropriated, transformed, and utilized by each of the TAs.

Tracking the Professional Development

We conjecture that the Vygotksy Space may be a useful framework for tracking the appropriation of the professional development (both formal and informal). The framework was meant to explain development in general (Harré, 1983) but it is also believed it could be used to investigate the ways in which the individual creates their own psychological world under particular "conversational forms and strategies from that discourse" (p. 245, Harré et al., as cited in McVee, Gavelek, & Dunsmore, 2007). In this framework, Harré (1983) identified two dimensions: the individual-social (collective) and the public-private. The superposition of these dimensions creates a two-dimensional space in which to describe the development of an individual over time as they move through the four quadrants.

For instance, the movement from the first quadrant to the second is known as appropriation. In this, the person is taking up the concepts that have been introduced and used in the public-social setting of the first quadrant. From there, the person moves towards the third quadrant, which is known as transformation. In this, the individual is taking the concept he or she has appropriated and is modifying it to fit their needs. When moving from the third quadrant to the fourth, the person is engaged in publication, which is when that person has made their meanings and strategies public for others to comment on. Finally, the movement from the fourth quadrant to the first is known as conventionalization. In this, the "individuals' public manifestations of thinking (i.e., their actions and their ideas) are incorporated as part of the community of discourse in which they participate" (Gavelek & Raphael, 1996, p. 188).

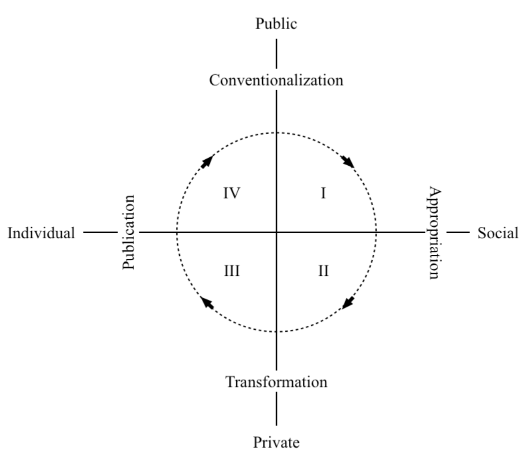

Figure 2: Vygotsky Space

We will have preliminary analysis and judgment of the suitability of this framework. The data used for this preliminary analysis will be from field-notes taken during classroom observations and weekly meetings as well as recordings of the debriefing conducted by the lead TA after his third observation during the fall semester. The poster session will provide a space to discuss this methodology.

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Mathematicians' grading of proofs with gaps

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In this study, we presented nine mathematics professors with three proofs containing gaps and asked the professors to assign the proofs a grade in the context of a transition-to-proof course. We found that the participants frequently deducted points from proofs that were correct and assigned grades based on their perceptions of how well students understood the proofs. The professors also indicated that they expected lecture proofs in the transition-to-proof course to have the same rigor as those demanded of students, but lecture proofs could be less rigorous than the rigor demanded of students in advanced mathematics courses. This presentation will focus on participants' rationales for these beliefs.

Key words: Mathematical Proofs, Assessment of Proofs, Transition to Proofs, Gaps in Proofs

In the United States, many mathematics majors are required to take a transition-to-proof course prior to taking proof-oriented courses such as real analysis and abstract algebra. A central goal of the transition-to-proof course is to help mathematics majors master the mechanics of proving so that they can produce acceptable proofs in their future advanced courses. There are, of course, a variety of strategies that a mathematics professor may use to achieve these goals, including being explicit about what types of inferences are valid (e.g., Alcock, 2010) and modeling good proving behavior (e.g., Fukawa-Connelly, 2012). There is currently a modest but growing body of research on how mathematics professors introduce students to proof-oriented mathematics in their lectures and their motivations for doing so (e.g., Alcock, 2010; Hemmi, 2006; Lai & Weber, 2014; Moore, 1994; Nardi, 2008; Weber, 2012). Recently, however, Moore (2014, submitted) identified an important facet of teaching that has received little attention from mathematics education researchers: professors' grading.

Moore (2014, submitted) found that mathematics professors viewed their grading, including both the marks they assigned and the commentary they provided, as essential parts of their teaching. As Moore observed, this raises important research questions. What student-written proofs in a transition-to-proof course constitute an acceptable product? What criteria do professors use for assigning grades? Do professors use the same criteria and assign similar grades to the same proofs? Or is there variance in the criteria that they use and the marks they assign? The purpose of this contributed paper is to further explore these questions. In particular, we presented nine mathematicians with proofs that contained gaps and asked them to grade these proofs. We were interested in how mathematicians would evaluate these gaps in their grading.

Related Literature

Mathematicians grading

As we noted in the introduction, there is little research on how mathematicians assign grades to students' proofs in a transition-to-proof course (or in any other course). Here we summarize the main findings from Moore's (2014, submitted) exploratory study on this topic. Moore asked four mathematics professors to assign grades to seven authentic student proofs with the aim of investigating the consistency, or lack thereof, in the marks that professors assigned. The main findings from Moore's study were that there was substantial variance in the scores they assigned to some proofs that could not be attributed to performance error (i.e., a participant overlooking a flaw in the proof). These disparities were based in part on the seriousness of the errors that stemmed from disagreement over whether an error was due to a mere oversight on the part of the student (which would receive only a small deduction in score or no deduction at all) or a significant misconception held by the student (which would receive a larger deduction). The professors all remarked that grading was an important part of their pedagogical practice.

Moore (submitted) called for expanding research in three directions: (i) conducting studies with more mathematicians to gain confidence in the generality of the findings, (ii) exploring how mathematics professors score different types of proofs, and (iii) looking into more depth about what criteria professors use to grade proofs. In this contributed report, we follow the recommendations of Moore. We presented nine mathematicians with a different type of proof grading tasks – looking at proofs with gaps – that allowed us to both replicate Moore's central findings and to explore them in more depth.

Proofs contain gaps in mathematicians' practice

A proof is sometimes defined as a deductive argument where each new statement in a proof is a permissible assumption (e.g., an axiom, a definition, a previously proven statement) or a necessary logical consequence of previous statements. In mathematical practice, it is commonplace for proofs to contain have *gaps* (Fallis, 2003). That is, the proof may not explicitly state exactly how a new assumption follows from previous assumptions, instead leaving this task up to the reader (Weber & Alcock, 2005). In many cases, this is because the author of the proof believed that the gap could easily be filled in by a knowledgeable reader. However, in other cases, the gap might be quite large and require the construction of a non-trivial sub-proof (Fallis, 2003; Selden & Selden, 2003). Mathematicians and philosophers have argued that gaps are both necessary and desirable in mathematical practice. Proofs would be impossibly long if every logical detail were included (Davis & Hersh, 1981) and supplying excessive logical detail would mask the main methods and ideas of the proof, which is a primary reason why mathematicians read published proofs in the first place (Rav, 1999; Thurston, 1994; Weber & Mejia-Ramos, 2011).

The pedagogical proofs that mathematics professors present to their students also contain gaps. Like published mathematical proofs, this is not only necessary for the sake of time and brevity, but also potentially beneficial, as students may learn mathematics from filling in some of the details of the proofs themselves (e.g., Alcock et al, 2015; Lai, Weber, & Mejia-Ramos, 2012). Prior research found a difference in how mathematics majors and mathematics professors regarded gaps in proofs presented in mathematics lectures: most mathematics majors believed that all the logical details should be specified in a well-written proof. In contrast, the majority of mathematics professors believed that even with a well-written proof, students would still be expected to justify some of the inferences within the proof (Weber & Mejia-Ramos, 2014). In this contributed report, we further explore the differences that mathematicians believe are needed with respect to rigor and gaps in lecture-based proofs and the proofs that students hand in for credit.

Mathematicians' evaluations of proofs with gaps

Prior research has explored how the existence of gaps affect the validity of proof in mathematicians' practice. In particular, mathematicians have not always agreed on the validity of specific proofs that contained gaps (e.g., Inglis, Mejia-Ramos, Alcock, & Weber, 2013; Weber, 2008; see also Inglis & Alcock, 2012). Mathematicians claimed that the permissibility of a gap was dependent upon the author of the proof, with some mathematicians claiming that they would be inclined to give an expert the benefit of the doubt when reading a proof with a large gap (Weber, 2008; Weber & Mejia-Ramos, 2013). In this paper, we illustrate how the mathematicians' estimation of the competence of the student plays a role in their grading when grading a students' proof with gaps.

Methods

Participants

Nine mathematicians in a mathematics department at a large state university in the United States agreed to participate in this study. These mathematicians represented a variety of mathematical subfields, including combinatorics, graph theory, number theory, partial differential equations, topology, and approximation theory. All were tenure-track or tenured at the time of the study with three professors representing each level of Assistant, Associate, and Full Professor. Six of the participants' had significant teaching and grading experience of over 10 years (three of which had over thirty years of experience) and the other three had four or more years of experience. We anonymized the data by referring to the first mathematician that we interviewed as M1, the second as M2, and so on.

Methods

In this study, we examined proofs of three theorems from number theory that might be proven in a transition-to-proof course. For each theorem, we generated two proofs. The *Gap Proof* is a proof that we designed to employ a logically correct line of reasoning but leaving some of the steps in the proof without a justification. The *Gapless Proof* is a modification of the Gap Proof such that the justifications for each of the steps were filled in. We refer to the three theorems as Theorem 1, Theorem 2, and Theorem 3. We refer to the two proofs of Theorem 1 as Gap Proof 1 and Gapless Proof 1, the two proofs of Theorem 2 as Gap Proof 2 and Gapless Proof 2, and the two proofs of Theorem 3 as Gap Proof 3 and Gapless Proof 3.

Procedure

Each participant met individually with the first two authors and was videotaped during a task-based interview. The interviewers made sure that each interviewee understood that the given proofs were from a transition-to-proof class. Each interview contained three phases. In the **Lecture Proof Evaluation** phase, the participant was told that a professor presented Gap Proof 1 in lecture, asked if they thought the proof was valid, and asked to comment on the pedagogical quality and appropriateness of the proof. This process was repeated for Gap Proof 2 and Gap Proof 3. In the **Student Proof Evaluation** phase, the participant was presented with Gap Proof 1 and told that a student submitted that proof for credit. They were asked to evaluate whether Gap Proof 1 was correct, assign a grade on a ten-point scale to that proof, and explain why they assigned that grade. They were then shown Gapless Proof 1 and asked to do the same thing. This process was repeated for the two proofs of Theorem 2 and Theorem 3. In the **Open-Ended Interview** phase, each participant was asked general questions about their pedagogical practice with respect to proof, with an emphasis on the grading of proof. One particular question was, "Do you expect the proofs that students hand in to have the same level of rigor as the proofs that the professors present in their lectures?"

Analysis

All interviews were transcribed. The research team engaged in thematic analysis as follows. First, each member of the research team individually read each transcript, flagging and commenting on passages that might be of theoretical interest. The research team met to discuss and compare their findings and identify themes that might be interesting to analyze in more detail. Next, each member of the research team individually read the transcripts again, searching for occasions in which a participant made a comment related to one of the themes in question and put this excerpt into a file related to that theme. The research team met again; for each theme, they used an open coding scheme in the style of Glaser and Strauss (1990) to create categories of participants' comments related to each theme.

Results

Summary of Evaluations

Although some participants were critical of the pedagogical quality of some of the Gap Proofs, they usually evaluated them to be correct. In the Lecture Proof Evaluation phase of the study, in all but one instance, the participants judged the Gap Proofs that they read to be correct. The one exception was when M8 could not decide if Gap Proof 1 was correct. In the Student Proof Evaluation phase of the study, none of the participants changed their evaluations of the correctness of the proof. Hence, there was only one instance (M8 evaluating Gap Proof 1) in which a participant evaluated the student proof as incorrect.

A summary of the grades that the professor assigned to the proofs that they evaluated in the Student Proof Evaluation phase is presented in Table 1. As can be seen from Table 1, there was substantial variance in the grades that the participants assigned from Gap Proof 1, with scores ranging from 6 through 10, thus replicating the findings of Moore (2014, submitted). There were 13 instances in which a Gap Proof received a score of less than 10; in 12 of those instances, the participant had judged the proof to be correct, with one score being as low as 6 out of 10. This illustrates how a correct proof is not guaranteed to receive full credit.

Table 1: Mathematicians' assessment of proofs with gaps authored by students

Proof	M1	M2	M3	M4	M5	M6	M7	M8	M9	Average
1 – Gap	6	9	8	9	9	7	10	6	8	8.00
1 – Gapless	9	10	10	10	10	9	10	9	9	9.56
2 – Gap	10	10	8	10	10	8	10	8	10	9.33
2 – Gapless	10	10	10	10	10	10	10	10	10	10
3 – Gap	10	9	10	10	10	10	10	10	8	9.67
3 – Gapless	10	10	10	10	10	10	10	10	10	10

Students' Proofs as a Model of Their Understanding

When discussing how they graded proofs, eight of the nine participants said that their grade was based on how well the student understood the proof that they handed in. For instance, M9 said[1],

M9: I think the way that I grade things is you're trying to see if the student understands and you believe he understands. Not so much that they have every period or word that you are looking for, but did they understand the concept … and if you could question them, then they could fill in the gaps, but they may have left them out.

[1] To increase the readability of the transcript, we lightly edited them by removing stutters, repeated words or phrases, and short fragments of text that did not carry meaning. We indicate where we have done so with an ellipsis (…). At no point did we add or alter words that participants said or change the meaning of the participants' utterances.

Similarly, in describing what he was looking for when he graded, M6 said, "I think it is… to see if they really understand what is going on. Careful proof and the steps, probably show that they understand the ideas and they can think – they can relate to one sentence to the other sentence and see the flow and how things put together".

The way that the participants modeled students' understanding influenced their grading in a number of ways. First, five participants indicated that if there was a gap in the proof but the inference could only be produced if the student had an adequate understanding of why it was true, they would not take off for it. For instance, in explaining why he didn't require a justification for the assertion that *n* is even in Gap Proof 1, M5 remarked:

M5: I guess I would say that if somebody was going to write a textbook and this was going to be a sample in the textbook, I would want them to say a little bit more. But anyone who says what is on the paper here, wouldn't say it without understanding it. So jumping from the fact that *n* squared is even to *n* is even, that does not bother me at all.

Similarly, five participants remarked that they would require more justification than was necessary for a proof to be correct to ensure that the student fully understood the proof and did not just copy or recall a proof that he or she saw elsewhere. For instance, consider the exchange between M4 and the interviewer when evaluating Gap Proof 1.

I: Do you think the proof is correct?
M4: Yeah.
I: If you had to, as we usually do as instructors, we grade things on some type of scale. Say you graded this on a zero to ten scale, what grade would you give this student?
M4: Well I would probably have to take off a little for not saying those [referring to an unjustified statement about why numbers are composite]. These two are obviously composite, composite numbers.
I: When you take off, a little off, what do you mean? How much is a little?
M4: Probably nine out of ten.
I: Would you make any comments on the students' papers?
M4: I would say why? Or explain why? And then I would think, did the student copy this somewhere? [The interviewer and M4 both laugh] Because it is sort of written in a mature style, leaving things out which are yes indeed. As I said before, compact proof, nicely done. Maybe too nicely.

Further, three participants indicated that they would take into consideration the past performance of the student when deciding whether to penalize a gap in the proof, with better students being more likely to get the benefit of the doubt. In discussing his strong students, M7 described:

M7: It is okay to leave some steps out because I might get to a point where I respect their mathematical minds enough so that I give them the benefit of the doubt that they understood what was going on without writing it down. So that is actually nice, a lot of people do that – in our own research we do that.

Level of Rigor in Lectures and Student Proofs

We asked the participants whether students' proofs should contain the same level of rigor as professor's proofs in lectures. Participants' responses frequently indicated that it depended on the course. For transition-to-proof courses, the answers were mixed, but on average was that the level of rigor in lecture proofs and student-generated proofs should be about the same. (Three participants remarked that they would expect *less* rigor for a student proof than a lecture proof based on their assumptions of what students would be capable at that point of their mathematical development). However, for upper-level courses like real analysis, six participants would expect students' proofs to contain *more* rigor. We summarize the participants' responses in Table 2.

Table 2: Rigor demanded in proofs that students submit for credit

Context	Less than lecture proof	Same as lecture proof	More than lecture proof
Transition-to-proof	M2, M3, M5	M1, M6, M7, M9	M4, M8
Advanced course	M5	M6, M9	M1, M2, M3, M4, M7, M8

M1 explained his justifications as follows. In the context of a transition-to-proof course, M1 exclaimed, "Yeah, at this level, especially if you are going to take off points, then I think the instructor owes it to the student to, you know, to write those things down carefully". For more advanced courses, M1 said:

M1: I think that is less important as the higher up you go and there is a difference in the venue of a lecture presented in class where there is just a, you know, for example for a graduate class, there well may be details or things that are clear from context that you would not want to spend time writing or take up class time for. You would like to focus on the mathematics itself.

Discussion

In this study, we replicated several of Moore's (2014, submitted) preliminary findings about proof grading. We found substantial variance in mathematicians' grading of Gap Proof 1 and we found that grading involved the professor's building models of how well they thought their students understood the proofs. Having these themes independently emerge in a study with a larger sample than Moore provides more confidence that his results would generalize to a larger number of mathematicians. Still, our sample size of mathematicians is rather small. As conducting qualitative analysis of interviews with a substantially larger number of mathematicians is impractical, we suggest further research might make use of the recent survey methodology that has been used to probe mathematicians' beliefs (e.g., Mejia-Ramos & Weber, 2014).

Our analysis builds on Moore's work by delving deeper into how and why participants use their models of students' understanding in assigning grades. We found that some participants would penalize students for gaps that would ordinarily be permissible, especially in proofs generated for a transition-to-proof courses, because they could not assume that students knew how to bridge these gaps. This helps explain why professors would assign scores of less than 10 to proofs that they judged to be correct. On the other hand, some participants would not penalize students for leaving a gap in the proof if they felt the student had a full understanding of the proof. Further, some gaps would be permissible for students who had previously earned the mathematical respect of the professor. Staples, Bartlo, and Thanheiser (2012) claimed that classroom proofs and mathematical proofs satisfy different needs and should be judged by different standards; in particular, in K-12 classrooms, the request for a proof is often used by a teacher as a lens to evaluate their understanding, something not ordinarily done in mathematicians' practice. The results of this study suggest that Staples, Bartlo, and Thanheiser's insight is relevant for transition-to-proof courses as well.

We also found that most participants believed an instructor of a transition-to-proof course should not expect students' proofs to be more rigorous than the proofs that he or she presents in lecture. However, for more advanced courses, most participants felt that the lecturer could be less rigorous in his or her lecture proofs than what he would demand of his or her students. This is significant for two reasons. First, this corroborates a finding reported in Lai and Weber (2014) that to mathematicians, proofs in advanced mathematics lectures have the purpose of communicating content while student-generated proofs are used for demonstrating students' capacities to write proofs. Second, previous research has shown that mathematics majors did not get this message (Weber, in press; Weber & Mejia-Ramos, 2014). Mathematics majors do not appear to read proofs as a tool to understand mathematical content or methods (c.f., Weber, in press) and they do not expect a good mathematical proof to have gaps (Weber & Mejia-Ramos, 2014). This has the context that they may ignore or misinterpret the most important ideas that a professor attempts to convey when delivering a lecture proof (Lew, Fukawa-Connelly, Mejia-Ramos, & Weber, in press). An implication of finding out how professors' standards of rigor in lectures and grading shifts from transition-to-proof courses to more advanced courses is that this information should be better conveyed to mathematics majors.

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Students' formalization of pre-packaged informal arguments

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We gave pairs of students enrolled in a graduate analysis class tasks in which they were provided with a video taped informal argument for why a result held and asked to produce a rigorous proof of this result. This provided a lens into students' formalization process and the various roles these informal arguments played in each pair's proving process. Comparing across several pairs of participants revealed 3 distinct roles informal arguments can play in proving, 1) solely as a starting point, 2) as a reference that can be continually returned to during the proving process, and 3) as a convincing argument that does not inform the proving process.

Key words: [proof, argument, formalization]

Introduction

Acceptable proofs must conform to norms that constrain which assumptions are appropriate starting points, which representations can be used and which inferences are allowed (Stylianides, 2007). These norms vary between mathematical and social contexts (Mamona-Downs, & Downs, 2010). However, regardless of the specific norms at play, it is important to note that the proof generation process need not be constrained by the norms that constrain proof (Boero, 1999). Students may generate an informal argument that provides justification for a particular mathematical result but does not conform to the norms of proof (Harel & Sowder, 1998). A number of researchers have touted the importance of using the formalization of such arguments as a mechanism for proof generation (Gibson, 1998; Raman, 2003; Weber & Alcock, 2004).

Researchers interested in formalization have primarily examined this phenomenon in contexts where it is not specifically prompted for (e.g., Weber & Alcock, 2004, Zazkis, Weber, & Mejia-Ramos, 2015). The major benefit of this approach is that the data are naturally occurring instances of formalization. However, the generation of informal arguments is not particularly common and the successful formalization of these arguments is even rarer. This means that researchers interested in formalization must often start with particularly large data sets that take years to generate in order to explore formalization (e.g., Pedemonte, 2007; Zazkis et al., 2015). Additionally, researchers have little direct control over which specific arguments the students that attempt to formalize start with.

An important under-explored avenue for studying formalization is to use researcher selected pre-packaged informal arguments as part of research tasks (e.g., Zazkis & Villanueva, in press). This gives the researcher the ability to ensure that a much greater percentage of their data is relevant to formalization phenomena and thus eliminates the need for starting with particularly large data sets. Zazkis et al. (in press) presented students with triples which consisted of one informal argument and two correct proofs of a mathematical result, only one of which was a formalization of the informal argument. They observed that a majority of the mathematics majors in their sample struggled to identify which of the two proofs in each triple was the distractor and which was the formalization of the informal argument. They observed that the students' difficulties with making correct assessments could be explained by their focus on a subset of the connections between proofs and informal arguments. This was interpreted as an indication that students had underdeveloped conceptions of what it means for an informal argument to be the basis of a proof.

Methods

In this work we build on Zazkis et al.'s (in press) findings by taking a different approach. We begin by presenting pairs of mathematics students' informal analysis arguments. We then instruct the pair to work cooperatively to turn the given informal argument into a proof. The advantage of this approach is that the entirety of our data set is relevant to formalization. From just 4 pairs of students we are able to generate 8 hours of data relevant to formalization. For comparison Zazkis et al. (2015) started with a data set of 73 students working on seven 15 minute tasks. This generated over 120 hours of data. But only about 8 hours of these data were relevant to formalization.

Participants were recruited from a master's level analysis course. However, some of these participants were undergraduates enrolled in this graduate course and the tasks were accessible at an undergraduate level. The interviews lasted from one to two hours and were conducted in pairs to encourage participants to verbalize their thinking.

The subject area of analysis was chosen because it often has graphical informal arguments that are accessible to students, but the formalization often takes a different form (i.e. epsilon-delta proofs). Two of the proofs used were borrowed from (Zazkis et al., in press). Two additional tasks were identified through informal conversations with faculty who had recently taught analysis.

Pairs of participants were given a statement to prove and then viewed a video clip containing an informal argument that justified why the statement is true. Participants were allowed to watch the video as many times as they needed to until they were confident that they understood the informal argument. Then they were asked to construct a formal proof based on the informal argument. Data was recorded using two video cameras: one recording the participants' gestures and facial expressions and one recording their written work.

Results

Below we discuss three of the four pairs of participants who worked to prove that $\int_{-a}^{a} \sin^3(x)dx = 0$, using the informal graphical argument that the area on each side of the *y*-axis will add to zero. We illustrate that each pair's approach to formalization, specifically the formalization of the oddness property, differ tremendously in the following vignettes. We use these differences as a launching point for discussing differences in the role informal arguments in general may play in proving.

<u>Pair 1 Vignette:</u> After watching the video, Mark immediately broke up the integral into two pieces, and noted that the two pieces represent the two corresponding areas in the video. They begin to try to manipulate the two integrals by flipping the bounds. They do not mention the oddness property at this point. Chris suggests, "So, as far as creating a rigorous proof, short of integrating sine cubed…" Mark adds, "Yeah, you could just do sine times sine squared and change sine squared to one minus cosine squared, distribute your sine then use u substitution to deal with the sine cosine squared part. That would be one thing."

The pair begin with the integral $\int_{-a}^{a} \sin^3(x)dx = \int_{-a}^{a} (1-\cos^2(x))\sin(x)dx$, and use the u-substitution $u = \cos(x)$ to correctly evaluate the integral. When asked how their solution relates to the informal argument, Chris said, "I would say this was not inspired by the video. If we hadn't watched the video, I think we would have come to the same conclusion in the same manner." Mark points out that their original idea to split the integral into two pieces was inspired by the video.

The interviewer asks them to pursue the two pieces idea in more detail. They show that $\int_0^a \sin^3(x)\,dx = -\cos(a)+1+\frac{\cos^3 a}{3}-1$ using the same calculation as before, and then show that the integral from $-a$ to zero was equal to the additive inverse of this expression by again repeating the method.

Throughout their work on the sin cubed task pair 1 seems to have no conception of the connection between the geometric and algebraic facets of the oddness property. Although they did indicate knowledge of the fact that $\sin(-x) = -\sin(x)$, they never used it in their argument, and it seemed to be a memorized fact unrelated to the graph. They were able to extract the fact the integral from zero to a was the additive inverse of the integral from –a to zero, however, the symmetry based reasons for this were not a feature of their discussion. Thus their original proof had nothing to do with the informal argument and their second proof was more inspired by their first proof than the informal argument itself.

Pair 2 Vignette: Heather immediately split up the integral and discussed the impact of the oddness property.

> Heather: I think If we are allowed to know this, split it up into the integral from -*a* to zero and zero to *a* is an idea.
> Rhonda: And maybe use that it's an odd function. Is it odd?
> Heather: Sine cubed odd. Wait, what do you mean by odd?
> Rhonda: They say it's even or odd, I think whenever you put in a negative, it's the same as negative of the regular one. So, like (sketches a graph of x^2) that… I feel like this one is even… because you get the same value.
> Heather: Then when it goes the other way it's odd.

Unlike the first pair discussed earlier, pair two worked to flesh out the graphical meaning of oddness. They flipped the bounds on one of the integrals, then Rhonda said,

> Rhonda: So, I was wondering. Now, since it's all negative here, we know that's equal to negative times what it is with just regular numbers, right? Positive numbers.
> Heather: Wait, what?
> Interviewer: What does it mean for something to be odd for you?
> Rhonda: I think it means whenever you put a negative number into sine cubed, it's the same of negative of that positive number into sine cubed. It's just the negative of the value of the positive (motions with fingers as though picking corresponding points on an odd function).
> Heather: So, if we are given a *k* in whatever, sine cubed of *k* is equal to negative sine cubed of negative *k*, right? Is that what you're going for? And since this is true... We know that this is the sum… the limit of sum of stuff… limit of the sum of stuff of the sine cubed *k* of things. Ok, so how would we do that formally?

Heather then began to talk about Riemann sums and chose corresponding rectangles on either side of zero. Heather explained, "This is nice 'cause it lets us work with the sine cubed value, whereas this (points to integrals) we can't really work with it as much, because it's already inside." Heather and Rhonda were able to talk through this proof construction, though they didn't flesh out the specific details required for a rigorous proof. In working out this proof sketch they commonly went back and forth between working with the graph of sin cubed and working with notation. We interpret this back and forth between the graph of sin cubed and analytic notation to be a back and forth between the informal argument and their proof sketch. Thus unlike pair 1, who almost entirely ignored the informal argument, the informal argument played a continuous role in pair 2's work. It is also worth noting that pair

4 created a similar argument based on Riemann sums, which is not discussed in this paper.

Pair 3 Vignette: Pair 3 began by discussing whether they should directly integrate (like pair 1) or whether they should appeal to the fact that sine cubed is an odd function (like pair 2). Gautam suggested that sine squared can be converted into one minus cosine squared. Cody said, “Yeah, I mean that would show it, but I think it's too much work. We could appeal to the fact that… So, I mean, clearly it's equal to this (splits up integral) and then we appeal to the fact that it is an odd function right here (points to the integrand in the first integral).” They began with the substitution $x = -t$. Gautam and Cody were able to correctly change the bounds and flip the bounds after some discussion. Then they appealed to the oddness property, saying “because *f* is odd, *f* of minus *t* is minus *f* of *t*.” They realized that they have missed a negative sign in their substitution, but they were able to find it and write a fully formalized proof.

After finishing this proof, Cody reflected on their proving process: “The oddness definitely made me think of, well, I guess having experience with proofs made me think of $f(x) = -f(-x)$, which is the step you need, along with, so you kind of use the symmetry of the function and the symmetry of the integral. Both of those are a very similar identity because they both involve a flip and a negative sign in some sense. So, you can flip the bounds which introduces a negative sign, and there's another negative sign introduced from substituting, uh, making this substitution.”

Pair 3 knew that they needed to use the oddness property in the informal argument. They were able to successfully translate the fact that sine cubed was odd into analytic notation. However, unlike pair 2, they didn’t draw any more pictures or give any evidence that they were continuing to think geometrically after they initially extracted the use of oddness and splitting up the integral from the informal argument. So the informal argument provided pair three with a viable starting point, after which they fleshed out the details without further reference to it. In Contrast, pair 2 continually went back to the informal argument during their proving process and pair 1 did not utilize the informal argument during proving.

Discussion

In this study we observed three different patterns in students’ usage of pre-packaged informal arguments during proving. It is unknown whether the existence of these patterns remains intact if the mathematical content is changed, or if students tackle the tasks individually. However, since other researchers have noted subject matter influences in other types of formalization studies (e.g., Pedemonte, 2008) we anticipate that these influences on the role of pre-packaged informal arguments are not insignificant. Additionally, how students’ usage of pre-packaged informal arguments differ from their usage of self produced informal arguments is an important question for future research.

It bears mentioning that we do not have evidence that any of the three usage patterns of informal arguments provides an advantage over others in relation to producing proofs. These utilizations may simply be different routes to achieve the same goal. We also believe that students are conscious of the role informal arguments play in their own proving. This belief is supported by our participants’ reflections on the role informal arguments played in their proving. For example, toward the end of the interview Heather from pair 2 commented on the usefulness of returning to informal arguments:

> Heather: I think sometimes it makes a difference. I think, like, maybe not from this, but sometimes when things are explained informally, it doesn't make sense. So, you play around with it, and after you've played around with it for awhile, then the explanation, if you go back to it, it makes more sense… So, I don't know if, like, watching it at the beginning doesn't help as much as watching it later.

Similarly, Chris from pair 1, after completing his initial proof for task 1, commented about his lack of usage of informal arguments during proving:

> Chris: It's funny, 'cause actually in watching the video. I was like, I don't think that will help [with writing a proof].

Chris and Mark both made similar comments after other tasks in this study. This is an indication that they realized the minimal role that the pre-packaged informal arguments played in their proving process. Further research is needed to explore whether students that treat pre-packaged informal arguments in this way treat their own self-generated informal arguments similarly, or perhaps avoid generating informal arguments altogether.

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Questions for the audience:

1) Is formalizing informal arguments a skill that all mathematics majors should acquire during their undergraduate work, or simply an attribute of some individuals' proving that is not necessarily beneficial for all majors?

2) What are the links between students' ability to generate and use their own informal arguments and their experience with formalizing pre-packaged informal arguments?

Opportunity to learn solving context-based tasks provided by business calculus textbooks: An exploratory study

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The purpose of this study was to investigate the opportunities to learn how to solve realistic context-based problems that undergraduate business calculus textbooks in the United States offer to business and/or economics students. To do this, we selected and analyzed examples and practice problems from six different textbooks that are widely used in the teaching of business calculus nationwide. There are three major findings from this study: (1) a majority of the tasks in all the textbooks uses a camouflage context, (2) all the tasks in all the textbooks have matching information, and (3) only three textbooks had reflection tasks. The findings of this study suggest that business calculus textbooks do not offer students rich and sufficient opportunities to learn how to solve realistic problems in a business and/or economic context.

Key words: opportunity to learn, textbook research, textbook analysis, business calculus, context-based tasks

Introduction

The concept of opportunity to learn (OTL) as it relates to mathematics instruction originated in the early 1960s. Carroll (1963) defined OTL as the time allowed for learning a particular topic. This study uses Husen's (1967) definition of OTL which, according to Floden (2002), is also the most common definition of OTL used in the mathematics education research literature. According to Husen, OTL refers to "whether or not … students have had the opportunity to study a particular topic or learn how to solve a particular type of problem" (pp. 162-163). Mathematics textbooks are one such opportunity from which students can learn how to solve certain types of problems. The role of mathematics textbooks as an opportunity to learn mathematics is well documented in the research literature on the learning of pre-university mathematics. Referring to mathematics textbooks, Reys, Reys, and Chavez (2004) argued "that the choice of textbooks often determines what teachers will teach, how they will teach it, and how their students will learn" (p. 61).

The existing research on pre-university mathematics focuses on students' opportunities to learn mathematical topics such as linear functions and trigonometry (Wijaya, van den Heuvel-Panhuzen, & Doorman, 2015), addition and subtraction of fractions (Alajmi, 2012; Charalambous et al., 2010), probability (Jones & Tarr, 2007), statistics (Pickle, 2012), reasoning and proof (Stylianides, 2009; Thompson et al., 2012), proportional reasoning (Dole & Shield, 2008), and deductive reasoning (Stacey & Vincent, 2009). Research on students' opportunities offered by mathematics textbooks to learn other mathematics topics, such as optimization, especially at the upper secondary and undergraduate level is lacking.

To our knowledge, only one study (Mesa, Suh, Blake, & Whittemore, 2012) examined students' opportunity to learn about the family of exponential functions, the family of logarithmic functions, and transformations of graphs provided by mathematics textbooks at the post-secondary level. The scarcity of research that examines the opportunity to learn how to solve context problems provided by college mathematics textbooks, is a motivation for this study. In particular, this study sought to examine the opportunity to learn how to solve realistic context problems (optimization problems) that occur in business and/or economics provided by undergraduate business calculus textbooks that are used nationwide. Our study sought to answer the following research questions:

1. *What type(s) of context (relevant and essential, camouflage, no context) is characteristic of tasks that are found in undergraduate business calculus textbooks?*
2. *What type(s) of information (matching, missing, superfluous) is characteristic of tasks that are found in undergraduate business calculus textbooks?*
3. *What type(s) of cognitive demand (reproduction, connection, reflection) is characteristic of tasks that are found in undergraduate business calculus textbooks?*

In this study, we viewed tasks as optimization examples and practice problems given at the end of each optimization section in each of the textbooks that we selected for the study. A description of the types of contexts, types of information, and types of cognitive demands is given in the methods section. Having rich opportunities to learn how to solve realistic optimization problems that are situated in a business or economic context is essential for students in several fields of study such as marketing, supply chain management, finance, and economics. According to Gordon (2008), over 300,000 students enroll in business calculus each year in the Unites States. We are not aware of any existing work where this particular context (business or economic) and type of problem (optimization) has been studied, which is the motivation for our study.

Relevant literature

The term, context, has been defined in several ways by researchers in mathematics education. Our view on the meaning of context is consistent with that given by White and Mitchelmore (1996). These researchers posited that "in calculus, the context of an application problem may be a realistic or artificial "real-world" situation, or it may be an abstract, mathematical context at a lower level of abstraction than the calculus concept that is to be applied" (p. 81). White and Mitchelmore's understanding of the term context is consistent with that of other researchers (e.g., Gravemeijer & Doorman, 1999; van den Heuvel-Panhuizen, 2005). According to Wijaya et al. (2015), mathematical tasks could have a realistic context, a camouflage context, or they could be bare (only mathematical symbols). Alajmi (2012) refers to bare mathematical tasks as tasks that are situated in a "purely mathematics context" (p. 243). Tasks with a camouflage context "are merely dressed up bare problems, which do not require modeling because the mathematical operations needed to solve the task are obvious" (Wijaya et al., 2015, p. 45). A realistic context is also referred to as a relevant and essential context in the research literature (e.g., de Lange, 1995; van den Heuvel-Panhuizen, 2005).

Several researchers (e.g., Maass, 2007; Maass, 2010; Wijaya et al., 2015) have identified three types of information that could be in a mathematical task: matching, missing, and superfluous. A mathematical problem with matching information is one in which all the information required to solve the problem is given in the problem statement. A mathematical problem has missing information if some of the information needed to solve the problem is not immediately available to the solver, that is, the solver has to deduce this information from the problem statement. A mathematical problem with superfluous information is one in which the problem statement not only contains the necessary information needed to solve the task but it also contains other extraneous or irrelevant information that may not be helpful in solving the given problem. Wijaya et al. (2015) argued that:

> Providing more or less information than needed for solving a context-based task is a way to encourage students to consider the context used in the task and not just take numbers out of the context and process them mathematically in an automatic way. (p. 45)

Maass (2010) recommended that students should be given opportunities to deal with these three different types of information.

A related line of research (e.g., Charalambos et al., 2010; Kolovou et al., 2009; Mesa et al., 2012; Wijaya et al., 2015) has investigated the types of cognitive demands in tasks that are presented in mathematics textbooks. The types of cognitive demands are: reproduction, connection, and reflection. These types of cognitive demands are similar to the levels of cognitive demands discussed by Stein, Grover, & Henningsen (1996). Reproduction tasks are routine problems that require the lowest level of cognitive demand to solve. These problems can be easily solved using memorized mathematical algorithms. Connection tasks are non-routine in nature and may require the solver to represent concepts in multiple representations: algebraically, numerically, graphically, and verbally. Mesa et al. (2012) analyzed, among other things, the cognitive demands of examples as well as the representations of these examples given in 10 college algebra textbooks. Five of these textbooks are used at community colleges, three textbooks are used at four-year institutions, and the other two textbooks are used at both community colleges and at four-year institutions. Mesa and colleagues found that "textbooks, independent of the type of institution in which they are used, present examples that have low cognitive demands, expect single numeric answers, emphasize symbolic and numerical representations, and give very few strategies for verifying correctness of the solutions" (p. 76). Reflection tasks require the highest level of cognitive demand to solve. These tasks "include complex problem situations in which it is not obvious in advance what mathematical procedures have to be carried out" (Wijaya et al., 2015, p. 46).

Theoretical framework

With a focus on the role of context in mathematical tasks found in business calculus textbooks, this study draws on the theory of realistic mathematics education (RME) which is both a theory of teaching and learning in mathematics education that originated in the Netherlands in the early 1970s. As a theory of learning, RME emphasizes that students should be asked to solve realistic contextual problems that are not only realistic in the sense of being connected to a real-world context but also that the context of these problems should be experientially real to the students. That is, students should be asked to solve "problem situations which they can imagine" (van den Heuvel-Panhuizen, 2000, p. 4). The economic or business context as it relates to optimization tasks may be experientially real for some students taking business calculus. This is especially true for students who take business calculus after having taken high school or college economics classes. In addition to the role of context, optimization tasks in business calculus textbooks may vary in terms of types of information and types of cognitive demands. These various types of information and cognitive demands are explained in the analytical framework presented in the next section.

Methods

The study followed a qualitative research design. Data for the study consisted of optimization examples and practice problems from undergraduate business calculus textbooks that are widely used in the teaching of business calculus in the Unites States.

Data collection and setting

To answer the research questions, we analyzed a total of 195 optimization examples and practice problems selected from the latest editions of six undergraduate mathematics textbooks that are widely used in the instruction of business calculus at large universities in the United States. Table 1 shows a list of the textbooks that were selected for the study. The textbooks were selected through google search using key words such as "business calculus textbook," "business mathematics textbook," "applied calculus textbooks," etc.

Table 1

Analyzed Textbooks

Textbook Name	Author (s)	Textbook Abbreviation	Section(s) analyzed	Textbook Publisher
Applied Calculus for the Managerial, Life, and Social Sciences (9th ed)	Tan, S. T. (2013)	TBK 1	Optimization I Optimization II (chapter 4)	Brooks/Cole (Cengage Learning)
Introductory Mathematical Analysis for Business, Economics, and the Life and Social Sciences (13th ed)	Haeussler, E. F., Paul, R. S., & Wood, R. J. (2011)	TBK 2	Applied Maxima and Minima (chapter 13)	Pearson Education
Applied Calculus (6th ed)	Waner, S. & Costenoble, W. (2014)	TBK 3	Applications of Maxima and Minima (chapter 5)	Brooks/Cole (Cengage Learning)
Applied Calculus (5th ed)	Hughes-Hallet et al. (2013)	TBK 4	Profit, Cost, and Revenue Average Cost (chapter 4)	Wiley
Calculus and its Applications (11th ed)	Bittinger, M. L., Ellenbogen, D. J., & Surgent, S. J. (2015)	TBK 5	Maximum-Minimum Problems; Business, Economics, and General Applications (chapter 2)	Pearson Education
Calculus for Business, Economics, Life Sciences, and Social Sciences (13th ed)	Barnett, R. A., Ziegler, M. R., & Byleen, K. E. (2015)	TBK 6	Optimization (chapter 4)	Pearson Education

Table 2

Analytical Framework

Task Characteristic	Sub-category	Explanation
Type of context	No context	-Refers to only mathematical objects, symbols, or structures.
	Camouflage context	-Experiences from everyday life or common sense reasoning are not needed. -The mathematical operations needed to solve the problems are already obvious. -The solution can be found by combining all numbers given in the text.
	Relevant and essential context	-Common sense reasoning within the context is needed to understand and solve the problem. -The mathematical operation is not explicitly given. -Mathematical modeling is needed.
Type of information	Matching	-The task contains exactly the information needed to find the solution.
	Missing	-The task contains less information than needed so students need to find the missing information.
	Superfluous	-The task contains more information than needed so students need to select information.
Type of cognitive demand	Reproduction	-Reproducing representations, definitions, or facts. -Interpreting simple and familiar representations. -Memorization or performing explicit routine computations/procedures.
	Connection	-Integrating and connecting across content, situations, or representations. -Non-routine problem solving. -Interpretation of problem situations and mathematical statements. -Engaging in simple mathematical reasoning.
	Reflection	-Reflecting on, and gaining insight into, mathematics. -Constructing original mathematical approaches. -Communicating complex arguments and complex reasoning.

Table 2: Analytical Framework reproduced from (Wijaya et al., 2015, p. 52)

Data analysis

The data (optimization examples and practice problems) were coded using the mathematics textbook analysis framework developed by Wijaya et al. (2015) shown in Table 2. In particular, there were three dimensions of analysis, namely, type of context, type of information, and type of cognitive demand. We coded a total of 195 tasks. We illustrate our coding with the following examples:

Acrosonic's total profit (in dollars) from manufacturing and selling x units of their model F loudspeaker systems is given by $P(x) = -0.02x^2 + 300x - 200{,}000 \quad (0 \le x \le 20{,}000)$. How many units of the loudspeaker system must Acrosonic produce to maximize its profit? (TBK 1, p. 301).

We coded this example as: (1) having a camouflage context because the operations needed to solve the problem are already obvious and the context can be ignored when solving this problem, (2) having matching information because it contains the exact amount of information needed to solve it, and (3) a reproduction task because the strategy required to solve it requires performing explicit routine procedures. In the same textbook, this example is also given:

Dixie Import-Export is the sole agent for the Excalibur 250-cc motorcycle. Management estimates that the demand for these motorcycles is 10,000 per year and that they will sell at a uniform rate throughout the year. The cost incurred in ordering each shipment of motorcycles is \$10,000, and the cost per year of storing each motorcycle is \$200. Dixie's management faces the following problem: Ordering too many motorcycles at one time ties up valuable storage space and increases the storage cost. On the other hand, placing orders too frequently increases the ordering costs. How large should each order be, and how often should orders be placed, to minimize ordering and storage cost? (TBK 1, p. 317).

We coded this task as: (1) having a relevant and essential context because reasoning within the context of the task is needed to understand and solve the problem, (2) having matching information because it contains the exact amount of information needed to solve it, and (3) a reflection task because the solver must construct original mathematical approaches e.g. the average inventory level of $x/2$ if x is the lot size. The results of the coding of all 195 tasks from the six textbooks are summarized in Table 3.

Results

There are three major findings from this study. First, a majority of the optimization tasks given in the business calculus textbooks reviewed in this study use a camouflage context. All of the textbooks except TBK 4 rarely had tasks with no context. Only TBK 5 has a significant number of tasks with a realistic (relevant and essential) context relative to the number of economic problems given in each of the textbooks we analyzed.

Table 3

Textbook Analysis Results

TA	NEP	Type of Context	Type of Information	Type of Cognitive Demand
TBK 1	24	No context: 0	Matching: 24	Reproduction: 19 (79%)
		Camouflage context: 19 (79%)	Missing: 0	Connection: 2 (8%)
		Relevant & essential context: 5 (21%)	Superfluous: 0	Reflection: 3 (13%)
TBK 2	29	No context: 0	Matching: 29	Reproduction: 28 (97%)
		Camouflage context: 26 (90%)	Missing: 0	Connection: 1 (3%)
		Relevant & essential context: 3 (10%)	Superfluous: 0	Reflection: 0
TBK 3	29	No context: 2 (7%)	Matching: 29	Reproduction: 25 (86%)
		Camouflage context: 26 (90%)	Missing: 0	Connection: 3 (10%)
		Relevant & essential	Superfluous: 0	Reflection: 1 (3%)

		context: 1 (3%)		
TBK 4	56	No context: 23 (41%)	Matching: 56	Reproduction: 43 (77%)
		Camouflage context: 31 (55%)	Missing: 0	Connection: 7 (13%)
		Relevant & essential context: 2 (4%)	Superfluous: 0	Reflection: 6 (11%)
TBK 5	32	No context: 8 (25%)	Matching: 32	Reproduction: 13 (41%)
		Camouflage context: 12 (38%)	Missing: 0	Connection: 19 (59%)
		Relevant & essential context: 12 (38%)	Superfluous: 0	Reflection: 0
TBK 6	25	No context: 0	Matching: 25	Reproduction: 12 (48%)
		Camouflage context: 21 (84%)	Missing: 0	Connection: 13 (52%)
		Relevant & essential context: 4 (16%)	Superfluous: 0	Reflection: 0

TA stands for textbook abbreviation. NEP stands for number of economic examples and practice problems in the section(s) of each textbook that we analyzed.

Second, all six textbooks have tasks which contain the exact amount of information students need to solve the tasks. As a result, students do not have to make sense of the context (if any) of the tasks in order to either deduce missing information or identify important information (in the case of superfluous information) that is necessary to solve the tasks from the problem statements. Third, only three textbooks (TBK 1, TBK 3, and TBK 4) had reflection tasks, that is, tasks with a higher cognitive demand. However, the number of such tasks was extraordinarily low, with only 13% (n=3) of the tasks in TBK 1, 3% (n=1) of the tasks in TBK 3, and 11% (n=6) of the tasks in TBK 4. Hence, the opportunity to learn from such tasks via textbooks is minimal. Reproduction tasks were common in all six textbooks.

Discussion and conclusion

The results of this study has some implications for different stakeholders, namely textbook authors, textbook selection committees, and instructors. Textbook authors need to include a much broader range of economic-based optimization examples and practice problems in terms of types of context, types of information, and types of cognitive demands to maximize the learning opportunities provided by their textbooks. Textbook selection committees need to select textbooks that contain a balance of optimization tasks in terms of types of context, types of information, and types of cognitive demands to avoid limiting students' opportunity to learn about optimization problems in an economic context to tasks with matching information, camouflage context, and tasks of low cognitive demand as the findings of this study suggest. Research (e.g., Reys et al., 2004) suggest "that the choice of textbooks often determines what teachers will teach, how they will teach it, and how their students will learn" (p. 61). Business calculus instructors may have to supplement the examples and practice problems given in business calculus textbooks to include tasks with superfluous (or missing) information and/or tasks of higher cognitive demands in order to maximize students' opportunity to learn from such tasks which are rare in the textbooks we analyzed.

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Graphs of inequalities in two variables

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In this study, I analyze how preservice secondary teachers represented and explained graphs of three inequalities—a linear, a circular, and a parabolic—in two variables. I then suggest new ways to explain graphs of inequalities, i.e. some alternatives to the solution test, based on the preservice teachers' thought processes and by incorporating the idea of variation. These alternatives explain graphs of inequalities as collections of rays or curves, which is similar to graphs of functions as collections of points in one variable functions and as collections of curves in two variable functions. I conclude the study by applying the alternatives to the solving of optimization problems and discussing the implications of these alternatives for future practice and research.

Keywords: Inequality, Variation, Graph, Preservice secondary teacher

Introduction

Mathematical inequalities are important in mathematics due to their connections to mathematical equations and their applications to real-life situations. There has however been a general lack of attention from the mathematics education community on inequalities. Furthermore, the vast majority of the studies on inequalities discuss the understandings and difficulties associated with solving algebraic inequalities in one variable (Almog & Ilany, 2012; Schriber & Tsamir, 2012; Verikios & Farmaki, 2010). Rsearch on inequalities in two variables is almost nonexistent.

In regards to graphs of inequalities in two variables, which is the main focus of this study, most secondary and post-secondary algebra textbooks (David et al., 2011; McKeague, 2008) explain graphs of algebraic inequalities through the solution test. For instance, for the graph of an inequality, $y < x+1$, a series of steps are performed: draw the graph of $y = x+1$; select one or more points from one of the two regions divided by the graph of $y = x+1$; and plug in the x and y coordinates of those points to the inequality, $y < x+1$. If the x and y coordinates of the selected points satisfy the inequality, $y < x+1$, the region from which the points were selected is the graph of the inequality. If not, the other region is the graph of the inequality.

Although the solution test might be the simplest way to explain the graph of $y < x+1$, it does not provide a valid justification for why the entire region below the line graph of $y=x+1$ is the graph of the inequality. This lack of justification may potentially impede students' sound development of conceptions about mathematical proof. As shown in several studies, many students and teachers erroneously derive the truth-value of a mathematical sentence from the truth-value(s) of one or more particular cases (Harel & Sowder, 1998). The solution test is not much different from the misconception for proof of students and teachers in that it determines the truth-values of $y < x+1$ for *all* points (x, y) in a region from the truth-values of $y < x+1$ for *some* points (x, y) in the region. As such, there is a need for a better justification for graphs of inequalities.

This study has two parts. In the first part, I analyze interview data in order to answer the research question: "How is preservice secondary teachers' instrumental and relational understanding of inequalities in two variables?" I then propose alternatives to the solution test based on the preservice secondary teachers' understanding, but by incorporating the concept of the variable as in the topics of one variable and two variable functions (see for example, Herscobics & Linchevski, 1994; Weber & Thompson, 2014, for the concept of variable in functions). I conclude with the benefits of the alternatives, including their connections to graphs of one or two variable functions and their application to the understanding and solving of optimization problems.

Framework

The theoretical framework related to this study is relational understanding and instrumental understanding by Skemp (1976). According to Skemp, there are two kinds of understanding under the same name, mathematics. One is instrumental understanding, which is knowing "without reasons," and the other is relational understanding, which is knowing both "what to do and why." With an instrumental understanding, for example, one might find a division of fractions, $(a/b) \div (c/d)$, by flipping c/d and by multiplying tops and bottoms to get at ad/bc, but she may be unable to explain why she flips or multiplies. Whereas, with a relational understanding, one might use the relationship between division and fraction and the equivalence relationship in fractions, and attain $(a/b) \div (c/d) = \dfrac{a/b}{c/d} = \dfrac{a/b \times (bd)}{c/d \times (bd)} = \dfrac{ad}{bc}$ using the relationships.

There are advantages of relational understanding over instrumental understanding: The former is more adaptable to novel situations, can grow like an organic substance, and helps learners to remember and sustain knowledge. It however has its drawbacks, such as the length of time needed to achieve understanding and in some cases, students' difficulties in obtaining such understanding. Accordingly, teachers often need to make reasoned choices between the two understandings. Skemp argues that relational understanding is the only adequate understanding for teachers, yet many teachers are equipped with only instrumental understanding.

For graphs of inequalities, the solution test is commonly used for instrumental understanding rather than for relational understanding, as it gives instructions on what needs to be done in order to draw graphs. The critical ideas of variation and the infinitude of points embedded in the algebraic and geometric representations of inequalities are omitted, or at least not salient. In the following investigation, I show how preservice secondary teachers represented and explained graphs of inequalities. The goal of the investigation was not only to examine their instrumental and relational understandings but also to investigate the ideas and difficulties involved in their explanations. The details follow.

Preservice Secondary Teachers' Understanding of the Graphs of Inequalities

Methodology

This investigation was performed as part of a larger project that studied the big ideas underlying learners' difficulties in making connections among representations. The participants of the project were 15 undergraduate mathematics majors on the secondary teaching track at a small doctoral comprehensive university in the Southeast. The level of participants' mathematics backgrounds varied—four taking a precalculus course, one taking Calculus I, and the other ten taking Calculus II or above. The participants were individually interviewed twice, for about one-and-a-half hours each time, in the form of a semi-structured clinical interview. The interviews were recorded with a video camera and were transcribed. Their written responses were also collected.

The interview items that were relevant to this study were two questions from the first interview:

Q1: (a) Find a solution of an inequality, $x+2y-32<0$.
(b) Represent all the solutions of the inequality above in the Cartesian plane.

Q2: (a) Find a solution of a system of inequalities, $y<x^2+1$ and $x^2+y^2>1$.
(b) Represent all the solutions of the system of inequalities above in the Cartesian plane.

The interview questions were designed and queried to invoke both the instrumental and relational understandings of the participants. In Q1 and Q2, I first asked the participants to find a "solution" of an algebraic inequality and of a system of inequalities, in order to examine their understanding of the meaning of the solutions of inequalities in symbolic forms. To graph an algebraic inequality or a system of algebraic inequalities is in fact to represent their solutions in the Cartesian plane. As such, it was important to examine their understanding of inequalities in both symbolic and graphical forms. In addition, studies show that students have difficulty understanding what a "solution" means in algebraic inequalities in one variable (Becarra, Sisrisaengtaksin, & Walker, 1999; Blanco & Garrote, 2007). As little is known about students' understanding of the meaning of solutions of algebraic inequalities in two variables, such an investigation is worthwhile.

During the interview, I asked the participants why they represented the solutions in certain ways or why the x, y coordinates of all the points in the region satisfied the inequalities. I also asked participants to represent mathematical statements that were in word or algebraic forms geometrically on the Cartesian coordinate plane. Some of those statements were created by themselves, such as "x is less than 32 when y=0," and some were created by me (the interviewer), such as "$y<x^2+1$ when x=0."

For analysis, I first used an open coding strategy (Strauss, 1987) to code the participants' mathematical behaviors and understandings. Most of the codes used in this stage were for the correctness of their work as well as the ideas, strategies, and difficulties shown in their written or oral explanations. The initial coding showed some patterns and similarities in their thinking and work, and hence yielded categories and subcategories that led to some hypotheses. I then performed the second stage of coding: reexamining and revising the prior codes, and at the same time performing an axial coding (Strauss, 1987) to focus on the categories and subcategories from the previous coding, and hence confirming or refuting the hypotheses.

The results below are some of the findings from the analysis above. The results included here pertain to the relational and instrumental understandings of preservice teachers and some of the characteristics of their mathematical behaviors that are relevant to the alternative explanations shown in the next section.

Results

Instrumental and relational understanding. For their instrumental understanding, I examined the correctness of their graphs, as instrumental understanding is essentially knowing what to do without reasons. For their relational understanding, I used the two factors—the idea of variation, at least to some degree, and the infinitude of points in their arguments—, as they are critical components in knowing why of graphs of inequalities, a relational understanding. An explanation using the solution test by testing one or multiple points was not considered to be relational understanding.

For the linear inequality $x+2y-32<0$, 7 out of 15 preservice teachers showed an instrumental understanding by correctly representing the graph as the lower half plane of the line graph of $y=-x/2+16$, with 2 of them showing relational understanding to some extent. For the circular inequality $x^2+y^2>1$, 5 of 15 showed instrumental and partial relational understandings to some extent by correctly representing the graph as the region outside the circle graph of $x^2+y^2=1$ and by providing a reasonable explanation for their graph. For the parabolic inequality $y<x^2+1$, 3 of 15 showed an instrumental understanding by correctly representing the graph as the lower part of the parabola graph of $y=x^2+1$, with 1 showing relational understanding to some extent (see Table 1).

None of the explanations by the preservice teachers fully used the idea of variation, as I show in the next section. It was also noteworthy that for the circular inequality, preservice teachers' graphical image of a circle corresponding to the algebraic form $x^2+y^2=r^2$ helped them to explain their graphs of inequalities

Table 1 Finding a solution in and representing the graph of an inequality

		Graph-Correct	Graph-incorrect
Linear: $x+2y<32$	One-solution (Correct)	6 (3*)	5
	One-solution (Incorrect)	1 (0*)	3
Circular: $x^2+y^2>1$	One-solution (Correct)	5 (5*)	4
	One-solution (Incorrect)	0	6
Parabolic: $y<x^2+1$	One-solution (Correct)	2 (1*)	7
	One-solution (Incorrect)	1	5

* The numbers inside parentheses indicate the number of participants who show relational understanding for graphs of inequalities to some extent.

The following are some examples that showed relational understanding to some extent:

- For the graph of $x+2y-32<0$, "my y-value is never going to be bigger than 16 when $x=0$, so it must be this whole region," and "anywhere other than the line there is going to be either less than or greater than. So if I choose one up here then I am going to have really big x and y so it will make the form greater."
- For the graph of $x^2+y^2>1$, "x square plus y square is equal to 2, which is greater than 1, then that would mean it is a circle slightly larger than the one," and "I guess it is because everything contained in x^2+y^2 with radius of 1. So nothing inside there is going to be greater than 1. It is all going to be less than 1, but everything outside should leave you something greater than 1."

- For the graph of $y<x^2+1$, “this is a parabola going out from here upward. So I am thinking either it is going to be inside the parabola or outside. You can choose negative numbers. Over here it is actually going to be awesome because x is squared but y won't. So y is going to be less than that. So the way I am seeing it is you can have x and y both being negative. So you can have the Quadrant 3.”

Knowing the meaning of solutions of an inequality versus representing the solutions graphically. The analysis also showed that knowing the meaning of solutions in algebraic inequalities was not sufficient for successful representations of graphs of inequalities (see Table 1). Although many preservice teachers—11 out of 15 for the linear inequality and 9 out of 15 for the circular and parabola inequalities—understood the meaning of solutions of algebraic inequalities in that they found one or more pairs of x and y values as solutions of inequalities, only about half of those (or fewer for the parabolic case) who provided a solution correctly in an algebraic inequality successfully provided its corresponding graph—6 of 11 for the linear inequality, 5 of 9 for the circular inequality, and 2 of 9 for the parabolic inequality.

There were many factors that contributed to this failure to transfer, from knowing the meaning of solutions in an algebraic inequality to representing the solutions of the inequality as a graph. The two most prominent, which are closely related to the alternative explanations suggested in this paper, were the following:

Lack of understanding that a graph of an algebraic inequality is a visual representation of the solutions of the algebraic inequality. Many preservice teachers who successfully found one or more solutions of the algebraic inequality $x+2y-32<0$ drew a graph of the line equation $x+2y-32=0$, but falsely claimed that the graph of $x+2y-32<0$ was the line, indicating a lack of understanding that the graph of an inequality is a collection of all points whose x and y coordinates satisfy the algebraic inequality. Yet when they were asked to explain why the points on the line were the solutions of the inequality, some were able to reflect on the meaning of the algebraic inequality and corrected their graphs. The following is an example of the characteristic.

Interviewer: So can you give me an example of the solutions?

Student: Well. $x+2y$ is less than 32. You still have a plenty of pairs of numbers that can satisfy that or not satisfy that. I mean I can give you a random x and y. If $y=2$ and $x=1$, that is certainly true.

Interviewer: Ok, now represent all the solutions in the x-y plane. How would you do that?

Student: That is why I was thinking about this. I was trying to solve in terms of x, so I have a number line because it should be anything on the line. Everything on the line should be a solution.

Interviewer: Why are they solutions?

Student: Everything on the line should be a solution. Hold on. If I plug in 0, 16, it is going to be 32. So, it is going to be this, everything below this line (pointing the graph of $x+2y$-32=0). I am trying to think now. Everything above this line, your y is going to be bigger, it is only this point that is not. Well, any point on this line won't work. I am sure there are still other examples to find. If I go below every value of y. Hold on. I have this whole region below. My y-values will never going to be bigger than 16 when x=0.

Lack of ability to represent an inequality when a variable is fixed as a constant. Most preservice teachers did not know how to transfer an algebraic or verbal inequality statement to a geometric object (which was a horizontal or vertical ray) when a variable in the inequality was fixed as a constant. This representational transfer was requested to the preservice teachers who generated such statement by themselves or to some preservice teachers who successfully provided graphs and/or explanations for graphs of inequalities. Out of 5 preservice teachers who were asked to do the representational transfer, only one of them was able to represent her statements as rays. As for the other 4 preservice teachers, they represented "$x+2y+32<0$ when $x=1$," "my y values will never going to be bigger than 16 when $x=0$," "when $x = 31$, y is less than 0," or "when $x=32$, all the solutions of this inequality would be when $y<0$" as the entire region below the line graph of $x+2y+32=0$ instead of as a vertical ray; 2 of those 4 teachers also represented "$y<x^2+1$ when $x = 3$" or "$y<x^2+1$ when $x=0$" as the entire region below the parabola graph of $y=x^2+1$ instead of as a vertical ray.

Graphs of inequalities in two variables: Alternative explanations

As shown in the Results section, some preservice teachers provided somewhat reasonable explanations for graphs of inequalities that included infinitely many points in their arguments. Their explanations however fell short in that they did not use the idea of variation systemically enough to explain their graphs. This study thereby suggests more systematical ways to explain graphs of inequalities in two variables. These alternatives utilize preservice teachers' ideas of fixing a variable by a constant and their use of the graphical image of curves as shown in the circular inequality, yet fill the gaps in their work by incorporating the idea of variation.

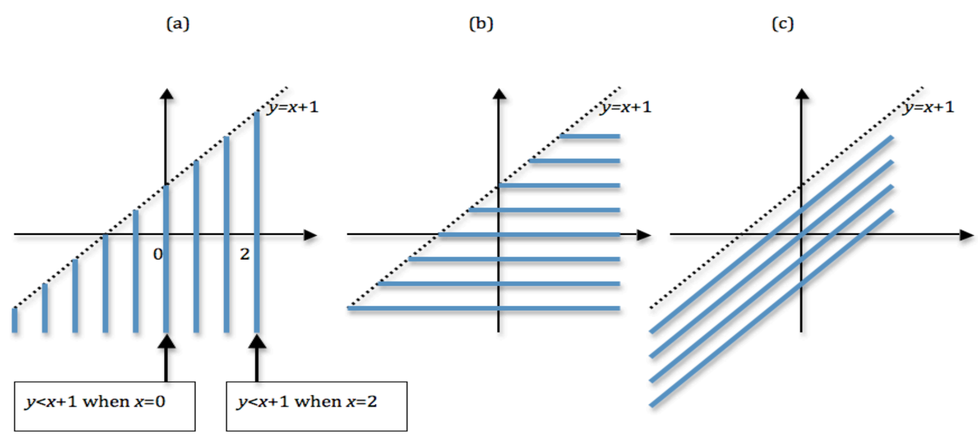

Figure 1 Graph of the inequality, $y < x+1$

Using the inequality $y < x+1$, as an example, the graph of an inequality in two variables can be understood as (a) a collection of vertical rays, $x=c$ and $y < c+1$, if the x variable is kept constant; (b) a collection of horizontal rays, $y=c$ and $c-1 < x$, if the y variable is kept constant; or (c) a collection of lines, $y-x=c$, with $c<1$, if $y-x$ is kept constant. In the first case, shown in Figure 1(a), the graph of $y < x + 1$ when $x=0$ is the collection of points, $(0, y)$, such that $y < 0+1$, which is an open vertical ray on the y-axis. Similarly, the graph of $y<x+1$ when $x=2$ is the collection of points $(2, y)$ such that $y< 2+1$, which is an open vertical ray on the line $x=2$. As x can be any real value, the graph of the inequality is the collection of all those open rays, which forms the entire lower half plane bounded by the line graph of $y=x+1$. The same line of thinking works for the second case, shown in Figure 1(b). Instead of the x variable, the y variable is kept constant; as such, the graph is a collection of open horizontal rays, which forms the entire lower half plane bounded by the line graph of $y=x+1$. In the third case, shown in Figure 1(c), $y < x+1$ is equivalent to $y-x <1$; as such $y-x$ is kept constant with a value less than 1. Whether $c =1/2$, 0, or -1, $y-x =c$ forms a line with the slope 1 and the y-intercept c; as such, the collection of such lines determines the lower half of the line graph of $y=x+1$, as the graph of the inequality $y < x+1$.

Discussion and conclusions

The suggested alternatives are more than merely different explanations for graph of inequalities in two variables. The true benefits of the alternatives are their connections to one and two variable functions as well as their applications to real-life optimization problems, which are one of the most importance uses of inequalities.

To elaborate, the graph of an inequality $y<f(x)$ as a collection of rays is an extended understanding of the graph of $y=f(x)$ as a collection of points, and an understanding that can lead to the graph of $z=f(x,y)$ as a collection of curves. In order to graph the equalities and inequalities, a learner performs an action of fixing a variable as a constant value, $x=c$ for example, and then finds the value of the other variable in the case of $y=f(x)$ or the relationship between the other variables for the case of $z=f(x,y)$. Such an action then yields a geometric object—a point $(c,f(c)$ in a plane in the case of $y=f(x)$; a ray, which is the graphical representation of $\{(c, y)|y<f(x)\}$, in a plane in the case of $y<f(x)$; and a curve $z=f(c,y)$ in a 3-dimensinal space in the case of $z=f(x,y)$. The graph is then a collection of all geometric objects, with the x variable running through all constant values in the domain. In this regard, the above alternatives provide consistency in mathematical thinking related to graphing through the concept of variables in functions, equations and inequalities.

The idea embedded in the alternatives can also help students to solve real-life inequality-related problems, such as those in the *Cookies* unit in the high school mathematics curriculum, *Interactive Mathematics Program*. When finding the maximum profit from the profit function, $f(p, i) = 1.5p + 2i$, with various constraints (represented as a region determined by linear inequalities), students can consider all points on a horizontal (or vertical) line segment by keeping p (or i) constant; they can then understand that the maximum can only occur at the upper boundary points of those segments, which consist of three linear equations. Students can then determine the maximal profit by considering the values of $f(p, i) = 1.5p + 2i$ with constraints given as linear equations, which are relatively easy. This approach not only is different from the strategies in the *Cookies: Teacher's Guide* but also brings a different kind of understanding to the problem. This line of reasoning also aligns with calculus ideas in that the partial derivatives, $f_x(x, y)$ and $f_y(x, y)$, with x or y kept constant, play a critical role in the determination of the extrema of $f(x, y)$.

The alternatives proposed in this study are suggestions based on preservice teachers' understanding of graphs of inequalities and on research on the graphs of one variable and two variable functions. Future research should examine the effects of or problems with implementations.

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Graphing habits: "I just don't like that"

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Students' ways of thinking for graphs remain an important focus in mathematics education due to both the prevalence of graphical representations in the study of mathematics and the persistent difficulties students encounter with graphs. In this report, we draw from clinical interviews to report ways of thinking (or habits) undergraduate students maintain for assimilating graphs. In particular, we characterize actions constituting students' ways of thinking for graphs that inhibited their ability to represent covariational relationships they conceived to constitute some phenomenon or situation. As an example, we illustrate that students' ways of thinking for graphs were not productive for their representing a relationship such that neither quantity's value increased or decreased monotonically.

Key words: Graphing, Covariational reasoning, Quantitative reasoning, Cognition, Function

"[U]nderstanding graphs as representing a continuum of states of covarying quantities is nontrivial and should not be taken for granted" (Saldanha & Thompson, 1998, p. 303). Saldanha and Thompson's call that educators not take for granted students' ways of thinking for graphs remains relevant given the difficulties students have with topics (e.g., function, rate of change and derivatives, and variables) that involve the use of graphs (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Ellis, 2007; Johnson, 2012; Oehrtman, Carlson, & Thompson, 2008; Thompson, 1994, 2013; Trigueros & Jacobs, 2008; Zandieh, 2000). In this paper, we respond to the need for a better understanding of students' way of thinking for graphs. Namely, we describe students' graphing actions on tasks we designed to afford tracking covarying quantities. We first provide relevant background information and describe our clinical interviews and methods. We then identify students' ways of thinking for graphs that were not productive for representing covariational relationships they had conceived some phenomenon to entail. We close by arguing that the identified ways of thinking for graphs are related to aspects that we perceive to be pervasive in U.S. school mathematics.

Background

Our core interest is characterizing students' covariational reasoning–"the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 354)–particularly in the contexts of phenomena (e.g., students taking a road trip) and graphs representing covarying quantities understood to constitute the phenomena. By quantity and covariation, we do not mean numbers (or measures) and numerical patterns of two sets (cf. Confrey and Smith (1994, 1995)). Rather, we use *quantity* to refer to an attribute (e.g., length) an individual conceives to constitute some situation or phenomenon such that the individual understands the attribute as having a measurable magnitude (Thompson, 2011), possibly with respect to numerous unit magnitudes (e.g., meters or feet). We draw attention to a distinction between a quantity's magnitude and its measure because it enables us to approach covariation in terms of the simultaneous coordination of magnitudes in flux with the *anticipation* that these magnitudes have specific measures (in an associated unit) at any moment (Saldanha & Thompson, 1998); one does not need specified measures at hand to reason covariationally.

We do not interpret this perspective to diminish the importance of understanding covariation in terms of quantities' measures and patterns in these measures. Such an understanding is important for understanding various function classes and rate of change. Our focus on covarying magnitudes provides a complementary lens that researchers, including ourselves, have found productive in characterizing students' images of covarying quantities, particularly their conceiving quantities in flux (Castillo-Garsow, Johnson, & Moore, 2013; Thompson, 1994). Most relevant to the current study, Moore and Thompson (2015) defined *emergent shape thinking* as a way of thinking about a graph as a locus or trace that is produced by the simultaneous coordination of two quantities' magnitudes. They explained, "emergent shape thinking entails assimilating a graph as a trace in progress (or envisioning an already produced graph in terms of replaying its emergence), with the trace being a record of the relationship between covarying quantities" (Moore & Thompson, in press). As Moore and Thompson highlighted, we cannot illustrate such a way of thinking due to the static medium of print, but we do provide instantiations in *Figure 1*. We emphasize that although a displayed graph is composed of points, a student thinking emergently constructs a displayed graph in terms of a projection of two coordinated magnitudes along the axes with the anticipation that these magnitudes correspond to specified measures.

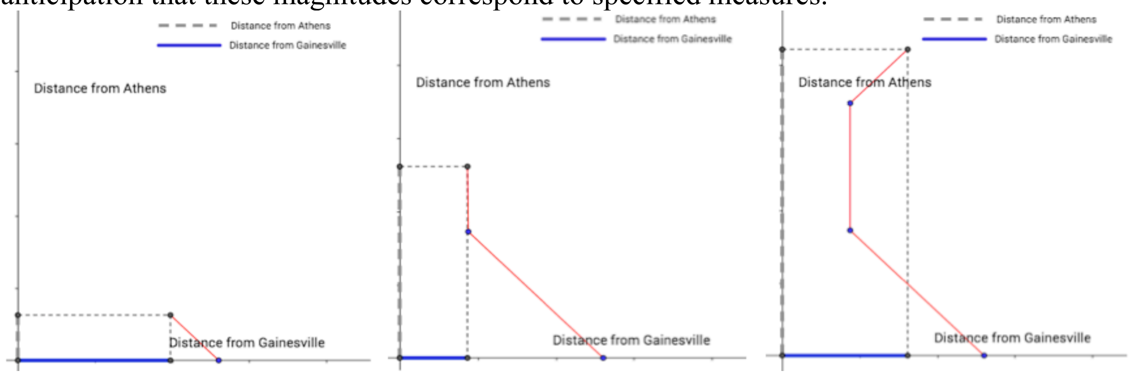


Figure 1. A graph as an emergent (snapshot) coordination of two magnitudes.

Subjects, Setting, and Methods

Our subjects were ten prospective secondary mathematics teachers (hereafter referred to as students) enrolled in an undergraduate secondary mathematics education program in the southeastern U.S. The students ranged from juniors to seniors in credits taken, and they had completed at least one mathematics course beyond an undergraduate calculus sequence. We chose the students on a volunteer basis. We conducted three semi-structured clinical interviews (Ginsburg, 1997; Goldin, 2000) with each student. Each interview lasted approximately 75 minutes. The lead author facilitated each interview, often with the aid of a member of the author team. We asked the students to talk-aloud as much as possible, and we asked open-ended questions during their progress to gain insights into the students' thinking with minimal guiding. The interviews occurred throughout one semester with approximately 1.5 months passing between each interview. The time between the interviews enabled us to design subsequent interviews based on retrospective analyses of prior interviews.

We video- and audio-recorded all interviews and digitized student work after each interview. The lead author and fellow interviewer also recorded observation notes after each interview. We analyzed the data using selective open and axial methods (Corbin & Strauss, 2008) in combination with *conceptual analysis*–an attempt to build models of students' mental actions that explain their observable activity and interactions (Thompson, 2008; von Glasersfeld, 1995). First, members from the research team identified instances that provided insights into the students' thinking. The research team then viewed these selected instances in order to characterize the students' thinking. As we developed these characterizations, we continually returned to previously viewed instances (across all students) to revise or provide alternative characterizations if necessary. We generated themes among our characterizations through this iterative process, including the ones that we report in this paper.

Task Design

We designed a series of six tasks (two per interview, with each student receiving the same sequence of tasks). Each task entailed a different context, but were similar in that we: (1) provided a dynamic, albeit often simplified, phenomenon through video; (2) did not include numerical values for attributes of the phenomenon; (3) prompted the student to graph a relationship between two quantities; and (4) often prompted the student to create a second graph, either between different quantities or the same quantities under different axes orientations. To illustrate, the second interview with each student included *Going around Gainesville* (*Figure 2*; see *Figure 1* for a solution to Part II), which entails a video depicting a car traveling back-and-forth between Athens and Tampa Bay. Reflecting (1) and (2), the task consists of a dynamic phenomenon depicted by a video without numerical information. In Part I, we prompted the students to graph a particular relationship between two quantities (i.e., (3)). In Part II, which we presented after the students completed Part I, we prompted the students to graph a different relationship with an imposed axes orientation (i.e., (4)).

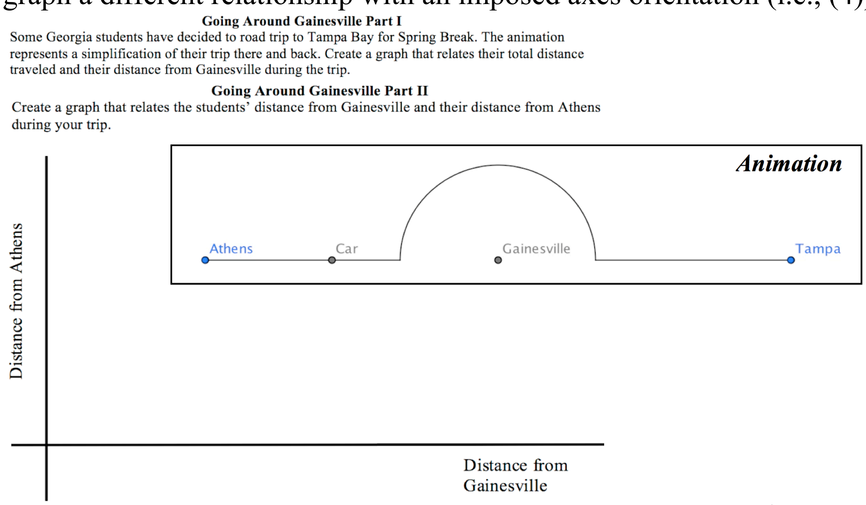


Figure 2. The *Going around Gainesville* task and video.[1]

In general, (1)-(3) each reflects our interests in students' covariational reasoning with particular attention to their coordinating magnitudes. Design goal (4) stems from two major findings from our previous work. First, we identified that students' ways of thinking for function and their graphs led to perturbations when graphing relationships in different axes orientations (Moore, 2014; Moore, Silverman, Paoletti, & LaForest, 2014). Although the quantities are slightly modified in Part II above (with us intending that one distance be accumulative and the other be displacement), we intended to gain additional insights into the perturbations that arose (or did not arise) when graphing the relationships in various orientations. Second, we designed the tasks to be what we perceived as non-canonical; our previous work led us to conclude that students encounter difficulties with such graphs (Moore, 2014). With respect to Part II, and because U.S. students nearly exclusively work with graphs such that the quantity represented on the vertical axis is a function of the quantity

[1] This task is a modification of the task provided by Saldanha and Thompson (1998). We strongly suggest that the reader work these tasks before continuing to read.

represented on the horizontal axis, we hypothesized that the students might experience perturbations graphing a relationship that did not have this property, thus providing us additional insights into their ways of thinking for graphs.

Results

We structure the results section around three interrelated ways of thinking for graphs: *graphs 'start' on the vertical axis*, *graphs are drawn or read left-to-right*, and *graphs pass the 'vertical line test'*. We present these ways of thinking separately but do not intend to imply they exist independently. For instance, we describe students' anticipating graphs drawn or read left-to-right, as well as their anticipating that graphs 'start' on the vertical axis. These schemes were related in that some students' ways of thinking for graphs involved the sequence of marking an initial point on the vertical axis and then drawing a graph from that point to the right. However, if one of these schemes constituted a student's ways of thinking for graphs, then it was not necessarily the case that the other scheme also constituted a student's ways of thinking for graphs.

Graphs 'start' on the vertical axis

We inferred from some students' activities that a necessary action constituting their ways of thinking when drawing a graph was beginning their drawing on the vertical axis. At times, 'starting' their graph on the vertical axis led to what *we conceived* to be contradictions between the relationship that the students claimed or intended the graph to represent and the relationship that we perceived the graph to represent. At other times, this way of thinking led to the students experiencing perturbations as *they conceived* 'starting' their graph on the vertical axis as incompatible with the relationship they intended the graph to represent.

As an example, we return to *Going around Gainesville, Part II*. A normative graph includes a point on the horizontal axis corresponding to paired magnitudes when starting the trip in Athens; the graph includes no points representing a magnitude of zero for the distance from Gainesville (see *Figure 1*). Upon orienting to a task, some students immediately marked a point on the vertical axis and anticipated drawing a graph from that point (Excerpts 1).

Excerpts 1. Two students 'starting' graphs on the vertical axis.

Paula:	Your distance from Athens starts at zero [*plots point at origin*] because you're in Athens. Um, so as you get. Mmm, no, you're gonna start up here [*plots point on vertical axis but not at origin*]. Ignore that [*covering origin*]. 'Cause, oh wait, no, stop [*crosses out second plotted point*]. No, you're here [*points to origin*].
Annika:	We're in Athens [*moves to paper, marks point at origin*], as we're moving away from Athens we're getting closer to Gainesville [*draws segment from the origin going up and to the right, Figure 3*].

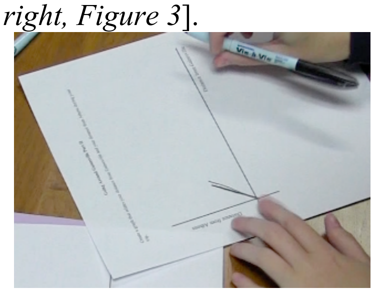


Figure 3. Annika starting the graph from the origin.

In anticipating the graph 'starting' on the vertical axis, the students' initial actions were to plot a point identifying the appropriate initial distance from Athens (and not the corresponding distance from Gainesville). Although Paula identified that the initial distance from Gainesville was non-zero, she maintained an alternative ‘starting’ point on the vertical axis and she quickly returned to plotting a point corresponding to the initial distance from Athens. As the students moved forward, they experienced perturbations due to their ‘starting’ point. Annika eventually noted that she did not take into account the initial distance from Gainesville and how to represent that distance as decreasing when first creating her graph. Paula experienced a more sustained perturbation, which we explain in the following section.

Graphs drawn or read left-to-right

Another action constituting some students’ ways of thinking entailed students understanding graphs as drawn or read left-to-right. When constructing a graph, students anticipated drawing the graph by starting at a point and exclusively moving their pen to the right while allowing for movements vertically. The vertical movements either connected previously plotted points (regardless of the order that these points were plotted) or captured some relationship that they intended the graph to represent (see Annika, Excerpts 1). To illustrate, we present two students’ activities (Excerpts 2). Karrie was working the task presented by Saldanha and Thompson (1998), which includes a prompt to graph how a traveler’s distances from two cities covary (see *Figure 4* for animation and a graph). Paula’s work is a condensed continuation of that in Excerpts 1.

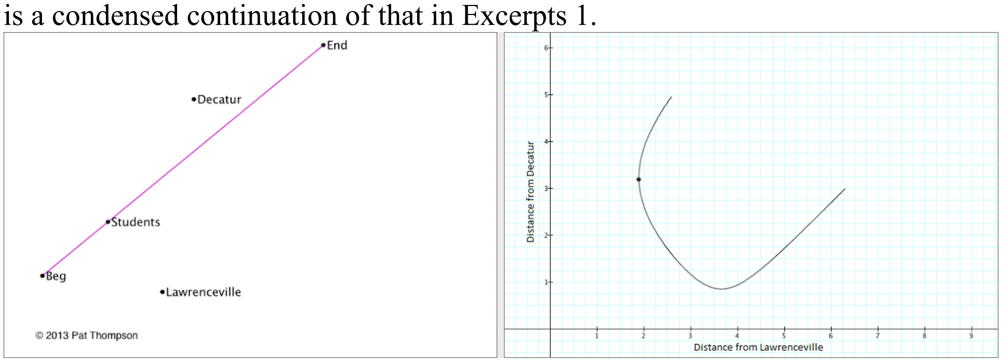


Figure 4. *City Travels* animation and graph (modified from Saldanha and Thompson, 1998).

Excerpts 2. Two students drawing graphs left-to-right.

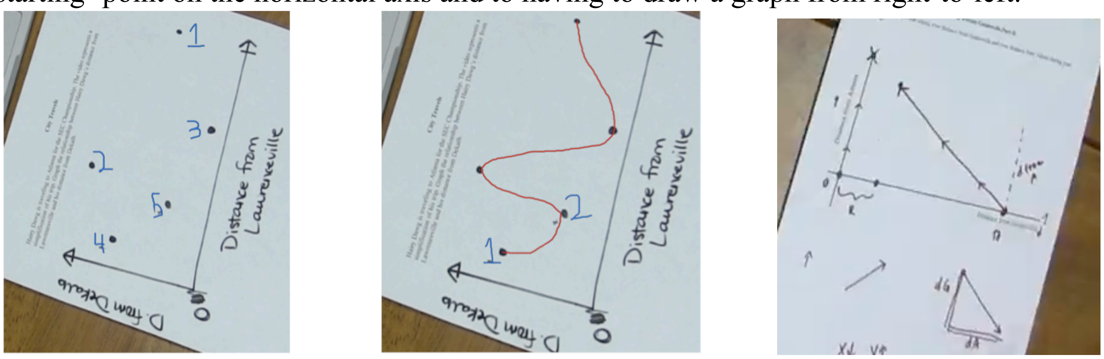
[*Karrie has plotted five points corresponding to locations during the trip in the order we have annotated in Figure 5a*]

Karrie: Okay, wait. This one [*pointing at the leftmost point she plotted*] was when he’s closest to Lawrenceville, which happens first [*labels the point ‘1’*], then this one [*labels the next leftmost point ‘2’, moves pen to the third leftmost point*] so it’s something like that [*making a sweeping motion indicating a curve connecting the points from left-to-right in the order we have annotated in Figure 5b*].

[*Paula is now focused on the initial point on the vertical axis that is not at the origin–see Excerpts 1–and anticipating drawing a segment sloping downward left-to-right from her initial point that she later crosses out–see Figure 5c*]

Paula: I wanted to show that the distance was decreasing [*motioning diagonally down and to the right from the point plotted on the vertical axis*], but that means that your distance from Athens is decreasing [*tracing along the vertical axis from the initial point to the origin*]…But your distance from Athens is growing. But your distance from Gainesville is decreasing. So, if that’s growing [*draws arrow pointing upward beside the vertical axis labe*] and that’s decreasing, so [*draws arrow pointing downward beside horizontal axis label and then an arrow pointing upwards beside the vertical axis label*]…[*the student works for six additional minutes before having an insight*]…Oh, what if I started it like here [*plots point on the right end of the horizontal axis*]…But I don’t want to start like, like I don’t like starting graphs. You know, I don't know. Work backwards. That's weird…[*draws in what we perceive to be a correct initial portion of the graph over the next minute and a half*]… my graph is from right-to-left, which is probably not right…Backwards is traveling from right-to-left. But I think my graph is just, I think I'm just not clicking. I think I'm missing something.

In Karrie's case, we highlight her immediate move to anticipate connecting the points from left-to-right after ordering two points from left-to-right (*Figure 5b*). Paula, on the other hand, did not plot points other than her initial point. Instead, she anticipated drawing a graph left-to-right from the initial point to indicate a decreasing *distance from Gainesville*. However, as she reflected on her anticipated graph with respect to the axes, she understood that such a graph would indicate a decreasing *distance from Athens*. Over the next seven minutes, Paula produced what we perceived to be a correct graph by thinking of the graph emergently (*Figure 5*c), but her resulting graph continued to perturb her due both to the 'starting' point on the horizontal axis and to having to draw a graph from right-to-left.



(a) (b) (c)

Figure 5. Karrie's annotated work (a-b) and Paula's work (c).

Graphs pass the 'vertical line test'

Some students' ways of thinking for graphs involved students anticipating that their drawn graph must pass the 'vertical line test' (i.e., a graph such that each abscissa value only corresponds to one ordinate value). In some cases, their anticipation was related to drawing graphs exclusively left-to-right. However, this way of thinking for graphs also emerged when students could anticipate graphs not drawn in this way. To illustrate, consider Angela's work on *Going Around Gainesville, Part II* (Excerpt 3). She did not encounter issues while drawing part of the graph right-to-left to indicate the distance from Gainesville decreasing as the distance from Athens increases. However, drawing a vertical segment, which she understood as representing the distance from Gainesville remaining constant as the distance from Athens increased, perturbed her to the extent that she both hesitated drawing a vertical segment and continued to question this vertical segment throughout working the task.

Excerpt 3. Angela anticipating that a drawn graph pass the 'vertical line test'.

[*Angela has plotted three points corresponding to positions on the semicircular path*]

Angela: So, that's weird [*motions pen indicating a vertical segment connecting the points*]. I don't wanna connect those dots, but, [*laughs softly*] I really don't like that.

Int.: What don't you like?

Angela: I just don't like that [*draws in a correct graph with a vertical segment*] my graph looks like this…I dunno. If I was taking a test and I drew that [*quickly motions the pen over the graph in the direction she had connected the points*] I'd feel like my answer was wrong. But I feel [*quickly motions pen back over the graph in the reverse direction*] like I graphed my points correctly…

Discussion

We find the above results notable for a few reasons. First and foremost, it is significant that undergraduate students who have completed mathematics courses beyond a calculus sequence hold ways of thinking for graphs that inhibit their ability to represent covariational relationships. Moreover, the students' difficulties did not stem from underdeveloped images of phenomenon, as is sometimes the case (Carlson et al., 2002; Moore & Carlson, 2012), but instead stemmed from ways of thinking for graphs that limited their ability to represent conceived relationships. We interpret this finding to corroborate researchers' (Moore, Paoletti, & Musgrave, 2013; Moore & Thompson, 2015) conjecture that ways of thinking that do foreground graphs as emergent traces of covarying quantities are more productive for accommodating novel phenomena and relationships than those ways of thinking that do not. A student thinking of a graph emergently maintains a focus on simultaneously coordinating magnitudes along axes with a trace emerging from this coordination; the mental operations that generate a trace are essentially equivalent regardless of the resulting trace and properties of its shape. On the other hand, ways of thinking for graphs that foreground recalling a repertoire of shapes and properties of these shapes (e.g., graphs passing the 'vertical line test'; graphs being traced left-to-right) are constrained to those phenomena or situations that are compatible with these shapes and properties.

We also find the results notable given the extent that some students remained perturbed after they had constructed graphs they conceived to represent a relationship compatible with the relationship they conceived to constitute some phenomenon. In these cases, and due to their resulting graphs being incompatible with particular ways of thinking they had previously constructed for graphs, the students questioned the correctness of their graphs (see Paula and Angela). We find the students' inability to reconcile their states of perturbation significant, especially because the students are prospective secondary mathematics teachers. Both researchers and policy authors (Carlson et al., 2002; Ellis, 2011; Johnson, 2015; Moore et al., 2014; National Governors Association Center for Best Practices, 2010; Thompson, 2013) have argued that covariational reasoning should underpin middle and secondary school mathematics (including precalculus and calculus). Our results raise questions about the extent that prospective teachers' ways of thinking support their capacity to heed this argument.

Closing Remarks

We close by noting that the aforementioned findings are, in retrospect, unsurprising given the traditional focus of U.S. mathematics curricula. For instance, U.S. mathematics curricula nearly exclusively limit the study of relationships to those relationships that are functions even if not explicitly defining such relationships as functions (e.g., the study of linear relationships in middle school). Such curricula afford students repeated opportunities to construct and re-construct ways of thinking compatible with those described here, possibly to the extent that these ways of thinking become habitual responses to graphing situations. For instance, if students only experience graphs such that the quantity represented along the horizontal axis is monotonically increasing, then there is little need for the student to maintain attention to variations in this quantity's value while attending to another quantity's value; students can merely focus on the latter quantity while assuming the other quantity's value is increasing. We conjecture that our findings have implications for curricular approaches to functions, relationships, and their graphs. Namely, students might benefit from opportunities to graph a wider range of relationships between covarying quantities.

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Interpreting proof feedback: Do our students know what we're saying?

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Instructors often write feedback on students' proofs even if there is no expectation for the students to revise and resubmit the work. However, it is not known what students do with that feedback or if they understand the professor's intentions. To this end, we asked eight advanced mathematics undergraduates to respond to professor comments on four written proofs by interpreting and implementing the comments. We analyzed the student's responses through the lenses of communities of practice and legitimate peripheral participation. This paper presents the analysis of the responses from one proof.

Keywords: Proof Writing, Proof Grading, Proof Instruction, Proof Revision, Student Thinking

Introduction and Research Questions

Rav (1999) claimed proofs "are the heart of mathematics" and play an "intricate role … in generating mathematical knowledge and understanding," (p. 6). Proof is perhaps the dominant feature of the advanced undergraduate mathematics curriculum. While practice writing proofs is certainly important in developing students' proficiency with proof-writing, without feedback students are unlikely to improve their proof-writing. Moreover, mathematicians act as if they believe that giving students feedback is critical to their learning, writing marks and notes on student proof-productions (Brown & Michel, 2010; Moore, 2014; Strickland & Rand, in press). Yet this feedback improves student learning only if students read, make sense of, and incorporate it into their future work. But few studies have examined the effectiveness of this process of giving feedback and asking students to revise their proofs. Thus, in this study we investigate the following questions:

1. How do students interpret professors' marks and comments on student-written proofs?
    a. How do students interpret and describe each mark or comment?
    b. How do students explain the rationale for making the proposed changes?
2. What changes do students make to the proofs in response to their interpretations of the comments?
3. How do students' responses to each of questions 1a and 1b above align with the way that is normative in the discipline (as described by mathematically enculturated individuals)?

Literature and Theory

Theoretical orientation

The theoretical orientation for this study is a version of social constructivism referred to as the emergent perspective (Cobb & Bauersfeld, 1995; Cobb & Yackel, 1996), drawing on ideas primarily from the social perspective. In particular, "The social perspective indicated is concerned with ways of acting, reasoning, and arguing that are normative in a classroom community" (Cobb, et al, 2001, p. 118). Students produce proofs as part of class, and the professor holds them to some standard and, we argue, communicates the normative ways of reasoning and arguing for the classroom via written comments on their proofs. That is, within the class she acts as the representative of the mathematical community with a goal of helping students to develop discipline-specific ways of writing. As a result, we draw on the notions of communities of practice and legitimate peripheral participation (Lave & Wenger, 1991).

We take a community of practice to be a group of people coming together in a process of collective learning in a shared domain, in this case learning about advanced undergraduate mathematics. There are two overlapping communities of practice of import to this study: the community of professional mathematicians and the community of advanced undergraduate mathematics students where the professor acts as a representative of the community of mathematicians. In such situations, we take learning to be a move towards fuller participation in the practices of the target community. Legitimate peripheral participation is a way to understand novices' attempts to participate as well as the means by which they learn. While to a mathematical expert, novice proof-productions are filled with errors of logic, grammar, syntax and more, under the lens of legitimate peripheral participation we instead view them as attempts to communicate in the style of the community where the rules are, at best, partially mastered and often tacit.

Student and mathematician misunderstandings

While we argue that professor comments on student proof-productions have a significant role in student learning to produce proofs, we also have reason to believe that students are likely to misinterpret them. In particular, regarding conceptions of proof, previous research, including that of Ko and Knuth (2013) and Selden and Selden (2003) have shown that students often fixate on the form, such as the ritualistic inclusion of particular features or the presence of mathematical symbols, rather than the content of the proof.

Research on student understanding of lectures suggests that students develop significantly different understandings of the presented material and meaning for professor actions than the professor intends (cf. Lew, Fukawa-Connelly, Mejia-Ramos, & Weber, in press; Weinberg, Weisner, & Fukawa-Connelly, 2014). We use this prior research on misunderstandings and "misses of understanding" to form hypotheses about how students are likely to interpret a professor's comments on proof-productions. In particular, we hypothesize that they are likely to:

- not apprehend some comments,
- develop only a surface-level understandings of comments, and
- interpret comments in ways that differ from what mathematical experts would do.

Moreover, we argue that the latter two of these hypotheses are supported by the construct of legitimate peripheral participation as described above because learners are likely to make exactly these kinds of production and interpretation mistakes.

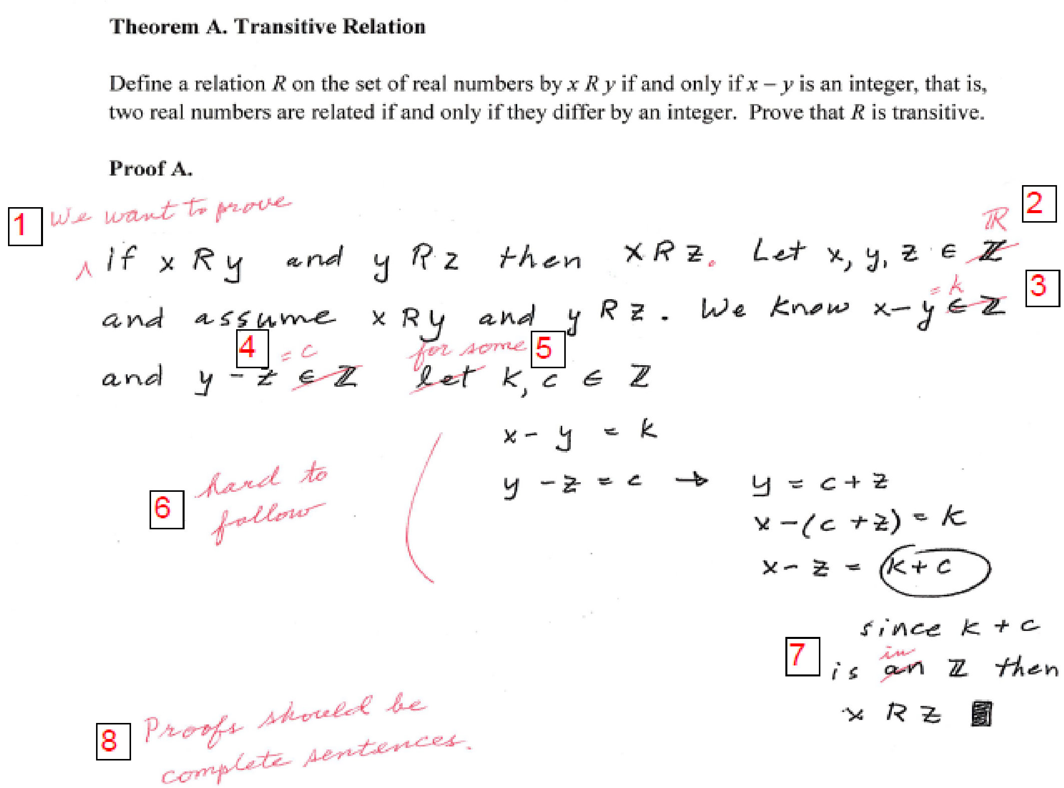
Methods

Participant selection

The participants were 8 students, 4 men and 4 women, with advanced undergraduate standing from two teaching-focused institutions, four from each institution. They had each taken at least 2 proof-based undergraduate mathematics classes, including a transition-to-proof course. We purposefully selected participants who had experience with writing proofs and receiving feedback from their professors so as to give the best possible chances for their success in understanding the proofs and interpreting the professor's comments in this study.

Data collection

We engaged each participant in a 90-minute task-based interview where the primary task was to describe and interpret a professor's comments about proofs. The interviews were audio-recorded and pencast with Livescribe pens. In each interview, we asked basic demographic questions and reflective questions about the participant's typical use of professor comments. Subsequently, we asked the participant to engage in the proof-comments task on a maximum of 4 proofs. We ensured that each proof had a mix of comments related to notation and presentation, and that some of the proofs also included comments that addressed logical issues. The proofs and professor comments were taken from a previous research project exploring professors' proof grading (Moore, 2014). An example proof, with comments, is shown in Figure 1.


Theorem A. Transitive Relation

Define a relation R on the set of real numbers by $x\ R\ y$ if and only if $x - y$ is an integer, that is, two real numbers are related if and only if they differ by an integer. Prove that R is transitive.

Proof A.

1 We want to prove

If $x R y$ and $y R z$ then $x R z$. Let $x, y, z \in \cancel{\mathbb{Z}}$ $\mathbb{R}$ 2

and assume $x R y$ and $y R z$. We know $x - y \cancel{\in \mathbb{Z}}$ $= k$ 3

and $y - z$ 4 $= c$ $\cancel{\in \mathbb{Z}}$ for some 5 ~~let~~ $k, c \in \mathbb{Z}$

6 hard to follow

$x - y = k$

$y - z = c \rightarrow y = c + z$

$x - (c + z) = k$

$x - z = k + c$

since $k + c$

7 is ~~an~~ in $\mathbb{Z}$ then

$x R z$ ∎

8 Proofs should be complete sentences.

Figure 1. An example of the proofs and professor comments used in the interviews

We presented the written proofs to the participant one at a time, told her it had been written by a student, and asked her to read and understand it as best she could. Then we presented a marked proof to the participant with a professor's feedback written in red ink. To determine whether the participant's interpretation of the mark or comment matched our own, we asked the participant to explain why the professor had made each individual mark or comment and what changes she thought the professor wanted. Finally, we asked the participant to rewrite the proof in order to allow us to further explore her interpretation of the comments and see how she implemented the professor's recommendations.

Data analysis

For each comment in each proof each of the researchers wrote a description of what change we believed the professor wanted in the proof and an explanation for the change. Based on these individual notes, we created a consensus description of what each proof-comment was asking the participant to change and the reason for the change.

For each interview we first transcribed the interview and then chunked it at a number of levels. We parsed the demographic and reflective questions in one piece and the participant's discussion of each of proof in additional pieces. We partitioned the discussion of each proof by identifying the participant's initial reading and the beginning- and endpoints of their conversation about individual comments. In cases where participants discussed multiple comments in the same utterance, we looked across interviews, and when it was common, we treated the comments as a single unit to parse all interviews similarly. We made a final block of the talk-aloud proof-writing process, for which we chunked their utterances around the comments and linked those to what they were writing in the proof.

To code the participant's utterances about each comment we first wrote a brief holistic summary. Then we developed a coding sheet identifying:

- What the participant identified as the part of the proof the comment was addressing,
- What the participant's response suggests should change in the proof,
- Any reasoning that the participant gave to explain the intended change,
- An inference about the participant's thinking about proofs,
- A summary of what the participant changed in his proof revision,
- A comparison of each of the above to our consensus expert-interpretation, and
- An explanation of how an unanticipated change exhibited during the proof writing could be understood as a logical interpretation of the professor's comments.

We then created summaries first by summarizing across participants within individual proof-comments and then by further aggregating within types of comments (e.g., logical issues) and describing the understanding of proof that was demonstrated by the related responses.

Results

We report three principal findings in response to the research questions. In the interest of space, we will limit our presentation of evidence to the proof displayed above. Overall, the participants were very successful at interpreting what a professor wanted them to do in response to any comment. We saw the following success rates: For each of the eight comments 100% of participants correctly identified an acceptable part of the proof to be changed, and they all executed a change in a manner logically consistent with our understanding of the comment. However, how they interpreted the comments and executed the requested changes was not always consistent with our understanding. In the sections that follow, we explore the participants' work and thinking about the professor's comments.

Students' revisions of specified changes

Six of the professor's eight comments on Proof A were very specific. Five of them indicated to the student that something in the proof should be crossed out and replaced. For example, comment 2 in the first line of the proof in Figure 1 specifies replacing $\boldsymbol{Z}$ with $\boldsymbol{R}$. In these instances, the participants unsurprisingly always identified what they believed the professor wanted them to revise and implemented the revisions in a way that conformed with expert understanding. Comment 1 was also specific but suggested the addition of new text, namely the phrase "We want to prove," rather than the replacement of existing text. In this case, seven of the participants added the recommended phrase as indicated, whereas one participant, Adam, showed some individuality by assuming that xRy and yRz and then writing, "We will prove that xRz." Thus, when the professor's comment was either a change of proof-text or an addition to the proof-text and the change was explicitly written out, the participants' identification and implementations of the recommended changes were largely consistent with the expert consensus, but according to the expert consensus, the participants did not always understand why the professor recommended a change.

Students' interpretations of the logic of specified changes

The participants were also asked to describe why they thought the professor had specified these changes to the proof-text, and their answers revealed a variety of ways of thinking about proof. With 8 participants and 8 professor comments on Proof A, there were 64 opportunities for students to explain their reasoning about the comments. Of these 64 potential explanations, 41 agreed with the experts' consensus interpretation, 18 did not agree (partial agreement was counted as half agreement, half disagreement), one student simply did not give explanations for 2 changes, and 3 explanations were unclear. Three students gave 7 or 8 explanations that aligned with the expert interpretation, suggesting they were very strong students in terms of proof comprehension. The difference between "let" and "for some" was where students most commonly gave explanations that did not align with the expert consensus.

For example, consider the direction to add, "We want to prove" to the beginning of the proof. The expert consensus was that the statement improved the readability of the proof by making clear that the first sentence expressed the goal of the proof. The experts all agreed that the clause the student wrote could have been deleted, but as written, the statement assumes the conclusion of the theorem. Bella explained that she understood the reason for the comment as, "that's just one of the proper ways to start a proof, that from what I've learned, yeah, it's just the way to start a proof." That is, her thinking appears to focus on the form of proofs, that they may begin with "we want to prove" rather than focusing on the function of the statement. This type of form thinking was relatively rare among the participants.

Five participants described the added phrase as entirely focused on clarifying the presentation. For example, Don said, "This is just to me good syntax. It's a way of setting it up to be understand better and to be read more easily." Yet our expert consensus was that the professor intended to point out a lapse in logic, as noted by two participants, including Genevieve, who explained: "I think the professor meant that at the beginning of this [proof] you have this statement which is, uh, it's the claim that you are seeking to prove at the end, and so if you are assuming that every statement in the proof is true, having said it, you ought to have something at the beginning that says we want to prove it." Thus, we suggest that the logic the professor intended to motivate by this comment was not successfully communicated to six of the participants. But at the same time, we agree that the first sentence is unnecessary for a successful proof, i.e., it could be omitted. Note that in terms of the revised proof-productions, it would be impossible to distinguish between a participant who included the phrase "we want to show" because "that's how proofs should start" from one who did so for reader clarity, or from one who understood the logical underpinnings.

The participants also initially showed mixed understanding of the reasons for the "cross out and replace" comments that the professor wrote, and there were differences in the participants' interpretations when the comments were logically necessary as opposed to more stylistic. For example, for comment #2, which specified changing $\boldsymbol{Z}$ to $\boldsymbol{R}$, the experts argued that the student's proof-text did not actually prove the statement for all real numbers x, y, and z, and thus the change was logically necessary. Six of the interviewed participants gave an explanation that approached that of the experts, including Genevieve who said, "there is no reason to believe that x, y, z are in the integers. The theorem never states that they are in the integers. [The theorem states] on the set of real numbers." Don gave a somewhat mixed explanation, initially saying "they [$\boldsymbol{Z}$ and $\boldsymbol{R}$] are both correct but the real numbers are more applicable, in most cases," but later in the interview noting that the theorem specifies that x, y, z are real numbers. In this case, we suggest that the fact that the statement of the theorem invoked both real numbers and integers increased the difficulty of parsing the domains and relating them to the appropriate piece of the definition of the relation, and as a result, Don's explanation for the requested change did not reject the original statement as inappropriate for the proof. Finally, we note that most participants gave a response that described normatively correct logic, yet none of them noted that the original proof attempt did not prove the theorem.

The changes requested in comments 3 and 4 were to change $x - y \in \boldsymbol{Z}$ and $y - z \in \boldsymbol{Z}$ to $x - y = k$ and $y - z = c$. The expert consensus interpretation describing the rationale for this comment was that the professor was attempting to give advice that would help the student revise the proof as written, rather than changing the structure of the subsequent argument. Thus, assuming that the remainder of the argument was to be preserved as much as possible, and because the student was asked to let $x - y = k$ and $y - z = c$ in that argument, the expert consensus was that this change is stylistic. As the statements are written originally, while possibly confusing due to lack of parentheses, they are correct and contribute to the argument, but they induce redundancies in the subsequent algebraic part of the argument.

Charles noted, as did five other participants, that "the student failed to define k and c, in my opinion, as necessary constants. It was a bit ambiguous…" Thus, these participants seemed to recognize that the professor's comment is directed at a logical issue, how constants should be introduced and defined. These participants wanted the constants to be defined earlier and specifically identified as being members of a particular domain.

In reference to comment 5, only two students, Don and Nancy, articulated the distinction between "let" and "for some," and only Genevieve gave a reason in her revised proof for the changes corresponding to comments 3, 4, and 5 by referring to the definition of the relation R. Thus, again we argue that the participants did not fully understand the motivation for the professor's comment.

Comments that did not specify the change

Two comments on Proof A were more general in that they did not ask for specific changes. Comment 6 was "hard to follow" and comment 8 was "proofs should be complete sentences." As for the first of these, the participants agreed with us that the main issue was that the algebraic steps lacked readability, and they succeeded in writing revised proofs that were more readable. In response to comment 8, four participants rewrote the entire proof in complete sentences, including the sequence of algebraic equations, whereas the other four displayed a sequence of algebraic equations. Both are reasonable interpretations of the professor's note and are stylistically acceptable.

Conclusion

The first significant aspect of this study is that it is the first study that describes and analyzes how students interpret and respond to the comments that professors write on proofs. We recognize that this is a single, exploratory case study with only analytical generalizations. Moreover, we note two significant limitations to this study that suggest the need for further work in this area. First, the participants were reading and writing proofs on mathematical topics that most of them had not worked with in some time, possibly since their introduction to proof class.

The second significant limitation is that we asked the participants to interpret comments on proofs they had not written, thus imposing a need to make sense of another student's proof attempt prior to interpreting the comments. These limitations raise questions of how students' ability to interpret comments relates to their proof-writing and proof-comprehension abilities generally. More research is clearly needed to explore these questions, yet without a body of empirical evidence, there is no basis for more theoretical work. Thus this first exploratory study provides direction for those further studies. We note one final limitation: initially the four experts did not always agree on the reasons for the changes. While we could come to a consensus interpretation, there were significant differences in our initial interpretations, which means that different researchers, or a different mix of researchers, might have arrived at a different consensus interpretation of the professor's comments. This is a limitation of the study and suggest an avenue for future research, motivated by Weber's (2014) argument that proof is a clustered concept. We hypothesize that while professors might share instructional goals about proof and use similar notes and language to communicate with students, in reality they may be attempting to convey very different content via the same notes, which has significant implications for students.

The first finding is that when participants revised the written proofs, they made all of the changes requested by the professor and very few changed anything that was not requested. In particular, when the professor specifically indicated a change in the way to write the proof, such as replacing a symbol or adding a phrase, most participants made the exact change that the professor suggested. When the professor's comment was more ambiguous, such as "write in complete sentences," the participants all complied with the request, but interpreted it in different ways, some writing a paragraph proof with equations in sentence form, and some writing a stack of displayed equations but adding text at more strategic points. Similarly, the phrase "hard to follow" prompted some participants to include reasons for each step in the revision while others made more minimal revisions. Only two participants revised a portion of the proof that the professor did not specifically indicate: two rewrote the beginning of the proof and one added a reason that the integers k and c must exist by the definition of the relation R.

The second finding is that the participants often gave explanations for the requested changes that did not align with how experts understood the reason for the changes. The participants were more likely to over-attribute the notion of "sounds better" or "that's what you do," which we interpreted as describing the cultural conventions of proof.

The third finding, which is closely tied to the second, reinforces the claim that participants often fail to understand professors' lectures in the intended way, or even in the way that mathematical experts do. Although the participants made the requested changes, they missed the professor's reasoning because the professor's comments did not convey the difference between logically necessary and stylistic changes, nor did the comments help the participants understand the logically problematic aspects of the original proof.

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Effect of teacher prompts on student proof construction

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Many students have difficulty learning to construct mathematical proofs. In an upper level mathematics course using inquiry based methods, while this is some research on the types of verbal discourse in these courses, there is little, if any, research on teachers' written comments on students' work. This paper presents some very preliminary results from ongoing analysis from Morrow's written prompts on students' rough drafts of proofs for an Abstract Algebra course. The teacher prompts will initially be analyzed through a framework proposed by Blanton & Stylianou (2014) for verbal discourse and the framework will be modified in the course of the analysis. Can we understand if a type of prompt is "better" in some sense in getting students to reflect on their work and refine their proofs? It is anticipated that teacher prompts in the form of transactive questions are more effective in helping students construct proofs.

Key words: Abstract Algebra, transactive questions, mathematical proof, Inquiry-Based Learning

Overview

Many students, who are quite successful at lower level undergraduate mathematics courses where calculations and applications are stressed, have difficulty as they learn to construct mathematical proofs, the focus of upper level mathematics courses (Harel & Sowder, 1998; Weber, 2001; Raman, 2003). Speer, et al. (2010) call for more research in the practice of mathematics teaching at the undergraduate level. As active learning approaches to teaching (as opposed to straight lecturing methods) become more prevalent in undergraduate mathematics classes, teacher skills of listening to students, and responding to their ideas becomes ever more important. There has been a little research on the verbal discourse that occurs in these classrooms that emphasize active learning and inquiry based learning methods (Blanton & Stylianou, 2014; Johnson, 2013; Remillard, 2014). Yet, teachers also interact with students when they (the teachers) comment on students' written work. At least in both authors' classrooms, we comment on student work and we expect students to read our comments and revise their work based on our comments. But little research has been found in mathematics education research (or physics or engineering education) that deals specially with the types of comments mathematics teachers make on written work, or the effect of these comments on revisions of student work. There is a body of research in the Rhetoric and Composition discipline on feedback but its applicability to the writing of proofs seems limited.

One of the active teaching methods used at the collegiate level is Inquiry Based Learning (IBL). In a common form of this, students are given a "list" of definitions and theorems, maybe some problems, that they work through and present to the class. In the Spring 2015, Morrow, in an Abstract Algebra class, had students prepare rough drafts of the proofs they would present the day before the proofs were to be presented in class. These drafts were hand written and submitted directly to Morrow. She had about four hours to provide short written prompts (comments) to students on their rough draft proofs. Copies of the drafts with comments were made and retained by Morrow as the initial artifact of the research. The drafts with comments were then given back to the students. It was the intention that the students would use these prompts to refine their proofs before class meeting. These second drafts of proofs were maintained in a portfolio by the students and collected at the end of the semester. Copies were made of the proofs corresponding to the rough drafts received. In reviewing the proofs in the portfolios, it seemed some of the prompts proved effective, some not so much.

Literature Review and Theoretical Framework

Mathematicians understand and believe that constructing proofs is a creative process that can involve imagery, heuristics and intuition (Raman, 2003). Mathematical creativity has been described as the process that results in insightful solutions to a given problem (Sriraman, 2004). IBL methods attempt to get students to practice this creative craft of mathematics. In an IBL approach to a classroom, teachers must be "active participants in establishing the mathematical path of the classroom community while at the same time allowing students to retain ownership of the mathematics." (Johnson, 2013) This ownership is in part in the form of creating their own proofs of the various statements, an often messy process. The IBL process, as implemented in Morrow's class, requires students to work alone without consulting tutors, fellow students or other resources. It not only calls for students to present their proofs but also calls for the other students to evaluate and validate (or not) the presented proofs. This is often done in whole class discussions.

Blanton & Stylianou (2014) have found that when transactive reasoning/discourse was promoted in whole class discussions, there were positive implications for the students' learning of proof. "Transactive reasoning is characterised by clarification, elaboration, justification, and critique of one's own or [anothers'] reasoning." (Goos, et al. 2002) In the study by Blanton and Stylianou (2014), teachers participated and focused the classroom discourse with various types of utterances. Transactive teacher utterances included requests for critiques, explanations, justifications, clarifications, elaborations, or strategies. They were in the form of questions that asked students for immediate responses requiring transactive reasoning. Other types of teacher utterances were facilitative (often rephrasing a student utterance), didactive (lecture), and directive.

Giving prompts on written work is asynchronous, as opposed engaging in verbal classroom discourse which occurs in real time (synchronously). Yet the goal is much the same, to get students not only to explain and justify where necessary, but to also reflect on their work and refine their proofs. In an inquiry oriented classroom, the asynchronous prompts and synchronous discussions are all part of the sociocultural approach to teaching (Goos, 1999; Goos, et al, 2002; Vygotsky, 1978). Looking at the type of prompts given in the pilot study, and the responses to those prompts lead us to believe that transactive prompts in the form of questions might be the most effective prompts in this context.

Framework

As far as we know, there is not an established framework within which to work, so we will be drawing on the work of Blanton and Stylianou (2014) in their verbal discourse analysis to start to categorize the types of written teacher prompts in the pilot data. We will initially look at the written prompts through a transactive/facilitative/didactive/directive lens. We will also look at the student pre and post work to identify whether the student appears to need to clarify or is on the wrong track entirely, drawing on Vygotsky's notion of zone of proximal development (ZPD). Finally we will see if there appears to be any effect of the type of prompt given. We want to identify the most effective type of written prompt.

Our research question is: Do different types of written prompts affect students' constructions of proofs?

Methods and Very Preliminary Data:

Data collected in the pilot study by the Morrow during the spring 2015 semester will be coded and analyzed by both authors. Approximately 130 rough (and not so rough) drafts of proofs with teacher prompts and their associated final proofs were collected from that spring 2015 class. Additionally, during the fall 2015 semester, the Morrow will collect some additional written proof sketches (with prompts) and final proofs from her introduction to proofs class. Also, during the fall, 2015 semester, Shepherd will collect copies of written proof sketches (with prompts) and final proofs from her Abstract Algebra—Groups class. The initial coding scheme will be based on the Blanton-Stylianou teacher utterances classifications (transactive, facilitative, didactive, directive) for the teacher prompts during whole class discussion. Additionally, a coding for the assumed need the student should address from the pilot study will be jointly developed guided by Vygotsky's ZPD ideas. Is the student on the "right track?" What is needed by the student to progress? Is the student is on a "wrong track" what does the student need to move toward a more productive line of thinking. This two track coding scheme, in addition to being used to analyze the data collected last spring, will be applied to the new data received this fall. Each teacher will initially code her own students, then the other teacher's students. Differences in coding will be discussed and adjustments made so that a consistency in coding can be developed. In addition, there will be an assessment of the students' final proof attempts to decide the effectiveness of the initial teacher prompt.

It is expected that the coding will have to be revised throughout the coding process as possible unanticipated patterns in prompts or responses occur. Very preliminary analysis of 5 examples from Morrow's spring 2015 Abstract Algebra class seems to indicate that transactive questions (utterances that are both transactive and posed as a question) are more effective. Two examples are given below.

In example 1 we see a case where a transactive question prompt was posed and the final proof was essentially correct, along the same idea as the initial sketch (as opposed to being very different and essentially what some other student presented in class), and seems to show the teacher prompt was effective.[1] This teacher prompt is considered a transactive question since it asks the student to clarify a statement.

> Example 1: **Problem 93**. *Suppose that G and H are groups, and that $\phi: G \to H$ is an isomorphism. Prove that if G is abelian, then H is abelian.*
>
> (initial student proof with prompts)
> Let $\phi: G \to H$ be an isomorphism.
> Let G be abelian.
> Then, for all $a, b \in G$, $ab = ba$.
>
> Let $c, d \in H$ such that $\phi(a) = c$
> and $\phi(b) = d$.

[1] The teacher prompts are the drawn and italicized parts.

Then, $cd = \phi(a)\phi(b) = \phi(ab) = \phi(ba) = \phi(b)\phi(a) = dc$.

Therefore, since $cd = dc$, H is abelian.

? Clarify ?
Where does this a come from.
Can you convince me that such an a exists?

(final student proof)
Let $\phi: G \to H$ be an isomorphism.
Let G be abelian.
Then, for all $a, b \in G$, $ab = ba$.

Let $c, d \in H$.
Since ϕ is an isomorphism, it is onto
So there exist $a, b \in G$ such that
$\phi(a) = c$ and $\phi(b) = d$.

Then, $cd = \phi(a)\phi(b) = \phi(ab)$.
Since G is abelian, $ab = ba$ so
$\phi(ab) = \phi(ba) = \phi(b)\phi(a) = dc$.

Thus, since $cd = dc$, H is abelian.

Example 2 shows a teacher prompt where the revisions are not as good as they should have been. Some of the errors are fixed. The first set of prompts, both involving the incorrect operation in the group G, would be considered a transactive question, and effective. The longer (italicized) prompt is more directive, telling the student what needs to be done and is not effective. It would appear the student did not re-engage with the material in a transactive fashion to address showing that any element of the group is a power of a^k.

Example 2: *Suppose $G = \langle a \rangle$, where a has finite stack-height (order) m. Prove that if k is an integer relative prime to m, then a^k is also a generator of G.*

(Initial student proof with prompts)
Let k be an integer relative prime
to m. Then there exists integers
s and t such that

$$sk + mt = 1$$

? operation in G?

Then,.

$$a^{sk} + a^{mt} = a^1$$

(?)

So, $a^{sk} + e = a$

$$a^{sk} = a$$

So $a^{ks} = a$ *aha*
$(a^k)^s = a$

Then, a^k is a generator
for G, since a^k to a power
is equal to the generator a.

Yes – that's the key idea…
but you you still need to turn it
into mathematical proof –
the main part is to show that
<u>IF</u> $x \in G$, then $x = (a^k)^n$ for some
integer n.

(final student proof)
Let $a \in G$.
Then $a^{sk+mt} = a$, note $a = a^1$.

So, $a^{sk} \cdot a^{mt} = a$
Then $a^{sk} \cdot (a^m)^t = a$
So, $a^{sk} \cdot (e)^t = a$
And $a^{sk} \cdot e = a$
So, $a^{sk} = a^1$, note that $sk = ks$.

Then $a^{ks} = a$

So $a = (a^k)^s$

Then, a^k is a generator for G
since a^k raised to a power is
equal to the generator a.

Preliminary results will be presented at the 2016 RUME Conference, and further questions and research will be designed for the spring 2016 semester so that a more complete framework can be constructed. It is anticipated that the data will show that teacher prompts in the form of transactive questions are more effective in helping students construct proofs.

Questions for discussion

1. Is the transactive, facilitative, didactive, directive scheme used the most appropriate for this type of analysis?
2. If indeed transactive questions are more effective, how can we train teachers and TAs in their use?

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Students' symmetric ability in relation to their use and preference for symmetry heuristics in problem solving

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Advanced mathematical problem solving is marked by efficient and fluid use of multiple solution strategies. Symmetric arguments are apt heuristics and eminently useful in mathematics and science fields. Research suggests that mathematics proficiency is correlated with spatial reasoning. We define symmetric ability as fluency with mentally visualizing, manipulating, and making comparisons among 2D objects under rotation and reflection. We hypothesize that symmetric ability is a distinct sub-ability of spatial reasoning which is more accessible to students due to inherent cultural biases for symmetric balance. Do students with varying levels of symmetric ability use or prefer symmetric arguments in problem solving? How does symmetric ability relate to insight in problem solving? Results from a pilot study indicate that, among undergraduates, there is high variation in symmetric ability. Further, students with higher symmetric ability tend towards more positive attitudes about mathematics. Methods, future research, and implications are discussed.

Key words: [Symmetry, Problem Solving, Heuristics, Cognitive Spatial Reasoning]

Background and Research Questions

Schoenfeld (1987) demonstrates that metacognitively aware problem solvers read, analyze, explore, plan, implement, and verify during their problem solving process. The "analyze" phase involves creating hypotheses of how a problem can be solved. Symmetry is often an easy path in problem solving and broadly appeals to students and mathematicians alike as a major convergence point of mathematics and beauty (Drefus, T., Eisenberg, 1990; Goldin & McClintock, 1980).

A typical American geometry curriculum is capped around age 16 with construction based proof geometry and trigonometry. Given this typical educational history it is unsurprising that undergraduate students have trouble with geometric transformations, including symmetrical relationships (Rizzo, 2013) a deficit which the CCSSM has addressed by including geometric transformations and symmetry of functions in its content suggestions (Initiative., 2011). Beyond this, however, the utility of symmetry as heuristic has application in multivariate calculus, organic chemistry, applied engineering and design, and physics. Expanding the bounds of a student's ability to use and conceive of multiple solution strategies serves to increase this "analysis" phase of problem-solving. Problem-solving and critical thinking are the main pillars of reformed K-12 curricula (Initiative., 2012; Mathematics., 2000) and are pervasive in PCAST reports (Holdren & Lander, 2012). It is the goal of this study to characterize the relationship between students' ability in and application of symmetry to problem solving. This will serve as a research basis for the continued curricular expansion of treatments of symmetry within the geometric transformations and provide insight into how students currently think about symmetry as a heuristic. Future research might seek to find out how one's symmetric ability can be built upon in challenging problem solving situations.

Previous research with in-service teachers shows that they generally do not use symmetric solution strategies, and are skeptical of the mathematical validity or sufficiency of such solution strategies when working on multiple solution tasks (Leikin, Berman, & Zaslavsky, 2000; Leikin, 2003). Similarly they believe that conventional solution strategies (relying on calculus, algebra, or geometric definitions) are more trustworthy and that they have more confidence in teaching them. This relationship, between ability to think symmetrically, the insightful recognition of when to use symmetric arguments, and preference for/against symmetric arguments among worked out solutions has not been investigated with students. Research indicates that affective/attitudinal factors greatly influence mathematical achievement (E. Fennema & Sherman, 1977) as well as selective processes. Meaning that one's attitudes about mathematics influence one's mathematics performance as well as one's decisions having to do with mathematics. This selection was studied as it related to career choice (Betz & Hackett, 1983). We propose to look at this in relation to problem scale mathematic preferences (ranking of solution strategies). In response to the research background and instructional significance presented here, this research project is designed to answer the following research questions:

1. How do students' attitudes about mathematics relate to their ability with symmetry?
2. How does students' symmetric ability relate to their use and preference for symmetric arguments in problem-solving?

Research Methods

In this section I describe pilot data that has already been collected and describe plans for future data collection. We have developed an instrument to measure students' symmetric ability defined as: a student's ability to mentally visualize, manipulate, and make comparisons among 2D geometric objects and as applied to cultural material in terms of reflectional and rotational symmetry. Our definition mimicks those that Olkun (2003) summarizes of, spatial ability in reasoning, relations, and vizualizations with an added cultural component influenced by research on ethnomathematics (Abas, 2004; D'Ambrosio, 2001; Eglash, Bennett, O'Donnell, Jennings, & Cintorino, 2006). Sample items can be seen in Figures 1-4.

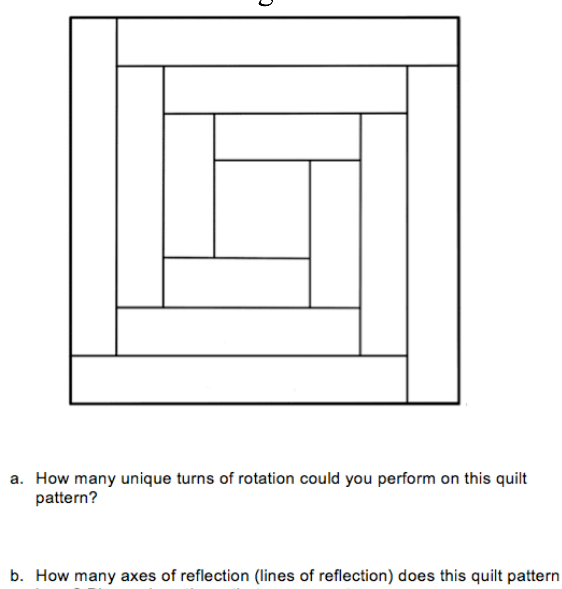


Figure 1. Cultural item from symmetric ability survey, one of five.

Cultural items were developed by the researchers with consideration of D'Ambrosio's (2001) findings on ethnomathematics: that using local cultural material in curricula, in this case flooring and quilt patterns, symbols and logos, and architecture, increases student engagement. Geometric 2D items were chosen from the literature on spatial reasoning (Ekstrom, French, Harman, & Derman, 1976; French, Ekstrom, & Price, 1963), and attitudinal items were drawn from the Fennema-Sherman confidence, beliefs, and effectance subscales (Fennema & Sherman, 1976). In error, only partial subscales were used in this pilot, future research will use the entirety of the confidence and effectance subscales.

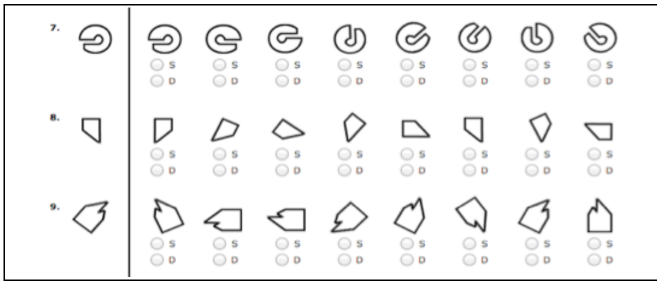
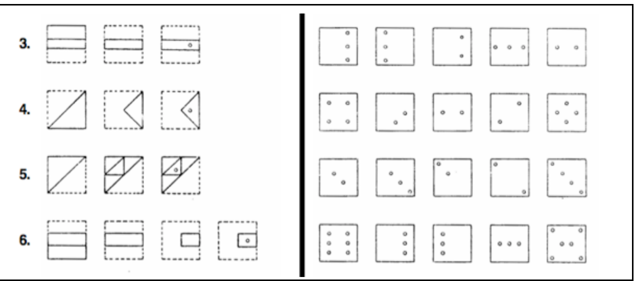



Figure 2. (left) Geometric 2D items from symmetric ability survey, tests rotational ability. Students mark whether the eight images are rotationally symmetric (S) or different (D) from the leftmost image. Three of ten.

Figure 3. (right) Geometric 2D items from symmetric ability survey, tests reflectional ability. Images to the left of the bold line show a series of folds and hole punches done on a square piece of paper. Students choose the image to the right that corresponds to the correct pattern of holes when the paper is unfolded in place. Four of ten.

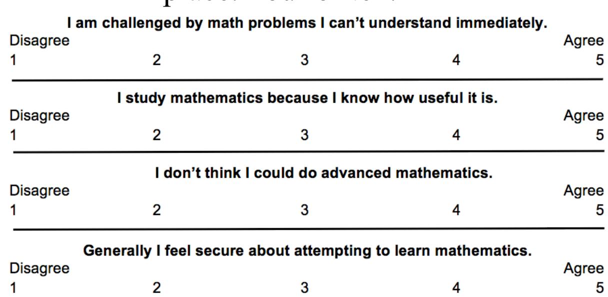
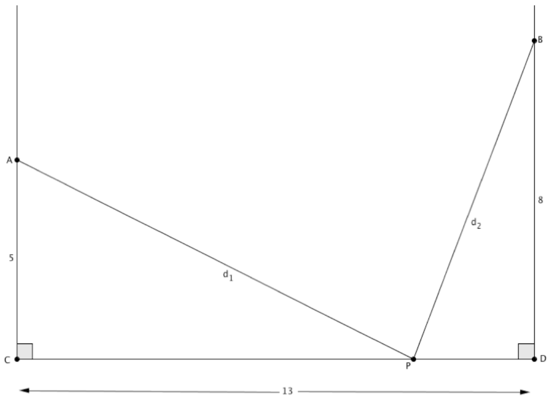


Figure 4. Attitudinal items from symmetric ability survey. Four of eleven.

Cultural and geometric problems were preceded by instructions with worked examples; students were encouraged to ask for clarification of these instructions as necessary. This timed survey was administered to a small (n=11) pilot sample of students in an introductory general mathematics course at a large northeastern university. Post survey interviews are in process to establish survey validity of the researcher-developed cultural items.

Primary data collection is set for the autumn of 2015. The symmetric ability survey (Fig. 1-4) will be administered to n~100 students enrolled in introductory calculus or more advanced math courses. A subsample, n~20, will be selected to take part in a think-aloud interview centering on multiple solution tasks. We will select students to ensure high variation of symmetric ability and attitude within the subpopulation. Example multiple solution tasks that assess use of symmetric heuristics, open response format, and preference for symmetric heuristic, ranking format, can be seen in figures 5-6. Sample prompts drawn from the cognitive interview protocol can be seen in figure 7. Students must have some access to ideas from calculus to complete preference questions; unfortunately calculus students were not available for the pilot. While students work through two problems of each type, use (Fig. 5) and preference (Fig. 6), prompts like those in Figure 7 will be used to elicit student thinking and understanding. Qualitative and grounded theory methods will be used to analyze audio/video recordings of these problem-solving interviews.

A TV game show contestant must run from her initial starting position A to a long table CD that is covered with chocolate cream pies. The table is 13m long and is 5m away from A. After picking up a pie from the table, the contestant is to race to her partner, who is 8m from the table at B, and give him a faceful of pie as fast as possible. What is the shortest distance in which she can accomplish this feat?



Which solution to the following system of equations is best?

$$\begin{cases} 3x + 2y + z = 30 \\ x + 3y + 2z = 30 \\ 2x + y + 3z = 30 \end{cases}$$

1. To solve a system of three equations with three variables we can solve by pairs of equations. Taking the first and second equations together and the second and third equations together we can remove *z* from both by multiplying out the *z* coefficient and subtracting equations as shown.

$$\begin{array}{ll} 3x + 2y + z = 30 & x + 3y + 2z = 30 \\ x + 3y + 2z = 30 & 2x + y + 3z = 30 \\ & \\ 2(3x + 2y + z) = (30)2 & 3(x + 3y + 2z) = (30)3 \\ 1(x + 3y + 2z) = (30)1 & 2(2x + y + 3z) = (30)2 \\ & \\ 6x + 4y + 2z = 60 & 4x + 2y + 6z = 60 \\ -\ 1x + 3y + 2z = 30 & -3x + 9y + 6z = 90 \\ \hline 5x + y = 60 & x - 7y = -30 \end{array}$$

We now have a system of two equations in two variables. Solving for *x* or *y* in either and substituting will yield an answer of *x* = 5, *y* = 5, and *z* = 5.

2. Notice that the coefficients of x,y,z rotate through the equations: {3,2,1} {1,3,2} {2,1,3}. For all of these permutations to equal 30, *x* = *y* = *z*. The first equation can be rewritten:

$$3x + 2x + x = 30$$
$$6x = 30$$

Therefore, *x* = 5, *y* = 5, and *z* = 5.

Figure 5. (left) Students may respond to this open question with several solution strategies. Expected solutions include parameterization and minimization, guess and check, and reflection of B about CD to form the straight path AB' and application of the Pythagorean theorem.

Figure 6. (right) Students respond by ranking their preferred solution strategy.

Use:
1. Can you explain how you decided to try solving the problem this way?
2. Can you think of any other ways to solve this problem?
3. How might a mathematician approach this problem?

Preference:
1. Can you rephrase this solution strategy in your own words?
2. How are you defining "best" in this question?
3. Can you think of any other ways to solve this problem?

Figure 7. Sample question prompts from the problem solving interview protocol.

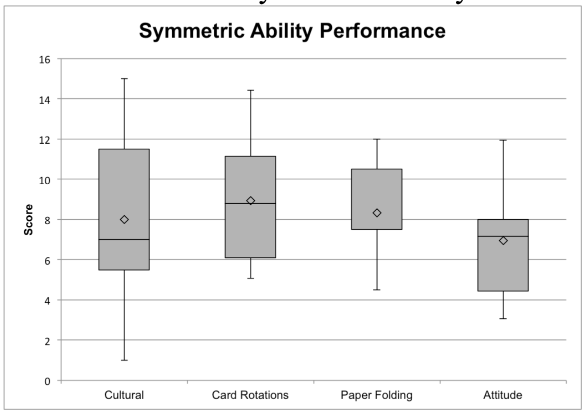
Data Analysis

A rubric was developed to establish this survey as a quantitative measure of symmetric ability and to provide insight into any relationships between cultural symmetric ability, geometric symmetric ability, and mathematics attitude. Cultural symmetry questions (e.g., Fig. 1) were scored out of three: one point for the correct number of rotations, one point for the correct number of reflectional axes, and one point for the correct placement of axes on the image. In cases where students responded with a valid answer (e.g., infinite rotations) but which were incorrect (because the question asked about only one 360 degree rotation), points were awarded when there was consistency of response across questions. A total of fifteen points were possible in this section. Card rotations questions (Fig. 2) and Paper folding questions (Fig. 3) were scored following the guidelines provided by the distributors (Ekstrom et al., 1976). Likert scale attitudinal data (Fig. 4) were scored using the reverse coding method (Field, 2009).

Future data analysis will: assure sampling validity by comparing the performance of the survey population to normative performance on the geometric tasks, search for trends within the card rotations and paper folding test having to do with angular difference (Cooper, 1975) and fold complexity, and establish inter-rater reliability.

Preliminary Results

In the initial sample population we see high variation in symmetric ability in three of the four measures (Fig. 8), indicating that this instrument can parse. Further, there seems to be a positive correlation between symmetric ability and mathematics attitude (Fig. 9).



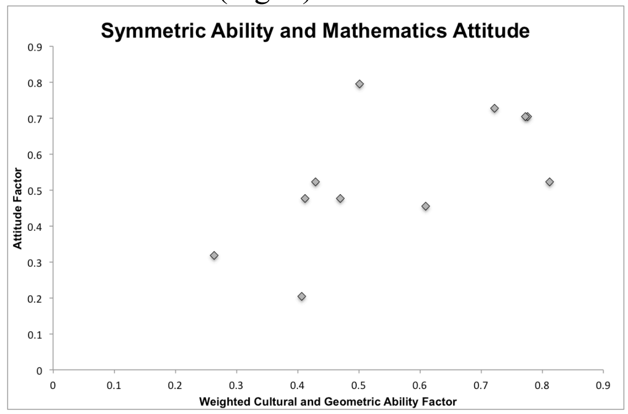


Figure 8. Summary statistics for each section of the symmetric ability survey. Note: scores on the geometric and attitude tasks have been scaled to 15 for comparison purposes.

Figure 9. The data appear to show that as symmetric ability increases (an equal weight given to cultural, card rotations, and paper folding items), so does mathematics attitude depicted as percent ideal response.

Anecdotal and preliminary interview analysis suggests that the cultural items are being interpreted as intended.

Implications and Future Inquiry

Based on preliminary findings it seems that there is a broad range of symmetric abilities among this population of undergraduate students. Further, that having higher levels of symmetric ability may correlate with more positive mathematics attitude. These results suggest that differences may exist between students with high or low symmetric ability. Future research plans include: expanding rigor and sample size of the symmetric ability instrument, and investigating the intersection of symmetric ability with problem solving through interviews. Possible interview findings include: high symmetric ability students prefer but do not natively use symmetric heuristics, low symmetric ability students do not prefer and do not natively use symmetric heuristics, or any combination therein. Further, this line of inquiry will provide a characterization of how students think about symmetry as a heuristic.

Discussion Questions

1. What are your thoughts on the interview tasks? Can you think of other useful tasks to consider?
2. Have you encountered students with high symmetric ability in your own teaching? Did these students have an advantage in your mind?

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Impact of Advanced Mathematical Knowledge on the Teaching and Learning of Secondary Mathematics

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There has been a longstanding debate in the mathematics and mathematics education communities concerning the knowledge secondary mathematics teachers need to provide effective instruction. Central to this debate is what content knowledge secondary teachers should have in order to communicate mathematics to their students, assess student thinking, and make curricular and instructional decisions. Many educators believe that mathematics teachers should have a strong mathematical foundation along with the knowledge of how advanced mathematics is connected to secondary mathematics (Papick, 2011). But according to others, more mathematics preparation does not necessarily improve instruction (Darling-Hammond, 2000; Monk, 1994). Therefore, it is important that, as a field, we investigate the nature of the present mathematics content courses offered (and required) of prospective secondary mathematics teachers to gain a better understanding of which concepts and topics positively impact teachers' instructional practice.

This exploratory study aims to advance our understanding of the nature of mathematics offered to prospective mathematics teachers by looking at mathematical connections. We investigate how in-service and pre-service teachers make connections between tertiary and secondary mathematics as well as if and how the understanding of connections influences teachers' thoughts about teaching and learning mathematics. The research questions for this study are as follows:

1) How does exposure to and instruction in tertiary mathematics impact the way teachers understand secondary mathematics?
2) How does exposure to and instruction in tertiary mathematics impact the way teachers approach secondary classroom instruction?

Conceptual Framework

We consider connections between tertiary and secondary mathematics to be ones that encompass both mathematical content and ways of thinking about and engaging with that content. To better understand these connections, we draw on three areas of research: mathematical knowledge for teaching (e.g., Ball, Thames, & Phelps, 2008), mathematical practices (e.g., Council of Chief State School Officers [CCSSO], 2010; RAND, 2003) and habits of mind (e.g., Cuoco, Goldenberg, & Mark, 1996). Mathematical knowledge for teaching (MKT) (Ball et al., 2008) incorporates both subject-matter knowledge and pedagogical content knowledge. One component in the larger domain of subject-matter knowledge is called *horizon content knowledge*. We believe this particular aspect of MKT is a potentially useful idea for thinking about what advanced content knowledge prospective mathematics teachers at the secondary level need for their teaching.

To expand the notion of subject-matter knowledge, it is also useful for us to consider what secondary teachers need to know beyond content and concepts and to encompass mathematical habits of mind (e.g., Cuoco et al., 1996) and engagement in mathematical practices (e.g., CCSS, 2010). These include looking for patterns, making conjectures, attending to precision, and connecting representations. Such habits and practices in mathematical thinking and learning extend across content areas and levels of mathematical study. Therefore, as we consider how advanced mathematical content impacts teachers' knowledge and understanding of the teaching and learning of secondary mathematics, it is important for us to consider habits and practices that may also influence how tertiary ideas are learned and interpreted for teaching.

We drew on these ideas to develop a unit for practicing middle school mathematics teachers that highlights a particular connection between advanced mathematics and secondary mathematics. This unit, described below, seeks to not only show how connections can be purely mathematical in nature and relate directly to subject-matter knowledge, but to also illustrate how connections can go beyond knowledge *of* mathematics and encompass engagement *in* mathematics through the lens of mathematical habits of mind and mathematical practices.

Research Methodology

Participants are 14 students in a master's level mathematics education course. Of this group, one is a special education teacher, two are pre-service teachers, and eleven are in-service teachers with one to fifteen years teaching experience.

There will be three data sources: a mathematics questionnaire, an instructional unit, and interviews. The researchers will first give participants a mathematical content knowledge questionnaire; the purpose of which is to gain insight into the level of mathematical content knowledge that the participants possess as well as their thoughts on the impact of tertiary mathematics on instruction. Responses to this questionnaire will also be considered when analyzing the video and audio data collected during the instructional unit. After completing the questionnaire, one researcher will teach one two-and-half hour lesson. The lesson will be filmed and all written artifacts will be collected. During the lesson, participants will engage in an instructional unit on solving equations; the purpose of which is to challenge teachers' understanding of procedures used for solving equations and to consider how attention to the algebraic structures and their properties may inform procedures and solutions. Following the classroom lesson, approximately four volunteers will be asked to participate in a follow-up interview; the purpose of which is to clarify ideas discussed in class and to probe students' thinking on the impact of tertiary knowledge on the understanding of secondary mathematics and instruction.

To analyze the data, the researchers will use initial coding (Saldana, 2009) of the video data transcripts to split the data individually coded segments. The researchers will then use theoretical coding as a way to "constantly compare, reorganize, or "focus" the codes into categories" (Saldana, 2009, p. 42). The goal is to code the data based on thematic or conceptual similarities with respect to how the participants make connections between tertiary and secondary mathematics as well as if and how the understanding of connections influences their thoughts about teaching and learning mathematics.

Applications and Implications

In this exploratory study, we investigate questions regarding mathematics teachers' content knowledge and preparation. In particular, we would like to better understand which concepts might positively impact teachers' instructional practice. We do this by considering how in-service and pre-service teachers make connections between tertiary and secondary mathematics as well as if and how the understanding of connections influences teachers' thoughts about teaching and learning mathematics. While many researchers of mathematics and mathematics education may intuitively understand how secondary mathematics teachers' deep knowledge of mathematics is related to the ability to be an effective mathematics instructors in secondary schools, the field still needs to understand how secondary teachers use their tertiary mathematics instruction in teaching secondary mathematics, which can lead to a better sense of the kinds of mathematics courses that can provide teachers with the content knowledge they need to make best use of these connections.

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Transforming graduate students' meanings for average rate of change

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This report offers a brief conceptual analysis of average rate of change (AROC) and shares evidence that even mathematically sophisticated mathematics graduate students struggle to speak fluently about AROC. We offer data from clinical interviews with graduate teaching assistants who participated in at least one semester of a professional development intervention designed to support mathematics graduate students in developing deep and connected meanings of key ideas of precalculus level mathematics as part of a broader intervention to support mathematics graduate students in teaching ideas of precalculus mathematics meaningfully to students. The results revealed that the post-intervention graduate students describe AROC more conceptually than their pre-intervention counterparts, but many still struggle to verbalize a meaning for AROC beyond average speed, a geometric interpretation based on the slope of a secant line, or a computation.

Key words: graduate student teaching assistant, average rate of change, precalculus

It may seem natural to assume that a graduate student in mathematics possesses strong meanings of foundational mathematics ideas because of their extensive experience studying mathematics. However, Speer (2008) and colleagues (Speer, Gutmann, & Murphy, 2005) reported that completion of more advanced mathematics courses does not necessarily improve a teacher's understandings and teaching practices. Other studies have advocated that teachers need productive meanings of the ideas they intend to teach (Carlson & Oehrtman, 2009; Moore, et al., 2011). Thompson, Carlson & Silverman (2007) claimed that:

> If a teacher's conceptual structures comprise disconnected facts and procedures, their instruction is likely to focus on disconnected facts and procedures. In contrast, if a teacher's conceptual structures comprise a web of mathematical ideas and compatible ways of thinking, it will at least be possible that she attempts to develop these same conceptual structures in her students. We believe that it is mathematical understandings of the latter type that serve as a necessary condition for teachers to teach for students' high-quality understanding (pp. 416-417).

In a recent study, Teuscher, Moore & Carlson (2015) report that a teacher's mathematical meanings provide a lens through which a teacher makes sense of student thinking. The teacher's model of students' thinking influences her subsequent instructional actions, including the nature of her questions, her questioning patterns, and the quality of the discussion she leads.

In a context of an intervention to support mathematics graduate students to act in productive pedagogical ways, we engaged graduate students preparing to teach precalculus at the college level in completing tasks aimed at developing their meanings of key ideas of precalculus level mathematics that are foundational for learning calculus. This study investigated mathematics graduate students' meanings of average rate of change (AROC) for the purpose of understanding graduate students' pre-intervention meanings of AROC and the degree to which our interventions impacted their meanings of this idea.

We describe what research has reported to be a productive meaning for the idea of AROC (Thompson, 1994). We follow this with a description of our intervention for shifting graduate mathematics students' meaning for AROC and conclude by reporting on results that point to the distinctions between pre- and post-intervention participants' meanings for AROC, and reveal the varied fluency among participants in *speaking with meaning* about AROC when probed in a clinical interview setting (Clark, Moore, & Carlson, 2008).

Theoretical Framework

We view an individual's *expressed meaning of an idea* as the spontaneous utterances that an individual conveys about an idea. From these utterances we can make inferences about how an individual has organized her experiences with the idea. The meaning held by an individual is then the organization of the individual's experiences with an idea, also referred to as a scheme. It is through repeated reasoning and reconstruction that an individual constructs schemes to organize experiences in an internally consistent way (Piaget & Garcia, 1991; Thompson, 2013; Thompson, Carlson, Byerley, & Hatfield, 2013). For example, an individual's meaning for the idea of *average rate of change* might consist of the calculation for the slope of a secant line, or simply $\Delta y/\Delta x$. An individual who has committed to memory that the average rate of change is the slope of a secant line does not possess the same meaning as someone who sees the slope of a secant line as the constant rate of change that yields the same change in the dependent quantity (as some original non-linear relationship) over the interval of the independent quantity that is of interest. These two individuals hold different meanings for the same idea, and the consequences of such differences can be profound.

An individual's meaning for the idea of *average rate of change* can be further developed through reflection, which occurs when she is faced with perturbations to her current meanings for average rate of change (Dewey, 1910).

A productive meaning for the idea of average rate of change

Constructing a rich meaning of *average rate of change* entails conceptualizing a hypothetical relationship between two varying quantities in a dynamic situation. Given a relationship between the independent quantity A and the dependent quantity B, and a fixed interval of measure of quantity A, the average rate of change of quantity B with respect to quantity A is the constant rate of change that yields the same change in quantity B as the original relationship over the given interval. In order to understand this complex idea meaningfully, an individual must first conceptualize the idea of *quantity* as a measurable attribute of an object (e.g., an airplane's height in feet above the ground, number of minutes elapsed since noon). Next, provided a situation in which two quantities vary in tandem, an individual must develop an understanding for what it means to describe the rate of change of one quantity relative to the other. Namely, the individual must conceptualize the multiplicative comparison of changes in the two quantities (the change in the output quantity is always some constant times as large as the change in the input quantity). In the special case that the relative size of changes in one quantity relative to the other remains constant, we say the quantities vary with a constant rate of change (CROC). Individuals with a robust meaning will draw connections between AROC and CROC and view those two connected ideas as a means for approximating values of varying quantities in dynamic scenarios.

The Intervention

The graduate students initially participated in a 2-3 day workshop in which they completed mathematical tasks that were designed and sequenced to support graduate students in constructing a productive meaning for the idea of *average rate of change*[1]. The graduate

[1] The idea of *average rate of change* is the culminating idea of the first instructional unit of the precalculus level course the graduate students will be teaching during the upcoming semester.

students confronted problems and questions designed to perturb their expressed meanings for AROC. The intent was to prompt reflection and subsequent shifts in their meanings of this idea. Concurrent with teaching the course during the fall or spring semesters, the graduate students attended weekly 90-minute seminars. The main goals of the weekly seminars were to support the graduate students in developing more productive meanings of the key ideas to be taught during the upcoming week, and to support them in clearly explaining their meanings for those ideas to others. As part of the work towards these goals, each of the graduate students decided on a lesson implementation plan detailing how they would engage their students in achieving their learning goals for that week prior to each seminar meeting. Participants further prepared mini-presentations of the material to practice talking about difficult or novel ideas. When preparing, participants used the Pathways Precalculus curriculum (Carlson, Oehrtman, & Moore, 2015), a research-based curriculum that incorporates student thinking and scaffolds the development of key ideas. The Pathways curriculum included materials designed to advance participants' understanding of AROC when preparing to use the student materials in their own classrooms.

Methods

We collected data from mathematics graduate students and instructors at three large, public, PhD-granting universities in the United States. Participants' teaching experience varied between zero and 11 years, at both the K-12 and tertiary level. We conducted semi-structured clinical interviews with 21 graduate teaching assistants, all of whom had at least one semester experience teaching the Pathways Precalculus course as lead instructor or recitation leader (Clement, 2000). Interviews addressed multiple issues, ranging from perceived shifts in beliefs about the roles of students and teachers to understandings of mathematical ideas to descriptions of teaching practices and goals. The lead author conducted these interviews, recording each using both a video camera and Livescribe technology to capture audio-matched written responses to sample teaching scenarios provided during the interviews. Interviews lasted 1-2 hours, and were transcribed and coded by three members of the research team. Members of our team analyzed videos in pairs at first, identifying themes of interest relative to our conception of a productive expressed meaning for AROC before working individually to continue coding and reconvening as a group to discuss our findings (Strauss & Corbin, 1990).

Results

We first share data that reveals the common meanings that the graduate students held for the idea of AROC when entering the program. We then share excerpts from clinical interviews with experienced participants to reveal the varied fluency participants demonstrated when describing the idea of AROC. The results show that the meanings for AROC conveyed by the novice and experienced groups are, in fact, categorically different; however, not all experienced participants shifted to speak fluently about the idea of AROC.

Pre-Intervention Meanings for AROC

As a warm-up activity for the start of a Summer 2015 teaching assistant workshop, we asked seven math graduate students to describe the meaning of "average rate of change." Each participant's response is recorded in Figure 1. Their responses align with the authors' prior experiences with both students and teachers at the secondary and tertiary levels; most of the participants provided geometric interpretations based on imagining a secant line between two points on the graph of a function. In particular, we see that Alan[2] described AROC both computationally (i.e., $\Delta y/\Delta x$) and geometrically as a line, instead of highlighting the key attribute of the line—its slope. Brian did mention slope, though he did so while conveying the idea that slope is an amount of change in the dependent quantity for each unit change in the independent quantity, a somewhat restrictive meaning for slope as it fails to support reasoning about variation when changes in the independent quantity have magnitude other than 1. Cassie and Diane spoke explicitly about steepness, a visual aspect of a graph that is simultaneously restricted to the Cartesian coordinate system and, in that setting, is potentially misleading when the coordinate axes do not have the same scale. Edgar provided two equivalent descriptions of how to *compute* the AROC over a given interval, but did not communicate what the result of that computation would *represent*. Greg commented on the "predictive" quality of AROC, making him the only participant to explicitly highlight the idea that AROC provides an alternate means for characterizing how two quantities change together. This thinking, however, is missing many elements of what we have previously characterized as a productive meaning for AROC.

Responses to the question: What does "average rate of change" mean to you?	
Alan:	A straight line between two points on a graph
Brian:	As one variable changes for every one unit, how much is the other variable changing. Slope.
Cassie:	Steepness of a graph, like how steep or how flat it is.
Diane:	Steepness of a graph. […] Uh, I don't have actual words.[…] Slope or derivative.
Edgar:	Rate of change over an interval
Frank:	The amount the dependent variable changes divided by the amount the independent variable changes. Delta y divided by delta x.
Greg:	I lost all the words…It's the predictive effect of changing one variable and the amount and how it's going to affect the other variable. One quantity affecting change in another quantity.

Figure 1. Pre-intervention participant descriptions of AROC

While most of the responses were accurate statements about AROC, the participants' expressed meanings were predominantly geometric or computational; moreover, only one of the seven participants spontaneously hinted at the idea that the AROC serves as a tool of approximation for rates of change within dynamic scenarios.

Post-Intervention Meanings for AROC

We have analyzed 11 clinical interviews with participants who experienced at least one summer workshop and one semester of our intervention. In contrast to the predominantly geometric and computational descriptions of AROC from our pre-intervention participants, 7 of the 11 participants attempted to describe a conceptual meaning for AROC. These descriptions can be classified as: the productive, general meaning described in our theoretical framework; a special case of that meaning for average speed; or, in one instance, a distinct interpretation the participant called "linearization." The other four participants offered explanations that fall into the last four categories described in Table 1.

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[2] Psuedonyms are used throughout the reporting to protect the identity of participants.

Table 1. Experienced participant descriptions of AROC

Expressed Meaning Category	*Sample Excerpts from Clinical Interviews*	*Number of Instances**
Productive – General	[Students] have to understand constant rate of change because the average rate of change is the constant rate of change someone else would have to go, and I'm talking about average speed now, to achieve the same change in output for a given change in input. So, if you don't have meaning for constant rate of change, well, then average rate of change is just this number.	4
Average Speed	[AROC] is a constant rate of change for that specific time and distance, or uh, you know how I mean…	6
Conceptual Other	I would like to say linearization. Right, this idea of approximating something that isn't linear in a linear fashion.	1
Computational	… this final minus initial over the outputs and this final minus initial over the inputs and that's a rate.	2
Geometric	Average rate of change is the constant rate of change to go between two points.	2
Incorrect	I want my students to understand that constant rate of change is a special case, I guess of average rate of change. It's this special case that exists when the corresponding changes in our two quantities are proportional.	2
None		1

* Total exceeds 11 because some interviewees conveyed more than one expressed meaning.

The excerpts in Table 1 highlight the fact that the impact of the intervention on participants is far from uniform. One participant failed to provide a clear statement of a meaning for AROC as he talked around the issue for 14 minutes during his interview. We found this surprising in light of the fact that this participant had taught the idea of AROC from conceptually oriented materials for the past 4 semesters. Two of the six participants who mentioned average speed did not convey a meaning for AROC beyond the context of comparing distance and time. The sample excerpt for an incorrect meaning suggests that the participant developed a meaning for AROC linked to CROC in a non-standard way; conventional treatment of the two ideas typically describes AROC as a CROC approximation instead of viewing CROC as a special case of AROC. Yet another participant proclaimed, "I will forever think of average rate of change as the slope of the secant line." The fact that these participants did not immediately produce the meaning for AROC supported both by the intervention and the curriculum materials points to the complexity of the idea of AROC and the difficulty that even graduate students had in modifying their strongly held geometric and computational images of the idea of AROC to a more robust scheme with connections that are rooted in a conceptual meaning that can be expressed in multiple representational contexts.

Nonetheless, many participants' expressed meanings did align with our productive meaning for AROC, even if only as the special case of average speed, focusing on a relationship between varying quantities. During the intervention, leaders encouraged participants to "speak with meaning" as a tool to support their students in reasoning about quantities. To "speak with meaning" means to use appropriate language, describe the underlying meanings of specialized vocabulary (e.g., explains "proportional" instead of just using that word), and offer multiple ways of explaining a concept. Consider the "Productive Meaning" excerpt from Table 1 that was conveyed by a participant with 3 years of experience with the intervention, first as a participant and more recently as a leader. Not only did she express a productive meaning for AROC, using appropriate descriptions that highlighted *changes* in quantities as opposed to *values* of quantities, she further made explicit the connection between CROC and AROC and described the mental imagery she hopes her students develop. She later elaborated the importance of students imagining a second object or scenario that displays a CROC relationship that would yield the same change in output over the given interval of the input quantity.

Similarly, Hannah stumbled slightly, but ultimately described AROC in terms of changes in quantities, as seen in the following:

> *I think one needs to understand that average rate of change means that [...] two quantities are varying but not necessarily at a constant rate of change—like the output quantity can, umm, not have a constant factor with respect to the input quantity. But the average rate of change of that relationship would be like if the…if there was a constant rate of change, the same output would be covered for a given amount of input. I think the easiest one for students to understand with that is like the example of like distance and speed. So if you're driving your car at a constant speed and I am like stopping and going and slowing down and speeding up, we will cover the same amount of distance in the same amount of time. And your—the constant rate that you go—is the same with my average rate. But I find that with that example it's really [...] hard for students to talk about things not in terms of time. I also find that using the word "average" is confusing to students.*

She continued to reflect on a driving context as a familiar example to support students' reasoning about AROC, but demonstrated an awareness of student thinking by highlighting that particular example as potentially problematic for students to generalize beyond contexts dependent on time. She also expressed an awareness of student difficulties with the multiple meanings of the word "average" appearing in the phrase *average rate of change*. Interestingly, though Hannah demonstrated a relatively high level of fluency in speaking with meaning about AROC, she pointed out that this particular idea is usually difficult for her to discuss with her students, saying:

> *I was struggling with it, and [...] it's just hard to word it in terms of input and output and varying quantities without having a concrete example. And so, to me I'm not even sure that [students are] not getting it so much as that they're not able to articulate it.*

Discussion

The vast majority of participants held weak meanings for the idea of AROC at the beginning of the study. Our findings further revealed that our interventions were only moderately effective in supporting the graduate students to acquire productive meanings for the idea of AROC. The initial impoverished meanings expressed by graduate students were widespread across all three institutions, suggesting that that the issues involved in shifting graduate students' meanings are not unique and require further investigation. These findings challenge the assumption that graduate students in mathematics have strong meanings of fundamental ideas of mathematics. Failure to act on this faulty assumption may have severe consequences for improving the predominantly procedural focus that exists in many introductory undergraduate courses in colleges and universities across the US.

Graduate mathematics students who hold a meaning for AROC that is strictly geometric (i.e., slope of secant line) will be unable to support their students in developing a quantitative meaning for AROC that could be leveraged to build ideas of accumulation from rate of change foundational to applications of calculus. Moreover, though several of the participants commented on how the Pathways materials exposed them to new ways of thinking about the mathematical ideas, these new ways of thinking do not necessarily translate to what the participants have as goals for their students' learning.

We also note that, though not described here, our experiences in working with the graduate students during the interventions produced encouraging anecdotal evidence that the opportunity to reconceptualize fundamental ideas may have a lasting impact on their image of what effective mathematics teaching entails. This leaves us optimistic that ours and other similar efforts might motivate mathematicians to engage in work to make undergraduate mathematics instruction more meaningful for students.

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Why students cannot execute their own global plans

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This qualitative study investigates undergraduate students' mathematical problem solving processes in time restraint situations by analyzing their global plans. The students in three undergraduate courses were asked to write their global plans before they started to solve problems in in-class quizzes and exams. Students' execution behaviors and their success or failure in producing correct solutions were explored by analyzing their solutions. Only student work that used clear and valid plans was analyzed, using qualitative techniques to determine students' success (or failure) in producing correct solutions, and also to identify the factors that were hindering their effort. Many categories of execution errors were identified, and how those errors affected students' efforts will be discussed. This study employed Garofalo and Lester's (1985), and also Schoenfeld's (2010) frameworks to understand students' problem solving behaviors. Their frameworks consist of some categories of activities or behaviors that are involved while performing a mathematical task.

Key words: [Global plans, Problem solving, Metacognition, Mathematics teaching, Mathematics writing]

Withdrawal and failure rates in undergraduate mathematics courses are higher than most other undergraduate courses (Steen, 1999, MAA notes; Gardner Institute, 2013). One of the most common ways that instructors assess students' mathematical understanding in colleges is grading their written work. Written work such as homework, quizzes, and exams mostly determine students' overall course grades in most instructors' grading schemes. Students' success or failure in undergraduate mathematics courses is usually determined by the weighted average of grades they obtain in a course by the end of the semester. Based on my own experience as a college mathematics instructor, many students' classroom interaction with the instructor and their peers indicate that they clearly seem to have grasped the understanding of the subject matter. But many of them cannot solve mathematical problems successfully on in-class quizzes and tests, and fail to earn the higher course grades that they actually deserve. Their failure to solve mathematical problems, and getting lower than expected grades could be a source of frustration among students and instructors alike. It is also reasonable to assume that students' poor performance in written assignments could be contributing to higher dropped, failed or withdrawn (DFW) rates in undergraduate mathematics courses. There is not enough information as to what factors are hindering undergraduate students' ability to solve mathematical problems successfully, even if they have the required understanding of the subject matter. This study is an attempt to look for reasons that might be hindering their effort to solve mathematical problems successfully in in-class assignments.

Students' writing is a source of information for instructors to assess how their students' think and learn mathematics. Writing can be considered as thinking aloud on paper, and therefore it provides rich data and a means of observing important processes that are difficult to identify using other methods (Flower & Hayes, 1981). It is not always sufficient to have our students spend time thinking about mathematical concepts; we should also require them to articulate their mathematical ideas in writing (Porter & Masingila, 2000). The purpose of this qualitative study is to study the connection between undergraduate students' global plans (articulation of their ideas) for solving mathematical problems in time restraint situations and their success in actually being able to produce correct solutions, by analyzing their written work. The study also attempts to compare the organization and execution behaviors of successful and unsuccessful problem solvers while performing a mathematical task, by examining their global plans and respective solutions of mathematical problems. More specifically, the study attempts to answer the following research questions:

1. How are students' global plans related to their success in mathematical problem solving?
2. What are some primary factors that lead to unsuccessful problem solving, even when students have a valid global plan for solving mathematical problems?

Literature Review and Theoretical Framework

This study employed Garofalo and Lester's (1985), and also Schoenfeld's (2011) framework to understand students' problem solving behaviors. Garofalo and Lester (1985) identified four categories of metacognitive activities involved in performing a mathematical task: orientation, organization, execution, and verification. According to them, the orientation phase pertains to strategic behavior to assess and understand the problem. The organization phase pertains to planning of behavior and choice of actions. Metacognitive behaviors during this phase include identification of goals and subgoals, global planning, and local planning (to implement global plans). The execution phase is related to regulation of behavior to conform to plans. This phase involves metacognitive behaviors such as performance of local actions, monitoring of progress of local and global plans, and trade-off decisions (such as speed vs. accuracy, degree of elegance). The verification phase is related to evaluations of decisions made and of outcomes of executed plans. Schoenfeld (2011) claimed that people's decision making and their success (or failure) in problem solving is a function of knowledge and resources, and beliefs and orientations. He also added that students' metacognitive activities or behaviors during problem solving also play a role in determining their success or failure in problem solving.

Time pressure might have some impact on student performance in in-class assignments. But studies show that extended time has no significant impact on students' test performance (Caperton, 2000; Orfus, 2008). It is not reasonable to expect our students to engage in high level of metacognitive activities during time restraint situation, but research shows that a metacognitive framework is evident in students' writing about their problem solving processes (Pugalee, 2004). It is therefore reasonable to assume that existing knowledge and metacognition both have roles in students' success or failure in problem solving in time restraint situations.

From a review of studies related to metacognition in problem solving, Simon (1987) found that monitoring, regulation, and orientation processes appear more frequently in the problem solving protocols of successful problem solvers. From a study with middle school students, Lester (1989) found that orientation to the problem actually has the most influential effect on students' successful performance in problem solving. Pugalee (2004) found that the students who construct global plans (stated or implied) are more likely to be successful at the problem solving tasks. In addition, he reported that the execution behaviors comprised the largest number of problem solving actions. It is therefore reasonable to assume that students' likelihood of making errors during the execution phase is somewhat higher, even if they have a valid global plan for solving the problem. Pugalee (2004) also found that most students do not check the accuracy of their final answers. Few or no metacognitive activities in the verification phase might also hinder students' effort in solving problems successfully, even if they have a clear conceptual understanding needed to solve the problems. Students' conceptual and procedural knowledge, therefore, might increase the likelihood of, but that alone does not guarantee, their success in problem solving. Schoenfeld (1985) found that effective problem solvers engage in self-regulation (or metacognitive) activities more often than others. Other studies have shown that successful problem solvers engage in metacognitive activities, and also have better understanding of mathematical concepts (Pugalee, 2004; Schur, 2002). Representation analysis of students' problem solving contexts in a recent study revealed that students who employed a nonsymbolic representation were more than three times more likely to solve the problems than the ones who employed symbolic representations (Yee & Bostic, 2014). From an analysis of the types of errors made by high school students in Algebra I, Pugalee (2004) found that 66.2% of all errors were procedural, 23% were computational, and 10.8% were algebraic. Based on the existing studies, an obvious assumption to this study is that students' lack of skills (both current and prerequisite) skills might be adding challenges to their effort to solving problems successfully.

Methodology

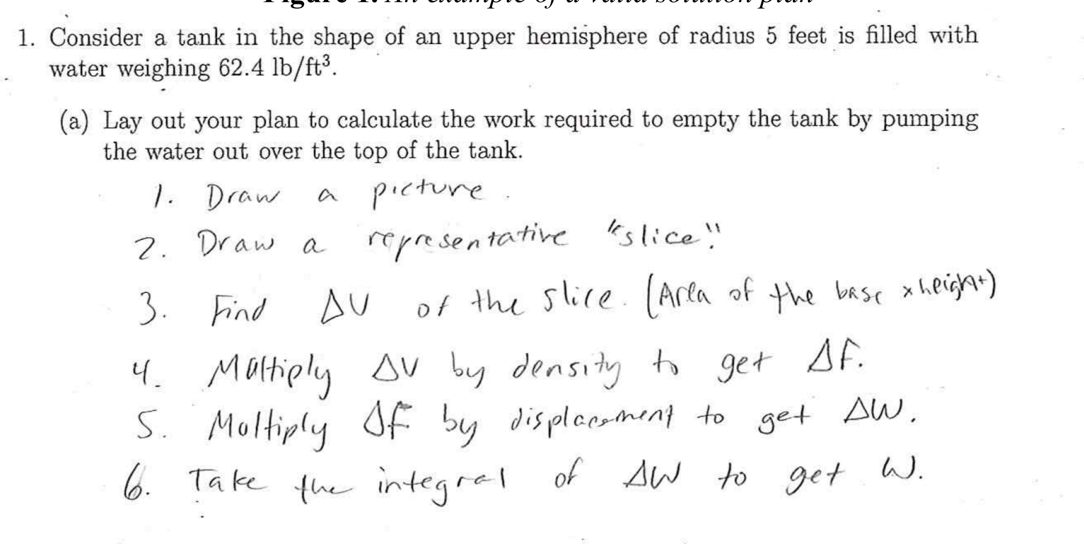
The researcher, who was also the instructor of the courses, collected data over the Spring 2015, Summer 2015 and Fall 2015 semesters from three undergraduate mathematics classes: Introductory Differential Equation (IDE) (three sections), Calculus I (one section), and Calculus II (three sections). The participants were traditional undergraduate students from a medium-sized university in the southeastern United States. The collected data is comprised of students' written work from in-class quizzes and tests. The researcher required all the students to write their brief global plans and follow the plans to solve mathematical problems. The problems were familiar or small deviation of the problems the students had seen or solved in the class. The problems were written for typical exams and quizzes, which included both procedural and word problems, but all these problems required multiple steps to solve. In other words, many of the problems were close to being exercises rather than actual "problems". In each quiz, the instructor announced that the students would be given 3-4 additional minutes on top of allocated time for a quiz (usually 12 minutes) to write their global plans. Lengths of exam times were similarly adjusted.

One of the reasons for requiring students to write their global plans was to see how they articulate their mathematical ideas to solve the problems and organize their problem solving processes. Students' articulation of ideas might serve as a form of thinking aloud on paper, which might be helpful to understand their planning behavior, understanding and thinking. Since some of the students did not attend classes regularly or stopped attending, the number of responses collected per student varies. The researcher also conducted audiotaped interviews with a few purposefully selected samples of students, asking them to describe their plans for solving the problems and have them solve. The selection is based on the analysis of student's written work. A representative sample of students having clear and valid global plans but failing to successfully solve the problems was selected based on the types of errors they made while solving the problems. The purpose of conducting audio or videotaped interviews was to illicit their detailed plan for solving problems, because it might be difficult for students to describe the detail of their plan in writing. Also, brief global plans might not provide enough evidence to know if the students have clearly understood the underlying concepts and the procedures for solving the problems. Interviews also allow the researcher the flexibility to ask follow-up questions.

Data Analysis

The first phase of data analysis involved the analysis of students' global plans and their solutions to mathematical problems from the written data. The data was analyzed using qualitative techniques: the constant comparative method (Strauss & Corbin, 1998)) and thematic analysis (Braun, 2008; Creswell, 2012). The data analysis involved both inductive and deductive approaches (Braun, 2008). Students' global plans were categorized into one of the two predetermined categories: *valid* or *invalid/unclear*. The solution plans that seemed capable of leading to correct solutions, provided that they were executed successfully, were categorized as valid. Figure 1 shows an example of a valid solution plan provided by a student in Calculus II in the fall 2015 semester. Some students did not know how to articulate their plans in writing, but their sketch clearly communicated their understanding and thought processes. Such global plans were also categorized as *valid*. Figure 2 shows an example of such a plan.

Figure 1. *An example of a valid solution plan*

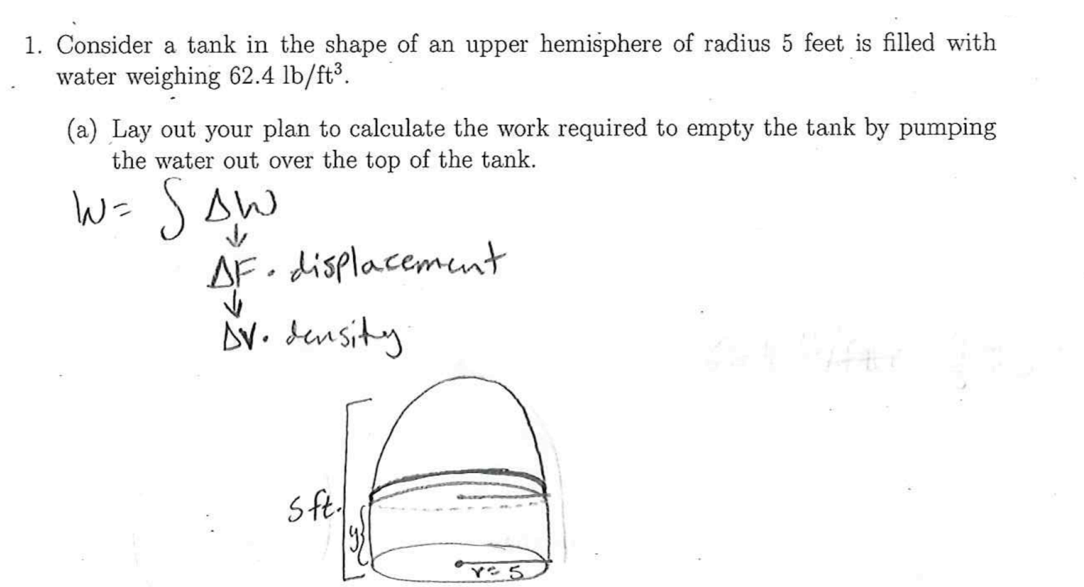


1. Consider a tank in the shape of an upper hemisphere of radius 5 feet is filled with water weighing 62.4 lb/ft^3.

(a) Lay out your plan to calculate the work required to empty the tank by pumping the water out over the top of the tank.

1. Draw a picture.
2. Draw a representative "slice".
3. Find ΔV of the slice. (Area of the base x height)
4. Multiply ΔV by density to get ΔF.
5. Multiply ΔF by displacement to get ΔW.
6. Take the integral of ΔW to get W.

Figure 2. *An example of a valid but not clearly articulated solution plan*



1. Consider a tank in the shape of an upper hemisphere of radius 5 feet is filled with water weighing 62.4 lb/ft^3.

(a) Lay out your plan to calculate the work required to empty the tank by pumping the water out over the top of the tank.

$W = \int \Delta W$

$\downarrow$

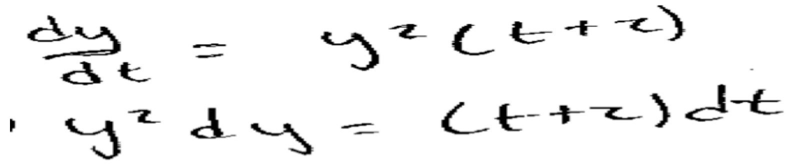
$\Delta F \cdot$ displacement

$\downarrow$

$\Delta V \cdot$ density

Students' errors in the solutions were also categorized into predetermined categories *algebraic, computational,* and *procedural* (Pugalee, 2004). See figure 3 for an example of an algebraic error found in one of the responses.

Figure 3. *An example of algebraic error in student work*



More error categories emerged during data analysis: *calculus, carelessness, conceptual, other prerequisite,* and *plan not followed.* Errors pertaining to only calculus I related concepts or skills were put into *calculus* category. An example of calculus error can be seen in figure 5. Errors due to lack of orientation (such as not reading the problem carefully) and concentration during writing solutions mostly comprised the *carelessness* category. See figure 4 for an example of error due to student's carelessness.

Figure 4. *An example of carelessness*

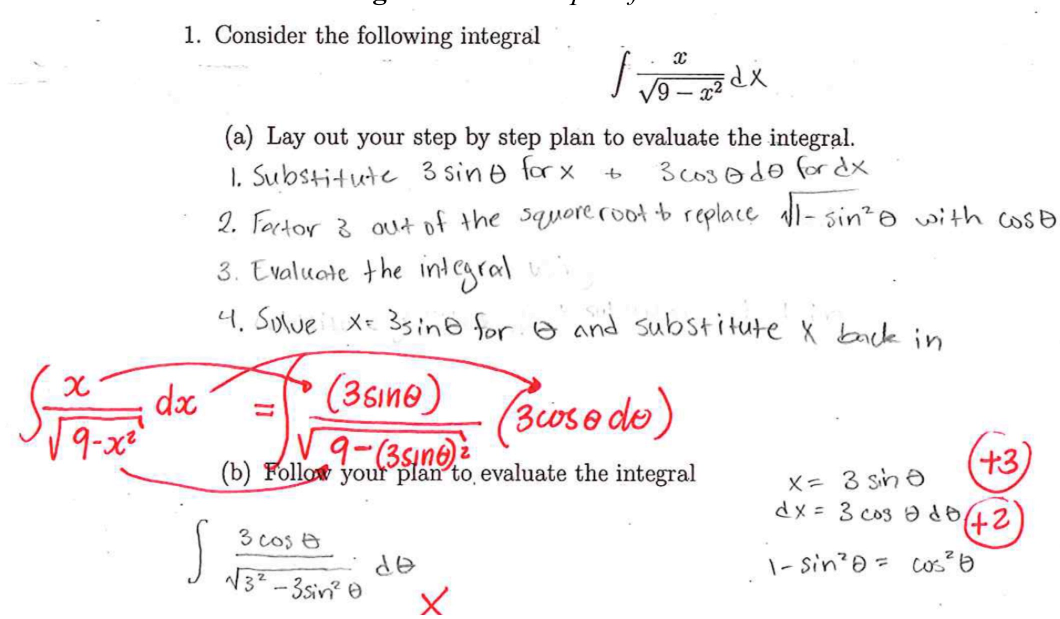
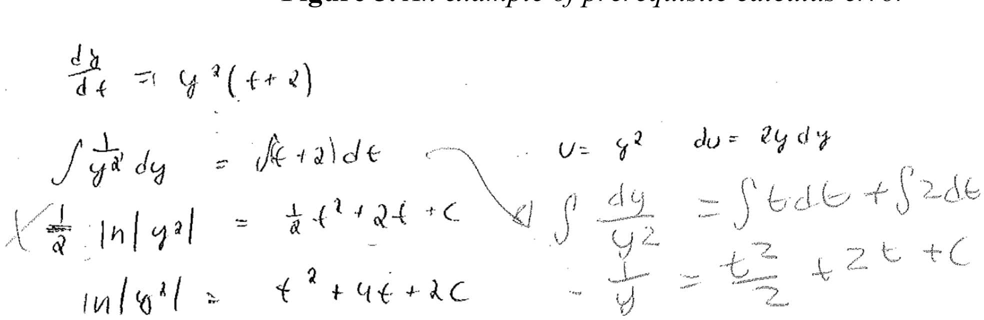


Figure 5. *An example of prerequisite calculus error*



Errors related to prerequisite concepts or skills (such as due to lack of trigonometry concepts or skills) other than algebraic, computational or calculus errors were put into the *other prerequisite* category. Some students gave legitimate plans but did not follow them and failed to produce correct solutions. Such errors were put into the category *plan not followed*. Errors that did not fall into any of the categories discussed above all fell into the *other* category.

Some of the students' work had multiple occurrences of errors that fell into the same category, but they were recorded collectively as 1 (detected) or 0 (not detected). Students' final answers or conclusions were categorized as *correct* or *incorrect*. Solutions with correct answers or conclusions supported by valid work were considered as solved successfully. The percentage of points that was taken off from each problem (out of the maximum possible score) due to one or a combination of all errors was also recorded. Solutions with correct answers not backed up by valid work did not receive full credit. This preliminary analysis includes data in the form of students' written work from two sections of Calculus II (one section each from the spring 2015 and fall 2015) and one section of IDE (spring 2015). A colleague was asked to grade and analyze a few student responses in order to establish intercoder reliability. There were small disagreements in categorizing a couple of student errors, as they would qualify to fall into more than one category. But a general consensus was reached by placing those errors under only one agreed-upon category, in order to avoid counting student errors more than once.

Results and Discussion

The number of students in each class was almost identical: 24 Calculus II-spring, 23 in IDT, and 24 in Calculus II-fall. But the number of responses collected was different; the number of assignments depended on the course as well as on the semesters and classroom dynamics. This section is focused primarily on those responses that had valid global plans but students were not able to execute them successfully.

Of the 272 responses collected from Calculus II-spring, global plans of 199 responses (73%) were coded as clear and valid. This means that 199 of the plans convinced the researcher that the students could solve the problems successfully if they followed their plans and did not make any execution errors. Of those 199 responses, solutions in only 97 responses (48.7%) were correct and in the remaining 102 responses were incorrect. Similarly, 118 written samples collected from IDE were analyzed. Of these, 98 responses (83%) had clear and valid global plans, and solutions in only 43 (43.9%) of these 98 responses were correct. Likewise, 333 responses were collected from Calculus II-fall, 215 (64.5%) of which had valid plans. Of these 215 valid plans, 113 (52.6%) responses had correct solutions. See Table 1 for the summary of the data.

Table 1. *Data summary*

Data	Cal II-Spring	IDE-Spring	Cal II-Fall
No of Stds	24	23	24
No of responses	272	118	333
Valid plans	199 (73%)	98 (83%)	215 (64.5%)
Correct solutions	97 (48.7%)	43 (43.9%)	113 (52.6%)

Calculus II-spring

There were 204 errors recorded in 102 responses with clear and valid plans but incorrect solutions; if any error type occurred more than once in a response, they were counted as one. Table 2 below summarizes the number (and percentages) of errors detected in those written samples that had clear and valid plans but incorrect solutions. The numbers in the column headings are the total number of responses with valid plans but incorrect solutions. The sum total of all procedural and conceptual errors was 49, which is 24% of all errors. The sum total of all errors related to required prerequisite knowledge was 96, which accounts for 47% of all errors. Of the 102 responses with clear and valid plan but incorrect solutions, 43 (42.16%) responses lost more than 20% of the maximum possible score. This means that they received a grade of C or lower.

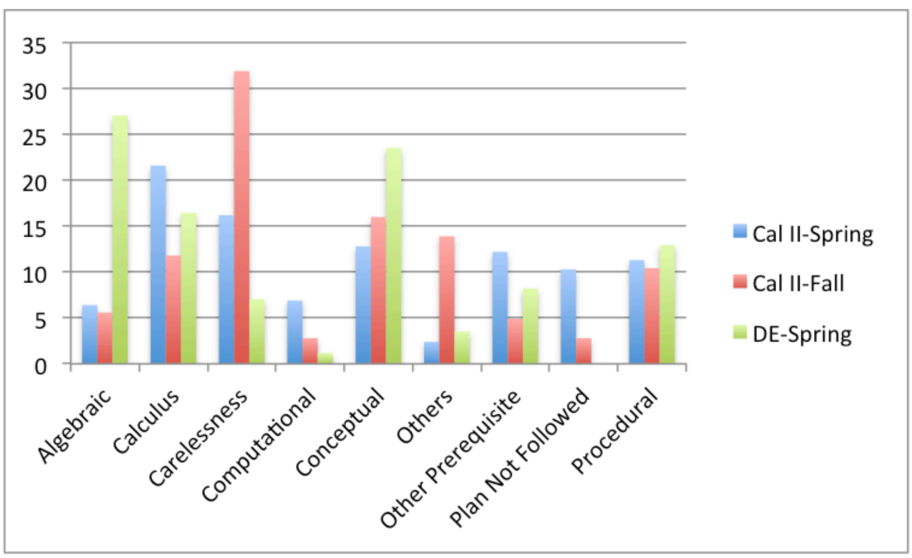
Calculus II-fall

There were 333 errors recorded in 102 responses that had valid plans but incorrect or incomplete solutions. Conceptual and procedural errors accounted for 26.42% of all errors. Required prerequisite knowledge-related errors altogether accounted for 45.5% of all errors. Among those 102 responses, 31 (30%) responses received a grade of C or lower.

Table 2. *Error counts in responses with clear and valid plans but incorrect solutions*

Types of Errors	Calc II-Spr (Total = 102)	IDE (Total = 55)	Calc II-Fall (Total = 102)
Algebraic	13 (6.4%)	23 (27.06%)	8 (5.6%)
Carelessness	23 (16.2%)	6 (7.06%)	17 (11.8%)
Calculus	44 (21.6%)	14 (16.47%)	46 (31.9%)
Computation	14 (6.9%)	1 (1.18%)	4 (2.8%)
Conceptual	26 (12.8%)	20 (23.53%)	23 (16%)
Others	5 (2.4%)	3 (3.53%)	20 (13.9%)
Other Prerequisite	25 (12.2%)	7 (8.23%)	7 (4.9%)
Plan not Followed	21 (10.3%)	0 (0.00%)	4 (2.8%)
Procedural	23 (11.3%)	11 (12.94%)	15 (10.42%)
Total	204 (100%)	85 (100%)	144 (100%)

Figure 6. *Comparison of error percentages by courses*



Differential Equations-spring

There were 85 errors recorded altogether from the 55 responses with a correct plan but incorrect solutions. The total number of errors related to all types of prerequisite knowledge was 45, which accounts for 53% of all errors. Among those 55 responses, 27 (slightly less than 50%) responses received less than 80% of available points. See Figure 6 to see how the percentages of errors compare in different courses.

Results show that 48.7% of all responses with clear and valid global plans actually solved the problems successfully in Calculus II-spring. The corresponding figures in Calculus II-fall and IDE were 52.6% and 43.9%. This shows that having clear and valid plans does not guarantee students' success in producing correct written solutions. If the students had not made required prerequisite knowledge related errors, many more would have performed better in the courses. Even though student responses had clear and valid plans for solving the problems, prerequisite knowledge related errors accounted for 47% of all errors in Calculus II-spring, 45.5% in Calculus II-fall and 53% of all errors in IDE-spring. Result indicates that lack of required prerequisite knowledge, especially the lack of precalculus concepts and skills, is hindering students' efforts to solve the problems successfully in in-class assignments.

Many students also seem to be not performing well because of carelessness and not following their own global plans. In Calculus II-spring, 44 out of 204 errors were due to either carelessness or not following their own plans, which accounts for 21.6% of all errors. The corresponding figure in Calculus II-Fall was 34.7%. But this figure is significantly smaller in IDE course; only 7.06% of all errors were due to carelessness, and none of the responses failed to follow their plans. Although the purpose of the research was not to compare the performance of students in these courses, it is noteworthy that 27.06% of all errors were algebraic in IDE, as compared to only 6.4% in Calculus II-spring and 5.6% in Calculus II-fall. On the other hand, 6.9% and 2.8% of all errors were computational in Calculus II-spring and Calculus II-fall as compared to only 1.18% in IDE. Obviously the problems posed in the courses were different, and a different level of algebraic or computational manipulations might have been necessary to solve them in two different courses. But it was interesting to see that 27.06% of all errors in the IDE course were algebraic, whose prerequisite course was Calculus II. This number could be much higher if we also counted the students' responses with incorrect and/or invalid global plans. Similarly, the percentages of prerequisite calculus I concepts or skills related errors were very high in all three sections: 21.6% in Calculus II-spring, 31.9% in Calculus II-fall and 16.47% in IDE-spring, which the students of both courses were expected to know.

Results show that overall less than 50% of the student responses with valid plans executed their plans successfully even though their global plans clearly indicated that they knew how to solve the problems. Results also confirm the findings from earlier research that most students did not even check the accuracy of their answers (Pugalee, 2004) even though many of them had enough time left. Very few students wrote in their global plans that they would check their answers, and fewer did so in the solutions. Although time restraints could be blamed as a cause of some of the errors such as those due to carelessness, not reading the problems carefully, not following their own plans, and not checking the accuracy of their answers, it would be interesting to see how often such errors would be detected if students were allowed to solve these problems in no time restraint situations.

It is also evident from the results that students' success in writing solutions is not only a function of knowledge of the current concepts being assessed, but also a function of students' prerequisite knowledge and skills. The combined total of prerequisite knowledge-related errors in each course was very high as compared to the combined total of conceptual and procedural errors (see Table 2). Many of the precalculus related errors or errors due to carelessness might not have big impact on students' understanding of the upper level (Calculus I and higher) mathematical concepts. It was found, however, that those students with better conceptual understanding also had better procedural and skills proficiency, as evident in students' global plans and solutions. These days skills proficiency is considered less important as compared to students' conceptual understandings of the subject matter. But the results of this study suggests that conceptual understanding does not come at the cost of sacrificing skills proficiency (Engelbrecht, Harding, & Potgieter, 2005). Also, it is difficult to assess students' learning using mathematical problems that require minimal amount of algebraic, trigonometric and computational concepts or skills. These prerequisite skills in first few steps set the stage for further steps in solving processes of many mathematical problems. Use of technology such as use of calculators could be helpful in reducing many of the precalculus concept or skills related errors. But technologies are not always helpful in enhancing students' conceptual understandings.

It has been widely documented that learning requires skills (Paris & Winograd, Wittrock, 1986; Paris, Washik, van der Westhuizen, 1988). But results from this study also show that communicating what has been learned in writing also requires skills. Detection of large number of errors in carelessness category suggests that orientation to the problem (such as reading the problem carefully), concentration, regulation and monitoring of progress are some of the skills that are helpful to the students. Understanding of mathematical concepts and being able to communicate the understanding in writing are two different aspects of learning process. The results confirm the evidence of existing research that successful problem solvers engage in metacognitive activities in addition to having better understanding of mathematical concepts (Pugalee, 2004; Schur, 2002). An implication to this finding is that students should also be encouraged to learn test-taking skills in addition to learning the mathematical concepts being assessed. Students' engagement in metacognitive activities while learning and solving (or writing solutions of) mathematical problems not only improves their learning (Schoenfeld, 1985), but might also improve their test-taking skills. In order to increase the likelihood of success in problem solving, metacognitive monitoring of progress becomes even more important in time restraint situations. These results indicate that students need to have more experience (or practice) in problem solving and writing solutions. At the present time, when online homework systems are taking the place of traditional homework assignments, these results indicate that students should be encouraged or required to communicate their understanding in writing more often.

Few student behaviors were observed while they were writing solutions, but these behaviors were difficult to document. First, many students (in all three sections) jumped into writing the solutions first before laying out their global plans. The instructor had to continuously monitor and remind the students to write their plans first and follow them to solve the problems. These students tended to describe how they solved the problems instead of describing how they would solve. This study does not have enough data or evidence to explain reasons behind such behavior of students. Many of them might have either difficulties in articulating their global plans in writing, or not even have clear big picture or path to follow and solve the problems in their minds. Second, some students erased their original plan and rewrote them after they realized that their original plan did not or would not lead to a correct solution. This could be taken as an indication of students being engaged in regulation activities while solving the problems. They might have changed their plans after they realized that their original plans did not or would not lead to correct solutions.

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Fostering teacher change through increased noticing: Creating authentic opportunities for teachers to reflect on student thinking

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This paper reports results from a case study focusing on a secondary teacher's sense-making as she was challenged to reinterpret her meanings for algebraic symbols and processes. Building from these opportunities, she redesigned lessons to gather information about how her students conceptualized quantities and how they thought of variables, terms, and expressions as representing those quantities' values. She then used this information to respond productively to her understanding of individual students' meanings and reasoning elicited during these lessons. We argue that this case study demonstrates the potential for coordinating quantitative reasoning with teacher noticing as a lens to support teacher learning and we recommend specific mathematical practices that can help teachers develop more focused noticing of students' mathematical meanings during instruction.

Key words: Professional Development; Quantitative Reasoning; Teaching Mathematics

Introduction and Context

Many states adopted the new Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) as part of an effort to improve student learning and achievement. But research suggests that substantial change, at least in the short-run, will be difficult for the following reasons: (1) teaching practices and teachers' conceptions of their curricula are more significant indicators of student achievement than curricula alone (Boaler, 2003; Thompson, 1985), (2) teaching practices are learned implicitly as a cultural activity (Stigler & Hiebert, 2009), (3) teachers have a limited tolerance for the discomfort they feel while trying to implement reforms (Frykholm, 2004), and (4) reform efforts challenge teachers' traditional images of efficacy (Smith, 1996).

Smith (1996) and Thompson, Phillip, Thompson, and Boyd (1994) argue that thoughtful lesson planning that includes carefully conceptualizing learning goals, generating conjectures about how a person might come to understand certain ideas, and considering alternative solution methods may alleviate some of the uncertainty teachers feel in shifting their practice and might positively impact student learning. But our experience is that shifting teachers' practices is difficult. Interventions that only target a teacher's personal mathematics knowledge, such as providing additional upper level math coursework, might not translate to teachers modifying their established curricula and lessons. Some reasons for this may be that mathematics coursework does not advance teachers' knowledge of student learning of specific mathematics content; nor does it address challenges teachers encounter when determining and implementing curriculum for a particular course. Interventions with broad goals, such as shifting the emphasis from direct instruction to student-centered activity, often fail to impact student learning positively if teachers make only superficial changes in their teaching (Stigler & Hiebert, 2009).

Theoretical Perspective

We agree with Thompson's (1993, 2013) assertion that meaning is not carried by symbols or pictures on a page but rather exists in the mind of an individual. Accordingly, at any moment during an interaction between two individuals there is a meaning in the mind of the person communicating and a meaning in the mind of the interpreter. In the moment of teaching, students' current mathematical meanings are critically important because these are the meanings students either build upon or transform to advance their mathematical understandings. We align with von Glasersfeld (1995) who argues that one person cannot transmit knowledge to another. An individual must construct knowledge and meanings for himself through personal experiences. Thus, it is ineffective to *tell* teachers what students think (even if we knew), how to respond to students' questions, or how to leverage student thinking to provide more meaningful mathematical experiences. Instead, we must help teachers to become fluent in revealing the variety of student reasoning existing in their classrooms and to recognize the utility in being sensitive to student thinking when planning and delivering instruction. Mason (2002) argues that professional development is always a personal endeavor and that change occurs when a person increases the scope as well as the specificity of what he notices while engaged in professional practice and uses this intentional noticing to inform his actions at relevant moments. The more a teacher engages in disciplined noticing, the more likely it is that the teacher constructs new images of how a student may conceptualize and represent a problem or idea.

If we can foster increased noticing of student thinking in teaching and learning situations, we believe that teachers will be more likely to set goals related to building models of student thinking and to see their own practice as a source of learning and personal professional development (Cobb & Steffe, 1983; Lage-Ramirez, 2011; Stigler & Hiebert, 2009; Teuscher, Moore, & Carlson, 2015; Thompson & Thompson, 1996). We further believe that a focus on what Thompson (1994, 2013) calls *quantitative reasoning* is a fruitful avenue for fostering increased and disciplined noticing. By focusing attention on quantities, their relationship to one another, and how to represent the values of quantities using expressions or reasonable operations for calculating the values (Thompson, 1993), teachers have a potentially small but useful set of guidelines for fostering the sociomathematical norm of *speaking with meaning* (Clark, Moore, & Carlson, 2008) whereby the classroom atmosphere is about making sense of others' thinking and representing one's own thinking. In such an environment, teachers are primed to be more disciplined at noticing and motivated to make use of the thinking they do notice.

Practices that Support Meaning-Making

Our experience with teachers and review of relevant literature suggest that it is difficult for teacher professional development to achieve measureable gains in student learning. Our hypothesis is that a small set of teaching practices focused on using quantitative reasoning to generate meaningful representations of quantitative relationships can foster more focused noticing of student meanings and improve a teacher's ability to react productively to student thinking. Synthesizing the works described in our theoretical perspective, we generated the following set of practices and expectations that we call *meaningful mathematical communication*, or MMC, practices.

1. An expectation that people speak clearly and meaningfully by avoiding the use of pronouns and referencing specific quantities in class conversations and written assignments (Clark et al., 2008; Mason, 2002).
2. An expectation that, when a person writes an expression or performs a calculation, she first writes out in words or says verbally what she intends to calculate. After a person writes an expression or performs a calculation, she justifies what the expression or calculation represents (Thompson, 1993; Thompson, 2013).

3. An expectation that when a person writes a formula or mathematical expression he is attempting to communicate his thinking. Therefore, each person in the course is responsible for being able to justify what his mathematical statements are intended to convey. If someone else creates a different representation (including mathematically equivalent statements that contain different expressions or orders of operations) then it may represent a fundamentally different way of thinking, and each person is responsible for trying to understand this thinking (Thompson 1993; Thompson et al., 1994).

The following example highlights why we think these are important classroom norms to establish and support. Consider an arithmetic series with an even number of terms. Figure 1 summarizes three methods one might use to efficiently determine the sum.

Method 1	(sum of the *n* terms of the series)	is	(number of pairs) (sum of each pair)	$S_n = \left(\frac{n}{2}\right)\left(a_1 + a_n\right)$
Method 2	(sum of the *n* terms of the series)	is	(number of terms) (average term value)	$S_n = (n)\left(\frac{a_1+a_n}{2}\right)$
Method 3	(sum of the *n* terms of the series)	is half of	((the number of pairs formed by combining the original series with the same series in reverse order) (sum of each pair))	$S_n = \frac{1}{2}\left[(n)\left(a_1 + a_n\right)\right]$

Figure 1. Different ways to conceptualize finding the sum of an arithmetic series.

The three formulas $S_n = \left(\frac{n}{2}\right)\left(a_1 + a_n\right)$, $S_n = (n)\left(\frac{a_1+a_n}{2}\right)$, and $S_n = \frac{1}{2}\left[(n)\left(a_1 + a_n\right)\right]$ produce the same sum for every arithmetic series. But it is not enough to treat the formulas off-handedly as merely mathematically equivalent statements and settle on one as a preferred method. Each formula derives from a different conception of efficiently calculating the sum and, when treated in this way, helps students understand that mathematical symbols and formulas are ways to communicate and represent their thinking. Explaining what someone intends to calculate before generating the mathematical representation helps to highlight this point. It also promotes synchronizing the reasoning process that generated the formula with the order of operations one performs while evaluating it and highlights how a different order of operations corresponds with a different way of reasoning.

We hypothesize that most teachers can implement the MMC practices regardless of the level they teach or their background in attempting to make student thinking an important part of their lessons. We also believe that the MMC practices are not overly ambitious in terms of the demands they place on teaches.

Research Questions

Based on our theoretical perspective and our conceptualization of the MMC practices, we designed a professional development intervention and conducted a study aimed at answering the following research questions. 1) How did the teacher come to make sense of quantitative reasoning as a lens for interpreting symbols and processes? 2) How, and to what degree, did the teacher attempt to recreate similar experiences for her students? 3) To what extent did the teacher adopt the MMC practices as classroom norms and use them to gather evidence about her students' meanings and reasoning? [In other words, did the teacher engage in *disciplined and productive noticing*? We say a teacher engages in disciplined and productive noticing when she intentionally chooses to gather evidence of student thinking and leverages what she notices to make *productive instructional moves* such as posing questions to help students build new connections, asking students to compare their reasoning to their peers' reasoning, or generate alternative representations of their thinking.]

Methods

This study involved one teacher in a large urban middle school in the Southwestern United States. Tracy was teaching an honors Algebra II course using the *Pathways Algebra II* curriculum (Carlson & O'Bryan, 2014). The study included five one-on-one professional development sessions and five classroom observations spread throughout the Fall 2014 semester and concluded with a final follow-up interview near the end of the Spring 2015 semester. The professional development sessions had two goals. First, to present tasks and contexts where thinking carefully about quantities' values being represented by mathematical expressions is useful and where mathematically equivalent statements can represent different ways of conceptualizing a context. Second, for Tracy to reflect and comment on anything she observed while implementing a similar focus during class sessions with students. We conducted classroom observations to determine if Tracy implemented the MMC practices and, if so, the effect they had on the quality of classroom discourse.

The first author video recorded and transcribed each professional development session and most classroom observations. A delay in receiving signed parental consent forms for videotaping classroom sessions necessitated using shorthand techniques to transcribe the first two classroom sessions in person. He later expanded these notes to a very close approximation of the class conversations. We analyzed the transcripts using open coding (Strauss & Corbin, 1990) and then looked for recurring and emergent themes over the course of the semester. In this paper we report results relative to one identified theme (increased noticing).

Results

Mason (2002) says that a teacher with a disciplined and systematic approach to noticing creates opportunities to gather important feedback from her students, and this feedback is used in planning for and acting in future learning moments. For Excerpt 1, the first author asked Tracy to explain anything that stood out to her while grading a recent quiz.

Excerpt 1.

1 *Tracy:* Um on number three…You leave a school and start walking home at a constant speed of four feet per second. Five minutes later you are a hundred feet from your house. Assume you're walking in a straight path to your house. Write a formula to define the relationship between the quantities distance from your house and time since leaving the school. And so then the first one was writing a formula telling them how to define the variables, and then the second two questions were so how far is your house from the school and how long did it take you to get there. What was interesting is almost everybody got b and c correct. A lot of people missed a. Because they were like I know you told me that the distance should be distance from my house, but what I would really like to do is distance from school. (*laughs*) So a lot of them used an incorrect reference point um or they, they said okay I'll use distance, but I'm gonna make it this other distance. So we really have done several problems like this after that…I've had them walk across the room. [Student] was on the little pretend bicycle …I have problems where I had them model okay so now back up what's happening?

And go forward, what's happening to the distance? And I also did several problems after this where I made them write both formulas. Okay, you really wanted to do this formula, fine do that one and do this one, and so they knew because they had to do two that they had to be different, and then we also talked a lot about that afterwards. How, one thing that I realized they were doing is your reference point has to match your variables, so like they would be using a reference point where the *d* in the reference point was distance from house, or school in this case, so they're using a reference point maybe of distance from school, but then they're trying to write a formula distance from house. Well, if your formula's tracking distance from house, your reference point has to be in distance from house, or if it's distance from school, it has to be in-, so we talked about the importance of, what my, what the meaning of the reference point and the variables that it's keeping track of, those need to be the variables that are appearing in the equation so then that got back to a discussion of and how important is it that we define (*laughs*) our variables accurately.

Tracy noticed a difference between how she and her students conceptualized a situation. This was an important moment for her because she saw a need to practice thinking about different ways to conceptualize a situation and how choosing different pairs of co-varying quantities to compare creates different relationships. After Tracy noticed that students did not conceptualize the situation as she intended, she modified her future instruction to account for this tendency and was primed to notice similar inconsistencies in the future.

In Excerpt 2, Tracy described her students' work to interpret the parts of the general explicit formula for geometric sequences.

5. We can create a general formula that serves as a model for the explicit formulas for *all* geometric sequences.
   a. What is the explicit formula that defines the value of a_n for a geometric sequence with an initial term value a_1 and a common ratio r?
   b. Explain what each of the following represents in the formula (be clear and specific).
      i. n ii. a_n iii. $n-1$ iv. r^{n-1} v. $a_1 \cdot r^{n-1}$

Figure 2. Exploring student meanings for expressions (Carlson & O'Bryan, 2014, p. 94).

Excerpt 2

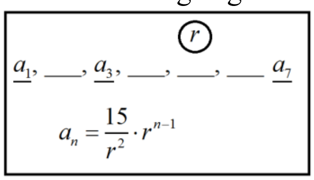
1 *Tracy:* [See Figure 2]. The one that was most concerning to me actually out of these five, cuz I feel like they got most of them, well like-like for example on this one [points to part (b. v.)], they said this is the first term multiplied by the ratio *n* minus one times, and I said, well, what does that find? Well, the n^{th} term. Okay, so this represents the n^{th} term. Then they said this- when I asked them what this was [points to part (b. iii.)] they said that's the position before the n^{th} position. They didn't see it as that's how far you need to change, that's how far your position is changing away from one. So that led to a discussion. And so, and then it had to be linked to this is multiplying by the ratio this many times but why? Because you're changing away that many positions.

According to Tracy, her students had little difficulty generating the formula $a_n = a_1 \cdot r^{n-1}$ inductively from several examples. But because Tracy asked them to describe their meanings for the different parts of the formula, she noticed that their meaning for at least one of the expressions differed from her own (and thus from the meaning she expected them to develop). Tracy said that in the past she was satisfied once students generated and could use the explicit formula, but she would not have realized that the students' conceptualizations of the formula differed from her own. After this experience Tracy created her own set of follow-up tasks designed to help students build a meaning for $n - 1$ as the value of the quantity "change in term position away from $n = 1$".

In the professional development sessions the first author encouraged Tracy to develop a personal appreciation for how different ways of conceptualizing a situation might lead to different (but equivalent) mathematical representations and that this point of view creates flexibility in modeling a situation. During the third such session, Tracy realized that she could use any term in the sequence as a reference point for relating the term values to their positions. That is, instead of $a_n = a_1 \cdot r^{n-1}$ representing the general explicit formula for a geometric sequence, she could think about the general formula as $a_n = a_j \cdot r^{n-j}$ where a_j is the j^{th} term in the sequence. Tracy said that she wanted her students to develop a similar understanding, and so she wrote and assigned the problem shown in Figure 3. She wanted students to understand that they could write the formula as either $a_n = 15 \cdot r^{n-3}$ or $a_n = 143 \cdot r^{n-7}$ with an appropriate choice of r. However, she was also sensitive to noticing alternative valid representations by working to make sense of the thinking behind these representations and wanted to help her students do the same.

Given a geometric sequence with $a_3 = 15$ and $a_7 = 143$, write the explicit formula. (*Note that you do not need to calculate the value of r.*)

Figure 3. Tracy's problem for defining a geometric sequence's explicit formula.



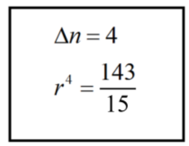


Figure 4 (left). Student response to the warm-up. *Figure 5 (right)*. Tracy's clarification.

Excerpt 3

1 *Tracy:* Yeah. Now take a look at S1's work. I want you to just read her work and think about it. [*Tracy shows Figure 4.*] S1, can you explain your work?

2 *S1:* Well, first we tried to find *r* but it was really messy. So we wanted to just leave it as *r*.

3 *Tracy:* So you did find *r*?

4 *S1:* Yeah. It was about nine point five three three three and then we had to take a root. It was kind of like the second one.

5 *Tracy:* So...wait. Tell me how you got nine point five three?

6 *S1:* The change in position was four, so *r* to the fourth is a hundred forty three divided by fifteen. [*Tracy writes Figure 5.*]…So then we focused on finding the first term, which is two positions away so we divided by *r* squared.

7 *Tracy:* [*long pause*] Should this work? [*pause*] Think back to yesterday, think of the formula we developed and the reference point. What did we use?

8 *S2:* One comma *a* one. The first term.

9 *Tracy:* Yes. And our goal was to write a formula that allows us to calculate the output *a n* for any input *n*. Think about the parts of the function and what they represent. What does *a* one represent?

10 *S2:* First term.

11 *Tracy:* Okay, *a* one is our first term. What does *r* represent?

12 *S3:* What we multiply by to get the next term.

13 *Tracy:* What is *n* minus one? We had some discussion yesterday about the possible interpretations. Some of you said that it was the position before the n^{th} position, but we talked about how that might not be the most useful interpretation here.

14 *S3:* It's the change away from one.

15 *Tracy:* Yes, the change in term position away from one. Now we don't know *a* sub one, instead we know *a* sub three, but by dividing by *r* to the second we can represent the value of *a* sub one even though we don't otherwise know what it is.

In another lesson, Tracy asked students to consider the contribution of a fictitious classmate who supposedly made the claim about finite geometric series shown in Figure 6.

> One of your classmates claims that he can calculate $S_6 - (3 \cdot S_6)$ without knowing the sums of the series. He says that $a_1 - 3a_6$ will have the same value as $S_6 - (3 \cdot S_6)$ and shows the following work. Is he correct?
>
> $$S_6 = 5 + \cancel{15} + \cancel{45} + \cancel{135} + \cancel{405} + \cancel{1{,}215}$$
> $$\underline{-\left(3 \cdot S_6 = \quad \cancel{15} + \cancel{45} + \cancel{135} + \cancel{405} + \cancel{1{,}215} + 3{,}645\right)}$$
> $$S_6 - 3 \cdot S_6 = 5 - 3{,}645$$
> $$S_6 - 3 \cdot S_6 = -3{,}640$$

Figure 6. Exploring a claim about geometric series (Carlson & O'Bryan, 2014, p. 106).

Tracy's goal was for students to see that the difference between the sum of the series and *r* times the sum of the series was the difference between a_1 and $r \cdot a_n$. As students worked on this task, Tracy asked them to think about the meaning of various expressions, and in doing so noticed that some students, seeing $S_6 - 3 \cdot S_6 = 5 - 3{,}645$, were confused because S_6 was not 5 and $3 \cdot S_6$ was not 3,645. We believe that a focus on the MMC practices helped Tracy notice and address this incorrect conception and reinforce what the various expressions represented in this context.

Discussion

We believe that teachers who state their learning goals in terms of student thinking (and not just in terms of performance objectives) are better equipped to monitor the development of students' meanings and to respond productively in the moment when they notice that these emerging meanings deviate from intended meanings. Excerpts from Tracy's teaching demonstrate that a teacher can leverage quantitative reasoning to make fine grained observations about students' mathematical reasoning. Specifically, Tracy worked to implement the MMC practices in her classroom on a daily basis and thus developed a disciplined practice of noticing that prompted her to adjust her instructional activities and trajectory to focus on and monitor students' mathematical meanings and make *productive instructional moves*.

We acknowledge that this case study involved only one teacher and that we do not have detailed observations of Tracy's teaching prior to her participation in the study. We only have Tracy's testimonial that the MMC practices were a significant contributing factor in what we witnessed and so we stop short of claiming that our professional development intervention in particular was responsible for Tracy engaging in *disciplined and productive noticing*. However, it was clear that Tracy leveraged quantitative reasoning in reinterpreting her own meanings for algebraic expressions and processes, designed lessons to create similar opportunities for her students, and leveraged quantitative reasoning to interpret and respond to students' classroom contributions. Thus, we believe that quantitative reasoning can serve as a useful tool to help teachers refine their noticing in the context of teaching mathematics. We hope that future research can develop and refine a framework to characterize teachers' quantitative reasoning and the connection between quantitative reasoning and disciplined and productive noticing.

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Changes in assessment practices of calculus instructors while piloting research-based curricular activities

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We report our analysis of changes in assessment practices of introductory calculus instructors piloting weekly labs designed to enhance the coherence, rigor, and accessibility of central concepts in their classroom activity. Our analysis compared all items on midterm and final exams created by six instructors prior to their participation in the program (355 items) with those they created during their participation (417 items). Prior exams of the six instructors were similar to the national profile, but during the pilot program increased from 11.3% of items requiring demonstration of understanding to 31.7%. Their questions involving representations other than symbolic expressions changed from 36.7% to 58.5% of the items. The frequency of exam questions requiring explanations grew from 4% to 15.1%, and they shifted from 0.8% to 4.1% of items requiring an open-ended response. We examine qualitative data to explore instructors' attributions for these changes.

Key words: [Calculus, Assessment, Cognitive Level, Representations, Problem-Solving]

One component of the recent national study of calculus programs in the United States (Bressoud, Mesa, & Rasmussen, 2015) examined the assessment practices of instructors of these courses. Tallman & Carlson (2012) analyzed the content of 150 Calculus 1 final exams sampled from a variety of post-secondary institutions in the larger study along three dimensions in their Exam Characterization Framework (ECF) detailing the cognitive orientation, mathematical representations, and answer format of each item. The study demonstrated that few final exam items required a demonstration or application of understanding of the material, primarily involved only symbolic representations, and rarely required explanation or involved open-ended responses. One explanation of these results may be that faculty assessment practices simply reflect the expectations of institutionally adopted curriculum. Lithner (2004), for example, found that a majority of exercises in calculus textbooks could be solved by choosing examples or theorems elsewhere in the text based on surface-level features and mimicking the demonstrated procedures.

We examined the assessment practices of pilot instructors implementing activities in their calculus courses designed to simultaneously enhance the coherence, rigor, and accessibility of student learning throughout the course. Project CLEAR Calculus provided weekly labs in which students participated in group problem-solving activities to scaffold the development of central concepts in the course along with instructor training and support to implement the labs. While the project did not address student assessment through exams, we hypothesized the conceptual focus in the labs and requirements of student write-ups would significantly impact the instructors' assessment practices. We address the following research questions:

1. How do the pilot instructors' exam questions compare to their previous exams along the three ECF dimensions?
2. What factors do the pilot instructors attribute for any shifts in their assessment practices?

Background

Limit concepts are at the core of mathematics curriculum for STEM majors, but decades of research have revealed numerous misconceptions and barriers to students' understanding. Building off of work by Williams (1991, 2001), Oehrtman (2009) identified several cognitive models employed by students that met criteria for emphasis across limit concepts and for sufficient depth to influence students' reasoning. Williams noted that frequently students attempt to reason about limits using intuitive ideas associated with boundaries, motion, and approximation. Oehrtman found that, unlike most other cognitive models employed by students, the structure of students' spontaneous reasoning about approximations shares significant parallels with the logic of formal limit definitions while being simultaneously conceptually accessible and supporting students' productive exploration of concepts in calculus defined in terms of limits. With this in mind, we contend that a false dichotomy exists between a formally sound, structurally robust treatment of calculus on the one hand and a conceptually accessible and applicable approach on the other. By adopting an instructional framework utilizing approximation and error analyses, we designed labs based on criteria listed in Figure 1 intended for weekly use in an introductory calculus sequence.

Design Criteria 1.	Language, notation, and constructs used in the labs should be conceptually accessible to introductory calculus students.
Design Criteria 2.	The structure of students' activity should reflect rigorous limit definitions and arguments without the language and symbolism of formal ε-δ and ε-N notation that is a barrier to most calculus students' understanding.
Design Criteria 3.	The labs should present a coherent approach across all concepts defined in terms of limits and effectively support students' exploration into these concepts.
Design Criteria 4.	The central quantities and relationships developed in all labs should be coherent across representational systems (especially contextual, graphical, algebraic, and numerical representations)
Design Criteria 5.	All labs should foster quantitative reasoning and modeling skills required for STEM fields.
Design Criteria 6.	The sequence of labs should establish a strong conceptual foundation for subsequent rigorous development of real analysis.
Design Criteria 7.	All labs should be implemented following instructional techniques based on a constructivist theory of concept development.

Figure 1. Design criteria for the labs.

When left unguided, students' applications of intuitive ideas about approximations are highly idiosyncratic (Martin & Oehrtman, 2010a, 2010b; Oehrtman, 2009). To systematize students' reasoning concerning approximation ideas and support an accessible yet rigorous approach to calculus instruction, throughout the labs students are engaged in contextualized versions of the questions in Figure 2. These questions develop coherence between structural components, reveal operations performed on these components, and highlight relationships among the operations, foundational for the generation of new understandings.

Question 1.	Explain why the unknown quantity cannot be computed directly.
Question 2.	Approximate the unknown quantity and determine, if possible, whether your approximation is an underestimate or overestimate
Question 3.	Represent the error in your approximation and determine if there is a way to make the error smaller.
Question 4.	Given an approximation, find a useful bound on the error.
Question 5.	Given an error bound, find a sufficiently accurate approximation.
Question 6.	Explain how to find an approximation within any predetermined bound.

Figure 2. Approximation questions consistent across most labs.

Exam Characterization Framework

Tallman and Carlson (2012) developed a three-dimensional framework to analyze a large sample of post-secondary Calculus 1 final exams and generate a snapshot of the skills and understandings that are currently being emphasized in college calculus. Their *Exam Characterization Framework* (ECF) characterizes exam items according to three distinct item attributes: (a) *item orientation*, (b) *item representation*, and (c) *item format*.

Item Orientation

Tallman and Carlson adapted the six intellectual behaviors in the conceptual knowledge dimension of a modification of Bloom's taxonomy (Anderson & Krathwohl, 2001) to characterize the cognitive demand of exam items. The six categories of item orientation are hierarchical with the lowest level requiring students to remember information and the highest level requiring students to make connections (see Table 1).

Table 1
Item orientation codes (adapted from Tallman & Carlson, 2012)

Cognitive Behavior	**Description**
Remember	Students are prompted to retrieve knowledge from long-term memory.
Recall and apply procedure	Students must recognize what procedures to recall and apply when directly prompted to do so.
Understand	Students are prompted to make interpretations, provide explanations, make comparisons or make inferences that require an understanding of a mathematics concept.
Apply understanding	Students must recognize recognize the need to use a concept and apply it in a way that requires an understanding of the concept.
Analyze	Students are prompted to break material into constituent parts and determine how parts relate to one another and to an overall structure or purpose.
Evaluate	Students are prompted to make judgments based on criteria and standards.
Create	Students are prompted to put elements together to form a coherent or functional whole; reorganize elements into a new pattern or structure.

Item Representation

The item representation domain of the ECF involves classification of both the representation of mathematical information in the task as well as the representation the task solicits in a solution (see Table 2). A task statement or solution may involve multiple representations. Since many tasks can be solved in a variety of ways and with consideration of multiple representations, we observed Tallman and Carlson's recommendation of considering only the representation the task requires.

Table 2
Item representation codes (Tallman & Carlson 2012)

Representation	Task statement	Solicited solution
Applied/ modeling	The task presents a physical or contextual situation.	The task requires students to define relationships between quantities or use a mathematical model to describe a physical or contextual situation.
Symbolic	The task conveys information in the form of symbols.	The task requires the manipulation, interpretation, or representation of symbols.
Tabular	The task provides information in the form of a table.	The task requires students to organize data in a table.
Graphical	The task presents a graph.	The task requires students to generate a graph or illustrate a concept graphically.
Definition/ theorem	The task asks the student to state or interpret a definition or theorem.	The task requires a statement or interpretation of a definition or theorem.
Proof	The task presents a conjecture or proposition.	The task requires students to demonstrate the truth of a conjecture or proposition.
Example/ counterexample	The task presents a proposition or statement.	The task requires students to produce an example or counterexample.
Explanation	Not applicable.	The task requires students to explain the meaning of a statement.

Item Format

The third and final dimension of the ECF is item format. The most general distinction of an item's format is whether it is multiple-choice or open-ended. However, there is variation in how open-ended tasks are posed. For this reason, Tallman and Carlson define three subcategories of open-ended tasks: *short answer*, *broad open-ended*, and *word problem*. A short answer item is similar in form to a multiple-choice item, but without the choices. A student can anticipate the form of the solution of a short answer item upon reading the item. In contrast, the form of the solution of a broad open-ended item is not recognizable upon immediate inspection of the item. Broad open-ended items therefore elicit various responses, with each response typically supported by some explanation. Word problems can be of a short answer or broad open-ended format, but prompt students to create an algebraic, tabular and/or graphical model to relate specified quantities in the problem, and may also prompt students to make inferences about the quantities in the context using the model. Also, tasks that require students to explain their reasoning or justify their solution can be supplements of short answer or broad open-ended items.

Exam Characterization Results of the National Sample

Tallman and Carlson coded 14.83% of items in their randomly-selected sample of 150 post-secondary calculus I final exams, collectively containing 3,735 items, at the "Understand" level of the item orientation taxonomy or higher. Their coding also revealed that 34.55% of items in their sample were not stated symbolically and required a symbolic representation in the solution. Also, Tallman and Carlson found that only 1.34% of items in their sample were broad open-ended questions.

Methods

Twelve instructors piloted up to 30 labs in 24 different first and second semester calculus classrooms at eight different institutions from Fall 2013 to Spring 2015. Training began with in-person and online meetings with pilot instructors before the start of the Fall semesters, and most of the instructors attended a three-day workshop outlining the goals, strategies, and activities of the project. We supported their implementation of the labs throughout the fall and spring semesters with online meetings with project personnel. The project website provided instructors with student materials, instructor notes for each lab, solutions, grading rubrics, and supporting handouts and virtual manipulatives. Support meetings frequently included discussions of assessing lab write-ups but did not include discussions of creating or grading exams.

To document changes in the pilot instructors' assessment practices, we collected mid-term and final exams from the calculus classes the instructors taught prior to implementing CLEAR Calculus labs and from the classes in which they were implementing the labs. Five of the instructors either had not previously taught calculus or were required to give exams that were created by other faculty, so these all exams from these instructors were removed from the comparative sample.

A lead researcher in the development of the ECF and its application in the national study trained two members of our team to code with the framework resulting in 89% agreement between coding the training sample. Subsequent training focused on discrepancies. One member of our team has coded 355 items from 21 exams given by six instructors prior to using CLEAR Calculus labs and 417 items from 22 exams given by the same instructors while implementing the labs. A small number of exams remain to be coded, and we will choose a random sample of items to be coded by the second team member and the trainer to determine agreement and resolve discrepancies.

We collected self-reported characterizations on the impact of pilot instructors' teaching and exams through their implementation of CLEAR Calculus labs. We are currently analyzing this data for themes in and for shifts in assessment priorities.

Preliminary Results

Our analysis of exams given by our pilot instructors prior to participating in the project revealed a pattern very similar to the national profile found by Tallman & Carlson (2012) as shown in Table 3. In contrast, while implementing the labs the instructors nearly tripled the frequency at which they asked questions requiring a demonstration or application of understanding (from 11.3% to 31.7%) and included representations other than symbolic expressions at over 1.5 times the previous frequency (36.7% to 58.5%). They asked for explanations nearly 4 times as often (4% to 15.1%) and included broad open-ended items over 5 times as often (0.8% to 4.1%).

Table 3
Shifts in CLEAR Calculus pilot instructor's assessment practices.

	Tallman & Carlson National Sample (3735 items)	Pilot instructors prior to CLEAR Calculus (355 items)	Pilot instructors with CLEAR Calculus (417 items)
Items requiring understanding or higher level reasoning	14.83%	11.3%	31.7%
Items involving representations other than symbolic	34.55%	36.7%	58.5%
Items requiring explanation	2.36%	4.0%	15.1%
Broad open-ended items	1.34%	0.8%	4.1%

Discussion Questions

As our coding is nearly complete our presentation of the complete analysis will be very close to the data shown above. In addition, we will present themes from our qualitative data on instructors' attributions for these changes as well as interesting patterns in the differences of the individual instructors. We will seek a discussion with the audience on questions of additional ways to analyze the ECF data to reveal other insights, potential follow-up questions to pursue with the instructors represented in this data, and additional data we may collect as we work with our third round of pilot instructors.

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Students Perceptions of Learning College Algebra Online using Adaptive Learning Technology

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Adaptive learning technology was used in the teaching of an online college algebra course. As students worked on the mastery goals set for them, the technology helped students identify content that they already understood and other content that they had yet to master. Goal orientation theory suggests that when learning is mastery-oriented, a student's motivation to learn may improve (Ames & Archer, 1988). Qualitative methodology was used to describe how students perceived the instruction of their college algebra course and their learning in the course. Preliminary findings suggested that an adaptive teaching approach may help build students' confidence because they can control the pace of instruction and chose where to focus their effort without drawing negative attention to themselves.

Keywords: adaptive learning, college algebra, blended learning

High failure rates in entry level mathematics courses continue to be problematic across college campuses in the United States (Haver, 2007). Low student motivation has been identified as a factor contributing to this high failure rate (Thomas & Higbee, 1999; Walter & Hart, 2009). Since most college students are required to take college algebra and many need additional mathematics courses such as trigonometry and calculus, it is important to design, implement, and evaluate instructional strategies that can increase a student's motivation to learn mathematics.

Theoretical Perspective

In order to combat low motivation, goal orientation theorists advocate interventions that utilize a mastery-oriented or goal-oriented learning approach (Ames & Archer, 1988). A goal oriented approach to learning focuses achievement on mastering a task, the learning process, and self-improvement, whereas a performance oriented approach emphasizes normative standards or getting the highest grades. Ames and Archer (1988) found that when a mastery goal oriented approach was perceived by students, students reported using more learning strategies, enjoying their class, and having willingness to tackle challenging problems.

Literature Review

Universities have continued to increase the number of online course offerings each year in an effort to accommodate the needs of today's student (Allen & Seaman, 2014). Adaptive learning technologies have emerged in online courses as a means to customize instruction to learners' backgrounds, experiences, and prior knowledge. Adaptive learning technologies have provided students with an opportunity to self-pace instruction and an opportunity to focus instruction on their individual needs rather than the collective needs of the whole class (Vandewaetere, Desmet, & Clarebout, 2011). As researchers have begun to evaluate the effectiveness of adaptive learning technologies, findings have suggested that adaptive learning has positively impacted both student learning and satisfaction (McKenzie, Perini, Rohlf, Toukhsati, Conduit & Sanson, 2013). The implementation of an adaptive teaching approach may provide instructors an opportunity to design an online college algebra course that addresses individual learner needs, supports a mastery oriented learning approach, and in turn motivates students to learn college algebra.

In order to effectively examine the efficacy of this teaching approach as a way to bridge the gap between perceived ability and actual mathematical ability, this study was conducted as part of a larger design and development research study. This paper focused on the outcomes of the study specifically relating to student perceptions of the adaptive teaching approach used in their college algebra course. The research question guiding this study was: What are the students' perceptions of the adaptive teaching approach used in their college algebra course?

Instructional Approach

For the purposes of this study, an adaptive learning courseware was implemented as the primary source of instruction for one section of an online college algebra course at Northeast University in the United States. The courseware implemented was an artificially intelligent assessment and learning system that used adaptive questioning to determine which topics students already knew and which topics students needed to learn.

Before the course began, the instructor identified which topics to include in the college algebra course. On the first day of class, students completed a pre-test in the courseware. Student performance on the test determined how many college algebra topics each student had yet to master. As students worked through topics in the courseware they were able to take advantage of several online resources. These resources included lecture videos, an e-version of the textbook, worked examples, and written explanations. In addition, the instructor held weekly office hours online and on-campus.

Mastery goals were set to encourage students to maintain an appropriate pace (see Table 1). By the end of week 1, students were to have mastered 20% of all topics in the course and by the end of week 6 they were to have mastered 100%. Grades were awarded each week. For example, if a student mastered 19% of the topics in week 1, his grade would be 95% for total mastery for week 1.

Table 1

Summary of Weekly Mastery Goals

Mastery goals	**Total course mastery**
Week 1	20%
Week 2	40%
Week 3	60%
Week 4	75%
Week 5	90%
Week 6	100%

The students' final grade was determined by their progress on specific mastery goals, a time goal, and two exams (see Table 2). Total mastery was worth 30% of their final grade and reflected the number of topics the student mastered out of the total number of topics (403 topics). Mastery of each objective was worth 35% of the final grade and reflected the number of topics mastered within each of the five objectives (Equations and Inequalities, Graphs and Functions, Polynomial and Rational Functions, Exponential and Logarithmic Functions, and Systems of Equations). The weekly time requirement was worth 10% of the final grade. Students were required to spend ten hours a week working in the courseware. A comprehensive midterm and final exam were given and worth 25% of the final grade.

Table 2

Components of Students' Final Grade

Component	Percentage of final grade
Total Mastery	30%
Mastery of each objective	35%
Weekly Time Spent	10%
Midterm/Final Exam	25%

Methodology

Participants included 27 undergraduate students and the instructor of record. The instructor was also the researcher and has been teaching algebra for the last 15 years. Qualitative methodology was used to analyze the empirical materials (Miles & Huberman, 1994).

An online anonymous survey with eight open-response questions was administered to all students during the last week of class. These surveys were used to provide students' an opportunity to assess their learning and the course. Twenty students completed the survey. In addition, a university course evaluation questionnaire was administered to all students during the last week of class. Question formats included Likert scale questions (both university-developed and instructor-developed) and open-response questions. Twenty-one students completed university evaluations.

Data analysis is still in a preliminary stage, however; the following steps have been taken. First, an initial reading of student responses was conducted. Starter themes emerged as the researcher was able to organize the responses into categories. Next, responses were coded according to each category. Sub-themes in each category were identified because of their frequent occurrence.

Results

Findings are described according to the themes and sub-themes that emerged during data analysis. Student responses were categorized into two themes: instructional approach and self-evaluation of learning. Sub-themes for theme one were: Adaptive learning technology, course structure, opportunity to self-pace instruction, and suggestions for improvement. Sub-themes for theme two were: self-discipline and overall learning in the course. Each theme and sub-theme is explained in detail in the following paragraphs. Specific quotations from the student surveys are provided to exemplify each theme.

Theme one: Instructional approach

Theme one included student responses from the anonymous survey and university course evaluations that specifically referred to the instruction of the course.

Sub-theme one: Adaptive learning technology

Students overwhelming felt that the adaptive learning courseware enabled them to work at their own pace. When asked, "What do you like best about the course?", 65% of the students mentioned that they liked working at their own pace. One student wrote, "I was able to move through the program at a pace that worked best for me." Another student wrote, "I like that it is self-paced." Thirty percent of the students provided comments related to how the adaptive nature of the instruction positively impacted their learning. One student commented, "I like the adaptive structure because it measures us on what we know instead of the class as a whole and I think this is an easier way to learn instead of having to keep up with everyone."

When asked directly, "Do you like the adaptive nature of the course?", 80% of the students answered "yes." Comments included, "yes, because I can spend more time learning things that I actually already need to work on and less time learning things I already know" and "yes, I struggle with math so I'm not holding anyone back." Five percent of students answered "no." Comments included, "no, there isn't enough time to work diligently on a topic you're struggling with otherwise you fall completely behind in the course." Fifteen percent of students answered "yes and no" citing reasons such as, "I like that it is online, but I struggle to learn all of this on my own."

Sub-theme two: Course structure

Students were asked, "What would you like to change about the course?" Twenty-six percent responded nothing. Seventy-four percent provided comments related to the structure of the course. Responses included references to pacing (the course took place during a 6-week summer session), lowering the weekly time goal, requiring office hours, providing more lecture videos, and not requiring the final exam to be taken on campus. Illustrative comments included, "I wish I was required to meet with my instructor once a week" and "If the course is only going to be in a 5-6 week time span, some of the material should be left out. If that can't happen, the course should be longer. Monday-Friday I spend an average of 5 hours a day working on the [Aleks] pie, yet I still struggle to meet the deadlines."

Theme two: Self-evaluation of learning

Theme two included student responses from the anonymous survey and university course evaluations that specifically referred to the instruction of the course.

Sub-theme one: Self-discipline

Students were asked, "What steps could you take to improve your own learning in this course?" Seventy-five percent admitted that they need to spend more time working on the course material. For example, one student said, "I should have invested more time and focused more and taken the opportunities to get help." Ten percent said that they were satisfied with their learning", and 15% identified resources that they did not take advantage of such as "office hours" and "online lecture videos".

Sub-theme two: Overall learning

The majority (81%) of students rated their overall learning in the course as good (19%) or excellent (62%). On the university course evaluations, students were asked to identify how much knowledge they have gained in the subject matter during the summer session. Forty-eight percent of students chose “quite a lot”, 24% chose “more than average”, 24% chose “some”, and 4% chose “very little”.

Implications

The purpose of this study was to describe students’ perceptions of the adaptive teaching approach used in their online college algebra course. Preliminary results indicated that students felt that the teaching approach allowed them to adjust the pace of the course to accommodate their individual learning needs. In addition, students liked that they could focus their energy on topics that they found difficult but did not have to spend time on topics that they already knew. However, many students still wanted their instructor to control the learning environment to some extent as they admitted to not taking the initiative to seek help or study on their own when it was not required.

Students appreciated the opportunity to work on topics while not feeling as though they were holding other students back. This outcome is consistent with findings from (Ames & Archer 1988) which found that when students were engage in the learning process and were working toward their own mastery goals rather than competing for high grades and out-performing other students, their motivation to learn improved. By focusing on mastery-goals, the instructor can provide an environment where potentially less motivated students feel safe. The incorporation of adaptive learning technologies should be studied more rigorously. Although this study has provided a glimpse into student perceptions regarding its use, more work is necessary to further examine a possible relationship between student motivation and the use of adaptive learning technologies in the instruction of undergraduate mathematics courses.

Questions for Audience

1. Do you have experience using adaptive learning technologies in your undergraduate mathematics courses? If so, do you have any recommendations for implementation?

2. Do you know of other studies that discuss the use of adaptive learning technologies in mathematics courses?

3. Do you know of other studies that connect the use of adaptive learning technologies with mastery-goal orientation?

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Covariational and parametric reasoning

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Researchers have argued that students can develop foundational meanings for a variety of mathematics topics via quantitative and covariational reasoning. We extend this research by examining two students' reasoning that we conjectured created an intellectual need for parametric functions. We first describe our theoretical background including different conceptions of covariation researchers have found useful when analyzing students' activities constructing and representing relationships between covarying quantities. We then present two students' activities during a teaching experiment in which they constructed and reasoned about covarying quantities and highlight aspects of the students' reasoning that we conjecture created an intellectual need for parametric functions. We conclude with implications the students' activities and reasoning have for future research and curriculum design.

Key words: Covariational reasoning; Quantitative reasoning; Parametric Functions; Cognition

An increasing number of researchers have made contributions to the literature base on students' quantitative and covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Carlson, Larsen, & Jacobs, 2001; Castillo-Garsow, 2012; Confrey & Smith, 1995; Ellis, 2007; Ellis, Ozgur, Kulow, Williams, & Amidon, 2012; Johnson, 2012a; Thompson, 1994a, 1994b) with respect both to students' understandings of various content areas (e.g., function classes, rate of change, and the fundamental theorem of calculus) and to their enactment of important mental processes (e.g., generalizing, modeling, and problem solving). Although maintaining the common intention of understanding students' covariational reasoning, researchers' treatments of covariation are varied. For instance, Confrey and Smith (1994, 1995) approached covariation in terms of reasoning about discrete numerical values, finding patterns in these values, and interpolating patterns between them. In contrast, Thompson and Saldanha (Saldanha & Thompson, 1998; Thompson, 2011) approached covariation in terms of coordinating changes in two continuous magnitudes thus not constraining covariation to the availability of numerical data or specified values.

In this report, we detail results from a teaching experiment in which students conceived of simultaneously covarying quantities in ways compatible with Thompson's and Saldanha's descriptions of covariation. We focus on the students' actions during the closing sessions of the teaching experiment to discuss how the students represented relationships that constituted some situation or phenomena using projected magnitudes with an associated coordinate system, which Moore and Thompson described as *emergent shape thinking* (2015, in preparation). In characterizing the students' reasoning, we highlight the parametric nature of their reasoning including the extent that students were explicitly aware of the parametric nature of their reasoning. We close by highlighting aspects of the students' reasoning that may have supported the students in developing an intellectual need (Harel, 2007) for parametric relationships and functions.

Theoretical Background

Researchers who draw from interpretations of Piagetian and radical constructivist theories of knowing and learning have developed definitions and frameworks they have found useful when describing the mental processes and conceptual structures entailed in reasoning about relationships between quantities (Carlson et al., 2002; Johnson, 2012a, 2012b; Moore & Thompson, 2015, in preparation; Steffe, 1991; Thompson, 1994a, 2011; Weber, 2012). Of importance to this report, Carlson et al. (2002) presented a developmental framework that allows for a fine-grained analysis of students' covariational reasoning. The authors identified mental actions students engage in when coordinating covarying quantities including coordinating direction of change (quantity A *increases* as quantity B *increases*; MA2), amounts of change (the *change* in quantity A *decreases* as quantity B *increases in equal successive amounts*; MA3), and rates of change (quantity A *increases at a decreasing rate* with respect to quantity B; MA4-5).

Also of importance to this report, Saldanha and Thompson (1998) described the developmental nature of images of covariation, "In early development one coordinates two quantities' values to think of one, then the other, then the first, then the second, and so on. Later images of covariation entail understanding time as a continuous quantity, so that, in one's image, the two quantities' values persist" (p. 298). Extending this description, Thompson (2011) provided a first-order model of such an understanding in which an individual conceives of a quantity's value, x, varying over (conceptual) time, t. The individual could then conceive of covering the domain of t-values using intervals of size ε, and consider the variation of x in these intervals (i.e. considering x_ε as the set of x-values $(x(t), x(t+\varepsilon)) = x(t_\varepsilon)$). Thompson (2011) concluded his description, "I can now represent a conception of two quantities' values covarying as $(x_\varepsilon, y_\varepsilon) = (x(t_\varepsilon), y(t_\varepsilon))$. I intend the pair $(x_\varepsilon, y_\varepsilon)$ to represent conceiving of a multiplicative object—an object that is produced by uniting in mind two or more quantities simultaneously" (p. 47). Apparent in both descriptions is the parametric nature of covariational reasoning; a student imagines two quantities varying with respect to (conceptual or experienced) time, eventually coordinating these two quantities with respect to each other to form a multiplicative unit.

Drawing on the parametric conceptions of covariation described by Thompson (2011) and Saldanha and Thompson (1998), researchers (Moore & Thompson, 2015, in preparation; Weber, 2012) have described *emergent shape thinking* as a student conceiving graphs in terms of an emergent trace constituted by covarying (projected) magnitudes. We use Figure 1 to represent instantiations of an emergent image of a graph representing the height and volume of liquid in a bottle covarying as liquid is poured into the bottle (i.e. $h_\varepsilon = h(t_\varepsilon)$ and $v_\varepsilon = v(t_\varepsilon)$ both increase as time, t_ε, increases). A student with such an image of a graph understands that the magnitude of the blue segment represents the height of liquid in the bottle and the magnitude of the red segment represents the volume of liquid in the bottle at a certain moment of (experiential or conceptual) time, and that the resulting trace is a product of tracking how these quantities covary with respect to (experiential or conceptual) time (i.e. understands the graph as representing $(h_\varepsilon, v_\varepsilon) = (h(t_\varepsilon), v(t_\varepsilon))$).

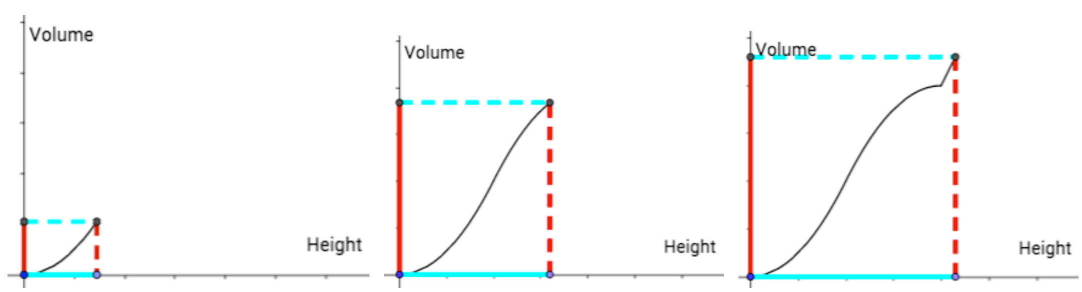


Figure 1: A representation of an emergent conception of covarying quantities.

Methods

We conducted a semester-long teaching experiment (Steffe & Thompson, 2000) with two undergraduate students, Arya and Katlyn (pseudonyms), to explore the ways of reasoning students engage in during activities intended to emphasize reasoning about relationships quantitatively and covariationally (e.g., if the students engaged in emergent shape thinking, what ways of reasoning supported this?). The students were enrolled in a secondary education mathematics program at a large state institution in the southern U.S. Both were juniors (in credit hours taken) who had successfully completed a calculus sequence and at least two additional courses beyond calculus. The teaching experiment consisted of three individual semi-structured task-based clinical interviews (per student) (Clement, 2000) and 15 paired teaching episodes (Steffe & Thompson, 2000). Each clinical interview and teaching episode lasted approximately 1.25 hours. We video and audio recorded the sessions and we captured and digitized records of the students' written work at the end of each episode.

When analyzing the data we conducted a conceptual analysis–"building models of what students actually know at some specific time and what they comprehend in specific situations" (Thompson, 2008, p. 60)–to develop and refine models of the students' mathematics. With this goal in mind, we analyzed the records from the teaching episodes using open (generative) and axial (convergent) approaches (Clement, 2000; Strauss & Corbin, 1998). Initially, we identified instances of Arya's and Katlyn's behaviors and actions that provided insights into each student's understandings. We used these instances to generate tentative models of the students' mathematics that we tested by searching for confirming or contradicting instances in their other activities. When evidence contradicted our constructed models, we made new hypotheses to explain the students' ways of operating and returned to prior data with these new hypotheses in mind for the purpose of modifying previous hypotheses or characterizing shifts in students' ways of operating.

Task Design

Throughout the teaching experiment, we provided Arya and Katlyn tasks that included prompts asking them to represent relationships between covarying quantities. We followed certain principles when designing these tasks. First, we designed tasks to include situations that would be familiar and accessible to the students, with most tasks including videos, applets, or images of phenomena (e.g., circular motion). Second, we avoided tasks that provided specific values for quantities to support the students in developing images of covariation that were magnitude based. Finally, we often asked students to construct multiple graphs related to a situation to explore if, and if so how, the students would leverage their images of the quantities and covariation between quantities when creating multiple graphs that may or may not differ in appearance.

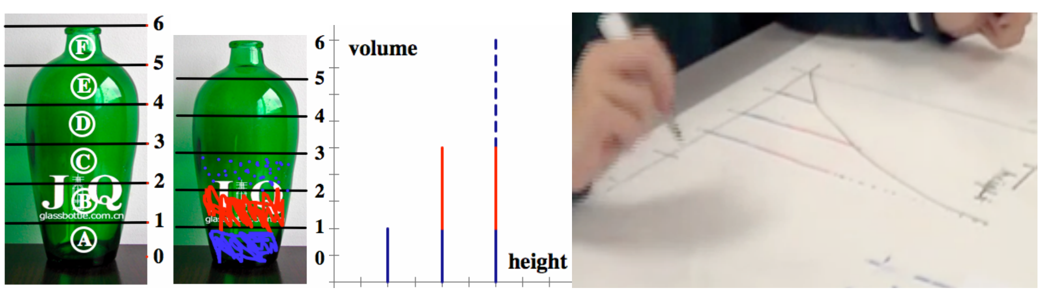
To illustrate, we used a variation of the *Bottle Problem*, which was designed by the Shell Centre (Swan & Shell Centre Team, 1985) and used by researchers investigating students' covariational reasoning (e.g., Carlson et al. (2002), Carlson et al. (2001), Johnson (2012, 2015)). We provided the students with a pictured bottle and asked them to imagine the experience of filling the bottle with liquid. We then asked them to graph the relationship between volume and height of liquid in the bottle as it filled with liquid. After they constructed a graph for a given bottle and a bottle for a given graph, we altered the prompt to ask the students to imagine liquid evaporating from the bottle. We then asked the students to represent the relationship between height and volume of liquid in the bottle for this new scenario.

Results

We first summarize the students' actions when creating graphs to represent how the height and volume of liquid covaried as a bottle filled with liquid. We then present their activities addressing liquid evaporating from the bottle in order to illustrate the students representing an additional aspect of the situation in their graph: the direction in which they imagined the graph being traced out. We conclude by highlighting the students' activities on a task that we implemented during the last clinical interview in which we explicitly asked the students to discuss a parametrically defined function for a graphed relationship.

Overview of students' activities addressing the Bottle Problem

As the teaching experiment progressed, the students exhibited activities indicative of reasoning about graphs as emergent traces representing two covarying quantities they conceived as constituting some situation (i.e. emergent shape thinking). For instance, during the first part of the Bottle Problem, each student coordinated how the volume and height of liquid in a bottle covaried in terms of direction of change (MA2) and amounts of change (MA3); each student conceived that the two quantities increase in tandem and then determined how the volume of liquid changes for equal successive increases in liquid height. Each student then created a graph while maintaining an explicit focus on how all drawn points and traces represented the relationship she conceived between the height and volume of liquid. As an example, consider Katlyn's activity as she created her graph (see Figure 2(c)). Describing why she was drawing the red segment longer than the blue segment, Katlyn stated, "'Cause this [*pointing to* (B) *in the picture of the bottle recreated in* Figure 2(a)] is so big compared to this [*pointing to* (A) *in the picture of the bottle*]." Katlyn then shaded in parts of her bottle (Figure 2(b)) corresponding to the segments in her graph, adding a dashed blue segment to represent the volume contained between tick 2 to tick 3 in her bottle (Figure 2(c)). Katlyn reasoned about the magnitudes of color-coordinated segments she constructed as representing amounts of volume within specific height intervals, understanding that each added segment corresponded to an amount of volume added to the total volume. Underlying this was Katlyn's understanding of the trace of her curve representing projected magnitudes as represented in Figure 1 (i.e. using Thompson's (2011) notation she conceived her graph as composed of the coordinate points $(h_\varepsilon, v_\varepsilon) = (h(t_\varepsilon), v(t_\varepsilon))$ with $h(t_\varepsilon)$ and $v(t_\varepsilon)$ representing height and volume as experiential or conceptual time, t_ε. elapses).

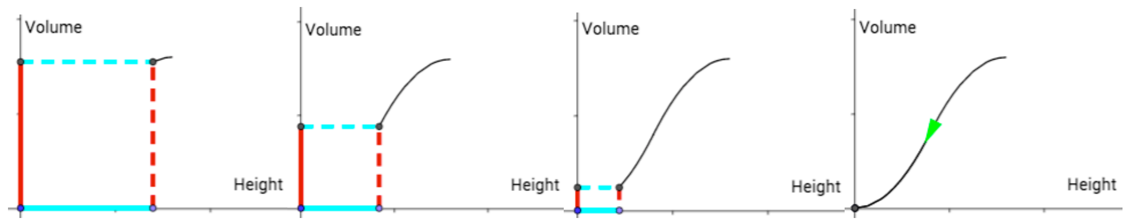


(a) (b) (c) (d)

Figure 2: (a) Katlyn's bottle (numbers and letters added for referencing), (b-c) Katlyn representing total volume with respect to height in the situation and a graph, and (d) Katlyn's resultant graph from tick 0 to tick 3.

Addressing water evaporating from the bottle

After the students had constructed a graph for the bottle in Figure 2(a), we asked them to graph the relationship between height and volume of liquid in the bottle as the liquid evaporated from the bottle. We asked them to complete the graph on the same whiteboard as a graph representing the relationship between height and volume of liquid in the bottle as the bottle filled. Indicating they did not anticipate that their previously completed graph might represent the posed relationship, the pair first drew a new set of axes. As they continued to consider the new scenario, Arya noted they should start at "full volume, full height." Katlyn then added, "It's going to look backwards… We can literally just travel this way instead [*motioning over the completed prior graph from the top-right most point back to the origin*]. [*To the interviewers*] We're done, we're just going to travel this way [*again motioning over the original curve from the top-right most point back to the origin*]." As the interaction continued, Katlyn's actions suggested she now conceived the prior graph as $(h_{2\varepsilon}, v_{2\varepsilon}) = (h(t_{2\varepsilon}), v(t_{2\varepsilon}))$ with $h(t_{2\varepsilon})$ and $v(t_{2\varepsilon})$ decreasing as experiential or conceptual time in this second situation, $t_{2\varepsilon}$, elapses (recreated in Figure 3(a)-(c)).



(a) (b) (c) (d)

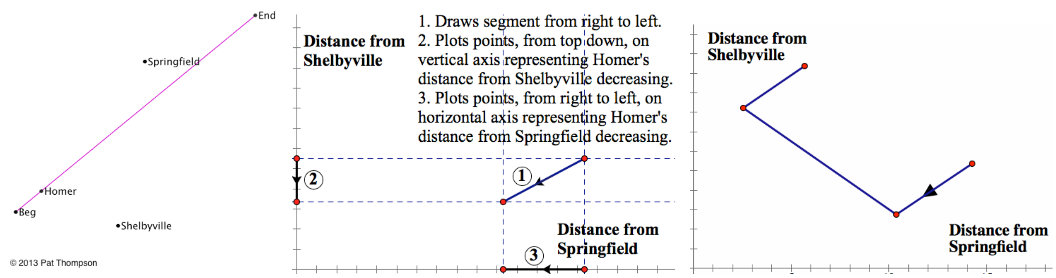
Figure 3: (a)-(c) A recreation of the students' graph as an emergent trace and (d) a recreation of their graph with the added arrow representing the direction of the trace.

To investigate if using the same curve for a new context created a perturbation for the students, we asked, "Is the situation the same? You're ending up with the same graph." Katlyn responded, "No, I just want to draw little arrows... we're going this way now [*draws an arrow on the curve pointing towards the origin, recreated in* Figure 3(d)]." As she addressed the displayed graph representing two (experientially) different situations, Katlyn differentiated the two situations by adding an arrow to indicate the direction in which the graph is traced out with respect to the second situation; Katlyn parameterized her graph (from our perspective) with respect to (experiential or conceptual) time to differentiate how it is traced out with respect to how the previous graph is traced out. Adopting Thompson's (2011) notation, Katlyn understood the displayed graph as composed of points (h, v) representing the appropriate magnitudes of height and volume of liquid in the bottle, regardless if liquid is entering or leaving the bottle. In the first scenario, she understood $(h, v) = (h_{1\varepsilon}, v_{1\varepsilon}) = (h(t_{1\varepsilon}), v(t_{1\varepsilon}))$ with $t_{1\varepsilon}$ representing (experiential or conceptual) time as liquid enters the bottle. In the second scenario she understood $(h, v) = (h_{2\varepsilon}, v_{2\varepsilon}) = (h(t_{2\varepsilon}), v(t_{2\varepsilon}))$ with $t_{2\varepsilon}$ representing (experiential or conceptual) time as liquid evaporates from the bottle.

The Car Problem

We conjectured that the students' actions addressing the Bottle Problem had the potential to support them in becoming explicitly aware of the parametric nature of their reasoning as well as possibly bringing to the surface parametric functions. We intended to explore the extent that we could support the students in bringing this reasoning to the forefront as they addressed the *Car Problem* that Saldanha and Thompson (1998) designed to investigate students' covariational reasoning. This task involves the students representing the relationship between an individual's distances from two cities as the individual travels back-and-forth along a road (see Figure 4(a)). Because the relationship is such that neither distance is a function of the other distance, we conjectured raising the idea of function after the students constructed their graphs might support them in reasoning about an explicitly defined parametric function.

Both students initially described the directional variation of each distance (e.g., as Homer moves from the beginning of his trip, the distance from each city decreases) (MA2). As Arya attempted to represent this relationship in her graph, she drew a segment from right to left getting closer to the horizontal and vertical axis (indicated by (1) in Figure 4(b)). After Arya re-described the directional relationship she conceived in the situation, she moved to her graph and marked points on each axis to confirm her graphed segment represented that Homer's distance from each city was decreasing (indicated by (2) and (3) in Figure 4(b)). As in previous situations, Arya conceived her graph as an emergent trace representing two projected covarying magnitudes, indicated by her careful attention to the axes when drawing this segment. Further, and similar to the students' activities addressing the Bottle Problem, Arya added an arrow to her completed graph (Figure 4(c)) to represent an additional aspect of the situation: how the graph was traced as Homer traveled along the road.



(a) (b) (c)

Figure 4: (a) The Car Problem applet, (b) a recreation of Arya's work, and (c) a recreation of Arya's final graph.

After Arya described that her graph did not represent distance from Springfield as a function of distance from Shelbyville or distance from Shelbyville as a function of distance from Springfield, and hoping to raise the idea of a parametrically defined function, a researcher asked, "What if your input was total distance traveled and your output was two-dimensional?" He then described the output as being composed of both the distance from Springfield and the distance from Shelbyville. Arya stated that this relationship represented a function as each total distance input corresponded to exactly one pair of distances.

Similarly, addressing whether the relationship with the same two-dimensional output but with 'distance on the path' as the input represented a function, Katlyn identified, "Well that's what [my graph] shows, right?" Katlyn stated that for any of Homer's distances on the path there was only one corresponding coordinate point on her graph, concluding that this relationship represented a function. Katlyn added, "I understand, like, what I've been drawing this whole time is like, how I'm traveling on like this purple path. But I don't, I never thought of that as my input, but it really is." Both students were able to assimilate a question concerning a one-dimensional input and two-dimensional output to consider a parametrically defined function after they had engaged in constructing the relationship via covariational reasoning and considered the graph as an emergent trace of this covariation.

Discussion

The students' activities here (and throughout the teaching experiment) provide examples of students who developed and maintained images of covariation we interpreted to be compatible with the descriptions of Thompson, Saldanha, and Moore. In addition, we conjecture the students' reasoning addressing the Bottle Problem raised an intellectual need for parametric functions, a need that we then capitalized on with the Car Problem. Harel

(2007) described, “The term *intellectual need* refers to a behavior that manifests itself internally with learners when they encounter an intrinsic problem—a problem they understand and appreciate” (emphasis in original, p. 13).

When addressing water evaporating in the Bottle Problem, the students’ actions resulted in their encountering an intrinsic problem (i.e. experiencing an intellectual need). Specifically, the students came to understand one curve as corresponding to two different experiential situations, which resulted in them seeking to determine how to differentiate between the two situations while using one curve. We conjecture that this problem, which was supported by their thinking about graphs as emergent traces of covarying quantities, was critical to the students considering the parametric nature of the relationships they represented. That is, by understanding one curve as representing two different emergent traces, the students became explicitly aware of their thinking about the curve in terms of two related quantities *and* (experiential or conceptual) time.

When addressing the Car Problem, we interpreted the students’ initial activities to indicate their reasoning parametrically about the relationship between Homer’s distance from the two cities covarying as Homer’s total distance or ‘distance on the path’ varied. However, the students did not explicitly conceive their graph parametrically until we asked the students to consider a relationship with a one-dimensional input and two-dimensional output as representing a function. Addressing this question, the students brought to the surface a particular conception of the graph, a graph as an emergent trace of covarying quantities, in relation to “function” (i.e. the uniqueness of a mapping). Both students described such a parametrically defined relationship as representing a function with Katlyn explicitly addressing the novelty of this reasoning to her (e.g., “I never thought of that as my input, but it really is”).

In one of the few studies examining students’ understanding of parametric functions, and parameters more generally, Keene defined dynamic reasoning as “developing and using conceptualizations about time as a dynamic parameter that implicitly or explicitly coordinates with other quantities to understand and solve problems” (2007, p. 231). The students’ reasoning was compatible with Keene’s (2007) definition of dynamic reasoning with their initial activities in each problem being compatible with Keene’s description of *implicitly* coordinating time with other quantities. Although the students engaged in reasoning that was parametric or dynamic in nature when responding to both tasks, the students did not exhibit activities to indicate they were *explicitly* aware of the parametric nature of their reasoning until they addressed later questions that we designed to focus in this area.

Unlike Keene (2007) and other researchers who have set out to examine students’ understandings of parameters and parametric function in differential equations or calculus settings (Stalvey & Vidakovic, 2015; Trigueros, 2004), in this study, we intended to examine students’ developing understandings of pre-calculus concepts through their quantitative and covariational reasoning; although this reasoning can be parametric in nature (e.g., emergent shape thinking) we did not expect to examine the students’ developing *parametric function* understandings. That fact that the students spontaneously engaged in reasoning that we interpreted as creating an intellectual need for parametric functions has both curricular and research implications. Future researchers and curriculum designers might examine how providing students with experiences in constructing graphs as emergent traces provide foundations for more explicit and formal introductions to parametric functions. For instance, and stemming from the current study ending before we could more extensively pursue the students’ reasoning about parametric relationships, researchers and educators should further explore how using different situations that result in students constructing and reasoning about the same displayed graph via different emergent traces has the potential to create an intellectual need for parametric relationships and functions.

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Prospective teachers’ evaluations of students’ proofs by mathematical induction

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This study examines how prospective secondary teachers validate several proofs by mathematical induction (MI) from hypothetical students and how their work with proof validations relates to how they grade their students’ proofs. When asked to give criteria for evaluating a student’s argument, participants wished to see a correct base step, inductive step, and algebra. However, participants prioritized the base step and inductive step over assessing the correctness of the algebra when validating and grading students’ arguments. All of the participants gave more points to an argument that presented only the inductive step than to an argument that presented only the base step. Two of the participants accepted the students’ argument addressing only the inductive step as a valid proof. Further studies are needed to determine how prospective teachers evaluate their students’ arguments by MI if many algebraic errors are present, especially in the inductive step.

Key words: Mathematical induction, Prospective secondary teachers, Proof validation, Proof grading

The proof method of mathematical induction (MI) is significant in the discipline of mathematics. In the *Principles and Standards for School Mathematics*, the National Council of Teachers of Mathematics (2000) asserts “students should learn that certain types of results are proved using the technique of mathematical induction” (p. 345). Secondary mathematics teachers are expected to teach MI (e.g., Australian Curriculum, Assessment, and Reporting Authority, 2012; California Department of Education, 2013; Korean Ministry of Education, Science, and Technology, 2012) and, therefore, are required to have a robust knowledge of MI as a prerequisite, including proficiency in reading and analyzing students’ arguments that use MI. Most of the previous studies on the learning and teaching of MI have focused on examining either the students’ or the teachers’ knowledge of MI, showing their difficulties with MI, especially in their proof production or while exploring the pedagogy of MI for better supporting students’ learning. Little research, however, has been devoted to how teachers read and reflect on students’ arguments using MI. In this study, I examine the characteristics of five prospective secondary teachers when validating and grading student arguments using MI. These arguments were presented in an interview setting and situated in the context of teaching at the secondary level.

Relevant Literature

Proof validation is an important mathematical activity, especially for mathematics undergraduates, prospective and practicing teachers, and mathematicians (Selden & Selden, 2003). Weber (2008) stated, “Teachers need to be able to determine if the justifications and proofs that students submit are acceptable and to provide feedback when they are not” (p. 4). Some researchers have begun to examine how undergraduate students, practicing teachers, and mathematicians validate proofs, but there have been few studies focusing on MI. Knuth (2002) found that some practicing teachers accepted an argument by MI as a proof by relying on its form (appearance) rather than understanding its reasoning. Dickerson (2006) found the same result in his study with two prospective teachers. In the process of examining both prospective secondary and elementary teachers' knowledge of proof by MI, Stylianides, Stylianides, and Philippou (2007) asked participants to validate two arguments, which were invalid. Stylianides et al. reported that although both groups had similar difficulties with MI, the prospective secondary teachers validated arguments more accurately than the prospective elementary teachers. In their study, participants who provided correct answers recognized that the first argument that they were asked to validate omitted the base step and judged the argument invalid. However, some of them were not able to explain the necessity of the base step. Stylianides et al. concluded that their participants focused on the form of proof by MI during proof validation.

Grading students' proofs is also an important teaching practice, but there has been little attention to how teachers assess and respond to students' written work. The process of *proof grading* includes judgments about a proof's validity, clarity, and readability. Previous studies (e.g., Inglis, Mejía-Ramos, Weber, & Alcock, 2013; Weber, 2008) showed that mathematicians used different criteria when evaluating students' proofs and disagreed on what arguments are considered valid proofs. These studies lead us to expect that mathematicians might also use different criteria when grading students' proofs and the scores that they give students might vary. Moore (2015), in his preliminary study, reported that the scores that four mathematicians assigned to students' proofs varied drastically even though they agreed on their overall evaluations of the proofs.

Theoretical Framework

Teacher learning occurs in multiple contexts such as "university mathematics and teacher-preparation courses, preparatory field experiences, and schools of employment" (Peressini, Borko, Romagnano, Knuth, & Willis, 2004, p. 69). According to the situative perspective, only relying on an individual's acquisition of knowledge without consideration of his or her participation in social contexts leads to difficulties in understanding his or her practices. A situative perspective is relevant for understanding how a teacher's knowledge can be recontextualized across situations. Borko et al. (2000) showed that the situative perspective assisted in understanding how a teacher, Ms. Savant, transferred her conceptions of proof as she participated in the multiple contexts of teacher education and in her actual teaching. Because I was interested in participants' conceptions of proof by MI in different situations, I used this situative lens (following Peressini et al., 2004) and situated my interview questions and proof tasks in participants' roles as teachers and students. The situative perspective was useful in making sense of the participants' responses. Because they had encountered MI as students and could imagine themselves encountering MI as teachers, the participants often referenced the settings of university and middle or high school mathematics classes when evaluating students' arguments and giving answers about their conceptions of proof by MI in school mathematics.

The activity of proof validation requires judging the correctness of arguments. Validating arguments is an important part of a teacher's work in assessing student work. A validator's judgment of whether an argument is a valid proof or not occurs mentally in his or her work on proof validation and, therefore, might not be observable. For analysis of the participants' proof validations, I referred to Selden and Selden's (2003) description of proof validation, which demonstrates it as a complex process by which someone reads and reflects on an argument in order to determine its validity. They suggested that the activity of proof validation includes such things as "asking and answering questions, assenting to claims, constructing sub-proofs, remembering or finding and interpreting other theorems and definitions, complying with instructions, and conscious feelings of rightness or wrongness" (p. 5). In this study, I examined the participants' behaviors in their proof validations and how they judged whether arguments by MI were valid.

Methodology

Five prospective secondary teachers, who were concurrently enrolled at the University of Georgia in either the undergraduate secondary mathematics teacher education program or the master's degree program leading to certification as a secondary school mathematics teacher, participated in this study from July to the middle of November 2014. Three of the participants were pursuing dual degrees in mathematics and mathematics education. All had taken Introduction to Higher Mathematics offered by the Mathematics Department in which they learn mathematical reasoning and proof writing, including proof by MI. Pseudonyms were used for identifying the participants –Emily, Jason, David, Brad, and Blain – to protect their anonymity. For this study, I conducted semi-structured interviews of about 80 minutes in length (one interview per participant). In the interviews, participants were asked to communicate their thoughts about the teaching and learning of MI, to prove two mathematical statements (an equation problem and an inequality problem), and to evaluate students' arguments purported to be proofs by MI that respond to the same statements (three arguments per statement). I created the student arguments used in this study by referring to literature (e.g., Baker, 1996; Harel, 2002) showing students' common mistakes in proving by MI (see Table 1 for a summary of the proof tasks). When validating arguments, the participants were asked what they thought about each argument and whether each argument was a valid proof and was convincing. Also, they were asked to assign a grade (out of 10 possible points) for each argument. The following are some of the questions I asked during the interview: *Is this argument a valid proof? Why? How many points would you assign each argument? What factors would go into your grading?* I conducted, video-recorded and transcribed all the interviews, and the transcriptions were checked by another person to verify their accuracy.

Table 1

A Summary of the Proof Tasks Presented to the Participants

Problem	Argument	Argument summary
Prove that for any positive integer *n*, $1 + 2 + \cdots + \mathrm{n} = \frac{\mathrm{n(n+1)}}{2}$	Rebecca's	No base step
	Shane's	No inductive step This argument concludes that the statement is true from three cases
	Polly's	One minor algebraic error in the inductive step
Prove that for any positive integer $\mathrm{n} \geq 4$, $2^{\mathrm{n}} < n!$	Kelly's	Incorrect base step This argument shows that the statement is not true for $n = 1$
	Garrison's	One equality/inequality error in the inductive step
	Laura's	In the inductive step, this argument addresses the induction hypothesis but does not show how $P(k+1)$ is derived from $P(k)$

For the data analysis, I used an open coding system (Strauss & Corbin, 1990). I first identified and coded parts of the data where participants talked about MI in general and then separated them from the parts in which the participants were working with the six arguments. After that, I summarized how each participant validated the students' arguments, including grades/scores they would give the arguments, and then compared their work across the participants.

Results

In this section, I report characteristics of the participants' work when validating and grading students' arguments that use MI and what relationships existed between their work with proof validation and their grading work on the students' arguments. Participants used similar criteria when evaluating students' arguments such as a correct base step, induction hypothesis, inductive step, and algebra. Some of the participants considered whether the arguments addressed the concluding statements and used the $P(n)$ notation, but they did not focus as much on these aspects in their evaluations.

The Equation Problem (Prove that for any positive integer *n,* $1 + 2 + \cdots + \mathrm{n} = \frac{\mathrm{n}(\mathrm{n}+1)}{2}$)

When analyzing Rebecca's argument, all participants recognized that there was no base step, and three of them concluded that this argument was not a valid proof. Two of participants, David and Brad, accepted this argument as a valid proof based on either their past learning experiences with MI or their perceptions of proof by MI. For example, when asked whether Rebecca's argument was a valid proof, Brad said, "It's a valid proof. Like I said, the only problem is, basically, there is no base case and there was no checking that the statement is true with that." When grading Rebecca's argument, participants took two or three points off (out of ten possible points) on average (See Table 2). For Shane's argument, all participants pointed out that there was no inductive step and concluded that this argument was not a valid proof. When grading Shane's argument, they gave him lower grades than they had given Rebecca's argument by observing that either the inductive step was an important part of proof by induction or that the inductive step was harder for students to understand than the base step. David, for instance, stated, "I think any students could be able to prove the base case, because that's not that hard, and any students could be able to look at the inductive step and then to say if the statement is true or not. But, actually defining a statement from a given problem and then taking out the inductive step takes a lot more careful effort and more cognitive demand. And, so that's why they put more emphasis on those parts of the questions." For Polly's argument, the participants checked each step of the argument, including whether the base step and inductive step were using algebra correctly. However, none of the participants recognized one minor algebraic error in the inductive step, even though four of them had correctly proven this statement before examining the students' arguments (Jason was not able to complete the inductive step). They determined that this argument was a valid proof in that everything – the base step, inductive step, and algebra – was correct and gave it full credit (see Table 2).

The Inequality Problem (Prove that for any positive integer $\mathrm{n} \geq 4$, $2^{\mathrm{n}} < n!$)

When validating Kelly's argument, all of the participants recognized that she used an incorrect base case, determined that this argument was not a valid proof and gave her a small amount of credit (less than 3 out of the 10 points; see Table 2). In evaluating Garrison's argument, two of the participants found one algebraic error in the inductive step, but they did not put as much emphasis on this minor error in their validation and even in their grading. Brad said, "I'm less concerned with so much of the algebra. I'm looking at the logic and the use of induction," while completing his validation work. The other three participants did not recognize the error. However, all of participants concluded that Garrison's argument was a valid proof, and all except David gave him full credit. David took one point off Garrison's argument by pointing out that "he did not define $P(n)$." David was the only participant who discussed the proper use of notation in his evaluation of Garrison's argument. As for Laura's argument, all of participants recognized that Laura addressed only the inductive hypothesis and did not show the inductive step. So, they concluded that this argument was not a valid proof and gave her 5.4 points (out of 10) on average (see Table 2).

Table 2

Scores Assigned to the Students' Arguments by the Participants

Participant	Argument (out of 10 possible points)					
	Rebecca's	Shane's	Polly's	Kelly's	Garrison's	Laura's
Emily	8	5	10	2	10	6
Jason	7	5	10	—*	10	7
David	8	2	10	0	9	4
Brad	8 or 9	3 or 4	10	2 or 3	10	6
Blain	6 or 7	2	10	1	10	4
Average score	7.6	3.4	10	1.25	9.8	5.4

* Jason did not assign a score for Kelly's argument, but instead, he stated, "I won't give her zero. I would say…just some kind of credit" when asked to assign a grade for Kelly's argument.

Conclusion

Overall, when grading the arguments, the participants gave more points to an argument that presented only the inductive step, rather than an argument that presented only the base step. Participants gave an argument full credit when they concluded that it included the correct base step, inductive step, and algebra. Even when they noted a minor algebraic error, most participants gave the student full credit, as was the case with Garrison's argument. Such criteria were also used when validating whether the arguments were valid proofs or not. When asked what criteria they used for proof validation, they wished to see the correct base step, inductive step, and algebra. All participants accepted the student arguments, recognizing three components as determinants of the proofs being valid or invalid. However, when given the student argument that addressed only the inductive step, two participants accepted that as a valid proof. Most participants compared the students' arguments to their own work when checking the correctness of the algebraic manipulations in the inductive step. However, some of the participants had difficulties understanding the students' algebraic manipulations and completely disregarded the algebraic manipulations in the proofs or presumed that all of the algebra in the inductive step was correct. Participants who recognized algebraic errors in the inductive step also did not put as much emphasis on the correctness of the algebra when validating students' proofs. Rather, both groups focused on the form of the arguments, whether they included the base case, inductive hypothesis, and inductive step, while validating and grading the proofs, without considering algebraic details. This finding raises questions about how the participants would evaluate student arguments if more algebraic errors were present. Future research should examine whether similar results can be found with other cohorts and how participants respond to student arguments by MI that include more errors in the algebraic manipulations.

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Re-claiming during proof production

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Abstract: In this research, I set out to elucidate the construct of Re-Claiming - a way in which students' conceptual understanding relates to their proof activity. This construct emerged during a broader research project in which I analyzed data from individual interviews with three students from a junior-level Modern Algebra course in order to model the students' understanding of inverse and identity, model their proof activity, and explore connections between the two models. Each stage of analysis consisted of iterative coding, drawing on grounded theory methodology (Charmaz, 2006; Glaser & Strauss, 1967). In order to model conceptual understanding, I draw on the form/function framework (Saxe, et al., 1998). I analyze proof activity using Aberdein's (2006a, 2006b) extension of Toulmin's (1969) model of argumentation. Reflection across these two analyses contributed to the development of the construct of Re-Claiming, which I describe and explore in this article.

Key words: *Mathematical Proof, Conceptual Understanding, Abstract Algebra*

Mathematical proof is an important area of mathematics education research that has gained emphasis over recent decades. The majority of empirical research in proof focuses on individuals' proof production (e.g., Alcock & Inglis, 2008), individuals' understanding of or beliefs about proof (e.g., Harel & Sowder, 1998), and how students develop notions of proof as they progress through higher-level mathematics courses (e.g., Tall & Mejia-Ramos, 2012). Researchers have also generated philosophical discussions that explore the purposes of proof (e.g., Bell 1976; de Villiers, 1990). Much of this latter discussion centers on the explanatory power of proof (e.g., Weber, 2010), with the primary focus being on the techniques and methods involved in a given proof (e.g., Thurston, 1996), rather than the development of concepts or definitions (Lakatos, 1976). Few studies, however, use grounded empirical data to explicitly discuss the relationships between an individual's conceptual understanding and his or her engagement in proof (e.g., Weber, 2005). In this research I set out to explicitly explore the relationships between students' conceptual understanding and proof activity.

Methods and Analytical Frameworks

Data were collected with nine students in a Junior-level introductory Abstract Algebra course, entitled *Modern Algebra.* The course met twice a week, for one hour and fifteen minutes per meeting, over fifteen weeks. The curriculum used in the course was *Teaching Abstract Algebra for Understanding* (TAAFU) (Larsen, 2013), an inquiry-oriented, RME-based curriculum, relies on Local Instructional Theories that anticipate students' development of conceptual understanding of ideas in group theory. Three individual interviews (forty-five to ninety minutes each) took place at the beginning, middle, and end of the semester, respectively. These interviews were semi-structured (Bernard, 1988) and used a common interview protocol so that each participant was asked the same questions as the others. Un-planned follow-up questions were asked during the interview to probe students' descriptions and assertions. The goal for each interview was to evoke the participants' discussion of inverse and identity and engage them in proof activity that involved inverse and identity. I developed initial protocols for these interviews, which were then discussed and refined with fellow mathematics education researchers.

Each interview began by prompting the student to both generally describe what "inverse" and "identity" meant to them and also to formally define the two mathematical concepts. Additional follow-up questions elicited specific details about what the participant meant by his/her given statements, figures, etc. The interview protocol then engaged each participant in specific mathematical activity aimed to elicit engagement in proof or proof related activity. Participants were asked to prove given statements, conjecture about mathematical relationships, and describe how he or she might prove a given statement. As with the questions about defining, each of these tasks had planned and unplanned follow-up questions so that all participants were asked at least the same base questions, but their reasoning was thoroughly explored. Throughout the interviews I kept field notes documenting participants' responses to each interview task. I also audio and video recorded each of the interviews, and all participant work and field notes were retained and scanned into a PDF format. I then transcribed all spoken communication during each interview with three of the participants (Violet, Tucker, and John), including thick descriptions of participants' gestures.

The retrospective analysis of the three participants' interview responses consisted of three stages, which I ordered so that each stage built upon the previous stages toward a resolution of the research question. This consisted of an iterative coding process to generate thorough models of the participants' conceptual understanding and engagement in proof and proof-related activity. I carried out this analysis separately for each participant, coordinating each data source chronologically so that the model of each participant's conceptual understanding corresponds with his or her conceptual development over the semester. I then investigated relationships between the participant's conceptual understanding and proof activity, exploring instances in which meaningful interactions between understanding and activity occurred.

Models of individual students' understanding

In this research I operationalize participants' conceptual understanding using Saxe et al.'s constructs of form and function (Saxe, Dawson, Fall, & Howard, 1996; Saxe & Esmonde 2005; Saxe et al, 2009). Throughout the literature, forms are defined as cultural representations, gestures, and symbols that are adopted by an individual in order to serve a specific function in goal-directed activity (Saxe & Esmonde, 2005). Three facets constitute a form: a representational vehicle, a representational object, and a correspondence between the representational vehicle and representational object (Saxe & Esmonde, 2005). Saxe focuses on the use of forms to serve specific functions in goal-directed activity as well as shifts in form/function relations and their dynamic connections to goal formation. Through this framework, learning is associated with individuals' adoption of new forms to serve functions in goal-directed activity as well as the development of new goals in social interaction.

The form/function analysis for participants' understanding consisted of iterative analysis similar to Grounded Theory methodology (Charmaz, 2006; Glaser & Strauss, 1967). This analysis is differentiated from Grounded Theory most basically by the fact that the purpose of this specific analysis was not to develop a causal mechanism for changes in the students' conceptual understanding, but rather that it was used to develop a detailed model of students' conceptual understanding at given moments in time. For each interview transcript, I carried out an iteration of open coding targeted towards incidents in which the concepts of inverse and identity were mentioned or used. In this iteration, I focused on the representational vehicles used for the representational objects of identity and inverse and pulled excerpts that afforded insight into the correspondence that the participant was drawing between the representational vehicle and object in the moment. Along with the open codes, I developed rich descriptions of the participants' responses that served as running analytical memos. After the open coding, I carried out a second iteration of axial coding using the constant comparative method, in which open codes were compared with each other and generalized into broader descriptive categories. These categories emerged from the constant comparison of the open codes and were used to organize subsequent focused codes until saturation was reached. Throughout this process, I wrote analytical memos documenting the decisions that I made in forming the focused codes and, in turn, providing an audit trail for the decisions made in the development of the emerging categories. This supports the methodology's reliability (Charmaz, 2006).

Documenting engagement in proof

In order to model the participant's proof activity, I use Aberdein's (2006a) adaptation of Toulmin's (1969) model of argumentation. Several researchers have adopted Toulmin's model of argumentation to document proof (e.g., Fukawa-Connelly, 2013). This analytical tool organizes arguments based on the general structure of claim, warrant, and backing. In this structure, the claim is the general statement about which the individual argues. Data is a general rule or principle that supports the claim and a warrant justifies the use of the data to support the claim. More complicated arguments may use backing, which supports the warrant; rebuttal, which accounts for exceptions to the claim; and qualifier, which states the resulting force of the argument (Aberdein, 2006a). This structure is typically organized into a diagram, with each part of the argument constituting a node and directed edges emanating from the node to the part of the argument that it supports (Figure 1).

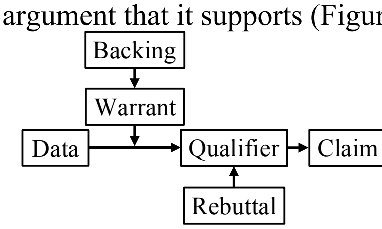

Figure 1. Visual representation of Toulmin models

Aberdein (2006a) provides a thorough discussion of using Toulmin models to organize proofs, including several examples relating the logical structure of an argument to a Toulmin model organizing it. Using "layout" to refer to the graphic organization of a Toulmin model, Aberdein includes a set of rules he to coordinate more complicated mathematical arguments in a process he calls combining layouts: "(1) treat data and claim as the nodes in a graph or network, (2) allow nodes to contain multiple propositions, (3) any node may function as the data or claim of a new layout, (4) the whole network may be treated as data in a new layout" (p. 213). The first two rules are relatively straightforward – the first focuses on the treatment of the graphical layout, as for the second, one can imagine including multiple data sources in the same data or claim node. The third and fourth rules provide a structure for combining different layouts and rely on organizational principles that Aberdein uses. He provides examples of combined layouts (Figure 2).

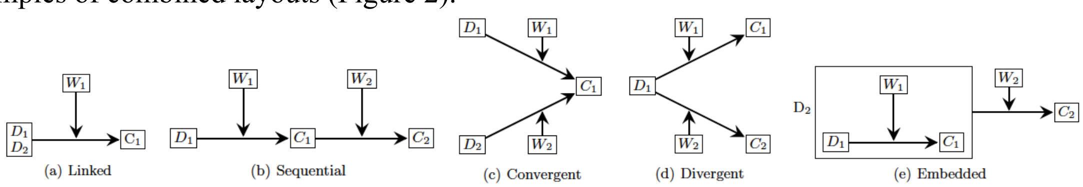

Figure 2. Five Ways of Combining Layouts (Aberdein, 2006a, p.214)

In this second stage of analysis, I first separated statements that conveyed a complete thought, initially focusing on complete sentences and clauses. I then reflected on the intention of each statement, focusing on prepositions and conjunctions that might serve to distinguish the intentions of utterances that comprise the sentence or clause. Following this, I compared these utterances to the model's constructs, focusing on which node an utterance might comprise. I constantly and iteratively compared each utterance relative to the overarching argument in order to parse out how the utterance served the argument in relation to other statements within the proof. For each proof, I then generated a working graphic organizer (i.e., a figure with the various nodes and how they are connected), including corresponding transcription highlighting the structure of the participant's argument. I then iteratively refined the graphical scheme to more closely reflect the structure of the argument as the participant communicated it. After this process, I completed a final iteration in which I compared the scheme to the participant's communication of the proof in its entirety to ensure that the model most accurately reflected the participant's communication of the proof. An expert in the field then compared and checked the developed Toulmin schemes against transcript of the interview in order to challenge my reasoning for the construction of the scheme, supporting the reliability of the constructions of the Toulmin schemes.

Relating conceptual understanding and proof

During the third and final stage of analysis, I focused on the participants' use of forms and functions within nodes of the Toulmin scheme, comparing the roles that specific forms and functions served in various nodes within the argument. I also focused on the shifts in which the participants' generated new, related arguments, specifically attending to concurrent shifts in forms and functions. I compared across arguments, looking for similarities and differences between the forms upon which the participant drew and the functions that the forms serve within the respective arguments. As in the previous stages, the analysis across conceptual understanding and proof centered on an iterative comparison of the patterns emerging across the analyses of the three participants' argumentation. In this comparison, I noted differences and similarities in the overall structures of Toulmin models for arguments. Further, I attended to the aspects of form/function relations that served consistent roles across similar types of extended Toulmin models. I continuously built and refined hypothesized emerging relationships through constant comparative analysis and memos. Through this process, I characterized constructs that unify the patterns found between the roles forms and functions of identity and inverse served across Toulmin schemes for the three participants.

Results

In this section, I discuss data from Tucker's second (midsemester) interview in order to demonstrate a broader construct of Re-Claiming that emerged during the third stage of analysis. I first discuss specific aspects of the form/function model of Tucker's understanding of inverse and identity relevant for discussing a selected part of his response to Question 7 of the protocol, which asked the participants to prove or disprove whether a defined subset H of a group G was subgroup of G (Figure 3). Specifically, Tucker's discussion throughout the interviews supported the development of three functions of inverse served by various forms of inverse (in this instance, the "letter" form of inverse): an "end-operating" *function* of inverse in which Tucker operates on the same end of both sides of an equation with a *form* of inverse, a "vanishing" *function* of inverse in which an element and its inverse are described as being operated together and are removed from an algebraic statement, and an "inverse-inverse " *function* of inverse characterized by an element serving a *function* of inverse in relation to its inverse. Throughout his proof activity in this excerpt, Tucker draws on the "letter" form of inverse to serve these functions.

"Prove or disprove the following: for a group G under operation * and a fixed element $h \in G$, the set $H = \{g \in G : g*h*g^{-1} = h\}$ is a subgroup of G."

Figure 3. Asking participants to prove about the normalizer of h

During part of his response to this part of the protocol, Tucker reads over his work and says, "I- you know what I might do actually?" (line 1078). He then begins an explanation, but pauses and restarts in order to explain his thinking more clearly, saying,

> So, right now, we have *g* star *h* star *g* inverse is equal to *h*. We want to get to somewhere that looks like- … Want to show. *g* inverse star *h* star *g* is equal to *h*. In

> order for the inverse of g to satisfy this (points to definition of H) right here. Cause that's what you do when you put in the g inverse. (lines 1084-1086).

With this excerpt, Tucker begins a subargument (Figure 4) for his broader, overarching proof in which he attempts to show that the set H contains inverses of its elements. He begins with the equation used to define H, saying, "right now, we have g star h star g inverse is equal to h" (line 1085), which serves as initial data (Data1.1) for the argument. He then describes wanting to show that $g^{-1}*h*g = h$, which serves as the claim in the subargument (Claim1). He supports this claim by explaining that this goal means that g^{-1} satisfies the given equation, saying, "Cause that's what you do when you put in the g inverse" (line 1087). This warrants the claim by reflecting Tucker's previous activity in which he replaced g in the equation used to define H with its inverse and drew on the "inverse-inverse" function of inverse to rewrite the equation ($g^{-1}*h*g = h$). This constitutes a shift in Tucker's description of what it would mean for the set H to contain inverse elements, anticipating a manipulation of the definition of H to result in the same equation.

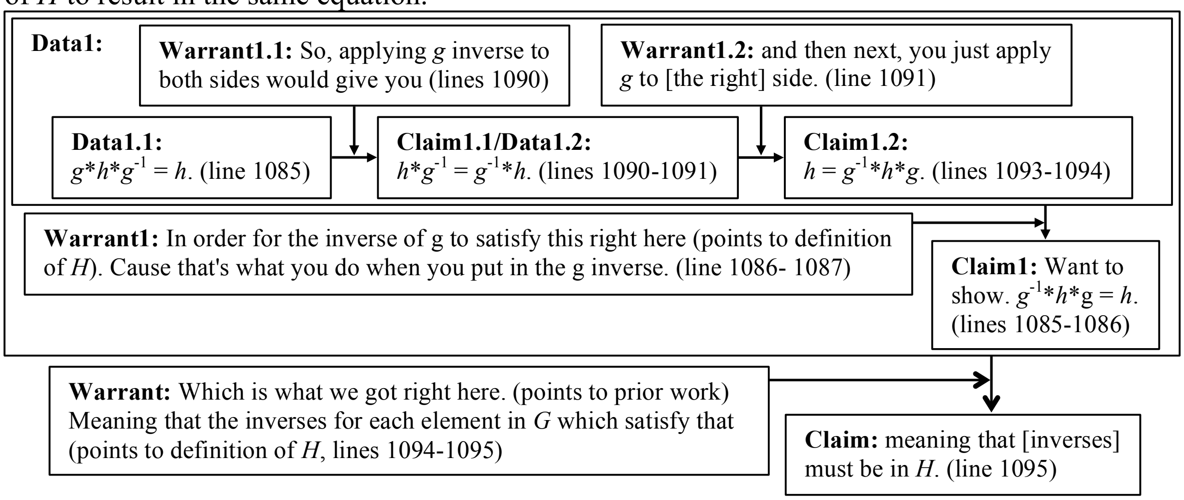

Figure 4. Tucker's inverse subproof in response to Interview 2, Q7

Tucker then continues, explaining how he might manipulate the first equation so that it looks like the second equation. Tucker begins by left-operating with g^{-1}, saying, "let's apply the g inverse to that. So, applying g inverse to both sides would give you h star g inverse is equal to g inverse star h" (Warrant1.1, lines 1089-1091). This process comprises a warrant that draws on the "end-operating" and the "vanishing" functions of inverse to support the claim that a new equation (Claim1.1/Data1.2) can be produced. This equation then serves as data as Tucker describes right-operating with g to produce the equation $h = g^{-1}*h*g$ (Claim1.2). Similar to the left-operation with g^{-1}, this draws on the "end-operating" and "vanishing" functions of inverse to warrant the new claim. However, this action also subtly draws on the "inverse-inverse" function of inverse in that Tucker is using the element g as the inverse of its own inverse in order to cancel the g^{-1} on the right end of the left-hand side of the equation. Tucker then interprets the result of this activity, saying, "Which is what we got right here. Meaning that the inverses for each element in G which satisfy that (points to definition of H), mean that must be in H" (lines 1093-1095), which comprises a warrant and claim for the overarching argument that H contains the inverses of its elements.

Tucker's work in this instance exemplifies a broader construct of re-claiming (Figure 5), which I define as the process of reframing an existing claim in a way that affords an individual the ability to draw on a specific form of identity or inverse and the functions that this form might be able to serve. In this study, it was often the case that re-claiming occurred when a participant was asked to prove or disprove a general statement and, in response, interpreted the general statement using a specific form to produce a new claim in terms of this form. An important part of successfully re-claiming is the consistency between the original claim and new claim. The individual must also be able to interpret any possible hypotheses or assumptions of the original claim with respect to the new form upon which they draw. Once the individual generates appropriate initial data from the given hypotheses and assumptions, he or she is then able to draw on the new form to serve specific functions, which affords the development of meaningful argumentation toward the new claim. Finally, after supporting the new claim, the individual should be able to provide a warrant for how or why this claim supports the original claim. More concisely, participants reinterpret a general claim by generating initial data in a specific form based on the original claim (in this case, being a proof by contradiction, they each draw on the "letter" form of identity to necessarily produce data contradictory to the original claim). They then draw on available functions of identity and inverse that this form serves in order to generate new. Finally, each participant interprets this claim to argument that it supports the original conjecture.

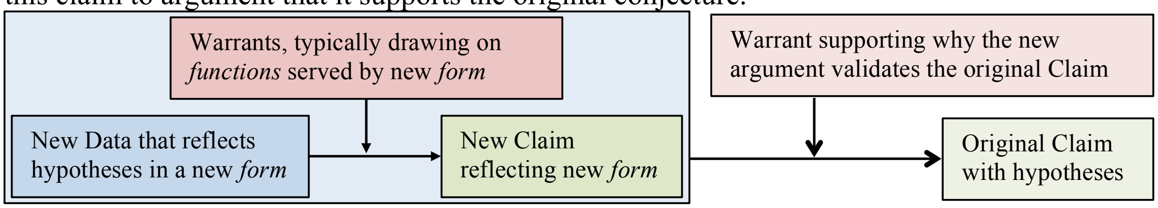

Figure 5. Toulmin scheme reflecting the general structure of re-claiming

A sense of the various facets involved in re-claiming can be drawn from the discussion of Tucker's proof activity. Specifically, in re-claiming, it is not sufficient, to only reframe a claim. Rather, one must likely also reframe its related (often hidden) hypotheses. These aspects of reclaiming reflect the frequently taught proof mantras of "what do I know?" and "what do I want to show?" In this case, Tucker describes needing to show that $g^{-1}*h*g = h$ and begins with the equation $g*h*g^{-1} = h$, which reflects the assumption that g satisfies the definition of H. In the context of the form/function framework, these restated hypotheses serve as initial data (drawing on a specific form of identity or inverse) in a new argument in which the participant is able to draw on the form of identity or inverse with which the data is reframed to serve appropriate functions of identity and inverse in support of the new claim. The individual should then be able to reason that this new argument supports the original claim. In this sense, Re-Claiming provides a type of proof activity in which an individual's conceptual understanding (forms upon which an individual draws and the functions that these forms are able to serve) informs the his or her proof approach. Specifically, the access to a form that is able to serve specific functions affords the individual an opportunity to generate a meaningful argument that he or she would likely not have been able to produce without Re-Claiming the initial statement. This activity is not necessarily an inherent necessity of a given conjecture, but rather depends on the individual's understanding in the moment. This reflects the importance of Balacheff's (1986) call to focus on students' understanding when considering their proof activity.

Conclusions

The current research was constrained by several factors. First, my focus on three students' responses to individual interview protocols limits analysis of the relationships between conceptual understanding and proof activity, warranting further analysis of different participants' conceptual understanding and proof activity. Also, although this analysis was informed by the broader contexts of the classroom environment, the focus on the individual interview setting affords insight into a specific community of proof in which argumentation develops differently than in other communities. For instance, the structure of the interview setting necessitated that participants developed their arguments solely relying on their own understanding in the moment and for the audience of a single interviewer. My early observations of and reflections on the development of argumentation in the classroom and homework groups included the mutual development of argumentation in which participants' argumentation was informed by their interactions. Accordingly, analysis of the classroom and homework group data is warranted.

This research contributes to the field by drawing on the form/function framework to characterize students' conceptual understanding of inverse and identity in Abstract Algebra. This affords insight into the forms upon which students participating in the TAAFU curriculum might draw as well as the various functions that these forms are able serve. The broader research also contributes to the field by providing several examples of how Aberdein's (2006a) extension of Toulmin's (1969) model of argumentation might be used to analyze proofs in an Abstract Algebra context. Further, this research draws attention to an aspect of the relationships between individuals' conceptual understanding and proof activity. These results situate well among the work of contemporary mathematics education researchers. For instance, Zazkis, Weber, and Mejia-Ramos (2014) have developed three constructs that also draw on Toulmin schemes to model students proofs in which the researchers focus on students development of formal arguments from informal arguments. These constructs provide interesting parallels with the three aspects of relationships between conceptual understanding and proof activity developed in the current research. Zazkis, Weber, and Mejia-Ramos (2014) describe the process of rewarranting, in which an individual relies on the warrant of an informal argument to generate a warrant in a more formal argument. However, the current research focuses more on the aspects of conceptual understanding that might inform such activity.

Moving forward from this research, I intend to analyze the data from other participants' individual interviews in order to develop more form and function codes for identity and inverse, affording deeper insight into the various form/function relations students in this class developed. Such analysis should also explore the proof activity of the other participants in the study, which would provide a larger sample of proof activity, in turn affording new and different insights into the relationships between mathematical proof and conceptual understanding. I also intend to analyze the sociomathematical norms and classroom math practices within the classroom. This will afford insight into the sociogenesis and ontogenesis of forms and functions at the classroom and small group levels in order to support and extend the individual analyses – which are focused on microgenesis – in the current research.

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Communicative Artifacts of Proof: Transitions from Ascertaining to Persuading

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With this poster, we wish to highlight an important aspect of the proving process. Specifically, we revisit Harel and Sowder's (1998, 2007) proof schemes to extend the authors' constructs of ascertaining and persuading. With this discussion, we reflect on the original theoretical framework in light of more recent research in the field and draw focus to a critical aspect of the proving process in which the prover generates the communicative artifacts of proof (CAP) critical to shifts between ascertaining to persuading. We also discuss possible ways in which an attention to the psychological and social activities involved in the development of the CAP might inform research and instruction.

Key words: Communicating Proof, Ascertaining, Persuading

With their influential work, Harel and Sowder (2007) outlined a perspective for viewing proof in which the authors distinguish between two primary subprocesses in the proving process – ascertaining and persuading.

> Seldom do these processes occur in separation. Among mathematically experienced people and in a classroom environment conducive to intellectual interactions among the students and between the students and the teacher, when one ascertains for oneself, it is most likely that one would consider how to convince others, and vice versa. Thus, proving emerges as a response to cognitive-social needs, rather than exclusively to cognitive needs or social needs - a view consistent with Cobb and Yackel's emergent perspective (p. 6, 2007).

As stated, the authors situate these subprocesses of proof relative to a broader community. This perspective emphasized both the individual's reasoning to gain conviction about the validity or invalidity of a conjecture and the individual's communication of his or her thinking. Importantly, the authors point out that ascertaining and persuading often occur simultaneously, underscoring the anticipation of communicating a person's reasoning in a community.

This aspect of the "proof schemes" framework connects well with more recent research emphasizing socially situated aspects of the proving process. For instance, Stylianides' (2007) provided a way of defining proof relative to the classroom community with three properties: "set of accepted statements…modes of argumentation…modes of argument representation" (pp. 291-292). In his discussion, Stylianides pointed out that individuals within a community may not agree on valid reasoning or types of arguments. While investigating mathematicians' practices, Weber (2008) also emphasized the importance of the community in which an argument is presented when considering the argument's validity. As Harel and Sowder stated in the excerpt, these points are consistent with an emergent perspective, which holds at the fore the development of mathematical practices as individuals' participate in mathematical communities. More recently, Weber (2010) maintained the emergent perspective by discussing the explanatory power of proof, focusing on the importance of the audience's interpretation of a proof as the source of any proof's explanatory power.

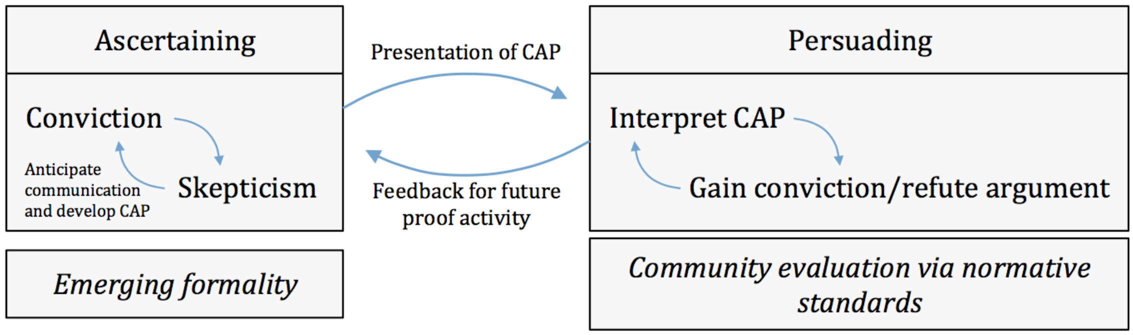

Figure 1: Diagram of interplay between individual and proof community

Our Theoretical Hypothesis on Ascertaining and Persuading

We envision the process of the development of a communicative artifact of proof as a result of the interplay between an individual and the proof community (Figure 1). While a version of this interplay may generalize to model any type of communication, we are focused on the dynamics involved in proof production. Specifically, we hypothesize that individuals' subprocess of ascertaining during proof production involves a cycle in which the individual balances conviction with skepticism. Throughout this process, the individual might anticipate the communication of their ideas within a broader community – anticipation that would likely inform the ascertaining subprocess, particularly during moments of skepticism, and persuading subprocess during the development and presentation of the communicative artifacts of proof (CAP).

The notion of CAP should inform the field's understanding and investigation of proof and the proving process by allowing proof researchers to distinguish between specific aspects of proof and focus on the specific proof activity in a participant's proving process. Further, these early notions of CAP can be developed to better explicate the types activity constituting the subprocesses of ascertaining and persuading. Refinement of the proving process may allow both researchers and instructors pinpoint hardships that students experience in their proving process, or may allow students to specifically target self-evaluation of their own proving.

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Analyzing Classroom Developments of Language and Notation for Interpreting Matrices as Linear Transformations

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As part of a larger study of students reasoning in linear algebra, this research analyzes how students make sense of language and notation introduced by instructors when learning matrices as linear transformations. This paper examines the implementation of an inquiry-oriented instruction that consists of students generating, composing, and inverting matrices in the context of increasing the height and leaning a letter "N" placed on a 2-dimensional Cartesian coordinate system (Wawro et. al., 2012). I analyzed two classroom implementations and noted how instructors introduced and formalized mathematical language and notation in the context of this particular instructional sequence, and then related that to the ways that language and notation were subsequently taken up by students. This work was conducted in order to enable me to build theory about the relationship between student learning and the ways in which language and notation are introduced.

Keywords: inquiry-oriented instruction, linear algebra, linear transformations, language and notation

The topic of matrix transformations is commonly taught in introductory linear algebra classes offered at many community colleges and four-year universities (Carlson, Johnson, Lay & Porter, 1993). The interpretations students need to develop and coordinate in the context of matrix transformations have been detailed in the literature (Larson and Zandieh, 2013). Studies have shown that students experience difficulties understanding functions (e.g. Oehrtman, Carlson, & Thompson, 2008); this work has the potential to inform our understanding of the difficulties students experience in coming to understand matrix transformations. This study has the potential to build theory about the relationship between how instructors introduce language and notation and how students make sense of that in the context of learning about matrix transformations.

Background

In the mathematics education community, researchers continue to find ways to support students' learning of concepts in ways that can be formalized into general definitions and theorems with instructional guidance. A hypothetical learning trajectory (HLT) is a theoretical model that is comprised of three components: learning objectives, a series of learning tasks, and a theorized learning process (Simon, 1995). This paper examines implementation of a particular HLT that consists of getting students to generate, compose, and invert matrices. This work is set in a 2-dimensional, geometric setting where students work to transform a letter "N" into a tall and leaner "N". Intuitively, students are investigating function mappings through a matrix transformation. Students' work in this context draws on ideas of matrix multiplication, noncommutativity of matrices, and invertible mappings.

Research Questions

When and how did instructors introduce and formalize language and notation in the context of this instructional sequence? How is the use of language and notation subsequently taken up by students?

Data Sources and Context

In order to explore these research questions, I analyzed two HLT classroom implementations that took place where students explored how to italicize the letter “N”. Data sources include video recordings of two different instructors implementing the instructional sequence at different institutions. The sequence took about three to four class periods, each of which were fifty minutes in length. My focus was on portions of the class in which there was whole class discussion and lecture.

Before students began working on the task sequence, the instructors both provided a review of three ways one can interpret the product of a matrix with a vector: as a linear combination, a system of equations, or a transformation. The third of these interpretations was highlighted and analogized to students’ previous work with functions as they begin to work on the three tasks. The first task consists of having the student figure out what matrix transforms a regular “N” to a tall and lean “N.” This task is aimed at supporting students to consider how the matrix representation of a transformation can be found by coordinating input vectors with output vectors. The second task requires students to consider the transformations from the previous task in two parts: one that stretches the “N” to make it taller and one that then skews/leans the taller “N” to make it look “italicized”. Students must coordinate this two-part transformation in a way that helps them conceptualize the composition of matrix transformations. The third task requires students to undo the italicization of the “N” by two ways: using a single matrix transformation and by using two separate matrix transformations. This is intended to give rise to the concept of invertible matrices, as students were instructed to find a matrix that ‘undoes’ the original transformation. In other words, students make sense of the definition of the inverse of an invertible square matrix A as that matrix B that “undoes” A so that $AB = I$ and $BA = I$ where I is the identity matrix.

Methods of Analysis

The first phase of my analysis involved developing content logs as I watched videos of both classroom implementations that were recorded. These logs contain detailed descriptions of the interactions between the students and the instructor. This information was organized in a table with time stamps for each key event. The type of interaction was be categorized as whole class discussion, student, group work, and lecture. I made note when language and notation was first introduced during the sessions. From there, I started a timeline of instances that summarizes when terminology and notation were introduced by the instructor and how those were subsequently used by students.

In the next phase of my analysis, I will first identify key terms and notation introduced by the two instructors, and develop categories for ways in which terms and notation were introduced, as well as categories for ways in which terms and notation were taken up by students. I will then trace the development of student thinking across the four days of instruction in each of the units. Finally, I will consider similarities and differences in the themes relating the categories for development of language and notation in the two classes. I will then discuss implications for when and how instructors might introduce definitions in order to bridge the gap between informal and formal mathematical language. I will also provide examples on my poster of the instances mentioned and speculate on patterns within and across these instances in order to address my research questions.

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Using the chain rule to develop secondary school teachers' Mathematical Knowledge for Teaching, focused on the rate of change: Secondary mathematics teachers' knowledge of the chain rule and its' impact on their teaching of the rate of change.

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The unit described in this study was designed to connect secondary and advanced mathematical topics. It focused on how the knowledge of chain rule impacts secondary teachers' understanding and teaching of rate of change so that they can address students' misconceptions. This project is informed by the idea of Mathematical Knowledge for Teaching, which encompasses both subject-matter knowledge and pedagogical content knowledge of teachers. The goal was to enhance secondary school teachers' teaching of the rate of change and the unit featured tasks connecting rate of change problems as seen in high school algebra to the concept of chain rule. The unit was designed to engage mathematics teachers in discourse about the content learned at the college level to content that is taught at the secondary school level.

Key words: [Chain rule, secondary school teaching, rate of change, Mathematical Knowledge for Teaching]

Introduction

According to the CCSSM standards, the concept of nonlinear models should be introduced in eighth grade but this introduction is limited to analyzing graphs to understand the functional relationship between two quantities (CCSS, 2013). This leads to several gaps in students' understanding of the rate of change of non-linear models. Furthermore since students learn the concept of linearity early in their primary and secondary school careers and this concept is reinforced they feel comfortable using it and tend to apply it without discretion (De Bock, 2002). Consequently, when students are introduced to higher level concepts, specifically in differential calculus, they struggle to develop conceptual understanding because of their assumptions of linearity (Brabham, 2014). It is important therefore to introduce rate of change as it relates to both linear and non-linear models so students get a deeper conceptual understanding of the rate of change. For this purpose we created an instructional unit as part of our class project.

Our unit was created with the goal to make connections between secondary and advanced mathematical topics. Specifically, our research question is: How does the knowledge of chain rule impact secondary school teachers' understanding and teaching of the rate of change?

Conceptual Framework

Our project is informed by the idea of Mathematical knowledge for teaching (MKT) (Ball, Thames & Phelps, 2008), which encompasses both subject-matter knowledge and pedagogical content knowledge. For example teachers need the knowledge to check correct answers, definitions and concepts but also specialized content knowledge to meet the demands of teaching mathematics. These include skills needed to pose questions, interpret students' responses, use multiple representations to provide explanations and most importantly make connections. Some MKT is a result of blending mathematics with other knowledge, like knowing the students, the curriculum, pedagogy etc. A recently added type of MKT is "horizon knowledge" which gives teachers a mathematical "peripheral vision" that is so important for effective teaching. It is "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p. 403). It provides a larger view of mathematics that gives the teachers a sense of where to place the content that they are teaching and how it is connected to higher level mathematics (Ball and Bass, 2009).

Research suggests that strong MKT is linked to certain habits of mind like careful attention to mathematical detail, reasoning skills, dexterity with various forms of mathematics curricula, working with students etc. Sometimes teachers develop this knowledge on their own by engaging in mathematics focused professional development but other times they need support (Hill & Ball, 2009). Based on the nature of MKT, there is a need to develop instructional guidance for teachers (Hill & Ball, 2009). For this reason, a unit was developed with the understanding that it will enhance secondary school teachers' teaching of the rate of change. The unit featured tasks connecting rate of change problems as seen in high school algebra to the concept of chain rule. Our goal in creating this unit was to engage mathematics teachers in discourse about the content learned at the college level to content that is taught at the secondary school level. Our unit employed tasks designed by Hill and Ball (2009) which focused on analyzing student errors, experiencing alternative solutions, choosing examples etc.

Research Methodology

Participants

The two participants in this study are fulltime students in a doctoral level mathematics education course. Of the two, one is currently a high school mathematics teacher in the United States and the other participant has several years of teaching experience in India.

Data Collection

The class was audio taped and observation notes of the lesson were taken focusing on participants' mathematical conversations and comments in regards to the lesson. Participants' work was also collected

Instructional Unit

The unit was designed to engage the teachers in discourse about the content learned and to make connections between mathematical content at the secondary and tertiary education levels. It posed a series of thought provoking questions which led the participants to examine the teaching of rate of change at the secondary level; specifically, the relationship between the instantaneous rates of change, the constant rate of change, and the chain rule. At the end of the lesson, the participants shared their reflection on the topic, both verbally and in writing.

Data Analysis

Our group analyzed participants' work to find recommendations for content and pedagogy to improve teaching of the concept of rate of change in secondary education. We also recorded any connections between secondary and tertiary mathematics as well as student misconceptions on this topic.

Implications

This pilot study focused on a unit developed as part of a class project. In the future, we plan to revise our unit and conduct this study with pre-service mathematics teachers.

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A national investigation of Precalculus through Calculus 2

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We present findings from a recently completed census survey of all mathematics departments that offer a graduate degree (Master's and/or PhD) in mathematics. The census survey is part of a larger project investigating department-level factors that influence student success over the entire progression of the introductory mathematics courses that are required of most STEM majors, beginning with Precalculus and continuing through the full year of single variable calculus. The findings paint a portrait of students' curricular experiences with Precalculus and single variable calculus, as well as the viewpoints held by departments of mathematics about that experience. We see that departments are not unaware of the value of particular features characteristic of more successful calculus programs, but that they are not always successful at implementation. However, our data also suggest hope for the future. Our work not only reveals what is currently happening, but also what is changing, how, and why.

Keywords: Census Survey, Precalculus, Calculus, Program success

There is a growing body of research pointing to why students are leaving STEM fields in general and first-year mathematics courses in particular. Contrary to common belief, introductory mathematics courses are not serving as a filter for students who are academically unprepared (Steen, 1988). Students who leave STEM majors are consistently shown to be as academically prepared as their persisting counterparts (Berrett, 2011; Rasmussen & Ellis, 2013; Reich, 2011; Taylor, 2011). Instead, students leaving STEM fields often cite poor instructional experiences in introductory level courses as the primary reason for their departure. These results are consistent with Tinto's integration framework, which emphasizes the effects of student engagement and integration on retention, especially in the first year of college (Kuh et al., 2008; Tinto, 1975, 2004). Integration occurs through a negotiation between the students' incoming social and academic norms and the norms of the department and broader institution. From this perspective, student persistence is viewed as a function of the dynamic relationship between the student and other actors within the institutional environment, including the classroom environment.

Literature focusing on student success in the pre-calculus to calculus sequence provides further insights into why students are leaving first-year mathematics courses (and therefore STEM fields). This research consistently indicates that: students are not learning what we want them to in these courses (Breidenbach et al., 1992; Carlson, 1998; Tallman et al., 2015; Thompson, 1994); these courses are not adequately preparing students for subsequent courses (Carlson, 1995, 1998; Selden & Selden, 1994; Thompson, 1994); students lose interest in STEM after taking these courses (Bressoud, Mesa, & Rasmussen, 2015; Seymour & Hewitt, 1997). These findings point to significant shortcomings in students' experiences. Unfortunately, many of these studies are focused on a limited number of institutions, a small number of students, or a single course.

What is currently missing is a national portrait of students' Precalculus through calculus curricular experiences and how these experiences relate to what is known about effective programs that support student success. In this paper we present initial findings from the *Progress though Calculus* (PtC) project, which builds on the insights from a recently completed five-year project, *Characteristics of Successful Programs in College Calculus* (CSPCC) (Bressoud, Mesa, & Rasmussen, 2015). The overall goal of the PtC project is to investigate, at a national level, department-level factors that influence student success over the entire progression of the introductory mathematics courses that are required of most STEM majors, beginning with Precalculus and continuing through the full year of single variable calculus. We refer to this sequence as Precalculus to Calculus 2 (P2C2).

As reported in Bressoud & Rasmussen (2014), the CSPCC study found that institutions with more successful Calculus I programs shared many of the following characteristics: (1) Calculus I was coordinated across sections and individual instructors contributed significantly to communal course decisions; (2) faculty used local data to check on the effectiveness of their program and make improvements; (3) for programs that made use of graduate, there was an extensive training of the Graduate-student Teaching Assistants; (4) faculty supported and encouraged active learning strategies; (5) the department had rigorous courses and high expectations for students; (6) the university offered many student supports, such as all day free tutoring centers and Supplemental Instruction; and (7) had adaptive placement systems that sought to place students in the highest course for which they could succeed.

In this report we address the following research questions:

1. How do mathematics departments prioritize the importance of the seven characteristics found in the CSPCC study?
2. How do mathematics departments characterize their implementation of the practices of successful programs identified in CSPCC study, what changes are being considered, and why?
3. What instructional format and structures (e.g., bridge courses, stretched out calculus) are currently in place in the P2C2 sequence and how common are they nationally?

Methods

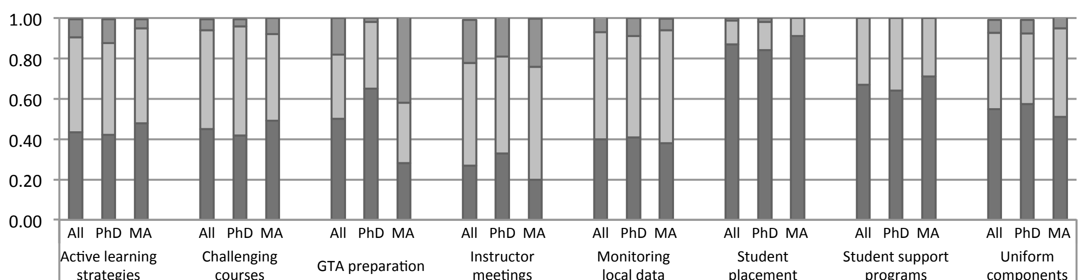
The five-year PtC project, which began in early 2015, is being conducted in two phases. Phase 1 is a census survey of all mathematics departments that offer a graduate degree in mathematics. These institutions were selected because they produce the bulk of STEM graduates while often struggling to find a balance between the demands of research and teaching. Phase 2 will consist of in depth longitudinal case studies. In this report we focus on initial findings from the census survey. In the United States there are a total of 330 institutions that offer either a Masters or PhD in mathematics. All 330 institutions, which included 178 PhD granting institutions and 152 Master's degree granting institutions, were surveyed. The overall response rate was 67.6%, with a response rate of 75% from the PhD institutions and 59% from the Master's institutions.

We designed the census survey to gather information on the implementation of the seven features of successful programs identified by the CSPCC project as well as to gain an understanding of the variety of P2C2 programs currently being implemented across the country, the prevalence of such programs, and what institutions are doing to improve their programs. The survey consisted of three main parts. Part I asked for a list of all courses in the mathematics department mainstream Precalculus/calculus sequence. “Mainstream” refers to any course in this sequence that would be part of student preparation for higher-level mathematics courses such as a sophomore- or junior-level course in differential equations or linear algebra. Part II asked about the departmental practices in support of the Precalculus/calculus sequence. Part III asked for detailed information about each course in the mainstream P2C2 sequence, including enrollment data and details about course delivery. The survey was closed mid August 2015.

Given the fact that the survey has only recently been closed, we begin analysis of the cleaned data set with descriptive statistics (counts, frequencies, means, standard deviations) which will then be followed by additional descriptive methods (e.g., Multiple Correspondence Analysis; clustering; Principal Components Analysis) to reveal patterns in the census data. Our aim is to identify models of existing P2C2 programs in their entirety rather than simply identifying patterns within individual components.

Sample Results

For research question 1, our data allow us to see how departments of mathematics view the practices identified in CSPCC as characteristic of successful institutions. Participants were asked to consider eight characteristics and group them by their importance to a successful P2C2 sequence. The results from this question are summarized in Figure 1. Note that in general, PhD- and MA-granting institutions agree on the importance of individual features, with the exception of GTA teaching preparation programs, in which case MA-granting institutions report the feature as less important than PhD-granting institutions. Of course many MA-granting institutions do not make extensive use of GTAs and so this difference is expected.

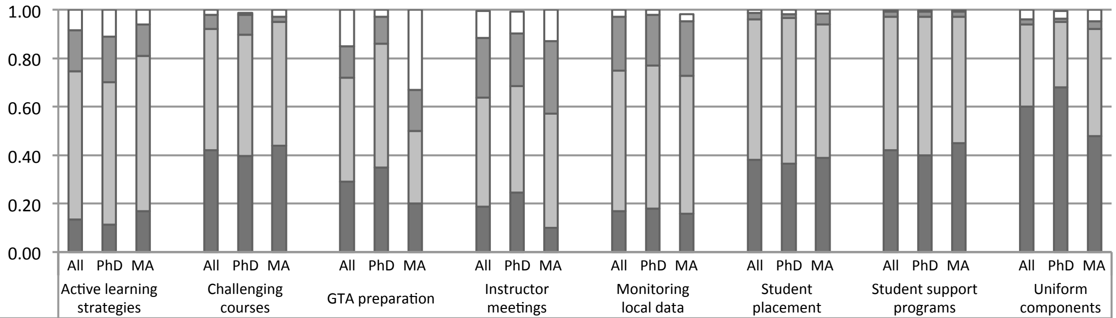


	Active learning strategies (All PhD MA)	Challenging courses (All PhD MA)	GTA preparation (All PhD MA)	Instructor meetings (All PhD MA)	Monitoring local data (All PhD MA)	Student placement (All PhD MA)	Student support programs (All PhD MA)	Uniform components (All PhD MA)
Not	0.09 0.12 0.05	0.05 0.04 0.08	0.18 0.02 0.43	0.21 0.20 0.24	0.08 0.09 0.06	0.01 0.02 0.00	0.00 0.00 0.00	0.06 0.07 0.06
Somewhat	0.47 0.46 0.47	0.49 0.54 0.43	0.32 0.33 0.30	0.51 0.48 0.56	0.53 0.50 0.56	0.12 0.14 0.09	0.33 0.36 0.29	0.38 0.35 0.44
Very	0.44 0.42 0.48	0.45 0.42 0.49	0.50 0.65 0.28	0.27 0.33 0.20	0.40 0.41 0.38	0.87 0.84 0.91	0.67 0.64 0.71	0.55 0.58 0.51

Figure 1. Reported importance of CSPCC features for successful P2C2 sequence. N_{all} = 219, N_{PhD}=132, N_{MA}=87.

Further, participants were asked how successful their program is with each of these features. The results from this question are summarized in Figure 2. Again we see general agreement between the institution types as to their relative success at implementation, with the exception of GTA teaching preparation, where MA-granting institutions reported a much higher rate of “NA.”

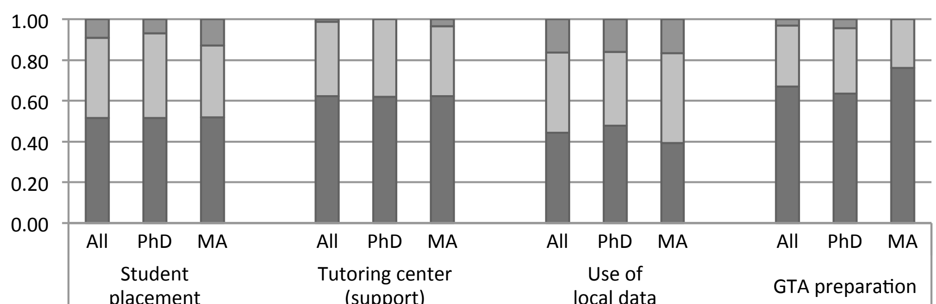
Student placement and student support programs are the two CSPCC features where the widest gap was observed between perceived importance and perceived success. Both were reported as very important to the success of a P2C2 sequence, but most participants reported that they were only somewhat successful at implementation of these features.



	Active learning strategies			Challenging courses			GTA preparation			Instructor meetings			Monitoring local data			Student placement			Student support programs			Uniform components		
	All	PhD	MA	All	PhD	MA	All	PhD	MA	All	PhD	MA	All	PhD	MA	All	PhD	MA	All	PhD	MA	All	PhD	MA
N/A	0.08	0.11	0.06	0.02	0.01	0.03	0.15	0.03	0.33	0.11	0.09	0.13	0.03	0.02	0.03	0.01	0.02	0.01	0.01	0.01	0.01	0.04	0.03	0.05
Not	0.17	0.19	0.13	0.06	0.08	0.02	0.13	0.11	0.17	0.25	0.22	0.30	0.22	0.21	0.23	0.03	0.02	0.05	0.02	0.02	0.02	0.02	0.02	0.03
Somewhat	0.61	0.59	0.64	0.50	0.50	0.51	0.43	0.51	0.30	0.45	0.44	0.47	0.58	0.59	0.57	0.58	0.60	0.55	0.55	0.57	0.52	0.34	0.27	0.44
Very	0.14	0.11	0.17	0.42	0.40	0.44	0.29	0.35	0.20	0.19	0.25	0.10	0.17	0.18	0.16	0.38	0.37	0.39	0.42	0.40	0.45	0.60	0.68	0.48

Figure 2. Reported success at CSPCC features for successful P2C2 sequence. N_{all} = 218, N_{PhD}=131, N_{MA}=87.

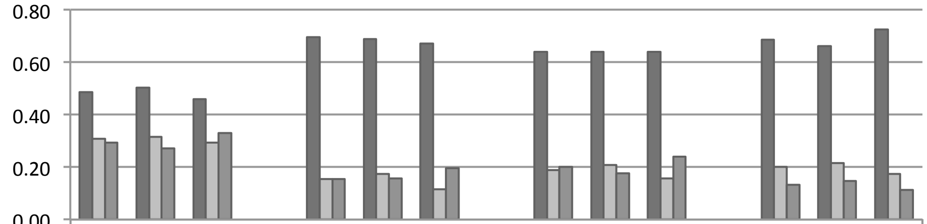
Regarding research question 2, our data also captures detailed aspects of how CSPCC practices are being implemented by departments of mathematics as well as what changes are being planned and why. In the full paper we will report on how departments initially place students into the P2C2 sequence, how they gather and use local data to monitor and modify the sequence, and what supports (in particular tutoring centers) are in place for P2C2 students. A separate proposal has been submitted that details how GTAs are prepared for their teaching roles. We also have information about satisfaction levels and the status of these features (i.e., if changes have recently occurred or are being planned). These results are summarized in Figures 3 and 4. Satisfaction ratings with the tutoring center and GTA preparation programs were collected only from departments reporting that they have these programs in place, while queries about status were asked of all participants.



	Student placement			Tutoring center (support)			Use of local data			GTA preparation		
	All	PhD	MA	All	PhD	MA	All	PhD	MA	All	PhD	MA
Not adequate	0.09	0.07	0.13	0.01	0.00	0.03	0.16	0.16	0.17	0.03	0.04	0.00
Could be improved	0.39	0.42	0.35	0.37	0.38	0.34	0.39	0.36	0.44	0.30	0.32	0.24
Adequate	0.52	0.52	0.52	0.62	0.62	0.62	0.44	0.48	0.39	0.67	0.64	0.76

Figure 3. Satisfaction levels with selected CSPCC features. Sample sizes for student placement and use of local data: N_{All}=217, N_{PhD}=132, N_{MA}=85. For the tutoring center: N_{All}=169, N_{PhD}=108, N_{MA}=61. For GTA preparation: N_{All}=160, N_{PhD}=118, N_{MA}=42.

Again we see that PhD- and MA-granting institutions report similar levels of satisfaction for each of these program features. However, the reports of being satisfied (program is adequate) are higher than reports of being "very successful" with these same programs. This appears to indicate that many departments are satisfied with being somewhat successful in their management of the P2C2 sequence.



	Student placement			Tutoring center (support)			Use of local data			GTA Preparation		
	All (218)	PhD (133)	MA (85)	All (167)	PhD (107)	MA (61)	All (213)	PhD (130)	MA (83)	All (210)	PhD (130)	MA (80)
No changes planned	0.49	0.50	0.46	0.69	0.69	0.67	0.64	0.64	0.64	0.69	0.66	0.73
Recent changes	0.31	0.32	0.29	0.16	0.17	0.11	0.19	0.21	0.16	0.20	0.22	0.18
Under discussion	0.29	0.27	0.33	0.16	0.16	0.20	0.20	0.18	0.24	0.13	0.15	0.11

Figure 4. Status of selected CSPCC features. Parentheticals indicate *N*. Options were not mutually exclusive.

The data regarding the status of individual CSPCC features indicates that most departments of mathematics are *not* planning changes to department-run tutoring centers, their use of local data, or GTA teaching preparation programs. While about half of participating schools indicate that no changes to their placement procedures are planned, it seems that this feature is the least static, tallying with our discovery that institutions across the nation feel that initial placement into the P2C2 sequence is important and that they are not entirely successful with this process. That most departments are not planning changes to their tutoring centers is more surprising, as it was widely reported that student supports are important to successful programs and departments do not report high rates of success. Note that in with all four features, the reports of "no changes planned" are higher than rates of "very successful." It appears that while many departments believe they are not entirely successful with their implementation of these CSPCC practices, they are not prepared to amend these processes.

In addition to the broad characterizations of satisfaction and status of department programs presented in this proposal, our presentation will include details of how the seven features identified in the CSPCC study are implemented across the nation with regards to the P2C2 sequence. For placement this will include initial placement procedures (e.g., AP exam results; MAA placement exam) and what (if any) procedures for revisiting and adjusting initial placement exist. Resources to support students include detailed information about the existence and format of tutoring centers for students in the P2C2 sequence, as well as supports available to students (e.g., online tutoring; arranged study groups) and any supports aimed particularly toward "at-risk" and/or underrepresented groups in mathematics (e.g., scholarships; targeted supplemental instruction). We will present also the types of local data that departments of mathematics collect and how departments have been using that data to inform decisions about their undergraduate program. In addition to reporting on the variety of implementation models, we will report on their relative frequency across institutions.

For research question 3, information about P2C2 instruction and structures in place across the nation was ascertained through Part III of the census survey. 201 institutions completed this section of the survey. Therein, participants were queried about details regarding the implementation of each course that is part of the mainstream P2C2 sequence at their institution.

Of particular interest are the data regarding primary instructional format. We collected detailed information about 904 P2C2 courses, and found that nearly 70% of these are reportedly taught in a lecture format and 15% are taught in a format that incorporates some active learning techniques alongside lecture. Around 10% of courses did not report a single primary instructional format, and fewer than 5% fell into the categories: mainly active learning (including flipped), lecture plus computer-based instruction, or "other." These values reflect the primary instructional format across all P2C2 courses, but the general pattern is the same for each category of courses (e.g., Precalculus courses, Calculus I courses). However, we note that the proportion of classes being taught in traditional lecture format increases through the sequence (from 57% to 75%), while all other formats decreased in frequency. That fewer than 20% of all P2C2 courses report incorporating any level of active learning in regular course meetings is remarkable, particularly in light of the fact that 44% of institutions noted that active learning strategies are "very important" to having a successful P2C2 sequence, and 75% reported being at least "somewhat successful" at implementing active learning strategies.

In the presentation we will provide more detail as to different P2C2 progressions that are in place as well as course-specific details such as enrollment data, DFW rates, instructor profiles, contact hour breakdown, prevalence and form of recitation sections, coordinated aspects of parallel sections, coordinator profiles, and the status of each course (e.g., if changes are being discussed). This will further illuminate what P2C2 sequences are experienced by undergraduates across the country.

Conclusion

This paper provides the first overview of the information we have gathered with regard to introductory undergraduate mathematics programs across the country. The findings paint a national portrait of students' curricular experiences with Precalculus and single variable calculus, as well as the viewpoints held by departments of mathematics about that experience. We see that departments are not unaware of the value of particular features, but that they are not always successful at implementation. However, our data also suggest hope for the future. Our work not only reveals what is currently happening, but also what is changing, how, and why. We note that many institutions reported in open-ended questions that they *want* to make improvements, but are not sure how. We believe that our work will not only describe what is happening in mathematics departments at the national scale, but will illuminate ways of reaching institutions interested in change – of which there are many. One institution wrote to us saying, "We should do more. This survey is giving me ideas." We suspect there are many other institutions ready for change. This report provides a first in its kind baseline of what is happening in the P2C2 sequence across the nation.

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On the Variety of the Multiplication Principle's Presentation in College Texts

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The Multiplication Principle is one of the most foundational principles of counting. Unlike foundational concepts in other fields, where there is uniformity in presentation across text and instruction, we have found that there is much variety in the presentation of the Multiplication Principle. This poster highlights the multiple aspects of this variety, specifically those with implications for the combinatorial research and education community. Such topics include the statement types, language and representation of statements, and mathematical implications.

Key words: Combinatorics, Multiplication Principle, Student Thinking, Textbook Analysis

Introduction and Research Questions

Combinatorics problems embody a duality of accessibility and difficulty for students at various levels. Because of the growing need for discrete mathematics in scientific fields, it is important for the mathematics education community to understand student conceptions of foundational counting principles and techniques. The multiplication principle (MP) is widely accepted as an important and fundamental principle in combinatorics, and serves as the basis for many basic counting formulas (Gersting, 1999; Mazur, 2009; Richmond & Richmond, 2009). We have experienced a variety in the presentation of the multiplication principle. This variety, and the importance of the principle motivated a formal analysis of a large sample of textbooks in combinatorics, finite, and discrete mathematics textbooks. This poster presents on the results of that study which sought to answer the following two research questions: 1) What is the nature and extent of the variation of statements of the multiplication principle presented in combinatorics, discrete mathematics, and finite mathematics textbooks?
2) What mathematical issues arise in comparing and contrasting different statements of the multiplication principle?

Relation to Literature

There have been recent studies that investigated student thinking in combinatorial contexts in which correct application of multiplication was a vital component of the learning process (eg. Lockwood & Coughman, 2015; Kavousian, 2008; Tillema, 2011; Tillema, 2013). There are also a number of researchers (e.g., Dubois, 1984; Fischbein & Gazit, 1988; Piaget, 1975) who have studied student discovery and application of counting formulas which rely heavily on the multiplication principle. While the above studies relate multiplication to counting, there is a lack of studies directly involving student thinking on the MP. This textbook analysis offers a glance at the pedagogical issues surround the MP that students are exposed to in their learning process.

Theoretical Perspectives

Researchers have examined textbooks to better understand how ideas are presented to students in the fields of linear algebra (Cook & Stewart, 2014; Harel, 1987), trigonometry (Mesa

& Goldstein, 2014), and abstract algebra (Capaldi, 2012). We adopt this examination to combinatorics texts.

We also utilize Lockwod's (2013) model for student combinatorial thinking in terms of sets of outcomes and counting processes. We combine Lockwood's model with Sfard's (1991) dual nature of mathematical conceptions. Her language reflects that students can think of mathematical concepts as objects (reflecting a structural conception) and process (reflecting an operational conception). This dualistic language proves vital to statement analysis.

Methodology

We selected textbooks for our analysis from a list of 76 colleges nation-wide. The list was made to include colleges from each state, as well as colleges of differing size and ranking. 6 colleges were excluded from the study. In total, we analyzed 32 textbooks that served as the assigned reading for 92 different courses from the 70 universities. We also then added textbooks from our personal libraries to make a total of 64 textbooks analyzed.

Our analysis followed Strauss and Corbin's (1998) constant comparative method of qualitative analysis. The data collected was scanned textbook sections introducing the MP and the surrounding narratives (Thompson, et al., 2012). In our initial glances at the data we noted emergent observed phenomenon, and built a coding scheme inductively. In each section we were specifically interested in statements of the MP, and so with each statement given we characterized the different statement types, the language used, and the representations given to accompany the statement. We were also interested in the mathematics of the statement types, specifically noting if each statement discussed independence of events, distinctness of composite outcomes, and subtleties involving the Cartesian product.

Results and Implications

This poster will demonstrate the variety across the different statements and textbooks. We categorized three inherently distinct statement types: structural, operational, and bridge. The former two statement types are in accordance with Sfard's dualistic concept notions and the latter merges the characterizations. We note, and will display, that there were differences in the combinations of these statement types in the textbooks. For instance, 6 discrete mathematics and 4 combinatorics books gave only structural statements. These different statement types further research in student thinking on the MP by providing researchers different conceptions of the MP to leverage when investigating combinatorial thought.

We will also display the extent of the diversity of the other considerations we accounted for in our analysis. We found that the languages and representations of the MP varied greatly. This variety is noteworthy to educators in that they may now be made aware of the kinds of presentations of the MP that exist in textbooks. Educators with this awareness can make more informed decisions when choosing the textbooks for their classes.

Finally, we found that statements accounted for differing combinations of the three mathematical considerations listed above. It is pedagogically important to note that these considerations can affect the accuracy of students' applications of the MP. For instance, not accounting for the distinctness of the composite outcomes may lead a student to misapply the MP and over-count when solving a particular counting problem. This discussion will be useful for the combinatorial instruction.

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Lockwood & Coughman Primus "set partitions"

Exploring tensions: Leanne's story of supporting pre-service mathematics teachers with learning disabilities

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This paper presents a case study of a mathematics teacher educator, Leanne, and her story of trying to support the development of two pre-service elementary school teachers with recognized learning disabilities. We analyze data through a lens of mathematical knowledge for teaching, focusing in particular on concerns and tensions about (i) maintaining academic rigor while meeting the emotional, cognitive and pedagogical needs of her students, (ii) seemingly opposing pedagogies between special education and mathematics education practices, and (iii) equitable opportunities for teachers with disabilities and the consequences for their potential pupils. We offer an analysis of Leanne's personal struggle, highlighting implications for teacher education and offering recommendations for future research.

Keywords: Mathematics Teacher Educators, Learning Disabilities, Mathematical Difficulties, Pre-service teachers, Pedagogical Content Knowledge

Preparing future elementary teachers in mathematics is often challenging, with a multiplex of considerations that aim to help prospective teachers transition from (and to) being engaged mathematics learners to being engaging mathematics teachers who can support the diverse needs of their future students. The learning needs within an elementary pre-service mathematics classroom itself can be just as diverse as the classes for which the pre-service teachers are being prepared. One consideration for teacher education that has not received much research attention is the preparation of future elementary teachers with learning disabilities and the pedagogical content knowledge it entails of teacher educators. This paper presents a case study of a mathematics teacher educator, Leanne, and her story of trying to support the pedagogical and mathematical development of two pre-service elementary school teachers with recognized learning disabilities. Leanne's story took place in a math-for-elementary school course, which focused on mathematical problem solving and content knowledge. We use the theoretical framework of mathematical knowledge for teaching (Ball, Thames & Phelps, 2008), specifically knowledge of content and students (KCS), to analyze the data. Analysis of the data revealed three areas of tension Leanne experienced while trying to meet the learning needs of her students, despite her background in special education. We discuss concerns and tensions about (i) maintaining academic rigor while meeting the emotional, cognitive and pedagogical needs of students, (ii) seemingly opposing pedagogies between special education and mathematics education practices, and (iii) equitable opportunities for teachers with disabilities and the consequences for their potential pupils. We offer an analysis of Leanne's personal struggle, highlighting implications for teacher education and offering recommendations for future research.

Mathematics Teacher Educators

Mathematics teacher educators are integral to the learning and growth of each generation of the pre-service and in-service teachers they teach and, by extension, to mathematics education reform as well (Tzur, 2001). Yet, there has been relatively little research into the learning and growth needs of mathematics teacher educators (Goos, 2014). This lack of research may echo the progression of mathematics education research where the original focus was on learners and not on teachers (Even, 2008). Just like teachers of mathematics, mathematics teacher educators require the subject matter and pedagogical knowledge of mathematics in order to teach children. However, mathematics teacher educators also require additional subject matter and pedagogical knowledge of mathematics in order to teach teachers (Jaworski, 2008). For this latter knowledge, teacher educators metamorphose from being "math teachers" to being "math teaching mentors", where mentorship involves preparing individuals for future teaching scenarios that may not be predictable(Lampert & Ball, 1999). Mason (2008) describes this accomplishment:

> "the effective teacher educator aims to direct attention so that participants' attention is drawn out of the actions of doing mathematics and also out of the actions of teaching mathematics, so that awarenesses become explicit. In this way, individuals and their social milieu may serve to educate that awareness, and thus inform actions in the future" (p.50).

Mathematical knowledge for teaching and KCS

Given the dearth of research on teacher educator's pedagogical content knowledge, we look to research conducted with teachers to frame our analyses. Ball and colleagues (2008) argued that the specific knowledge required to teach mathematics may be qualitatively different than the knowledge needed to teach other school subjects. They, thus, posited the theoretical framework, mathematical knowledge for teaching, as an extension to Shulman's (1986) pedagogical content knowledge framework. Ball and colleagues found that the knowledge required for teaching mathematics was complex, with multiple layers of knowledge required. They saw mathematical knowledge for teaching as having two main general categories, with each category further broken down into three subcategories: Subject Matter Knowledge (SMK) consists of content knowledge, specialized content knowledge and knowledge at the mathematical horizon, and Pedagogical Content Knowledge (PCK) consists of knowledge of content and students (KCS), knowledge of content and teaching, and knowledge of curriculum. We focus on KCS.

At the root of KCS is knowledge about student learning of mathematics, and the specific background knowledge that allows a teacher to anticipate, recognize and mediate likely misconceptions and errors in students' learning of mathematics. Using knowledge of the mathematics curriculum and knowledge about potential errors, a teacher can then create lessons that have at their center the goal of (re)mediation of the potential errors (Hill, Ball, & Schilling, 2008). An important distinction between KCS and common content knowledge is that the former requires teachers to *anticipate* and *prepare* for mistakes while the latter entails responsiveness to such mistakes (Ball et al., 2008). For a rich KCS, teachers require more than knowledge of mathematics, but also knowledge of how learners' may interpret, respond to, or represent mathematical ideas (Hill et al., 2008).

Leanne's Story

Leanne teaches a required mathematics content course for elementary pre-service teachers. The purpose of the course, like others of its kind (Goos, 2014), is to develop much needed content knowledge and problem solving skills for pre-service teachers. The course uses the text Thinking Mathematically by Mason, Burton and Stacey (2010) and is structured around mathematical tasks. Leanne's academic background is in mathematics education, and her professional background includes special education teaching in the elementary school. Through a series of informal interviews, Leanne reflected on tensions she felt when trying to support two students with identified learning disabilities in mathematics. She discussed what she felt were successes and failings in her attempts to meet the needs of these students. In what follows we analyze Leanne's reflections with an eye toward what the construct of KCS could mean for teacher educators.

Results

Maintaining academic rigor while meeting the emotional, cognitive and pedagogical needs of her students:

Leanne: I want to practice what I preach, and even in a university class I tried to differentiate, but there is always a stress about academic rigor.

Leanne: I really want my students to succeed, these students were so labor intensive… spending time with me outside of class time… they took class time and all my office hours and then some… The population itself is already riddled with its own problems, and layered on top of that are students with special needs who have had negative experiences with mathematics. It is a circular attempt to help them… I do not have enough information about how to help special needs populations, but we are in the class now so it is a trial and error mode.

Here, Leanne expresses tensions concerning meeting the emotional needs of her students while maintaining academic rigor. Leanne wanted to support her students with special needs in the same way she supported her students when she was an elementary teacher. However, Leanne faced barriers of knowledge and tried to compensate by spending more time inside and outside of class with her students. Leanne expressed she did not feel she was progressing in helping her students; describing her attempts as "circular."

Seemingly opposing pedagogies between special education and mathematics education practices:

Leanne: It is not the same in elementary school. There, you are supporting them for doing well on a test… not for this. You can use all these strategies to help the students because the question is, if the kid passed the test, not if they know the material. Of course I wanted my students with learning disabilities to know, but we need to help them pass and we don't know how to get them to know.

Leanne: In math research, I don't know why, we have the ideal student and it is definitely not the LD kid. In math research, there is so much of a focus on conceptual understanding and abstractness… but not in special education. There the focus is on procedures. Procedures, and they don't care if there is understanding.

Here, Leanne's views of teaching from special and mathematics education perspectives seem to oppose each other. In Leanne's view mathematics educators and special educators have different purposes. Mathematics education teaches for understanding beyond the classroom, and special education teaches for success in a classroom. Leanne finds herself bending towards the special education perspective as "we need to help them pass." However, this view is in direct disagreement to the purposes of the course she is teaching.

Equitable opportunities for teachers with disabilities and the consequences for their potential pupils:

Leanne: The other student didn't ask for any help. As soon as we started looking at anything resembling mathematics, she disappeared. She would be there at the beginning of class, I think she was trying, hoping each day was different, that maybe we wouldn't do anything resembling math that day... So I spoke to her…I think I was successful for her and the other student because in the end they had a more positive attitude towards mathematics. Some might argue that they should not be teaching because of their low content knowledge but we could lose a great teacher and what does this mean for equity?

Leanne: I know that the best teachers for kids with special needs are often those with special needs themselves. But kids also need teachers who have knowledge of the content.

Leanne deeply believed that all of her students had the right to become teachers and that great teachers may develop with a variety of different backgrounds, abilities, and needs. However, she also had pressing concerns about the subject matter knowledge demonstrated by her students and how to prepare them to meet the mathematical demands of the profession while accommodating their special needs and disabilities.

Discussion

Similar to Mason's (2008) observations, Leanne seemed to take for granted that her experience of being a special education teacher would be a seamless transition to mentoring students with special educational needs in a university setting. However, it was fraught with difficulties and tensions. Leanne tried to use the strategies she had acquired from teaching in special education, however those strategies were meant for scholastic achievement in a school setting and not for supporting the development of *math-for-teaching*. Leanne described a lack of KCS to teach her students with special needs. KCS would have allowed her to anticipate their difficulties and to create a program around those difficulties. Instead, and not dissimilar to what happens in elementary schools with children with special needs, Leanne was frantically trying to support her students to achieve after the fact.

Students with learning disabilities can learn mathematics but learn differently (Lewis, 2014). What Leanne was feeling relates to how little we know of how to help students who learn differently in mathematics and especially in a university setting. Leanne associated her tensions with academic rigor, however, the issues may have stemmed from a mismatch of settings. Just as the knowledge needed to teach mathematics is different from the knowledge needed to teach mathematics teachers (Simon, 2008), so too, here the knowledge needed to help those who learn differently in an elementary setting is different than the knowledge needed to help those in a university setting.

The differences in special education needs in the university and in the elementary classroom may also have their roots in the differences between the fields of special education and mathematics education (Sfard, 2007). The elementary school system is structured so that a child can do well on a final exam or a state test and be ready to progress to the next grade. The question is not if the child "understands the mathematics" or sees the aesthetic beauty of the mathematics, or if she can use the mathematics in the outside world. The child has passed the test and is ready to move on. In university, and in this course in particular, the purposes diverge. One of the many goals of a course like this is to "help students, who do not see the world as examples and non-examples of the operation, to do so" (Simon, 2008, p.21). In other words, the students in the mathematics course that Leanne is teaching will have to go out and aid their own students in making sense of the mathematical world. They will have to use what they learned from the course as a tool to help their own students. Thus, the way remediation is used in elementary schools cannot be duplicated in universities where the purposes differ. However, one might argue that with the new reform efforts towards understanding in elementary mathematics classrooms, they too require new strategies for remediation.

Many universities require their pre-service elementary teachers to take and pass some iteration of a mathematics course in order to graduate. For many students this course may stand in the way of their aspirations of becoming teachers. However, the content, delivery and theoretical underpinnings that frame these math courses, like Leanne's, vary across universities. Thus, mathematics acts as a gatekeeper for teaching, in different ways to different students in different spaces. In this case, as Leanne reflected, there were many facets to the problem of equity: there is the pre-service teacher who is faced with a barrier; the future student who deserves to have access to mathematical content knowledge; and there is also the mathematics teacher educator who requires her own specialized KCS.

Remarks and Questions

Elementary pre-service teachers already arrive at pre-service programs with a variety of needs in regards to (re) learning mathematics. We would argue that an additional need, not given attention in the literature is the knowledge needed to teach students with learning differences. It is notable that Leanne experienced tensions, despite her training in special education. This stresses the importance for special attention to be paid to KCS for helping special needs populations *understand* mathematics at the university level. Mathematics teacher educators need the knowledge of misconceptions, errors and difficulties and how to create lessons that address them for even their most different students. In this way, mathematics teacher educators can create more equitable opportunities for all their students and themselves. We propose the following questions:

- *In what ways might a teacher educator's KCS differ from that of a school teacher's?*
- *How can teacher educators use-to-advantage their KCS such that they can adequately support pre-service teachers' development of math-for-teaching?*
- *In what ways do learning disabilities impact pre-service teachers' development? What are the challenges? What are the advantages? How can teacher educators better support the learning and professional needs of this community?*

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On the use of dynamic animations to support students in reasoning quantitatively

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This study addresses the well-documented issue that students struggle to write meaningful expressions and formulas to represent and relate the values of quantities in applied problem contexts. In developing an online intervention, we drew from research that revealed the importance of and processes involved in conceptualizing quantitative relationships to support students in conceptualizing and representing quantitative relationships in applied problem contexts. The results suggest that the use of dynamic animations with prompts that focus students' attention on conceptualizing and relating quantities can be effective in supporting students in constructing meaningful expressions to represent the value of one quantity in terms of another, and formulas to define how two co-varying quantities change together.

Key words: Quantitative Reasoning; Problem Solving; Online Learning; Dynamic Imagery

Introduction

This study investigated student learning in the context of online lessons that were designed to support students in conceptualizing and relating quantities in applied contexts. It is well documented that students have difficulty knowing how to approach word (or applied) problems (Schoenfeld, 1992; DeFranco, 1996; Geiger and Galbraith, 1998; Carlson & Bloom, 2005; Moore & Carlson, 2012). A major difficulty for students in defining meaningful formulas or functions to model how quantities are related in an applied context results from their lack of effort to conceptualize the quantities in the problem and consider how they are related and change together (Moore & Carlson, 2012). Writing formulas to represent how two varying quantities change together further requires that students conceptualize variables as a means of representing the varying value that a quantity assumes, as opposed to only seeing a variable as an unknown value to solve for (Trigueros & Jacobs, 2008; Jacobs, 2002). This study leveraged these research findings to design an instructional intervention to support student learning in an online instructional environment. Furthermore, past research findings informed the development of research-informed and adapted instructional sequencing to support students in employing reasoning abilities needed to construct formulas and graphs that are meaningful to students as ways of conveying how two varying quantities change together in applied problem contexts. In this article we describe the online instruction. We then report the results of a study that examined a student's thinking as he interacted with the online dynamic animations, responded to instructional prompts, and viewed videos designed to support students in conceptualizing and relating quantities and expressing these relationships symbolically.

Theoretical perspective

The orientation phase in problem solving has been generally described by Polya (1957) and Carlson & Bloom (2005) as making sense of the problem, organizing relevant information, and developing a plan for producing a solution. In more recent studies it has been revealed that the mental process of orienting to a problem context involves initially conceptualizing the quantities in the situation and imagining how the relevant quantities are related and change together (Carlson & Moore, 2015; Moore and Carlson, 2012). The conceptualization of quantities in a situation and how they are related has been described by

Thompson (2002) as quantitative reasoning; the act of analyzing a situation into a network of quantities and relationships between quantities. Thompson further describes a *quantity* as a conceived attribute of an object that one envisions as being measureable. Integral to reasoning quantitatively is the act of quantification, "the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship … with its unit" (Thompson, 2013). Upon deciding on a unit of measure, the *value* of a quantity is the numerical value assigned to the measurement of that quantity. According to Thompson (2013) it is important that students are able to perform quantitative operations - mental operations on quantities for the purpose of characterizing new quantities - as opposed to only performing numerical operations on real numbers. An example of a quantitative operation is comparing the lengths of Car A and Car B additively to determine how much longer Car A is than Car B, using subtraction to determine (numerically) this difference, and knowing that *the length of Car A – the length of Car B* represents the amount by which the length of Car A exceeds the length of Car B. An example of a numerical operation is performing the calculation 5–12 for the purpose of finding a number.

Writing meaningful formulas further requires that one consider how the two varying quantities to be related with a formula change together. Carlson et al. (2002), Saldanha & Thompson (1998), and Thompson (1992) have described *covariational reasoning* to be the cognitive activities involved in coordinating two varying quantities while simultaneously attending to the ways in which they change in relation to each other. In the context of this study, we examine covariational reasoning in the context of two continuously varying quantities. In general, we use the term *covariational* as an adjective to describe an entity that involves two quantities varying simultaneously.

Conceptual analysis and design

Stemming from the work of Von Glasersfeld (1995), Thompson (2008) presents two meanings for the term *conceptual analysis* – to build a model of knowing that might help the researcher understand how a person might know an idea, and to devise a way of understanding an idea such that if a student had such a way of understanding, it would likely support that student in dealing mathematically with his or her environment. We refer to these as conceptual analyses of the first and second type, respectively. In this section, we use the term in the latter manner – to propose a way of thinking that may be powerful for supporting a student in building particular mathematical meanings.

Lesson 1 was designed to support students in identifying and relating quantities in a given applied problem-context. The over-arching student-learning goal for Lesson 1 of the intervention is for students to be able to write an expression that describes the value of one quantity in terms of the value of another quantity. Past research has shown that the act of conceiving of a situation in terms of quantities and determining how the conceived quantities are related and vary together is imperative for writing meaningful expressions and formulas (Moore & Carlson, 2012; Carlson & Moore, 2015; Moore, 2013). As such, in this lesson the student watches an animation of a dynamic situation in which at least two quantities are varying together (See Figure 1). They are prompted to determine values of one quantity when specific values of another quantity are known, and are then asked to consider changes in one of the varying quantities when given that the other varying quantity changes from one specific value to another. We have evidence from past studies that the mental imagery required to respond to these types of questions requires students to conceptualize how the value of some known fixed quantity can be combined with specific values of one varying quantity to determine values of the other varying quantity (e.g. Moore & Carlson, 2012; Carlson & Moore, 2015). Therefore, we provide the student with multiple occasions to engage in covariational reasoning and imagine combining a fixed quantity and a varying quantity to obtain another varying quantity in various non-complex problem contexts.

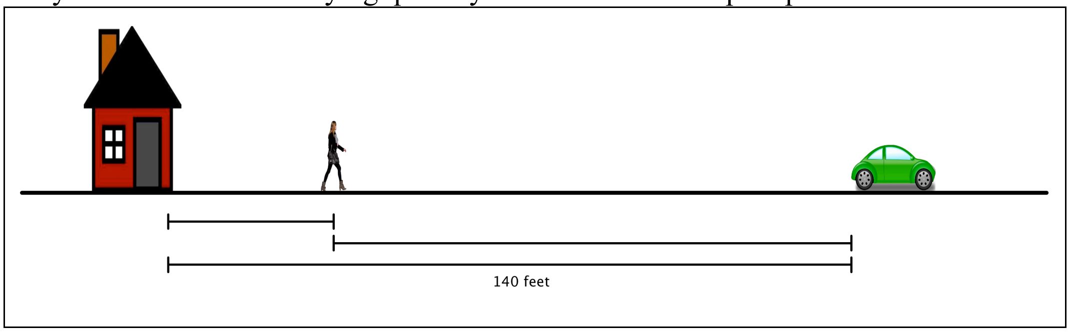

Figure 1: Jo Walking to Car

Lesson 2 introduced ideas of variable, expression, and formulas. Since past studies have documented that students have difficulty interpreting what is being asked when a problem requests students to define "one variable in terms of another," specific prompts were included to support students in developing this meaning. Past studies have also revealed that students have a strong tendency to view variables as always representing a single unknown value to solve for (Trigueros & Jacobs, 2008). A student who only thinks about a variable as an "unknown to solve for" will likely have difficulty using variables meaningfully to write formulas that define how two co-varying quantities change together. We introduce a variable as a letter or symbol that represents the varying value that a quantity can assume, and provide an emphasis on the varying nature of variables. The use of variables is motivated by the notion that trying to precisely relate the values of varying quantities via words can quickly become very cumbersome. Subsequently, the student is engaged in describing the meaning of variables in dynamic problem contexts. The student is also provided with tasks that involve determining how the value of one variable changes when the value of another variable changes from some initial value to some final value. Once the student has been provided with an intellectual need to use variables, we introduce the notion of *formula* as a mathematical statement that both expresses the value of one quantity using the value of another quantity and describes how the values of two quantities co-vary. In this intervention, we present formulas in a function-like manner. We reserve the phrase "in terms of" to describe the directionality of the formula – that is, to describe which quantity's value is being explicitly represented using the value of the other quantity. For example, we take a formula to express x in terms of y if the formula determines *the* value of x given any valid value of y; as a convention, such a formula is in the form x = <some expression in y>. The student is provided with instructional videos discussing the idea of formula, as well as the surrounding conventions. Amongst the probing questions in this lesson, the student is given practice computing the value of one quantity given the value of another quantity, and writing formulas to express one quantity "in terms of" another.

Methods

An online intervention was developed to support students in first conceptualizing quantities in an applied context and then consider how two varying quantities in the context change together. The design of the intervention was guided by past research on writing meaningful formulas to relate two co-varying quantities. To justify the content and scaffolding of the online intervention we performed a conceptual analysis (of the second type) of the mental processes involved in: conceptualizing and relating relevant quantities in a problem context, defining variables, and writing meaningful formulas to relate the values of two co-varying quantities. The online intervention in this study consisted of two consecutive "lessons." Each lesson was of a similar structure, consisting of carefully scaffolded instructional videos, mathematical animations, interactive applets, and probing questions. Once a learning trajectory was developed, I designed animations and applets using GeoGebra (Hohenwarter, 2001), a dynamic geometry software. We then recorded the instructional videos using a screen-recording software. We embedded these videos, animations, applets and probing questions into a math-assessment and course platform, IMathAS (Lippman, 2006).

The lessons were piloted with two Pre-Calculus level students who completed a series of three clinical interviews in which they worked through the online intervention while the interviewer watched the student and periodically asked questions to elicit the student's thinking. After the interview data was collected, we analyzed the data and performed a conceptual analysis (of the first type) for the purpose of building a model of the student's thinking. We then made inferences about the ways in which the online intervention shifted the student's meanings. We conclude by offering suggestions for using these research findings to guide future revisions of the online instructional materials. The subject discussed in this manuscript is Alex[1], an undergraduate student studying pre-law at a large university in the Southwestern United States.

Results

The first page of Lesson 1 is grounded in the following context: "*Jo walks the 140 foot distance from the front door of her house to her car*." The student was given the animation portrayed in Figure 1. He was then asked a series of questions designed to support him in conceptualizing and relating pairs of related quantities in the problem context. The first two tasks prompted the student to select from a list of quantities that vary and those that are constant within the given problem context. Alex had no difficulty identifying the constant and varying quantities in this situation. The next task provided the student (consecutively) with three values for the distance from Jo to her front door (in feet), and asked the student to compute the corresponding distance from Jo to the car (in feet). The resulting interactions are provided in Excerpt 1.

Excerpt 1:

1	Alex	How far is Jo when she is 40 feet from her front door. She's 40 feet from the door [on the animation Alex uses mouse to point from front door to somewhere between the door and the car] and this is 140 feet [points to line segment with 140 marked on it] means she is 100 feet from her car [with mouse, points from somewhere between the door and the car to the car].
2	Alex	When she is one-hundred and fifteen feet. [1-second pause] One hundred and forty minus one hundred and fifteen is twenty-five.
3	Alex	[Alex computes 140 – 55.3 in head] Normally I would be writing this down. But the thought process is one-forty minus that number.

In Line 1 of Excerpt 1 Alex used the mouse to point at the quantities in the animated diagram that he used to determine Jo's distance from her car. While doing so he noted the 40 feet Jo had walked from the front door and the 140 feet between the front door and car, and

[1] Alex is a pseudonym.

realized that the distance Jo needed to walk to her car could be determined by considering what he must add to Jo's distance from her front door (40 feet) to obtain the total distance of 140 feet. We consider this mental process to be a quantitative operation, wherein Alex envisioned comparing two quantities to obtain a new quantity; subtraction was the appropriate numerical operation to evaluate the value of the quantity resulting from the quantitative operation. Lines 2 and 3 of Excerpt 1 then show Alex performing the same numerical operation without reference to the diagram. We interpret this as Alex generalizing the quantitative operation and resulting numerical operation he enacted in Line 1 to other situations wherein the distance from Jo to the front door is known and the distance from Jo to the car is to be determined. Alex then went on to correctly answer a prompt to select a worded expression that represents the quantity "the distance from Jo to her car." After Alex completed Page 1 of Lesson 1, the interviwer (Grant) asked him for his reactions to the page. This interaction is displayed in Excerpt 2.

Excerpt 2

1	Grant	Alright, good. So what. What are your initial thoughts on this page? Is there anything that seemed confusing to you other than using words instead of x and y?
2	Alex	No. I thought the diagrams really, really helped. And that's kind of what I was visualizing before. A constantly moving person, but the 140 stays fixed. Knowing those two things grounds pretty much everything else that I thought about in this problem.
3	Grant	Okay, good. So what about… what do you think of these line segments that change as she walks [moves mouse to the diagram in the instructional video]?
4	Alex	I really like that. Because it shows that you have two different variables changing as she's walking, or moves. And I like that in relation to that 140, which doesn't move. In my mind, I was taking that a step further and picture numbers on the two distances, one increasing and one decreasing. To show the relationship between the two, not just in lines but also in distances.

In Lines 2 and 4 of Excerpt 2 Alex expressed that the quantitative images provided by the animation supported him in reasoning about the problem context. In Line 2, Alex's comments suggest that knowing and visualizing how the three relevant distances varied in relation to one-another grounded his thinking for "pretty much everything else that I thought about in this problem." In Line 4, Alex's comments suggest that he visualized the moving distance segments as representing "two different variables changing as she's walking," one whose value is increasing while the other's is decreasing. We infer that the animated diagram supported Alex in both conceptualizing the relevant quantities as well as imagining how they co-vary.

The first page of Lesson 2 is grounded in the following context: "A 14-inch long candle is lit and steadily burns until it is burned out." This page starts with an instructional video designed to motivate the usefulness of variables as a means of representing the varying values that a varying quantity assumes. In this context the variable b was defined to represent the varying number of inches that have burned away from the candle; the variable b is portrayed on the animated diagram next to a steadily increasing line segment representing the burned length of the candle. This video was followed by prompts to explain the meaning of the variable b and the expression $14 - b$. The last of these tasks prompts the student: "According to the video, what does the letter b represent in the context of the candle-burning problem?" Alex's verbal reaction to this task is presented in Excerpt 3.

Excerpt 3

1	Alex	…the length that has been burned. And before I'm looking at the answers,

		I’m thinking that’s what I have in my head. And it is an unknown value, so [option] A looks right. But I’m going to scan the other answers. It’s not the remaining length. Oh, but it’s not! I did that wrong. I was focusing on the ‘unknown’ versus ‘known,’ but that’s not the contrast. The contrast is a single unknown versus a varying value, and I am looking for a varying value. That would be the burned length.

In Excerpt 3, Alex’s response revealed that he conceived of the variable *b* as representing varying values, rather than a single unknown value. The second instructional video describes how one could use a value of *b* to determine the remaining length of the candle (in inches). The variable *r* is also defined to represent the values of the remaining length of the candle (in inches). Following this video is a series of tasks designed to engage the user in determining how the value of *r* changes if *b* changes from some initial value to some final value. Alex responded to the prompt “As *b* varies from $b = 7.5$ to $b = 12$ inches, how does *r* vary?” by determining that the remaining length of the candle would decreases from 7.5 to 2 inches. The next task on this page began with the prompt: “What does your answer to the above question mean?” The student is then given a sentence with four missing words that can be selected via dropdown select-menus (displayed in Figure 2 below). Alex’s response to this task is presented in Excerpt 4 below.

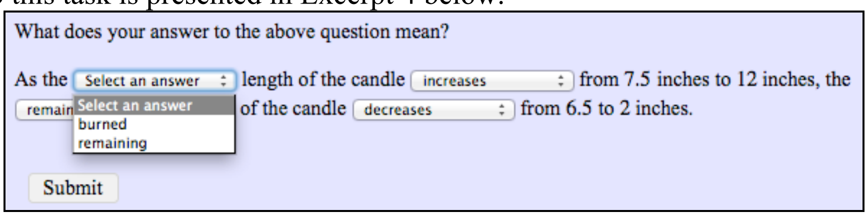

Figure 2: Candle Burning Task

Excerpt 4:

1	Alex	So what does the above answer mean? So I look at the above answer and try to predict first, as opposed to looking at the answers. So that means as the burned length of the candle increases, the length, *r*, decreases. I’m seeing a similar structure… And that is how we initially conceptualized it. And that makes sense, because you’re essentially just saying as one variable goes up the other variable goes down.
2	Grant	And does that make sense to you… like if you imagine the candle, does that make sense?
3	Alex	Yes, especially from the video, when they showed how one variable was increasing while the other one was decreasing. That wording is how I would have worded that from the video.

In Line 1 of Excerpt 4, Alex determined that as the value of *b* increases, the value of *r* decreases. In Line 3, Alex explained that this statement aligned with his dynamic image of the variables changing together. In Line 3 Alex mentioned the dynamic diagram and how it “showed how one variable was increasing while the other was decreasing,” suggesting that the dynamic diagram supported Alex in conceptualizing how the relevant quantities vary together. From Line 3 of Excerpt 4 we infer that the dynamic diagram supported Alex in conceiving of the variables *b* and *r* as varying while imagining how they change together as the candle burns. Although it may be the case that, without the dynamic diagram, Alex could have conceived of *b* and *r* as varying wherein *r* decreases as the value of *b* increases, Excerpt 4 suggests that the dynamic diagram strongly supported Alex in bolstering these conceptions. A few minutes after the interaction in Excerpt 5, Alex responded to the task prompt: “What does the expression 14 – b represent in the context of the candle burning question?" by saying,

Excerpt 5:

1	Alex	So 14 is the initial length of the candle. b we know is the burned length. So that equation is going to be varying because we know b is a varying variable. And if you have that in an equation and you don't have b, you don't have a varying value. So, we know that A is correct because 14 minus b gives you the remaining length as a varying value.

Line 1 of Excerpt 5 suggests that Alex conceived of the expression 14 – b as representing the varying values of the remaining length of the candle. Alex did not only conceive of 14 – b as representing the remaining length of the candle, he also conceived of it as varying as the value of b varies. Alex continued working through this Page of Lesson 2, and was able to determine a formula that expresses r in terms of b and a formula that expresses b in terms of r.

Conclusions and implications

The findings suggest that the dynamic diagrams presented in the intervention supported Alex's construction of a coherent and useful image of the quantities in the problem context, and how relevant quantities change together. These images emerged as Alex conceptualized the quantities in the situation and determined how they were related. This suggests that the dynamic animations with accompanying prompts that support students in conceptualizing the quantities in a problem context, and how they change together, can be effective in helping students construct expressions and formulas that are meaningful to them.

From these results we draw the following implications relative to computer-based instructional design.

i. Dynamic diagrams can be useful tools for supporting students in conceptualizing relevant quantities in problem contexts. To further support students in conceptualizing relevant quantities in a problem context, these animations should be supported by questions that probe the student to consider various quantities in the context and describe how they vary together.
ii. The dynamic nature of the diagrams can be leveraged to support students in thinking in dynamic ways. For example, the candle burning animation was effective in supporting the student in conceptualizing variables as varying, not as single unknown values. This can be extended to other ideas, such as constant rate of change and proportionality, that students often think of statically but are perhaps more productively thought of as dynamic.
iii. Computer-based tools can be used to simultaneously represent the co-variation of two quantities via different representation systems. This capability can be used to support students in "seeing" the meaning of the symbols that they write on their paper, in the sense that in the student's mind, the symbols and expressions they write represent the values of quantities within the context.

Although the implications we mention above are framed in the context of computer-based instructional activities, we believe that such implications are relevant to in-class practice. Dynamic diagrams and applets can be used to center classroom conversation around particular mathematical ideas while simultaneously providing a dynamic view of how multiple quantities might vary together and how one might represent the varying values of these quantities and how they are related.

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PHYSICS STUDENTS' CONSTRUCTION AND USE OF DIFFERENTIAL ELEMENTS IN MULTIVARIABLE COORDINATE SYSTEMS

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As part of an effort to examine students' understanding of non-Cartesian coordinate systems when using vector calculus in the physics topics of electricity and magnetism, we interviewed four pairs of students. In one task, developed to force them to be explicit about the components of specific coordinate systems, students construct differential length and volume elements for an unconventional spherical coordinate system. While all pairs eventually arrived at the correct elements, some unsuccessfully attempted to reason through spherical or Cartesian coordinates, but recognized the error when checking their work. This suggests students' difficulty with differential elements comes from an incomplete understanding of the systems.

Key words: Coordinate Systems, Differential Elements, Physics, Vector Calculus

Introduction

Various physics education researchers have explored student difficulties with the mathematics applied in Electricity and Magnetism (E&M). These studies have assessed student understanding of integration and differentials (Doughty *et al.,* 2014; Hu & Rebello, 2013; Nguyen & Rebello, 2011); have identified difficulties in applying Gauss's and Ampère's Laws, two integral components of E&M courses that involve a surface integral and line integral, respectively (Guisasola, 2008; Manogue, 2006; Pepper, 2012); and have addressed calculation, understanding, and application of gradient, divergence, and curl in both mathematics and physics settings (Astolfi & Baily, 2014; Bollen, 2015).

A key factor in the application of these mathematical concepts and operations in E&M is a working understanding of the spherical and cylindrical coordinate systems appropriate for the symmetry of most physical situations. In order to solve problems, students are expected to use differential line, area, and volume elements, as well as position vectors that describe the locations of charges distributed over volumes, surfaces, and lines, in order to set up appropriate integrals. A further complication is that the differential line and area elements are vector quantities and thus have a specific direction, while the volume elements are scalar. Given the importance of these differential elements – in different coordinate systems – to the calculations, the main research questions of this study are:

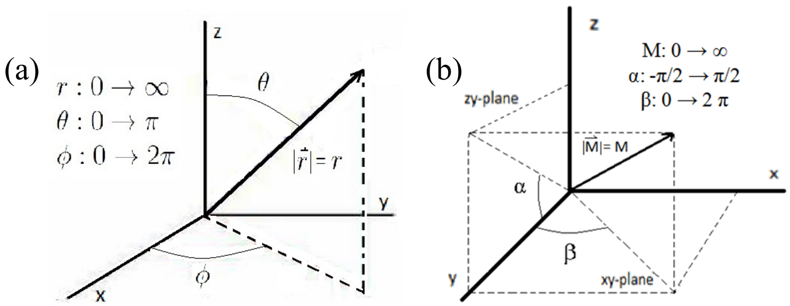
- How do students make sense of and work with coordinate systems, specifically cylindrical and spherical coordinates?
- How do students construct differential vector elements within a given coordinate system?

While disciplinary conventions (e.g., ϕ and θ angle labels switch from math to physics) can be an obstruction to student understanding early in the course (Dray and Manogue, 2003; 2004), even when these are addressed, students have difficulty constructing these differential elements.

Methods

Clinical think-aloud interviews were conducted with pairs of students (N=8) at the end of the first semester of a year-long, junior-level E&M sequence. Pair interviews allowed for a more authentic interaction and sharing of ideas between students with minimal influence from the interviewer. This report focuses on a task in which students were given an unconventional spherical coordinate system. Students were asked to conclude whether the system was feasible, and to build and verify the differential line and volume elements. As students work through these tasks, we are able to see how they reason about the differential elements in a specific coordinate system, thus giving insight into the choice and use of these elements in their problem solving.

Our initial analysis has identified student specific difficulties (Heron, 2003) and successes. We are currently connecting these to aspects of student *concept images* (Tall & Vinner, 1981) of the differential elements and of the non-Cartesian coordinate systems. Similar analysis has been done for student difficulties with divergence and curl in electrodynamics (Bollen *et al.,* 2015).



(c)

$$\vec{dl} = dr\hat{r} + rd\theta\hat{\theta} + rsin\theta d\phi\hat{\phi}$$
$$d\tau = r^2 sin\theta dr d\theta d\phi$$

(d)

$$\vec{dl} = dM\hat{M} + Md\alpha\hat{\alpha} + Mcos\alpha d\beta\hat{\beta}$$
$$d\tau = M^2 cos\alpha dM d\alpha d\beta$$

Figure 1: (a) Conventional (physics) spherical coordinates; (b) an unconventional spherical coordinate system given to students, for which they were to construct differential length and volume elements. The correct elements for each system are in (c) and (d), respectively.

Results

Results shed a unique light on how students build differential elements within a coordinate system. None of the interview pairs determined the correct elements at first, often attempting, incorrectly, to map from a conventional coordinate system rather than constructing the necessary differential elements geometrically. However, all of the pairs correctly attempted to verify the volume element with a spherical integral, which is when they recognized any error(s) in their differential elements.

Some pairs attempted to recall how to decompose the vector **M** into its Cartesian components. Two pairs, trying to map directly to a spherical system, incorrectly included a *sin*(α) in the β component of their differential length rather than the appropriate *cos*(α); this is reminiscent of "*x,y* syndrome" (White & Mitchelmore, 1996), wherein students remember expressions in terms of symbols used rather than in terms of the concept. Another pair had no trigonometric function in their components.

Regardless of the elements determined, all pairs attempted integration to obtain the spherical volume formula. All pairs eventually realized the need for *cos*(α) because of the projection into the *xy*-plane. In some cases the cosine term arose in an attempt to obtain the correct formula by integration, while in other cases the need for the vector to project into the *xy* plane was recognized first, and the cosine term was inserted or substituted into the differential.

Our results suggest students do not have a robust understanding of how to build differential elements, but are able to check the validity of these elements and adjust terms appropriately.

This work is preliminary; subsequent data interpretation will use perspectives that have been productive in describing student understanding of mathematics in physics contexts, including *layers* (Zandieh, 2000; Roundy *et al.*, 2015) and *symbolic forms* (Sherin, 2000; Jones 2013). Additional plans are to develop instructional resources that improve student understanding of the construction of differential elements in multivariable coordinate systems in physical contexts.

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Student interpretation and justification of "backward" definite integrals

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The definite integral is an important concept in calculus, with applications throughout mathematics and science. Studies of student understanding of definite integrals reveal several student difficulties, some related to determining the sign of an integral. Clinical interviews of 5 students gleaned their understanding of "backward" definite integrals, i.e., integrals for which the lower limit is greater than the upper limit and the differential is negative. Students initially invoked the Fundamental Theorem of Calculus to justify the negative sign. Some students eventually accessed the Riemann sum appropriately but could not determine how to obtain a negative quantity this way. We see the primary obstacle here as interpreting the differential as a width, and thus an unsigned quantity, rather than a difference between two values.

Key words: Definite integrals, Calculus, Differential

In this preliminary report, we examine the role of the differential in the "backward" definite integral, $\int_b^a f(x)dx$ where $a < b$. The definite integral is a fundamental concept in calculus, with applications throughout mathematics and science. Studies of student understanding of definite integrals reveal several difficulties (Bajracharya, Wemyss, & Thompson, 2012; Bezuidenhout & Oliver, 2000; Jones, 2013; Lobato, 2006; Sealey, 2006, 2014; Sealey & Oehrtman, 2005). The existing literature on definite integrals tends to support a specific approach to developing an understanding of the definite integral, specifically by recognizing it as the sum of infinitely small products, which are formed via Riemann sums (Jones, 2013; Meredith & Marrongelle, 2008; Sealey, 2008, 2014). Additionally, Sealey (2006) and Jones (2013) point out that recognizing the Riemann sum as a sum of products of the function value $f(x)$ and the increment on the x-axis (Δx) is necessary for students to understand the meaning of the *area under the curve,* which is, arguably, the most prominent metaphor/interpretation of the definite integral. On the other hand, reasoning about a definite integral as area under the curve may limit students' ability to apply the integral concept (Norman & Prichard, 1994; Sealey, 2006; Thompson & Silverman, 2008).

Another aspect of the definite integral that leads to student difficulties is the meaning of the differential itself. Students treat the differential as an indicator of the variable of integration rather than a fundamental element of the product in integration of both single- and multivariable functions (Hu & Rebello, 2013; Jones 2013). This could stem from a failure to understand the product layer of the integral (Sealey, 2014; von Korff & Rebello, 2012). Other recent work has shown students treating dx as a width rather than a difference or change, both for positive and negative integrals (Bajracharya et al., 2012; Hu & Rebello, 2013; Wemyss, Bajracharya, Thompson, & Wagner, 2011).

Interpreting the sign of the integral has been shown to be difficult for students. In particular, definite integrals that have a negative result are of particular difficulty geometrically. Students often do not treat the area as a negative quantity, effectively associating it with spatial area rather than the quantity represented by the product of $f(x)\,dx$. This is true for integrals for which $f(x)$ is negative, i.e., below the x-axis (Bezuidenhout & Oliver, 2000; Lobato, 2006), as well as those for which dx is negative, i.e., the direction of integration is in the negative direction (Bajracharya et al., 2012). The former type of negative integral is more common, but the latter also has relevance to applications in physical situations (e.g., finding thermodynamic work during the compression of a gas). Bajracharya et al. (2012) found that students could justify the sign of a negative integral represented graphically by overlaying a physical context on the graph.

The notion of dx as a signed quantity is somewhat controversial, depending on the way one defines the differential. The perspective here, which is consistent with applications in physics and other fields, is that dx is defined as an infinitesimal *change* in the quantity *x,* akin to the limit of the change in x for the products in a Riemann sum: $\Delta x = \frac{b-a}{n}$; $dx = \lim_{n\to\infty} \frac{b-a}{n}$. This is consistent with von Korff & Rebello (2012), who argue that infinitesimal quantities and infinitesimal products are important for an understanding of the meaning of definite integrals. Generally the sign of these quantities is not of interest, since $b>a$ in most cases. However, if $b<a$, then Δx, and thus dx, are negative. In Stewart's (2007) most recent text, he explains that the backward integral is negative because Δx is negative, but does not explicitly refer to dx as a signed quantity.

Given the prior work in this area, we wanted to explore the facets of students' concept image (Tall & Vinner, 1981) of the definite integral that apply to the sign of the integral. In particular, the role of the differential in a backward integral, $\int_b^a f(x)dx$, is crucial in interpreting the sign. We suspected that students would not recognize the fact that the differential would be negative for backward integrals. Thus the backward integral had the potential to illuminate students' understanding of the meaning of differentials, definite integrals, and to some extent, the Riemann sum, beyond what has been seen in the literature to date.

Methods

During clinical interviews, students were asked a series of questions about the relationship between forward and backward integrals. As this was a pilot study, we chose to interview five students at various levels: two second-semester freshmen (both double majors in math and physics and concurrently enrolled in a second-semester calculus course), one junior math major, one senior math major, and one first-semester Ph.D.-level graduate student in math/math education. Interviews were videotaped and transcribed. The interview subjects were volunteers who were either former students or teaching assistants of one of the authors. Interviewees received a \$10 gift card at the conclusion of the interview. Prior to the interviews, we developed an interview protocol and agreed upon the order in which the questions would be asked of the students, starting with the open ended general expressions shown below and concluding with a physical example. In each case we gave the forward integral first, then asked about the backward integral of the same expression.

1. General expressions: $\int_a^b f(x)dx$ and $\int_b^a f(x)dx$
2. Specific expressions: $\int_1^3 2x\,dx$ and $\int_3^1 2x\,dx$
3. Physical scenario: Work required to stretch a spring, $\int_{x_1}^{x_2} F\,dx$, where $F = kx$

Data and Results

All five students were able to use the Fundamental Theorem of Calculus (FTC) to justify why $\int_a^b f(x)dx = -\int_b^a f(x)dx$. Specifically, they were able to state that $\int_a^b f(x)dx = F(b) - F(a)$, where $F(x)$ is the antiderivative of $f(x)$, and then that $\int_b^a f(x)dx = F(a) - F(b)$, which would have the opposite sign. Graphically, the students had much more difficulty. In the preliminary analysis, several student difficulties were observed; two of these are discussed in more detail here. We are still in the process of analyzing the data and determining plans for future data collection.

Student thinking about the differential

Most of the students were able to think about *dx* in at least two ways. Many of the students mentioned that the *dx* refers to the variable of integration, and most also were able to discuss the *dx* as the width of individual rectangles under a curve. Subsequent data analysis will note which concept image for *dx* was evoked in different circumstances, which concept image was evoked first, and if/when the students changed the way in which they thought about the *dx*. Of particular interest to us is whether or not the students can conceive of *dx* as a signed quantity, as either a negative width, or as a negative value obtained from $x_2 - x_1$. According to our preliminary analysis, none of the students thought about *dx* as a signed quantity on their own accord, but with prompting from the interviewers, some were able to do so.

Anna, a senior math major, had no trouble thinking about Δx as a negative width, but did not seem comfortable thinking about *dx* being positive or negative. Her explanation of why the backward integral was negative was because the width was negative, and explained, "You're going to have that negative width times a positive value, which is going to give you a negative number, so you're going to get the addition of a bunch of negative numbers." Much later in the interview, one of the interviewers asked Anna if it was possible for *dx* to be positive or negative, and Anna responded, "I've actually never thought of that. So I'm not sure. I mean I guess it could, but I just always viewed the *dx* as the indication of what term to integrate to. So I'm not actually sure, I guess."

Similar to Anna's response, Matt, a junior math major, eventually was able to think about Δx as a negative quantity and described *dx* as the limit as Δx approached zero. After many attempts from Matt, the interviewer asked him if *dx* could be negative. His response indicated that he was not confident in his answer, but responded, "That's probably the hidden spot that I couldn't figure out before. Yeah I would say that this *dx* would be negative (from *a* to *b*) and this one would be positive (from *b* to *a*) because it's approaching 0 so this (from *a* to *b*) would still stay positive like stay right north of 0. And this one (from *b* to *a*) would stay under, yeah I'm going to say this *dx* here (from *b* to *a*) is negative and this *dx* is a positive *dx* (from *a* to *b*), and I guess that's where it's hidden and that's what their difference is? I don't know."

Nick, a mathematics graduate student, focused his explanation as to why the backward integral was negative on direction. He said that the *dx* represents a change, and that change implies motion. He seemed to be thinking about the variable *x* representing time, and mentioned more than once that the backward integral would be like playing a movie in reverse. On another note, Nick spent a great deal of time during the interview talking about the two terms that made up the product in the definite integral, namely the 2*x* and the *dx* in $\int_3^1 2x\,dx$. He knew that when multiplying two quantities to obtain a negative result, exactly one of the terms multiplied must be negative. He debated if the *x* turned negative or the *dx* turned negative. He "voted" for the *dx* to be negative, but didn't seem confident of his answer. He said to be sure, he would have to go back to the definition of Δx in the textbook to see if he was right.

Using area under the curve and the Fundamental Theorem of Calculus

All of the students seemed comfortable discussing the integral as the area under the curve. While they were able to consider the total area as the sum of small rectangles (or trapezoids), their calculation of the total area ended up being an interesting part of our analysis.

Sara, a sophomore mathematics and physics double major, evaluated $\int_1^3 2x\,dx$ by finding the area of the large triangle (Fig. 1a) and subtracting the area of the small triangle (Fig. 1b) to obtain the desired area (Fig. 1c). She noticed that these calculations corresponded to the values she obtained when applying the FTC to the same problem: the area of the large triangle corresponded to $F(3)$, and the area of the small triangle to $F(1)$. Then, when computing $\int_3^1 2x\,dx$, she reversed the order of her subtraction, subtracting the area of the large triangle (Fig. 1a) from the area of the small triangle (Fig. 1b), and said, "But I'm not sure why that order is. I mean I know why for the integral [symbolically] because it's written that way, but if you were to solve this geometrically, I don't know why you would change the order of the subtraction."

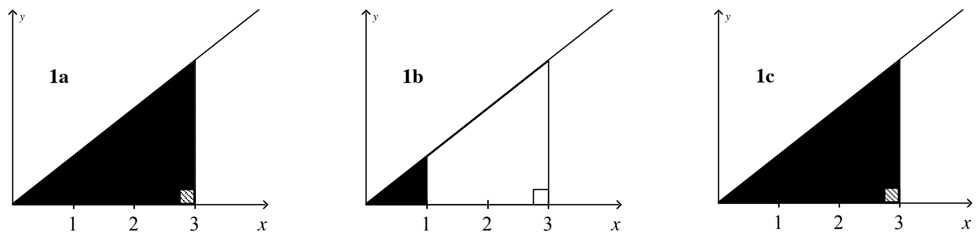

Figure 1: Sara's method of computing the area

Matt also was able to justify the relationship between the forward and backward integral symbolically using the FTC, but also struggled to justify the result graphically. When computing the area under the function $2x$ between $x = 1$ and $x = 3$, he recognized it as a trapezoid. Instead of using Sara's method of subtracting the smaller triangle from the larger triangle (Fig. 1), Matt added the area of the lower rectangle (Fig. 2a) to the area of the upper triangle (Fig. 2b) to obtain the total area (Fig. 2c).

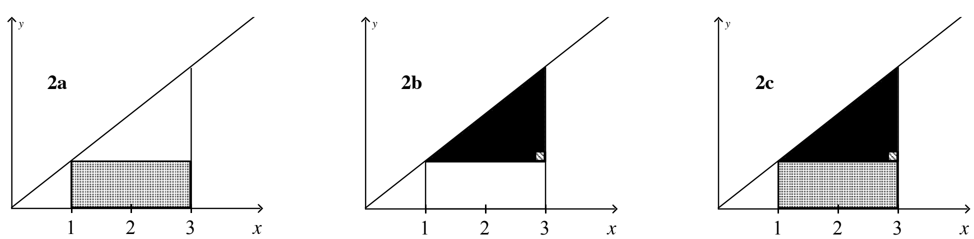

Figure 2: Matt's method of computing the area

Matt's solution is perfectly valid, but did not mimic the calculations from the FTC, as did Sara's method. Matt tried several different ways to graphically justify the negation of the backward integral but was never completely content with his justification. He noted that the backward integral represented the same area as the forward integral, but the backward integral would have to be negative since the limits were reversed "because I already know that, like as a fact, that it's a negative if you want to flip the bounds." He did state that he believed there *should* be a graphical justification, but he did not know what one would be.

We do not mean to imply that Sara's solution was in some way better than Matt's, but simply note the connection to the FTC in Sara's solution. In fact, both Sara and Matt used solutions that sidestep the need for thinking about the Riemann sum and the *dx* specifically. Near the end of Sara's interview, we pushed her to consider each rectangle under the curve, which she had described at the beginning of her interview. Sara was comfortable with $f(x)$ being negative or positive, depending on if it was above or below the *x*-axis, but said, "Well no, I don't think *dx* would ever be negative because it's just a distance, it's not like an actual value."

Discussion

Students recognized the negative value of the backward integral based on the FTC/antiderivative difference formula, but when asked for a geometric interpretation, most said they hadn't thought about it before and had difficulty making a reasonable interpretation on their own. Most students' graphical explanation of why the backward integral yields a negative result seemed to be invoking the direction of the integration, treating the area as a macroscopic negative quantity, but failed to recognize the role of the differential in generating that sign. We know from the literature and our own prior research (Bajracharya et al., 2012; Sealey, 2006; Thompson & Silverman, 2008) that students often lack an understanding of *why* or *how* area under a curve is a representation of a definite integral. Our subjects, who we acknowledge may be more advanced than the average calculus student, did not seem to have this difficulty and were able to describe the definite integral as the sum of the areas of very small rectangles, and adequately described the product layer that makes up these small rectangles. They could all explain that $f(x)$ represented the height of the rectangles and that Δx (and sometimes *dx*) represented the width of the rectangle.

However, thinking about the backward integral adds another level of difficulty to describing the definite integral in terms of area. The students did not always recognize that Δx and *dx* could be negative values. Instead of thinking about Δx as a difference, (e.g. as $(x_{i+1} - x_i)$ or as $\frac{b-a}{n}$), they initially thought of Δx as the width of a rectangle, and usually assumed it was always a positive value.

We certainly do not mean to imply that Δx and *dx* should never be thought of as a width. In fact, research by Hu and Rebello (2007) suggested that *dx*-as-width is an important perspective for problem solving in physics. Instead, we emphasize the necessity for being able to think about *dx* as positive or negative widths *and* the change between two quantities. With moderate prompting, most of our research subjects were able to do this, and our future research will examine what type of instruction or intervention enables students to make this connection.

Discussion Questions

1. We have some examples in physics where one might consider the backward integral (stretching/releasing a spring). Are there other examples *in mathematics* where it makes sense to consider $\int_b^a f(x)dx$?
2. Where do you think this difficulty might be best addressed? Calculus 1? Calculus 1? Real analysis? Physics?
3. Student functional understanding of the differential seems to be the underlying cause of several difficulties with students (in our work as well as other studies in the literature). Do you have recommendations for why and/or how this can be improved?

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An example of a linguistic obstacle to proof construction: Dori and the hidden double negative

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This paper considers the difficulty that university students' may have when unpacking an informally worded theorem statement into its formal equivalent in order to understand its logical structure, and hence, construct a proof. This situation is illustrated with the case of Dori who encountered just such a difficulty with a hidden double negative. She was taking a transition-to-proof course that began by having students first prove formally worded "if-then" theorem statements that enabled them to construct proof frameworks, and thereby, make initial progress on constructing proofs. But later, students were presented with some informally worded theorem statements to prove. We go on to consider the question of when, and how, to enculturate students into the often informal way that theorem statements are normally written, while still enabling them to progress in their proof construction abilities.

Key words: Transition-to-proof, Proof Construction, Informally Worded Theorem Statements, Proof Framework, Unpacking

This paper considers linguistic obstacles[1] that university students often have when unpacking informally worded mathematical statements into their formal equivalents. This can become especially apparent when students are attempting to prove such statements. We illustrate this with an example from Dori, who was taking a transition-to-proof course that began by having students construct proofs for formally worded "if, then" theorem statements. Early on, she was introduced to the idea of constructing proof frameworks (Selden, Benkhalti, & Selden, 2014; Selden & Selden, 1995) and was successful. Later, she encountered difficulty when attempting to interpret and prove an informally worded statement with a hidden double negative. First we will introduce our theoretical perspective and the idea of proof frameworks.

Theoretical Perspective

We adopt the theoretical perspective as described in Selden and Selden (2015); that is, we consider a proof construction to be a sequence of mental or physical actions, some of which do not appear in the final written proof text. Each action is driven by a situation in the partly completed proof construction and its interpretation (Selden, McKee, & Selden, 2010). For example, suppose that in a partly completed proof, there is an "or" in the hypothesis of a statement yet to be proved: *If A or B, then C*. Here, the situation is having to prove this statement. The interpretation is realizing that *C* can be proved by cases. The action is constructing two independent sub-proofs; one in which one supposes *A* and proves *C*, the other in which one supposes *B* and proves *C*.

We also note that a proof can be divided into a formal-rhetorical part and a problem-centered part. The *formal-rhetorical* part is the part of a proof that depends only on unpacking and using the logical structure of the statement of the theorem, associated definitions, and earlier results. In general, this part does not depend on a deep understanding of, or intuition about, the concepts involved or on genuine problem solving in the sense of Schoenfeld (1985, p. 74). Instead it depends on a kind of "technical skill". The remaining part of a proof has been called the *problem-centered* part. It is the part that *does* depend on genuine problem solving, intuition, heuristics,

[1] The idea of linguistic obstacles to learning mathematics is not new to mathematics education research (Boero, Douek, & Ferrari, 2002; Ferrari, 1999). However, to our knowledge, no one has previously discussed hidden double negatives.

and a deeper understanding of the concepts involved (Selden & Selden, 2013).

One might suppose that the problem-centered part of a proof is the most important part, and as students make progress in their proof construction ability, this may be true. However, for students of a one-semester transition-to-proof course, constructing the formal-rhetorical part of a proof can be non-trivial, yet easier to learn than the construction of the problem-centered part. Furthermore, first writing some of the formal-rhetorical part of a proof is often helpful for constructing the problem-centered part of the proof because the formal-rhetorical part exposes the "main problem" to be solved (Selden & Selden, 2009).

Proof Frameworks

A major feature that can help one write the formal-rhetorical part of a proof is a *proof framework*, of which there are several kinds, and in most cases, both a first-level and a second-level framework. For example, given a theorem of the form "For all real numbers x, if $P(x)$ then $Q(x)$", a first-level proof framework would be "Let x be a real number. Suppose $P(x)$. … Therefore $Q(x)$," with the remainder of the proof ultimately replacing the ellipsis. A second-level framework can often be obtained by "unpacking" the meaning of $Q(x)$ and putting its (second-level) framework between the lines already written for the first-level framework. Thus, the proof would "grow" from both ends toward the middle, instead of being written from the top down. In case there are subproofs, these can be handled in a similar way. A more detailed explanation with examples can be found in Selden, Benkhalkti, and Selden (2014). A proof need not show evidence of a proof framework to be correct. However, we have observed that use of proof frameworks tends to help novice university mathematics students write correct, well-organized, and easy-to-read proofs (McKee, Savic, Selden, & Selden, 2010).

The Formal-Informal Distinction and Linguistic Obstacles

An *informal statement* is one that departs from the most common natural language version of predicate and propositional calculus or fails to name variables. For example, the statement, "differentiable functions are continuous," is informal because a universal quantifier is understood by convention, but is not explicitly indicated, because the variables are not named, and because it departs from the familiar "if-then" expression of the conditional. Such statements are commonplace in everyday mathematical conversations, lectures, and books. They are not ambiguous or ill-formed because widely understood, but rarely articulated, conventions permit their precise interpretation by mathematicians, and less reliably, by students. In our experience, mathematicians, including those with no formal training in symbolic logic, move easily between informal statements and their equivalent more formal versions.

We conjecture that an informal version of a theorem will often be more memorable, that is, more easily remembered and brought to mind, but also be more difficult to prove, and also, given a proof, be more difficult to validate, than a formal version. This suggests the question: Can undergraduates who have taken a transition-to-proof course reliably unpack an informally stated theorem into its formal version? Our earlier paper on students' unpacking of the logic of statements (Selden & Selden, 1995) indicates that the answer to this question is often no. Because validation is difficult to observe directly, data were collected to determine whether the participants could reliably unpack, rather than validate, informally written statements, many from calculus, into their formal equivalent versions. (See Selden and Selden, 1995, for details.)

Because the inability to unpack an informally written theorem statement into a formal version can often prevent a student from constructing a proof, we think that the informal way that a theorem is stated can be a *linguistic obstacle*. Such an obstacle need not be a mistake or misconception (i.e., believing something that is false). Indeed, the obstacles mentioned in the Selden and Selden (1995) paper are related to difficulties with unpacking the logic of informally worded mathematical statements. In what follows, we extend this work and examine a linguistic obstacle of a somewhat different kind.

The Case of Dori and the Hidden Double Negative

What happens when a student is confronted with a hidden double negative in a theorem statement and wants to construct a proof? We report, using our field notes and photos, on a mathematics graduate student, Dori, who in a tutoring session at the end of an inquiry-based transition-to-proof course, was confronted with the task of proving: *A group has no proper left ideals*. (This was in a semigroup setting, in which L is a left ideal of a semigroup S if $L \subseteq S$ and $SL \subseteq L$.) Dori had already experienced proving theorems on sets, functions, real analysis, and abstract algebra (semigroups). She had available to her the course notes, with all previous definitions and theorems. In the same tutoring session, she had just taken 40 minutes to prove, with some difficulty and backtracking, that group inverses are unique. Specifically, she had just proved: *Let G be a group with identity* 1. *If g,* g,' *$g'' \in G$ with $gg' = g'g = 1$ and $gg'' = g''g = 1$, then $g' = g''$.*

Dori, who was working at three seminar room blackboards, next began to prove a theorem about left (semigroup) ideals by writing the theorem statement on the middle board . She wrote: *A group has no proper left ideals*. Dori then looked up various definitions, such as that of left ideal and proper, in the course notes. We then talked with her about what "not proper" means, after which she wrote $GI \neq I$ and $GI \subset I$ on the right board and suggested doing a proof by contradiction. We were surprised at this suggestion, and now speculate this might have been because of the word "no" in the theorem statement. At the time, however, realizing that this would not be a productive approach, we suggested that Dori write a proof framework as she had been accustomed to doing in the past. She continued writing, below the theorem statement on the middle board:

Suppose G is a group and I is a left ideal of G.
…
Then $G=I$.
Therefore, I is NOT a proper left ideal of G.

We then suggested that Dori write in her scratch work the properties of a group and of a left ideal of a semigroup. She wrote these additional observations correctly on the right board. These included noting that G has an identity and inverses, that I being a left ideal means that $GI = \{gi \mid g \in G, i \in I\}$, $GI \subseteq I$, and $I \neq \varnothing$. Dori also noted the existence of the identity element, $1 \in G$ and that there is an $i \in I$ and so $i \in G$. In addition, Dori drew an appropriate diagram of the situation, with one circle labeled I contained in a larger circle labeled G, with $i \in I$ and 1in the space between the two circles. (See Figure 1.)

The emphasis, in Dori's scratch work, on what it means for I to be a proper ideal may not have been helpful, as she, according to her proof framework, was trying to show that I was *not* proper, namely, the negation. It is often difficult for university students to form proper mathematical negations; instead, they often formulate the opposite, as they would in everyday life (Antonini, 2001). Somehow, Dori did *not* note, at this point, that in order to show that $G=I$ (the penultimate line of her proof framework), all she needed to show was $G \subseteq I$. One can speculate on why this might have been.

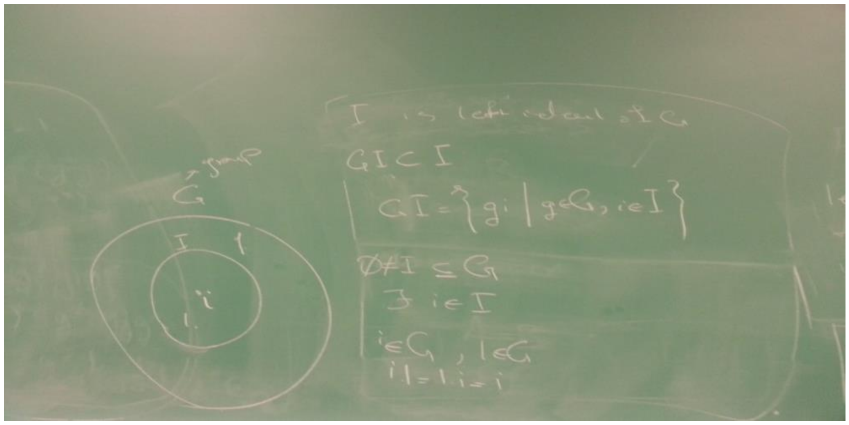

Figure 1. Some of Dori's scratch work with the diagram.

Difficulties Inherent in Converting the Theorem Statement to its Formal Version

As Dori was working diligently on her scratch work, it appeared to one of us that the informal wording of the theorem statement might be causing Dori difficulty. So, while Dori continued her scratch work, this one of us decided to try to translate the theorem into "if-then" format, judging that it might be easier to comprehend. It became clear that there were two negations involved in the phrase "no proper". The first was contained in the word "no". The second was hidden within the word "proper", which means that the ideal, *I*, is a proper subset of G, namely, that $I \neq G$. Thus, there is a double negation in the statement of the theorem. Having noted this, we went on to use this observation to write the theorem statement in a positive "if-then" way as, *If I is a left ideal of G, then I=G,* on the left blackboard. We went over this version of the theorem statement with Dori. The positive "if-then" formulation of the theorem has the following apparent advantages: (1) The notation has been introduced. (2) It is in the formal "if-then" form, from which a proof framework can be written in a straightforward way. (3) It does not have a hidden double negation, but rather is entirely positive and straightforward.

Dori had had no trouble introducing the notation. Perhaps this was because of the theorem on inverses that she had proved earlier that day; it already contained the notation *G* for a group and 1 for the identity element. With encouragement from us, towards the beginning of her proof attempt, Dori had written a proof framework, introducing the letters *G* for the group and *I* for a left ideal of *G*, and scrolling to the bottom, had written $G=I$ in the penultimate line and had concluded in the final line that I is not a proper left ideal of *G*, as well as having produced some scratch work (Figure 1). After discussing with her the positively worded version of the statement, namely, *If I is a left ideal of G, then I=G*, we suggested that she "Suppose $1 \in I$" to see what happens. Dori wrote "Let $1 \in I$" and also, "Let $g \in G$, $i \in I$, so $gi \in I$. Let $i=1$, so $g \cdot 1 = g \in I$." This essentially completes the argument that $G \subseteq I$, and hence, proves the theorem. From start to finish, this entire proving episode took 45 minutes.

To recapitulate, to prove the theorem, one observes, as Dori had, that ideals are non-empty, so there is an $i \in I$, that $i^{-1} \in G$, and hence, $i^{-1} \cdot i \in I$ because *I* is a left ideal. That means $1 = i^{-1} \cdot i \in I$, But if $1 \in I$, then $g = g \cdot 1 \in I$ for any $g \in G$. So $G \subseteq I$.

The Hidden Double Negation

Did the presence of a hidden double negation in the informal version of the theorem statement cause Dori difficulty? We cannot say for sure. However, it seems quite clear that the informal version of the statement, like many such informal versions, while definitely memorable, is difficult for students to unpack into its formal (positive) version. It is well-known to cognitive psychologists that negations are hard to decode and understand. Pinker (2014) reasoned as follows:

> The cognitive difference between believing that a proposition is true (which require no work beyond understanding it) and believing that it is false (which requires adding and remembering a mental tag), has enormous implications for a writer [and a reader]. The most obvious is that a negative statement like *The king is not dead* is harder on the reader than an affirmative one like *The king is alive*. Every negation requires mental homework, and when a sentence contains many of them the reader can be overwhelmed. Even worse, a sentence can have more negations that you think it does. Not all negation words begin with *n*; many have the concept of negation tucked inside them, such as *few, little, least, seldom, though, rarely, instead, doubt, deny, refute, avoid,* and *ignore*. The use of multiple negations in a sentence … is arduous at best and bewildering at worst …" (pp. 172-173).

The word "proper" in the above informally worded theorem statement has the concept of a double negation "tucked" inside it, and according to Pinker, would be arduous and bewildering.

Transitioning Students from Proving Formally Stated Theorems to More Informally Stated Theorems

When we write our course notes, we begin by including all notation and write the statements of theorems in "if-then" format, which allows students to write at least a first-level proof framework without difficulty. In addition, if they can "unpack" the conclusion (the final line of their emerging proof), they can produce a second-level proof framework. This goes a long way to exposing the real mathematical problem to be solved in order to construct the rest of the proof. Eventually, during our course, we begin to transition students to less formal ways of stating theorems, by for example, having them prove*: Every semigroup can have at most one identity element and at most one zero element*. Here the difficulty is not in introducing notation, but in deciding what "at most" means and how to structure a proof of it (namely, by assuming there are identity elements e and f and using the definition of identity in a clever way).

Perhaps a better progression would be to have students first prove a number of "if, then" theorems with all notation included, and then to have them introduce the notation and reformulate an "easy" informal statement into its formal version. For example, we might ask students to prove: *The composition of two* 1-1 *real functions is* 1-1, omitting the names of the two functions. Here it is relatively easy to introduce notation, f and g, for the two functions, and to put the statement in "if-then" form." In addition, composition has been defined in the course notes so there are no decisions to make on how to structure a proof, provided students can unpack the definition of composition. We feel such a rearranged course design would help increase student success and the early building of a sense of self-efficacy (Bandura, 1994, 1995), while gradually transitioning students to more informally worded theorem statements that are more difficult to unpack.

We anticipate that further theoretical and linguistic comments and conjectures will be included in the presentation.

Discussion Questions

1. What sorts of problems do students have in unpacking informally worded theorem statements, other than: (1) Suitable notation has not been included and has to be introduced. (2) It is not in "if-then" format, so it is not clear how to structure a proof (i.e., it is unclear how to construct a proof framework) . (3) Lack of positive phrasing (e.g., hidden double negations)?
2. What are some possible progressions that would help undergraduate mathematics students transition to being able to interpret informally stated mathematical theorems into their formal equivalents in order to construct proofs of them? How would one research their effectiveness?
3. Would it be an interesting research project to examine a variety of undergraduate textbooks to determine how many theorems are stated in an informal, possibly confusing, way?

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The complement of RUME: What's missing from our research?

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The Research in Undergraduate Mathematics Education (RUME) community has generated a substantial literature base on student thinking about ideas in the undergraduate curriculum. However, not all topics in the curriculum have been the object of research. Reasons for this include the relatively young age of RUME work and the fact that research topics are not necessarily driven by the content of the undergraduate curriculum. What topics remain largely untouched? We give a preliminary analysis, with a particular focus on concepts in the standard calculus sequence. Uses for this kind of analysis of the literature base in the education of novice researchers and potential future directions for further analyses are discussed.

Key words: student thinking, calculus, literature review

Overview

The inherently applied nature of research in undergraduate mathematics education (RUME) means it is natural for non-researchers seeking information about teaching and learning to desire (or perhaps even expect) that members of the RUME community carry out research that spans the undergraduate curriculum. This desire and/or need for research-based information about how student learn particular content, vetted teaching practices, and validated assessment measures can prompt college mathematics instructors to ask of their mathematics education colleagues, "Is there any research on [topic X]" where "topic X" might be anything in the undergraduate curriculum from limits to infinite-dimensional vector spaces. Although our field is generating research related to many topics, we are far from having a complete catalog of findings on all topics taught in undergraduate mathematics courses.

In a field as young as RUME, it is to be expected that considerable effort is focused on the development and refinement of theory as researchers work to identify and characterize factors that shape learning. There is, however, value in periodically taking stock of where we are. As a contribution to this effort, we pursued the following questions: What topics from the undergraduate curriculum have been the objects of research with direct links to practice? In particular, if one looks through the table of contents for typical texts for the calculus sequence, which items are associated with research findings about student thinking and which have yet to be examined? In short, what's the complement of our existing literature base? In this Preliminary Report, we offer our answers to these questions based on literature reviews we conducted in the process of writing a book on student thinking for novice mathematics instructors. Our findings indicate that the contents of the complement include some topics known to be challenging for both students and instructors and some topics with strong connections to concepts from secondary mathematics. Although we provide a rationale for this work below and some findings in subsequent sections, we consider this work "preliminary" because we are seeking feedback on potential uses for and ways of representing/communicating the products of this kind of literature review.

Rationale and Relevant Literature

Others (e.g., Schoenfeld, 2007) have noted tensions that can exist between theory development and the desire to address practical or applied issues in fields such as education research. On the one hand, from economic and other societal perspectives there are significant and pressing needs to improve the teaching and learning of undergraduate mathematics (Holdren & Lander, 2012) and to do so we need research-based answers to questions derived from practice. On the other hand, the relative newness of the field of RUME means that we have significant needs for basic research and the associated development of theories. We see two primary ways in which an "inventory" of topics already addressed and missing can aid the field in balancing these needs and in making progress in ways that address both the applied and theory-focused needs in the field.

The first rationale is connected to the fact that we are not currently operating in a time of "normal science." Times of "normal science" (Kuhn, 1970) are characterized by having methods for conducting research in a field with substantial track records of accepted use as well as theory to guide investigations that has accumulated and withstood examination over time. Although there have been substantial advances in these areas over the past several decades, the field of RUME is still in a phase of significant theory-building and development of tools for research. Although this description characterizes the field overall, some areas have been the object of more research than others. Therefore, it is productive to identify areas in which the most development has occurred because those areas may be where methods and theory are most mature. This can help researchers identify theory and methods that might be good candidates for use when conducting studies in less well-developed areas in RUME.

The second reason an inventory could be useful has to do with the issue of problem identification. There are many reasons for selecting a focus for research and that decision need not be shaped only by where no research has yet been done. However, in seeking out areas for research, it can help to know where the gaps are. Informed decisions can then be made about whether questions of significance can be pursued by venturing into an area where literature is scarce or by focusing instead on issues in an already somewhat established area. This is of particular importance for those who mentor new researchers (e.g., graduate students) because "beginning researchers need to learn how to identify and frame workable problems– meaningful problems on which legitimate progress can be made in a reasonable amount of time" (Schoenfeld, 1999, p. 170). And unfortunately, "For most students, *problem identification* is not part of the research apprenticeship process" (Schoenfeld, 1999, p. 168).

Research Design

This is the story of what happened when we set out to write an instructors' guide to student thinking about topics spanning the undergraduate curriculum. Our goal was to provide novice instructors with insights into student thinking that were based on research findings and to do that for as many topics from the calculus sequence and core major courses as possible. To locate relevant literature, we did the following:

- Generated list of topics, basing it on a "standard" course sequence, with particular emphasis on the calculus sequence because of our primary target audience for the book (graduate students and novice faculty instructors). We consulted tables of contents from widely used textbooks (e.g., Stewart, 2015).
- Set out to find published research, utilizing Google Scholar and literature reviews or summaries (e.g., Carlson & Rasmussen, 2008). We created lists of relevant articles for each of the sections as part of the preparation for drafting chapters of the book.

- Sent draft chapters to expert scholars in the area of the chapter and asked if we were missing anything. In particular, we asked them: Have we faithfully reported the literature, remaining true to each researcher's intent while still making it accessible to those outside of the education world? Have we missed any significant findings or citations that should be included? We then revised chapters based on the feedback, incorporating additional references suggested by reviewers.
- We gave up on writing some chapters or sections because of lack of literature.
- After all chapters were drafted, reviewed, and revised, we returned to our original topic list. We compared that to the final table of contents and noted topics that were missing.

Our goal was to generate text that illustrated how students think (productively and unproductively) about key ideas in the undergraduate curriculum. Because of this particular focus, we were seeking research of a particular sort—research that either had an explicit focus on describing student thinking about the topic or where there were clear implications from the findings to students' ways of thinking. This focus means that certain kinds of work that are incredibly valuable to the RUME community (e.g., teaching studies, theory development and testing, assessment and evaluation studies) were outside the scope of our search.

Findings

For the purposes of this proposal, we present the findings based on our analysis of the literature review done for just a few chapters. In particular, we focus on topics in calculus: limit, derivative, application of derivatives, integral, and sequences and series. The references listed are representative of the literature we located for each topic – however, they are not comprehensive of all existing literature on the topic.

Limits

This is perhaps the most extensively covered topic in the calculus sequence. We found research findings related to both student thinking about core ideas of limit and computations done to determine limits. These findings include, for example, ways students think (incorrectly) about limits of functions by treating all functions as if they are continuous (e.g., Bezuidenhout, 2001; Tall & Vinner, 1981), thinking of a limit as an unreachable bound (e.g., Davis & Vinner, 1986; Grabiner, 1983), viewing all limits a monotonic (e.g., Davis & Vinner, 1986), and believing that testing a few values is sufficient to evaluate a limit (e.g., Williams, 1991).

Derivatives

We located research on student thinking about derivative as well as about computations to generate derivatives and uses of derivative on applied problems. This literature addressed various specific topics, including:

- student difficulties with the building blocks of derivative such as understanding the secant line representation for limits of difference quotients (e.g., Carlson, 1998; Habre & Abboud, 2006; Monk, 1994; Orton, 1983);
- difficulties with applying various procedures for calculating derivatives (e.g., Cipra, 2000; Horvath, 2008; Smith & Ferguson, 2004; Zandieh, 2000);
- trouble with graphical representations of derivatives (e.g., Ferrini-Mundy & Graham, 1994; Nemirovsky & Rubin, 1992; Orton, 1983);
- thinking associated with linking features of first and second derivatives (positive, negative, increasing, decreasing, etc.) with properties of the original function (e.g.,

increasing, decreasing) (Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, 2002; Zandieh, 2000).

What's missing from the literature on derivatives?

In comparing the sections of commonly-used texts to the list of topics for which we could locate research findings, several stood out as missing from the existing literature. The topics that are in the "complement" of the existing literature on derivatives include:

- implicit differentiation, in particular examinations of what sense students make in the transition from df/dx to d/dx and the idea of differentiation as an operator;
- student thinking and sense making about linear approximation and differentials;
- connections between trigonometric functions (as ratios of lengths of triangle sides) and the calculus of them;
- Newton's method, in particular what sense students make of the process.

Some topics that have been addressed in only a few studies could benefit from substantially more attention from researchers include: chain rule, related rates, and optimization.

Integrals

As with derivatives, the literature on student thinking about integrals provides insights into how students think about the concept of integration and how they think about computations used to determine the value of integrals. The research articles we located addressed various topics, including the following:

- definition and meaning of integrals (e.g., Abdul-Rahman, 2005; Fuster & Gómez, 1997; Gonzalez-Martin & Camacho, 2004);
- integral as a measure of accumulation (e.g., Anaya & Cavallaro, 2004; Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, 2002; Thompson & Silverman, 2008);
- Riemann Sums (e.g., Bezuidenhout & Olivier, 2000; Oehrtman, 2009; Thompson & Silverman, 2008);
- anti-derivative computations (e.g., Hirst, 2002; Orton, 1983b).

What's missing from the literature on integrals?

Substantial topics from a standard treatment of integration for which we were unable to locate research include:

- integration techniques involving the very commonly-used method of substitution;
- other integration techniques and what sense, if any, students make of this topic;
- volumes of revolution.

Sequences and Series

This very challenging topic has been the object of a fair amount of research on student thinking about the following:

- characteristics of the real numbers and how that intersects with understanding sequences and series (e.g., Anderson, Austin, Barnard, & Jagger, 1998);
- ideas related to limit and their impact on thinking about sequences and series (e.g., Li & Tall, 1993; Tall, 1977);
- issues of representation, in particular, coordinating algebraic and other views of sequences and series (Cornu, 1991; Tall & Vinner, 1981);
- definition of convergence and proofs of it (e.g., Harel & Sowder, 1998; Pinto & Tall,

1999).

What's missing from the literature on sequences and series?

Topics where research is very limited included:

- power series, especially the question of what sense students make of the overarching idea of approximating one function with other functions;
- Taylor and Maclaurin Series and what students think the core ideas are behind the computations we ask of them.

Conclusions, Implications and Ideas for Further Research

Efforts of the RUME community have resulted in research-based answers to the question of how students think productively and unproductively about a wide range of topics in the undergraduate curriculum. From our analysis of that literature and the topics typically addressed in that curriculum, there are ideas for which we still lack insights into student thinking. Looking just at the calculus sequence, limit appears to have been a rich and productive arena for research, resulting in findings about key ideas as well generating and refining theories about student thinking. For differentiation and integration, the literature does not seem to address the topics as uniformly. Although the existing research provides valuable insights into student thinking about some ideas, there are some noticeable gaps. These include the challenging ideas and techniques associated with implicit differentiation, linearization, various techniques of integration, and applications such as volumes of revolution. The community might also benefit from additional investigations into student thinking about power and other kinds of series. As with topics such as implicit differentiation and some techniques of integration, we suspect that there is much to be learned about what sense students do (and do not) make of the processes used in these sections of the course. The areas in which significant amounts of research have already occurred might be useful sources of theories that could be tested out in the less-researched areas.

Our search for literature was limited to research on student thinking and learning but further analyses could be conducted of research related to teaching, curriculum development or other genres of studies found in RUME. In addition, a separate kind of analysis could be done of just the works represented in RUME proceedings. This might provide insights into trends in the field or areas of investigation that have not yet found their way into journal publications.

Describing the complement of what exists in the literature may be useful in guiding graduate students or others new to the field. Knowing where theory development and findings are scarce or plentiful can help researchers (and those who advise them) to know whether their chosen topic is apt to take them into well-understood territory or whether they will encounter few studies and perhaps only limited theoretical frameworks to guide their efforts.

Questions to be posed to the audience:

- What are the reasons behind the dearth of research on various topics? Are these simply areas the last to be visited – or is there something about some topics that make them less interesting or productive for researchers?
- Might the RUME community benefit from this type of high-level view of its research – including who is studying what mathematical topics? What is the best way to assemble such information – and disseminate it?

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SUPPORTING PRESERVICE TEACHERS' USE OF CONNECTIONS AND TECHNOLOGY IN ALGEBRA TEACHING AND LEARNING

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The Conference Board of the Mathematical Sciences recently advocated for making connections and incorporating technology in secondary mathematics teacher education programs, but programs across the United States incorporate such experiences to varying degrees. This study explores preservice secondary mathematics teachers' opportunities to expand their knowledge of algebra through connections and the use of technology and to learn how to use both to support teaching and learning of algebra. We explore the research question: What opportunities do secondary mathematics teacher preparation programs provide for PSTs to learn about connections and encounter technologies in learning algebra and learning to teach algebra? We examine data collected from five teacher education programs chosen from across the U.S. Our data suggest not all secondary mathematics teacher preparation programs integrate experiences with making connections of different types and using technology to enhance learning across mathematics and mathematics education courses. We present overall findings with exemplars.

Key words: Algebra, Technology, Connections, Secondary Teacher Training

Algebra plays a prominent role in mathematics education reform efforts because it is valued as a foundational subject in mathematics. Particularly in the United States, preparing future secondary mathematics teachers to teach algebra has gained importance as, in response to algebra-for-all initiatives, more states include algebra as a high school graduation requirement (Teuscher, Dingman, Nevels, & Reys, 2008). Due to these new requirements, not only are more secondary mathematics teachers teaching algebra in their first professional position, but these new teachers are also expected to teach algebra to a more diverse population of students than ever before (Stein, Kaufman, Sherman, & Hillen, 2011). Hence it is imperative that we study how teaching programs prepare preservice teachers (PSTs) for teaching algebra to this diverse population. Particularly important is attention to how future mathematics teachers are supported in developing a deep understanding of algebra.

This presentation is situated within a larger project that has used several different perspectives in exploring opportunities PSTs have to learn algebra and learn to teach algebra in teacher preparation programs. In this presentation, we have chosen to focus particularly on two ways that can support PSTs in deepening their own understanding of algebra, as well as supporting them in developing strategies for supporting their future students' algebra learning.

Standards for both secondary mathematics content and teacher preparation have emphasized the importance of developing PSTs' abilities to make connections and to use appropriate educational technologies in their own mathematical learning and in their future mathematics teaching. Particularly with respect to PSTs' mathematics courses, *Mathematics Education of Teachers II (METII)* recommended that instructors of mathematics courses support PSTs in "forming connections" (p. 56) and that experience with technology "should be integrated across the entire spectrum of undergraduate mathematics" (CBMS, 2012, pp. 56-57).

Standards developed for teacher preparation program accreditation agencies, such as Interstate Teacher Assessment and Support Consortium [InTASC] and National Council for Accreditation of Teacher Education [NCATE], have emphasized the importance of developing PSTs' abilities to see mathematics as a complex, connected system woven through other non-mathematical disciplines as well as a way to make sense of the real world (Council of Chief State School Officers [CCSSO], 1995; National Council of Teachers of Mathematics [NCTM], 2012). PSTs must think about mathematics as a "whole fabric" as they make connections among mathematical topics and in relation to others (NBPTS, 2010). To support this view of mathematics, PSTs need to make connections within algebra, and between algebra and other mathematical fields, while linking algebra with real-world situations. PSTs should prepare to teach using "rich mathematical learning experiences" and provide their future students with opportunities to "make connections among mathematics, other content areas, everyday life, and the workplace (NCTM, 2012). Further, PSTs should also be able to prepare to support their future learners in reflecting "on prior content knowledge, link[ing] new concepts to familiar concepts, and mak[ing] connections to learners' experiences" (CCSSO, 1995).

Teacher preparation standards have emphasized the importance of PSTs' own learning of mathematics content using technologies as both a "practical expedient" as well as to enhance learning. Teachers also need support in critically evaluating and strategically using technology in mathematics teaching and learning (CBMS, 2012; CCSSO, 1995; NCTM, 2012). In addition, *METII* emphasized the importance of PSTs' preparation for using a variety of technologies, including problem-solving tools and tools for exploring mathematical concepts (CBMS, 2012).

This study explores opportunities provided by secondary mathematics teacher preparation programs for PSTs to expand their knowledge of algebra by making connections and using tools and technology and to learn how to incorporate their own use of connections and technology when they teach algebra. We explore the following research question: "What opportunities do secondary mathematics teacher preparation programs provide for PSTs to learn about connections and encounter tools and technologies in learning algebra and learning to teach algebra?" Making connections in the service of algebra teaching and learning might include making connections within algebra, between algebra and other mathematical fields, between algebra and non-mathematical fields, and between ideas in advanced algebra and school algebra. Encounters with technology in the service of algebra teaching and learning might include using or learning about a variety of algebra-appropriate technologies, as well as thinking critically about technology use. In this study, we define tools and technology broadly as electronic tools and software, as well as physical tools such as manipulatives.

Method

This study is part of *Preparing to Teach Algebra (PTA),* a mixed-methods study that explores opportunities provided by secondary mathematics teacher preparation programs to learn algebra and to learn to teach algebra. The *PTA* project consists of a national survey of secondary mathematics teacher preparation programs and case studies of five universities. The current study is a qualitative analysis of the case studies focusing on the opportunities provided to PSTs to encounter technology and to make connections in learning algebra and learning to teach algebra.

The *PTA* project purposefully chose secondary mathematics teacher preparation programs at five universities to explore. We refer to these universities as Beta, Gamma, Kappa, Sigma, and Zeta Universities. Beta, Kappa, and Zeta Universities have Carnegie classification of Master's L (Master's-providing Colleges and Universities – larger programs). Gamma and Sigma Universities are doctorate-granting institutions with Carnegie classification of RU/VH (Research Universities – very high research activity). Beta, Gamma, and Kappa Universities are located in the Great Lakes region of the U.S., while Sigma is in the Southeast and Zeta is in the Far West.

We compiled data by conducting two focus groups of 3-4 PSTs and 10-13 instructor interviews at each site (except Zeta, where we conducted three interviews) and collected corresponding instructional materials from each instructor we interviewed. In the instructor interviews at each site, we included required mathematics, mathematics for teachers, mathematics education, and general education courses selected for potential algebra content.

Among other interview questions, we asked instructors which types of tools and technologies they used in a particular course and how they supported PSTs in making connections in algebra. Similarly, we asked PSTs in focus groups to identify required courses that incorporated opportunities to make connections or to use technology in learning algebra or learning to teach algebra. We asked PSTs explicitly about their required or shared experiences with connections and with technology.

Prior to considering the data for mentions of connections or technology, the *PTA* project team had coded data for algebraic content. In analyzing data, four researchers worked in pairs, reading the interview and focus group transcripts and discussing potential opportunities reported by instructors or PSTs.

For connections, the two researchers individually coded data sources based on the major four types of connections (e.g., connections within algebra, connections between algebra and mathematics) and met to make consensus on the coding. We then developed summary documents of each university, including tables of the number of opportunities and quotations in each course. We will analyze the quotations to document different types of opportunities that were reported (e.g., algebraic topics that PSTs were exposed, specific activities that PSTs engaged with, or/and opportunities to help PSTs learn to teach connections).

For tools and technology, the two researchers have so far only considered instructors' interviews and instructional materials. We captured types of tools or technologies mentioned by course instructors and PSTs, as well as details of the experiences the rationale (if any) given by the instructor detailing why (or why not) tools and technology were used (e.g., "dulls the mind" or "representations help students understand quantitative situations"). Based on previous research, we will analyze instructors' and PSTs' reports of technology use to understand why opportunities are or are not provided in particular mathematics or mathematics education courses, and to understand the types of experiences provided, whether the experiences are as a "practical expedient," or to "advance learning," or to provide opportunities for PSTs to think critically about choice and use of tools and technology by engaging with potential affordances and limitations (CBMS, 2001).

Results

For the purposes of this proposal, and due to space limitations, we focus on finding exemplars of *types* of experiences provided to PSTs by two of the five different programs (i.e., Beta and Kappa) and focus on experiences in Abstract Algebra, Linear Algebra, and Secondary Mathematics Methods courses. We are not evaluating the programs; rather, exploring strengths and challenges of each program to understand what rich experiences across a program's offerings could look like, and to understand the challenges that arise.

Connections

Beta University. Linear Algebra instructor reported opportunities related to all four types of connections, while Abstract Algebra instructor provided examples of three types (except for the connections between algebra and non-mathematics) and Secondary Mathematics Methods instructor provided two types of connections (within algebra and between algebra and non-mathematics). To be specific, Linear Algebra instructor reported that he discussed the meaning of solving an equation connected to distributivity and associativity (within algebra), probability through Markov chains (between algebra and other math), population dynamics through modeling (between algebra and non-math), and connections between solving systems of linear equations and college algebra (e.g., identities, inverse). Abstract Algebra instructor reported that he emphasized common structures and themes behind different number systems, discussed connection between ring isomorphisms and graph morphissms in Discrete Mathematics course, and discussed the relationships between high school level division algorithm and machinery in the division algorithm. Secondary Mathematics Methods instructor focused on how PSTs made connections rather than how the instructor made them. The instructor said that PSTs made algebraic connections when they created lesson plans and participated in reading workshops.

Kappa University. Instructors of the three courses made different types of connections: Linear Algebra and Secondary Mathematics Methods instructors reported that they made the major types of connections except for connections between college and school algebra; and Abstract Algebra instructor reportedly provided opportunities except for connections between algebra and non-mathematics. Specifically, Linear Algebra instructor mentioned that PSTs studied how to solve systems of equations, connected them with the topics in the course, and learned how technology could best assist them. Abstract Algebra instructor reported PSTs' opportunities to learn about abstract proofs that are related to college algebra and the usefulness of number theory and set theory. Secondary Mathematics Methods instructor provided a specific activity (border problem) where PSTs discussed the meaning of the variable in context and generalized the situation by using both words and symbols, which provided them the opportunity to connect different representations and use geometry.

Tools and Technology

Beta University. The Abstract Algebra instructor reported using little technology in his course. He did provide experiences using instructional technologies to facilitate communication, however, by asking students in the course to collaboratively develop class agendas using GoogleDocs. For example, as part of the agenda, the instructor asked students to post questions on readings and add checkmarks to questions posted by classmates that they also had. Use of technology was extensive in Linear Algebra, as the instructor reported using Maple and targeted Java applets in weekly computer lab activities. When talking about the computer activities, the instructor used phrases like "they discover the concept" and "they develop intuitive understanding." The Secondary Mathematics Methods instructor reported focusing more on supporting PSTs in thinking about "the appropriate use of technology and helping their students [with] the appropriate use of technology" based on experiences that the PSTs have in their student teaching classrooms. For example the instructor reported discussing the possibility of the PSTs asking their students, "Here's a calculator. You need to tell me which five problems you want to use the calculator on and why." As a part of the course, the PSTs also keep a blog to communicate their experiences with each other and receive feedback on ideas they're trying out.

Kappa University. The Kappa Abstract Algebra instructor also reported using little technology in his course, but he also used instructional technologies to facilitate course communication. He used the Blackboard Learning Management System to communicate to students and asked students to write course or homework questions in the discussion section of Blackboard. He said that he also has recently begun using his iPad to record his voice and writing as he answers students' questions during office hours. He then posts those videos on Blackboard so that the student who asked the question "can go back and play it over and it's there for them" but also other students with similar questions can see his responses. The Linear Algebra instructor did not report using technology explicitly in his course, although he did provide access to Mathematica for his students. He did report using unsharpened pencils as physical tools in class to represent vectors and vector operations, and said he sometimes sees his students bring their own pencils to exams. The Secondary Mathematics Methods I course instructor reported using, "SMARTBoards, algebra tiles, pattern blocks, TI-83 calculators, Fathom, TinkerPlots, and GeoGebra." The instructor reported emphasizing "not using the technology and the resources for the sake of using them but making sure that there is a purpose and a reason behind why are we using this technology." The instructor reported many discussion about potential pedagogical uses of different tools and technology, especially focused on having students use their resources in mathematical investigations.

Discussion

Our preliminary results show different types of opportunities that PSTs were provided related to the learning of algebraic connections and the use of technology to learn and learn to teach algebra. There was a wide range of opportunities that instructors provided related to algebraic connections: some instructors provided lists of topics and ways that they made connections (e.g., Linear Algebra at Beta); others reported specific activities that engaged PSTs to make connections (e.g., Secondary Mathematics Methods at Kappa). At Beta University, mathematics instructors described how they emphasized different types of connections, while the mathematics education instructor focused on how PSTs made connections in his class. At Kappa University, instructors described connections among not only algebraic topics (e.g., systems of equations, variables), but also practices that can be used in different courses and grade levels (e.g., proofs, generalization), along with how technology can be used to make such connections (e.g., Linear Algebra instructor). We heard concerns from both mathematics and mathematics education instructors that technology could impede PSTs' learning. Some mathematics education instructors argued, to the contrary, that use of technology enabled PSTs to increase their understanding of algebra topics in ways that were not possible otherwise. At each university there was at least one opportunity for PSTs to think critically about their future educational use of technology, but experiences varied.

Our preliminary results on the other cases (i.e., Gamma, Sigma, Zeta) show patterns among the major types of connections and uses of technology in different courses, which we plan to share during the presentation. We will additionally provide detailed examples of how instructors provided PSTs with opportunities to make algebraic connections and use technology as recommended by policy documents (e.g., CCSSO, 1995; NCTM, 2012), along with different types of opportunities that can help other educators assess their own programs.

In our presentation, we plan to ask participants: From our preliminary analysis, what do you find surprising? What recommendations would you make for our analysis or what would you like to see in published reports? One perspective on technology use in mathematics is that technology should mainly be used as a practical expedient to support applied mathematics projects. We plan to analyze our data specifically for opportunities for PSTs to engage in experiences that combine mathematical modeling, technology, and making connections. What would be interesting to you as results? How can we approach the analysis to make our results stronger?

Endnote

This study comes from the *Preparing to Teach Algebra* project, a collaborative project between groups at Michigan State (PI: Sharon Senk) and Purdue (co-PIs: Yukiko Maeda and Jill Newton) Universities. This research is supported by the National Science Foundation grant numbers DRL-1109256/1109239 and by the National Science Foundation, Spencer Foundation.

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Investigating college students difficulties with algebra

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Algebra is frequently referred to as the "gateway" course for high school mathematics. Even among those who complete high school Algebra courses, many struggle with more advanced mathematics and are frequently underprepared for college level mathematics. For many years, college instructors have viewed the final problem solving steps in their respective disciplines as "just Algebra", but in reality, a weak foundation in Algebra maybe the cause of failure for many college students. The purpose of this project is to identify common algebraic errors students make in college level mathematics courses that plague their ability to succeed in higher level mathematics courses. The identification of these common errors will aid in the creation of a model for intervention.

Keywords: Algebra, Common Errors, Symbolic World

Algebra is often referred to as a gateway course because it is foundational and fundamental to STEM subjects and it is clearly here to stay. At college level, algebra content is considered as assumed knowledge and the professors are not expected to re-teach it. Calculus curricula are demanding and fast moving leaving no extra time to resolve basic algebra issues. However, it seems that many college level instructors are only concerned with one side of the coin and are somewhat disconnected from students' prior experiences, let alone the psychological effects and possible negative experiences that originated many years ago. Although, Stacey, Chick and Kendal (2004) in their edited book titled: *The Future of the Teaching and Learning of Algebra*, discussed the main problems on Algebra in school Algebra, very little mentioned in the way of consequences in college level.

A survey published by the National Center for Education Statistics reported that nationwide, in 2000, 28% of incoming freshmen took a remedial class (U.S. Department of Education, National Center for Education Statistics (NCES), 2004). Beyond those who find themselves underprepared for college level mathematics coursework, the majority of students struggle due to incomplete or insecure understandings of many important Algebraic topics. The impact of weak or incomplete mathematical understanding at the middle school and high school level, and Algebra in particular, has a profound impact on the future mathematical success of students and their educational possibilities.

This research will employ Tall's (2008, 2010, 2013) framework of embodied, symbolic and formal mathematical thinking in an effort to construct a model of mathematical thinking for investigating students' understanding of algebra concepts. Tall (2010) defines the worlds as follows: The *embodied world* is based on "our operation as biological creatures, with gestures that convey meaning, perception of objects that recognise properties and patterns...and other forms of figures and diagrams" (p. 22). Embodiment can also be perceived as giving body to an abstract idea. The *symbolic world* is based on practicing sequences of actions which can be achieved effortlessly and accurately as operations that can be expressed as manipulable symbols. The *formal world* is based on "lists of axioms expressed formally through sequences of theorems proved deductively with the intention of building a coherent formal knowledge structure" (p. 22). Through in-depth qualitative research we anticipate ascertaining more about each of the three worlds as well as the blending of relationships *between* the worlds, and ultimately proposing a model that is applicable to interventions for the algebra skills needed for success in calculus.

In an effort to identify common student errors and gain insight about how best to develop appropriate interventions, this project is focused on the following research questions: 1) In what areas of algebra do the major difficulties occur? 2) How would algebra interventions affect a student's understanding of calculus and help them with more symbolic ways of interpreting mathematics? 3) As students advance into various STEM disciplines and encounter more formal mathematics, how does the lack of understanding algebra affect them and what interventions would help them? 4) What are some of the pedagogical challenges related to the symbolic world of mathematics?

For the purpose of this poster presentation we will only answer the first research question.

Method

Our interdisciplinary, multi-institutional team includes two mathematics educators, two mathematicians, a cognitive psychologist who specializes in children's algebra thinking process, an elementary school algebra teacher and two graduate students. To achieve the project goals and answer the research questions, a pilot study was conducted at a mathematics department at a large research university in South-West of the United States. Data were gathered from approximately 2500 students' final exams from the following five different math courses: College Algebra; Pre-Calculus and Trigonometry; Pre-Calculus for Business, Life and Social Sciences; Calculus I for Business, Life and Social Sciences; and Calculus and Analytical Geometry. A small sample of the pilot data were analyzed in order to provide preliminary results. We plan to use the results of this study to develop a model for intervention. Data will be collected before and after the intervention to further our understanding of both the common errors and how best to help students overcome them.

Preliminary Results

The initial analysis of the pilot study data show several themes emerging among the types of common errors made by students while encountering *fractions* and *exponents* and dealing with *variables* and *mathematical properties*. The frequent occurrence of these categories of errors amplify significantly in calculus courses and will have a negative impact on students' overall performance. For the purpose of this proposal, we have provided two examples from the data that reflect the common errors while *encountering fractions* (see Table 1).

Table 1. Sample students' common errors causing significant disruption in solving problems.

Calculus I problems	*Errors*
*Algebra (ii) **[6 marks]** $g(x)=\sqrt{x}+\frac{x+1}{x-1}-\frac{x+1}{x^2+1}$ $g(x)=\sqrt{x}+\frac{x+1}{x-1}-\frac{\cancel{x+1}}{(\cancel{x+1})(x+1)}=\sqrt{x}+\frac{x+1}{x-1}-\frac{1}{x+1}$ $g(x)=\frac{\sqrt{x}(x-1)(x+1)}{(x-1)(x+1)}+\frac{(x+1)(x+1)}{(x-1)(x+1)}-\frac{1(x-1)}{(x+1)(x-1)}$ $\boxed{(-\infty,-1)\cup(-1,0]\cup[0,1)\cup(1,\infty)}$	*Simplifying fractions causing difficulties in finding the domain*
$\left[\frac{f(x+h)-f(x)}{h}\right]$ $\frac{\cancel{x}x-x^2+\cancel{x}h-\cancel{x}x-x^2}{\cancel{x}}$	*Simplifying fractions causing difficulties in finding the derivative*

More than a century ago, De Morgan (1910) wrote about the difficulties students face in learning mathematics noting common errors related to arithmetic and rational number computation. Since that time, other researchers have catalogued common errors in computation and algebra (Ashlock, 2010; Benander & Clement, 1985; Booth, Barbieri, Eyer & Pare-Blagoev, 2014). Connecting with the existing body of knowledge on students' persistent problems related to algebra our preliminary findings seem to parallel some of these categorizations and document that these errors persist in college level. In many cases, students are working to learn concepts that are new to them in college calculus courses and the results on assessments, formative or summative, are often more reflective of student difficulties with algebra than the newer concepts. The frequency of such errors creates frustration for both students and their instructors and may create barriers to student advancement in college level mathematics. We believe that the model generated by this project will be generalizable and can be used to examine the effect of students' understanding of Algebra in other sciences e.g. Physics and Chemistry. Moreover, the study will have an enormous impact on our understanding of the symbolic world and its pedagogical complexities.

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The Intermediate Value Theorem as a starting point for multiple real analysis topics

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In this paper I argue that the proof of the Intermediate Value Theorem (IVT) provides a rich and approachable context for motivating many concepts central to real analysis, such as: sequence and function convergence, completeness of the real numbers, and continuity. As a part of the development of a local instructional theory, an RME-based design experiment was conducted in which two post-calculus undergraduate students developed techniques to approximate the root of a polynomial. They then adapted those techniques into a (rough) proof of the IVT.

Key words: *RME, instructional design, design experiment, real analysis, limit*

Introduction

The concept of limit has served as the theoretical foundation for the calculus and its applications ever since the work of Cauchy, Bolzano, and others in the early and mid-19th century (Grabiner, 1981). Specifically, in a standard analysis class, the ideas of limit and convergence lie at the heart of such topics as sequences, continuity, derivative, integral, and the completeness of the real numbers. It follows that a formal understanding of the limit concept is essential to any investigation of the theoretical underpinnings of the calculus.

Presented here is a description of how the context of the Intermediate Value Theorem (IVT) can serve as a natural launching point for many topics in a real analysis course, starting with formalizing the concepts of limit and convergence. The IVT provides such a context in two ways: 1) using said theorem to approximate the root of a polynomial and 2) adapting that approximation technique into a formal proof. I will report on an RME-based design experiment, the goal of which was to investigate the following questions:

- *What student strategies anticipate the formal limit concept?*
- *What problems or tasks can be used to elicit these strategies?*
- *How can these strategies be leveraged to develop more formal understandings of the limit concept?*
- *What student strategies suggest avenues for developing other real analysis topics?*

Literature Review

Student understanding of the limit has received a great deal of attention from the mathematics education research community. A great deal of research has focused on investigating the struggles students face in working with limits and the tools they use to deal with those struggles (Bezuidenhout, 2001; Cornu, 1991; Davis & Vinner, 1986; Moru, 2009; Oehrtman, 2009; Sierpińksa, 1987; Szydlik, 2000; Tall & Schwarzenberger, 1978). Briefly, students employ intuitive metaphors (e.g. “limit as motion”, “collapsing dimension”, “limit as unreachable boundary”) that can be problematic in more formal endeavors. The other main area of focus has been investigating the process of students formalizing their understanding of limit (Cottrill, et al., 1996; Oehrtman, Swinyard, & Martin, 2014; Swinyard & Larsen, 2012; Williams, 1991); that is, coming to understand and work with limits in a way that is consistent with the standard formal definition(s)[1]. One important development in our understanding has

1 There are many logically equivalent formulations of these definitions, so saying “the standard formal definitions” is perhaps misleading. By “standard” I refer to the ε-δ (or ε-N) characterizations found in most

been the recognition that formal definitions of convergence, and therefore formal work with limits, serve a markedly different purpose than informal work with limits (Swinyard & Larsen, 2012). Specifically, tasks in the calculus sequence generally involve *finding* or *evaluating* limits, while more formal tasks focus on *verifying* limit candidates, or constructing proofs given the existence of certain limits. Motivating this shift in character, while still building on intuitive knowledge gained in the calculus sequence, heavily influenced the development of the task sequence and local instructional theory for this design experiment.

Theoretical Framework

This paper reports on an RME-based design experiment, which represents the early stages of curriculum development for a real analysis course. Design experiments should inform both instructional design and theory development (Cobb, et al., 2003; Gravemeijer, 1998). The design heuristics of Realistic Mathematics Education (RME), namely *guided reinvention, emergent models*, and *didactic phenomenology,* guided the development and implementation of the experiment as well as the underlying theory. The contribution of each of these heuristics will be discussed briefly below.

Guided Reinvention

On a macro level, the heuristic of guided reinvention motivated my overall instructional goal of having the students develop their own formal definitions of convergence, rather than working to make sense of the standard formal definitions. In RME, the goal is not that everything be strictly reinvented by the students, but rather that, "formal mathematics would be experienced as an extension of [students'] own authentic experience" (Gravemeijer & Doorman, 1999). That is, instructional activities should be designed and sequenced so that the formal mathematics emerges from students' informal mathematical activities, so that students feel a sense of ownership over the mathematics developed. While guided reinvention provides a macro-level structure for instructional design, other RME heuristics are more useful at filling in this structure.

For actual task generation, sequencing, and refinement, I relied largely on the design heuristics of *didactic phenomenology* and *emergent models*.

Didactic Phenomenology

In order to find an intuitive context that could evoke potentially useful student strategies, the heuristic of didactic phenomenology suggested that I look to the origins of the formal definition, paying particular attention to the didactic implications (i.e. consequences for instruction) of those origins. From where did our modern formal definition of convergence come? What problems did it solve for mathematicians at the time? Approximations of various kinds played a pivotal role in the historical development of the limit concept (Grabiner, 1981). Mathematicians (especially Lagrange) of the late 18th and early 19th centuries had made great strides in techniques of approximation and error-bounding in applied contexts. Cauchy is credited with developing the first ε-δ style definitions of convergence, and there is strong evidence to suggest that he took inspiration from these approximation techniques (Grabiner, 1981). Further, both he and Bolzano developed formal proofs with these definitions by adapting those same approximation techniques[2]. Prior to these developments, the mathematical community, including Newton and Leibniz, had only been able to justify limits with vague (by

analysis textbooks, e.g.: *For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.*

2 For a clear, thorough demonstration of this, see Grabiner, 1981, especially pp. 69-76.

today's standards) statements about "vanishing quantities" and "infinitesimals". The work of Cauchy and Bolzano put calculus on a firm, well-defined foundation for the first time.

Additionally, Gravemeijer and Terwel interpreted didactic phenomenology to suggest that, "situations should be selected in such a way that they can be organized by the mathematical objects which the students are supposed to construct" (2000, p. 787). That is to say, in order to support students in reinventing a formal definition of convergence, a curriculum designer should seek contexts and tasks in which the students would able to reason intuitively, and in which a formal definition would have power to organize and solve problems. Inspired by the works of Cauchy and Bolzano, I conjectured that approximating the roots of a polynomial using the Intermediate Value Theorem (IVT), and then constructing a formal proof of the theorem[3], would be just such a context.

The Intermediate Value Theorem:
If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) \neq f(b)$, then for every y between $f(a)$ and $f(b)$ there exists a $c \in (a, b)$ so that $f(c) = y$.

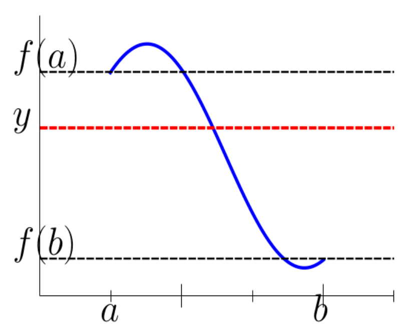

Figure 1: The IVT for continuous functions.

There are a few features that make the IVT such a context. First, the IVT is a fairly intuitive result which students will likely assume even if they have never been exposed to the formal theorem (Figure 1). This context requires students to draw on their concept images of functions and limits, and so builds on their intuitive knowledge gained from the calculus sequence. Second, the IVT provides an incredibly rich context for investigating the properties of real numbers and convergence. If we follow Cauchy's example and adapt our proof from the approximation techniques of Lagrange, then a rigorous proof of the IVT requires formal definitions of sequence convergence, continuity, and the limit of a function at a point. Some form of the Completeness axiom of the real numbers is also necessary, and so this context could motivate investigations in that direction as well. (Exploring these possibilities will be the focus of ongoing analysis.) In this way the context of approximating roots using the IVT, and then constructing a proof of the IVT for continuous functions, is a mathematically rich context that provides the students with a need to develop the desired formal definitions of convergence, as

3 Technically, if we restrict ourselves to establishing the existence of *roots* of continuous functions, then we are only proving a special case of the IVT (sometimes referred to as *Bolzano's Theorem*). But the proof is easily adapted to the general case by a simple vertical shift.

well more formal understandings of continuity and the completeness of the real numbers. The specific development of students Brad and Matt in this design experiment will be outlined below.

Emergent Models

The heuristic of emergent models provides one way to describe the process by which formal mathematics might emerge from informal student activity in these contexts. The use of "models" in RME is not restricted to physical drawings or tools. In describing a local instructional theory for the development of the quotient group concept, Larsen conjectured that, "the quotient group concept could emerge as a model-of students' informal mathematical activity as they searched for parity in the group D8 (the symmetries of a square)" (Larsen & Lockwood, 2013). Thus "model" in this sense can also refer to a concept or structure that the teacher or researcher recognizes as a model of the students' mathematical activity, but of which the students themselves may not be aware. Continuing with Larsen's example, once the students had begun to reflect on their activity with parity and other group-like partitions of groups, conjecturing and verifying common properties, the concept of quotient group became a model for their reasoning in this new mathematical reality; a "model of" informal mathematical activity had become a "model for" more formal mathematical reasoning. "This shift from model of to model for concurs with a shift in the students' thinking, from thinking about the modeled context situation to a focus on mathematical relations" (Gravemeijer, 1999, p. 162). In RME-based instruction, this progressive mathematization is the primary mechanism by which students develop more formal mathematics and create new mathematical realities for themselves.

The modern formal definition of convergence can be seen as a model of the approximating activity of the mathematics community in the 18th and 19th centuries. A formal definition of convergence emerged from these activities of approximating and error-bounding, first for Cauchy and then for the rest of the mathematical community. In this way the historical development of the concept of limit suggested that a formal definition of convergence could emerge as a *model of* student activity centered around approximations. By reflecting on and organizing this approximating activity, such a formal definition could emerge from their activity and serve as a *model for* more formal mathematical reasoning about limits and convergence.

Students' informal understandings of approximations and error-bounding have also been used as a foundation for instruction of the calculus sequence. Research suggests that this foundation has supported students in formalizing their concept of limit (Oehrtman, 2008; Oehrtman, Swinyard, & Martin, 2014). In this way formal characterizations of convergence can be seen as a useful model for describing and supporting students' progressive mathematization.

Methods

The design experiment involved two students, Brad and Matt, working together on a sequence of tasks over the course of 10 sessions, approximately 60-minutes each. Data consisted of the video/audio recordings of each session, researcher notes, and student-generated summaries from the conclusion of each session. After each session an outline of the students' progress was made, with key segments being analyzed in greater depth. This analysis focused on finding student strategies and statements on which to build toward the larger goal of formalizing their understanding of limits, which in turn supported the ongoing development of the task sequence.

Brad and Matt begin their investigation by working on the following task:

Does p(x) have a root in [0,3]?

$$p(x) = x^4 - 4x^3 - 7x^2 + 22x + 10$$

This polynomial was intentionally constructed to have only irrational roots, so that students would not be able to use algebraic tools (e.g. factoring, the quadratic formula, polynomial division, the rational roots theorem, etc.) to find the exact roots and would have to find a way to approximate. Subsequent tasks had the students approximating the root to different degrees of accuracy, and then working to generalizing their technique. The task was then to prove a version of the IVT which they had postulated, which in turn motivated the development of formal definitions of multiple types of convergence.

Preliminary Analysis and Results

On the first task, Brad and Matt developed an approximation strategy wherein they iteratively bifurcated the given interval to get more and more accurate approximations for the root. Through the course of constructing a proof of the IVT from this approximation technique, Brad and Matt were tasked with developing their own formal definitions of sequence convergence, function limit at infinity, continuity, and function limit at a point. Below I have included their first and their final definitions for what it means for a function to have a limit of zero as x tends to infinity.

Def 1b: $\forall$ 1/ε, $\exists$ n s.t. $f(n) < 1/\varepsilon$. ε, n in **R.**

Def 3: $\forall$ 1/ε $\exists$ an interval (x_a, ∞) s.t. $|f(x)| < 1/\varepsilon$ $\forall$ x in (x_a, ∞)

Current analysis is focusing on explaining how these reinventions were supported by the students' activity in the starting task.

Subsequent analysis will focus on identifying fruitful starting points, within the proof of IVT task, for follow-up tasks investigating other real analysis topics. Brad and Matt had some very interesting conversation about continuity which were not fully capitalized on. Further, their approximation strategy suggested many possible approaches to the idea of the completeness of the real numbers, including the Monotone Convergence Theorem, the Nested Interval Property, and even the Least-Upper Bound Property. Designing and implementing these tasks will also be the focus of future design experiments in the further development of this real analysis curriculum and local instructional theory.

Questions:

- What do you consider to be central topics in an introductory real analysis course?
- What student strategies presented here suggested possible paths for further development of other topics?
- What role should counter/pathological examples play in an introductory real analysis course?

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Secondary teachers confronting mathematical uncertainty: Reactions to a teacher assessment item on exponents

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Teaching is inherently uncertain, and teaching secondary mathematics is no exception. We take the view that uncertainty can present opportunity for teachers to refine their practice, and that undergraduate mathematical preparation for secondary teaching can benefit from engaging pre-service teachers in tasks presenting uncertainty. We examined 13 secondary teachers' reactions to mathematical uncertainty when engaged with concepts about extending the domain for the operation of exponentiation. The data are drawn from an interview-based study of items developed to assess mathematical knowledge for teaching at the secondary level. In our findings, we characterize ways in which teachers either denied or mathematically investigated the uncertainty. Potential implications for instructors include using mathematical uncertainty to provide an opportunity for undergraduates to learn both content and practices of the Common Core State Standards. The proposal concludes with questions addressing how undergraduate mathematics instructors could use uncertainty as a resource when teaching preservice teachers.

Key words: Preservice Secondary Teachers, Uncertainty, Algebra, Teacher Assessment

Overview and Research Questions

This study attends to how cases of mathematical uncertainty can be used to elicit teacher thinking in ways that might help them learn to engage in and model mathematical practices for their students. The research questions are:

- How did teachers respond to a mathematically challenging teacher assessment item?
- How the student work in the item influence the teachers' responses?

In this paper, we discuss an example case of a teacher assessment item that creates mathematical uncertainty, and a set of teacher responses illustrating patterns of thinking that emerged as those teachers reasoned about the situation.

Theoretical Perspective: Uncertainty

"Teaching is evidently and inevitably uncertain" (Floden & Buchmann, 1993, p. 373). Uncertainty in thought, encountered while teaching, can be conceptualized as cognitive "perplexity, confusion, or doubt" (Dewey, 1933). Sources of uncertainty are ubiquitous in teaching and the teaching environment, and range from instructional content or teacher authority (Floden & Buchmann, 1993) to student traits or school culture Labaree (2003).

While the literature reflects clear consensus that uncertainty is inevitable because of the complexity of teaching (Cohen, 1988; Floden & Buchmann, 1993; Helsing, 2007; Labaree, 2003; Zaslavsky, 1995), there is less agreement about whether either the existence of or the ways that teachers respond to that uncertainty should be considered a good thing (Helsing, 2007). Helsing (2007) describes two schools of thought, one of which characterizes uncertainty as a deficiency and another that characterizes it as productive in teacher learning. In other words, uncertainty could be a signal of systemic dysfunction or teacher underpreparation, but uncertainty can also be something that would be productive to celebrate rather than avoid. Denying uncertainty may restrict teachers' opportunities to look for alternative teaching methods and in turn limit students' learning (Cohen, 1988; Helsing, 2007). By pretending everything is certain and under control, teachers potentially lower their standards so to mask potential ineffectiveness, establishing routines that increase predictability, accepting status quo to maintain security, and blaming students, other teachers, parents, or society for students' failures (Helsing, 2007). These attempts decrease the opportunities teachers have to enhance their own practices. When a teacher confronts uncertainties and accepts that teaching is open and fluid, the chance to develop their practices and their subject matter knowledge increases (Floden & Buchmann, 1993; Labaree, 2003). And we might expect this to be as true for more expert teachers than for more novice teachers; Floden and Chang (2007) suggest the metaphor of a jazz score for teaching, where certain frames of reference can be nailed down and others are, of necessity, open to creativity and interpretation, and in fact a sign of expertise is the ability to make use of uncertainty rather than the ability to avoid uncertainty.

In this paper, we focus our attention on teachers' mathematical uncertainty. Most literature on uncertainty in teaching attends to general sources that have less to do with subject matter knowledge and more to do with the complexity of teaching itself (exceptions include Rowland (1995) and Zaslavsky (1995)). This may be due to the perception that teachers can deal with and resolve uncertainty around the subject matter by further studying it and simply learning the subject better. However, as Lakatos (1976) argues persuasively, mathematical knowledge can be under continual revision. In other words, the subject matter itself may be uncertain beyond whatever uncertainty a teacher may have due to not understanding it fully. Therefore, mathematical uncertainty can be as irreducible as other uncertainties. For the purposes of this paper, we conceptualize mathematical uncertainty following Zaszlasky (1995) as any mathematical situation in which competing claims, an unknown path or questionable conclusion, or non-readily verifiable outcome occurs. Uncertainty is both a condition of the situation (that something cannot be known) and the associated emotions. As Mason (1994) posits, "emotion is harnessable". Here we examine one instance of how the experience of uncertainty may be harnessable for learning to attend to a particular mathematical issue, that of domain extension.

Data, Method, and Analysis

We drew on a subset of data from a larger study focused on validation of secondary-level items for assessing teachers' mathematical knowledge for teaching (MKT) (Ball & Bass, 2003; Ball, Thames, & Phelps, 2008). Researchers in the larger study conducted retrospective cognitive interviews (Ericsson & Simon, 1985) of more than 20 secondary mathematics teachers on a subset of assessment items, in which the participating teachers were asked to talk aloud about their reasoning processes in responding to the items. The interviews were audio recorded and transcribed.

For this analysis, we focused on 13 teacher interview transcripts responding to a particular item from the set. This item (see Figure 1) asks teachers to examine the validity of two samples of student work where students have been asked to evaluate the expression $((-9)^{1/2})^2$.

Ms. Williams is reviewing a set of homework problems on which students were asked to evaluate exponential expressions, including the expression $\left((-9)^{\frac{1}{2}}\right)^2$. Ms. Williams asks two students to share their work.

For each of the following student's work, indicate whether it demonstrates a valid application of the laws of exponents to solve the problem.

	Valid Application	Not Valid Application
Craig said: I used the exponent rule to change the order of the squared and the one half because that way you don't have to take the square root of a negative number. $(-9)^2 = 81$ Then the square root of 81 is 9.		
Katlynn said: I did it an easier way. 2 and $\frac{1}{2}$ cancel, so it's just -9.		

Figure 1. Ms. Williams item. (Copyright @ 2013 Educational Testing Service)

In terms of the underlying mathematics, the item seeks to assess whether the respondent knows that the law of exponents $(x^a)^b = x^{ab}$ does not hold in general, for instance when x is a negative number and a or b is non-integer. One student work sample has reached the 'correct' answer of –9 and the other has reached an incorrect answer of +9, but both have over-generalized the above law of exponents and applied it in a situation where it is not appropriate. In other words, the student whose answer is correct reached a correct answer coincidentally and not by use of a valid method, by overgeneralizing a rule that applies when both the base and the exponent are positive integers. The situation is additionally complex in that, while one student has reached the 'correct' answer of –9, the mathematical justification for this being the 'correct' answer depends on the use of complex numbers, and the item describes the students as not yet familiar with complex numbers. Arguably, if limited to real numbers, it might be more 'correct' to state that the expression is not defined. This example resembles one of Zaslavsky (1995)'s examples of competing claims, taken from Tirosh and Even's (1997) study, which discusses different mathematically-substantiated possibilities for $(-8)^{1/3}$. In the case here regarding squaring the 1/2 power of –9, the student work represents competing claims.

Responses to this particular item, because of the embedded mathematical uncertainty, and because many of the participants reported experiencing uncertainty or showed evidence of uncertainty, presented an ideal situation in which to examine patterns of reasoning in response to uncertainty. Transcripts were coded using grounded theory (Suddaby, 2006) for evidence of uncertainty, to characterize the nature of the response overall, and more specifically, to characterize the nature of the response to uncertainty. We paid particular attention to strategies that teachers engaged in as methods of resolving uncertainty, and to the valance they assigned that uncertainty when they noted it.

Potential Implications for Future Research

In this paper, we examined teachers' experiences of uncertainty, in particular, in the face of competing claims. One interesting observation about the teachers' different experiences was the range of certainty about the verifiability of the claims. Some teachers felt more certain than others about the verifiability of the answer than others. The engagement of those that felt less certain suggests that the perception of verifiability may play a role in how productive a situation of competing claims can be for learning. If there are competing claims, but one is perceived to be automatically more correct to the point of disregarding flawed reasoning to a valid conclusion, then engagement with the claims may be less productive than if claims are seen to be less automatically verifiable. Indeed, the point of this task was to engage in verifying the claims so as to be aware of how students may have overreached on the domain of the laws ostensibly applied.

Feedback and Suggestions for Future Directions for the Research

- One of our conclusions is that perceived verifiability may play a role in the productiveness of a task structured around competing claims. Does this conclusion seem plausible to you? What experiences have you had as a researcher or undergraduate instructor that might support or counter this conclusion? (The purpose of this question is to see whether this conclusion is reasonable enough to dig deeper into the analysis with this idea in mind.)
- If we were to investigate the role of verifiability further, what analytic frameworks or coding strategies would you suggest for looking into the data further?

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Effects of dynamic visualization software use on struggling students' understanding of calculus: The case of David

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Using dynamic visualization software (DVS) may engage undergraduate students in calculus while providing instructors insight into student learning and understanding. Results presented derive from a qualitative study of nine students, each completing a series of four individual interviews. We discuss themes arising from interviews with David, a student exploring mathematical relationships with DVS who earns a C in calculus. David prefers to visualize when solving mathematical tasks and previous research suggests that such students, while not the 'stars' of their mathematics classroom, may have a deeper understanding of mathematical concepts that their non-visualizing peers. Using modified grounded theory techniques, we examine evidence of uncontrollable mental imagery, the need to refocus David on salient aspects of the animations, instances when David's apparent conceptual knowledge is neither fully connected to nor supported by procedural knowledge, and David's failure to transfer knowledge when DVS was not offered during assessment.

Key words: calculus, visualization, dynamic visualization software

The urgent need for the United States to produce an additional one million graduates studying science, technology, engineering and mathematics (STEM), the fact that students to choose to leave the sciences often cite uninspiring introductory courses as the reason (President's Council of Advisors on Science and Technology (PCAST), 2012) and our knowledge that students who fail to obtain a deep understanding of calculus abandon their quest for a STEM degree (Carlson, Oehrtman, & Thompson, 2007) lead us to consider innovative methods for student engagement in college calculus that may also provide insight into student learning of the subject. Dynamic Visualization Software (DVS) facilitates visual investigation of important mathematical relationships and may assist students in exploring topics in a way that promotes conceptual understanding. Presmeg (2006) found that students in the high school mathematics classroom who prefer to visualize when solving mathematical tasks are not the 'superstars' of the classroom but that their understanding of the concepts and ideas of mathematics may be stronger than those of their non-visualizing peers. Presmeg defines visualization as including, "the processes of constructing and transforming both visual and mental imagery and all the inscriptions of a spatial nature that may be implicated when doing mathematics," (Presmeg, 2006, p. 3). This case study of David, is part of a larger study that explores how student interactions with DVS influence the learning and understanding of calculus for undergraduate STEM majors (Sutton, 2015).

Study Design

In this study we investigate how the use of DVS influences student understanding of derivative as a rate of change of one quantity with respect to another and how the experiences with DVS affect student understanding of derivative at a point as well as a student's graphical, analytical and conceptual understanding of derivative. We also explored student understanding of the relationship between continuity and differentiability.

We completed this study during the Fall 2013 semester at a large university in the Southwestern United States. Participants came from a single section of calculus with 110 students enrolled. Fifty students enrolled in this section also participated in an intervention program that met twice weekly and focused on problem solving. Funding guidelines required that students in the intervention program be first-time, first-semester freshmen US Citizen or permanent resident, majoring chemistry, engineering, physics or mathematics.

During the first week of classes, all students enrolled in the selected section of calculus completed a background demographic survey and Presmeg's (1985) Mathematical Processing Instrument (MPI). Scoring of the MPI only provides information about the student's preference to visualize when solving mathematical tasks. However, when reviewing the MPI we saw that not all students had adequately solved the problems in the instrument and we decided to assign each student two scores: one indicating the preference to visualize (0-24) and an additional score showing the number of questions that student correctly answered (0-12). We did this in an effort to select participants most likely to complete the course successfully. Students with a correctness score less than eight were not invited for participation. We invited eight visualizers (MPI score above 15) and seven non-visualizers (MPI score less than 8) to participate in a series on four individual interviews. Nine students (five visualizers) completed three one-hour interviews and a thirty-minute Exit Interview.

We placed each student in one interview group: DVS or Static. Students experiencing the DVS interviews explored the mathematical relationships highlighted during each interview using pre-designed visualizations called sketches. Students in the static interview group worked on problems adapted from a calculus textbook and answered questions analogous to those from the DVS interviews. We provided students in both groups a basic scientific calculator, paper and a writing instrument. Interviews were video recorded and smartpen technology captured real-time voice and written data. We transcribed the interviews and analyzed common themes using open coding techniques (Corbin & Strauss, 2007).

David

David, an eighteen year-old black male majoring in mechanical engineering, graduated from a large (more than 2100 students) urban high school in 2013. The only Advanced Placement course David completed was Calculus AB. He correctly answered 10/12 items on the MPI and has an MPI visualization score of 18/24, classifying him as a visualizer. David participated in the intervention program offered for calculus and he earned a C in the course. He participated in the DVS interview group for this study.

Interview I

During Interview I David, explored relationships between tangent and secant lines and how they corresponded to the relationship between average rate of change over an interval and instantaneous rate of change at a point contained within the interval.

The first sketch in Interview I presented David with the graph of a quadratic function. The sketch includes a fixed point, A, corresponding to $\left(x_a, f(a)\right)$, a dynamic point, B, corresponding to $(x_b, f(b))$, and the secant line containing both A and B. During the interview, David manipulated B as he collected data in a dynamic table. The data in the table included the values for $x_a, x_b, f(x_a), f(x_b)$, and $\frac{f(x_b)-f(x_a)}{x_b-x_a}$. Eventually, David moves B sufficiently close to A and the secant line disappears from the screen. The interviewer asks David why he thinks this happened, "The change in x is equal. When x - - I mean, when $A = B$ the change is 0 'cuz… the same thing so… there is no average speed." When asked if there is a relationship between the function's average rate of change over $[x_a, x_b]$ when the points are close together and the function's instantaneous rate of change at point A, David says that there is a relationship, but the instantaneous rate of change at point A is undefined.

The disappearance of the secant line when points A and B are sufficiently close together leads David to believe that the instantaneous rate of change of the function at point A is undefined.

The final sketch of Interview I focused on investigations related to $f(x) = e^x$. Before he began exploring, the researcher asked David about his knowledge regarding such functions. David responds that, "… like, it's never touching, whatever, like at the 4 and 0." While making this statement, David makes hand gestures indicating that the function has a vertical asymptote at $x = 4$ and a horizontal asymptote at $y = 0$. The researcher asks for more information regarding his statement that it never touches at 4. David continues to gesture and states, "it's like it's going up, so there's an asymptote and, and asymptote right there." He clarifies that the function has a horizontal asymptote at $y = 0$ and, "…a vertical asymptote at… I'd say 4 'cuz it passed right through it."

David's ability to use the software did not hinder his exploration of the graph $f(x) = e^x$. In fact, on a previous sketch he asked if he could explore and used a dynamic point on the function graph that he found interesting to do so. His assertion that the graph of $f(x) = e^x$ has a vertical asymptote at $x = 4$ is a powerful referent that he continues to hold through this and subsequent interviews. Prior to beginning Interview II, the researcher asks David what he remembers from Interview I; he states that the exponential function has a vertical asymptote.

While exploring the exponential function in a similar manner as he did with the quadratic function, David appears to gain some insight into the relationship between the function's average rate of change over an interval and its instantaneous rate of change at a point within the interval. He states that when two points A, corresponding to $(x_a, f(x_a)$ and C, corresponding to $\left(x_c, f(x_c)\right)$, on a graph are "really close" together the secant line connecting them could represent, "the tangent line. Oh! The speed! The speed." The researcher continues to ask probing questions and David says that the slope of the tangent line corresponds to the instantaneous speed of a particle whose position with respect to time is determined by the exponential function. Using another dynamic table to collect data, and through some probing questions from the researcher, David eventually relates, "the y value of A," to the particle's speed point A. "So, yeah. If you, if you have your y value then it would be equal to the instantaneous speed of the particle [at that time]."

During Interview I, David experiences several transitions in thinking about the relationship between average rate of change over an interval and instantaneous rate of change at a point within the interval. Initially, David believes that the instantaneous rate of change at point A does not exist, or at least he believes that he cannot find it, when points A and B coincide. However, after further exploration (with the initial quadratic function, a quartic function and, finally, the exponential function) David appears to make a connection between the decreasing size of $[x_a, x_c]$ and the slope of the secant line as an appropriate estimate for the instantaneous rate of change at point A. After stating this connection, David discusses the slope of the tangent line at point A as the value of the function's instantaneous rate of change at the point. Though he may be building upon previous conceptual knowledge, it was not evident that he possessed this knowledge prior to the interview. It may be that the DVS evoked this knowledge that he did not demonstrate initially. We did not observe evidence of procedural knowledge or skills during this interview.

Interview II

Prior to beginning Interview II, David recalls the relationships he explored during Interview I. David uses his hands to illustrate that he understands relationships between secant lines, tangent lines, average rate of change over an interval and instantaneous rate of change at a point contained in the interval and he draws a sketch (prompted by the researcher) illustrating that he understands the role of interval size in this relationship. However, when asked what he knows about derivatives David's response is rambling and nonsensical. He says that, "derivative is the velocity… and the derivative of velocity is… well…"

The first sketch in Interview II presents the graph of a cubic function, a dynamic point P, corresponding to $(x_p, f(x_p))$, and a line tangent to the graph at point P. As David manipulates P, he mentions that the value of $f'(x_p)$ corresponds to, "the slope of the point on the graph at that instant." He also collected data in a dynamic table listing the values $x_p, f(x_p)$, and $f'(x_p)$. When asked what information from the table would be needed to plot a point lying on the graph $y = f'(x)$, he struggled to answer. After answering some probing questions, he eventually realizes that the point $(x_p, f'(x_p))$ lies on the graph of the derivative of f, though he struggles to equate $f'(x_p)$ with the instantaneous rate of change at point P. Using the software, David checks his hypothesis and, by manipulating point P along the function graph, he traces out the derivative graph of f.

The final sketch presented during Interview II revisits the graph of $f(x) = e^x$. He explores the function using a dynamic point P corresponding to $(x_p, f(x_p))$. He notes that, "it [the instantaneous rate of change at point P] is always positive," and that, "as it moves farther [in the positive direction of the x-axis] it changes faster." David collects data in a dynamic table listing the values of $x_p, f(x_p)$ and $f'(x_p)$. He immediately notices, "wherever it is on the y-axis it's the same as the slope—the slope of the tangent line," and states that the point $(x_p, f(x_p))$ lies on the graph of f'. David says that he learned in class that, "the derivative of e^x is just e^x. When asked if other exponential functions, say $f(x) = 5^x$, have this same property, David is unsure. He reasons that, "if you have e^x and you add an ln it would just be x," but he remains unsure what this means mathematically or how to even write it. In the end, David says that he, "just knows" that if $f(x) = e^x$ then $f(x) = f'(x)$.

During Interview II, David continues to make conceptual connections about the relationship between a function's instantaneous rate of change at a point and the derivative value at the point. He struggled, but succeeded, in giving the coordinates of points lying on the graph of f' when provided with the graph of f. Evidence of David's weak procedural knowledge emerges in his inability to show or explain why $f(x) = f'(x)$ for $f(x) = e^x$.

Interview III

Unlike Interviews I and II, Interview III consisted of a single sketch showing the graph of a piecewise-defined function on a closed interval (see Figure 1). This interview focused on the Extreme Value Theorem and the relationship between continuity and differentiability.

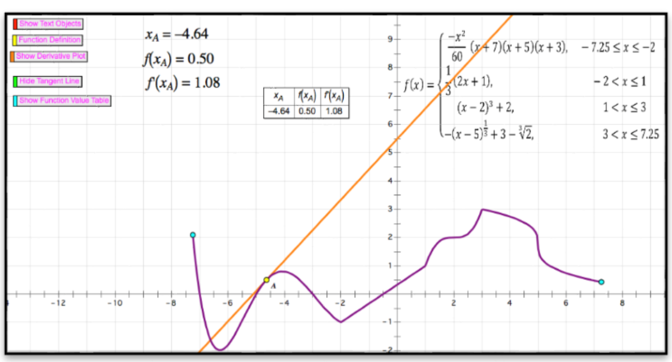

Figure 1 Screenshot of DVS Interview III.

Prior to working with the DVS for Interview III, the researcher asks David what it means for a function to be continuous on its domain. He responds, "… that it will go through all the

x values. The positive ones and negative ones… there's no holes and asymptotes or no, like stops in the graph." He relates differentiability to a lack of "corners or cusps" on the graph of the function. When the researcher probes about what David means by this he responds that, "it's like 0 at the corner, I'm guessing. You can't find the derivative of 0. I just know like - - I just remember that if there's a corner or a cusp you can't… it's not differentiable."

Once he begins exploring properties of the function graph using the dynamic point A corresponding to $(x_a, f(x_a))$, David easily identifies the maximum function value and minimum function value on the given domain and to state that the function is defined on a closed interval. He is unable, however, to write an inequality guaranteed by the EVT for all function values compared to the maximum function value. After further questioning from the researcher, David eventually concludes that $3 \geq f(x)$ for all x in $[-7.25, 7.25]$, though he in unable to explain an analogous inequality for the function's minimum value. He also states that if $f(x_d)$ is the function's minimum or maximum function value the $f'(x_d) = 0$, because, "… it changes from increasing to decreasing… or the other way."

The researcher asks David to use the DVS capability to collect data in a dynamic table and to mark points on the graph where he estimates that the instantaneous rate of change is greatest given several closed intervals. For each x_i he indicated, David is asked about the value of $f'(x_i)$. He states that the points $(x_i, f'(x_i))$ would correspond to "relative maxes," on the graph of f' and that $(x_i, f''(x_i))$ would correspond to zeros on the graph of f''.

David struggles to understand why the derivative at a point corresponding to a sharp corner on a function graph does not exist. Initially, David believes that, "you can't set the derivative equal to zero," at such a point. However, after investigating on the graph (see Figure 1) near $x = -2$ he realizes, "… so the derivatives from both sides aren't equal." David continues to investigate his notion near $x = 1$ and $x = 3$ on the same graph and concludes that his statement also applies there. The final question in Interview III required David to explain the relationship between continuity and differentiability; he responds that, "… like it can be continuous but that doesn't mean that it is differentiable everywhere."

The transcripts for Interview III contain several examples of David acquiring conceptual knowledge, or experiencing transitions to existing conceptual knowledge. He demonstrates how changes in one quantity result in changes in another quantity as he relates $f(x_i), f'(x_i)$, and $f''(x_i)$. However, it is unclear if he adjusted his conceptual knowledge relating a point, $x_a, f(x_a)$, that corresponds to a maximum function value to include that $f'(x_a)$ may equal 0 or be undefined. Through probing questions from the researcher and exploration with DVS, David makes conceptual connections about the derivative at sharp corners of the graph of f.

Exit Interview

The Exit Interview for all students was administered in a static fashion. No DVS was offered for exploration and the interview protocol was identical for all student participants.

The first question asks students to say what comes to mind when they hear the word derivative. David's responses only include lists of specific derivative rules and examples. "x^2 derivative equals $2x$. $\cos x$ derivative is $-\sin x$…"

The second task included the graphs of two polynomial functions with the point (2,3) marked on each graph. David is asked to compare the instantaneous rate of change at $x = 2$ of each function. His response, "you take the derivative and plug in 2," while correct, relied upon the function definitions, but only graphs were given. He does make some comparisons, "… where it goes from increasing to decreasing, the point where it does that, the point where it switches the 0, the instantaneous rate of change, which is the derivative of the function would be 0." He reasons that the instantaneous rate of change at $x = 2$ is the same value as the slope of the line tangent to each graph at (2,3) and attempts to find this. Yet, for one graph he chooses (2,0) and (2,3) to find the slope of the tangent line and becomes confused.

David also struggles with the idea that a continuous function may not be differentiable over the entire interval. He states that "…it's continuous and differentiable," when asked about what continuity implies about differentiability. He also struggles with the relationships between the function value at a point and the derivative at a point and, at times, is unsure which he is referencing. After drawing a graph similar to the graph of $y = |x|$, David appears to clear up his confusion and he states that the derivative is undefined at a point making a sharp corner, but he amends this statement at the end of the interview and states the derivative would be zero. When asked why at a point, $\left(x_a, f(x_a)\right)$, corresponding to a sharp corner on the graph of f, $f'(x_a)$ would not exist, David replies, "I've never thought about that before." His understanding of why the derivative does not exist at the point he indicated is limited to an incorrect, rudimentary procedural understanding that, "… Because it's - - it's a 0 or not 0, undefined so. Never thought about that. Cuz the - - what's it called? The slope at that point is like, no - - I guess it would be 0. And you can't find the derivative of 0 so - -."

Overall, David exhibited a weak ability to complete the tasks presented to him during the Exit Interview. He was, in general, able to make correct, or partially-correct, conceptual statements. Even when his statements suggested that he possessed the procedural knowledge necessary to complete a task it was a challenge for him to do so. David's inability to accurately explain why $f'(x_a)$ exists when $(x_a, f(x_a))$ corresponds to a point making a corner on the graph of f is puzzling, as he was able to explain this during Interview III.

Course Performance

The course grades for calculus at this institution are based heavily (80%) on departmental exams. These exams included minimal visualization and very few conceptual questions. Instead, the exams are heavily procedurally based. Given the evidence of David's weak procedural ability, his grade of C in the course is unsurprising.

Literature Review and Discussion

We interweave the supporting research literature into the discussion of the themes emerging from the open coding of our interview transcripts.

Throughout David's interviews the researcher refocused his attention toward the particular mathematical relationship highlighted in the sketch when he seemed unsure where to direct his attention, when he overlooked the mathematical relationship presented or when he simply needed further guidance. During Interview III, the researcher asks David to investigate near $x = -2$, using the dynamic point and data table. Even when David explains why $f'(-2)$ is undefined, he is again refocused toward places on the function graph with similar characteristics. This need to refocus the learner's attention to the highlighted mathematical relationship is called focusing phenomena (Lobato & Burns-Ellis, 2002). Without such refocusing it is possible that many of the conceptual gains noted in David's interviews would either be fewer in number or quality or not present at all. This underscores the importance of the instructor's role in refocusing attention when needed, especially in an environment where greater numbers of students may interact with dynamic visualizations as part of online homework while working alone.

David's struggle during Interview II to give the coordinates of a point on the graph of f' illustrates how scaffolding, instructor probing, and refocusing resulted in the student successfully completing the task. Only after David answers probing questions can he state that the points on the graph of f' all had the form $\left(x_p, f'\left(x_p\right)\right)$. He is then able to validate his hypothesis using the software and additional scaffolding included in the sketch. This supports Henningsen & Stein's (1997) work on the need and importance of scaffolding during problem-solving tasks and suggests that it may play an equally vital role in DVS exploration as well as work regarding student validation routines (Walter & Barros, 2011).

A balance of both conceptual and procedural knowledge is necessary for student success in calculus (Gray, Loud, & Sokolowski, 2009; White & Mitchelmore, 1996; Hardy, 2009; Lithner, 2004; Szydlik, 2000). David made many statements in each interview suggesting that he either possessed conceptual knowledge relevant to the topic being discussed, or he made statements of a conceptual nature in a procedural manner.

There are instances where David makes a statement showing evidence of conceptual knowledge that is either not replicable or that does not transfer to newly encountered situations. These episodes suggest that, for David, the interactions with DVS are possibly not resulting in the creation of connected schema between concepts. It is possible that his lack of access to DVS for exploration during the Exit Interview also contributed to this. Had DVS been allowed, his responses may have reflected the conceptual knowledge present in earlier interviews. However, it is possible that David's isolated conceptual remarks that were unsupported by procedural knowledge may be statements learned from lecture, lab or the intervention workshops but forgotten due to the lack of connections with which to form schema (Cooley, Baker, & Trigueros, 2003). Possibly for David, working with DVS enabled him to communicate his understanding of concepts, but the absence of the tool, limited his access to connections needed to complete the task in the Exit Interview (Lobato, Rhodehamel, & Hohensee, 2012). David's weak procedural knowledge failed when he was unable to access his understanding of mathematical relationships in the absence of DVS and he could not apply his previous knowledge to the new situation.

David's experience with uncontrollable mental imagery (Aspinwall & Shaw, 2002; Aspinwall, Shaw, & Presmeg, 1997) is important to note as he carried the incorrect notions regarding the graph of $f(x) = e^x$ having a vertical asymptote at $x = 4$ with him throughout the interviews. Though David's previous statements suggest his comfort with exploring using DVS, he chose not to do so when faced with probing questions regarding his observation about the graph. Situations where this occurs should be carefully discussed and addressed to address student thinking and understanding in an effort to address uncontrollable mental imagery and to limit possible misconceptions introduced by DVS.

Conclusions

The case of David, a struggling C student in calculus, raises important issues regarding the use of DVS in calculus learning. DVS accompanied by instructor guidance or embedded scaffolding questions may enhance conceptual gains and limit possible drawbacks in using DVS. Also, static assessments may not accurately reflect understanding for students who use DVS in learning the concepts. We observed that David needed consistent refocusing and additional probing questions from the researcher throughout the interviews. Without the presence of scaffolding or the focusing phenomena, it is unlikely the outcomes regarding conceptual knowledge would be the same. We also observed that, when not offered DVS as a tool, David's assessment results indicate a below average understanding of calculus (his grade of C in the course) and that he may be unable to transfer his knowledge.

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Effect of emphasizing a dynamic perspective on the formal definition of limit

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ABSTRACT. We attempt to determine the efficacy of using an alternate, equivalent formulation of the formal definition of the limit in a first-year university calculus course in aiding the understanding of the definition and of alleviating the development of common misconceptions concerning the limit.

Introduction

Students face many difficulties in learning calculus, and in learning the concept of limit in particular (see Tall, 1993; Williams, 1991, for example). One difficulty that is commonly experienced by students is in resolving their intuitive, dynamic, step-by-step view of the limit of a function with the static, continuous point of view required for understanding the formal ε-δ definition. This difficulty is exacerbated by the fact that the language that both textbooks and instructors use in discussing limits is rife with dynamic terms like "approaching," and dynamic notation like the $\to$ symbol (see Monaghan, 1991).

In making formal definitions, mathematicians usually have foremost in mind utility for theoretical development, with a generous helping of an appreciation for elegance and conciseness. Such goals can be at odds with the tasks of teaching and learning. The formal definition of limit in calculus is a notoriously difficult topic for introductory calculus students to grasp, let alone develop deep understanding. It captures the notion of limit in a succinct and mathematically rigorous manner, but for neophytes the connection between informal idea and formal definition is often lost in a soup of Greek letters, mathematical symbols, and nested logic. Given that it is often possible to formulate a formal definition in multiple equivalent ways, might it not be preferable for mathematics educators, at such an introductory level, to choose a definition most conducive to teaching and learning rather than a definition conducive to doing mathematics in the style of professional mathematicians? In this study, we attempt to investigate whether a definition using convergence of sequences might meet this criterion. To begin to study the effectiveness of such a definition, we study its effect on the development of common misconceptions concerning limits.

Methodology

In the fall term of 2014, pre- and post-tests were administered to students across four sections of two first-year introductory calculus courses, and results from 134 participants were retained. The first author was the instructor for two sections (henceforth referred to as 'Section A1' and 'Section A2') of 'Course A,' a standard introductory calculus course attended by students both with and without prior calculus exposure at the high school level. The second author was the instructor for a third section (henceforth 'Section A3') of Course A and for the sole section (henceforth 'Section B') of 'Course B,' a slightly more challenging introductory calculus course with prior calculus exposure at the high school level as a prerequisite. In Course A, the usual ε-δ formal definition of limit was used, while in Course B, an equivalent definition using convergence of sequences

Key words and phrases. Calculus, Limit, Formal Definition, Dynamic Perspective.

was introduced. Students in Course B did not have a preceding in-depth unit on sequences and convergence, relying only on a brief introduction to sequences sufficient for their use in the study of the limit. The relevant definitions are given below.

Definition. *A sequence* $\{x_n\}$ ***approaches*** *c (or* ***converges to*** *c) if for every possible error bound E, no matter how small, all but a finite number of terms lie within* $c \pm E$.

Definition. *We say that* ***the limit of*** f ***as*** x ***approaches*** c ***is*** L*, and write* $\lim_{x\to c} f(x) = L$*, when* every possible *input sequence* $\{x_n\}$ *with* $x_n \to c$ *(but* $x_n \neq c$*) produces an output sequence* $\{y_n\}$ *with* $y_n \to L$.

In the definition of convergence of sequences above, we chose a definition appropriate for an introductory calculus course, rather than the more technical ε-N version typical of more advanced analysis courses. Also, the notation $c \pm E$ represents the interval $(c-E, c+E)$, and was used to be consistent with the notation of error bounds in other scientific disciplines.

Table 1. Concepts and misconceptions evaluated on the pre/post-tests.

Question	Concept/Misconception
1	Limit versus value of a function.
2	A function's values must approach but never equal the limiting value.
3	A finite table of values is enough to determine a limit with certainty.
4	The values of a function always monotonically approach the limiting value.
5	Input-output order reversal/confusion in the formal definition.
6	Negation of the formal definition.

In each test, students were given six multiple choice questions designed to test common misconceptions about the limit. The pre-test was administered shortly after the concept of limit was introduced. The formal definition of the limit and all discussion regarding formal justification of limit properties were then delayed until the end of the unit on limits. The post-test was administered some time after discussion of the formal definition of the limit, to give the definition and its relationships to the idea and calculation of limits time to sink in. The post-test was identical to the pre-test, with the exceptions that the order of the questions, the order of the possible answers for each question, and some of the numbers involved in the questions were changed. The students did not know beforehand that the questions on the post-test would be essentially the same as the questions on the pre-test, and students were not given answers to either test until both tests were completed. The concept or misconception that each question was designed to measure is listed in Table 1.

Results

Question 1 asked students whether it is possible for $\lim_{x\to c} f(x) = f(c)$ to be true for a given function f. From experience with elementary functions, students often are under the misconception that this equality is *always* true. Or, they believe that the equality cannot hold because a function cannot actually "reach" its limit. In Table 2, we summarize the student responses to this question. In the table headings for this table and all subsequent tables, the symbols ✓and ✗ are used as shorthand for "correct" and "incorrect," respectively, and an arrow indicates a change in response between the pre- and post-tests.

Question 2 asked students about the relationship between a known limiting value L of a function f, the value $f(c)$ (if any) at the point of interest c, and the values $f(x)$ at points x near c. Students

Table 2. Student responses on question 1.

Section	N	Pre ✓	Post ✓	✗→✓	✓→✗
A1	37	35%	76%	49%	8%
A2	29	38%	83%	55%	10%
A3	35	57%	66%	23%	14%
A (all)	101	44%	74%	42%	11%
B	33	70%	85%	21%	6%

often believe that the values at neighbourhood points must be close to L, but should not equal either L or $f(c)$. Student responses to Question 2 are summarized in Table 3.

Table 3. Student responses on question 2.

Section	N	Pre ✓	Post ✓	✗→✓	✓→✗
A1	37	8%	30%	27%	5%
A2	29	3%	7%	7%	3%
A3	35	6%	11%	9%	3%
A (all)	101	6%	17%	15%	4%
B	33	27%	15%	3%	15%

In Question 3, students were given a finite table of values for a function at points near a point of interest c, both to the left and to the right, where the values appeared to be approaching a specific common value from both sides of c. They were then asked about possible inferences that could be made about the limit of the function at c. The insufficiency of a finite table of values in establishing a limit with certainty is a fundamental issue addressed by the formal definition of the limit. Table 4 summarizes the student responses to this question.

Table 4. Student responses on question 3.

Section	N	Pre ✓	Post ✓	✗→✓	✓→✗
A1	37	16%	14%	11%	14%
A2	29	38%	28%	0%	10%
A3	35	17%	34%	26%	9%
A (all)	101	23%	25%	13%	11%
B	33	30%	24%	3%	9%

In Question 4, students were given a limiting value and a nearby value of a function, and were asked about possible inferences that could be made about the value of the function at a point in between the two given values. Students often conceptualize a limit as always being monotonically approached by the values of the function nearby, and this question aimed to expose such misconceptions. The student responses are summarized in Table 5.

Questions 5 and 6 aimed to probe students' understanding of the formal definition in plain language. In Question 5, students were required to choose the phrase that correctly justified a limit, where the incorrect phrases contained various forms of input-output reversal. Because students are used to the input-output order of function evaluation, they often struggle with the process of working in the reverse order, starting with an arbitrary interval of y-values around the limiting value and working backwards to an appropriate interval of x-values around the point of interest.

Table 5. Student responses on question 4.

Section	N	Pre ✓	Post ✓	✗→✓	✓→✗
A1	37	14%	11%	11%	14%
A2	29	7%	17%	14%	3%
A3	35	3%	11%	9%	0%
A (all)	101	8%	13%	11%	6%
B	33	18%	12%	6%	12%

In Question 6, students were required to choose the phrase that correctly justified why a proposed value could *not* be the limit of an example function. Because of the nested logic involved in the formal definition of the limit, students struggle with understanding the negation of the definition. It was hoped that the convergence version of the formal definition would aid with this task, as in this version the nested logic broken out into the separate definitions of convergence and limit via convergence.

Responses to Questions 5 and 6 are summarized in Tables 6 and 7.

Table 6. Student responses on question 5.

Section	N	Pre ✓	Post ✓	✗→✓	✓→✗
A1	37	14%	38%	30%	5%
A2	29	28%	38%	21%	10%
A3	35	14%	17%	14%	11%
A (all)	101	18%	31%	22%	9%
B	33	15%	21%	18%	12%

Table 7. Student responses on question 6.

Section	N	Pre ✓	Post ✓	✗→✓	✓→✗
A1	37	22%	32%	22%	11%
A2	29	41%	14%	7%	34%
A3	35	23%	11%	11%	23%
A (all)	101	28%	20%	14%	22%
B	33	24%	30%	27%	21%

Overall, the initial results from the pre- and post-tests are disappointing: on most questions, students who learned the formal definition via sequences (Section B) performed more poorly on the post-test than on the pre-test. Question 6 in particular was one on which it was expected that unchaining the nested logic of the formal definition into separate definitions would be of benefit to the students. However, while the percentage of students in Section B answering this question correctly increased from the pre-test to the post-test (and the percentage in the other sections actually decreased), we see that almost as many students in Section B moved from correct thinking to incorrect between tests as did vice versa.

Conclusion

In this study, we attempted to determine the efficacy of using an alternate, equivalent formulation of the formal definition of the limit in aiding the understanding of the definition and of alleviating the development of common misconceptions concerning the limit. On both counts, the alternate definition did not seem to produce any improvement in student performance over the traditional definition, and perhaps could be construed to actually have produced a decrease in student performance.

These results might be explained by factors other than the definitions used. First, the instructors of the courses involved in the study have many years experience teaching first-year calculus using the traditional formal definition, and with that experience comes the knowledge of typical student difficulties with the material and strategies to mitigate those difficulties. But for the instructor of the experimental Section B, teaching a formal definition in terms of sequences was a new experience. Second, sequences were only introduced in Section B at the end of unit on limits, to facilitate the discussion of the formal definition. The sequence version of the formal definition might be more relevant and attractive to the students if the entire unit on limits was infused with sequences.

On the other hand, it may be the case that the formal definition of the limit via sequences, while unchaining the nested logic of the traditional definition into two separate definitions, has merely pushed the conflict between informal, dynamic conception and formal, static definition to the definition of convergence of sequences. Or, it could be that briefly introducing sequences merely introduces a significant, preliminary learning barrier, beyond which students were unable to progress to the learning of the formal definition of the limit.

Some questions, the discussion of which would help further this research, follow.

- Would it be worthwhile to reverse the set up of the experiment and have students in Course A (who mostly have not previously studied calculus) exposed to a sequences version of limits?
- Should the pre- and post-tests focus more on understanding of the formal definition itself?
- In the experimental version, formal understanding of limit is dependent on formal understanding of convergence of sequences. Should understanding of sequences be simultaneously tested, to be correlated with understanding of limits?

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Investigating the role of a secondary teacher's image of instructional constraints on his enacted subject matter knowledge

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I present the results of a study designed to determine if there were incongruities between a secondary teacher's mathematical knowledge and the mathematical knowledge he leveraged in the context of teaching, and if so, to ascertain how the teacher's enacted subject matter knowledge was conditioned by his conscious responses to the circumstances he appraised as constraints on his practice. To address this focus, I conducted three semi-structured clinical interviews that elicited the teacher's rationale for instructional occasions in which the mathematical ways of understanding he conveyed in his teaching differed from the ways of understanding he demonstrated during a series of task-based clinical interviews. My analysis revealed that that the occasions in which the teacher conveyed/demonstrated inconsistent ways of understanding were not occasioned by his reacting to instructional constraints, but were instead a consequence of his unawareness of the mental activity involved in constructing particular ways of understanding mathematical ideas.

Key words: Mathematical Knowledge for Teaching; Enacted Knowledge; Instructional Constraints; Secondary Mathematics.

Introduction

Mathematics educators have devoted increased attention in recent years to: (1) identifying categories of knowledge that teachers must possess to effectively support students' mathematics learning (e.g., Ball, 1990; Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008; Hill & Ball, 2004; Hill, Ball, & Schilling, 2008; Hill, Schilling, & Ball, 2004; Shulman, 1986, 1887), (2) characterizing the specific mathematical ways of understanding that allow teachers to engage students in meaningful learning experiences (e.g., Ma, 1999; Yoon et al., 2014), (3) discerning how teachers might construct such ways of understanding (e.g., Harel, 2008; Harel & Lim, 2004; Silverman & Thompson, 2008; Tallman, 2015), and (4) developing instruments to measure teacher's knowledge and its effect (e.g., Hill et al., 2008; Hill, Rowan, & Ball, 2005; Thompson, 2015). These initiatives, while essential to the enterprise of improving students' mathematics learning, do not ensure teachers will utilize the full extent of their knowledge in the act of teaching. Teachers must recognize the knowledge they possess as appropriate to employ in the process of achieving their goals and objectives in the context of practice. This recognition is subject to a host of cognitive and affective processes that have thus far not been a central focus of research on teacher knowledge in mathematics education (Day & Qing, 2009, p. 17; Hargreaves & Shirley, 2009, p. 94; Meyer, 2009, p. 89; Schutz et al., 2009, p. 207). Identifying the factors that condition the knowledge teachers utilize in the context of teaching, and understanding the effect of such factors on the quality of teachers' enacted knowledge, is imperative for improving students' mathematics learning and for fashioning well-informed teacher preparation and professional development programs and educational policies that take seriously the effect of teacher knowledge and the factors that compromise it. It is to this end that the present paper seeks to contribute. In particular, I address the following research questions:

RQ1: Are there incongruities between a teacher's subject matter knowledge and his enacted subject matter knowledge? [1]

RQ2a: If so, in what ways does the teacher's image of instructional constraints condition the subject matter knowledge he utilizes while teaching?

RQ2b: If not, how is the teacher appraising and/or managing what he perceives as instructional constraints so that these constraints do not condition the mathematical knowledge he enacts while teaching?

Theoretical Framing

The "image of" qualifier in the title of this paper suggests my radical constructivist approach to defining instructional constraints. I take the position that environmental circumstances per se in the absence of a teacher's construal of them cannot constrain his or her practice, but the teacher's construction and appraisal of environmental circumstances can and often does. For this reason, I contend that particular circumstances cannot maintain an ontological designation as instructional constraints, however consensual are teachers' construction and appraisal of such circumstances. Therefore, in consonance with radical constructivism's skeptical position on reality, I define instructional constraints as an individual teacher's subjective construction of the circumstances that impede the teacher's capacity to achieve his or her instructional goals and objectives. Such subjective constructions are the only "constraints" that maintain the potential to influence teachers' instructional actions. Accordingly, I locate instructional constraints in the mind of individuals, not the environment. This conceptualization stands in stark contrast to the common perception of instructional constraints as external pressures that exert influence on the quality of teachers' instruction. According to this view, the pressure comes from without instead of from within. My interest in understanding how a secondary teacher's image of instructional constraints conditioned the mathematical ways of understanding and ways of thinking he utilized in the context of teaching necessitated my constructing a model of the teacher's construction of those circumstances he appraised as constraints on his practice.

As a result of my view that instructional constraints are subjective constructions that reside in the minds of teachers, I consider anything that a teacher appraises as an imposition to achieving his or her instructional goals and objectives to be an instructional constraint. The appraisal need not even be of an external circumstance. A teacher may appraise internal characteristics such as his or her mathematical self-efficacy, social endowments, creativity, tolerance, attitude, perseverance, temperament, empathy, confidence, etc., as imposing limits on the quality of his or her instruction. Since a teacher's appraisal of such intrinsic characteristics is a subjective construction in the same way that a teacher's appraisal of external circumstances is, both types of appraisals have the capacity to influence teachers' practice in the same way.

[1] I note that the identification of incongruities between the teacher's subject matter knowledge and the subject matter knowledge he invokes while teaching is from my perspective. Similarly, characterizing the effect of a teacher's image of instructional constraints on his enacted mathematical knowledge is also a characterization from my perspective.

Methods

My experimental methods proceeded in three phases. In the first phase, I conducted a series of nine task-based clinical interviews (Clement, 2000; Goldin, 1997; Hunting, 1997) that allowed me to construct a model of the participating teacher's (David's) mathematical knowledge of various topics associated with trigonometric functions. In the second data collection phase, I used video data from 37 classroom observations to construct a model of the mathematical knowledge David utilized in the context of classroom practice. Finally, I employed a phase of three semi-structured clinical interviews to construct a model of David's perception of instructional constraints and to discern the role of this image on the quality of his enacted mathematical knowledge.

The goal of the series of task-based clinical interviews was to facilitate my construction of a model of David's *ways of understanding* and *ways of thinking* (Harel, 2008) relative to angle measure, the outputs and graphical representation of sine and cosine, and the period of sine and cosine. Constructing a model of an individual's cognition by projecting or imputing one's cognitive schemes to the individual constitutes developing a *first-order model* (Steffe & Thompson, 2000). This is in contrast to developing a *second-order model*, in which the researcher attempts to make sense of the individual's actions by interpreting them through the lens of his or her model of the individual, not through his or her own cognitive schemes (ibid.). It is important to note that the goal of the series of task-based clinical interviews I conducted was to construct a second-order model of David's mathematical knowledge. Although I constructed a second-order model of David's mathematics, this model does not constitute a direct representation of David's knowledge, but rather a viable characterization of plausible mental activity from which his language and observable actions may have derived. Constructing such a model involved my generating prior to, within, and among task-based clinical interviews tentative hypotheses of David's ways of understanding that explained my interpretation of the observable products of his reasoning. I developed these provisional hypotheses by attending to David's language and actions and abductively postulating the meanings that may lie behind them. I designed and modified tasks for subsequent interviews to test, extend, articulate, and refine my tentative hypotheses of David's mathematical knowledge.

All task-based clinical interviews took place in David's classroom after school on the days that best suited his schedule. I attempted to schedule the interviews so that there was at least one day between each to accommodate for ongoing analysis, and accomplished this with the exception of the last two task-based clinical interviews. In each interview, I obtained video recordings that captured David's writing, expressions, and gestures. I also created videos of my computer screen via *QuickTime Player* to capture the *didactic objects* (Thompson, 2002) David and I discussed as well as any work David completed on the computer. Additionally, I collected and scanned all written work that David produced during the interviews.

I collected daily video recordings of two of David's Honors Algebra II class sessions over a seven-and-a-half-week period, which resulted in 37 videos of classroom teaching. The only days I did not intend to collect videos of David's teaching were those days students were testing or the days David was teaching content unrelated to the angle measure, sine, or cosine. While the classroom observations did not demand the type of ongoing analysis that was part and parcel of the series of task-based clinical interviews, I documented, in the form of memos, the mathematical understandings and ways of thinking David afforded his students the opportunity to construct. I must emphasize that I characterized the ways of understanding and ways of thinking David *allowed* his students to construct, and not the understandings and ways of thinking his students *actually* constructed. In essence, I documented the understandings that I would be able to construct, and the ways of thinking that I would be able to develop, were I an engaged student in the class with sufficient background knowledge, uninhibited by unproductive understandings or disadvantageous ways of thinking.

The objective of the third phase of my experimental methodology was to obtain data that allowed me to construct a model of David's image of those aspects of his environmental context that he appraised as constraints on the quality of his instruction, and to determine the way in which this image conditioned the mathematical knowledge he employed in the context of teaching. Constructing such a model and determining the effect that David's image of instructional constraints had on his enacted subject matter knowledge involved my conducting a series of three semi-structured clinical interviews after David completed his instruction of trigonometric functions.

The content of these semi-structured clinical interviews was heavily informed by my analysis of the data I obtained from the series of task-based clinical interviews as well as from David's teaching. Based on my analysis of this data, I selected video clips to discuss with David during the clinical interview sessions to discern the role of David's image of instructional constraints on the quality of his enacted mathematical knowledge. I devoted particular attention to ascertaining David's rationale for those instructional actions in which the mathematics he allowed students to construct differed from the mathematical ways of understanding he demonstrated during the series of task-based clinical interviews. It is essential to point out that I did not assume David recognized the discrepancies I noticed in the videos excerpts I selected to discuss. Therefore, after having presented pairs of videos to David that I believed demonstrated him conveying/supporting discrepant meanings, I asked him to compare the ways of understanding he communicated in both videos. My rationale for doing so was to determine if David recognized the same inconsistencies that I noticed in the ways of understanding he demonstrated/conveyed.

Analytical Framework

I leveraged explicit formalizations of *quantitative reasoning* (Smith & Thompson, 2007; Thompson, 1990, 2011) in the design of the present study and my analysis of its data. A growing body of research (e.g., Castillo-Garsow, 2010; Ellis, 2007; Moore, 2012, 2014; Moore & Carlson, 2012; Oehrtman, Carlson, & Thompson, 2008; Thompson 1994, 2011) has identified quantitative reasoning as a particularly advantageous way of thinking for supporting students' learning of a wide variety of pre- and post-secondary mathematics concepts. Additionally, this body of research has demonstrated the diagnostic and explanatory utility of quantitative reasoning as a theory for how one may conceptualize quantitative situations.

Quantitative reasoning is a characterization of the mental actions involved in conceptualizing situations in terms of *quantities* and *quantitative relationships*. A *quantity* is an attribute, or quality, of an object that admits a measurement process (Thompson, 1990). One has conceptualized a quantity when she has identified a particular quality of an object and has in mind a process by which she may assign a numerical value to this quality in an appropriate unit (Thompson, 1994). It is important to note that quantities do not reside in objects or situations, but are instead constructed in the mind of an individual perceiving and interpreting an object or situation. Quantities are therefore conceptual entities (Thompson, 2011).

Conceptualizing a quantity does not require that one assign a numerical value to a particular attribute of an object. Instead, it is sufficient to simply have a measurement process in mind and to have conceived, either implicitly or explicitly, an appropriate unit. *Quantification* is the process by which one assigns numerical values to some quality of an object (Thompson, 1990). Note that one need not engage in a quantification process in order to have conceived a quantity, but must have in mind a quantification process whereby she may assign numerical values to the quantity (Thompson, 1994). Defining a process by which one may assign numerical values to a quantity often involves an operation on two other quantities. In such cases we say that the new quantity results from a *quantitative operation*—its conception involved an operation on two other quantities. Quantitative operations result in a conception of a single quantity while also defining the relationship among the quantity produced and the quantities operated upon to produce it (Thompson, 1990, p. 12). It is for this reason that quantitative operations assist in one's comprehension of a situation (Thompson, 1994). It is important to note the distinction between a quantitative operation and a numerical or *arithmetic operation*. Arithmetic operations are used to calculate a quantity's value whereas quantitative operations define the relationship between a new quantity and the quantities operated upon to conceive it (Thompson, 1990).

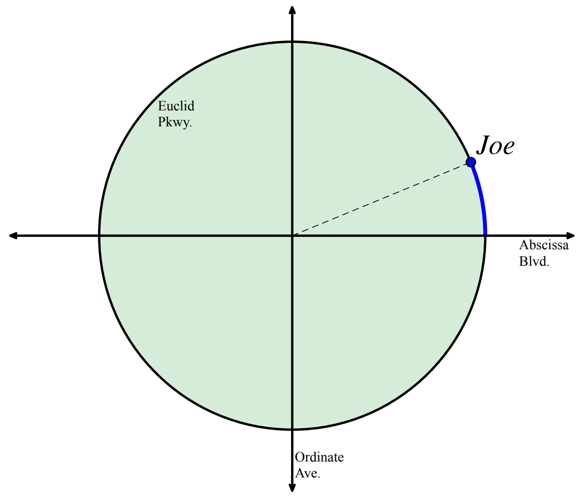
Results

On several occasions David demonstrated ways of understanding during the series of task-based clinical interviews that were inconsistent or incompatible with the ways of understanding his instruction supported. I selected three such occasions to discuss with David during a phase of clinical interviews I conducted after David completed his instruction of trigonometric functions. Specifically, I presented David with three pairs of videos, each containing an excerpt from the series of task-based clinical interviews and an excerpt from his classroom teaching. From my perspective, these pairs of videos exemplified David communicating discrepant mathematical meanings. My purpose in presenting David with these pairs of videos was to determine if he willingly compromised the quality of his enacted mathematical knowledge in response to the circumstances and events he appraised as instructional constraints. The following is a presentation of my analysis of our conversation around one of these three pairs of video excerpts. I do not discuss my analysis of David's and my conversation around the second and third pair of video excerpts since the conclusions drawn therefrom are consistent with those I present below.

I presented David with a video from the fourth task-based clinical interview in which he used an applet (see Table 1) to successfully approximate the values of sin(0.5) and cos(¾). During this interview David interpreted the task of approximating the value of sin(0.5) as, "Estimate how many radius lengths is Joe north of Abscissa Boulevard when the angle traced out by his path is 0.5 radians." In particular, David interpreted the 0.5 as representing the number of radius lengths that Joe had traveled along Euclid Parkway and sin(0.5) as representing Joe's distance north of Abscissa Boulevard in units of radius lengths. David similarly interpreted the task of approximating the value of cos(¾) in the following way: "Estimate how many radius lengths Joe is to the east of Ordinate Avenue when his path has traversed an arc that is $3/4^{\text{ths}}$ times as long as the radius of Flatville." David's response to the task of using the applet in Table 1 to approximate the values of sin(0.5) and cos(¾) suggests that he had constructed the outputs of sine and cosine as quantities; that is, as measurable attributes of a geometric object. After David watched the video except from the fourth task-based clinical interview, he described the way of understanding he demonstrated in a way that was consistent with my interpretation.

Table 1
Applet Designed to Support a Quantitative Understanding of Sine and Cosine Values

Suppose Joe is riding his bike on Euclid Parkway, a perfectly circular road that defines the city limits of Flatville. Ordinate Avenue is a road running vertically (north and south) through the center of Flatville and Abscissa Boulevard is a road running horizontally (east and west) through the center of Flatville. Assume Joe begins riding his bike at the east intersection of Euclid Parkway and Abscissa Boulevard in the counterclockwise direction.



After David watched the video excerpt from the fourth task-based clinical interview, I presented him with a video excerpt from Lesson 7 (which occurred four days after the fourth task-based clinical interview) in which he defined the outputs of sine and cosine relative to the following two cases: (1) when the radius of the circle centered at the vertex of an angle has a measure of one unit and (2) when this radius does not have a measure of one unit. Specifically, in the video excerpt David claimed that if the radius of the circle has a measure of one unit, then the sine and cosine values of the angle's measure are respectively equal to the y- and x-coordinates of the terminus of the subtended arc. David then explained that if the radius of the circle centered at the angle's vertex does not have a measure of one unit, then the values of sine and cosine are given by the respective ratios of the y- and x-coordinate of the terminal point to the length of the radius. It is noteworthy that David's explanation did *not* support students in conceptualizing sine and cosine values as the measure of a quantity in a particular unit. In other words, David's explanation in Lesson 7 did not support students in being able to answer the question, "What are the attributes to which sine and cosine values may respectively be applied as measures and in what unit are these attributes being measured?" In contrast to the quantitative way of understanding the outputs of sine and cosine David demonstrated in the fourth task-based clinical interview, during Lesson 7 David conveyed sine and cosine values as respectively representing y- and x-coordinates of the terminal point, or as arithmetic operations (i.e., $\sin(\theta) = y/r$ and $\cos(\theta) = x/r$).

After David viewed the two video excerpts, I asked him to determine if the way of understanding he supported in the excerpt from Lesson 7 differed from the understanding he employed to approximate the value of sin(0.5) and cos(¾) in the excerpt from the fourth task-based clinical interview (Excerpt 1).

Excerpt 1

Michael:	Is there any way that the understanding of sine and cosine you convey in this clip (*Lesson 7*) is different from what you did here (*Task-Based Clinical Interview 4*)?
David:	Only in the units of measure that we started with to obtain the ratio, but in the end we end up with an output that is a proportion of the entire radius. So in the end, no [they aren't different]. In the end they end up giving me the same thing.

David did not appear to recognize the way of understanding he demonstrated in the first video excerpt as being fundamentally different from the way of understanding he conveyed in the second. David's remark in Excerpt 1 focused primarily on the outcome of his application of two discrepant (from my perspective) ways of understanding instead of attending to the ways of understanding themselves. Like several occasions in other interviews in which David demonstrated an incapacity to attend to ways of understanding—either his own or his students'—his remarks in Excerpt 1 demonstrate that he had not achieved clarity relative to the mental activity involved in his own ways of understanding, nor of those he intended to support in his teaching. Had David done so, he would likely have been positioned to notice the discrepant meanings he conveyed in the videos I presented. David similarly failed to identify the inconsistent meanings he communicated in the other two pairs of video excerpts I presented to him.

Discussion

To investigate the role of David's image of instructional constraints on his enacted subject matter knowledge, I provided opportunities for him to rationalize occasions in which the ways of understanding he supported in his teaching differed from the ways of understanding he demonstrated during a series of task-based clinical interviews. My analysis of our conversation around all three pairs of video excerpts revealed that David failed to notice the discrepancy in the ways of understanding he conveyed/demonstrated in these excerpts. David's inability to recognize such discrepancies suggests that he was not consciously aware of the mental actions that comprise the meanings he intended to promote in his teaching, as such awareness would likely have equipped David with the cognitive schemes to recognize the inconsistent and often incompatible ways of understanding he conveyed in the excerpts we discussed. My analysis further revealed that the occasions in which David conveyed/demonstrated discrepant ways of understanding were not occasioned by his reacting to his image of instructional constraints.

The results of this study suggest that inconsistencies between mathematics teachers' subject matter knowledge and their enacted subject matter knowledge do not necessarily result from teachers' making conscious concessions to the quality of their enacted knowledge in the process of accommodating for the circumstances and events they appraise as constraints on their practice. Such inconsistencies may be a byproduct of teachers' unawareness of the mental activity that constitute their ways of understanding mathematical ideas. Pre-service mathematics teacher educators and in-service professional development specialists should therefore take care to provide opportunities for teachers to have explicit answers to questions like, "When my students read the symbols 'sin(θ)' what do I want them to imagine?" and "When my students look at an angle and think about measuring it in radians, what do I want them to visualize in their minds?" Providing opportunities for teachers to achieve such conscious awareness of the mental activity involved in particular ways of understanding may minimize the potential that teachers will not leverage the full extent of their subject matter knowledge to support students' mathematics learning.

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Physics: Bridging the embodied and symbolic worlds of mathematical thinking

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Physics spans understanding in three domains – the Embodied (Real) World, the Formal (Laws) World, and the Symbolic (Math) World. Expert physicists fluidly move among these domains. Deep, conceptual understanding and problem solving thrive in fluency in all three worlds and the facility to make connections among them. However, novice students struggle to embody the symbols or symbolically express the embodiments. The current research focused on how a physics instructor used drawings and models to help his students develop more expert-like thinking and move among the worlds.

Keywords: Embodied and symbolic worlds of mathematical thinking; visualization; physics

Introduction

Mathematics educators have long been fascinated by the power of visualization for learning and teaching mathematics. For example, researchers have cited the theoretical framework in Tall and Vinner's (1981) *Concept image and concept definition* paper over 1600 times. Presmeg (2006) reviewed over 20 years of papers from the *Psychology of Mathematics Education (PME) Proceedings* and found that there is great interest in the topic of visualization. For example, Dreyfus (1991b) stated during his plenary address at PME-15: "It is therefore argued that the status of visualization in mathematics education should and can be upgraded from that of a helpful learning aid to that of a fully recognized tool for learning and proof" (p. 33). Presmeg's review concluded with the statement that: "An ongoing and important theme is the hitherto neglected area of how visualization interacts with the didactics of mathematics. Effective pedagogy that can enhance the use and power of visualization in mathematics education is perhaps the most pressing research concern at this period" (p. 227). Almost two decades later, Presmeg's proposed list of 13 "Big Research Questions" pertaining to the topic of visualization still remains unanswered. Some of her questions include: "How can teachers help learners to make connections between visual and symbolic inscriptions of the same mathematical notions? How may the use of imagery and visual inscriptions facilitate or hinder the reification of processes as mathematics objects? How may visualization be harnessed to promote mathematical abstraction and generalization? What is the structure and what are the components of an overarching theory of visualization for mathematics education?" (Presmeg, 2006, p. 227).

The overarching aim of the current paper is to find out how an expert visualized mathematical ideas and how he subsequently helped his novice students hone their own visualization skills. In this research, we used physics as a case study to investigate how an instructor engaged over 200 students to use visual representations, in particular diagramming physics problems, during lecture and assessments.

Theoretical Framework

We employed Tall's (2013) three-world model of mathematical thinking (conceptual embodiment, operational symbolism, and axiomatic formalism) to describe the possible tensions that novices may face while learning physics. The embodied world involves mental images, perceptions, and thought experiments; the symbolic world involves calculation and algebraic manipulations; the formal world involves mathematical definitions, theories and proofs. In Tall's (2008) view, "all humans go through a long-term development that builds through embodiment and symbolism to formalism" (p. 23). Bridging between the embodied and symbolic worlds is of critical importance according to Tall: "A curriculum that focuses on symbolism and not on related embodiments may limit the vision of the learner who may learn to perform a procedure, even conceive of it as an overall process, but fail to be able to imagine or 'encapsulate' the process as an 'object' (p. 12).

In the current research, we operationalized the embodied world as demonstrations, real world examples, and models that represent real phenomena. We operationalized the symbolic world as the mathematical operations and computations, such as vectors and calculus, used to solve physics problems. Finally, we considered the formal world to be the rules, laws, and abstract quantities of physics, such as conservation laws, concepts of fields, and energy. In this work we focused on the embodied and symbolic worlds, although using the formal structures of physics was an important course goal held by the physics instructor we studied.

In our perspective, physics must "bridge" the embodied and symbolic worlds of mathematical thought (Figure 1). Expert physicists and engineers embody problems by visualizing them with diagrams, graphs, and schematics prior to solving them symbolically, while novice physics students will "plug and chug." Students have limited experience relying on visualizations to help them "make sense" of problems, perhaps due to expectations developed from computations in their mathematics classes. We have depicted the bridge that one experienced physics instructor created for his students to move between the embodied and symbolic worlds. The instructor put several connected support pillars in place, including classroom demonstrations of physical phenomena, a student response system that allowed real-time communication with the instructor, and peer instruction. The experienced instructor acted as a guide for his novice students as they traversed unfamiliar territory.

Our current research question investigated how physics bridges the symbolic and embodied worlds of mathematical thought: How does an expert physics instructor construct a bridge between the embodied and symbolic worlds of mathematical thought and help his novice students cross this bridge?

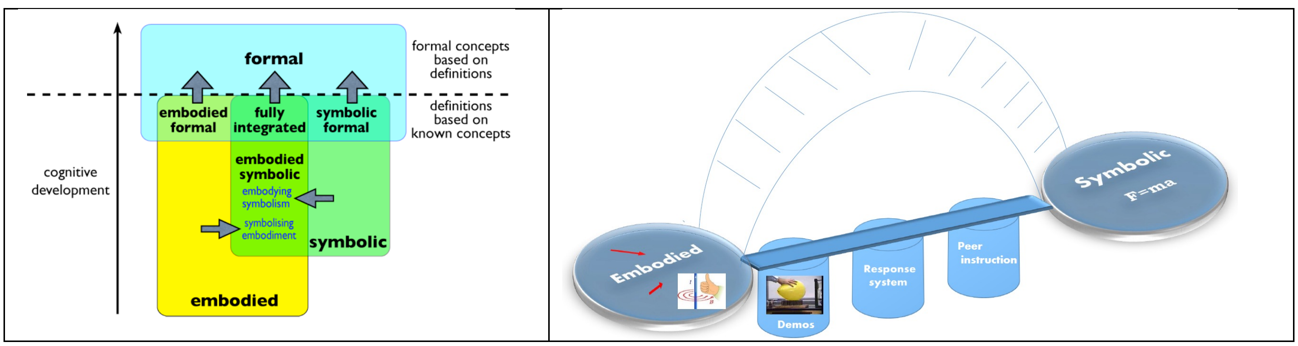


Figure 1. (a) Cognitive development through three worlds of mathematics (Tall, 2008, p. 9). (b) The process of embodying the symbolism and symbolizing the embodiment in physics.

Novice Versus Expert Understanding in Physics

Classic research in cognitive psychology suggests that physics experts and novices approach problems differently (Chi, Feltovich, & Glaser, 1981). For example, experts categorize physics problems on the basis of the underlying physics principle involved, whereas novices categorize the problems on the basis of superficial similarities found across problems. For instance, novices may view all inclined plane problems as equivalent. Experts have a rich, intertwined, hierarchical structure to their knowledge base, whereas novices rely on isolated facts that are not highly structured (Van Heuvelen, 1991). This lack of knowledge structure makes it quite difficult for students to identify the "conceptual unity" in the physics they are taught (Van Heuvelen, 1991). Physics experts effortlessly switch between representations—physical, graphical, schematic, and algebraic—as they reason about problems. It is difficult for novices to learn to identify and use representations that will improve their accuracy and problem solving efficiency (Dreyfus, 1991a; Siegler, 1996).

Van Heuvelen (1991) argues that to get students to think like an expert physicist, students should (1) construct qualitative representations, (2) reason about physical processes through the use of diagrams, (3) construct mathematical representations by referencing the diagrams, and (4) solve the problems using quantitative methods. It is more common for students to attempt a means-end analysis by finding an equation that appears to be appropriate for the problem than it is for them to first employ a sense-making strategy, such as drawing a diagram, to understand how the physical system in question is behaving (Maloney, 2011). Students may resist using diagrams because they do not fully understand the concepts represented in the diagrams, the students have minimal opportunities to develop and practice creating diagrams because they are often passive observers as their instructors create the diagrams, and students' preconceived notions about the way the world works may conflict with what they are being taught in class (Van Heuvelen, 1991). There is educational value in using multiple representations. "The ability to identify and represent the same thing in different representations, and flexibility in moving from one representation to another, allows one to see rich relationships, develop a better conceptual understanding, broaden and deepen one's understanding, and strengthen one's ability to solve problems" (Even, 1998, p. 105). However, visualization is not a trivial task for novice physics students: "We consider the ability to translate between physical and mathematical descriptions of a problem and to meaningfully reflect on or interpret the results as two defining characteristics of a physicist, yet these are areas where our students struggled most" (Wilcox et al., 2013, p. 020119-11).

Sometimes novices who are immersed in a physics course where the instructor values the use of multiple representations to solve problems (e.g., visualization strategies as well as computational strategies) will draw pictures to help them solve problems because that is the course norm. They may not actually understand why they should use the drawings to help them solve problems (Kohl & Finkelstein, 2008). Kohl and Finkelstein (2008) noted that many introductory physics courses do not teach meta-level problem-solving skills, such as highlighting the importance of using diagrams to solve problems. Students may solidify these skills after taking several physics courses, and it is unclear how to effectively teach these skills in an introductory course.

Method

Our qualitative research investigated the ways an expert physics instructor made instructional decisions to help his novice introductory physics students bridge the gap between the embodied and symbolic worlds. The research team consisted of three members: a mathematician, who specializes in mathematics education research (second author), a cognitive psychologist, who investigates learning and transfer of knowledge in the domain of mathematics (first author), and a physicist, Bruce (third author), who focuses on physics education. We examined the daily teaching journals that Bruce kept as he taught a 240-person introductory physics course in the Spring 2014 semester. An innovative aspect of this research is that Bruce is not only a participant in our qualitative research, but he is also an integral member of our research team. Bruce was a consultant throughout the research process from research question design, to data collection and coding, to data analysis and dissemination of results. Bruce was able to confirm whether we had accurately portrayed his teaching techniques and decision-making processes. The research team met weekly to discuss the contents of Bruce's teaching journals and to give Bruce a chance to expand on his weekly lesson plans. Transcripts of these weekly meetings were another source of data.

With Bruce's help, the team qualitatively coded his journals based on the following themes (arranged from most-to-least frequently mentioned): (1) teaching (126 instances, or 35%: goals, real time feedback, question creation, examples, philosophy/best practices, lecture only, sequencing, pedagogical content knowledge), (2) reflections (97 instances, or 27%: on instructor, student understanding/effort, class quality), (3) questions (48 instances, or 13%: instructor questions, peer instruction, in-class quizzes, pre-class questions, formal assessment, qualitative, calculation, homework), (4) visualizations (34 instances, or 9%: student abstract/concrete, instructor abstract/concrete, kinesthetic) (5) students (26 or 7%: real time feedback, questions asked, engagement level), (6) demonstrations (19 instances, or 5%: interactive, illustrative, affective), and (7) mathematics (15 instances, or 4%: qualitative-sense making, quantitative-calculation, use in physics). There were 365 total coded instances. An instance was one or several sentences. Multiple codes could be applied to each instance.

Additional data was drawn from interviews with Bruce in which he created a table of the most difficult concepts that students face in his class. A student in Bruce's course kept a daily journal about his course experiences, and we also investigated answers on an end-of-semester exam and responses submitted during class to Learning Catalytics (http://www.learningcatalytics.com), an online polling system. Students used Learning Catalytics to answer in-class questions, such as multiple choice and free-response questions, to submit drawings for discussion, and to send backchannel questions to Bruce.

The Course Structure and Bruce's Teaching Philosophy

The course that we examined was the second introductory physics course in a two-course sequence focused on thermodynamics, electricity and magnetism, and simple circuits. One of the course goals mentioned on Bruce's syllabus highlighted the importance of bridging the embodied and symbolic worlds: "improve problem solving skills by approaching new problems in a systematic way, plotting out strategies for solution, building and using models, and developing critical thinking skills."

In Bruce's view, the purpose of classes is for students to learn, not for instructors to "teach," and learning is an active process that requires student engagement and effort. Bruce engaged students in active learning activities that included student predictions and explanations. Students prepared for class by completing pre-class reading questions and often worked on peer instruction activities during class (Mazur, 1997).

Bruce's philosophy characterized his beliefs about the connections between math and physics. Physics is about real things that behave in predictable ways, and symbols and numbers are used to represent the properties of these things. If students do not make connections to things, they are doing abstract math, not physics. Students did not need to memorize equations in Bruce's class, but they were expected to understand and use the equations to solve physics problems. His philosophy also characterized the role of models in his physics course. Physics, and all science, is done using models. As approximations to the real world, models allow scientists to consider what is important and ignore what is not. Models can be represented by drawings, graphs, equations, and basic physical concepts. Identifying and applying correct models and representations are important skills to learn.

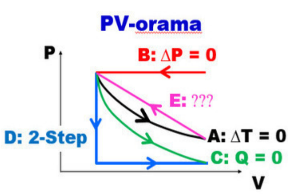
Results and Discussion

Bruce taught this particular course many times, so he could anticipate the concepts that students would struggle with the most. There are many well-known, well-researched stumbling blocks faced by physics students. We asked Bruce to list his lesson plans for all of the difficult course topics that contained some element of visualization. Table 1 provides an example of the use of embodied (e.g., demonstrations, simulations, and visualizations) and symbolic (e.g., visualizations and mathematics) representations by the instructor and students for a challenging concept, using Pressure-Volume (PV) diagrams and models of gas processes to describe real phenomena.

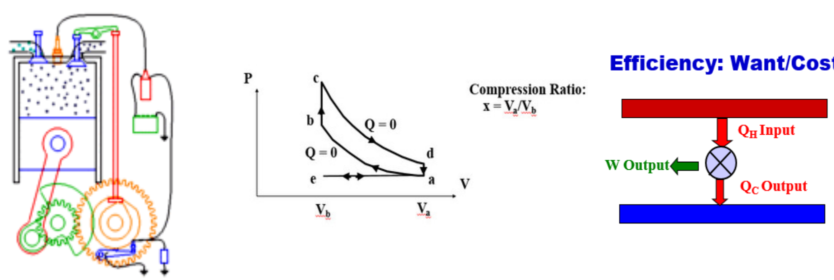
Table 1: Example representations and mathematical tools used for thermodynamics.

Representation	Example	Description	Physics
Demonstration	Heat Engines	Display of Steam and Stirling heat engines. (Class)	Heat exhaust and relative efficiency
	Hard Sphere Model	Styrofoam balls in a cylinder, agitated by a motor. (Class)	Pressure, volume, and energy in an ideal gas.
Simulation	Ideal Gas Model	Simulation of gases with visible bouncing molecules. (Class, Homework)	Pressure, volume, and temperature in gas processes.
	Engine Models	Animated engine illustrations with PV Diagrams. (Class)	Physical applications involving ideal gas processes.
Visualization	PV Diagrams	Draw and interpret pressure vs volume graphs for standard processes. (Class, Homework, Tests)	Sign and magnitude of work, heat, and energy change.
Mathematics	Integration	Work as area under a PV graph. (Class, Homework, Test)	Work and related energies from PV representations.
	Functions	Work, heat, and entropy as logarithms or power laws. (Class, Homework)	Shape and relative slope of curves. Connection with physical properties.

Figure 2 shows visualizations from thermodynamics lessons. The first and last are static images that students create. The visualizations of an operating engine are animations of these static images. Bruce first asked students to make connections between the animated engine and a real engine that was running at the front of the classroom throughout his lesson.



Instructor-provided diagram for in-class questions on the physics of gas processes, asking about energy and temperature.



Three representations of a heat engine showing, from left to right, (1) a schematic real-world operation, (2) details of the gas processes used for quantitative analysis, and (3) a high-level abstraction of energy transfer.

Figure 2: Instructor-created visual representations in thermodynamics.

Students are expected to use these visualizations on assessments, both for group problem solving and on a summative exam. Figure 3 shows student-generated representations of an adiabatic gas process, where $PV^{\gamma} = \text{Constant}$, collected through Learning Catalytics. These give real-time insight into student thinking and use of visual representations. About 60% of students submitted diagrams similar to the first (correct) or second (mostly correct) examples. The next three diagrams illustrate various types of confusion. The final example shows that a few, but not many students attempted to solve the problem, but gave up. Bruce encouraged a fun learning environment. After students submitted their PV Diagrams, some were shown to the class (anonymously) for further discussion by small groups and the entire class to help foster students' use of drawings as they attempted to solve physics problems.

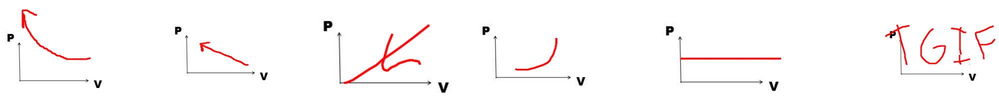


Figure 3: Student submitted in-class drawings for an adiabatic gas process.

In our weekly meetings, Bruce identified that part of students' difficulty with generating the PV diagrams was that many students do not realize that P and V are variables (physics) like X and Y are variables (math): "Let's make P equal to Y and V equal to X, so Y is equal to C over X. Draw that. That one they can do. So this is the ability to generalize that P and V are variables just like you use in math, and therefore you can do the same operations. That's sort of more abstract thinking that a lot of them are working through."

Bruce often broke more complex problems down into smaller, more manageable pieces. "I sort of lead them through it. The first thing I asked them to do is to tell me for this process is the final temperature going to be bigger or smaller? Is the final pressure going to bigger or smaller? It's purely qualitative so that there's no calculation necessary. Then, I would say draw a PV diagram. Then, they actually calculate something. They start out doing it by themselves, and then I have them discuss it. When I'm actually having them calculate things, I have them work together. Then [I'll] step them through the calculation and then sometimes finish up with another qualitative question."

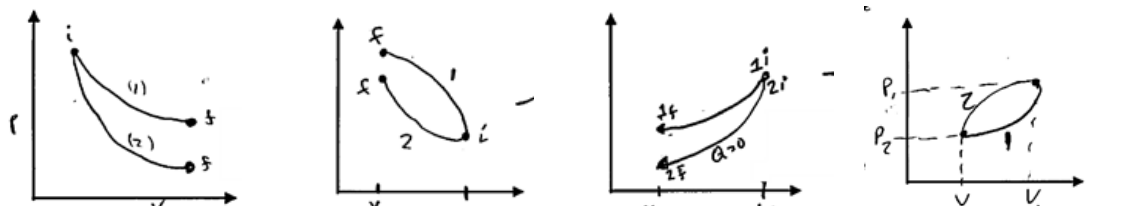
For the visualizations that Bruce showed in class, he allowed the drawing to unfold a little bit at a time: "If it's something that is a picture that I would expect them to create, I try to animate it [on my slides], bring it in piece by piece so that they see how they would build it. So first you draw the axis, you draw this arrow, you draw this arrow, then you label this."

Bruce noted the importance of students creating visualizations on a regular basis: "There are things that I've probably gotten away from a little bit [that] I got back to which is them actually drawing and submitting their drawings. So I'm going to do more of that for the rest of the semester. I kind of got away from that [having them draw pictures and discussing them as a class], and I think that was a mistake. Draw representations that work and then apply those representations to get and organize the equations that will allow you to solve it."

Sometimes students' main obstacle to crossing the embodied-symbolic bridge is simply a lack of mathematical knowledge: "I wish I could guarantee that my students had vector calculus when we were talking about some of this." Sometimes algebra poses a difficulty: "Frankly, we need to get them better at doing algebra. Not really basic algebra, most of these students can do the really simple stuff, but ratios and powers they can't. The music analogy is: algebra is like doing your scales. You know it's just the machinery that you have to be able to do. That should be well practiced, but it's not."

How do students view the experience of crossing the embodied-symbolic bridge? One student agreed to keep a journal about his experiences in Bruce's class. His entries showed the role that diagrams played in facilitating an "a-ha" moment as he bridged the embodied and symbolic worlds. "Often, the diagrams will still be stuck in my head hours later, even if I can't recall the expressions or even much of what was said, and I will keep thinking about them. Then after reading something on Wikipedia, or Googling the material, or reading something in the textbook, the diagram will "resolve" and I suddenly understand why it's that way. Rarely do I understand this in class when it is taught. However, Bruce's depictions were absolutely crucial to my understanding of physics. I just could never make sense of it in class. The most frequent "a-ha" moments for me occur during the homework. As I labor to understand the expressions, I'll recall the diagrams or models and try to understand how the expression describes the model/diagram."

Students' self-generated drawings on the final exam (Figure 4) indicated that crossing the bridge is a protracted process: the drawings show gaps in students' embodied understanding even though their overall exam grades showed that they had a firm grasp on how to symbolically solve related problems. Figure 4 shows a problem that is a comparison of isothermal (logarithmic) and adiabatic (power law) processes. Students did well on this question overall, but some responses still showed student misunderstanding. These examples are from students who received a B on the test, and thus had a reasonable grasp on symbolic solutions to these problems.



Expansion rather than compression | Incorrect curvature and temperatures | Decreasing temperatures | Incorrect temperature and curvature

Figure 4: Student submitted drawings from an exam comparing isothermal and adiabatic gas processes.

Conclusions

This study analyzed teaching diaries and weekly meeting transcripts from an expert physics instructor, who encouraged students to create and submit diagrams through an online response system during his large, introductory physics lectures. Our intention was to begin an investigation of how expert instructors may help novice students navigate the worlds of symbolism and embodiment. Novice students do not possess the dense, interconnected web of physics knowledge that experts have at their disposal (Van Heuvelen, 1991). Students need to be guided or trained as they attempt to cross the bridge between the symbolic and embodied worlds of mathematical thinking. In closing, we offer recommendations that might help physics students make connections between the embodied and symbolic worlds. For example, math instructors may provide students with concrete examples relevant to real world phenomena covered in science and engineering courses. For instance, when instructors are discussing the reciprocal function (Y = 1/X), they could provide an example of the physics concept, Boyle's law (V = C/P), that states when temperature is constant (C), the volume of a gas (V) is inversely proportional to pressure (P). Physics instructors may need to help students hone their drawing and visualization skills, and this is a skill that should be practiced regularly. Instructors may remind students that their drawings are models, but they are important for identifying the relevant elements of the symbolic and formal worlds for different physical problems. Models are approximations of actual physical systems that can highlight "conceptual unity" and help answer the question "What's the Physics?" These models help problem solvers to approach a correct answer. Of course, the use of the model must be confirmed through computations and comparison with the real world.

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Mathematicians' ideas when formulating proof in real analysis

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This report presents some findings from a study that investigated the ideas professional mathematicians find useful in developing mathematical proofs in real analysis. This research sought to describe the ideas the mathematicians developed that they deemed useful in moving their arguments toward a final proof, the context surrounding the development of these ideas in terms of Dewey's theory of inquiry, and the evolving structure of the personal argument utilizing Toulmin's argumentation scheme. Three research mathematicians completed tasks in real analysis thinking aloud in interview and at-home settings and their work was captured via video and Livescribe technology. The results of open iterative coding as well as the application of Dewey's and Toulmin's frameworks were three categories of ideas that emerged through the mathematicians' purposeful recognition of problems to be solved and their reflective and evaluative actions to solve them.

Key words: proof construction, Toulmin argumentation scheme, inquiry, real analysis, mathematicians

Writings of mathematicians and mathematics education researchers note that the mathematical proving process involves a formulation of ideas; specifically, for mathematicians, there is a reflection, reorganization of ideas and reasoning that "fill in the gaps" so a proof will emerge (Twomey Fosnot & Jacob, 2009). Byers (2007) described an idea as the answer to the question "what's really going on here?", and Raman, Sandefur, Birky, Campbell, and Somers (2009) observed three critical moments in the proving process in which there were opportunities for a proof to move forward. Tall and colleagues (2012) gave a description of proof for professional mathematics that "involves thinking about new situations, focusing on significant aspects, using previous knowledge to put new ideas together in new ways, consider relationships, make conjectures formulate definitions as necessary and to build a valid argument" (p. 15). Rav (1999) stated that the term "proof" can describe the written product used to "*display the mathematical* machinery for solving problems and to justify that a proposed solution to a problem is indeed a solution" (p. 13, italics in original); however the process of constructing proof involves informal and formal arguments to find methods to attack the problem as well as incomplete proof sketches (Aberdein, 2009). Despite these writings, little research describes the context around the formulation of ideas that a professional mathematician finds useful and how these ideas influence the development of the mathematical argument. This study focused on describing mathematicians' development of these ideas when constructing proofs in real analysis made evident in changes in the structure of the argument (Toulmin, 1958/2003) utilizing Dewey's (1938) theory of inquiry to describe the problem situation.

Research Questions

Part of a larger project, this report focuses on the findings for the research questions: What ideas move the argument forward as a professional mathematician's personal argument evolves? What problem situation is the mathematician currently entered into solving when s/he articulates and attains an idea that moves the personal argument forward?

Theoretical Perspective

This research conceived of the mathematical proving process as an evolving personal argument. The *personal argument* is a subset of one's total cognitive structure associated with the proof situation (described as a *statement image* by Selden and Selden (1995)) that the individual deems relevant to making progress in proving the statement. The personal argument is graded in that some aspects of the statement image may be central and others may lie on the periphery. The personal argument evolves or moves forward when an individual develops an idea that s/he sees as useful in making progress in proving the statement. The focus of this study was to describe the ideas incorporated and the inquirential context surrounding that development.

Toulmin's (1958/2003) argumentation model provided a means of describing structurally the evolution of the personal argument as the individual incorporated new ideas. The framework notes the content of the statements given in the argument (either explicitly or not) as well as the purposes that those statements serve. The framework classifies statements of an argument in six different categories. The claim (C) is the statement or conclusion to be asserted. The data (D) are the foundations on which the argument is based. The warrant (W) is the justification of the link between the grounds and the claim. Backing (B) presents further evidence that the warrant appropriately justifies that the data supports the claim. The modal qualifiers (Q) are statements that indicate the degree of certainty that the arguer believes that the warrant justifies the claims. The rebuttals (R) are statements that present the circumstances under which the claim would not hold.

New ideas result from periods of ambiguity or when engaged in non-routine problem solving (Byers, 2007; Lithner, 2008). John Dewey (1938) posited in his theory of inquiry that new knowledge or ideas are developed when one is engaged in active, productive inquiry into a problem. An individual engaged in the cyclical process of inquiry reflects on problem situations, selects and applies tools to the situations, and evaluates the effectiveness of the tools (Hickman, 1990). Dewey's framework provided for understanding the context surrounding the emergence of new ideas from the participant's point of view.

Related Literature

This research followed the lead of other researchers who have conceived of the proof construction process as a particular type of problem solving (i.e. Savic, 2012; 2013; Weber, 2005). Selden and Selden (1995; 2013) maintained that there is a close relationship between problem solving and proof, and that two kinds of problem solving could occur in proof construction: solving the mathematical problems and converting an informal solution into a formal mathematical product. Building upon extensive work in understanding the problem solving process and investigating the problem solving processes of twelve mathematicians, Carlson and Bloom (2005) developed a Multidimensional Problem Solving framework providing a description of the cyclical progression through the phases of problem solving (orientation, planning, executing, and checking), cycling, and problem-solving attributes. Savic (2013) found that the four phases of Carlson and Bloom's framework could be used to code and describe most portions of the proving process. However, he found some differences including the mathematician cycling back to orienting after a period of incubation and one participant not completing the full cycle; Savic hypothesized additional problem solving phases could be added.

Some research has been conducted and documented the existence of and provided initial descriptions of the types of ideas that this study sought to describe. Raman (2003) characterized three types of ideas involved in the production of a proof: heuristic ideas (ideas based on informal understandings linked to private aspects of proof), procedural ideas (ideas based on logic and formal manipulations), and key ideas (heuristic ideas that can be mapped to formal proofs). In later work Raman and colleagues (Raman, Sandefur, Birky, Campbell, & Somers, 2009) identified the potential for three critical moments when constructing proof (1) attaining a *key idea* (later termed conceptual insight; Sandefur, Mason, Stylianides, & Watson, 2012) that gives a sense of why the statement is true; (2) gaining a *technical handle* for communicating a key idea, and (3) the culmination of the argument into a standard form. The potential for a key idea to exist apart from a technical handle exists when a prover is engaged in some informal mathematical reasoning. Although they did not describe them as ideas, Ingils, Mejia-Ramos, & Simpson (2007) found mathematics graduate students used warrants based on both formal mathematical deductions (*deductive warrants*) and non-deductive reasoning including inductive reasoning (*inductive warrants)*and intuitive observations or experiments with some kind of mental structure (*structural-intuitive warrants*). Noting these ideas' existence is interesting but calls for further research into descriptions of how these ideas are developed and what kinds of ideas are deemed important when formal or informal reasoning is utilized.

Methods

Three professional mathematicians with faculty appointments at four-year universities who specialized in researching or in teaching courses in real analysis served as the participants for this study. Each participant worked on a task or tasks in a "think-aloud" interview setting, continued to work on the tasks on their own, turned in their at-home work captured via Livescribe technology, participated in a follow-up interview replaying the video and Livescribe capture of their previous work, and repeated this process with new tasks in the next interview. Each participant worked on three to four tasks in total.

Data analysis proceeded in two phases. In the preliminary analysis of the participants' work on the tasks, I noted moments where participants articulated insights, observations, or hypotheses, and these acted as markers in the transcripts. I hypothesized Toulmin models of the participants' personal argument as well as the inquiential context while these ideas were formulated (perceived problem, contributing actions and tools, and anticipated outcomes of applying the tools) prior to and following these markers. These hypotheses informed the questions asked at the follow-up interview. In the primary analysis, the follow-up interviews provided information to complete and modify the initial analyses. For each task, I wrote stories of the participant's complete work on the task sectioned by the ideas in order to capture the evolution of the argument. I conducted open iterative coding of each idea, the problem situation encountered, the tools that influenced the generation or articulation of the idea, and the anticipated outcome of said tools. Most analysis was inductive; however, I borrowed language from the literature when elements fit the descriptions given by other authors. I analyzed across the ideas of each participant and across participants along the common tasks to look for emerging themes and patterns. I report some findings regarding the types of ideas formulated and the problems encountered when ideas were articulated.

Results

In presenting these results, I first give an overview of the characteristics of the ideas that moved the argument forward and then brief descriptions of each idea type and sub-type. I describe the problems that participants were entered into solving when they developed these ideas and finally illustrate these themes through one participant's, Dr. C's, work on a task.

The ideas that moved the argument forward either were accompanied by a structural shift in the personal argument captured by a Toulmin diagram, provided a means for the participant to communicate their personal argument in a logical manner, gave a participant a sense that his way of thinking was fitting, or were explicitly referred to by the participant as a useful insight. While pictures, examples, or individual actions were not included as ideas, the insights extracted from performing and reflecting upon these tools or a collection of tools were included. Ideas were coded in terms of the work they did for the participant. In total, I identified fifteen sub-type ideas grouped into three categories: ideas that focus and configure, ideas that connect and justify, and monitoring ideas (see Table 1). An action or evaluation of that action from one particular moment could solve multiple problems or give rise to multiple feelings. Therefore, multiple idea-types at times characterized a single moment. For example, an insight that provided a *deductive warrant* could also give the prover a sense of *I can write a proof*. Note that three of the idea sub-types that connect and justify are meant to keep in the spirit of the descriptions given by Inglis et al. (2007).

No distinct pattern involving the types of problems and tools that contributed to the generation of certain ideas. There was a discernable pattern of a participant proposing or articulating an idea or tool, testing the usefulness of the proposed idea or tool or the prior ideas against the consequences of the new idea, and then articulating a new idea or evaluation. This process involved the passing through, perhaps multiple times, the inquirential cycle of reflecting, acting, and evaluating against the ideas' abilities to solve a perceived problem. The participants transitioned through the following four phases of problems to tackle or tasks to complete in order to finish the construction of the proof.

1. Understanding the statement and/or determining truth
2. Determining a warrant of some kind
3. Validating, generalizing, or articulating those warrants
4. Writing the argument formally

At times the participants proceeded linearly through the four phases; however, there were instances where participants needed to cycle back to a previous phase when a proposed idea or tool was not fitting or if no tool could be found to solve the current problem (see Figure 1). Aside from these major problems to solve, the participating mathematicians also tackled problems parallel to or embedded within these problems such as dealing with a found problem with a tool. Writing the argument formally typically was not problematic for the professional mathematician once they had developed a deductive warrant.

To illustrate these themes, consider Dr. C's work on the task: *Let f be a function on the real numbers where for every x and y in the real numbers, $f(x + y)= f(x)+ f(y)$. Prove or disprove that f is continuous on the real numbers if and only if it is continuous at 0.*

Upon his initial reading of the problem, Dr. C declared that he believed the statement was true for the rational numbers but not generally true for the real numbers.

> Dr. C: I was thinking about the well-known fact that the only continuous linear functions in the reals to the reals are those of the form y equals mx for some fixed m. And one shows that those are continuous on the rationals fairly easy - linear functions are continuous on the rationals pretty easily by doing some induction.

This idea was in response to the problem of determining the truth of the statement and was coded as a truth proposal, informing the statement image, and a structural-intuitive warrant since he was basing his conjecture on a connection between the additive property and linearity and his conceptual knowledge.

Table 1
Descriptions of ideas that moved the argument forward sub-types

Idea sub-type	Description
Ideas that focus and configure	Ideas that gave a sense of what was relevant, what claims to connect to the statement, fitting strategies to achieve connections, and how to structure and articulate the argument
Informing statement image	Ideas that broadened or narrowed the conception of the situation.
Task type	Assessments about what tools or ways of approaching developing connections between the conditions and the claim would be fitting
Truth proposal	Participant-generated conjectures about the validity of a given claim based on a warrant of any type
Identifying necessary conditions	A sense that "The statement can't possibly be true unless this condition is fulfilled"
Envisioned proof path	A proposal of a series of arguments that will lead to a solution that may be missing connections
Logical structure & representation system of proof	Decisions regarding structuring and communicating the formal argument
Ideas that connect and justify	Warrants and backing, the means of connecting data with claims
Deductive warrant*[1]	Reasoning based on generalizable logical statements
Inductive warrant*	Reasoning based on specific examples
Structural-intuitive warrant*	Reasoning based on a feeling that is informed by structure or experience
Syntactic connection	Symbolic manipulations deemed useful to connect given evidence to a claim that may not be supportable by deductive reasoning or attend to the mathematical objects that the symbols represent
Proposed backing	Proposed support for previously identified non-deductive warrants or vague senses of what would underlie a possible warrant
Ideas that monitor the argument evolution	Ideas or feelings about the mathematicians' progress
Truth conviction	Personal belief as to why a statement must be true
"I can write a proof"	A feeling of formulating the connections necessary to communicate the argument in a final proof
Unfruitful line of inquiry	An idea that persuaded the participant that the tools or actions pursued or considered were not optimal for achieving the set goal
Support for line of inquiry	A sense that one's actions were fitting

[1] The asterisks indicate that the titles of idea sub-types of deductive warrant, inductive warrant, and structural-intuitive warrant borrow from the descriptions of reasoning given by Inglis et al. (2007).

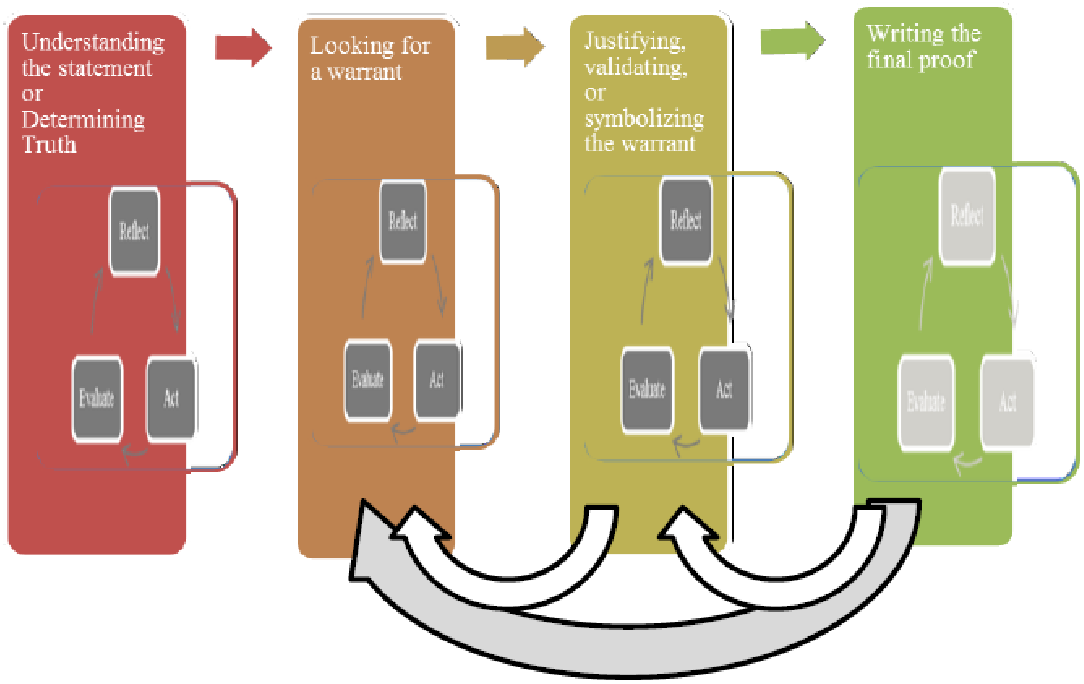

Figure 1. Illustration of the problem phases observed and the potential to cycle back.

Dr. C then set about determining a deductive warrant by proposing a counterexample function that was continuous at zero and the rational numbers but discontinuous on the reals, namely, the piecewise defined function that has an output of zero when the input is rational and the value of the input otherwise. He then tested this function and found it to not possess the additive property and concluded that the given statement might be true.

Dr. C: It turned out that didn't work. And if the easier ones didn't work, then the harder ones probably wouldn't either. Matter of fact, if the easier one didn't work, then it seemed likely that none of the harder ones would work.

I: Okay. So I was going to ask about that. So after you found that it didn't work, it didn't satisfy it. You paused for a while. Was it because you were trying to think of different examples, or were you convincing yourself that it-

Dr. C: Yeah. I was trying to convince myself that if this didn't work, then nothing would.

Dr. C recognized an *unfruitful line of inquiry*, moved back to the problem of determining the truth of the situation, and gave a new *truth proposal* based on the generated example function coupled with his knowledge of functions (an *inductive warrant*). He then moved to try to prove the statement was true (look for a deductive warrant). In exploring, he developed a string of inequalities based on instantiations of the definition of continuity and logical mathematical deductions, and he identified the *necessary condition* that $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$. He recalled a proof that $f(0) = 0$ and that the function was given to be continuous at zero to fulfill the condition. Dr. C symbolically evaluated that his written assertions were correct and declared a sense that he could now write the proof based on his deductive warrants. Because his work in proving the task was based on deductive warrants within the representation system of proof, the writing of the proof did not require the formulation of any new ideas.

Discussion and Conclusions

Every participant on each task identified ideas from each of the three idea categories. As was described above with Dr. C, the evolution of the personal argument was not linear in identifying focusing and configuring ideas, identifying connections and justifications, and then making monitoring decisions. The process of articulating ideas, testing the new idea or previous ideas against these new ideas, and then proposing new ideas was apparent. The process of testing ideas varied by idea-type, but the process involved active, productive inquiry in that ideas were tested against their abilities to do work in solving a perceived problem.

The four identified phases of understanding the statement or determining truth, looking for a warrant, working to validate, generalize, justify or articulate their warrant; and writing the formal proof are reminiscent of findings of other researchers. The following aspects have been identified as part of the proof construction process: understanding the statement or described objects (Alcock, 2008; Alcock & Weber, 2010; Carlson & Bloom, 2005; Savic, 2013); determining the truth of the statement (Sandefur et al., 2012); determining why the statement is true (Raman et al., 2009; Sandefur et al., 2012); translating ideas into analytic language (Alcock & Inglis, 2008; Alcock & Weber, 2010; Weber & Alcock, 2004); and justifying a previous idea (Alcock, 2008; Alcock & Weber, 2010). This research is unique in its specific efforts to identify the problems encountered as participants developed new ideas and in its use of Dewey's theory of inquiry to explain how ideas were developed and tested against these problems. The mathematicians progressed through these four phases but needed to cycle back to a previous phase when the ideas that the mathematicians had previously incorporated into the personal argument were insufficient in resolving a situation in a later phase.

The choice to conceive of the proof construction process as involving an evolving personal argument was made due to a desire to talk about all the ideas, relationships, concepts, pictures, and so on that an individual personally judges as important to providing a final proof and the relationships amongst these elements at various points in time. This conception allowed for attending to moments when ideas were generated that the prover saw as useful which broke the construction process into significant events to illustrate the story of the argument's evolution. As researching the proving process in this manner is relatively unexplored, many avenues of research are open to explore how these ideas develop, how they are tested, and the consequences their development provides for the evolution of the argument. The findings of this study were descriptive and exploratory and the fifteen idea sub-types found may or may not be salient in other studies. It is probable that varying the mathematical content area or narrowing the research questions would provide new and clarifying findings to refine the categorizations or provide insight as to how the proof construction process compares across mathematical content.

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College-educated adults on the autism spectrum and mathematical thinking

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This study examines the mathematical learning of adults on the autism spectrum, currently or formerly undergraduate students. I aim to expand on previous research, which often focuses on younger students in the K-12 school system. I have conducted various interviews with current and former students. The interviews involved a combination of asking for the interviewee's views on learning mathematics, self-reports of experiences (both directly related to courses and not), and some particular mathematical tasks. I present some preliminary findings from these interviews and ideas for further research.

BACKGROUND ON AUTISM-RELATED RESEARCH

The Autistic Self Advocacy Network (2014) states that autism is a neurological difference with certain characteristics (which are not necessarily present in any given individual on the autism spectrum), among them differences in sensory sensitivity and experience, different ways of learning, particular focused interests (often referred to as 'special interests'), atypical movement, a need for particular routines, and difficulties in typical language use and social interaction. Over the past few decades, there have been many research studies about learning in students on the autism spectrum, such as those reviewed by Chiang and Lin (2007). A large portion of these studies focus on K-12 students, and particularly elementary students, but some of the ideas and procedures in those studies lend themselves to use in a post-secondary context.

INTERVIEW PROCEDURES

After an initial period of background information and anything else in particular my interviewees wished to share about their perspectives on mathematics, I gave various mathematical tasks to elicit more specific responses. Some of these were directly related to specific courses, such as the example-generation tasks used by Bogomolny (2006) and the Magic Carpet Ride sequence used by Wawro et al. (2012). I have also given more general tasks, such as the paradoxes examined by Mamolo and Zazkis (2008); one reason for this was the interplay between visual and algebraic explanations seen in some student responses to these paradoxes.

THEORETICAL FRAMEWORK

There were several reported characteristics of people on the autism spectrum which I thought could be promising for mathematics education research. In particular, I was interested in details of prototype formation, special interests, and geometric approaches. I will detail each of these with a comparison to the particular findings relevant to them in Joshua's case.

PROTOTYPE FORMATION

I started looking into prototype formation after reading a study by Klinger and Dawson (2001). It suggested that people on the autism spectrum did not form prototypes of objects when given tasks asking about group membership, instead taking an approach based on lists of rules. Although this is presented as a problem, like many other autism-related studies, I suspected that this approach could be helpful for more abstract or proof-based mathematics. I have found many other students having trouble with mathematical questions that appear to result from a prototype-based approach, and this is particularly true when the course focuses on mathematical proof. In fact, I found a very similar division reported in mathematics education research by Edwards and Ward (2004), phrased as lexical or extracted definitions versus stipulative definitions. This did not appear to be the case for Joshua; he reported having this kind of thinking in the past, but was quite focused on "big picture" ideas today (this was, in fact, a recurring phrase in the interviews).

GEOMETRIC FOCUS AND VISUALIZATION

Particularly due to the work of Temple Grandin, one of the most famous people on the autism spectrum, there is often an association between the spectrum and visualization or spatial reasoning (Grandin, Peterson, and Shaw, 1998). While I would caution against being too broad with an association like that, I did find a strong preference for visual, spatial, or geometric reasoning in the interviews I conducted with at least one student. My suspicion is currently that there may be stronger variance or preference in types of reasoning, but that it is not all necessarily toward the geometric type.

PARADOXES

I have also presented several paradox tasks during my interviews. Like many of the students in previous studies, the people I interviewed found these to be strange and paradoxical. However, the response was notably more positive than those from most students. I also found it notable that I did not see any tendency toward rejecting the mathematical facts after they had been presented, unlike in many of the students in the prior studies.

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Students' concept image of tangent lines compared to their understanding of the definition of the derivative

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Our research explores first-semester calculus students' understanding of tangent lines and the derivative concept through a series of three interviews conducted over the course of one semester. Using a combination of Zandieh's (2000) derivative framework and Tall and Vinner's (1981) notions of concept image and concept definition, our analysis examines the role that students' concept image of tangent lines plays in their conceptual understanding of the derivative concept. Preliminary results seem to indicate that students are more successful when their concept image of tangent includes the limiting position of secant lines, as opposed to a tangent line as the line that touches the curve at one point.

Key Words: Tangent Line, Derivative, Conceptual Understanding

"Conceptually, the role of visual thinking is so fundamental to the understanding of calculus that it is difficult to imagine a successful calculus course which does not emphasize the visual elements of the subject" (Zimmerman, 1991, p. 136). This quote encapsulates the relevance and motivation of our research efforts in studying students' understanding of tangent lines in first-semester calculus. Given the crucial role that tangent lines play in the visual aspects of the derivative concept, it is pertinent to consider how misconceptions about tangent lines may contribute to a lack of conceptual understanding of the derivative concept. Our preliminary analysis seems to reveal that students who consistently defined a tangent line as the limiting position of secant lines were also able to graphically explain the definition of the derivative, while those who used other definitions for a tangent line, such as a line intersecting the graph at only one point, were not able to do so.

Literature Review

It has been well documented that students' early experiences with tangency in geometry have the potential to negatively affect their understanding of tangency in subsequent settings (Fischbein, 1987; Tall, 1987; Winicki & Leikin, 2000; Biza, Christou, & Zachariades, 2008). In addition, Biza's (2011) study demonstrated that developing a concept definition of tangency characterized by its use in analysis is both difficult and non-intuitive for students. Vincent, LaRue, Sealey, and Engelke (2014) identified misconceptions first-semester calculus students may have concerning tangent lines, such as believing that several tangent lines may exist at a single point or confusing the notion of a tangent line with the tangent function ($y = \tan x$).

We know that students typically prefer working with the derivative concept algebraically and often struggle with the visual aspects (Asiala, Cottrill, Dubinksky, & Schwingendor, 1997; Habre & Abboud, 2006). Similarly, Hahkionieme's (2006) research project accentuated the difficulties students have interpreting what the formal definition of the derivative means graphically. Other studies have highlighted students' confusions concerning the relationship between tangent lines and the derivative, such as confusing the y-coordinate of the point of tangency with the derivative or equating the tangent line to the derivative (Orton, 1983; Amit & Vinner, 1990).

Theoretical Perspective

Our research incorporates a blend of two theoretical perspectives: Tall and Vinner's (1981) *concept image* and *concept definition* and Zandieh's (2000) derivative framework. The *concept image* represents the overall cognitive structure constructed by the learner and includes all the mental pictures and associated properties and processes that an individual has built up over time. Any *concept image* has a related *concept definition,* which is a learner's description of his or her understanding. Based on a student's language and written work, interpretations can be made about pieces of knowledge that do or do not belong to the concept image.

According to Sfard (1991), *processes* are operations on previously established objects that can be *reified* into *objects* and acted on by other processes. This forms what Zandieh (2000) calls *process-object* pairs. Zandieh's derivative framework (Table 1) is made up of three layers (Ratio, Limit, and Function) of process-object pairs and is meant to describe what the mathematical community means by the concept of the derivative (within four key contexts) at the first-year calculus level. Part of an individual's understanding may be noted within the grid when he or she mentions a context and any component of the layers of the derivative concept in response to interview questions. A circle in the grid indicates that the student has at least demonstrated a pseudo-object (an object with no internal structure) understanding of the row and column that intersect at that box. A shaded circle represents that the student has also demonstrated an understanding of the underlying process of the layer.

Process-object layer	Graphical (Slope)	Verbal (Rate)	Physical (Velocity)	Symbolic (Diff. Quotient)	Other
Ratio					
Limit					
Function					

Table 1. Zandieh’s derivative framework.

Methodology

Our study took place during the spring 2015 semester with twelve first-semester calculus students enrolled in a large public university. Each participant completed a series of three interviews over the course of one semester: beginning half of the semester (immediately following instruction on tangent lines and the definition of the derivative), midterm, and end of term. Each interview focused on the concept of tangent lines- students’ personal concept definition as well as tasks involving construction of tangent lines. Interviews 2 and 3 additionally consisted of tasks involving the derivative- sketching the graph of $f'(x)$ given the graph of $f(x)$ and interpreting the formal definition of the derivative graphically.

We are currently in the process of analyzing the data. At this stage, we have completed a detailed analysis for two of the twelve participants (Jamie and Andy) and a surface level analysis of the remaining ten. Jamie and Andy's series of interviews have undergone multiple viewings along with detailed notes and have been transcribed and coded. To code the data related to the derivative concept, we used Zandieh's derivative framework (Table 1, above). The concept of tangent lines is situated within the first two layers of this framework, and so, a modified version of the framework (Table 2) was used to code interview data related to tangent lines. Comparing these coded sections of the data, we are interested in examining relationships between students' concept images of tangent lines and the derivative.

Process-object layer	Graphical (Slope)	Verbal (Rate)	Physical (Velocity)	Symbolic (Diff. Quotient)	Other
Ratio					
Limit					

Table 2. Modified version of derivative framework used to code data on tangent lines.

Results

The results presented in this section focus on the preliminary analysis of one of twelve of the participants (Jamie), but we will also make reference to some of the other participants as well as discuss general themes found in the data, thus far. We will specifically focus on analysis of Jamie's concept image of tangent line (Interviews 1, 2, and 3) and her graphical understanding of the definition of the derivative (Interview 3), identifying relationships between the two.

Throughout all three interviews Jamie consistently defined a tangent line in terms of its "one point" relationship with the graph. When constructing tangent lines and justifying her work, she mainly reasoned about the location of the point of tangency and whether or not the tangent line should be "above" or "below" the curve. She never defined a tangent line in terms of the limiting position of secant lines. Due to her unstable concept definition, she almost always constructed a tangent line at places where one should not have existed. Table 3 shows Jamie's coded chart for her responses to the question "What is a tangent line?" from Interviews 1, 2, and 3.

Process-object layer	Graphical (Slope)	Verbal (Rate)	Physical (Velocity)	Symbolic (Diff. Quotient)	Other
Ratio		○			
Limit	○				

Table3. Jamie's summary chart for definition of tangent line.

The open circle in the limit row represents Jamie's definition of a tangent line as "where it hits one point on a graph." Since she did not discuss the limiting process, this circle is not shaded in, and since she did not mention the notion of slope in any of her explanations, there is not a code in the slope row.

During Interview 3 Jamie was asked to graphically interpret the formal definition of the derivative concept. Table 4a below shows an excerpt from her transcript.

199	Jamie	Maybe like when you graph it you determine the tangent line. Maybe if you want to find like one exact point I guess on it. I don't know.
200	Int.	Ok. So if you come back to this for a second. So, you're not sure about, like if you take the limit part away do you know what this portion of the definition of the derivative represents?
201	Jamie	No. No.
205	Jamie	Oh! The *h* might be a slope of zero.
206	Int.	So, what do you mean by that?

207	Jamie	Like, here it'd be zero and here would be zeroes [constructs horizontal tangents].

Table 4a. Jamie. Example response. Definition of derivative

Jamie did not discuss the role of tangent lines on her own initiative. Her ideas in line 199 were a response to the interviewer's question about the role, if any, tangent lines may play in the graphical interpretation of the derivative. She mentioned the idea of finding "one exact point," and this response was coded with an open circle in the limit row (Table 4b). Jamie was also uncertain about the meaning of h in the definition of the derivative, and because h is going to zero, the idea of zero slope (horizontal tangents) was triggered in her concept image. The open circle in the ratio row demonstrates her understanding that slope is somehow involved in the derivative concept, but this circle is not filled in because again, she does not discuss the process of determining slope, such as rise over run or change in y over change in x. Table 4b shows Jamie's coded chart for her entire response to the graphical derivative question posed during interview 3.

Process-object layer	Graphical (Slope)	Verbal (Rate)	Physical (Velocity)	Symbolic (Diff. Quotient)	Other
Ratio	○			x	
Limit	○			x	
Function				x	

Table 4b. Jamie's summary chart for definition of the derivative.

It is important to note that although Jamie did not mention the notions of rate of change or velocity during her explanations, other students did. Since participants were given the symbolic definition of the derivative and were not responsible for generating it on their own, we were unable to code the symbolic column of the chart and have labeled it with x's.

Although Jamie's concept image for tangent line and her concept image for derivative have similar structures and seem to influence one another, throughout the interviews she demonstrated that she was unaware of such connections (Table 5). For example, she often could not mathematically justify her work, stating: "it's not coming to me yet", or "cause this is how we learned it in class", or simply "I don't know". She also reasoned that it was possible to contruct tangent lines at places where the derivative didn't exist. So, even though we see similar structures in Jamie's concept images according to Zandieh's framework, she is not aware of the connection between these to concepts.

216	Int.	Ok. Um, so you said that this is the definition of the derivative and then I asked you about tangent lines and their relationship and what was your... what do you think about that?
217	Jamie	Maybe there is. Maybe there isn't.
196	Int.	Ok. And are tangent lines...how do tangent lines...what role do they play in this whole thing?
197	Jamie	I don't know.

Table 5. Example of disconnect between concept images.

Considering Tables 3 and 4b, we see that pseudo-objects in Jamie's concept image of tangent line transferred to pseudo-objects in Jamie's concept image of the graphical derivative (or vice versa). We see similar results with Andy (Table 6)- structural understandings in the layers of tangent line transferred to structural understanding in the layers of the graphical derivative.

	Definition of Derivative		Tangent Line	
Process-object layer	Graphical (Slope)	Verbal (Rate)	Graphical (Slope)	Verbal (Rate)
Ratio	●		●	○
Limit	●		●	
Function	○		N/A	N/A

Table 6. Andy*'s s*ummary *c*hart *of the d*efinition *of the d*erivative *and tangent lines*.

These preliminary results reveal a connection between students' personal concept definition of tangent line and their graphical understanding of the derivative concept. In reviewing the data of all twelve participants, only two participants were able to graphically explain the definition of the derivative. The remaining ten were unsuccessful in their attempts and also consistently (in their definitions and justifications) used other definitions for tangent such as a line intersecting the graph at only point and did not use the limiting position of secant lines definition**.**

In comparing Jamie to Andy, we see Andy exhibited an understanding of the processes involved in the layers of both concepts. Andy often indicated he was "searching" his concept definition, using phrases such as, "hold on let me think about this." Jamie never demonstrated such activity. In contrast, she exhibited that her reasoning mainly flowed from her concept image, using phrases such as "this is how we did it in class" or referencing "similar homework problems". Vinner (1991) discussed that students most often reason from their concept image rather than their concept definition. We are interested in further exploring these ideas and their implications as we progress in our analysis.

Our preliminary results indicate that students' understanding of the graphical derivative may be strongly influenced by their concept definition of tangent lines. While this connection may not be surprising, it *is* surprising how many tasks Jamie was able to complete with her "one point" concept definition of tangent line. Even though she was able to sketch most tangent lines accurately, she was never able to connect the definition of the derivative with the tangent line. Andy, however, was able to make these connections, and our preliminary analysis suggests that his concept image of tangent line as the limiting position of secant lines played a big part in his success. Additional data analysis is necessary to further explore this conjecture.

Implications for teaching

Shortcut definitions for tangency, such as "the line touching the graph at one point" or even "the line whose slope is equal to the derivative" are helpful but should not replace the definition of tangency as the limiting position of secant lines. Consistent classroom use of shortcut definitions may result in the creation of pseudo objects within students' concept images of tangent line and the derivative. These definitions do not accentuate the underlying processes involved in the layers of Zandieh's framework. We do not imply that consistent use of the secant line definition of tangency will magically result in structural understandings, but our results do seem to imply that pseudo objects within the tangent line concept image transfer over to the derivative concept image, and likewise for structural understandings.

Questions for the audience

1. What are your thoughts on the modified version of the derivative framework for tangent lines?
2. What role should the *series* of interviews play in data analysis?

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Examining Student Attitudes and Mathematical Knowledge Inside the Flipped Classroom Experience

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Flipped classrooms or hybrid online courses are becoming increasingly prevalent at the undergraduate level as institutions seek cost-saving measures while also desiring to implement technological innovations to attract 21st century learners. This study examined undergraduate pre-calculus students' (N=427) experiences, attitudes and mathematical knowledge in a flipped classroom format compared to students in a traditional lecture format. Our initial results indicate students in the flipped format were more positive about their overall classroom experiences, were more confident in their mathematical abilities, were more willing to collaborate to solve mathematical problems, and achieved slight higher gains in mathematical knowledge. Contrary to prior research, this study indicated that a majority of students in the flipped classroom would take the class again in the same format, but of concern is the gender disparity, indicating that female students are much more likely to resist taking a class in a flipped format.

Key Words: *Flipped Classrooms, Technology Enhanced Learning, Pre-Calculus, Student Attitudes*

The development of online math education has made huge strides in recent years with the creation and wider availability of open source math tutorials such as Khan Academy, Udacity, and Coursera. This has lead traditional institutions to seek time and money saving measures by developing pre-recorded lectures and utilizing problem-based education inside the classroom (Bacow & Bowen, 2012; Mehaffy, 2012); however, little consideration is given to the effects that these changes will have on students' attitudes and academic performance toward the subject of mathematics. One of the key-concepts behind the "flipped classroom" or the "inverted classroom" approach is using technology to offload traditional style lectures to allot more classroom time for problem based exploration and applied learning (Lage, Platt, & Treglia, 2000; Sams & Bergmann, 2012).

Review of the Literature

There is a limited amount of peer-reviewed research available on flipped classroom approaches; however, studies have been steadily increasing in recent years. Preliminary reports seem to suggest that students in flipped classrooms show improved academic success and achieve greater learning outcomes as compared to students in traditional classroom models, (Baepler, Walker, & Driessen, 2014; Love, Hodge, Grandgenett, & Swift, 2014; Mason, Shuman, & Cook, 2013; Wilson, 2013) or at worst does no harm (Mason et al., 2013; McCray, 2000, Bagley, 2014).

In addition, student attitudes are fairly consistent and show students view the flipped classroom as promoting their learning (Arnold-Garza, 2014; Scida & Saury, 2006), increasing confidence in their abilities (Baepler et al., 2014; Kim, Kim, Khera, & Getman, 2014) encouraging social engagement with students and teachers (Baepler et al., 2014; Jaster, 2013; Love et al., 2014), as more relevant to their future career goals (Love et al., 2014) and appreciate the flexibility allowed by online videos (Jaster, 2013); however there is evidence that given a choice, students prefer a traditional model of learning (Arnold-Garza, 2014; Jaster, 2013).

Although recent studies support the use of flipped classrooms, most studies thus far have used small samples sizes, and with the exception of a few conference proceedings (Overmyer, 2013; Wasserman, Norris, & Carr, 2013; Bagley, 2014) most are not specific to the subject of undergraduate mathematics. Since the research on the effectiveness of this pedagogical approach is limited, there are clear gaps in the literature that this study hopes to address. Accordingly, this study is a first step in determining how do students in a flipped learning undergraduate math course compare to students in a traditional lecture course in their:

- Attitudes (motivation, enjoyment and confidence) and beliefs about learning mathematics?
- Experiences and opinions of the course activities and interactions?
- Perceived learning gains and mathematical knowledge?

Research Design and Methodology

Participants were students from four undergraduate pre-calculus II course sections offered at a large Research University in the Midwest. Two of the courses used the flipped learning model (FL) for instruction and two used the traditional lecture model (TL) for instruction. Each of the course sections met for three hours a week of classroom time and one hour for a Q&A section lead by a graduate assistant. The TL courses used the traditional classroom time to lecture on the classroom material with limited interaction between teacher and students. In comparison, The FL classes used online video tutorials that features a voiceover PowerPoint to present the lecture material outside of classroom (https://goo.gl/DMyd6q) and classroom time was then used primarily to complete group (3-4 students) based worksheets with low level practice problems combined with mathematical proofs to derive trigonometric formulas in an active learning classroom.

The research instruments and design methodology parallel the research conducted by Laursen et al. (2014) regarding inquiry-based learning. The first survey instrument referred to as the attitudinal assessment, consisting of 54 questions using a seven point Likert-scale, and was used to measure changes in student's affect (motivation, enjoyment, and confidence), and beliefs and strategies about learning. The second survey instrument is based on a subset of the mathematically focused Student Assessment of their Learning Gains, referred to as the SALG-M and measures student's experiences and learning gains using a 5-point Likert scale from (1 –No gains) to (5-Great gains) for each item. The attitudinal assessment pre-survey was administered at the start of the second week of the course and the attitudinal post-survey and SALG-M were administered in the last week of the course. Scores from the multiple choice section of the mathematics department common final examination were used to assess student's mathematical performance. In addition, demographic information including gender, race, class year, college major, previous math courses taken, and GPA were requested.

Results

We received 427 responses (87.5% of enrolled students) from the pre-survey and 300 responses (61.5% of enrolled students) from the post survey. Using the unique identifier we were able to match 214 (43.8% of enrolled students) pre- and post-surveys. Based on prior research from Laursen et al. (2014), a factor analysis was performed on each of the survey items to create composite variables to measure changes in students affect (motivation, enjoyment, confidence), beliefs about learning, and strategies for problem solving problems (See Table 1). In addition composite variables were determined to assess students perceptions of the classroom experiences, and self-reported learning gains as a result of the course (See Table 2). A summary of the composite variables and reliability ratings are reported in Table 1 and Table 2.

Table 1: Composite Variables of Attitudinal and Learning Behaviors in Mathematics

Variable	Description	Reliability *Cronbach alpha*	
		Pre	Post
Motivation	Motivation to learn mathematics	.761	.771
Interest	Interest in learning and discussing math outside of the classroom	.749	.774
Math degree	Desire to pursue a math major/minor	.838	.822
Math future	Desire to pursue and study for additional math courses.	.536	.672
Teaching	Desire to teach mathematics	-	-
Enjoyment	Pleasure in doing and discovering mathematics	.893	.908
Confidence	Confidence in math and math teaching ability	.828	.859
Math confidence	Confidence in own mathematical ability	.805	.852
Teaching confidence	Confidence in teaching mathematics	.682	.745
Beliefs about learning			
Instructor-driven	Exams, lectures, instructor activities	.687	.689
Group work	Small group presentation and critique of math	.639	.629
Exchange of ideas	Active exchange with other students	.765	.728
Strategies			
Independent	Find one's own way to think and solve problems	.450	.640
Collaborative	Work with other students to brainstorm and solve problems	.717	.683
Self-regulatory	Review and organize one's own work; check one's understanding	.562	.647

Table 2: Composite Variables for Student Experiences and Learning Gains

Variable	Description	Reliability
		Post
Experiences of course practices		
Overall	Overall experience, workload, and pace of the course	.797
Active participation	Participating in discussion, group work, and explanation of work.	.800
Individual work	Studying on your own	-
Lectures	Listen to lectures	-
Assignments	Tests, homework, feedback on written work	.603
Personal interactions	Interacting with peers, TAs and instructors	.667
Cognitive Gains		
Math concepts	Understanding concepts	.906
Math thinking	Understanding mathematical thinking	.819
Application	Applying ideas outside math, making math understandable for others.	.828
Affective Gains		
Positive attitude	Appreciation of math	.812
Confidence	Confidence to do math	.889
Persistence	Persistence, ability to stretch mathematical capacity	.781
Social Gains		
Collaboration	Working with others	.773
Teaching	Comfort in teaching	-
Independent Gains	Ability to work on your own	.828

Linear regression analysis was performed on each of the composite variables to determine the main effect of classroom format. The results of this analysis, which are displayed in Figure 1, indicated significant differences for students experiences in the classroom, math confidence, and collaborative strategies for problem solving. In addition there were significant differences in self-reported affective, cognitive, and social learning gains, but no difference in independent learning gains (See figure 2). We subsequently discuss the themes that emerged from this initial analysis.

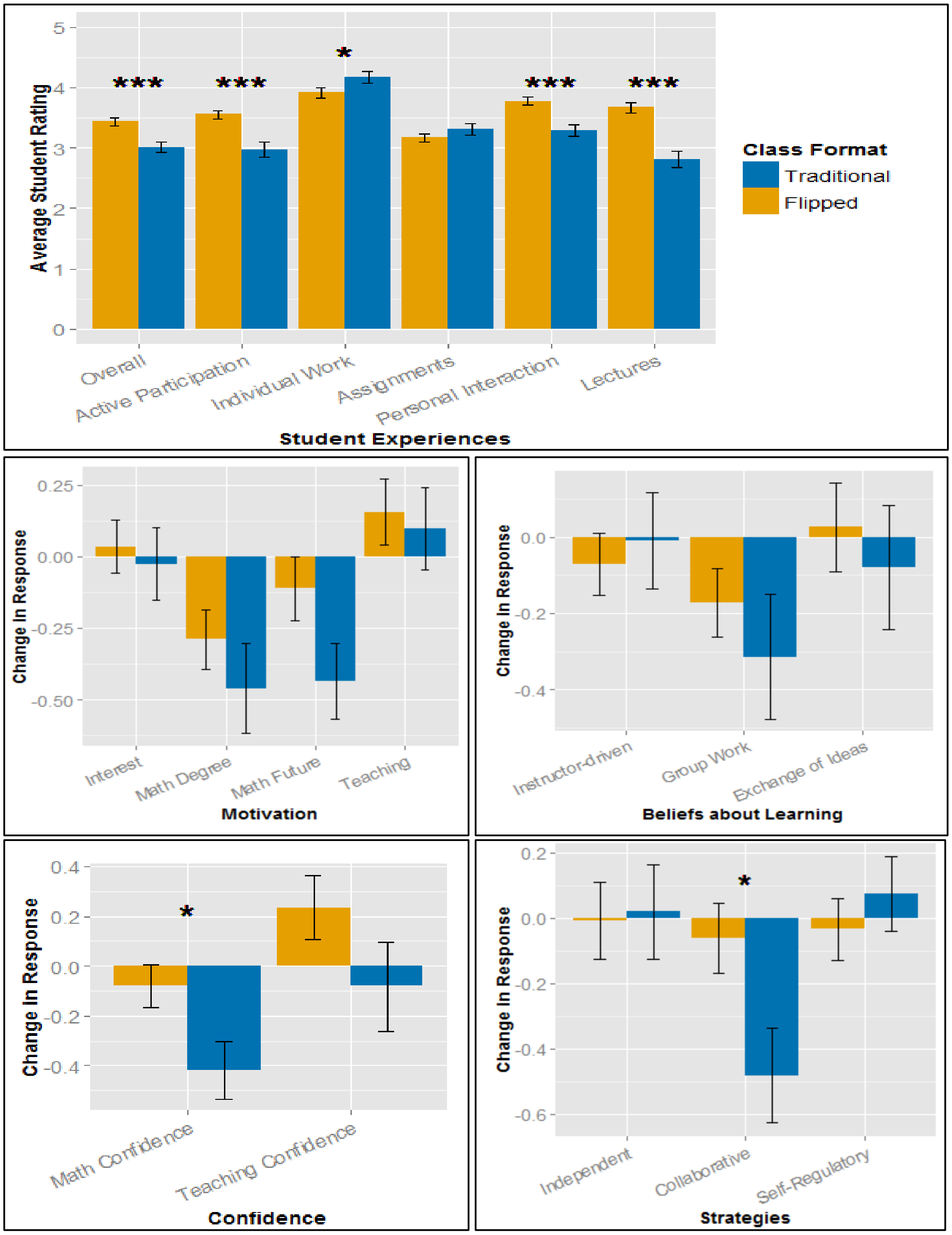

Figure 1. Average Classroom Experiences and Changes in Pre and Post Survey Attitudinal Variables Based on Classroom Format with Standard Error Bars.

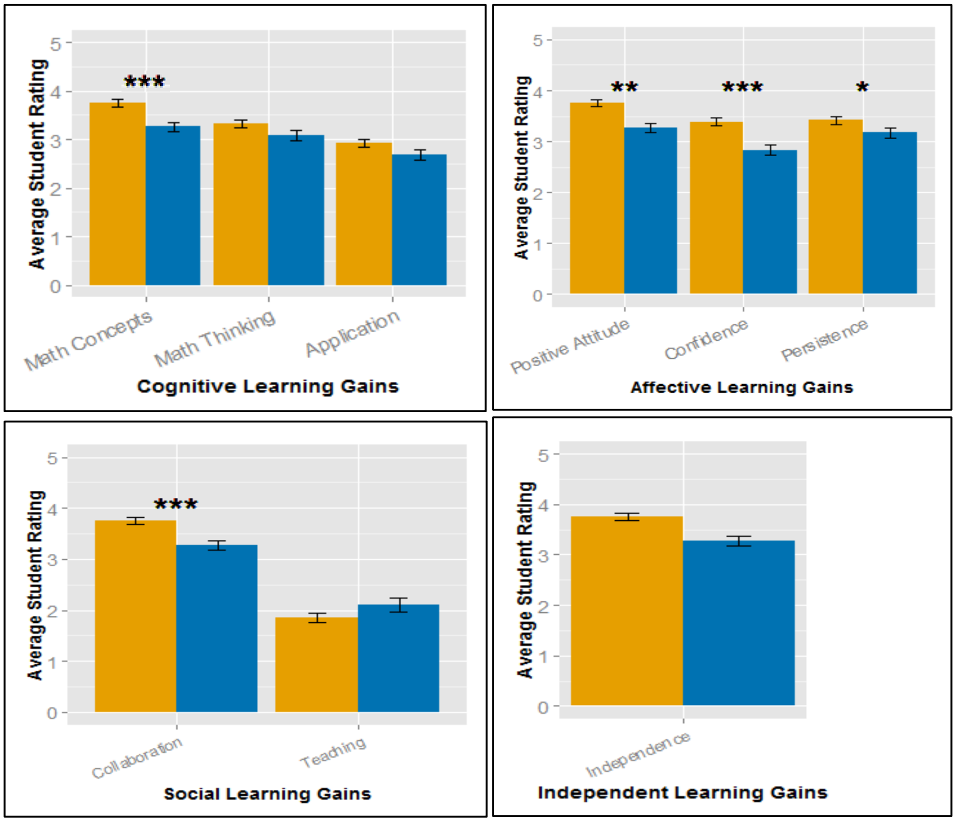

Figure 1. Average Rating with Error Bars for Learning Gains Based on Classroom Format.

Classroom Experiences

As suggested by prior research, students in a flipped format viewed the overall experiences in the course (workload, pace, and overall approach to the course) as a significantly greater help to their learning than students in a traditional format; however, the research goal was to further investigate the specific components of the course that may have contributed to the overall differential experiences of students in the FL versus the TL format. Active participation (class discussions, group work, explaining work to other students, and listening to other students explain their work), personal interactions (with the instructor, teaching assistant, and peers in the course) and lectures were seen as a greater help to students in the FL format, while individual work such as studying on your own was seen as a greater help to students in the TL format. Assignments were viewed as equally supportive for students in either the FL or the TL format.

In addition to questions about classroom experiences, students were asked, "Would you recommend taking another course offered in the SAME FORMAT as this one?" Contrary to the findings of Arnold-Garza (2014) and Jaster (2013), a large majority of the students (67%) in the FL courses would take the course again in the same format given the choice, compared to a similar but smaller percentage of TL courses students (54%) who said they would take the course again in a traditional lecture format. Further investigation into the make-up of students who would not recommend taking a flipped classroom format, showed a significant difference (χ^2 $(1, \mathrm{N} = 182) = 8.12, \mathrm{p} = .004$) in the gender composition with a larger proportion of women (N=40) saying they would not recommend the format as compared to men (N=15). The same difference was not present in the traditional class ($\chi^2(1, N = 118\,) = .145, p = .70$). Although gender and gender interactions with flipped learning were not significant for any of the composite variables, the fact that women were almost three times as likely to indicate a preference for not take the course again in flipped learning format warrants further investigation.

Affective and Learning Strategies Changes

Our results from the attitudinal assessment mirror the results of the MAA national study (Bressoud, Carlson, Mesa, & Rasmussen, 2013) indicating overall students are less confident in their mathematical ability after the completion of the course, but notably students in the FL course had significantly smaller declines in mathematical confidence ($F(1, 210) = 5.44, p = .02$). In addition FL students as a result of the course reported higher affective learning gains including positive attitude ($\beta = -.39, t(282) = -2.92, p = .004$), confidence ($\beta = -.56, t(284) = -4.65, p < .001$), and persistence in mathematics ($\beta = -.25, t(283) = -1.98, p = .048$). We conjecture that there are two contributing elements that resulted in the smaller declines in confidence for the FL students. One notable difference between the FL and TL courses, was the implementation of ten proficiency based quizzes that students had to master in order to pass the course. This mastery based learning approach gives students the opportunity to assert that they fully understand the core topics in the course. In addition to the mastery quizzes the availability of having the online lectures, which our log data shows a majority of students watched multiple times, also provides students with increased scaffolding to support understanding and learning of the course topics.

Students in the FL course also show attitudinal changes in the benefit they see in using collaborative strategies toward learning indicating that they are more likely to seek help from others and share information with other peers ($F(1{,}211) = 5.39, p = .02$). This change in collaborative learning strategies we attributed to the reported social gains in collaboration ($\beta = -0.53, t(259) = -2.48, p = 0.01$) due to the course, where FL students reported higher gains in their ability to work well with others, willingness to seek help from others and appreciation of difference perspectives as a result of the course.

Mathematical Knowledge

Results from student performance on the common math final indicate modest gains in academic performance for students in the FL course (M=67.2) compared to students in the TL course (M=64.7) format ($F(407{,}1) = 3.38, p = .067, d = .18$). Although it was not possible to obtain prior mathematical ability, the two course formats had no significant differences between the GPA's, number of college math courses taken, and highest high school math taken for the students, indicating that the prior mathematical ability among the two course formats were roughly equal. This information coupled with the reported higher cognitive learning gains for math concepts ($\beta = -.48, t(285) = -4.25, p < .001$) for the FL students, indicates the FL format was beneficial for student learning. Future studies should examine if the increases in collaboration and confidence for FL students will translate to better knowledge of higher level mathematical concepts, since we were only able to assess lower-order mathematical thinking on final exam multiple choice items.

Conclusions and Future Studies

Results from this study are promising for the future implementation of flipped style learning in undergraduate mathematics education. Students generally respond positively to flipped classroom learning experiences, and as a result show increased gains in confidence and willingness to collaborate with others in solving mathematical problems. In addition students show modest gains in mathematical knowledge. These positive trends indicate that flipped learning not only does no harm, but actually benefits students academically and attitudinally.

The next phase in this study will assess the qualitative data obtained through the survey instruments as well as course artifacts in order to understand with greater richness the experiences students had throughout the course, and answer some of the questions raised through our initial quantitative analysis. We seek to understand what factors contributed to the gender disparity in preference for taking a flipped course and whether there exist gains in higher-order mathematical knowledge as a result of using the flipped format. Additionally, we will be collecting longitudinal data to assess the impact this course had on persistence in STEM fields and student performance in subsequent math courses.

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ANALYZING STUDENTS' INTERPRETATIONS OF THE DEFINITE INTEGRAL AS CONCEPT PROJECTIONS

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This study of beginning and upper-level undergraduate physics students extends earlier research on students' interpretations of the definite integral. Using Wagner's (2006) transfer-in-pieces framework and the notion of a concept projection, fine-grained analyses of students' understandings of the definite integral reveal a greater variety and sophistication in some students' use of integration than previous researchers have reported. The dual purpose of this work is to demonstrate and develop the utility of concept projections as a means of investigating knowledge transfer, and to critique and build on the existing literature on students' conceptions of integration.

Key words: Definite integral, Knowledge transfer, Physics, Knowledge in pieces

This article, rooted in Wagner's (2006) transfer-in-pieces framework, considers the problem of knowledge transfer from mathematics into physics, although the implications extend to other disciplines as well. A distinguishing characteristic of this perspective is that knowledge flexibility and transfer at all levels of expertise are supported not by a purely abstract quality of the knowledge in question, but by its ability to adapt to and accommodate contextual differences. In this sense, knowledge is said to be context sensitive. Wagner (2006, 2010) argued that applying a single mathematical principle or concept across a variety of contexts, for example, may require the knower to construct a variety of collections of knowledge resources known as *concept projections*. By this means, seeing and using the "same" concept in different circumstances requires the use of different (though perhaps overlapping) combinations of knowledge resources.

Because the definite integral lends itself to a variety of different conceptual interpretations, it is a rich area for the study of knowledge flexibility and transfer. A recent series of papers by Jones (2013, 2013/2014, 2015a, 2015b) categorized students' conceptions of the definite integral and argued that different conceptions are more productive than others in the study of physics. The purpose of this paper is twofold. First, using interview data of both beginning and upper-level undergraduate physics students, it will demonstrate the utility of a concept projection as a theoretical construct for knowledge flexibility across levels of expertise. Second, it will expand on Jones' work by examining data revealing both novices' and upper-level physics students' understanding of the definite integral. These findings suggest how concept projections might function to support expert understanding, and point toward opportunities for additional research.

Background: Physics Students' Use of Mathematics

Challenges in transferring mathematics into physics

Both physics and mathematics educators have long observed that even students who have a considerable background in mathematics do not readily use it or apply it in the context of learning physics. Researchers have documented students' challenges in applying ideas from calculus (Christensen & Thompson, 2010; Cui, Rebello, & Bennett, 2006, 2007; Doughty, McLoughlin, & van Kampen, 2014; Nguyen & Rebello, 2011a, 2011b), trigonometry (Ozimek, 2004), and algebra (Torigoe & Gladding, 2011) to concepts and problems in physics. Although some of these researchers have pointed to deficits in students' understanding of mathematics, Yeatts and Hundhausen (1992) and more recently Dray, Edwards, and Manogue (2008) have suggested that students' difficulties result from a "mismatch" or a "gap" between what is taught in mathematics classrooms and what students actually need to use in their study of physics.

Students' understanding and use of the definite integral

A large portion of the research on students' understanding of the definite integral has revealed the limitations in their understanding even after completing a several semesters of calculus. Ferrini-Mundy and Graham (1994) showed the fragility of a student's concepts of integration and other topics in calculus that more recent research continues to find. Most students, it would seem, complete a course in integral calculus with some degree of proficiency in evaluating a definite integral and some knowledge of its applicability to computing the "area under a curve," but students' overall knowledge of procedures, definitions, and underlying concepts are often weak and disconnected (Grundmeier, Hansen, & Sousa, 2006; Mahir, 2009; Rasslan & Tall, 2002).

Evidence suggests the primary interpretation that many students place on the definite integral is an area under a curve (Bezuidenhout & Olivier, 2000; Jones, 2015b), which may limit students' ability to apply integration to other contexts (Sealey, 2006; Jones, 2013, 2015a). Increasingly, researchers have argued that interpreting the integral as a Riemann sum, a sum of (infinitesimal) products, or an accumulation is advantageous to understanding how to use and apply integration to contexts outside of mathematics (e.g., Doughty, McLoughlin, & van Kampen, 2014; Jones 2013, 2015a; Nguyen & Rebello, 2011a, 2011b; Sealey, 2006, 2014; Thompson & Silverman, 2008). These results have emerged in parallel with a growth of research directed toward supporting the development of such understandings in students (Carlson, Smith, & Persson, 2003; Doughty, McLoughlin, & van Kampen, 2014; Engelke & Sealey, 2009; Kouropatov & Dreyfus, 2014).

A recent series of studies by Jones (2013, 2015a, 2015b) has documented a variety of interpretations and understandings that students use to make sense of integration, the definite integral, and its notation. In particular, he found that the most frequent interpretations of the integral used by students could be categorized as *area under a curve*, *antiderivative*, or *multiplicatively based summation* (Jones, 2015b; see also Jones 2013), and of these, the area and antiderivative interpretations were by far the most common. Jones (2015a) further argued that the multiplicatively based summation conception is more productive for sense-making in applied contexts. In this paper, Jones' (2013, 2015a, 2015b) research is used as a basis for critique and development. Further discussion of his work is placed in the analytical sections below.

Theoretical Framework

This study adopts a cognitive, constructivist framework rooted in diSessa's (1993) knowledge- in-pieces epistemology. Wagner (2006) took advantage of the knowledge-in-pieces framework's attention to the context sensitivity of knowledge to use it as a basis for a new understanding of transfer, transfer in pieces. He argued that, contrary to traditional approaches to transfer that presume it takes place due to some abstract nature of knowledge, transfer actually occurs as a learner develops, (re)organizes, and integrates varieties of knowledge resources to accommodate rather than overlook contextual differences.

Wagner (2006) described a concept projection as "a specific combination of knowledge resources and cognitive strategies used by an individual to identify and make use of a concept under particular contextual conditions" (p. 10; see also diSessa & Wagner, 2005; Wagner 2010). From this perspective, recognizing or using a concept (such as a definite integral) in a particular context requires an individual to engage a specific collection of knowledge resources, but the makeup of that collection of resources may vary when the same individual makes use of the same concept in a different contextual situation. The current work takes the applicability of concept projections further by showing that a single individual may use different concept projections in order to "see" or interpret different manifestations of the same concept, in this case, the definite integral, in a single context.

Methods

Student Participants

Students who took part in this study were enrolled in a large public university using a quarter system of eight-week terms. Volunteers included eight beginning students enrolled in an introductory calculus-based physics course focusing primarily on classical mechanics and seven third-year physics majors who had already completed two terms of multivariable (vector) calculus, and at least one additional course in advanced mathematics. For ease of presentation, students will be referred to by a letter and number combination, with beginning students identified as B1- B8 and upper-level students as U1-U7.

Interviews

Students were interviewed individually by the author every other week during the course of an eight-week term, with each interview typically lasting 45-60 minutes. Questions and problems involved conceptual and procedural aspects of integration, differentiation, and other aspects of calculus, some in purely abstract mathematical form, and others in applied contexts. Each segment of an interview typically began with a written question or problem that the student was asked to read aloud, and the student was then asked to respond, thinking aloud as much as possible and explaining his or her thinking as clearly as possible. Further questioning was open-ended and free-flowing. Except in rare circumstances, the interviewer avoided taking on an instructive role. The interviews were audiotaped and videotaped using two cameras and an additional audio recorder.

Data analysis

Analysis of the data took part in stages, using primarily qualitative methods. For the present research, the student's responses and explanations were analyzed and classified according to the type of interpretation of the integral that the student used. In the transfer-in-pieces framework, these categories of interpretation were understood as constituting classes of concept projections. Careful attention was given to students' reasoning strategies and their patterns of use, particular use of language and gesture, the use of intuitive and naive knowledge, and changes and patterns of reasoning across contexts to infer marked characteristics of the different concept projections students used to interpret definite integrals. The goal was not to attempt to specify the entire make-up of any single concept projection, but, by highlighting characteristic knowledge resources that constitute a particular concept projection, to infer differences in concept projections used by an individual under different contextual circumstances, as well as differences in concept projections used by different individuals.

Students' Concept Projections for the Definite Integral

Jones (2013) examined undergraduate students' conceptions of the definite integral and found three principal ways that students interpreted integration and corresponded to normative reasoning. He named these *perimeter and area*, *function matching*, and *adding up pieces* (*multiplicatively based summation*). Jones (2015b) later added one more: *average*. In this paper, I will interpret these categorizations as distinct classes of concept projections. Due to constraints on length, I focus on only two.

The integral as a measure of change

Jones' (2013) *function matching* category for students' reasoning was later identified as an *antiderivative* conceptualization in Jones (2015b). Under his analysis, this interpretation refers to students' perception of the integral as a process of finding an "original function" from which the integrand was derived through differentiation, followed by a process of evaluating the difference of the original function's values at the two endpoints of integration. Although Jones (2015b) permitted the possibility of "a modest layer of meaning" in students' antiderivative conceptualization, he appeared to suggest that this represented a deficient conceptualization:

> However, what is striking is the high prevalence of the anti-derivative conceptions for the definite integral, when anti-derivatives do not actually compose the underlying meaning of the definite integral. It is simply a tool used for calculation purposes à la FTC. (p. 9)

Although it is quite possible that some students hold understandings of the definite integral that support very little sense-making beyond the procedural, the current study found a number of students who made entirely good conceptual and contextual sense of the use of the definite integral to retrieve an "original function."

The RPM Problem
The durability of a car engine is being tested. The engineers run the engine at varying levels of "revolutions per minute" for a period of time. Denote the number of revolutions per minute at time t by $R(t)$. Interpret the following:

$$\int_0^{600} R(t)\,dt$$

Figure 1. *The RPM Problem* (adapted from Jones, 2013). The statement of the problem inadvertently omitted units in which time was measured. In all cases, students either made an arbitrary choice of units, or they were told to assume that t was measured in minutes.

All of the students in this study were asked to consider the RPM Problem, shown in Figure 1. Beginning student B2 quickly concluded that the integral would determine "how many revolutions there were between time 0 and time 600." He offered the following explanation:

B2: R(t) would be the change in revolutions, in change of time, and if I were to integrate that-. Like that's a form-, it's like a derivative of some function. And if I were to integrate that it would just become a function that was the revolutions rather than the change of revolutions in-, per minute, for example. Revolutions per minute indicates that it's like a ratio of revolutions and minutes. So whatever the integral of this is, it's going to be just an equation that gives you revolutions. And if you were to plug in these values, 600, you would get how many revolutions there were at time 600. And then if you subtracted off revolutions at 0, you would get how many revolutions there were between these bounds.

The student's explanation stands in contrast to Jones' (2015) assertion that "anti-derivatives do not actually compose the underlying meaning of the definite integral." To the contrary, B2 had constructed a more sophisticated antidifferentiation concept projection for the definite integral that allowed him to interpret the antidifferentiation process in a conceptually meaningful manner. His concept projection was composed not only of those procedural resources identified earlier, but also of additional interpretive resources. In addition to understanding differentiation and integration as reversible procedures, he invoked resources that enabled him to reverse the interpretation of the derivative of a quantity as the rate of change of that quantity. In this way, he could conclude that integrating turned a rate of change of something ("the change of revolutions in-, per minute") into a function that gave an amount of that something. Further resources allowed him to interpret the substitution and subtraction procedure as a means of finding how much that something changed over the interval of integration.

In a pure math context, or faced with a need simply to evaluate an integral, B2 would probably not need to invoke all of the resources at his disposal. But in a contextually meaningful situation, he gathered a rich collection of interpretive resources to construct a concept projection that gave meaning to the function matching procedures. B2 was not alone. Among all the participants of this study, three beginning students and two upper-level students all demonstrated an ability to use an equally meaningful concept projection for the antidifferentiation process. Although these concept projections are elaborations on *function matching*, they may deserve to comprise a class of their own, which I call *integration as change*.

The example of B2 offers a nice opportunity to highlight the context-sensitivity of concept projections. Although B2 (and others) could be observed using an *integration as change* concept projection in some circumstances, both B2 and another student could not use it to interpret integrals whose integrand was not known to them as a derivative or a rate of change. When asked to consider an integral whose integrand was a position function, B2 concluded that the integrand made no sense because there had to be "a change in something else." He concluded, "I don't really know if that correlates to real life." Similarly, another beginning student who showed herself able to use an *integration as change* concept projection in some circumstances denied that one could integrate a force function, because she believed one could integrate only "a function that gives a rate," and she did not perceive a force as a rate. Such examples demonstrate the delicate relationship between mathematical knowledge and contextual understanding.

One should not conclude, however, that an *integral as change* concept projection is somehow poor or deficient. It serves a perfectly fine purpose in appropriate contexts, and it demonstrates that students who may appear to be using a purely procedural function matching interpretation of the definite integral may well be trying to engage in meaningful sense-making.

The integral as a weighted average

Jones (2015b) added the notion of an integral as an *average* to his earlier list of categories for integral interpretations. In this conceptualization, the integral is interpreted "as a process that takes a non-uniform function and 'smooths' it out over the domain to make it as though it were uniform" (p. 11). The uniform function represents the average value of the integrand over the interval of integration, and the value of the integral is the area under the graph of the average value over the same interval. Jones's data did not permit him to investigate the basis for this conception, however, and he initially framed it as potentially in conflict with "the *underlying meaning* of the integral," and he suggested that it might be the result of "interesting circular reasoning" (p. 12, emphasis original).

One of the upper-level students in the present study, U2, also demonstrated an ability to conceptualize the definite integral using an averaging process, and, over the course of several interviews, he offered substantial explanations for his understanding that are not rooted in circular reasoning. I argue that he developed a *weighted average* concept projection for the definite integral that is entirely sensible, based on variation of a *sum of products* concept projection, and, perhaps surprisingly, powerful in its ability to enable him to make intuitive sense of an integral that is otherwise quite difficult to interpret. A thorough analysis of U2's understanding of integration requires lengthy and detailed investigations of his thinking over several problems and several interviews. What I provide here will necessarily be an insufficient summary of that analysis.

In U2's discussion of the definite integral in the abstract (ie, apart from a specific problem or application), he demonstrated an ability to interpret the integral as sum of products. He resisted the notion of the differential as an "infinitesimal," however, and so he preferred to avoid the language of products, concluding instead that "you're doing a two dimension sort of summing that "coupled" the integrand with the differential in a way that measures how "present" the values of the integrand are over the interval of integration. Several times throughout the interviews, he referred to the "presence" of the integrand, usually using the gesture of air-quotes to highlight his use of the word. In further discussion, U2 proposed the following:

> U2: I wonder if it could be explained with like a weighted average kind of, where you are weighting each number in the range by its, kind of like, "presence" [indicates air quotes] in the range, where its-, each number has an infinitely small presence in the range of this like weighted average. But we can still compute it, because integrals allow us to do that.

U2 used the term *range* to refer to all the possible values of the integrand over the interval of integration. When I asked him what, specifically, was doing the weighting, he replied, "the weight of each value is the width of that value," and referred to a graph of a Riemann sum area approximation, clearly indicating that the "width" referred to the rectangles, or the role of the differential in the integral. He superimposed a horizontal line over an existing graph of a non-constant function, calling it the average value of the function, and concluded, "if you just find the area of this shape [shades the rectangular part], you would have the area of the original shape." This is precisely what Jones (2015b) reported observing with some of his students.

From a mathematical perspective, the overall argument that U2 made is entirely sensible, if accompanied by some clarifying detail. Technically, for example, finding the (weighted) average of the integrand requires weights equivalent to the width of the subintervals divided by the width of the interval of integration, but since the actual average is never found, U2 never approached the question at that level of detail. The point is that U2 constructed a concept projection for the integral that permits a way to imagine how the value of the integral is found (not simply, as Jones suggested, how to interpret its answer). In a follow-up discussion, U2 also indicated that he was aware that the actual value of the average value of the function is never actually found or needed, rather, "It's kind of like a conceptual tool…, so there is no average that ever happens."

U2's developed notions of interpreting the integral as measuring the "presence" of values of the integrand were pervasive throughout his interviews. He spontaneously explained at one point, that if one were to integrate the constant function $f(x) = 7$, the result could be interpreted as “the sevenness that's done between these bounds.” He reaffirmed it: “Its seven-ness. That makes sense to me.”

This way of looking at the integral is not simply novel; it actually has some power behind it. I asked all students in this study to consider the meaning of an integral of a position function over an interval of time. In practice, this integral has no common interpretation, and I was primarily interested in whether or not students could deduce its units. Nonetheless, U2 was the only student who was able to give a rather insightful interpretation of the integral:

> U2: So the units of your integral are going to be distance times time, since you're integrating over time. And so [...], so I guess, yeah, my brain can't interpret the physical-. I'm trying to think of like a real world problem that would do something like this, and, I don't know, like, "awayness," [uses air quotes] like, you wanted to figure out how far a particle was from a location where both distance and time are important.

U2’s interpretation of the integral as a measure of “awayness” clearly comes from his weighted average concept projection. He paralleled his language of “sevenness” used above, and he reintroduced the air quotes he used in the past when he spoke of the integral as a measure of the “presence” of the integrand. His concept projection allowed him to offer the only conceptual interpretation of the integral of distance with respect to time suggested in this study. To my eyes, it is a lovely interpretation, capturing, as he noted, the sense that, when one is away, “both distance and time are important.”

Discussion

Concept projections offer a way to consider how a variety of different meanings can be (are!) constructed to interpret and make sense of integration. Which meaning is most advantageous or useful to any individual in any particular circumstance can and will vary. In many cases, for both students and experts, interpreting an integral using an *integral as change* concept projection is entirely sufficient, without any appeal to Riemann sums. It is a simpler, perhaps more direct, way of making sense of integration through antidifferentiation, it holds up mathematically, and it appeals to the meaning of the antiderivative. Furthermore, using a *weighted average* concept projection is also a legitimate way to see meaning in the integration process, and one that carries its own interpretive advantages. It is true that nowhere in the process of integration does one actually find or use the average value of the integrand. It is, as U2 indicated, a conceptual tool. But it is equally true that nowhere in the integration process carried out through antidifferentiation does one actually find rectangles, infinitesimals, or sums. Riemann sum interpretations are also conceptual tools. There is no single “meaning” of the definite integral.

There are ongoing efforts being made to expand opportunities for students to learn more conceptually sophisticated interpretations of the definite integral, particularly Riemann sum-based interpretations. I believe that thinking of the educational task at hand as supporting students in developing a variety of concept projections for the integral can be helpful in developing and measuring the success of these efforts. The repeated message that comes through research based on the knowledge-in-pieces and transfer-in-pieces frameworks is that contextual differences that experts have learned to think about as irrelevant, only appear irrelevant after engaging with them enough to construct the cognitive resources required to accommodate them.

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STUDENTS' OBSTACLES AND RESISTANCE TO RIEMANN SUM INTERPRETATIONS OF THE DEFINITE INTEGRAL

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Students use a variety of resources to make sense of integration, and interpreting the definite integral as a sum of infinitesimal products (rooted in the concept of a Riemann sum) is particularly useful in many physical contexts. This study of beginning and upper-level undergraduate physics students examines some obstacles students encounter when trying to make sense of integration, as well as some discomforts and skepticism some students maintain even after constructing useful conceptions of the integral. In particular, many students attempt to explain what integration does by trying to interpret the algebraic manipulations and computations involved in finding antiderivatives. This tendency, perhaps arising from their past experience of making sense of algebraic expressions and equations, suggests a reluctance to use their understanding of "what a Riemann sum does" to interpret "what an integral does."

Key words: Definite integral, Riemann sums, Knowledge transfer, Physics

Researchers have argued that the Riemann sum-based interpretation of the definite integral is perhaps the most valuable interpretation for making sense of integration in applied contexts, particularly physics (Doughty, McLoughlin, & van Kampen, 2014; Sealey, 2014). Generally, a "Riemann sum-based interpretation" refers to imagining the definite integral as a sum of products, in which one of the factors is an infinitesimal or a "very small amount." Despite the utility of Riemann sum-based interpretations, many students do not develop such reasoning in their calculus courses, despite having studied Riemann sums (Jones, 2015b). It is likely that at least part of the reason for this situation is that Riemann sum-based interpretations are not generally emphasized in traditional calculus classrooms, where procedural methods and "area under the curve" ideas dominate (Jones, 2015b). I argue here, however, that there are also psychological factors at work that can interfere with students' ability to adopt Riemann sum-based reasoning, and that even among students who do adopt such reasoning, these factors continue to lead students to doubt its legitimacy.

Background

Earlier research has shown that what typical students do know about integration when they complete a course in integral calculus is often confined to the procedural knowledge necessary for solving certain integrals symbolically, while their conceptual understanding is limited to interpreting the definite integral as an "area under a curve" (Bezuidenhout & Olivier, 2000; Jones, 2015b). Nevertheless, evidence suggests that area interpretations are often not the most useful for using the integral in applied contexts (Sealey, 2006; Jones, 2013, 2015a). Furthermore, although many students do not have a well-developed understanding of the definite integral as a sum, several researchers have argued that a Riemann sum interpretation is perhaps the most useful to support students in making sense of many applications of integration (Doughty, McLoughlin, & van Kampen, 2014; Jones 2013, 2015a; Nguyen & Rebello, 2011a, 2011b; Sealey, 2006, 2014; Thompson & Silverman, 2008). This situation exists even though most traditional calculus textbooks and curricula in the United States include presentations of

Riemann sums and their limits, as well as numerical methods for approximating areas based on a Riemann sum understanding.

Some have suggested that there is a mismatch between what is emphasized in traditional calculus curricula and what is actually most useful to students in applications to disciplines outside of the mathematics classroom (Dray, Edwards, & Manogue, 2008; Yeatts & Hundhausen, 1992). I think that this is likely to be true. I argue in this paper, however, that learning to interpret the definite integral using a Riemann sum conception offers particular psychological challenges to students that may, in fact, be exacerbated by traditional pedagogical approaches. In other words, I claim that the problem is not merely a pedagogical one, but also a psychological one rooted in students' inability to reconcile a Riemann sum interpretation with the symbolic manipulations involved in computing the value of a definite integral using the Fundamental Theorem of Calculus.

Interpretive framework

Although students and experts alike commonly use an "area under a curve" interpretation of the definite integral, it is important to note that this is really a means of interpreting what the value of the integral might represent. Riemann sums, however, can be used as part of a process through which that value is found. Of interest to this research is how students understand the process of using Riemann sums, and how they understand the process of integration. That is, from a student's perspective, what do these two processes do and what do they have to do with each other?

What does a Riemann sum do?

Under standard definitions of the embedded symbols, the definite integral can be expressed (or defined) as a Riemann sum:

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

As a mathematical, algebraic statement, the Riemann sum representation of an area as a combination of rectangular areas has explanatory sense built into it, in that a meaningful algebraic and geometric process for finding area can be mapped onto the symbols of the expression, and the process can be directly modeled and investigated. The area of each rectangle is computed by multiplying its height and width, and is represented by $f(x_i)\Delta x$. The summation indicates that all the rectangles are to be added together and combined into a total area. The limiting process mathematizes the notion of letting the width of the intervals get "very small" (or, equivalently, letting the number of rectangles increase without bound). Furthermore, the fact that the limiting process has a completion point, the limit itself, supports the notion of the rectangles themselves becoming "infinitesimally thin." In short, the Riemann sum process algebraically does what it says it does. There is a clear way in which algebraic and geometric meaning can be mapped on to the algebraic syntax.

What does a definite integral do?

The situation with regard to the definite integral, however, is quite different from the perspective of a student who considers a definite integral to be the means of calculating an area using antidifferentiation. Although the symbols used in expressing a definite integral lend themselves to being interpreted as the sum of the products of lengths and widths of rectangles, the actual process of computing the definite integral is entirely different. Consider a simple example:

$$\int_0^2 x^3 dx = \frac{1}{4}x^4 \Big|_0^2 = \frac{16}{4} - 0 = 4$$

In this case, what the integral "does" is transform the integrand, x^3, into its antiderivative, $\frac{1}{4}x^4$, through a process that cannot be subjected to algebraic sense-making. The power rule for finding such an antiderivative can be readily proven, of course, but the computation takes advantage of a known pattern associated with antiderivatives of polynomials, and the algebraic manipulation is immune to any sort of geometric explanation or metaphorical algebraic interpretation, except perhaps in the simplest cases. In addition, no actual "summation" of any sort takes place, nor does anything "get very small," and the differential *dx* appears simply to be extraneous and to evaporate in the solution process. (Indeed, a number of the beginning students wondered aloud why the differential was used at all.) Riemann sum-based reasoning may fit the syntax of the original integral expression, but it cannot be used to explain or extract meaning from the computational process.

The central thesis of this paper is that the algebraic solution process for finding an area as a limit of a Riemann sum is inherently different from the solution process for finding an area through the computation of an antiderivative, and that this difference can cause varying levels of confusion and puzzlement to students. At the very least, nothing would suggest that importing a Riemann sum-based explanation into the process of integration via antidifferentiation ought to be automatic or natural for students, at least, not for students who are accustomed to trying to make sense of their mathematical activities. Nothing they do when computing a definite integral is at all related to Riemann sum-based ideas. I am not aware of any other circumstances in typical mathematics curricula prior to the study of calculus that asks students to use the semantics of one mathematical process to interpret the syntax of another. The purpose of this research is to document how this conflict between syntax and meaning is manifested in the reasoning used by undergraduate physics students.

Methods

Data for this study are taken from extensive interviews between the author and individual undergraduate physics students. Eight beginning students were selected from among volunteers in an introductory calculus-based physics course, focusing primarily on classical mechanics. These students, representing a variety of majors, had all completed at least one academic quarter each of differential and integral calculus. Seven upper-level students were also selected from a third-year cohort of physics majors, each of whom had completed two quarters of multivariable (vector) calculus. Students were interviewed using open-ended interview methods about every other week during the course of an eight-week term. All students completed at least two interviews, and most completed four. The questions presented to students involved primarily problems and concepts of calculus, both in abstract mathematical form and in applied problem contexts. The interviewer asked students to answer questions and solve problems, sometimes using mechanical devices, thinking aloud as best they could. Additional questioning continued until the interviewer believed he understood the reasoning the students were using, but, in general, the interviewer refrained from evaluating the students' ideas or offering instruction. All interviews were audio- and video-recorded for later analysis. The portions of the interviews transcribed for and relevant to the present research all involved students' use and interpretation of definite integrals in both abstract and applied contexts.

Searching for meaning in integration

A thorough presentation of the data requires a careful and extended analysis of a large number of lengthy interview transcripts. In the limited space allotted here, I offer only summaries of the central findings of this work, absent the detailed transcripts that support these findings. A more comprehensive paper is in preparation.

At the time of the interviews, none of the beginning students demonstrated an ability to use Riemann sum-based reasoning to interpret the definite integral. All of these students, however, had studied both Riemann sums and definite integration. When asked about the relationship between the two topics, many students indicated an awareness of some relationship, but none could articulate it. About half identified them as two different ways of finding the same thing, an area. The other half focused on Riemann sums as a means of *approximating* an area, while a definite integral could find it *exactly*. As such, integrals were preferred, and Riemann sums were invoked only as a last resort when an integral could not be directly computed. When asked *why* solving a definite integral through antidifferentiation found an area, none of the beginning students (and only one of the upper-level students) could answer the question.

Most of the beginning students, either spontaneously or under direct questioning, indicated that their knowledge of Riemann sums did not in any way inform their understanding of the definite integral. Throughout the interviews, however, many students showed evidence of searching for meaning in the procedures used to compute definite integrals through antidifferentiation and the Fundamental Theorem of Calculus. This search for meaning arose in at least three different aspects of the solution process.

Searching for meaning in symbol manipulations

I described above how the algebra of Riemann sums directly supports algebraic and geometric sense-making for finding areas. Several students in this study showed evidence of looking for or expecting similar sense in the symbolic manipulations of the antidifferentiation process. These students, for example, questioned why the power rule for finding the antiderivative of a polynomial should result in a function that gives area, or why that same rule used in an applied context resulted in a change of units between the original function and its antiderivative. They specifically noted an expectation of meaning to be found in the symbolic manipulations:

> I don't know why, like, bringing up a constant in the exponent, or whatever you have to do to solve it … I don't know why that means that it's now revolutions instead of revolutions per minute, if I was integrating revolutions per minute.

Even upper-level students appeared to expect to find sense in the symbolic manipulations. One upper-level student, also discussing the power rule, suggested that there should be a geometric explanation for it:

> I know it's the power rule, but I guess they never showed me why behind the power rule, or like, the visual, a connection between the graph-.

The data lead me to conclude that some students are treating the symbolic manipulations of antidifferentiation as though they are algebraic manipulations, and should be subject to algebraic interpretation and sense-making. Both physics and mathematics educators alike have emphasized the importance of developing conceptual understanding to underlie algebraic skills (Kieran, 2007; Sherin, 2001), but we have not yet directed attention to the possibility of students' subsequent search for such meaning in the context of integration and antidifferentiation techniques where it cannot be found in the same way.

Searching for meaning in substitution procedures

Some students showed evidence of finding significance in the substitution procedures using the limits of integration at the end of the integral evaluation process. The appeal to the substitution procedures typically arose as students were trying to explain why the units of an integrand changed in the process of integration; for example, why the antiderivative with respect to time of a function with units of revolutions per second was a function with units of revolutions. A student using Riemann sum-based reasoning could appeal to a cancellation of units between the integrand and the differential. Without such Riemann sum-based reasoning skills, these students appealed, instead, to the substitution process as the source of the change in units. For these students, the antiderivative, in itself, essentially maintained the units of the original integrand. They reasoned, however, that during the substitution procedure, the units of the limit were inserted into the antiderivative, thereby resulting in a change of units. One student gave an extensive explanation for this and maintained his reasoning for the final units of "revolutions" under repeated questioning: "Because you're adding in the time component. You're substituting in." He argued that, during the substitution process (and only then), the units of the original integrand cancelled with the units of the limits of integration.

Searching for meaning in the geometry of the antiderivative

In their attempt to explain why a definite integral could be interpreted as an area, some students sought geometric structure within the algebraic form of the antiderivative. They reasoned that, since finding an antiderivative is necessary for finding the area under the curve, there ought to be a way to uncover the area calculations within the algebraic structure of the antiderivative. This is, in a sense, an attempt to construct a direct parallel to finding the underlying geometric structure within a Riemann sum, where heights, widths, and sums of rectangles are all represented in the algebra.

One beginning student made a considerable effort to deconstruct an antiderivative algebraically in an attempt to match its algebraic structure to her graph of the area she knew it was used to find. In the end, she exhibited some satisfaction in mapping her calculations to two area regions, the difference of which gave her the final answer. She could not, of course, explain why the algebra produced the correct areas, and she quickly realized that she still could not explain why the antiderivative process should be used at all. At that point, she returned to a written expression of the power rule, appeared to try to make sense of why it should yield an area, and soon gave up: "I don't know."

Recall that none of these students ever invoked Riemann sum-based reasoning to interpret an antiderivative, and most gave convincing evidence that they could not do so. In these circumstances, attempts to explain why integration does what it does led students to bring out the only other tools they had at their disposal: algebraic and geometric reasoning tools that serve well in other circumstances. What was lacking was an awareness that antidifferentiation procedures are not subject to algebraic reasoning.

Skepticism of Riemann sum reasoning

In contrast to the beginning students, all of the upper-level students demonstrated competence in using Riemann sum-based reasoning to interpret definite integrals. They all, in fact, used it quite well for both abstract mathematics problems and contextually rooted physics problems. Nonetheless, a number of the students clearly and repeatedly expressed skepticism in the validity of using such reasoning.

One student interrupted his otherwise clear Riemann sum-based explanation for an integral he set up with a humorous expression of embarrassment:

> Yeah, I do it. I don't-, I'm not proud of it, but I hope there's some way to justify it.

Asked to explain his comment, the student indicated that such reasoning seemed to him to be "kind of a trick," but that he could not justify it mathematically, and he did not know if it *could* be justified mathematically. He seemed particularly troubled by interpreting a differential as an infinitesimal, calling such an identification "hokey."

Another upper-level student, equally skilled at demonstrating Riemann sum-based reasoning when asked to do so, went out of his way to *avoid* using such reasoning. His explicit reason was that the Riemann sum explanation did not reflect what integration actually does. He observed, correctly, that integration via antidifferentiation involved a function transformation, but that this transformation took place through an entirely mysterious process that was not subject to sense making:

> Like, it's impossible to actually accurately explain what this integral is conceptually. It's impossible to do it…. It's not possible … to talk about an infinitesimal volume and an infinitesimal density. That doesn't make sense.

This student's case is particularly striking because, unlike the beginning students, he understood that the antidifferentiation process could not be subjected to algebraic or geometric interpretation, but he equally rejected a Riemann sum-based explanation because he could not accept that one could reason sensibly about infinitesimals. What all of these students have in common, however, is an inability to reconcile reasoning about Riemann sums with the actual computational process of calculating a definite integral through antidifferentiation.

Discussion

A number of researchers in both physics and mathematics education have observed that many students, even those with a strong calculus background, fail to use Riemann sum-based interpretations of the definite integral, despite its unique value to supporting sense-making in many applied contexts. There can be little doubt that part of the reason for students' unfamiliarity with such reasoning process is that they are not given much emphasis in traditional calculus curricula. What I hope this research demonstrates, however, is that addressing this situation is more complex than it may first appear.

I am arguing that there are psychological explanations for why students do not quickly pick up Riemann sum-based reasoning, and why such reasoning may seem puzzling or suspect to them even when they have been taught to use it. I hope it is clear that this paper should not be interpreted as another exposition of "student deficits." To the contrary, the heart of the argument is that most of the students who took part in this research were actively trying to make sense of the mathematical activities that make up the integration process. The problems they ran into, however, exist because of the peculiar marriage that must take place between the reasoning of Riemann sums and their limits, and the algebraic symbols and symbolic manipulations that represent the process of integration by means of antidifferentiation. At face value, there is no obvious reason that students can find for using Riemann sums to interpret antidifferentiation procedures that not only appear to be, but actually are, algebraically unrelated to the complex limit and summation procedures they have learned for Riemann sums. The algebra of Riemann sums readily supports reasoning about area computations; the procedures on which the Fundamental Theorem of Calculus is based do not. We should not be surprised that students question the validity of using the reasoning for one to interpret the computational results of the other. If they are to be successful, increased attempts to introduce students to the use of Riemann sum-based reasoning will need to accommodate these peculiar psychological hurdles that students will encounter.

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Enriching Student’s Online Homework Experience in Pre-Calculus Courses: Hints and Cognitive Supports

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Abstract: As part of reforming our Pre-Calculus courses, we realized that reforms to instruction needed to be accompanied by reforms to the homework. We utilized the online homework system WeBWorK but recognized our students wanted more support on missed questions. WebWorK “hints” provided us an avenue to ask students leading questions to prompt thinking over procedures. Preliminary data show many students are using these hints and the hints are working as intended. We plan to expand hints beyond our Pre-Calculus courses. The open source nature of WeBWorK provides an opportunity for hints to be implemented on a wide scale.

The Department of Mathematics at the University of Nebraska-Lincoln (UNL) in the midst of reforming high-enrollment (first-year) mathematics courses. The reforms include implementing an active learning model for instruction; common activities, exams and lesson plans; and a blended course format using the WeBWorK online homework system to supplement in-class instruction and activities.

The reform efforts began in fall 2012, and the first attempts prompted much more extensive involvement by faculty and a more comprehensive research study. The extra support of instruction and additional data allowed us to both experience greater success and better understand the positive contributors to the success. While the levels of success have increased substantially (from 62% to 80%), there is still room for improvement. During focus group interviews of students in fall 2014, the biggest complaint students had is the way the online homework system works.

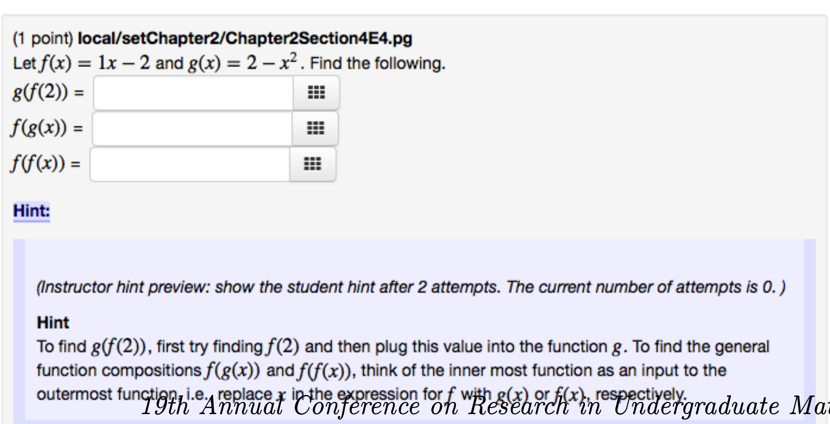
In 2005, Hauk and Segalla conducted an extensive study of student perceptions of web-based homework using the WeBWorK online homework system. They found

> as a facilitator for engaging in mathematical self-regulation WeBWorK is involved only as a monitor for correctness…the web based tool does some monitoring but the responsibility for metacognitive control (response to the monitoring), problem-solving heuristics, and the impact of mathematical beliefs rests on the student. (p. 241)

Thus, while an effective tool, WeBWorK lacks in a key aspect of the triadic reciprocity proposed by Bandura (1986) and modified by mathematics educators (e.g., Cohen, Raudenbush, & Ball, 2002). WeBWorK lacks the environmental interaction with a subject expert to provide the cognitive apprenticeship. This project aims to improve these interactions through direct modification of elements of the triadic reciprocity within our courses. Specifically, we attempted to improve interactions between the students and the mathematics content (WeBWorK), the students and the teachers (instructors, learning assistants and tutors), and the teachers and the mathematics content, all within the UNL context.

This poster will talk about the collection of modifications to the WeBWorK system and problems.

One benefit of online homework is that students are given immediate feedback regarding the correctness of their answer, and are allowed multiple attempts, on the exact same problem, to get a correct answer. One problematic aspect of this type of system is minor errors are treated the same as more egregious errors. Additionally, knowing an answer is incorrect almost never helps a student determine how to correctly complete a problem. Hauk and Segalla (2005) quote students as having reported “I prefer getting feedback from the professor because he could help me understand what I did wrong” (p. 244).



Thus, we attempted to leverage a new WeBWorK “Hint” feature. Our hypothesis was that using the “Hint” button would provide students with focused questions to prompt higher-order thinking about problems the student has answered incorrectly. In the figure, the hint button only works after the student has attempted the problem and input an incorrect answer. The hint also involves more than just showing students and example problem but leverages the questioning one might expect from an actual instructor.

In social cognitive theory, people are said to learn from their social environment. According to Schunk (2004), learning can occur vicariously through engagement with electronic materials. Adding a feature that allows a hint to be given will further enhance the interaction students have with the electronic materials on WeBWorK. We note that while some online homework environments (such as MyMathLab) offer hints in the form of similar problems worked out correctly, our hints are in the form of questions designed to help students think about common errors related to each particular problem. Providing student with "cookie cutter" examples increases the likelihood students will try to learn through memorizing procedures, rather than reinforce our active learning philosophy that students need to learn through higher-order thinking and making sense of mathematical problems.

After creating the hints in the summer of 2015, students were given the opportunity to use the hints in the fall of 2015. We surveyed students about their use of WeBWorK hints and recorded results from 274 students (27% response rate). Among those surveyed, 84% reported accessing the hints at least once, with responses evenly split among "more than once a week", "less than once a week" and "a few times a semester". We further asked students to report on the usefulness of the hints. Students reported "sometimes (42%)" or "rarely (37%)" finding the hints helpful. We did not find this surprising since the students accessing the hints were already stuck on the problem and may have needed more help than a few questions could provide. In fact, only 12% of the respondents stated that they never found the hints helpful. Nine percent of the respondents reported "usually" finding the hints helpful. We claim this supports the goal of creating a hint that prompts higher order thinking instead of walking students through procedures. Students often want to be walked through a procedure, and see anything short of this as unhelpful. Therefore, we do not expect students to identify the hints as "always" helpful. Our hints are designed to maintain the cognitive demand of the questions while also helping students to clear up any common misunderstandings. We thus view a successful hint as one in which some students find the hint helpful but not every student found the hint satisfactory.

All indications show that in our university online homework is here to stay. Further research needs to be conducted into how to support students most effectively. We have found the hints to be a valuable resource in supporting student thinking and plan to extend the use of hints into other courses beyond our Pre-Calculus courses. Due to the open nature of WeBWorK, other instructors using WeBWorK homework can also make our hints available to their students.

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A case for whole class discussions: Two case studies of the interaction between instructor role and instructor experience with a research-informed curriculum

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This paper presents case studies of two instructors implementing a research informed multivariable calculus curriculum. The analysis, structured around social constructivist concepts, focuses on the interactions between the roles of the instructor in facilitating student discussions and the instructors' experiences with the activities. This study is a part of an effort to evaluate and improve the project's effectiveness in supporting instructors in implementing the activities to promote rich discussions with and among students. We find these instructors to be focused on their roles as facilitators for student-centered small-group discussion and that they choose not to have of whole class discussions. We argue that initiating whole class discussions would address concerns and negative experiences reported by the instructors.

Keywords: Multivariable calculus, Active engagement, Instructor roles, Whole class discussions

The *Raising Calculus to the Surface* (RC) project uses physical manipulatives, group work, and open-ended questions to encourage students to discover multivariable calculus concepts and use different representations of multivariable functions. During lab activities, the materials are designed for the instructor to move between two roles: (a) acting as a **small-group discussion facilitator** or questioner to help students within a group resolve difficulties, and as a (b) **whole-class discussion facilitator** to help the whole class together engage in meaningful discussion about the newly discovered concepts. As a small-group discussion facilitator**,** instructors do not inject content, but by asking questions, they help groups move toward meaningful discoveries while maintaining the group's autonomy and ability to struggle meaningfully. As a whole-class discussion facilitator, the instructor shapes the discussion productively by prompting students with important questions and ideas.

In contrast to the instructor's roles of lecturer, question answerer, or answer checker, the instructor's role in these activities often requires much less active talking. This is sometimes a challenge for instructors to adopt, in part because it is so easy to focus on only one role instead of moving between roles and is uncomfortable for some lecture-first instructors. The project includes several support mechanisms for instructors, including (a) a training workshop, (b) an instructor's guide, and (c) an online support website. This paper explores the implementation of RC materials by two instructors in order to better understand how the project team can support instructors in having rich discussions with students in group and whole-class discussions.

Theoretical Perspective and Methods

Theoretical perspective

Our overall research, similar to active-learning research studies, is structured around social constructivist concepts (Ernest, 2010; Vygotsky, 1978 & 1986). This conceptual lens regards individuals and their learning environment as an interconnected unit where learning of an individual occurs both in sociocultural activity and in the mind of individuals. With this perspective in mind, all class interactions (group work, instructor communication and interaction, etc.) play an important role on the students' development of mathematical concepts. Students' in-class engagements on tasks with more knowledgeable and capable others such as the instructor and peers help them develop the mathematical concepts through communication. Thus, it is important to focus on instructors' interaction and the ways in which they communicate with their students.

In this paper, we specifically focus on how instructors enact discussions with students during student-centered activities. Both personal characteristics (e.g. knowledge of different teaching strategies and beliefs about how to support student learning) (Ball, et al., 2008) and situational factors (e.g. expectations of content coverage, class size and room layout) (Henderson and Dancy, 2007) influence such actions.

Methods

Data for this paper were accumulated from pre/post surveys for two instructors, Rickie and Janos, about their participation in the project (summer 2014, December 2014, and spring 2015 if implemented a second time), a post-survey about the training workshop (summer 2014), electronic weekly reports submitted by adopting instructors (fall 2014 and spring 2015), an evaluation interview (summer 2015), and some student data. The instructor data are in the form of responses written by the instructors on surveys and reports, an audio recording and transcript of Janos's interview, and detailed notes from Rickie's interview (no recording was made). The student data are written work and audio recordings of student groups in Janos's class working on an activity and scanned copies of Rickie's students' activity worksheets. The pre/post surveys included items about the instructors' attitudes and beliefs about student learning in multivariable calculus and items about practices. The interviews were semi-structured and included logistical questions about how the activities were used and reflection questions about how the activities went and the instructors' opinions about the geometric and contextual features of the activities.

The authors surveyed the corpus of data and took notes on information interesting to the RC project. The authors revisited the data and took additional notes focusing on how the instructors' implemented the activities and how the instructors supported and managed students' discussions. These notes were used to create descriptions of the instructors' implementation of the activities and how they supported and managed discussions during the activities.

Both instructors have characteristics for adoption: they both voluntarily chose to adopt the activities, attended an off-site training workshop, describe themselves as valuing active learning, and have in the past used active learning classroom strategies apart from the RC materials. Additionally, they both have been/are participants in a professional organization which focuses on helping mathematicians grow professionally in their early years in a tenure-track position. We choose to highlight these two instructors because, despite these commonalities, the role of the instructor in these implementations was conceived of and put into practice differently. In examining these two cases, we hope to understand how to support instructors in facilitating productive discussions across a variety of implementations.

The descriptions presented here are not an attempt at a comprehensive picture of these instructors' experiences in implementing the activities. They are focused on one aspect of the implementation: the instructors' roles during the activity in facilitating student discussions.

Additionally, these instructors were not selected to be representative of the average adopting instructor, but rather to illustrate a range of instructors included in the project.

Descriptions of Implementation for Two Instructors

Rickie teaches at a large public Master's-granting university, used the project materials in two sections of multivariable calculus in Fall 2014 and another two sections in Spring 2015. He was able to attend the full workshop, had previously taught multivariable calculus at his home institution for multiple semesters, and incorporated the project materials in both fall and spring terms with two sections each semester.

In contrast, Janos teaches at a medium private Ph.D.-granting university and used the project materials in one section of a third-semester calculus course. Janos attended only the first three hours of the nine hour workshop, had a curriculum mismatch with the project materials (multivariable functions were introduced in the previous, second semester calculus course), and hence chose to implement the materials on the last day class.

Implementation by Rickie

Rickie initially attempted to change his instructional practice to a more student-centered approach by attempting to "not talk" to the entire class and to not provide answers to students. He then expressed frustration at students missing opportunities to discuss important aspects and not reflecting on their answers. He modified the student worksheets to prompt these discussions/reflections. At the end of the second semester, he reported that he highly valued the active, social nature of his class as a result of using the activities, consistent with student-focused approach, but wished the worksheets contained extension questions that challenged students to reflect on their answers.

The curriculum is designed for students to discover mathematical content in small groups, prior to lecture. This process often requires that instructors give students the space to make and correct mistakes. Rickie conscientiously attempted to achieve this goal. After implementing the first activity, Rickie wrote that "I focused as much as possible on not talking. That is obviously difficult" (weekly report, 9/1/2014). When asked what went well in the first activity, Rickie wrote "Letting them take over their own learning. Good prep for the next lab" (wr, 9/1/2014). When asked what could have gone better, Rickie said "Too little of me helping each group. I have 9 groups making it very tough to always help. Or prompt" (wr, 9/1/2014). We interpret these comments as meaning that Rickie felt the groups generally needed assistance from him to be productive, but that he was trying to give students space to explore by not addressing the entire class and providing explanations, or "not talking," an unnatural instructional mode for him. He mentions "not talking" in his reports of the first two activities in both semesters.

The instructor expressed a desire for students "to have someone in the group question responses that were incorrect" (wr, 1/17/2015). Students are not likely trained in this 'cynic' role, but it is an important part of authentic mathematical practice. The instructor can model this by prompting or promoting student discussion during either small-group discussion or whole class discussion. By doing it with the whole class, all of the students can see how this role contributes to and shapes the discussion.

In his evaluation interview, Rickie also noted that he particularly valued the wrap-up questions on the student worksheets which challenge students' new understandings of mathematical content. On the second activity, a wrap-up question intended to create a two-sided debate between students based on relying upon (or not upon) their intuition. Done in the whole class, this question provides the instructor multiple students for each side who can argue using knowledge gained from the activity. By putting the question on the activity sheet, Rickie gained an ability to address issues with students at their pace at the small-group level. He altered the question to address the conflict between the surface's height referring to lead level and not elevation. He diverted a whole class discussion into a small group discussion. This decision removed opportunity for students to contribute to the class's collective understanding. Rickie expressed a desire to offload whole class discussions onto the worksheet on several occasions.

One benefit of employing whole-class discussion is it offers flexibility to instructors during the activity. Rickie noted on the fourth activity in the fall semester that "I think if at least one of the pos[itive] x or pos[itive] y direction was such that the slope was negative, the activity would have more 'bang' to it in terms of discussion" (wr, 9/27/2014) He also noted that "A question like 'where should you draw the gradient vector...on the surface or on the grid paper'? Do it. Why? might be nice. Better yet, have them do it on the contour map for that surface" (wr, 9/27/2014). Rickie provided this constructive feedback to the project as a way to improve the activity sheet. Mathematically, specific scenarios like the first suggestion described above cannot be guaranteed. Hypothetical questions on an activity sheet are confusing; such questions fit better in discussion. The whole class discussion allows students with different surfaces and different scenarios to share their work and find the common mathematical ideas common across cases. This sharing of knowledge across groups helps focus students on underlying math content, not studying their specific surface.

In the post-course survey, Rickie notes the project's impact on the classroom and student learning: "The social interactions are much improved in my mind. The early labs really help break the ice and make the class more lively (both during class and after hours). That alone is worth consideration for continuation" (2nd post survey, spring, 2015).

Implementation by Janos

In a manner similar to Rickie, Janos "spent most of my time answering their [the students] questions" (weekly report, 12/8/2014) in small groups and did not engage the students in a whole class discussion.

Janos worked diligently to help students address their concerns and understandings during the class period, consistent with the intended small-group discussion facilitator role described above. Audio recordings of student group discussions revealed that the instructor had meaningful discussions with students and used questioning techniques with students. He noted that as a result, "I was exhausted by the end of the class!" In a separate part of the report, the instructor noted "Running this activity took a lot of energy and I could not get to each of the seven groups quick enough" (weekly report, 12/8/2014).

We believe two aspects of the implementation contributed to the fatigue reported by Janos: focusing on his small-group discussion facilitator role and inadequate preparation (reported by the instructor). First, when moving between student groups, an instructor will encounter the same student difficulties/questions several times. Another strategy would be for the instructor to orchestrate a whole-class discussion when encountering questions/difficulties common to several student groups. This strategy has several advantages, including being more energy efficient for the instructor, providing opportunity for students in other groups to suggest or share resolutions to difficulties (promoting student authority), and allowing for a common language and consensus to develop among the groups (engaging in authentic mathematical practice).

Second, Janos used a lot of intellectual energy addressing students' questions. He reported instances of not being quite sure about how to address some of the students' questions and attributed this to a lack of preparation on his part. "I feel like I should have read the instructor [guide]. I thought that going through it once with my advanced physics kids would be enough but, um, it wasn't. I should have--later, after it happened, I went and I looked at all the different material and I thought, 'Oh, I really should have read this' because **…** they [the researchers] saw … common students' responses and questions, and those definitely came up [in my class]." (evaluation interview).

Janos mentioned having a really interesting conversation while practicing with two former students in preparation for class. He attributes the opportunity for this to happen to his role as a more equal member of the group. "I remember we started talking about functions and '1:1' and 'onto', I think that's what it was--and inverses, but with respect to, you know, in three space. And, I thought that was a really interesting conversation I had with them because it was like the three of us were in a group together, working on this together, you know. I wasn't acting like the professor guiding them; I was learning with them … but that conversation did not come up in class in any of the groups that I witnessed… I think that if … we had someone IN each group… who can kind of guide the conversation then it might have gone there, but since I was bouncing around...I wasn't able to really spend that deep time with any of the groups" (evaluation interview). One possible way to address this facilitation challenge is to allow time for a whole-class discussion. In a whole-class discussion, the instructor can participate as a more equal member of the class than when moving from group to group where his participation in each group is necessarily transient.

Discussion and Questions

These case studies describe instructors who focus on the role of small-group discussion facilitator and do not facilitate whole-class discussions. The case studies above include instances where having a whole-class discussion may result in less work for the instructor and increase opportunities for all groups to participate in important discussions. Although the workshop modeled whole class discussions and the instructors guides provide support for instructors to conduct whole class discussions, these cases suggest that this support does not go far enough. It seems that these instructors may have ideas about student-centered instruction, or about the specific RC curriculum, that do not include a role for whole class discussions. Given this backdrop, we ask the following questions:

a. What aspects of a training workshop can support instructors in initiating and moderating whole class discussions?

b. What kinds of instructor discourse promotes student autonomy in whole class discussions?

c. What impact do whole class discussions have on student learning and mathematical practices in student-centered curricula?

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Assessing Students’ Understanding of Eigenvectors and Eigenvalues in Linear Algebra

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Many concepts within Linear Algebra are extremely useful in STEM fields; in particular are the concepts of eigenvector and eigenvalue. Through examining the body of research on student reasoning in linear algebra and our own understanding of eigenvectors and eigenvalues, we are developing preliminary ideas about a framework for eigentheory. Based on these preliminary ideas, we are also creating an assessment tool that will test students’ understanding of eigentheory. This poster will present our preliminary framework, and examples of the multiple-choice-extended questions we have created to assess student understanding.

Key Words: Linear Algebra, Eigenvector, Eigenvalue, Assessment

The study of linear algebra is highly useful to students in science, technology, engineering and mathematics (STEM) fields and is often introduced in the first or second year of university. The use of linear algebra extends into upper-division university studies as well, in courses such as quantum physics. One crucial, and particularly valuable, concept encountered by students in linear algebra is that of eigentheory. As part of a larger study investigating how students reason about and symbolize concepts related to eigentheory in quantum physics (*Project LinAl-P*), we are (a) creating a preliminary framework for student understanding of eigentheory, and (b) developing an assessment to examine students’ understanding of eigentheory. These research activities go hand-in-hand because, as we strive to develop a way to measure students’ rich and nuanced understanding of eigentheory, our measurement tool (a collection of multiple-choice-extended questions) must be grounded in and aligned with a research-based framework that characterizes what it means to understand eigentheory (Izsák, Lobato, Orrill, & Jacobson, 2011).

Literature and Preliminary Framework

Although other researchers have examined students’ understanding of eigenvectors and eigenvalues (Gol Tabaghi & Sinclair, 2013; Salgado & Trigueros, 2015; Sinclair & Gol Tabaghi, 2010; Stewart & Thomas, 2006; Thomas & Stewart, 2011), a comprehensive framework encompassing and connecting the elements necessary to conceptually understand eigenvectors and eigenvalues and their uses (such as in diagonalization) does not currently exist. To begin our preliminary framework, we consulted this literature base. In particular, we drew from delineations of conceptual understanding of eigenvectors and eigenvalues through genetic decompositions (Salgado & Trigueros, 2015; Thomas & Stewart, 2011); these papers mainly focused on the mental constructs necessary to understand the standard algorithm for calculating eigenvalues and eigenvectors, rather than geometric or structural modes of reasoning. We also examined a Quantum Mechanics textbook (McIntyre, 2012) to investigate what skills related to eigentheory, such as diagonalization, were crucial to applications within that discipline.

As we progressed, we noted the compatibility of our work with that of Sierpinska (2000), who distinguished three modes of reasoning – synthetic-geometric, analytic-arithmetic, and analytic-structural – available to students in linear algebra corresponding to three interacting languages. These languages are: “the ‘visual geometric’ language, the ‘arithmetic’ language of vectors and matrices as lists and tables of numbers, and the ‘structural’ language of vector spaces and linear transformations" (p. 209). While our framework is still a work in progress, it will include delineations across: comprehending calculations involved in finding the eigenvalues and eigenvectors of a given matrix and why they work; understanding eigenvectors, eigenvalues, and eigenspaces geometrically when working within $\mathbb{R}^2$ and $\mathbb{R}^3$; using eigenvectors and eigenvalues in applications (e.g., diagonalization, long-term behavior of dynamical systems, Markov Chains); and drawing structural inferences from known information such as algebraic and geometric multiplicities. By the time of the conference, we aim to have this framework more fully fleshed out and organized into a structure useful for examining student understanding of eigentheory.

Assessment Instrument

Our current work in student understanding of linear algebra in physics draws its foundation from a larger research project in the teaching and learning of linear algebra; one research product from that project is an assessment instrument for measuring student understanding of key linear algebra concepts (Zandieh et al., 2015). This instrument contains closed-ended questions in an adapted multiple-choice format, which we call *multiple-choice-extended* (MCE). This format, which is just appearing in physics education research, is based on work by Wilcox and Pollock (2013), who adopt questions from the valid and reliable electricity and magnetism diagnostic to explore "the viability of a novel test format where students select multiple responses and can receive partial credit based on the accuracy and consistency of their selections," to allow for "preserving insights afforded by the open-ended format" (p. 1). Questions written in a MCE format begin with a multiple-choice element and then prompt students to justify their answer by selecting all statements that could support their choice. In *Project LinAl-P*, we have been working to create a MCE-style assessment instrument for measuring and characterizing students' understanding of eigentheory in linear algebra. Figure 1 contains an example of a MCE question from the pilot version of *Project LinAl-P*'s assessment instrument.

An eigenvalue of the matrix $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ is:

(a) $\lambda = 2$
(b) $\lambda = 3$

Because … (select ALL that could justify your choice)

(i) This eigenvalue is a solution to the characteristic equation of A.
(ii) This eigenvalue makes $\det(A - \lambda I) = 0$ a true statement.
(iii) This eigenvalue makes $\det(A - I)\boldsymbol{x} = \boldsymbol{0}$ a true statement.
(iv) When acted on by matrix A, all vectors in $\mathbb{R}^2$ are stretched by the amount of this eigenvalue.
(v) When acted on by matrix A, there is a line of vectors in $\mathbb{R}^2$ that are stretched by the amount of this eigenvalue.
(vi) $A\boldsymbol{x} = \lambda\boldsymbol{x}$ is equivalent to $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$, and this eigenvalue makes it possible for $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ to have a nontrivial solution.
(vii) $A\boldsymbol{x} = \lambda\boldsymbol{x}$ is equivalent to $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$, and this eigenvalue makes it possible for $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ to have only the trivial solution.
(viii) $A\boldsymbol{x} = \lambda\boldsymbol{x}$ is equivalent to $(A - \lambda)\boldsymbol{x} = \boldsymbol{0}$, and this eigenvalue makes it possible for $(A - \lambda)\boldsymbol{x} = \boldsymbol{0}$ to have a nontrivial solution.

Figure 1. Example of an MCE question from the eigentheory assessment instrument.

In Fall 2015, we interviewed two students using pilot versions of eight MCE questions, which led to minor question revisions. In January 2016, we will administer the questions in written format to approximately 20 students entering a quantum physics course. Data and analysis from both of these sources will be included on the poster.

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Support for proof as a cluster concept:
An empirical investigation into mathematicians' practice

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Abstract. In a previous RUME paper, I argued that proof in mathematical practice can profitably be viewed as a cluster concept in mathematical practice. I also outlined several predictions that we would expect to hold if proof were a cluster concept. In this paper, I empirically investigate the viability of some of these predictions. The results of the studies confirmed these predictions. In particular, prototypical proofs satisfy all criteria of the cluster concept and their validity is agreed upon by most mathematicians. Arguments that satisfy only some of the criteria of the cluster concept generate disagreement amongst mathematicians with many believing their validity depends upon context. Finally, mathematicians do not agree on what the essence of proof is.

Key words: Cluster concept; Mathematical practice; Proof

Mathematics educators agree that an important goal of mathematics education is to improve students' abilities to write proofs. Unfortunately, there is also a consensus that mathematics educators do not agree on what a proof is (Balacheff, 2002; Reid & Knipping, 2010; Weber, 2009). In a previous RUME theoretical report (Weber, 2014), I suggested that proof in mathematicians' practice might profitably be viewed as a *cluster concept* in the sense of Lakoff (1987). Essentially, this means that there may not be a precise definition that distinguishes a proof from a non-proof; rather, proof is actually a cluster of characteristics where a proof was expected to satisfy most or all of the characteristics but an argument might still be a proof if any one or two of the characteristics were not. I claimed that this had the following testable hypotheses:

(i) Mathematicians would believe that an argument that satisfied all characteristics of the proof cluster would be regarded as a proof by all mathematicians and would not be viewed as controversial.

(ii) Arguments that satisfied some, but not all, of the characteristics of the proof cluster would be viewed as controversial by mathematicians. There would not be a consensus on whether these arguments were proofs and such evaluations would be context-dependent.

(iii) Mathematicians would not agree on what the true essence of proof was.

In the studies reported in this contributed report, I specifically test whether these hypotheses were true.

Theoretical perspective

The goal of this paper is to test the viability of the theoretical perspective that proof is a cluster concept. I begin by briefly summarizing the arguments from Weber (2014). I start with the presumption made by many mathematics educators: we want our definition of proof to be descriptive and align with mathematical practice[1]. That is, the arguments that we define to be proofs must include the proofs that mathematicians actually read and write. As

[1] A mathematics educator need not adopt this presumption (c.f., Staples, Thanheiser, & Bartlo, 2012). This article offers consequences for those who accept the presumption that our delineation of proof should be accountable to mathematicians' practice, which I believe constitutes the majority of mathematics educators who are studying proof.

educators, we are not satisfied with define proofs as types of formal derivations that would exclude nearly all of the proofs in the published literature (CadwalladerOlsker, 2011). As Lakoff (1987) observed, when we try to define categories such as proof, we naturally try to list a set of properties that all proofs satisfy. However, Lakoff also argued that most real world concepts and many scientific ones cannot be defined in this way. In Weber (2014), I present arguments for why proof is an example of a concept that cannot be defined by properties that all proofs satisfy. One alternative that Lakoff (1987) suggested is that some concepts are cluster concepts which occurs when as "a number of cognitive models combine to form a complex cluster that is psychologically more basic than the models taken individually" (p. 74).

I suggested that proof might be a cluster concept with six components: (i) a proof is a convincing argument; (ii) a proof is a perspicuous argument; (iii) a proof is deductive and non-ampliative; (iv) a proof is sufficiently transparent so that a knowledgeable mathematician can fill in any gaps; (v) a proof is written in a representation system with agreed upon methods of inference; and (vi) a proof is an argument that is sanctioned by the mathematical community. In Weber (2014), I give a more detailed account and rationale for these criteria. I also claimed that no single criterion above is sufficient to define proof. For each criterion, we can find arguments accepted as proofs by (most) mathematicians that fail to satisfy that criterion. (e.g., computer-assisted proofs are not perspicuous and we do not expect a knowledgeable mathematician to be able to complete each of the gaps contained within that proof).

If proof can productively be conceptualized as a cluster concept, then this makes three predictions. First, an argument that satisfies all elements of the cluster concept should be viewed as prototypical and non-controversial. Mathematicians should all agree that such an argument is a proof independent of context and expect their colleagues to agree with them. Second, if an argument satisfies some but not all elements of the cluster concept, it should be viewed as an atypical proof whose validity is questionable. There should be variance in mathematicians' responses and they should be aware of the controversial nature of these arguments. Third, proof does not have an "essence". That is, mathematicians should not agree on which criterion in the proof concept is most important.

Citing philosophers of mathematical practice, in Weber (2014), I argued that each of the above hypotheses is plausible. In the current paper, I complement these theoretical arguments with an empirical study. As I will discuss in the contributed report, it is somewhat problematic to rely on writings about mathematical practice as the claims in the literature are often contradictory. Take, for example, the claim that computer-assisted proofs are controversial proofs that are ultimately accepted by mathematicians, a claim that has been made by philosophers (e.g., Aberdein, 2009) and mathematics educators (e.g., Dreyfus, 2004). In this contributed report, I provide empirical support for this claim, which might lead a skeptic to say that I am merely verifying the obvious. Hence, it is important to note that there are philosophers and mathematicians who write about computer-assisted proofs as being *uncontroversial*, arguing that they are clearly epistemologically on par with more conventional proofs (Fallis, 1996; Montano, 2012) and even that debates about their validity are "anachronistic" as this issue has been decided years ago (Fallis, 1996). On the other side, there are those who claim that computer-assisted proofs- in spite of being undeniably correct- are *not* recognized as proofs by the mathematical community (Rota, 1997) and others who believe that computer-assisted proofs are fundamentally unreliable for obtaining conviction (Jean-Pierre Serre; as cited in Raussen & Skau, 2004). With the exception of Serre, each of the authors cited in this paragraph used their assumptions about computer-assisted proofs as starting points to deduce strong conclusions about mathematical practice; they cannot all be right. The main point here is that citing philosophers and mathematicians to justify empirical claims about mathematical practice is problematic as there is internal disagreement between the groups; by carefully choosing who is cited, a researcher can find grounds to justify both a strong claim, its negation, and qualified version of that claim[2].

Methods

Participants. These studies are comprised of two internet-based survey in which mathematicians were asked to evaluate the validity of five purported proofs. The rationale, validity, and methodolgoy of using the internet to obtain a large sample of mathematicians has been discussed elsewhere (Inglis & Mejia-Ramos, 2009; Lai, Weber, & Mejia-Ramos, 2012) and is not discussed in detail here for the sake of brevity. For the first study, e-mails were sent to the secretaries at the mathematics departments at 25 large state universities in the Great Britain. In these e-mails, the secretaries were asked to forward a request to participate in the study with a link to the study's website to the faculty members of their department. Through this process, 95 mathematicians agreed to participate in the study and completed the survey. For the second study, the same process was completed with 25 large state universities in the United States, yielding a total of 110 mathematician participants

Procedure. In the first study, which I will call the **proof evaluation** survey, participants were told that they would be asked to make validity judgments on five mathematical arguments from number theory. The participants were told that the focus of the study was on the type of reasoning within the argument and that no attempt was being made to deceive them. They were then told that each proof was published, each sentence in the argument was true, and each calculation was carried out correctly. These provisions were put in place because my previous research has shown that validating proofs in number theory can be a time-consuming process for those who did not specialize in that area (Weber, 2008), which would limit the number of mathematicians who would invest the time to complete this survey. Further, I wanted to avoid generating disagreement amongst mathematicians due to performance errors (c.f., Inglis et al, 2013)-- that is, I did not want mathematicians to disagree on whether a proof was valid because some mistakenly thought a true statement was false. I was interested in the *types* of reasoning mathematicians considered valid in a proof rather than their evaluations of particular arguments.

The participants were then shown five arguments in a randomized order and told the publication source from where the argument came. The five arguments were:

- Prototypical Proof 1 (PP1): A conventional proof that "The n^{th} prime p_n satisfies $p_n \leq 2$^(2^{n-1}) for all $n \geq 1$" taken from Jones and Jones (1998) *Elementary Number Theory* textbook that was published by Springer.
- Prototypical Proof 1 (PP2): A conventional proof that, "if n is a number of the form $6k$-1, then n is not perfect" by Holdener (2002) that appeared in the *American Mathematical Monthly*.
- Empirical Proof (EP): An empirical argument to support "if n is an odd integer, then n^2 is an odd integer" based on verifying the claim for n = 1, 3, and 5. The participants were told this appeared in Weber (2003)[3].

[2] To be clear, quantitative studies are certainly not without their limitations as well. The point is that it is better to have a good theoretical argument *and* quantitative evidence to support it. This is especially true in the case of investigating mathematical practice, where the leading theoretical experts do not agree about factual claims about what arguments mathematicians accept.

[3] The argument did appear, but as a common type of invalid student proof. However, based on recent studies (Iannone & Inglis, 2010; Weber, 2010), I no longer think these proofs are that common amongst mathematics majors in proof-based courses.

- Visual Proof (VP): A visual proof of the claim that "if *n* is an odd integer, then n^2 is congruent to 1 (mod 8)" by Nelsen (2008) that appeared in *Math Horizons*.
- Computer Assisted Proof (CAP): A modification of a computer-generated proof that $\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right)$ given by Adamchik and Wagon (1996) in the *American Mathematical Monthly*.

After each proof was presented, participants were asked to make four judgments:

- On a scale of 1-10, how typical was the reasoning used in this proof of the proofs that they read and write?
- In their estimation, was this argument a valid proof? (yes/no)
- What percentage of mathematicians did they think would agree with their judgment? (>90%, 71-90%, 51-70%, <50%)
- For a more nuanced judgment on validity, did they think that: (i) The proof was valid in nearly all mathematical contexts, (ii) I think the proof is valid but there are some mathematical contexts in which it would be invalid, (iii) I think the proof is invalid, but there are some mathematical contexts in which it would be valid, and (iv) The proof would be invalid in nearly all mathematical contexts.

The prediction is that the two conventional proofs, which satisfied all the criteria in the cluster concept, would not be controversial. They would be regarded as prototypical proofs (scoring high on the first judgment), widely recognized as valid (most participants would answer "yes" to the second judgment) independent of context (most participants would answer (i) for the fourth judgment), and most would believe that the mathematical community would agree with them (most participants would answer >90% on the third judgment). Likewise, the empirical argument that satisfies none of the criteria of cluster concept would also not be controversial. Most participants would say this was not a proof, independent of context, and would expect their colleagues to agree.

The visual proof and computer-generated proof satisfy some, but not all, criteria of the cluster concept. Visual proofs are not written in a conventional representation system and computer-generated proofs are not perspicuous and contain gaps that could not be necessarily filled in by a knowledgeable mathematician. Hence the prediction is that these proofs would be controversial. Mathematicians would find these to be atypical of the proofs that they read (scoring low on the first judgment), would disagree on their validity (there would be a significant percentage of participants who answered yes to the second judgment but also a significant percentage who answered no), would be aware that there was disagreement (most participants would not answer >90% on the third judgment), and would think the validity of the proof was contextual (most participants would answer (ii) or (iii) for the third question).

In the second survey, which I call the **proof essence** survey, participants were asked what they believed the essence of a proof was and were given nine options to choose from:

1. A proof provides a mathematician with certainty that a theorem is true
2. A proof provides a mathematician with a high degree of confidence that a theorem is true
3. A proof is a deductive argument with each step being a logical consequence from previous steps
4. A proof is a blueprint from which a mathematician could write a complete formal proof if he or she desired
5. A proof, in principle, can be translated into a formal argument in an axiomatized theory
6. A proof explains why a theorem is true
7. A proof convinces a particular mathematical community that a result is true

8. None of the above captures the essence of proof
9. There is no single essence of proof

If proof is a cluster concept, then we would predict that there is no single criterion that captures the essence of proof. Hence, we would not expect the majority of participants to choose any one of these responses.

Results

Proof	Mean Typicality Rating	Validity Judgment		Anticipated Level of Agreement			
		Valid	Invalid	91-100%	71-90%	51-70%	0-50%
PP1	7.4	99%	1%	90%	9%	1%	0%
PP2	6.8	98%	2%	78%	20%	2%	0%
VP	2.6	62%	38%	14%	46%	33%	7%
CAP	2.7	39%	61%	10%	41%	37%	12%
EP	1.6	0%	100%	92%	0%	1%	6%

Table 1. Participants' judgments on the validity of the five proofs that they read

Proof	Valid proof in nearly all contexts	Valid proof but invalid in some contexts	Invalid proof but valid in some contexts	Invalid proof in nearly all contexts
PP1	94%	5%	1%	0%
PP2	79%	20%	0%	1%
VP	21%	33%	40%	6%
CAP	10%	33%	42%	15%
EP	1%	1%	3%	95%

Table 2. Participants' judgment on the more fine-grained question on utility

The results of the proof evaluation survey are presented in Tables 1 and 2. The results of the study confirmed the predictions. For PP1 and PP2, the large majority of participants claimed the arguments were valid, valid in nearly all mathematical contexts, and thought most of their peers (>90%) would agree with them. The median score for how representative these proofs were of what they actually read and wrote was about seven. For VP and CAP, there was substantial disagreement amongst the participants, the participants were mostly aware that at least 10% of their colleagues would disagree with them, and the majority thought the validity of the proof depended on context.

For the Essence phase of the study, no participant chose "none of the above" and 11% chose 9, that there was no single essence of proof. No choice gathered the majority of the participants; the fourth choice (that proof was a blueprint where a knowledgeable mathematician could fill in every gap) was the most popular, chosen by 25% of the participants, and the first choice (that proof provided certainty) being chosen by 22% of the participants. Every option aside from 8 (none of the above) was chosen by at least three participants.

Discussion and significance

The proof evaluation phase of the study

There are philosophers and mathematics educators who claim that there is a very high rate of agreement amongst mathematicians as to whether a particular argument is a proof or not (e.g., Azzouni, 2004; Selden & Selden, 2003). However, there are also philosophers and mathematics educators who challenge this claim (e.g., Aberdein, 2009; Auslander, 2008; Dreyfus, 2004; Inglis et al., 2013; Rav, 2007; Weber, Inglis, & Mejia-Ramos, 2014). The data presented here offer a potential approach to resolve this discrepancy. For *typical proofs*, mathematicians may indeed usually agree on their validity. Disagreements may arise due to performance errors (e.g., a reviewer overlooks a flaw in the proof), but this could presumably be resolved in a conversation between mathematicians, as Selden and Selden (2003) suggested. The disagreements do not concern the legitimacy of the *type* of reasoning being used. Hence, those who highlight mathematicians' "unusual degree of agreement about the correctness of arguments" (Selden & Selden, 2003, p. 7) seem to be correct in the following sense: for the proofs that mathematicians *typically* encounter in their working lives, it may well be the case that there is usually a high level of agreement amongst mathematicians on the validity of these proofs.

However, for *atypical proofs*, arguments that satisfy some but not all criteria of the cluster concept, disagreement on validity is common and mathematicians are aware of it. Importantly, the majority may think validity judgments about these proofs are *contextual*. (A good follow-up study would be to interview individual mathematicians to get a better sense of what these contexts are). Hence, those who challenge the claim that mathematicians share the same standard of proof are right to note that there are classes of proofs where this is not so.

The finding about the validity of atypical proofs being contextual has a useful consequence for methodological design. In a sense, we can say that asking someone whether a visual argument is a proof is not a well-formed question. The majority of the participants in this study felt the answer depended on mathematical context. In general, asking individuals to judge whether an imperfect argument without a fatal flaw is a proof to make a binary judgment on the argument's validity might be asking an artificial and unreasonable question. It might be better to ask *in what sense* is the argument a proof (and in what sense is it not) and *in what contexts* the argument would be acceptable (and in what contexts would it not be acceptable).

The essence phase of the study

The data on the proof essence phase of the study offer a strong challenge to a researcher who wants to describe what proof essentially is to mathematicians. For instance, take the claim that proof is, at its essence, a convincing argument- an assertion made by numerous mathematics educators (e.g., Balacheff, 1987; Harel & Sowder, 1998; Mason, Burton, & Stacey, 1982) and some philosophers (e.g., Davis & Hersh, 1981). If this were so, we might expect that for the essence question, most participants in this study would have chosen option 1 (proofs provide a mathematician with certainty), 2 (proofs provide a mathematician with a high degree of confidence), or 7 (a proof convinces a mathematical community). Perhaps some participants might have chosen 9 (there is no single essence of proof) on the grounds that a proof needed to be convincing *and* something else. Yet if we add the number of participants who chose 1, 2, 7, or 9, we only reach 41%. It seems difficult to claim that mathematicians essentially view proof as a convincing argument if the majority of mathematicians chose another facet of proof that proof is essentially about (in particular, choices 3, 4, 5, and 6). To avoid misinterpretation, no single study can be offered as a definitive rebuttal to the claim that many mathematicians view proof as something other than a convincing argument. What I do contend is that those who want to claim that proof is essentially about conviction (or explanation or anything else) should at least be held to *account* for these empirical findings.

This offers a practical suggestion for teachers or researchers who desires that proof in their classrooms to be epistemologically consistent with mathematicians' practice. They should not take conviction, explanation, social acceptance, or deduction as the primary criteria for what constitutes a proof. Different mathematicians place different weight on the importance of each of these. My contention is that good proofs satisfy *all* of these roles and I would encourage classroom research to reflect that.

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Obstacles in Developing Robust Proportional Reasoning Structures: A Story of Teachers' Thinking About the Shape Task

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This paper presents some initial findings of an investigation focused on mathematics teachers' ways of thinking about proportional relationships, with an emphasis on multiplicative reasoning. Deficiencies in proportional reasoning among teachers can be serious impediments to the development of robust reasoning among their students. As such, this study focuses on how mathematics teachers reason through tasks that involve proportional reasoning by addressing the following two research questions: (1) In what ways do teachers reason through a specific task designed to elicit proportional reasoning? and (2) What difficulties do teachers encounter while reasoning through such tasks? This paper discusses the construction of a robust proportional reasoning structure in the context of a specific task and discusses one particular obstacle, which impedes the construction of such a structure.

Key words: Proportional reasoning; Multiplicative reasoning; Within-measure comparison; Across-measure comparison

Recent reform efforts to institute the Common Core State Standards for Mathematics (CCSSM, 2010) have called for an increased emphasis on multiplicative and proportional reasoning, particularly in the middle grades. According to Lesh et al. (1988), proportional reasoning is the capstone of elementary school mathematics and the cornerstone of high school mathematics. One of the most critical elements of proportionality is the ability to make sense of the multiplicative relationships among the relevant quantities. Multiplicative reasoning is rooted in the ability to reason quantitatively and make sense of contexts involving multiplicative structures. The CCSSM standards themselves call for students to be able to "describe a ratio relationship between two quantities" (CCSS.math.content.6.rp.a.1). Yet historically, the mathematics traditionally taught in K-12 has emphasized additive reasoning and ill-conceptualized procedures for multiplicative situations, rather than building productive ways of thinking about quantities, relationships among quantities, ratios, multiplicative comparisons, and proportional relationships. This paper describes an investigation conducted with middle school teachers who are participating in a large-scale professional development program designed to improve their conceptual understanding of mathematics in the middle grades. Specifically, this study describes teachers' conceptions of proportionality through the lens of a proportional reasoning structure and highlights the challenges that teachers encountered.

A Discussion of the Literature

Centrally nested in the idea of multiplicative reasoning is the ability to first conceive of the quantities that need to be compared multiplicatively. In this study, the notion of quantity is aligned with Thompson's (1993) definition of quantity: "a person constitutes a quantity by conceiving of a quality of an object in such a way that he or she understands the possibility of measuring it" (p. 165). Thompson (1994) refers to the mental operation of conceiving one quantity in relation to another as a quantitative operation. Thompson also points out that "a quantitative operation creates a structure – the created quantity in relation to the quantities operated upon to make it work" (p. 185). The mental structure created as a result of a quantitative operation ultimately supports images of other numerical operations.

Reasoning about quantities is necessary for reasoning about proportional relationships. Cramer et al. (1993) outlined several components involved in proportional reasoning: (1) understanding the multiplicative relationships that exist within proportional situations, (2) being able to differentiate proportional situations from non-proportional ones, (3) realizing the existence of and relationships between multiple solution pathways, and (4) being unaffected by the situational context or the types of numbers in the task. Kaput and West (1994) found that the context of the problem, the language of the task, the kinds of quantities involved, and the numerical values of the quantities all impact student thinking.

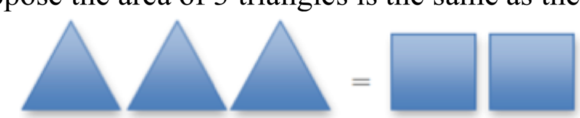
Methodology

This investigation focuses on nine middle school mathematics teachers who were recruited for this investigation based on their participation in a large-scale, two-year professional development program. Leveraging Goldin's (2000) principles, this study incorporated semi-structured, task-based interviews for investigating teachers' thinking when working through tasks involving proportional relationships. The teachers participated in five, one-hour videotaped interviews. The research team analyzed all interview sessions with the lens of characterizing teachers' thinking and reasoning as they grappled with the tasks. The design and implementation of this study was guided by the following two research questions: (1) *In what ways do teachers reason through a specific task designed to elicit proportional reasoning?* and (2) *What difficulties do teachers encounter while reasoning through such tasks?*

Creating a Robust PR Structure

In situations where two quantities are proportional, there exists an opportunity to construct a structure that can be utilized when addressing missing value proportion problems. A proportional reasoning (PR) structure is a network of multiplicative relationships that exist among the values of proportional quantities. This section of the paper presents one PR structure that is robust and founded on meaningful reasoning. Consider the Shape Task, which was used in this study:

The Shape Task: Suppose the area of 3 triangles is the same as the area of 2 squares.



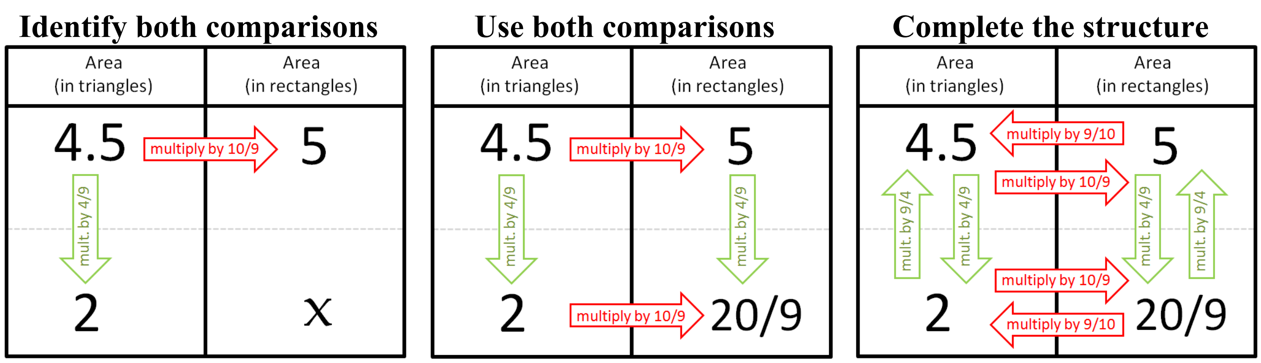
Also, suppose the area of 3 squares is the same as the area of 5 rectangles.

What is the area of 2 triangles, measured in the rectangle areas? Explain your reasoning.

In the Shape Task, it is important to recognize that the value of an area, measured in triangles, is proportional to the value of the same area measured in rectangles. Most of the teachers in the study were able to deduce through various methods that the area of 4.5 triangles is the same as the area of five rectangles. However, it was not trivial for many teachers to subsequently determine the amount of rectangles that is equivalent to two triangles, and a few were not able to overcome this challenge. We present data in this paper that highlights this one particular obstacle.

For the Shape Task, the constant of proportionality is 10/9 (found by computing 5÷4.5); which results from a multiplicative comparison of the two area measurements, five rectangles to 4.5 triangles. Kaput & West (1994) refer to this comparison of 10/9 (or its reciprocal of 9/10) as an *across-measure comparison* because it is a multiplicative comparison of two distinct ways to measure one quantity (i.e. area in triangles versus area in rectangles). We interpret the across-measure comparison of 10/9 as 10/9 rectangles for every one triangle, just as we interpret 9/10 as 9/10 of a triangle for every one rectangle.

Another approach is to construct a scale factor within the same measure (e.g. scaling one area measured in triangles, to a new area measured in triangles). By multiplicatively comparing two triangles to 4.5 triangles, the scale factor of 4/9 (found by computing 2÷4.5) can be constructed and then applied to the second measure (area in rectangles) to maintain the proportional relationship. Kaput & West (1994) call the comparison of 4/9 (or its reciprocal of 9/4) a *within-measure comparison* because it is a multiplicative comparison of two values within the same measure space, each value expressed using the same unit. We interpret the within-measure comparison of 4/9 as representing that the area of 2 triangles is 4/9 times as large as the area of 4.5 triangles. A robust PR structure includes both ways of reasoning – across-measure and within-measure – as well as the associated reverse operations. The construction of a robust structure is depicted in the figure below.



The ability to construct a PR structure as described above depends on the refinement of other ways of thinking about mathematics. For example, one should be able to use division to evaluate multiplicative comparisons as instinctively as one might use subtraction to evaluate additive comparisons. Our data indicates that utilizing division to evaluate multiplicative comparisons is not trivial for some middle school mathematics teachers. Unless one is able to meaningfully determine the across- and within-measure comparisons, one will not be able to construct the PR structure described.

Where is the relationship of cross-products: $(\mathbf{4.5})\left(\frac{\mathbf{20}}{\mathbf{9}}\right) = (\mathbf{2})(\mathbf{5})$?

The PR structure that we describe deliberately omits the cross-product relationship because it is not a necessary component of a robust proportional reasoning structure. Research has shown that students and teachers who leverage the procedure of cross-multiplying as a strategy for solving proportional tasks often lack the conceptual knowledge to explain why this strategy works (Cramer et al., 1993). An important goal in mathematics is to help students develop the ability to make sense of their world and to reason through problems. As supported by NCTM's Principles to Action (2014), procedural fluency should emerge from conceptual understanding. When trying to reason why cross-multiplication is effective, it can be challenging to explain the meaning of the cross-products. In the Shape Task, we have difficulty making sense of the product of an area (measured in triangles) and an area (measured in rectangles). Consequently, we claim that techniques involving cross-multiplication (or other procedures) should only be introduced after a solid foundation of proportional reasoning is constructed.

Discussion of Findings

In this study, initial data have revealed that the difficulties teachers encountered while solving the Shape Task were consistent with past research findings about student thinking (Kaput & West, 1994; Thompson, 1994). This study contributes to the field by investigating obstacles that *teachers* encounter while reasoning about situations that involve proportional relationships. The following is a discussion of the data from two teachers who grappled with the Shape Task, each of whom demonstrated difficulty in answering the mathematical question: *What do I need to multiply this by to get that?*

The Case of Ellie:

Within the first couple of minutes of engaging with the task, Ellie deduced that 4.5 triangles was equivalent to five rectangles. She recognized the need to scale 4.5 triangles to two triangles, but she was unable to determine the scale factor by which to do so.

> Ellie: I have to divide nine halves by, to get to two, I have to divide it by two ninths? No, that's going to give me one…What I was trying to do was, okay, I have to get down to two triangles (points at the 4.5 triangles drawn on the page)…

Ellie's inability to determine how to scale 4.5 triangles to two triangles led her to abandon a sensible way of thinking – a way of thinking that is essential to the construction of the PR structure set forth in this paper. During another attempt to answer the question, she again encountered difficulty when trying to scale 1.5 triangles to two triangles.

> Ellie: So then I've confused myself again.
> Interviewer: How have you confused yourself? What are you thinking?
> Ellie: …How do I get to two from one and a half? What do I have to multiply by? And I could not, for the life of me, think of what that would be. But it would have to be (long pause) four thirds? Does that work?

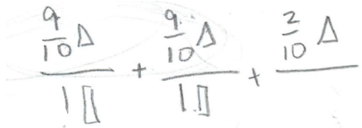
Although successful in determining how to scale 1.5 triangles to two triangles, the cognitive load was heavy and she ultimately relied on algebraic methods – writing down and then solving the algebraic equation $\frac{3}{2}x = 2$.

The Case of Anne:

Like Ellie, Anne quickly deduced that 4.5 triangles were equivalent to 5 rectangles. Anne unitized this relationship to 0.9 triangles per one rectangle, but struggled to leverage this information productively.

Anne: So two full triangles would be…Oh now for some reason I'm getting stuck and I know all I have to do is enlarge it. What do I do? Okay, um, to get to two full triangles…

Unable to multiplicatively scale 0.9 triangle to two triangles, Anne relied on additive reasoning to combine the amounts of triangles (see written work below).



This initial approach was eventually abandoned since Anne could not determine how many rectangles were equivalent to 2/10 of a triangle. After moving on with other tasks in the interview, Anne returned to the Shape Task and successfully completed the task using a modified strategy of scaling three triangles to two triangles, which did not seem to pose a challenge.

Anne: But I want two triangles. I have three triangles. So I'm gonna multiply this, I'm just gonna multiply this whole thing by 2/3, will let me say two triangles.

Anne's initial challenge with the task could be indicative of issues pertaining to scaling with fractional numbers – scaling from a whole number to another whole number is less cognitively demanding than scaling from a fraction to a whole number. According to Cramer et al. (1993), Anne does not have a robust ability to reason proportionally because the numbers in the task affected her reasoning.

Conclusion and Discussion Questions

The initial data reveal that several teachers struggled to evaluate multiplicative comparisons, which is a severe hindrance to the construction of the robust PR structure that we have described. Also, the data reveal that teachers in this study have inconsistent – and sometimes incoherent – ways of thinking about the quantities and their proportional relationships. At times, the teachers in the study lose track of the quantities they are relating together and they experience difficulty in describing how the quantities are related. Other instances have revealed that teachers rely on additive reasoning in order to cope with an inability to compare two quantities multiplicatively. A PR structure that sensibly relies on multiplicative comparisons may have provided the teachers with a conceptual understanding of proportionality to further facilitate their thinking and mitigate their struggles. As part of the presentation, the following questions will be posed and discussed to further the direction of this research investigation: (1) Have other PD researchers encountered similar obstacles to proportional reasoning and, if so, how have they addressed them? (2) How can we develop video coding frameworks to investigate proportional reasoning? (3) Are there researchers who already have such coding frameworks that we could adopt?

Acknowledgements

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Students' sense-making practices for video lectures

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Abstract

There has been increased interest in the use of videos for teaching techniques such as "flipped" classrooms. However, there is limited evidence that connects the use of these videos with actual learning. Thus, there is a need to study the ways students experience and learn from videos. In this paper, we use sense-making frames as a tool to analyze student's video-watching. We describe preliminary results from interviews with 12 students who watched short videos on introductory statistics and probability concepts and discuss implications for student learning.

Key words: Video Lecture, Statistics Education, Sense-Making

Introduction & Background

In the past decade, the ideas of "flipped" classrooms, "blended" classrooms, and massive open online courses ("MOOCs") have been increasingly hailed as effective teaching strategies and innovative ways to deliver content to students (e.g.,White House, 2013; USA Today, 2012). Most MOOCs and many flipped classrooms rely on video-recorded lectures to deliver their content. Despite the increasing interest in these pedagogical techniques, relatively little is known about how students watch and learn from these videos.

There have been numerous studies that have described the the positive influence of flipped classrooms—and, indirectly, video lectures—on student learning (e.g., Bergmann & Sams, 2008; Day, 2008; Demetry, 2010; Franciszkowicz, 2008; Frydenberg, 2012; Fulton, 2012; Gannod, Burge, & Helmick, 2008; Green, 2012; Lage, Plat, & Treglia, 2000; Lockwood & Esselstein, 2013; McGicney-Burelle, Jean, & Xue, 2013; Moravec, Williams, Aguilar-Roca, & O'Dowd, 2010; Seltzer, Gladding, Mestre, & Brookes, 2008; Toto & Nguyen, 2009; Warter-Perez & Dong, 2012; Wasserman, Norris, & Carr, 2013). However, there are few studies on flipped classrooms that provide empirical data to support their claims. Even those that do tend to suffer from several significant methodological issues. First, the data sources tend to consist of surveys in which students self-report their own engagement and learning; when studies use more objective measures of learning, these have tended to be very broad, such as students' scores on in-class exams and standardized state tests such as ACT scores. Second, many of the studies failed to use blinding or randomization when comparing groups of students in different types of classrooms, and did not account for variables such as instructor enthusiasm, instructor planning, and the effects of the novelty of the pedagogy. Third, most of the studies do not determine the degree to which the students are using out-of-class resources (in particular, watching videos).

In addition to these methodological issues, the studies generally do not attempt to separate learning that might occur in the classroom from learning that might occur from utilizing the out-of-class resources. Consequently, these studies have not established a connection between what the students do outside of class and what the students learn. Thus, it is essential for us to begin to investigate what students learn from watching video lectures, independently of class time.

The research questions we are attempting to answer are:

1. How do students make sense of video lectures?

2. What do students learn from video lectures, and how does this relate to their sense-making practices?

Theoretical Framework

We use the idea of sense-making frames (Weinberg, Wiesner, & Fukawa-Connelly, 2014) to describe the aspects of video lectures that students attend to and the ways students make sense of these aspects. A *conceptual frame* is "a mental structure that filters and structures and individual's perception of the world by causing aspects of a particular situation to be perceived and interpreted in a particular way" (Weinberg, Wiesner, & Fukawa-Connelly, 2014, p. 169). From this perspective, a student who is watching a video lecture experiences and seeks to organize a collection of phenomena; the student uses his or her prior knowledge and experience to create a conceptual frame, and this frame then determines which phenomena are noticed and how they are interpreted. While watching the video, students encounter *gaps*, which are "questions that must be answered in order for the student to engage in or construct meaning for the mathematical situation or activity" (Weinberg, Wiesner, & Fukawa-Connelly, 2014, p. 170). When the student answers the question, we say that she or he has constructed a *bridge*. There are four basic types of sense-making frames:

- *Content-oriented*: Students notice mathematical aspects of the situation (e.g., symbols, definitions, facts, and concepts) and encounter gaps about the meaning of the mathematical content or how to use it in an example that is being presented.
- *Communication-oriented*: Students notice the instructor's spoken, written, and gestural actions for organizing and presenting mathematical ideas and seek to understand the ways the instructor is categorizing or connecting ideas, the ideas communicated by board layout, and the instructor's organizational cues.
- *Situating-oriented, mathematical purpose*: Students notice mathematical aspects of the situation and seek to determine why the concept is useful or why it is mathematically significant.
- *Situating-oriented, pedagogical purpose*: Students notice communicational aspects of the situation and seek to understand how the instructor's pedagogical actions and decisions—such as choosing and ordering lecture content—are related to the meaning or significance of the mathematical ideas.

Methodology

The goal of sense-making research is to elicit the student's perspective and experience of watching a video lecture. Thus, the methodology focuses on providing students an opportunity to directly experience a situation (in this case, by watching a video lecture); to identify and discuss the gaps they encounter; and to investigate the ways they bridged the gaps. To do this, we used *message q/ing* and *abbreviated timeline* methods (Dervin, 1983; Glazier & Powell, 1992; Spirek, Dervin, Nilan, & Martin, 1999) as part of an interview protocol:

- In message q/ing, participants are asked to read a text and stop at places where they have a question to engage in an in-depth analysis. In order to generate stopping points, we asked students to take notes while they watched the video and to write a question mark in the margins of the paper when they felt that there was an aspect of the video that was unclear or confusing.

- In abbreviated timeline, the researchers select excerpts from the video and have students discuss these chronologically. We identified numerous points in each video that we thought included an interesting description of a mathematical concept, an interesting aspect of the way the concepts were presented or organized, or aspects of mathematical concepts that illuminated an aspect of a "big idea."

We wanted to know how the students' sense-making might be influenced by the mathematical content, conceptual focus (i.e., focusing on conceptual or procedural aspects), and presentation style of the video. For the mathematical foci of the videos, we selected two concepts from introductory statistics: the five-number summary and basic probability computations using counting, addition, multiplication, and complements. We selected these topics because they require relatively little background mathematical knowledge. Some research suggests the presentation style might influence students' engagement with the video lecture (e.g., Guo, Kim, & Rubin, 2014) and there is some evidence that explicitly addressing conceptual difficulties in videos might improve student understanding (e.g., Muller, Bewes, Sharma, & Reimann, 2007). Thus, for each content area and conceptual focus, we decided to use videos that had one of three presentation styles: A two-person discussion that explicitly addressed potential areas of confusion; a "talking head" video with an instructor drawing on a tablet or writing on a board; or a "Khan academy" style video with an instructor narrating a drawing or Powerpoint slides.

Methods

To recruit students, we visited all of the introductory mathematics classes at our institution (a mid-side, comprehensive Northeastern college) and invited all students who had not previously taken a statistics class to participate; students were offered a $20 gift certificate as compensation. Twelve students expressed interest in participating and all were interviewed.

In order to find videos that fit each of the twelve categories described above (i.e., two content areas, three presentation styles, and two conceptual foci), we searched various online sources (e.g., Coursera). We were unable to find any Discussion-style videos, so we created these ourselves, attempting to make the content and examples roughly equivalent to those presented in the other videos.

Each interview lasted approximately one hour, which was divided into a two half-hour blocks, the first one focusing on the measures of spread video and the second focusing on the probability video. In each block, the students were asked to describe their prior experience with the content area in order to gauge their background knowledge. Then, the students watched the video and took notes using the message q/ing method. The students then summarized the main ideas of the video and worked on several conceptual and procedural problems. After answering the questions, the students identified each place in the video where they had written a question mark, describing what was happening, what aspect they thought was unclear or confusing, and how they had eventually understood what was happening. If there was time remaining in the block, the interviewer "rewound" the video to several of the pre-selected excerpts and asked the students to describe the mathematical content, the significance of the content, and/or the instructor's reasons for including or explaining the concepts in a particular way.

The entire interview was audio-recorded; the student's note-taking was recorded with a Livescribe pen; the video was played on a tablet using Coach's Eye software (which allowed the student to draw on the video) and the student's playback of the video was recorded using Camtasia software. The audio recordings were transcribed and used as a basis for analysis.

The members of the research team initially worked independently to identify the sense-making frames that the students used. We each coded each student's questions and responses to each excerpt using the theoretical framework. We also categorized each question the interviewer asked as suggesting a particular frame; for example, when one video indicated that $0 \leq P(A) \leq 1$, the question "what does this mean?" suggests that the use of a content frame, whereas "why did the instructor introduce this notation?" suggests the use of a pedagogy frame. After applying the codes individually to one of the interview transcripts, the members of the research team compared codes and used differences in the coding to refine the coding manual. This process of refinement occurred until over 80% agreement was reached, and then we individually coded all of the transcripts.

Preliminary Results and Discussion

Although we are still in the early stages of analysis, we have already noticed four interesting aspects of the data related to identifying gaps and constructing bridges.

First, no two students identified the same gaps while watching the videos. This suggests that it is not possible to design a video that all students would experience in the same way or that would be an equally effective learning tool for all students.

Second, students identified relatively few gaps while watching the videos. Out of the twelve students, only three wrote more than one question mark in the margins of their notes; most of the students felt that they had constructed bridges for the gaps as they watched. However, all of the students had difficulty responding to many of the interviewer's questions. Student 4 summarized this at the end of his interview:

> I think, well until, like I said like before just watching the videos, that was all fine. And then when you actually broke down the video and then asked me like why do you think, like motives behind certain things that he did I kinda was like, kind of stumped I mean because I don't really know, I don't really know his teaching methods or his styles so I didn't know if it was something mathematical based like you were saying or if that's just the way he teaches to kind of give us a further understanding.

In addition to not recognizing when parts of the video didn't make sense, this student's description suggests that this may be, in part, a consequence of not attending to, at various points, the mathematical content, the instructor's way of presenting the content, or the big picture ideas. One way to interpret this is that productively interpreting the video requires the student to use and switch between multiple sense-making frames.

Third, all of the students encountered the issue of only recognizing aspects that they didn't understand when the interviewer asked them specific questions. For example, Student 1 did not make any question marks in the margins of her notes, and stated that the video made sense while watching it. When she was later asked what is meant by a "random variable," after a long pause she responded that she "definitely didn't" understand the term. There are two ways we might interpret this result. First, the student might experience a gap, but various constraints—such as the need to quickly attend to subsequent parts of the video—might prohibit the student from consciously recognizing the gap and constructing a bridge. Second, the gap might not exist until the researcher helps the student notice particular aspects of the video and choose an appropriate sense-making frame.

Fourth, in addition to not recognizing aspects of the video that they didn't understand, most students also experienced gaps and constructed bridges that, when questioned by the interviewer, appeared to have flaws. For example, Student 12 described how he was able to interpret the symbol string $P(Y)=2$ while taking notes, but subsequently was unable to understand what it meant:

> *Interviewer*: So here he says Y is the total number of heads, P Y equals two. What does that mean when he says P Y equals two?
> *Student 12*: Your guess is as good as mine. I think it's the possi... let's pretend that P equals two and I don't know what two would mean. Yeah I have no idea. I wrote it down and it made sense when I was writing it down here, but I have absolutely no—I can't fathom what it is.

There are several ways we might interpret this result. First, the constraints described above might prohibit the student from fully examining his bridge and recognizing its limitations. Second, the student might have been using a sense-making frame that did not enable him to construct a "robust" bridge. Third, the student might not match the *implied reader* (Weinberg & Wiesner, 2011) of the video and does not possess the necessary background knowledge or ways of interpreting aspects of the video that are required to construct an accurate bridge.

These last two results have important implications for students' opportunities to learn from watching videos. If one of the benefits of video use is that students are able to pause and rewatch sections that are confusing or aren't making sense, then an implicit assumption is being made that students are able to recognize when this is happening and either identify concepts with which they need help or construct a correct understanding of the concepts. Our data suggest that students may have difficulty recognizing these moments and, when students do recognize such moments, they might not realize when their understanding is insufficient. Consequently, students may not be able to take full advantage of the potential benefits that video use may provide.

As indicated above, we have not yet completed the analysis of our data. In the future, we plan on identifying patterns in the students' use of various sense-making frames; the role that background and cross-disciplinary knowledge play in sense-making; what the students learned from watching the videos; and how the students' sense-making practices are connected to their learning. We hope to use these results to make recommendations for structuring students' video-watching practices to help them use videos effectively as learning tools.

Discussion Questions

- What aspects of videos might influence the ways students make sense of the videos?
- How might we structure students' video-watching to support their learning?
- What are the limitations of sense-making frames as a theoretical tool?
- What additional tools might be useful for analyzing this data?

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Exploring pre-service teachers' mental models of doing math

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This preliminary study explores the mental models pre-service teachers hold of doing math. Mental models are cognitive structures people use while reasoning about the world. The mental models related to mathematics would influence a teacher's pedagogical decisions and thus influence the mental model of mathematics that their students would construct. In this study, pre-service elementary teachers drew images of mathematicians doing math and of themselves doing math. Using comparative judgements, they selected an image that best represented a mathematician doing math. Most images of mathematicians doing math were of a man in front of a blackboard filled with mathematical symbols. The mathematicians appeared happy. In contrast, many images of participants showed them to be unhappy or in confused states. The preliminary results suggest that their shared mental model of doing math is naïve and shaped by limited experiences with mathematics in the classroom.

Key words: Mental Models, Drawing Research, Pre-service Teacher Mathematics Beliefs

In a recent article of the *MAA FOCUS* magazine, Francis Su, newly installed president of the MAA was asked the following question, "What is your earliest memory of doing mathematics?" Dr. Su spoke of solving arithmetic problems on worksheets, prior to being of kindergarten age, given to him by his father. He further clarified that, at that time, this was what he believed mathematics to be (Peterson, 2015). What does it mean to do math? Being able to better understand this notion and clearly explain it is imperative, especially for those involved in education, as beliefs about it drive curricular and pedagogical decisions. Additionally, lack of a clear vision hampers efforts to help people learn mathematics and to recruit future mathematicians into the field, endangering mathematics as a whole. This current study aims to explore mental models held by pre-service elementary teachers to better understand their perceptions of what it means to *do* math.

Mental Model Theory

Mental model theory is a theory of how people reason about the world. A mental model is a cognitive structure that is constructed by an individual as a representation of a possibly real, imaginary, or hypothetical external reality (Gentner, 2002; Jacob & Shaw, 1999; Johnson-Laird, Girotto, & Legrenzi, 1998; Jones, Ross, Lynam, Perez, & Leitch, 2011). Due to cognitive limitations of an individual, a model cannot contain every detail of the reality and thus are not complete or technically accurate representations (Gentner, 2002; Jones et al., 2011; Norman, 1983/2014). However, a model will have structural features in common with the represented domain and be as iconic as possible (Johnson-Laird, 2004). Thus, structural relations present in the reality will have analogous representations in the individual's mental model (Johnson-Laird, 1998).

An individual constructs a mental model through experience, by perceiving or imagining the reality, or by understanding discourse and gaining formal knowledge (Jacob & Shaw, 1999; Johnson-Laird et al., 1998; Jones et al., 2011). Because of how models are constructed, a mental model is contextually bound, constrained by an individual's experiences with the represented domain (Norman, 1983/2014). In addition to experience, goals and motives for construction of the model influence the structural aspects of the reality that end up being represented in the model (Jones et al., 2011). An individual uses mental models as conceptual frameworks through which to interpret, understand, and reason about the world (Gentner

2002; Jacob & Shaw, 1999). New information filters through the model (Jones et al., 2011), and the individual reasons about situations, leading to predictions and decisions through mental manipulations of the models (Johnson-Laird, 2005).

Reasoning about unfamiliar situations occurs as an individual constructs a new mental model by appealing to existing models. An individual imports structural relationships from a mental model of a similar domain via analogical thinking. As described by Collins and Gentner (1987), "People construct generative models by using analogy to map the rules of transition and interaction from known domains into unfamiliar domains. Analogy is a major way in which people derive models of new domains" (p. 263). Thus, new mental models are created from existing mental models of situations that appear analogous to the new situation.

Closely related to mental models is the notion of a mental image. Mental models are considered to be more general than a mental image. As some features of a mental model may not be visualizable, a mental image refers to the visualizable aspects. Hence, underlying any mental image is a mental model, with the image being the projection of the mental model's visualizable aspects (Johnson-Laird, 1998; Johnson-Laird, Girotto, & Legrenzi, 1998).

Although mental models are typically explored as an individual's model, research has been conducted into models shared by cultural groups. One such model is the *shared* mental model, which is the "overlapping mental representation of knowledge by members of a team" (Van den Bossche, Gijselaers, Segers, Woltjer, & Kirschner, 2011, p. 285). It is generally accepted that for teams to function in an effective manner, the members of the team must share a mental model (Langan-Fox, Code, & Langfield-Smith, 2000).

In addition to representing physical aspects of a particular domain, mental models also incorporate an individual's beliefs related to the domain. Therefore, mental models are reflective of belief systems (Libarkin, Beilfuss, & Kurdziel, 2003; Norman, 1983/2014). The connection to belief systems can be used as a means to explore an individual's mental model. Due to being internal constructs, mental models are difficult to explore. While one method of exploration is the direct questioning of an individual's beliefs, people generally have difficulty clearly articulating their beliefs (Gentner, 2002). As a result, novel methods are necessary to construct an external representation of an internal mental model (Jones et al., 2011).

Efforts continue in order to improve methods for constructing external representations of internal mental models. For example, some recent studies have explored mental models via participant-made drawings. Included among the models explored via drawings are elementary and middle school students' mental models of circuits (Jabot & Henry, 2007), pre-service teachers mental models of themselves as teachers of science (Thomas, Pederson, & Finson, 2001), pre-service agriculture teachers' mental models of effective teaching (Robinson, Kelsey, & Terry, 2013), and pre-service teachers' mental models of the environment (Moseley, Desjean-Perrotta, & Utley, 2010). While not explicitly using mental model theory, other studies have used a drawing methodology to explore pre-service elementary teachers' visual images of themselves as mathematics teachers (Utley & Showalter, 2007) and middle and secondary students images of mathematicians at work (Aguilar, Rosas, Zavaleta, & Robo-Vazquez, 2014; Picker & Berry, 2000; Rock & Shaw, 2000).

In their work, Picker and Berry (2000) suggested a model for how a cultural image of mathematicians and their work is formed. A young learner, someone unfamiliar with the stereotypical cultural view of mathematics, begins school. Through exposure to cultural stereotypes and memes via media, adults, and peers, through interaction with teachers lacking rich images of mathematics who could otherwise challenge stereotypes, through a pedagogy that reinforces stereotypes, and through the lack of clear intervention by the mathematics community, students begin forming their mental model of mathematics. Stereotypes fill the void left vacant by desirable alternatives, and the student's mental model is validated through experience. The student can now take his or her place in the culture and perpetuate the shared mental model. Teachers play a key early role in inculcating students into the stereotypes of mathematics and thus could rescue students from entering the vicious cycle. However, the teachers would need to hold a healthy model of mathematics themselves to have much effect, as a teacher's beliefs influence the mathematical experiences they have with their students and so influence the model that the students form (Mewborn & Cross, 2007). If students do not have healthy images of mathematics, they may choose to pursue other vocations, potentially robbing society of valuable mathematical innovation. Thus, exploring pre-service teachers' mental models related to mathematics is of importance.

***Doing* Math**

From a survey of twenty-five post-secondary mathematics professors, Latterell and Wilson (2012) formulated a working definition stating that in order to be considered doing math, mathematicians must be *creating* new mathematics. Schoenfeld (1994) stated, "research – what most mathematicians would call *doing* mathematics – consists of making contributions to the mathematical community's knowledge store" (p. 66). As a result of their definition, Latterell and Wilson specifically excluded teachers of mathematics from being considered as mathematicians and only included mathematics professors as being mathematicians doing math if they were engaged in research mathematics. The general populace does not necessarily hold to this same definition.

Through a survey of children in grades K-8, Rock and Shaw (2000) determined that the students believed mathematicians did the same kind of math the students did in the classroom, only with larger numbers. Students also tended to believe mathematicians solved the hard problems no one else wanted to do. Many images drawn by the participants showed a mathematician in a classroom setting. Picker and Berry (2000) found similar results when they explored the images that 12-13 year olds had of mathematicians at work. About one-fifth of the drawings were of a teacher. The images of mathematicians adhered to some stereotypes found in the research of images of scientists; most of the images were of men, and some of the drawings resembled Einstein. In a follow-up prompt, the plurality of students mentioned that mathematicians were hired to teach math, suggesting that students actually do not have a clear idea of what mathematicians did. As a result, Picker and Berry suggested that mathematicians and their work were basically invisible to the students. Fralick, Kearn, Thompson, and Lyons (2009) studied the drawings of middle schoolers to explore perceptions of scientists and engineers at work. Approximately one-fourth of the drawings of engineers contained no discernible action. Other drawings showed engineers more in the role of a "worker bee" rather than suggesting that engineering required mental efforts, leading to the conclusion that students' images of engineers and their work were naïve or incomplete.

From a study of images of mathematicians at work created by high-achieving high school students attending a mathematics and science school, Aguilar, Rosas, Zavaleta, and Romo-Vázquez (2014) discovered that while the images were mostly male figures and contained many images of teachers, the students had a richer conception of what mathematicians did. They suggested this richer view developed from more exposure to advanced mathematics. Also, since many of the images contained items found in school settings, the students' limited interactions with math, mainly in the schools, heavily influenced their image of doing math.

Due to the important role that teachers and the school setting play in the formation of a student's mental model of mathematics, this study will explore the following questions.

1. What shared mental model of *doing mathematics* is held by pre-service elementary teachers in a mathematics content course?

2. To what extent do pre-service elementary teachers' mental models of themselves doing mathematics align with this shared model?

To address these questions, this study will use mental model theory to explore drawings of "doing math" generated by the participants. In this study, drawings created by the participants are taken to be external representations created by them of their own mental images, which are in turn the projections of the visualizable aspects of the corresponding internal mental model. An individual's mental model is influenced by the culture to which he or she belongs and thus forms a shared mental model. The formation of the shared mental model occurs in a fashion as described by Picker and Berry (2000).

Methodology

The study was conducted at a regional university in the southeastern United States. Participants in the study were undergraduate students in a teacher preparation program. The students were enrolled in one of three sections of a mathematics content course for pre-service teachers. The course was the third in a sequence of four mathematics content courses required by the program. Forty-six students were enrolled in the sections. The students were divided between two disciplines, early childhood education (31, 67.4%) and special education (15, 32.6%). Of these students, 4 (8.7%) are male and 42 (91.3%) are female. Additionally, 2 are Hispanic (4.3%), 10 are African-American (21.7%), and 34 (73.9%) are Caucasian.

During the sixth week of classes, students responded in an at-home activity to the following four prompts: 1.) Draw a picture of a mathematician. 2.) Draw a picture of a mathematician doing math. 3.) Draw a picture of you doing math. 4.) Draw a picture of one of your students doing math. Students had approximately one week to create the drawings. The drawings were subsequently collected and scanned to create electronic files.

The *Mathematician* and *Mathematician doing Math* drawings were uploaded to the No More Marking website (nomoremarking.com). During the ninth week, for an at-home activity, students were invited to perform comparative judgments on the two sets of drawings with the following questions, respectively: 1.) Which best represents who a mathematician is? 2.) Which best represents a mathematician doing math? Furthermore, participants were instructed to compare each drawing and choose the one they believed best answered the questions, to give honest responses, and to not judge the pictures on artistic merit. Each participant made 40 comparisons per data set.

Based upon the results of the comparative judgments, the top 11 drawings for each set were compiled into files. During week eleven, students reviewed each set of drawings and listed features common among the drawings. They then compared and contrasted their drawing they made of *You doing Math* to the *Mathematician doing Math* images, explaining why they believed their drawings were either similar or dissimilar.

Finally, during the twelfth week, for an at-home activity, students were shown the image selected through comparative judgment as the best representative of a *Mathematician doing Math* and answered the following prompts: 1.) Why do you believe this picture was selected as the best representation of a mathematician doing math? 2.) To what extent does this picture align with your beliefs of what it means for a mathematician to do math? 3.) To you, what does it mean to be a mathematician?

Participant drawings and responses will be explored for common themes using an open coding procedure. The drawing of a mathematician doing math selected through comparative judgement as the most representative will be used as an initial model to construct the shared mental model the class had of a mathematician doing math. The themes from the drawings of themselves doing math will be used to construct a shared mental model of the pre-service teachers doing math. The themes from the drawings and written responses will be used to

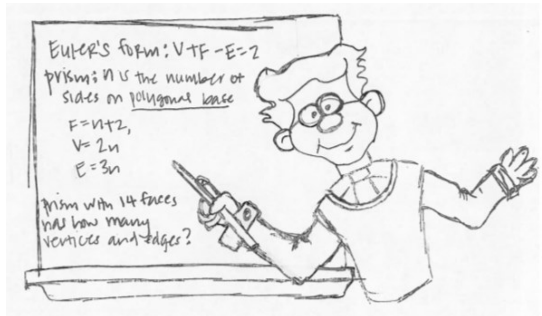

Figure 1. Image selected as best representing a mathematician doing math.

triangulate the results. A comparison will be made of the mental models of a mathematician doing math and the pre-service teacher doing math.

Discussion and Implications

As data analysis is currently ongoing, early results and implications will be offered. Figure 1 shows the image selected through comparative judgement of a mathematician doing math. The selected image apparently shows a teacher discussing Euler's formula and prisms. This suggests that experience and context played a key role in the students' mental model. For them, a math teacher teaching represented a mathematician doing math. Additionally, the content on the board can be explained by the content of the mathematics course the students were enrolled in, geometry. This image of a mathematician doing math fits in with the students' previous experiences with mathematics as being a classroom subject involving formulae and facts to be memorized.

When considering the next ten highly selected drawings, every single one showed a person standing in front of a chalkboard covered in math symbols. The symbols were all related to mathematical content that would have been experienced within a classroom. The person was either apparently teaching the content or pondering the problem. Moreover, while over 90% of the participants were female, very few participants actually drew mathematicians that could be considered female. Of the top 11 selected drawings, only 2 could potentially represent a female mathematician.

While the drawings of the mathematicians generally appeared to be happy or in a pensive state, many of the drawings that participants made of themselves doing math showed people who were dismayed, stressed, or upset. Even drawings that suggested the participant as a teacher lacked confidence. Also, some drawings showed participants in more of a student role or working with very basic math, suggesting students had trouble viewing themselves as potential mathematicians.

Overall, the images suggest that students' limited experience with math and mathematicians has led to a mental model that is very naïve. Unless these students experience an intervention, this model will continue to be reinforced as they experience struggles with math, commiserating with each other, while witnessing the apparent ease with which their teacher interacts with math.

Audience Questions

1. Should teachers of mathematics be considered mathematicians?
2. What implications would there be in comparing images of mathematicians doing math and the participants doing math, and would these implications be worthwhile?

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Investigating student-learning gains from video lessons in a flipped calculus course

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Abstract

The flipped classroom has garnered attention in post-secondary mathematics in the past few years, but much of the research on this model has been on student perceptions rather than its effect on the attainment of learning goals. Instead of comparing to a "traditional" model, in this study we investigated student-learning gains in two flipped sections of Calculus I. In this session, we will focus on the question of determining learning gains from delivering content via video outside of the classroom. In particular, we will compare student-learning gains after watching more conceptual videos versus more procedural ones. We will share qualitative and quantitative data gathered from surveys and quizzes, as well as results from in-class assessments.

Keywords: Flipped Classroom, Video Lessons, Learner-Centered Teaching, Calculus

Background

Learner-centered or active classrooms are those which change the role of the instructor from "sage on the stage" to "guide on the side" and encourage students to construct their own meaning while engaging in authentic problem-solving. Recent research has consistently showed that active classrooms improve student learning in a variety of fields; for example, in 2014 the National Academy of Sciences published a meta-study of 225 studies on student performance and failure rates in undergraduate science, technology, engineering, and mathematics (STEM) classrooms employing active learning components. Their analysis suggests that students in traditional lecture classrooms are 1.5 times more likely to fail than students in classrooms including any type of active learning techniques (Freeman et al., 2014).

The flipped classroom structure is one example of an active learning method that has become increasingly popular. This classroom structure takes on many forms, but the common trait is that most of the initial content delivery happens outside of the classroom while in-class time is spent solving problems, often in small groups, to assimilate the new knowledge and to deepen understanding. Some instructors deliver content through assigned readings from a text or other source, while others use videos that they made or found online. The core idea is to use classroom time for challenging problem solving where students can draw support from their peers and instructor; this design more effectively uses the experience and knowledge of the instructor to guide students through the topic at hand.

Literature Review

As the flipped classroom has gained popularity among undergraduate STEM educators, more literature has appeared. Much of the initial literature on flipped classrooms only described the varying structures of such classrooms or the particular technologies employed by teachers using a flipped classroom. The controlled studies published on this classroom model have often focused on student perceptions of and attitudes towards the structure rather than its impact on the attainment of learning goals. For example, Foertsch, Moses, Strikwerda, and Litzkow (2002) described the use of a specific video streaming software in an engineering classroom, and reported student opinions of the videos and software, and Ford (2015) described her activity structure in a math course for pre-service elementary teachers. Strayer (2007) gathered data on a traditional and flipped introductory statistics classroom to evaluate the learning environment of each structure, and found that students enjoyed the innovation and cooperation in the flipped class, but had a low “comfortability” with the learning activities in this environment. Roach (2014) found that 76% of his economics students believed that video lectures helped them learn, and the same percentage would take another class using the flipped format.

While lecturing as been a staple of academia for close to a millennium, the flipped classroom structure might be seen as a return to an even older system of teaching where classroom time was centered around academic debate and discussion rather than the transmission of information, just using newly available technologies. This recent resurgence dates to at least the mid-1990s when Eric Mazur, a physics professor at Harvard, started using team learning and in-class activities as ways to stop lecturing (Mazur, 1996). Jonathan Bergmann and Aaron Sams (2012) started using video lectures in the mid-2000s and are often credited with pioneering the flipped classroom and its current popularity. Since then, many educators in a variety of fields and at a wide range of institutions have started using this structure. For example, Gaughn (2014) wrote about her experiences running a flipped history classroom, and Findlay-Thompson and Mombourquette (2014) published research from their flipped business classroom. Bishop and Verlager (2013) did a meta-analysis of the literature on flipped classrooms in all areas of STEM, as well as economics and sociology. Additionally, the research ranges from high school level (Johnson, 2013; Moore, Gillett, & Steele, 2014) to upper division medical courses (Sharma, Lau, Doherty, & Harbutt, 2015). Education-focused video repositories like Khan Academy are available on the web, and many have spoken about their experiences with various forms of the flipped classroom at local and national professional meetings (e.g., in 2014 the Joint Mathematics Meetings included a session titled *Flipping the Classroom* with 37 different talks).

More recently, research studies used classroom data to evaluate the success of flipped classrooms. Lape et al. (2014) and Mason, Shuman, and Cook (2013) compare grades on individual assessment questions in engineering between flipped and traditional sections of the same course and found few cases of statistically significantly higher scores in the flipped classroom, but no cases where students in a lecture section outperformed students in a flipped section. Wilson (2013) found that students in a flipped section of statistics did outperform their lecture counterparts on exams and the course post-test. In mathematics in particular, McGivney-Burelle and Xue (2013) flipped a unit in a Calculus II course and showed that student grades on exams and homework were higher for the flipped section than the traditional section. Love, Hodge, Grandgenett, and Swift (2014) found that students in a flipped linear algebra course had greater improvement in exam scores than those in a traditional section, and outperformed them on the final exam. Additionally, PRIMUS has a forthcoming special issue on research in flipped classrooms that will increase the literature within mathematics education.

Research Question

Since students in the flipped classroom model do introductory learning of topics outside of the classroom, it is prudent to investigate the effectiveness of the content delivery method. The classroom in our study most often introduced new content outside of class through the instructor's own video-recorded lessons. Studies about using video have been conducted previously; for example Zappe, Leicht, Messner, Litzinger, and Lee (2009) investigated how students used online lecture videos to learn in an undergraduate engineering course, including the percentage of videos watched, students reviewing unclear segments, and time spent per video. In this paper we investigate the effectiveness of these videos on the learning gains made by students enrolled in two sections of a standard first semester calculus course. In particular, we explore student-learning gains from watching videos outside the classroom to determine students’ development of conceptual understanding and procedural skills in calculus.

Methods

Participants

The participants were undergraduate students in a first semester calculus course at a large university in the Mid-Atlantic United States. Of the 59 students in the study, 51 (86%) were freshmen, 5 (8%) were sophomores, 2 were juniors, and 1 was a senior. The majority of the students were male (64% male, 36% female). Four students withdrew from the course before the end of the semester. More than 80% of the students had previously had a course in calculus, generally in high school. The majority of the students were majoring in STEM fields. The students were divided into two sections (34 students in one section, 25 in the other) and generally covered the same material on the same days.

Classroom

This was the third semester that the instructor had run a flipped Calculus I classroom. Before each class, students would have a pre-class assignment, such as watching a video or completing a reading. Nearly all class sessions started off with a short quiz related to their pre-class assignment. The majority of class time was spent on group-work activities. The students worked in groups of 2–4 and the instructor would interact with the groups one-on-one. Students were also given homework and practice problems to be completed outside of class.

Data Sources

Over the course of the semester, we gathered qualitative data from the students, including student feedback about specific video lectures (for example, questions like "What did you find confusing?" or "What helped clear up confusion?"), student answers to post-video or post-activity questions or problems (calculus content questions to evaluate learning gains), and student surveys about their perceptions of the class structure and their learning gains. Aggregate quantitative data, such as assessment scores and course grades, were also recorded. We used video recording on certain class days to help the instructor objectively evaluate and improve student-teacher interactions in the classroom. This data was used to make changes to course attributes in order to increase potential learning gains, as well as to consider the general effectiveness of the class model.

Analysis

We created rubrics to analyze the written feedback from students. For example, the rubric shown in Table 1 was used to analyze responses to a question asking students to describe L'Hôpital's Rule. We then used two-tailed pairwise comparisons ($\alpha = 0.05$) to compare groups of students (e.g., students who had previously viewed a more conceptual video about the mathematical content versus a more calculational video) or to compare pre- and post-test results. Written responses were also categorized so that we could view trends in the data.

Table 1.
Rubric Used for Scoring Responses to Conceptual L'Hôpital's Rule Question

Score	Explanation
0	Answer was blank or made no mention of tangent lines.
1	Answers either lack "functions act like their tangent lines", or say something about tangent lines but neither "slope" nor "compare".
2	Answer states that functions act like their tangent lines near a point, and that one can find limits of $f(x)/g(x)$ (or compare $f(x)$ and $g(x)$) which have indeterminate forms by comparing the slopes of their tangent lines.

Students were also given some in-class surveys consisting of Likert-scale questions. The surveys generally asked students about their perceptions of the class structure and their learning gains. Aggregate quantitative data, such as assessment scores and course grades, were also used to look for prevailing student trends.

Results

While data analysis is still ongoing, in this section we share a subset of results from our study. In particular, we share students' overall opinions about video use and data around one class period specifically designed to help us see differences in the ways students learn conceptual and procedural content via video.

Several times throughout the semester, students were given surveys where they could voice their opinions about the structure of the class. When asked to compare learning a new topic outside of class via reading assignment versus watching a video, students overwhelming preferred videos (86%). However, when asked what part of their class structure had the greatest positive impact on their learning, 56% of students said the pre-class videos and readings, whereas 46% said the in-class activities and interactions.[1] We also asked the students to state their beliefs on how the videos increased both their conceptual understanding and computational skills in the class (see Table 2). For both questions, the majority of the class believed the videos greatly or significantly helped their mathematical understanding and skills, although more of the students found video helpful for their conceptual understanding than their computational skills.

Table 2.
Students' Beliefs About Video Usage

	Greatly	Significantly	Moderately	Slightly
Conceptual understanding	38%	38%	24%	0%
Computational skills	20%	40%	30%	10%

So the students believed the videos were helpful, but what objective evidence for learning gains could be seen in the students' work in the classroom? Prior to an in-class activity about L'Hôpital's Rule, we had the students watch an introductory video about the topic. However, we split the classes into two groups: one group watched a more conceptual video, and the other watched a more procedural video (n = 23 for each group). At the beginning of class, the students were given an assessment about L'Hôpital's Rule, with one question asking for a more conceptual explanation and the other asking for a more procedural explanation. We then had the students form groups of 2–3 so that each group contained at least one student who had watched each video. We videotaped the class session. At the end of class, students were given the same assessment as before to help us see what changes in their understanding occurred due to their group discussions.

Video data analysis is still ongoing. However, preliminary analyses seem to indicate that students gained knowledge from watching the videos and were able to share that knowledge with other students. We have completed scoring their responses to the pre/post assessment using rubrics like the one described above (0–2 scale). The students' average results can be found in Table 3. Results indicate that students who watched the more conceptual video were able to answer the more conceptual question on the pre-class assessment, but were not able to answer the more procedural question. The opposite was true for the students who had watched the procedural video.

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[1] Percentages add up to more than 100% because students could choose more than one answer.

Table 3.
Average Scores on L'Hôpital's Rule Assessment

	Conceptual Question		Procedural Question	
Group	Pre	Post	Pre	Post
Watched conceptual video	1.39	1.48	0.09	1.26
Watched procedural video	0.04	1.35	1.74	1.57
Significantly different? (p-value)	Yes $p < 0.001$	No $p = 0.210$	Yes $p < 0.001$	No (barely) $p = 0.057$

After working in groups, both groups of students were able to answer the conceptual and procedural questions. No statistically significant differences were found in the two groups' post-class assessment average scores. However, the difference in their post-assessment scores for the procedural question was just barely insignificant.

Implications and Discussion

In reviewing the pre-class assessment results, we were not surprised by how well the students did on the question that related to the type of video they had viewed. However, more than 80% of the students in the class had taken at least one calculus class before. As such, we thought some students would be able to answer both questions successfully, which was not the case. Also, we were surprised by how well both conceptual and procedural understanding was improved by the students working in groups. Our results seem to indicate that students learned conceptual and procedural content from the videos and were able to share that knowledge.

However, there still are some open questions from the data. The post-class assessment scores on the procedural question were just barely insignificantly different and students felt the videos helped them more with conceptual knowledge than with learning procedures. This means we need to take into consideration what content educators teach via video. However, because of the small number of students in this study, more research needs to be done to determine if there is a statistically significant difference in learning gains from more procedural videos than more conceptual ones.

Last, teachers thinking about using videos in their classes should know that students will get at least a basic understanding from videos, whether the videos be more conceptual or procedural. Video lessons alone, however, are not enough; the content from the videos should be clarified and reinforced in class through discussion with peers.

Open Questions

- What balance of conceptual and procedural videos should be used to have the greatest impact on student-learning gains?
- What effect does the use of video-recorded lessons have on specific populations of students (e.g., gender, course of study, non-traditional students, etc.)?
- In what ways are students using the videos? Are they actively engaging with the video lessons (instead of just passively listening like with a lecture)? How can we make the videos more useful and productive for the students?

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Student Interest in Calculus I

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This reports on a secondary analysis of data collected by the Mathematical Association of America's Characteristics of Successful Programs in College Calculus (2015). Survey data were collected from more than 700 instructors, and roughly 14,000 students making these data ideal for multiple level analysis techniques (Raudenbush & Bryk, 2002). Here, these data are used to analyze students' interest in Calculus I. Results suggest that students with higher frequencies of presenting to their classmates, collaborating with peers, working individually, explaining their work, and taking Calculus I with an experienced instructor tend to be more interested in class.

Key words: Student interest; College calculus; Multiple level modeling

This report presents results from a secondary analysis of the data collected as part of the *Characteristics of Successful Programs in College Calculus* (CSPCC) project[1] headed by the Mathematical Association of America ([MAA], 2015). Here, student interest in Calculus I is investigated and associations with different student-level and instructor-level predictors are analyzed using multiple level modeling techniques with SAS 9.4 (Raudenbush & Bryk, 2002).

The CSPSS project administered surveys to Calculus I students, their instructors, and department heads from a nationwide stratified random sample. Roughly 14,000 students, 700 instructors, and 212 institutions participated. Here, only 5,278 students and 378 instructors are analyzed due to the variables used in this analysis (described below).

Methods

This analysis aims to address two research questions: (1) What are the effects of components of classroom activity (described below) on student interest in Calculus I? (2) Do these effects depend on instructor experience with teaching Calculus I?

Variables and Centering

Participants took two surveys (pre- and post-semester) pertaining to their experiences as a student or instructor of Calculus I at the college level. One item on the student survey was, "My instructor makes class interesting." Students were instructed to rank their beliefs of this item on a 6 point scale, where 0 represents "Strongly disagree" and 5 corresponds to "Strongly agree." Here, *interest* will serve as the dependent variable. Students were also asked to rank how often their instructors allowed them class time to collaborate with their peers, present solutions, explain their work, and work individually. Each of these items was also ranked on a 6 point scale, where 0 represents "Not at all" and 5 represents "Very often." These classroom activities will be used as student-level predictor variables.

Instructors were asked to indicate the number of terms they taught Calculus I during the previous five years. This item was reported with a scale of ranged values (e.g., 3-5 times), so a linearized variable was created using the central value from each range. This is the instructor-level predictor variable used in this analysis. All predictor variables were grand-mean centered

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(Raudenbush & Bryk, 2002) to provide meaningful interpretations of results.

Analysis

To begin an unconditional model was conducted. This model is used to partition variation in interest between instructor-level (level 2) variation and student-level variation (level 1). This model is also used to establish a baseline of the overall variation in interest present in the sample.

The second model conducted contains the four student-level classroom activity predictors, instructor experience, and all cross-level interaction variables. This model is constrained; meaning the variation around slopes was constrained to zero. When compared to similar models with unconstrained slopes, the constrained model was a better fit (Singer, 1998).

Level 1: $interest_{ij} = \beta_{0ij} + \beta_{1ij}(COLLAB) + \beta_{2ij}(PSNT) + \beta_{3ij}(EXPN) + \beta_{4ij}(INDV) + r_{ij}$

Level 2:
$$\beta_{0i} = \gamma_{00} + \gamma_{01}(IEXP) + u_{0i}$$
$$\beta_{1i} = \gamma_{10} + \gamma_{11}(IEXP)$$
$$\beta_{2i} = \gamma_{20} + \gamma_{21}(IEXP)$$
$$\beta_{3i} = \gamma_{30} + \gamma_{31}(IEXP)$$
$$\beta_{4i} = \gamma_{40} + \gamma_{41}(IEXP)$$

Results and Discussion

Results from the unconditional model indicate that 76% of the overall variation in student interest resides at the student-level and the remaining 24% at the instructor-level. Also, on average, students slightly agree that their instructors make class time interesting ($\gamma_{00} = 3.26, t = 74.23, p < .001$). Significant variation at level 2 suggests that further analysis is appropriate.

Peer collaboration ($\gamma_{10} = .06, t = 4.45, p < .001$), opportunities to explain ($\gamma_{30} = .35, t = 28.20, p < .001$), and time for individual work in class ($\gamma_{40} = .08, t = 7.01. p < .001$) are all positively associated with students' interest in Calculus I. Opportunities to present ($\gamma_{20} = .008, t = .52, p = .60$) are not associated with student interest. Recent experience with teaching Calculus I is also associated with an increase in student interest ($\gamma_{01} = .11, t = 5.97, p < .001$).

Additionally, the relationships between frequency of collaboration and interest ($\gamma_{11} = .01, t = 2.46, p = .014$) and frequency of student explanations and interest ($\gamma_{31} = -.01, t = -3.14, p < .001$) both depend on instructor experience, such that the effects are intensified in classes taught by instructors with low experience (more than one standard deviation below the sample mean). The relationship between frequency of presentations and interest depends on instructor experience ($\gamma_{21} = -.01, t = -2.61, p = .009$) such that frequent student presentations are detrimental to student interest in classes taught by instructors with high experiences (more than one standard deviation above the sample mean). However, the opposite occurs with frequent presentations in classes taught by instructors with low experience. The relationship between frequency of individual work and interest does not depend on instructor experience ($\gamma_{41} = -.005, t = -1.25, p = .21$). This model explains 24% of student-level variation and 34% of instructor-level variation (Snijders & Bosker, 2011) in student interest in Calculus I.

Conclusion

Classroom activities such as student presentations, peer collaboration, student explanations, and time for individual work were all identified as factors of *ambitious teaching* (Sonnert & Sadler, 2015). These results suggest that instructors, especially those with low experience, should attempt to implement elements of ambitious teaching, and departments should encourage instructors to teach Calculus I often in order to positively affect student interest.

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Student characteristics and online retention:
Preliminary investigation of factors relevant to mathematics course outcomes

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There is evidence that students drop out at higher rates from online than face-to-face courses, yet it is not well understood which students are particularly at risk online. In particular, online mathematics (and other STEM) courses have not been well-studied in the context of larger-scale analyses of online dropout. This study surveyed online and face-to-face students from a large U.S. university system. Results suggest that for online courses generally, student parents and native-born may be particularly vulnerable to poor online-versus-face-to-face course outcomes. The next stage of this research will be to analyze the factors that are relevant to online versus face-to-face retention in mathematics (and other STEM) courses specifically.

Key words: online learning; retention; student characteristics

As more and more courses move to online formats, higher education is undergoing a virtual transformation. By 2013, over 40 million college students took online classes worldwide; by 2017, that number is expected to reach over 120 million post-secondary students globally (Atkins, 2013). On the positive side, online courses may provide increased access to college, removing impediments to college progression by providing the flexibility that "non-traditional" students need. However, because they often have higher attrition (the reasons for which have yet to be determined), online courses may also be detrimental to degree completion (Jaggars, 2011).

Many questions remain about factors that impact course outcomes in online versus face-to-face courses. Further, the factors that impact course outcomes in the online versus face-to-face medium may be different for mathematics courses than for courses in other subjects, yet almost no larger-scale studies have focused on online mathematics courses specifically. In order for policies and advisement to be grounded in research evidence, mathematics education research must address the rapid growth in online learning and the need to focused research on factors impacting outcomes.

Research questions

This initial exploratory study seeks to determine the relationship between student characteristics and online course-taking in order to inform later research as to which factors may impact mathematics course retention and grades specifically:

1. Which student characteristics make a student more likely to enroll in online than face-to-face courses?

2. Which student characteristics exacerbate or mitigate differences in rates of online versus face-to-face course retention and successful course completion?
3. Are there specific groups (e.g. women, racial/ethnic minorities, "non-traditional students") that are particularly successful or particularly vulnerable when taking courses online?
4. How do these patterns differ when comparing mathematics courses to other STEM or non-STEM courses?

Theoretical framework and prior research

Doubling from just under a decade ago, thirty-two percent of U.S. college students enrolled in online courses in 2011-2012 (U.S. Department of Education, Institute of Education Sciences, National Center for Education Statistics (NCES), 2008). Further, since 2010, online enrollment has increased 29% (Allen and Seaman 2010; 2013; CCRC 2013). Numerous studies, including a 200 study meta-analysis, found no significant difference in learning outcomes in online versus face-to-face courses (e.g.(Bernard et al., 2004; Bowen, Chingos, Lack, & Nygren, 2012). Despite these findings, online course dropout in the U.S. ranges from 20-40%, and online attrition rates are reported as 7-20 percentage points higher than those for face-to-face courses (e.g. Hachey, Wladis & Conway, 2013; (Nora & Snyder, 2009; Patterson & McFadden, 2009).

Further, recent research suggests that the gap in attrition between the same courses offered online versus face-to-face can be larger for STEM than for non-STEM courses (Wladis, Hachey, & Conway, 2012). There is also some research that suggests that this gap may be larger for mathematics than for English gatekeeper courses, although differences in the gap between subjects was not tested for statistical significance (Xu & Jaggars, 2011). This may mean that there are factors in the online environment which impact mathematics and other STEM courses differently or more strongly than courses in other subjects. However, previous findings on student characteristics cannot necessarily be generalized to mathematics and other STEM courses specifically. In addition, there is currently little rigorous research on factors affecting retention in online STEM courses specifically. Given the rapid growth in online courses and the already high rates of dropout in many mathematics and STEM courses, it is essential to identify which students are at higher risk in online mathematics (and other STEM) courses, in order to target appropriate support services.

Previous research has found that online learners are more likely to be female, older, married, active military or to have other responsibilities (such as full-time work and/or children) (Shea & Bidjerano, 2014; C. Wladis, Hachey, & Conway, 2015). Additional studies have also found that online students tend to have higher G.P.A.'s, to be white, native English speakers, and to have applied for or received financial aid (Conway, Wladis, & Hachey, n.d.; Jaggers & Xu, 2010; Xu & Jaggars, 2011). Further, online student are more likely to have other "non-traditional" characteristics (e.g. delayed college enrollment; no high school diploma; part-time enrollment; financially independent) (e.g. Shea & Bidjerano, 2014; C. Wladis et al., 2015), and to be first-generation college students (Athabasca University, 2006). And non-traditional characteristics have been shown to be more significant as predictors of online enrollment for STEM than for non-STEM students (Wladis, Hachey, & Conway, 2015).

However, research on demographic variables is conflicting (Jones, 2010) and it remains unclear how different characteristics interact with each other to affect retention in online courses. For instance, Bernard, Brauer, Abrami and Surkes (2004) found that self-direction and beliefs were significant positive predictors of online course grade, however, the evidence showed that G.P.A. was a stronger predictor of online course outcomes. Waschull (2005) reports that self-discipline/motivation was significantly correlated with online course grades, but these same factors may predict success in both online and face-to-face classes. Aragon and Johnson (2008) found that online completers were more likely to be female, enrolled in more classes, and had a higher G.P.A., but unlike Abrami and Surkes (2004), they found no significant difference in academic readiness or self-directed learning.

In a similar vein, other investigations of student characteristics have also been inconclusive. Several studies investigating gender found no differences, whereas others report that females outperform males in terms of outcomes (for a review, see (Xu & Jaggars, 2013)). Angiello (2002) and Xu and Jaggars (2013) report differences in outcomes based on ethnicity while Welsh (2007), Aragon and Johnson (2008) and Wladis, Conway and Hachey (2015) found that ethnicity did not have an impact on online course outcomes. G.P.A is cited as a significant factor impacting online course outcomes in some studies, (e.g. Xu & Jaggars, 2013), but was found to be non-significant in others (e.g. Hachey, Wladis, & Conway, 2012). In one study on STEM courses specifically, older students did significantly better in online STEM courses, and women did significantly worse (although still no worse than men) online, than would be expected based on their outcomes in comparable face-to-face STEM courses, but there was no significant interaction between the online medium and ethnicity (C. Wladis et al., 2015).

Studies focused on online mathematics courses specifically have tended to compare student outcomes across mediums, without attempting to assess which factors predict or contribute to those outcomes (see e.g. Ashby, Sadera, & McNary, 2011; Bowen & Lack, 2012), or they have tended to explore factors that predict successful outcomes in online mathematics classes, without comparing them to face-to-face courses, so that it is impossible to determine whether the factors studied are relevant to learning mathematics online or just to learning mathematics more generally (see e.g. Kim, Park, & Cozart, 2014).

To accurately assess whether a factor puts a student at greater risk in the online environment, it is critical to analyze the *interaction* between that factor and course medium, while simultaneously controlling for self-selection into online courses. This is the only way in which it is possible to determine the extent to which particular factors are important in the online medium specifically, and not simply predictors of academic outcomes more generally. This study addresses an important gap in the literature by doing just that. It is an initial step in determining which factors may need to be explored as impacting outcomes in online mathematics courses specifically.

Methodology

Data source and sample

This study uses a sample of 9,663 students with 37,442 course records from the 18 two- and four-year colleges in the City University of New York (CUNY) system in the U.S. Students were selected if they were enrolled in a course in the sample frame, which consisted of all online and comparable face-to-face courses offered during the 2014 fall semester at one of the CUNY colleges. Of the courses that determined the sample frame, roughly 25% were STEM courses and roughly 10% were mathematics courses. At the end of the semester, students in the sample were invited to participate in an online survey.

Measures

Two measures of student outcomes were utilized: *course retention*, defined as whether a student dropped a course (officially or unofficially); and *successful course completion*, defined as whether the student successfully completed a course with a C- or higher (chosen because it is the typical standard to receive major or transfer credit).

The main independent variable (IV), course medium, was dichotomized to face-to-face or fully online, based on Sloan Consortium definitions (Allen & Seaman, 2010); fully online courses have 80% or more of the course content online, and face-to-face courses have 33% or less of the content online. Previous studies contend that students who take hybrid courses (33-80% online content) are similar to students who take face-to-face courses and further, that their outcomes are similar (Xu & Jaggars, 2011).

The other IVs investigated were chosen because there is evidence that they may: 1) predict online course enrollment; 2) be related to course outcomes more generally 3) be related to face-to-face mathematics and STEM course outcomes specifically; or 4) be significant predictors of outcomes in the online medium. Covariates included in the study are: whether the student had a child (and age of youngest child); gender; race/ethnicity; age; work hours; income; parental education; developmental mathematics, English, and ESL course placement; marital/cohabitation status; immigration generational status; native speaker status; college level (two-year, four-year, or graduate); G.P.A; and number of credits/classes taken that semester. During preliminary analyses, different non-linear versions of variables were explored (e.g. converting credits to part-time/full-time status, squaring age), but these did not produce significantly different results. Also included in the study survey are scales measuring: motivation to complete the course; course enjoyment/engagement; academic integration (i.e. interaction with faculty/students outside class); self-directed learning skills; time management skills; preference for autonomy; and grit (i.e. perseverance and passion for long-term goals). As much as possible, these scales were based on previous instruments that had already been tested for reliability and validity (Duckworth, Peterson, Matthews, & Kelly, 2007; Macan, Shahani, Dipboye, & Phillips, 1990; Pintrich & de Groot, 1990; U.S. Department of Education, Institute of Education Sciences, National Center for Education Statistics, 2009; Vallerand et al., 1992). However, they were shortened and modified for use in this study. Confirmatory factor analysis using structural equation modeling (SEM) was used on the full dataset to model items for each scale as predictors of a single latent construct. Error covariance terms were added between some individual items based on theory, prior to estimation. Some items from the motivation and grit scales were eliminated because of poor performance during SEM. For the final scales, average variance extracted (AVE) was 0.50 or greater, indicating convergent validity, and composite reliability (CR) ranged from 0.77 to 0.89, indicating good reliability (Hair, Anderson, Tatham, & Black, 1998); SRMR ranged from 0.000 to 0.059, supporting the operationalization of each scale as a single factor structure (Hu & Bentler, 1999).

Analytical Approaches

Courses for which valid grades did not exist (e.g. not submitted by instructor, course was audited) were dropped. Multivariate multiple imputation by chained equations was used to impute values for survey questions with missing responses, using all IVs chosen for subsequent analyses. Binomial, ordered, or multinomial logit models, or predictive mean matching on three nearest neighbors was used for imputation depending on variable type. A median of 2.6% of data were missing in each imputed variable in the dataset. After preliminary tests for stability of model estimates, 35 imputations were used.

Propensity scores, indicating the probability of online enrollment, were generated using logistic regression and included all of the IVs used in the subsequent analyses. The scores were averaged across imputed datasets. Because this approach yielded the best balance on covariates based on the standardized bias for each imputed variable averaged across imputations, matched datasets were generated using single nearest-neighbor matching with replacement. The median standardized bias across variables was 2.6%, showing a good balance on all covariates based on Rubin's (2001) rule of thumb. Distribution of propensity scores was evaluated both before and after matching, and there was significant overlap in the region of common support.

The imputed dataset was used to run multilevel mixed-effects logistic regression models with course as the first-level and student as the second-level factors, in order to control for unobserved heterogeneity between students. The KHB decomposition method (Kohler, Karlson, & Holm, 2011) was used to calculate direct and indirect effects, in order to explore the relationship between online course outcomes, student characteristics, and subsequent college persistence. Standard errors during KHB decomposition were computed using clustering by course, to account for the multi-level data structure.

Results

Initial models were run on the whole dataset, including mathematics and non-mathematics courses, in order to look for baseline patterns. Both unmatched and matched datasets were analyzed. For the full dataset consisting of non-STEM, STEM non-mathematics, and mathematics courses, the most consistent predictor of both course retention and successful course completion was being foreign-born. Native-born students were at greater relative risk online compared to foreign-born students, and this was particularly true for native-born students for whom both parents were also native-born. Native-born students with one or no native-born parents were also at increased relative risk online, but the difference was less pronounced. Having a child under six years old was associated with higher risk of unsuccessful course completion or dropout online. No other factors tested were consistently significantly correlated with differential online-versus-face-to-face course retention or successful course completion across different models of the dataset.

The next step in this research project is to repeat these analyses with non-STEM courses, STEM courses, and mathematics courses specifically, to see if different patterns emerge for each group. If there are differences in the patterns observed between groups, then these differences will be tested for significance in an attempt to determine the extent to which different factors are relevant for online STEM and mathematics courses specifically. Further surveys and interviews of online mathematics and STEM students are also being conducted to explore other factors that may be relevant to online mathematics course-taking. Another round of data has recently been collected that explores the following additional constructions: computer and internet self-efficacy (Eastin & La Rose, 2000; Torkzadeh, Chang, & Demirhan, 2006); ethnic/gender identity and stigma consciousness (Picho & Brown, 2011); mathematics (and other STEM subject) self-efficacy and domain identification (May, 2009; Picho & Brown, 2011); sense of belonging (Osterman, 2000); and achievement orientation, e.g. fixed/growth mindset (Dweck, 2006). Interviews are also being conducted with online students; roughly 45 students have been interviewed so far.

Implications

The results of this study will have strong practical applications. If specific factors can be identified that make students particularly at risk of dropout or failure when they take an online mathematics or STEM course, then these students could be targeted for additional interventions (e.g. tutoring, advising, technical assistance) when they enroll in an online mathematics or STEM course. Future research could test the efficacy of various interventions in improving course outcomes for these at-risk groups.

Questions

There are several questions that we see as important as we move this research forward. Specifically:

- Are there particular factors that might be relevant to outcomes in online versus face-to-face STEM and mathematics courses specifically that we have not yet considered?
- We plan to run parallel analyses on non-STEM courses; STEM courses; and mathematics courses. But are there other analyses relevant to mathematics courses specifically that may not be parallel to analyses that would make sense in the context of non-mathematics courses?
- Is there anything else that we have overlooked in our choice of variables and analytical approaches, or in our overall study design, that has not yet been addressed?

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A Comparative Study of Calculus I at a Large Research University

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Abstract: In this report, we describe the results of analyzing data collected from 502 Calculus I students at a large research university in the U.S. Students were enrolled in one of five different versions of Calculus I offered at the university. We are interested in (i) whether the different characteristics of each version of the course affect students' attitudes toward mathematics and (ii) how each course might affect students' intentions of pursuing science, technology, engineering and mathematics (STEM) degrees. We examine data related to these two issues gathered from students via surveys during one semester in Calculus I.

Key words: Calculus, Persistence, Attitudes, Enjoyment

Introduction

There is an urgent need for science, technology, engineering, and mathematics (STEM) graduates (PCAST, 2012), yet studies show that many STEM intending students are dropping out from their majors (Seymour & Hewitt, 1997; Ellis, Kelton & Rasmussen, 2014). In the US, Calculus I is the first college mathematics class many STEM major students must take for their degree program. Results from a recent national study showed that student experiences in Calculus I have significant effects on decisions about pursuing a STEM major as well as their attitudes toward mathematics (Bressoud, Carlson, Mesa & Rasmussen, 2013).

Within our institution, we offer five different versions of Calculus I: a slower paced two-semester course with precalculus (1A/B), an engineering-focused course (E), a non-engineering course (NE), a course for honors students (H), and an Emerging Scholars Program (ESP) course. The courses are offered in different formats as described in Table 1.

Table 1
Course formats for five versions of Calculus I

Calculus I Version	Weekly Course Format	Weekly Contact Hours
1A/B	Three days of lecture, one day of lab	4
E	Three days of lecture, two days of recitation	6
NE	Three days of lecture, two days of recitation	5
H	Four days of lecture	4
ESP	Three days of two-hour inquiry-based learning sessions	6

Theoretical perspective

Tinto (2004) states that “interaction across academic and social geography of a campus shape the educational opportunity structure … and … both student learning and persistence” (p. 92). According to Tinto’s framework of persistence (1975), a satisfaction in the integration of social and academic life on a campus has a significant impact on persistence. We hypothesize that Calculus I experiences of students will significantly change their attitudes toward mathematics and decisions in continuing in a STEM major.

Methodology

We collected data using the same surveys administered by Bressoud et al. (2013) to specifically investigate student enjoyment of mathematics, desire for more mathematics, self-confidence and plans for pursuing more mathematics. Students received a survey between second and third week of the beginning of a fall semester and a follow up survey two weeks before the end of the semester. Extra credit for the completion of the surveys was given to the participating students differently for each course as determined by the coordinator of each version of the course.

Results

We observe a positive shift in students’ confidence in 1A/B, E & NE while there is almost no change in attitudes for those in H & ESP. T-tests show that the positive shifts in E and NE are statistically significant, but not in 1A/B. Moreover, we see a negative change in students’ enjoyment in the same three classes, 1A/B, E and NE, and again almost no change in ESP & H, possibly due to a small number of responses. The t-tests show that the changes are statistically significant in E and NE (both with $p < .0001$). Furthermore, we see a decrease in desire for more mathematics in 1A/B, E, ESP and H, but an increase in NE. The t-tests show that the changes in E ($p = .0065$) and NE ($p < .0001$) are statistically significant. We also see changes with respect to students’ intentions of persisting in a STEM field as in Table 2. Specifically, the contribution to STEM intending from undecided, respectively, in 1A/B, NE, and E is: 18, 3, 15; the contribution to non-STEM intending from undecided respectively in 1A/B, NE, and E is: 12, 16, 3.

Table 2
Changes in Student’s STEM Intentions

	1A/B		NE		E		ESP		H	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
STEM Intending	85	91	67	55	232	232	2	2	5	5
Non-STEM Intending	14	32	17	44	4	16	0	0	0	0
Undecided	34	10	24	9	18	6	0	0	0	0

Conclusions

Results indicate that student experiences in several versions of Calculus I at our institution have an effect on both their attitudes toward mathematics and in their plans for continuing to pursue (or not) a STEM degree. Specifically, we see attitude shifts in the engineering, non-engineering and slower paced two-semester student populations, though not as much in the honors or ESP versions. These findings provide baseline data needed to document and analyze change in these factors as the courses pursue interventions to retain talented STEM majors.

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Graduate students' pedagogical changes using iterative lesson study

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Abstract

Researchers at two universities implemented an iterative lesson study process with ten graduate student instructors (GSIs), five from each university's mathematics department. Over the span of two weeks, each group of GSIs met with a facilitator to collaboratively plan an undergraduate mathematics lesson, implement the lesson, revise their lesson plan, reteach the lesson to another class of students, and complete a final reflection. Using a multiple case study qualitative methodology, we thematically coded GSI consistencies and revisions to lesson planning during the iterative process according to the Principles to Actions national mathematical teaching practices. At both universities there were specific teaching practices that GSIs used throughout the iterative lesson study and specific teaching practices that GSIs revised. Identifying these teaching practices offers insight into the utility and value of iterative lesson study with graduate student instructors.

Keywords: *Lesson Study, Graduate Student Instructors, Multiple Case Study, Measurable Goals, Classroom Tasks*

Given that graduate student instructors (GSIs) serve as instructors of record for hundreds of thousands of undergraduate mathematics students each semester (Belnap & Allred, 2009; Lutzer, Rodi, Kirkman, & Maxwell, 2007), they can significantly impact the quality of mathematics instruction for freshmen and sophomores. Although many mathematics departments acknowledge the need to support mathematics GSIs' learning to teach (Belnap & Allred, 2009; Latulippe, 2009; Speer, Gutmann, & Murphy, 2005; Speer & Murphy, 2009), research on classroom practices of GSIs is severely limited (Speer, Smith III, & Horvath, 2010), and there are only a few studies that examine GSIs' classroom practices (e.g., Gutmann, 2009; Rogers & Steele, in press; Speer, 2008).

Lesson study is a well-established Japanese systematic inquiry into teaching practices where teachers collaboratively create, teach, revise, and reteach lessons to continually grow as educators (Fernandez, 2002). In the last decade, collegiate instructors have begun implementing lesson study (Cerbin & Kopp, 2011; Kaplan, Cervello, & Corcoran, 2009), noting its collaborative format helps instructors develop rich lesson plans and reflect on teaching practices with others. To aid GSIs in developing similarly valued teaching practices at the collegiate level, this study implemented an iterative lesson study with novice[1] GSIs where GSIs had to revise and reflect on their teaching. We posit that the reflections and revisions will actively engage GSIs with teaching theories, offering GSIs an experiential and collaborative foundation for understanding how to address pedagogical concerns that arise during teaching. To bolster current research in GSI pedagogy, this study identifies mathematical teaching practices (NCTM, 2014)

[1] In this paper, we reference *novice* to mean first-year graduate student instructor.

that GSIs noticed and changed when revising their lesson during the iterative lesson study process. Thus this research addresses the following question: *During the iterative lesson study process, how did novice GSIs revise their lesson design and what mathematical teaching practices did they use?*

Framework

Lesson Study Logistics

Lesson study encourages teachers to methodically examine and improve their effectiveness in the classroom (Fernandez, 2002). To this end, a group of teachers will collaboratively create a lesson plan, teach the lesson plan while collecting observable data from the class, and discuss and revise the lesson plan from the observations. These lessons typically focus on instruction (curricular and classroom management issues), students (prior knowledge, student engagement, and anticipatory reasoning), goals (measurable long and short term goals), and content (key concepts and tasks; Stepanek et al., 2007). Usually one member of the group will teach the class, while the rest observe; the group is encouraged to iterate this cycle.

A crucial difficulty associated with implementing lesson study stems from the group logistic dynamics: necessitating a common space and time to collaborate, a common time and setting to teach and observe the lesson, and an asynchronous lesson to allow for iterations (Stepanek et al., 2007). Typically, preservice secondary teachers cannot iterate the process due to logistical issues (Fernandez, 2002; Perry & Lewis, 2009). Working with mathematics GSIs helps alleviate these logistic difficulties because (1) they teach and learn on the same campus, (2) their availability is more predictable (many first-year GSIs take similar mathematics classes), and (3) mathematics educators facilitating lesson study can reserve a space and time for GSIs to meet. Thus lesson study is a viable professional development option for GSIs.

Lesson Study Collaboration

Lesson study also has the potential to develop GSIs' collaborative teaching practices. That is, researchers have shown that, after participating in a lesson study process, prospective and practicing secondary mathematics teachers were more likely to collaborate in the future concerning pedagogical issues (McMahon & Hines, 2008). Lesson study allows cooperation and collaboration to become part of the teaching process, which opens up avenues to creating a community of practice amongst teachers (Stepanek et al., 2007). This is a valuable tool for GSIs because often their learning of mathematics is evaluated by individual homework, course exams, and qualifying/comprehensive exams. Helping GSIs understand that their teaching can be more collaborative than assessments of their learning in their graduate mathematics courses can help develop a *community of practice* (Hart, Alston, & Murata, 2011) in mathematics departments.

Lesson Study with Graduate Students

Despite lesson study being initially adapted for K-12 classrooms, recent studies have included lesson study in many STEM graduate programs. In natural science labs, for example, the use of lesson study processes developed biology GSIs' inquiry-based teaching practices (Miller, Brickman, & Oliver, 2014) and chemistry GSIs' pedagogical content knowledge (Barry & Dotger, 2011). Using lesson study had a positive impact on undergraduate mathematics education by giving mathematics GSIs experience similar to a teaching practicum (Alvine, Judson, Schein, & Yoshida, 2014) and allowing GSIs to critically reflect on teaching (Deshler, 2015). However, in these studies, the lesson study process stopped without iterating the teaching process—without a *revise and reteach* opportunity. To address this significant lack of opportunity in the literature, this study offers the field a lesson study design (Table 1) that includes the iterative process to focus on GSIs' revisions as a means to improve pedagogy.

Lesson Study Measurable Goals

The use of clear instructional objectives is an important feature of the lesson study process and one that the facilitators emphasized because GSIs are asked to collaboratively plan a lesson when they often have little experience planning or teaching college mathematics. First-year mathematics graduate students also typically have limited prior experiences taking pedagogical courses (Speer, Gutmann, & Murphy, 2005). Hiebert, Morris, and Glass (2003) emphasize learning to teach by treating lessons as experiments, suggesting novice teachers need to have clear and measurable goals. A primary feature of this lesson study process revolved around GSIs defining goals they would observe and measure during their lesson to require GSIs to actively engage with student learning and not focus solely on their teaching presentation.

Method

Mathematics educators from two universities designed and facilitated this lesson study process, implementing the same process and analysis for both of the cases.

Participants

Ten GSIs from two different universities volunteered to participate. These GSIs formed two groups of five, and all participants and names for the groups are pseudonyms. From one university, five novice mathematics graduate students, who were also recitation instructors, volunteered to participate to help guide their transition to instructors of record the following semester. This group is called the *Calc* lesson study group.

The other group consisted of five novice mathematics and statistics graduate students from another university who participated as part of a mathematics pedagogy course. Although the completion of the lesson study process was required for course credit, participation in the research study was voluntary, and all graduate students participated. These GSIs were preparing to be instructors of record in the following semester and formed the *Stats* lesson study group.

Lesson study process

Researchers followed the same lesson study design (Table 1) using identical handouts for each session and framework to collect GSI data. Data sources included video, audio, and (undergraduate and graduate) student work. Through regular meetings over the span of two weeks, GSIs met with the facilitator to go through the lesson study process, implement the lesson, revise the lesson plan, reteach the lesson, and complete a final reflection (Table 1).

Table 1

Lesson Study Process with Graduate Student Instructors

Session	Time (hrs)	Description	Outcome of Session
Introduction	1	Introduce lesson study process (Stepanek et al., 2007), sign consent forms, & discuss logistics for teaching.	GSIs determined the course & section for the lesson.
Goal Writing	3	GSIs learn about conceptual and procedural goals, identify measurable goals for their lesson, determine how they will measure those goals, & identify how they will collect data to see if each goal is achieved.	GSIs stated goals, metric for each goal, & data collection methods written clearly.
Mathematical Task	3	Identify high and low level mathematical tasks (Smith & Stein, 1998), create appropriate tasks for each lesson goal, & integrate measurements for goals with tasks.	Task(s) created and aligned with learning goals, metrics for each goal refined in light of mathematical task, & sketched lesson design.
Lesson Plan	2	Integrate goals and tasks with lesson design using four-column technique (Matthews, Hlas, & Finken, 2009).	Four-column lesson plan with activities, anticipated student responses, anticipated teacher responses, & alignment with goals.
Initial Teaching of Lesson	1	Video recorded and observed by other GSIs. GSIs also walk around and take notes of student work.	GSI notes on lesson & measures of goals via observation form.

Revision	2	Discuss what went well, reviewed sections of initial lesson looking for changes, & identify what goals were met, why or why not, making necessary changes.	Identify if goals were met by measurable data. Modified lesson plan intended to meet all desired goals for second iteration.
Second Teaching of Lesson	1	Video recorded and observed by other GSIs. GSIs also walk around and take notes of student work.	GSI notes on lesson & measures of goals via observation form.
Reflection	1	Discuss what went well, what goals were met, & if changes were effective or not and why. Also, some reflection on the entire process.	Identify if goals were met by measurable data. Reflect on the value of the lesson study and what was learned through the process.

Both universities used the same lesson study sessions, handouts, and observation forms to compare GSIs' lesson-plan revisions and mathematical teaching practices. To focus on the research question, researchers kept track of how GSIs changed the lesson plan according to pedagogical issues through the iterative lesson study, as follows: Using a multiple case study qualitative methodology and *naturalistic inquiry* (Lincoln & Guba, 1985), researchers themed GSI revisions relative to their goals and tasks. When theming revisions, researchers referenced the eight mathematical teaching practices (MTPs) as described in the *Principles to Actions* (NCTM, 2014) because these nationally recognized practices are designed to "provide a framework for strengthening the teaching and learning of mathematics" (p. 9).

Results

For structure, each case study includes a brief description of the teaching setting, a table summarizing the lesson goals, revisions, and observations, and a description of each group's maintenances and changes in mathematics teaching practices. A commonality across both groups is that they did not change their goal statements, but modified the lesson design and planned questions to try to better meet their objectives.

Calc Case Study

Teaching Setting. Due to scheduling demands, Calc decided to teach a lesson on area between curves to students in Calculus I because 60% of the GSIs would be running recitation for this content in two weeks. First, Alfonzo taught the lesson to a lecture-sized class of 64, using group work with 16 groups of four. Second, Aaron taught the revised lesson to his recitation class of 32 students. Table 2 describes the three goals related to mathematical tasks the GSIs designed their lesson to measure.

Thematic Revisions. Calc decided that to observe and measure students' work during class, a group structure would be the most efficient. After the initial lesson, Calc chose not to modify the goals or group structure, only the examples and tasks, to address two pedagogical issues. Abe stated, "In the second example, Alfonzo gave them the intersection point which limited their understanding of how to find an intersection." Anna agreed saying, "If he hadn't given them the intersection points, they would have struggled in a good way." Since students struggled when finding intersection points, they ran out of time to evaluate their integral to find the area between the two curves (Goal 3).

Table 2

Calc Lesson Goals, Revisions, and Observations

Measureable Goal	Did GSIs Conclude the Goal was Met After the Initial Lesson? How did they know?	Revisions	Did GSIs Conclude that the Goal was Met After Second Lesson? How did they know?
Goal 1: Students will identify intersection points and determine which function is on "top" versus "bottom."	Partially. In a task, a majority of students were able to identify and discuss "top" and "bottom" functions, but were not able to identify intersection points because the intersection points of the two curves were given as the endpoints of the interval.	The endpoints for the tasks' intervals changed. One task became an open-ended question where students had to determine the endpoints and the intersection points of the functions.	Yes. The open-ended question forced every group discussions on how to determine intersection points of graphs which lead to meaningful group conversations about determining intersections.

Goal 2: Students will be able to switch the "top" and "bottom" functions to set up the region of integration.	Partially. Polling after a task, a majority of students understood the need to switch the "top" and "bottom," but to set up the integral, they had to know the points of intersection (Goal 1)	Modified the instruction to illustrate how to more clearly identify intersection points of graphs algebraically.	Partially. Polling indicated half the students still struggled with multiple integrals and multiple points of intersection.
Goal 3: Students will be able to evaluate the desired integral to find the area between curves.	Inconclusive. Limited time had many groups of students not get to the last part of a task addressing this goal; results could not be determined.	More time given for the open-ended task and time was removed from the task with the modifications to the endpoints (Goal 1)	Yes. Seven of the eight groups evaluated the integral they created in the first task appropriately using the power rule.

To resolve these issues, Calc chose to make the first task an open-ended question to provoke meaningful discussions about what interval was appropriate to find the area between curves and how to identify intersection points of curves. Aaron hypothesized this change would better develop students' understanding of when and how to set equations equal to determine intersection points of curves. Aaron's hypothesis was proven true when all eight groups were heard discussing how to find the intersection points. As a result, Aaron's students spent more time on the first task, but then applied their understanding of intersections to move more quickly through the remaining tasks, accomplishing Goals 1 and 3.

Calc held to their measurable goals, the use of mathematical tasks, and observational data to measure their goals throughout the entire lesson study process. Thus Calc maintained established mathematical goals to focus learning (MTP1), implemented tasks that promote reasoning and problem solving (MTP2), and elicited evidence of student thinking (MTP8). Calc's main revisions stemmed from their observations of student work on tasks. Thematically, their revisions modified the first task to be an open-ended question and more cognitively demanding. The modified task promoted small-group discussion and GSIs saw how modifications of tasks can facilitate meaningful mathematical discourse (MTP4), which helped efficiently facilitate other tasks and goals by building procedural fluency from the conceptual understanding (MTP6) of how to find intersection points of two curves. Although difficult at first, the revised tasks encouraged productive struggles in learning mathematics (MTP7).

Stats Case Study

Lesson Setting. Due to logistics and timing, Stats taught an introductory statistics lesson on linear correlation. Sam was the instructor for the first lesson (50mins with 20 undergraduates) and Steve taught the second lesson (50mins with 21 undergraduates). Their study lesson included four goals and one mathematical task that incorporated four main activities (Table 3).

Thematic Revisions. Stats had students work in a variety of group structures: in pairs, in groups of four (by pairing the pairs together), and as a whole class so students could make sense of content with their classmates and Stat could observe and measure students' work during class. After Sam's teaching, Stats also did not change the goals or overall structure of the lesson (Table 3). They focused, instead, on modifying the lesson plan to address two pedagogical issues: student misconceptions and pacing of the lesson. First, Stats realized that students expressed an unanticipated misconception: Sarah observed that "a couple of people were confused that . . . it doesn't matter what the slope is. Students' reason for a graph with $r = 0$ was 'because it's a horizontal line' . . . not because the points were really spread out." Other Stats members agreed that students considered r-values as the slope of the line of best fit rather than the descriptor for the strength of the correlation, leading to confusion about how to tell the strength of correlation in a scatterplot (Goal 3). To address this unanticipated misconception, Stats discussed how "it could have been explained better" (Sam). Suzie suggested incorporating an example during the introduction of the activity for Goal 3 where you have "two lines of best fit, both with the same slope but with different r-values, and see that the r-value is higher for the one with the points closer to the line." Steve incorporated this suggestion in the revised lesson and students seemed to follow along with this explanation. However, Stats did not modify the associated task, nor their way of measuring if Goal 3 was met. Thus students still needed clarification to more thoroughly understand the differences between $r = -0.42$ and $r = -0.72$.

Second, Stats recognized a number of places where "the big thing for the lesson is . . . time management" (Suzie) because Sam's discussions took longer than anticipated, leaving insufficient time for the closure activity (Goal 4); thus, they revised the lesson plan to address the pacing of the lesson (e.g., explicitly announcing when groups needed to wrap up part of the activity, grouping students up to make certain transitions smoother, and polling the class to indicate if it was OK to move on). Steve incorporated many of the group's suggestions (e.g., explicitly pairing and then grouping up students), which encouraged students to interact with their classmates and ask additional questions during the second iteration of teaching. Although pacing of the second study lesson improved, students still had less time than planned (~4mins) for the closure. Realizing they were running behind, Steve modified the participatory structure of the closure on the spot, encouraging students to work with their partner, thereby shortening the final activity in hopes of still addressing the final goal.

Table 3

Stats Lesson Goals, Revisions, and Observations

Measureable Goal	Did GSIs Conclude that the Goal was Met After Initial Lesson? How did they know?	Revisions	Did GSIs Conclude that the Goal was Met After Second Lesson? How did they know?
Goal 1: Students will be able to explain that correlation is not causation.	Partially. Correct responses elicited, but only from a handful of students (volunteered or called on). GSIs unsure if a majority of the class understood confounding variables and appropriate conclusions.	Additions to lesson plan for instructor: (1) ask students to explain their reasoning during whole-class discussion, and (2) poll the class to ask everyone to indicate (thumb up or down) agreement with conclusions or reasoning shared.	Partially. Discussion involved a wider variety of participants; students asked clarifying & contextual questions. Instructor pressed students for more examples, but felt pressed for time forgetting to use the polling strategy. More data needed to conclude how a majority of the class understood.
Goal 2: Students should recognize the correlation must be linear to calculate r.	Yes. Instructor called on each group, and each group shared at least one answer and all answers were correct.	Managing group dynamics: have instructor explicitly group pairs of students (from earlier in lesson) into groups of four to facilitate group work time.	Yes. The transition into groups of four went more smoothly, and responses from each group as they were called on were correct.
Goal 3: Students will be able to associate r-values with scatterplots	Partially. Groups mostly provided correct answers during class discussion, but expressed confusion about how strongly correlated the graph was and how to tell.	Amended lesson plan to include more explicit instruction about slope of a line being different from the r-value before students worked on this question to address the students' apparent source of confusion	Partially. The added explanation seemed to help guide the activity more clearly, and most answers from groups were correct values. Some confusion remained as to how to determine if a given scatterplot had an r-value closer to -0.42 or -0.72, for example.
Goal 4: Using provided technology (excel or statcrunch), students will be able to calculate r from two lists of numbers.	Inconclusive. Groups ran out of time and many only calculated some of the r-values. Many groups did not attempt to answer the questions about what conclusions could be drawn.	Timing recommendations: stressed areas where instructor could give students explicit instructions about how much time was left to move earlier parts of the lesson along faster.	Inconclusive. Time was still a factor so instructor encouraged students to work in pairs (fewer r-values to calculate), working through the open-ended closure more quickly. Insufficient data about a majority of students' understanding.

As was also observed for Calc, Stats maintained established mathematical goals to focus learning (MTP1), implemented tasks that promote reasoning and problem solving (MTP2), and elicited evidence of student thinking (MTP8). Stats' revisions stemmed primarily from their need to address an unanticipated misconception and desire to build in enough time to provide opportunities for students to generate their own understanding of relevant content relevant. The need to elicit and use evidence of student thinking (MTP8) was further reinforced by the fact that it was only through observing students' responses during Sam's teaching that the group was aware of the slope vs. *r*-value misconception. During Steve's teaching, his incorporation of specific examples pertaining to that misconception further reinforced the need for the group to elicit more evidence of students' thinking (MTP8) than what they planned to determine if this change helped address their goal. Through the modifications for the pacing of the lesson, Stats observed how modifications in student participatory structures could support meaningful mathematical discourse (MTP4) because students interacted with their group members more during Steve's lesson and raised additional questions about the material (Table 3, Goal 1, Column 4). Finally, it is important to note that Steve recognized the need to modify the closure activity on the spot because the group stressed the importance of making more time for the closure to help address Goal 4. Steve modified the closure activity in a way to encourage productive struggles in learning mathematics (MTP7) by maintaining the open-ended, cognitively demanding features of the activity, but encouraging students to talk with one classmate (instead of three) to make sense of the statistical ideas.

Discussion

By comparing two universities' iterative lesson study process, this study (1) identified *mathematical* teaching *practices* (NCTM, 2014) GSIs used consistently and revised, and (2) demonstrated the utility of collaboratively reteaching and revising lessons as a GSI professional development tool. The revisions and second teaching iterations have rarely been examined in lesson study literature. This iterative process provided an opportunity for both groups to implement the changes they deemed necessary to more clearly address their goals. Table 4 states the mathematical teaching practices GSIs used and revised during this iterative lesson study process. In Table 4, after each mathematical teaching practice, there is a reference to the GSI choices in Tables 2 and 3 that justify the coding.

Table 4

Iterative Lesson Study's Mathematics Teaching Practices Thematic Revisions

	Calc Lesson Study	Stats Lesson Study
MTPs with GSIs Throughout Both Lessons	(MTP1) Establish mathematics goals to focus learning (MTP2) Implement tasks that promote reasoning and problem solving (MTP8) Elicit and use evidence of student thinking	(MTP1) Establish mathematics goals to focus learning (MTP2) Implement tasks that promote reasoning and problem solving (MTP8) Elicit and use evidence of student thinking
MTPs that Changed with Revisions	(MTP4) Facilitate meaningful discourse, *Table 2, Goal 1 Revisions* (MTP6) Build procedural fluency from conceptual fluency, *Table 2, Goal 3 Revisions* (MTP7) Support productive struggle in learning mathematics, *Table 2, Goal 1 Revisions*	(MTP4) Facilitate meaningful discourse, *Table 3, Goal 1, Column 4* (MTP7) Support productive struggle in learning mathematics, *Table 3, Goal 4, Revisions & Column 4* (MTP8) Elicit and use evidence of student thinking, *Table 3, Goal 3, Revisions & Column 4*

To answer our research question, these results demonstrate this iterative lesson study process encouraged GSIs at both universities to consistently (1) establish mathematical goals to focus learning (MTP1), (2) implement tasks that promote reasoning and problem solving (MTP2), and (3) elicit evidence of student thinking (MTP8). At both universities, GSIs revised their lessons to facilitate meaningful discourse (MTP4) and support productive struggle in learning mathematics (MTP7). Useful future research could more specifically examine how the use of measurable goals, highly cognitive tasks, and the iterative process lead to similar mathematical teaching practices at both universities.

The results of this study indicate how revisions with the iterative lesson study lead to specific teaching practices being addressed. Thus, GSI educators can use a lesson study process to actively involve GSIs in learning about pedagogical concerns in undergraduate mathematics education prior to, or in conjunction with, GSIs learning about specific pedagogical topics and theories; thereby reinforcing or supporting GSIs' understanding of collegiate mathematics pedagogy. This study provides GSI educators with a format for iterative lesson study for GSIs as well as specific teaching practices that GSIs can gain via this teaching practicum.

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Students' Experiences and Perceptions of an Inquiry-Based Model of Supplemental Instruction for Calculus

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The Inquiry-Based Instructional Support (IBIS) workshop model is part of an innovative degree program designed to prepare elementary mathematics teachers. The reason behind IBIS workshops was to support students enrolled in "historically difficult" mathematics courses, such as Calculus I and Calculus II. The design of IBIS workshop was framed and guided by Peer-Led Team Learning (PLTL) (Gosser & Roth, 1998) and Complex Instruction (Cohen, 1994). During workshop, students work in small groups and engage in "groupworthy" mathematical tasks that promote their conceptual understanding of Calculus topics (Cohen, 1994). A pilot study was conducted to evaluate the workshop structure and these tasks. In order to assess students' workshop experiences, follow-up interviews were conducted. Students' responses indicated that their workshop experiences helped to promote the development of their problem solving skills and highlighted the critical roles of thinking and reasoning in learning Calculus with understanding.

Keywords: [Inquiry-based learning, groupwork, Peer-Led Team Learning, Complex Instruction, Calculus]

Noyce @ Montclair is an innovative degree program designed to provide outstanding preparation for prospective elementary mathematics teachers. One of the enhancing components of this degree program is a series of Inquiry-Based Instructional Support (IBIS) workshops for students enrolled in "historically difficult" mathematics courses, such as Calculus I and Calculus II. Two existing models of academic support, Peer-Led Team Learning (PLTL) and Complex Instruction (CI), informed the development of IBIS. A pilot study was conducted in the fall of 2014 to examine what students learned from their participation in IBIS workshops.

Theoretical Framework

Peer-Led Team Learning (PLTL)

PLTL is an education intervention that included the use of well-trained peer leaders to facilitate small study groups into a part of the course structure (Gafney & Varma-Nelson, 2007). PLTL consists of six critical components: (1) workshop is integrated part of the course; therefore attendance is mandatory; (2) course faculties are closely involved in workshop organization and peer leader training; (3) all peer leaders are selected, well trained, and supervised; (4) workshop materials, also called modules, are challenging and appropriate; (5) workshops are designed for small groups of six to eight students; and (6) department and institution support, encourage, and acknowledge contemporary learning and teaching (Varma-Nelson, Cracolice, & Grosser, 2004).

Complex Instruction (CI)

CI is an instructional approach that utilizes cooperative groupwork for effective teaching in diverse classrooms. Complex Instruction emphasizes the value of groupworthy tasks, which are designed to facilitate the development of students' conceptual understanding. Initially, group members engage in training activities to begin to develop proper groupwork skills. Inside of a CI classroom, student interactions are viewed as a learning resource (Cohen, 1994).

Groupworthy Tasks

Buell, Greenstein, and Wilstein (*to appear*) proposed five considerations for designing problems and tasks that are groupworthy. The first consideration focuses on the high cognitive demand nature of tasks. This requires students to look beyond procedures and examine the underlying conceptual ideas. The second consideration refers to tasks with multiple entry points that allow students with different levels of understandings to get access of the tasks. With the third consideration, tasks should "open up the space" to allow for multiple pathways reach possible solutions. The fourth consideration suggests for tasks to be thought-revealing in order to promote discussions and collaboration among students. With the last consideration, tasks should be realistics to the students so the contexts are meaningful to them (Buell, Greenstein, & Wilstein, *to appear*). In order to develop workshop modules that focus on the development of students' conceptual understanding of Calculus, these five considerations were used to guide the development of our workshop modules.

Methods

Participants

There were nine students that attended IBIS workshops. These students were enrolled in Calculus I. Out of these nine students, four participated in the follow-up focus group interviews conducted at the end of the semester.

Data Collection

During the pilot study, each workshop session was videotaped and students' work on the workshop module was collected at the end of each workshop session. The follow-up focus group interviews conducted at the end of the semester were also videotaped. The purpose of this interview was to gain a better understanding of how student experienced and perceived IBIS workshops.

Data Analysis

The video interviews were transcribed. The interview transcripts were first individually coded by four researchers. The codes were then shared and discussed amongst the researchers to achieve consensus. The researchers took turns to interpret the codes and each interpretation was justified and explained.

Results

Instead of focusing on the correctness of the solutions, students were encouraged to think and reason about their approaches to the module problems. This experience is evident in the result of the interviews, as students' responses indicated that during IBIS workshop opportunities were provided for them to promoted the development of their problem solving skills, which is something that they found lacking in their Calculus classes. Students acknowledged that IBIS provided an environment that promoted, engaged, and focused on the development of their conceptual knowledge. Further, students identified that they need to think and reason in order to understand and learn Calculus. They also expressed that in IBIS, they are encouraged to be persistent to overcome obstacles and challenges. This resulted from students' interactions with their group members and the modules. Students were always encouraged to help each other while working on modules that have high cognitive demands.

Conclusion

The purpose of this study was to investigate what students learned from participation in IBIS workshops. The results suggest that workshop experiences helped to promote students' problem solving skills, conceptual understanding, and perseverance on solving challenging mathematical problems. Further research is required to examine the possible impacts that IBIS has on both workshop students and peer leaders with a larger population. This is our first step to building a sustain inquiry-based support system for students to success in the Calculus sequence.

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Service-learning in a precalculus class: Tutoring improves the course performance of the tutor.

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We have introduced an experiment: as part of a Precalculus class, university students have been tutoring algebra prerequisites to students from the community via an academic service-learning program. The goal of the experiment was to improve university students' mastery of basic algebra and to quantitatively describe benefits of service-learning to students' performance in mathematics. At the end of the experiment, we observed 59% decrease of basic algebraic errors between experimental and control sections. The setup and analysis of the study have been informed by the theoretical research on service-learning and peer learning, both grounded in the constructivist theory of John Dewey.

Key words: Precalculus, design experiment, service-learning

Introduction and Research Questions

Academic service-learning consists of two integral components: a useful service to the community, and a meaningful learning opportunity to the students, which is relevant to the material covered in the course (Hadlock, 2013). Astin, Vogelsang, Lori, Ikeda, and Yee (2000) found that service-learning shows positive effects on academic performance (GPA, writing and critical thinking skills) and values of participating students. Service-learning in mathematics courses has recently been gaining prominence (Hadlock, 2005), and our present study was motivated in part by the need for a quantitative analysis of the benefits of service-learning to students' mathematical performance.

Our second motivation was the ever-present need to improve student success and retention in Calculus (Bressoud, Mesa & Rasmussen, 2015). Edge and Friedberg (1984) show that solid algebra skills are one of the main factors determining success in Calculus. Success of the service-learning project raises students' fluency in algebra and leads to a stronger chance of their mastering Calculus, and staying within their chosen technical field.

Our study explored the following research questions:

Question 1: Will students who engage in tutoring algebra pre-requisites to middle-school and returning students demonstrate fewer 'fundamental' mistakes than students from the control section without the tutoring experience? By 'fundamental' mistakes we mean the following:

1. *Mistakes that result from misunderstanding the addition/subtraction algorithm of*
    a. *numerical fractions*
    b. *rational expressions*
2. *Cancellation mistakes in*
    a. *numerical fraction arithmetic*
    b. *rational expressions*
3. *Mistakes in operations on radicals*
4. *Mistakes in operations with exponents.*
5. *Mistakes in basic factoring using formulas.*
6. *Other – may be added after consultations with other mathematics faculty, or after marking the final exam.*

Question 2: What will be the reaction of students to the service-learning experience introduced in a scientific course that has not traditionally been associated with community work at this and other institutions?

Theoretical Perspective

Our framework for the present study follows a standard pseudo-experimental setup as described by McKnight, Magid, Murphy, and McKnight (2000): baseline performance for experimental and control sections is determined via a diagnostic test; the two sections receive equivalent instruction for the duration of the course, except for the difference in the tutoring service-learning component. The two sections are given identical final exam, and their performance is analyzed via a rubric. Qualitative data is also compared. To our knowledge ours is the first study that quantitatively analyses benefit to mathematical performance of service-learning students engaged in tutoring.

Our idea to use tutoring as a means to help student-tutors learn mathematics is rooted in the long-standing theory that underlies Peer Learning in general. From a well-known saying 'I hear and I forget. I see and I remember. I do and I understand', to the theoretical work of Allen and Feldman (1976), tutoring has been shown to benefit the tutor, as well as the tutee.

When designing and implementing the service-learning structure, we closely followed the suggestions and project design outlined in the Special Issue on Service-Learning in Mathematics, PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies (2013), particularly Schulteis' (2013) experience of building a course with university students' satisfying the tutoring needs of local institutions and non-profit organizations.

In order to fully benefit from the service-learning experience, students must have an opportunity to engage in structured reflection and connect for themselves the tutoring experience with the content of the course (Bringle & Hatcher, 1999). In building the theoretical foundations of service-learning on the basis of the experimentalism of John Dewey, Giles and Eyler (1994) name reflection as the means of converting experience into knowledge. An integral part of the experimental service-learning section was a weekly guided reflective diary of tutoring experiences, helping students analyze mathematical, as well as social and pedagogical, aspects of their work with the students from the community.

Methodology

Our work took place at Seattle University: a medium-sized urban Catholic university in the heart of Seattle, WA, with a long tradition of incorporating service-learning and community work into students' coursework and extra-curricular education. The setting for our study was two sections of a standard Precalculus course that served as a pre-requisite for the science and engineering track Calculus sequence. The course focused on advanced algebra material and served as a mathematics refresher for students whose ACT and SAT scores would not allow them to be placed directly into a Calculus I course. Both sections numbered 21 students each. The experimental section of the Precalculus course involved a service-learning component, while the other section served as control and consisted of standard in-class instruction only.

The students in the experimental section took part in the established university Service Learning program. They spent 2-3 hours per week tutoring basic algebra and sometimes arithmetic to middle school students, students at immigrant assistance centers, as well as adults returning to complete their education at a local community college. The students put in a total of 18-21 hours of tutoring work during the 12-week duration of the quarter.

The control section was identical to the service-learning section in every aspect of the course syllabus, such as the topics covered, the number of exams, attendance and make-up policy, etc. The only difference was the lack of the tutoring component in the control section, as well as a slight difference in grading weights assigned to exams and homework. The control section also received some amount of extra homework intended to balance the additional workload faced by experimental section.

We established a baseline of the students' prior knowledge and preparation by using a diagnostic pre-test. The pre-test was given to both sections on the first day of class and covered the pre-requisite material including arithmetic with fractions and radicals, basic factoring, and solving basic equations. In order to make sure the task was taken seriously, the students received a small amount of credit for completing the pre-test.

To connect the tutoring experience to the algebra content, the students in the experimental section kept a weekly reflective diary which included mathematical and non-mathematical components. Mathematical reflection helped the students analyze the mathematical component of the tutoring experience and reflect on the following and similar questions:

- What problem did you discuss with your student? What mathematical concept did you address?
- What piece of knowledge was missing from the students' understanding that prevented them from moving forward?
- What method did you use to approach the solution and how did you explain the material?
- Did you discover any gaps in your own mathematical knowledge? What steps did you take to address them?
- Did you discover any parallels between the topics you tutored and our mathematical lectures and problems from class?

The free-form non-mathematical reflection was intended to help the students process the human aspect of their experience with service-learning and tutoring. The following guiding questions were suggested to the students: "What do you think is holding this student back and what can be done to help the student succeed? How was your tutoring week? What non-mathematical problems did you encounter? Any thoughts on what you are seeing and experiencing while tutoring? Any questions you would like to ask me, or your fellow students, or the management of the organizations where you tutor?"

In addition, the experimental section held two in-class reflection meetings, offering the participants an opportunity to discuss the pedagogical issues raised by the students themselves, through the diaries or in class. The students also had a chance to address practical matters of service-learning, such as transportation, time commitment, communication with the community partners.

At the end of the course, the students submitted a typed anonymous reflection where they were free to comment on any aspect of their experience, to offer suggestions for improvement, and to voice any additional concerns regarding the course.

At the end of the quarter, both sections took a standard final exam with identical questions. Relevant statistics were computed and compared for both sections. The number of fundamental mistakes (see Introduction) was determined via a special rubric.

Results

Our diagnostic pre-test indicated that the experimental and control sections were comparable in preparation and abilities and showed similar score distributions, with the experimental section showing a slightly better average, but the difference between the two sections not being statistically significant.

Research question 1 was answered affirmatively. We compared the number of fundamental mistakes in the final exams for both sections: there were only 13 fundamental mistakes made by the 21 students of the experimental section, while the 21 students of the control section made 32 fundamental mistakes. Thus, there were 59% fewer fundamental mistakes in the experimental section than in the control one.

The course average for the experimental section was higher, due to the difference in the weights given to individual course components.

Data from the reflective diaries and the end-of-term anonymous reflection indicate that the answer to the second research question was also positive: out of 21 submitted anonymous reflections, 20 were positive, reflecting a sense of accomplishment and a clear understanding of the privilege of university education, as well as the appreciation of new friendships. Similarly to Butler (2013), we observed a number of service-learning benefits that went far beyond the original goal of the project, including an increased level of confidence in oral communication skills mentioned by the international students. After the quarter ended, several students voluntarily continued their work with the community partners.

In their diaries, the students enthusiastically pointed out multiple connections between the mathematical concepts covered in class and the topics they had explained in the tutoring sessions. Students rediscovered for themselves that the underlying concepts and definitions were in fact the same for the polynomial graphs and the radical equations covered in class, and the basic linear graphs and equations their tutees had studied in middle school. As Roscoe and Chi (2007) point out, peer tutors manifest highest levels of tutor learning as a result of explaining conceptual rather than process-based questions to the tutees. In our case, reflection diaries worked as a tool to reinforce mathematical knowledge gains made by the tutor as a result of the tutoring session.

Conclusions and Implications for Mathematics Education

Our research statistically establishes a number of tangible benefits of service-learning to students' mathematical performance in class. The non-mathematical benefits have been widely explored, and they are confirmed by our study. Our research opens venues to further exploration of the long-term academic and non-academic benefits of service-learning to the university students, as well as to students from the community. Service-learning requires commitment of time and sometimes additional funding: our findings may encourage Mathematics departments and university administration to promote service-learning in mathematics courses.

For discussion

We will be grateful for any comments and ideas regarding the following topics.

- Please suggest additional theoretical frameworks for exploring the setup and results of the experiment.
- Please suggest instruments and experiments to assess changes in students' knowledge and academic performance in Calculus courses, following their service-learning experience in the Precalculus course.

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Symbolizing and Brokering in an Inquiry Oriented Linear Algebra Classroom

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The purpose of this paper is to explore the role of symbolizing and brokering in fostering classroom inquiry. We characterize inquiry both as student inquiry into the mathematics and instructor's inquiry into the students' mathematics. Disciplinary practices of mathematics are the ways that mathematicians go about their profession and include practices such as conjecturing, defining, symbolizing, and algorithmatizing. In this paper we present examples of students and their instructor engaging in the practice of symbolizing in four ways. We integrate this analysis with details regarding how the instructor serves as a broker between the classroom community and the broader mathematical community.

Key words: symbolizing, brokering, inquiry, linear algebra, disciplinary practices

Creating and sustaining engaged classrooms in which students learn particular mathematics and develop positive mathematical dispositions that transcend course-specific concepts is a daunting and challenging endeavor. For instructors, these challenges include (a) creating or selecting tasks that afford opportunities for students to learn mathematics by doing mathematics, (b) leading and facilitating small group and whole class discussions in which student ideas are shared and valued, and (c) relating students' intuitive, informal, or blossoming ideas to conventional and more formal mathematics. We refer to classrooms where these challenges are realized as "inquiry-oriented." Prior research (e.g., Laursen, Hassi, Kogan, & Weston, 2014; Rasmussen, Kwon, Allen, Marrongelle, & Burtch, 2006) points to the power of engaging students in typical mathematical practices through inquiry in undergraduate mathematics. As such, the need exists to further investigate and understand the relationships between what students do, what the instructor does, and the role of tasks in these inquiry-oriented classrooms. This report, using symbolizing in an inquiry-oriented linear algebra classroom as a case study, makes a contribution toward this need.

Literature and Theoretical Framing

We operationalize the notion of inquiry, using the definition put forth by Rasmussen and Kwon (2007), both in terms of what students do and what instructors do in relation to student activity. On the one hand, students learn mathematics through inquiry as they work on challenging problems that engage them in typical mathematical practices, which we refer to as disciplinary practices. Disciplinary practices of mathematics are the ways that mathematicians go about their profession and include practices such as defining, theoremizing, symbolizing, and algorithmatizing (Rasmussen, Wawro, & Zandieh, 2015; Rasmussen, Zandieh, King, & Teppo; 2005). On the other hand, instructors engage in inquiry by listening to student ideas, responding to student thinking, and using student thinking to advance the mathematical agenda of the classroom community (Rasmussen & Kwon, 2007).

Brokering

In addition to characterizing what constitutes disciplinary practices, over the years we have also developed and refined the work of instructors in leading inquiry-oriented classrooms (e.g., Rasmussen & Marrongelle, 2006; Rasmussen, Zandieh, & Wawro, 2009; Wawro, 2014; Zandieh

& Rasmussen, 2010). As part of this work, we adapted the idea of broker from the communities of practice literature (Lave & Wenger, 1991; Star & Griesemer, 1989; Wenger, 1998) to help make sense of the difficult work of inquiry-oriented instruction. By definition, a broker is someone who can facilitate communication and fluidity of practices between different communities and who has membership status in the different communities. Here we consider two different communities: the local classroom community and the broader mathematical community. Typically, the instructor has membership status in both communities. More importantly for us, brokers link practices (in our case defining, conjecturing, proving, etc.) between communities and are able to promote learning by introducing into the classroom community elements of practice from the broader mathematical community.

In previous work, we examined the case of students reinventing a bifurcation diagram in a first course in differential equations and the role of the instructor in this process. This analysis revealed three different types of instructor brokering moves: creating a boundary encounter, bringing participants to the periphery, and interpreting between communities (Rasmussen et al., 2009). In this paper we highlight the first and third of these brokering moves.

Creating a boundary encounter refers when a broker (i.e., the instructor) sets up an indirect interface between the classroom community and the broader mathematical community. A boundary encounter involves a boundary object, typically a well-chosen task or sequence of tasks, that provides an opportunity for students to engage in one or more disciplinary practices. In the sections that follow we delineate features of such tasks in our case study that opened a space for students to engage in the disciplinary practice of symbolizing.

Interpreting between communities is a brokering move in which the instructor coordinates students' mathematics with the more conventional or formal mathematics of the broader mathematical community. This type of brokering move typically occurs when the instructor inserts notations, symbols, graphs, diagrams, or provides other information that enables students to transcend the idiosyncrasies of their local classroom community. Interpreting between communities is significant because it shows how the instructor can connect student thinking to the well-developed mathematical culture. Moreover, it facilitates the sense of ownership of ideas and belief that mathematics is something that can be reinvented and figured out.

Symbolizing

Not all classroom activity is characterized by participation in disciplinary practices, even in inquiry-oriented classrooms. For instance, classroom mathematical practices capture the emerging content-specific mathematical progress of a local classroom community (Cobb, 2000; Rasmussen et al., 2015), whereas disciplinary practices capture how that progress might reflect what professional mathematicians do that transcends specific content. Symbolizing is the disciplinary practice of creating and using symbols to communicate mathematical ideas. Symbols include graphs, diagrams, and analytic expressions such as letters, numbers, and vectors. By engaging in their own symbolizing, students act like mathematicians – notating processes and connections between ideas with shorthand expressions that allow for efficiency of processing.

In this paper we highlight symbolizing of the following four types that we have found to be prevalent in inquiry-oriented classes:

(S1) Notating steps in a calculation or process,
(S2) Stating a relationship between two or more mathematical objects,
(S3) Creating a connection across two different representations (notating in new symbolism what has already been explained or described in a different way), and

(S4) Creating a unifying inscription, a graphic or diagram that illustrates multiple relationships at once.

In the results section, we call explicit attention to these four types of symbolizing in the context of students' solving problems and communicating their reasoning.

Research Setting and Methods

Our research over the last decade in the teaching and learning of linear algebra has been grounded in the design-based research paradigm of classroom-based teaching experiments (Cobb, 2000). This involves a cyclical process of (a) investigating student reasoning about specific mathematical concepts and (b) designing and refining tasks that honor and leverage students' mathematical ideas towards accomplishing the desired learning goals (Gravemeijer, 1994; Wawro, Rasmussen, Zandieh, & Larson, 2013). One product of this design-based research is the Inquiry-Oriented Linear Algebra (IOLA) curricular materials, designed to be used for a first course in linear algebra at the university level. At present, three units comprise the IOLA materials: Unit 1: Linear Independence and Span (Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012; Wawro et al., 2013); Unit 2: Matrices as Linear Transformations; and Unit 3: Change of Basis, Diagonalization, and Eigentheory. Many of the tasks in the IOLA materials are created to facilitate students engaging in experientially real task settings in such a way that their mathematical activity can serve as a basis from which more formal mathematics is developed.

The data presented in this paper come from a classroom teaching experiment in a first course in linear algebra during Fall 2014 at a large public mid-Atlantic university. The data sources were classroom videos that capture small-group work and whole-class discussion, as well as students' written work from class. In addition, the four walls of the physical classroom were almost entirely whiteboards, which the instructor took advantage of by encouraging students to work in their small groups together at the whiteboard; as such, another primary data source was photos of student and teacher work on classroom whiteboards.

In this paper we focus on the class's mathematical development through the first task of Unit 3, "The Stretching Task" (see Figure 1). The first task builds from students' experience with linear transformations in $\mathbb{R}^2$ to introduce them to the idea of stretch factors and stretch directions and how these create a non-standard coordinate system for $\mathbb{R}^2$. This is the beginning of a larger sequence in which students reinvent the diagonalization equation $A = PDP^{-1}$. We analyzed the data to determine what types of symbolizing the students and the instructor were engaged in. These became our four types of symbolizing (S1) – (S4). In addition we examined the data for instances of each of the three types of brokering moves (of which we found examples of two of the types). Analysis of this data allows us to illustrate examples of students engaging in the practice of symbolizing, and we integrate this analysis with details regarding how the instructor serves as a broker between the classroom community and the broader mathematical community.

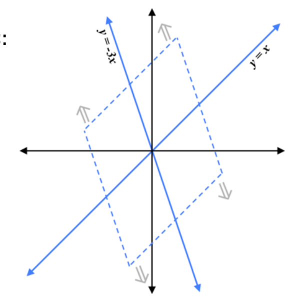
Results

In this section we provide examples of both student and teacher symbolizing activity. This symbolizing activity falls into the aforementioned four categories: (S1) - (S4). We begin with a description of the task itself as a boundary object and then follow with a description of student and teacher use of the four types of symbolizing as they occur. Symbolizing serves as a means for students to record and communicate their inquiry into the mathematics. Communicating through symbolizing also serves as a means for brokering within and across different groups of students in the classroom as well as for brokering between the classroom community and the mathematical community.

In Unit 3 Task 1 (referred to as "The Stretching Task"), students are asked to describe the result of a transformation given in terms of what the transformation does to two lines (See Figure 1). One goal of the task is to create a means for students to engage with ideas that will facilitate their learning about stretch factors and stretch directions and the possibility of using these stretch directions as a grid for their work in the plane. Although the formal definitions of eigenvector and eigenvalue arise later (Task 3 of this unit) for students, we purposely use the terms stretch direction and stretch factor here to immerse students in the geometric interpretation of these terms. In choosing this task, the instructor serves as a broker by presenting the students with a task that can serve as a boundary object between the classroom community and the mathematical community. The task serves as a boundary object in that it provides an opportunity for students to engage in the disciplinary practice of symbolizing in ways that begin to align with how the mathematical community uses symbolizing in this context.

The Stretching Task

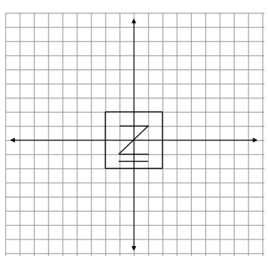
Imagine a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ that has the following properties:

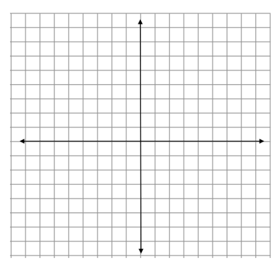


In the direction along the line $y = -3x$, the transformation stretches all points by a factor of two.

In the direction along the line $y = x$, the transformation keeps all points fixed.

1. Use the space on the right to sketch what should happen to the image shown on the left when it is stretched according to the transformation described above. You may use a combination of intuition or calculations, as well as any additional sketches below or on your group's whiteboard.





2. Determine what will happen to $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and to $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ under this transformation. Use an initial estimate from your sketch in problem 1. Then try to do a calculation that will determine these locations more precisely.

3. Determine a matrix that allows you to calculate what happens under the transformation to any point on the plane. Use it to check your sketch or improve its accuracy.

Figure 1. Task 1: The Stretching Task.

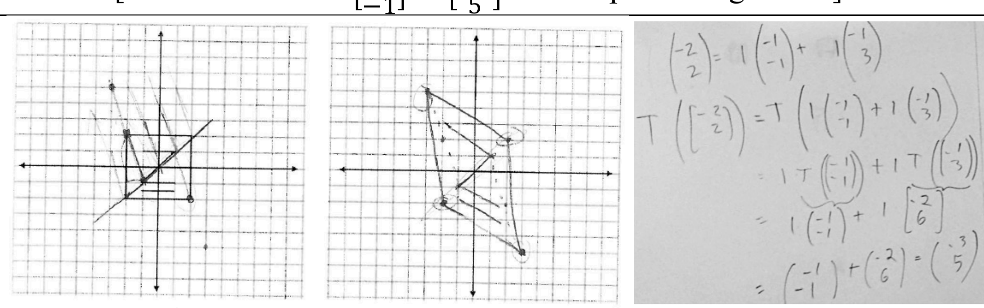
In what follows we present data from a particular classroom implementation of the materials in Fall 2014. Symbolizing of various types occurred during the parts of two class periods that students worked on Task 1. Student activity on Task 1 began with engagement within the graphical symbolization that they had been given in the introduction to the Stretching Task and in problem 1. This symbolizing is of types S1 and S2 in that students were notating steps in their graphing process and recognizing relationships that would help them create a transformation of the Z-box. Because this initial work occurred at the end of class, it was on the following class day that students and the instructor made connections between this graphical work on the task and other symbolic notation to describe the transformation (symbolizing type S3). In addition, the instructor introduced a unifying graphic based on student work to aid students in working with the transformation (symbolizing type S4). These symbolizing examples and the role of the teacher as a broker in these examples are detailed in the next four sections.

Within the graphical representation (S1 and S2)

Students initially engaged in the task by symbolizing within the graphical and verbal description that they had been given at the beginning of Task 1 and in Problem 1. Many students began by notating the points that stay fixed and then estimating the images of the points that stretch. Figure 2a illustrates the work of Donald, who presented at the board to explain his graphical symbolizing process (S1):

> *So you know the points along y = x are the same and, like, that'd be these points along that line. So you know like you get two of the corners, you know these points and these points are gonna stay the same. And then you also know that this stretches along the y = -3x line, which is like any of these. But this can be moved, like, kind of like a linear combination of this, where like you start along this line. And it stretches like up that way. And this corner point happens to, like, coincide with this point here which you know stays the same. So that's along the line y = -3x and then you just double it to get that point, which comes over here. And you do the same for down here [the lower right corner]. And then once you get the 4 corners, you can just like figure that everything else is gonna stretch kind of similarly*. [See Figure 2a]

From the video of that day of class we can reconstruct his explanation. First, he explained that the points along the line $y = x$, in particular the corner points $\begin{bmatrix}-2\\-2\end{bmatrix}$ and $\begin{bmatrix}2\\2\end{bmatrix}$, will "stay the same." He continued by noting the lines parallel to $y = -3x$ all stretch in the same way, away from their "start along this line," i.e., starting from the $y = x$ line. [See the line segments Donald drew in the left of Figure 2a]. Note also Donald's use of the phrase "linear combination." He did not explain the algebra of this phrase but symbolized this graphically as a stretch from the $y = x$ line along a line parallel to $y = -3x$. Donald continued by discussing the specific case of the corner point, $\begin{bmatrix}-2\\2\end{bmatrix}$. He described that the corner point "happens to coincide with this point," i.e., the corner point $\begin{bmatrix}-2\\2\end{bmatrix}$ is on the same line segment as $\begin{bmatrix}-1\\-1\end{bmatrix}$. The line segment is one of those he drew parallel to $y = -3x$. Then "you just double it," i.e., double the segment from $\begin{bmatrix}-1\\-1\end{bmatrix}$ to $\begin{bmatrix}-2\\2\end{bmatrix}$. In doing this he marks the point $\begin{bmatrix}-3\\5\end{bmatrix}$, which is the result of the transformation on the upper left corner of the box [See the line from $\begin{bmatrix}-1\\-1\end{bmatrix}$ to $\begin{bmatrix}-3\\5\end{bmatrix}$ in each part of Figure 2a.]



(a) (b)

Figure 2. (a) Donald's work on Problem 1 and *(b)* The instructor's record of Donald's work.

Donald's description exemplifies symbolizing in a graphical context to share steps in a process (S1) and show connections between pieces of graphical information (S2), i.e., how to combine the information that $y = x$ stretches by one and $y = -3x$ stretches by two. Clearly, verbalizations accompanied and were need to communicate his graphical approach. However, the focus remained on working within a graphically represented system of ideas. Next we discuss how other symbolizations allowed students to incorporate other ways of exploring this problem.

A transition to vector notation (S3)

On the next day of class, the instructor included a scanned copy of several examples of student work on this problem, including Donald's work in Figure 2a (and the work in Figure 3a and 3b in the next section). As a follow up to Donald's explanation, the instructor used a linear combination of vectors to record the student idea that $\begin{bmatrix}-2\\2\end{bmatrix}$ could be reached by going to $\begin{bmatrix}-1\\-1\end{bmatrix}$ and then travelling $\begin{bmatrix}-1\\3\end{bmatrix}$ (see the first line of Figure 2b). The students' idea that the vector $\begin{bmatrix}-1\\-1\end{bmatrix}$ stays fixed but $\begin{bmatrix}-1\\3\end{bmatrix}$ doubles under the transformation, T, is indicated by lines 2 though 4 of Figure 2b. Finally, the last line of Figure 2b indicates the fact that combining the fixed $\begin{bmatrix}-1\\-1\end{bmatrix}$ with the doubled $\begin{bmatrix}-1\\3\end{bmatrix}$ (now $\begin{bmatrix}-2\\6\end{bmatrix}$) reaches the point $\begin{bmatrix}-3\\5\end{bmatrix}$. The instructor's choice of symbols helps interpret between the student ideas and the standard mathematical notation. In particular, the symbols for a linear combination of vectors and the distributive properties of a linear transformation were familiar to the students from earlier work in the course, but they had not previously seen the application of a linear combination in the sense of line 2 of Figure 2b.

The symbolizing by the instructor connected the graphical reasoning of the student to a symbolic vector notation, creating a connection across representations (symbolizing type S3). In addition, the symbols written by the teacher served as what Rasmussen and Marrongelle (2006) define as a transformational record. Transformational records are "notations, diagrams, or other graphical representations that are initially used to record student thinking and that are later used by students to solve new problems" (Rasmussen & Marrongelle, 2006, p.389). In other words, they record student inquiry in a way that provides a symbolization for future inquiry.

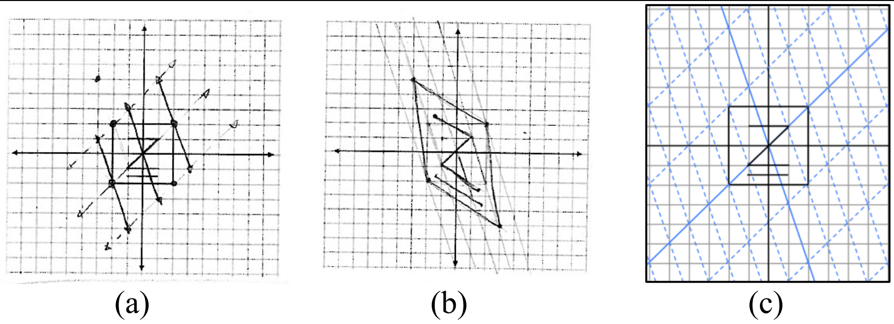

(a) (b) (c)

Figure 3. (a), (b) Two student examples showing a gridding of the plane using stretch directions. and *(c)* The instructor's graph of gridding using the stretch directions.

Creating a unifying graph (S4)

The instructor also included in her presentation examples from two other students on Problem 1 (Figure 3a, 3b). These examples show students creating a grid when trying to determine how the Z-box transformed. Note that each of these examples has similar features to

Donald's work, but more extensive gridding of the plane using lines parallel to $y = -3x$ and $y = x$. Each also has points marked at $\begin{bmatrix}-2\\2\end{bmatrix}$ and at $\begin{bmatrix}-3\\5\end{bmatrix}$ indicating the doubling of a vector along that line to find the new corner point.

The instructor emphasized the gridding in the student work and introduced the gridding of Figure 3c. In this way the instructor acted as a broker between the developing graphical symbolizing of the class and more sophisticated ideas from mathematics community.

The graphical representation in Figure 3c illustrates two ways to grid the plane. One (in grey) is the standard familiar grid and the other (in blue) is based on lines parallel to the stretch lines described in Task 1. This gridding can be used as a way to more easily see the doubling along the lines parallel to $y = -3x$. This is how Donald explained that $\begin{bmatrix}-2\\2\end{bmatrix}$ stretches to $\begin{bmatrix}-3\\5\end{bmatrix}$, but now, with the complete grid available, that graphical method is available for any point. In addition to this practical result that connects to student thinking, the grid sets the stage for the student exploration in Task 2 of more sophisticated ideas of change of basis and diagonalization as described below. The graphic of Figure 3c then is unifying of students' current work and thus serves as an example of type S4 symbolizing. In addition, this graphic is key to the activity in Task 2 that leads to the creation of $A = PDP^{-1}$.

Conclusion

In this paper we explored student and instructor inquiry in the context of the disciplinary practice of symbolizing. Student inquiry in Task 1 involved exploring a graphical situation, creating symbols that expressed their emerging ideas about the mathematical situation, and symbolizing their graphical activity using vectors and vector equations. The inquiry involved: (a) creating or choosing appropriate ways to symbolize mathematical processes (S1) and relationships (S2) within particular representations, (b) making connections between different symbolizations of mathematical content (S3), and (c) creating a unifying graphical representation (S4). Each of these reflects a facet of the disciplinary practice of symbolizing, characterized through types S1-S4. Student inquiry into the mathematics of this task created a necessity for mathematical symbolism that students used to express their emerging ideas about the mathematical situation as well as to create more powerful and efficient solutions. The vector representations, vector equations, and unifying graphic (Figure 3c) provided additional tools for inquiry that were further developed and used in the subsequent tasks of this unit.

Instructor inquiry into students' mathematical thinking involves them making sense of and leveraging student insights so that they can appropriately connect the mathematics being developed by students in the classroom with the mathematics of the mathematical community. This is the notion of brokering. For the instructor to truly serve as a broker between the two communities, the instructor must participate in, recognize, and understand how students are engaging in the mathematics. A broker is not someone who simply relays information from one community to another, like a messenger; rather, a broker negotiates mathematical meaning between two communities. Furthermore, what tasks are selected and how they are used also plays a part in the brokering process, as tasks provide an interface through which the classroom community will encounter situations that can serve as a basis from which the more formal notions of the broader mathematical community can be developed. With the help of the instructor as broker, and boundary objects carefully chosen by the instructor, students can begin to act as mathematicians do. They can progress in their ability to engage in disciplinary practices in ways helpful not only for learning particular mathematical ideas but also for applying in other settings.

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Personification as a lens into relationships with mathematics

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Personification is the attribution of human qualities to non-human entities (Inagaki & Hatano, 1987). Eliciting personification as a research method takes advantage of a naturally occurring means through which (some) people discuss the nuanced emotional relationships they have with those entities. In this paper, we introduce the eliciting personification method for exploring individuals' images of mathematics, as well as discuss an initial set of approaches for analyzing the resulting data. Data from both pre-service teachers and research mathematicians are discussed in order to illustrate the method.

Key words: [personification, relationship with mathematics, conceptual blending]

The available research on people's relationships with mathematics predominantly relies either on assessment instruments (e.g., Likert scale surveys, concept mapping, responses to vignettes or videotapes, and linguistic analyses) or case studies (see Philipp, 2007). Although these are reliable research tools, as we argue later in this manuscript, the results they generate are not conducive to empathizing with others' mathematical experiences. The method presented in this article, *eliciting personification*, comes closer to this goal. We suggest this method complements existing approaches, adding new dimensions to our understanding of individuals' perceptions of, and experiences with mathematics.

Method and participants

Although personification may occur naturally (e.g., Hill, 1930; Inagaki, & Hatano, 1987; Piaget, 2007), it is also possible to elicit personification data from participants. The personification data discussed in this work come from two sources. The first is an assignment given to 36 pre-service elementary/middle school teachers that invited participants to personify mathematics and describe this imaginary character and their relationship. The second is a series of interviews with a convenient sample of 9 research mathematicians that elicited similar data. Below is the prompt given to the pre-service teachers:

> *Your assignment is to personify Math. Write a paragraph about who Math is. This paragraph should address things such as: How long have you known each other? What is he/she/it look like? What does he/she/it act like? How has your relationship with Math changed over time? These questions are intended to help you get started. They should not constrain what you choose to write about.*

We present the story of our research in two parts: we begin with a consideration of specific data, which we use to frame our introduction of analytic methods appropriate for such an approach to research; we then analyze this data and that of research mathematicians; we conclude with a comparative discussion that attends to the benefits and possibilities of the eliciting personification method.

Pre-service teachers' personification of mathematics

Below is an excerpt from one of the pre-service teacher participants, which we use as a launching point to discuss approaches to analyzing elicited personification data:

> *Mathonious was a very sensible young boy from Athens, Greece. Not many people liked him but at age 6 he became the best of friends with a young girl named Kukla.*

> *Every day they would hang out together and while Mathonious was a sensible young boy, Kukla began to notice that over the years he was becoming more and more complex. Kukla had noticed this and suggested that they see the oracle in order to find a solution. The oracle was known for simplifying and clarifying things for people in order to better their lives and though the oracle did great things, there were always consequences for those who do not listen to her advice. Mathonious met with the oracle and she told him that though he thought his complexity was a good thing it was confusing and hurting those closest to him; she warned him that if he did not revert to his more sensible simple self soon he would lose those closest to him and become a terrible beast; feared by many. He returned to Athens to tell Kukla his prophecy and when he did he was not very serious about it. In fact he did not seem to care about the oracles' advice or warning at all. Because he did not simplify himself to those around him the consequences of the prophecy came true and a horrid exiled beast he did become. He was indeed feared by many. The people feared him so much that they dehumanized him and called him MATH, which stood for mental abuse to humans. Despite his awful new nature, Kukla wanted to try to understand him desperately so that maybe he could return to the boy he once was and they could be friends. However, every time she attempted he would cast her away.*

The above paragraph describes the relationship between Kukla, the character the author attributes to herself, and Mathonious, who is a personification of mathematics. It paints a rich picture of the author's relationship with mathematics. However, since personification data is novel in mathematics education, there is no well-defined set of approaches for performing an analysis of a set of such elicited personification data.

Analyzing data using character summaries

The initial approach to analyzing personification discussed in this article is to use open coding (Strauss & Corbin, 1990) to summarize each participant's Math-character. The compilation of a list of all characters produced by a particular group then serves to summarize the types of relationships with mathematics present in that group. For example, in the above excerpt, the writer describes Mathonious in terms of two characters. The first character is a young boy that Kukla befriends. However, this relationship deteriorates and Mathonious becomes a former friend with whom Kukla is trying to rekindle a friendship. The second character is a terrible beast, feared by many. Both characterizations, the terrible beast and the former friend, concisely encapsulate the writer's relationship with mathematics.

After the pre-service teachers' personification excerpts were sorted into similar character categories, three themes emerged from the scripts. The first and most common theme that emerged was that of a monster or other evil creature. The terrible beast from the excerpt was subsumed under this category along with other goblins ghouls, and nasty things. This theme depicts mathematics as a cruel, unattractive, and unforgiving entity that often takes pleasure from the suffering of others. The second common theme was that of a former friend. Mathematics was described as someone with whom the pre-service teacher once had a healthy and sometimes even happy relationship, but at some point the relationship had soured. This theme also occasionally involves descriptions of repeated attempts on the part of the pre-service teacher to mend the relationship, while mathematics 'resists'. The last type of character, which only occurred once, involved a lover who is loathed by friends, family and even strangers. The lover character might resonate with readers who have encountered deleterious comments when discussing their profession (for a detailed discussion of the lover excerpt see Zazkis, 2015).

We interpreted the Kukla excerpt as drawing upon both the monster and former friend themes. However, we sought a more detailed analysis of participants' relationship with mathematics than that which is afforded simply by identifying common character themes. We develop such analysis below.

Personification as a conceptual blend

Conceptual blending involves taking the elements of two (internal or external) input spaces and blending them together to form new inferences which are said to exist in a newly formed (internal) output space (Turner & Fauconnier, 2002). It has been used to analyze a number of mathematics education related phenomena, including proof construction (Zandieh, Roh & Knapp, 2014), task design (Mamolo, Ruttenberg-Rozen, & Whiteley, 2015), and the concept of infinity (Núñez, 2005). Conceptual blending is a crucial way in which people make-sense of, and communicate, complicated and multi-faceted phenomena (Fauconnier & Turner, 2008). For example, in order to make sense of the statement, "My karma ran over my dogma," one needs to blend a road-kill input space in which a car runs over a dog and a theology input space in which the words karma and dogma are defined. The resulting blend allows for the interpretation of the sentence's meaning—my karma overcame my dogma.

Personification can be conceptualized as a kind of conceptual blend (Fauconnier & Turner, 2008). In the case of eliciting personification of mathematics, a mathematics space and a human relationship space are blended to form a space that allows for the communication of one's complex emotional relationship with mathematics. The rich experiences and vocabulary associated with the human relationship space serve as a platform for participants to discuss emotional relationships with mathematics, and as a lens for the researcher to interpret the complexities of individuals' affective experiences with mathematics. This relationship would otherwise be difficult to discuss or interpret with the same level of depth and detail since vocabulary and images associated with emotion are primarily housed in the human relationship space, not the mathematical space.

Analyzing pre-service teachers' elicited personification using conceptual blending

As mentioned earlier, the Kukla excerpt describes mathematics as both a (former) best friend and a terrible beast. These two characterizations are quite different and coincide with different categories in the character summary analysis. So we use separate blending diagrams to describe each. First, the best friend: this characterization in the human relationship space maps to comfort with, and enjoyment of, mathematics in the mathematics space. A best friend characterization does not imbue the same level of passion for mathematics that a lover characterization would. However, it still portrays the author as someone who likes to spend time with mathematics and portrays mathematics as someone who likes to spend time with the author. This personification of mathematics provides a level of detail in regard to how much, and in what ways, the author enjoyed mathematics. For example, a relationship with a lover would have a closer degree of intimacy, than that with a friend. One would spend more time with the former, compromise differently for him or her, and feel more deeply emotions of elation, frustration, or despair.

Some of the details about the best friend are also revealing. Mathonious the best friend is sensible, a human trait that can be interpreted to map to the logical coherence and understandability of mathematics, since a reasonable definition of sensible is "having sound judgment" and "readily perceived". Additionally, Mathonious is presented as male, while Kukla is female. This is in line with research that points to mathematics being perceived as a male dominated discipline (e.g., Keller, 2001; Picker & Berry, 2000), and may also be indicative of perceived power structures between the author and her "friend". Lastly, there is a timeline of Kukla's relationship with Mathonious. This timeline can be assumed to coincide with the timeline of the writer's relationship with mathematics – as Kukla learned more about Mathonious, he seemed to become more inaccessible, less friendly, hurtful and indifferent to his effect on his former friends. Details of this blending analysis are summarized in Figure 1.

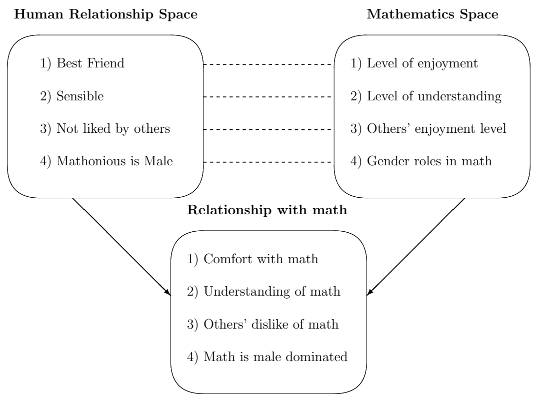

Figure 1. Best friend conceptual blending diagram.

The excerpt describes Mathonious getting progressively more complex and confusing, causing Kukla and Mathonious to grow apart. Complexity and confusion can be assumed to be a part of the mathematics space that stands in opposition to the previously mentioned sensibility from the best friend part of the human relationship space. Complexity, a mathematical trait, is not generally associated with emotions. However, describing this complexity in association with a personification of mathematics allows the excerpt's author to describe the complexity as "confusing and hurting." Mathonious' pride in his complexity (he refuses to follow the oracle's advice, despite 'consequences'), and his indifference towards its effects on others, are described as the root causes of the deterioration of Kukla's relationship with Mathonious.

After the falling-out, Mathonious is re-characterized as a terrible beast who exiles himself. This replaces the positive emotions associated with a best friend with the fear and repulsion associated with a beast. This characterization, much like the best friend characterization that preceded it, provides a level of detail in regard to the emotions involved. The excerpt's author could have chosen to describe simply growing apart, which would entail a level of indifference toward mathematics. However, the author instead chose to use a "terrible exiled beast" and draw upon the fear and repulsion that this characterization entails.

Interestingly, the excerpt's author describes repeated attempts to rekindle the friendship with the 'old' Mathonious, which can be mapped onto attempts to return to a state of understanding mathematics. However, notice that Mathonious casts her away repeatedly, placing the blame for the poor relationship with mathematics on Mathonious, not Kukla. What this means in terms of the mathematics space, is that the excerpt's author seems to attribute blame to mathematics, an abstract entity, for her lack of understanding and enjoyment of the subject. It is Mathonious who should, in the eyes of Kukla, return to his former "sensible" self, and it is only he who values his complexity (as the oracle identifies). Figure 2 summarizes this analysis.

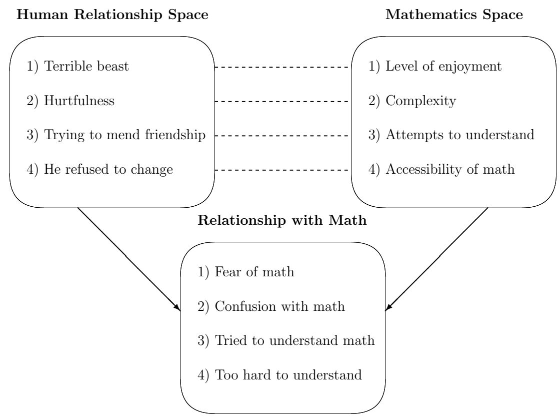

Figure 2. Terrible beast blending diagram.

Mathematicians personification of mathematics

We now turn to personification data of mathematicians. Unlike the 36 pre-service teachers whose character summaries fit into three categories, the 12 mathematicians we interviewed personified mathematics in more diverse ways. This included descriptions of mathematics in sexual-relationship terms as spouse, ex-spouse, forbidden lover or mistress. Additionally, some mathematicians described mathematics as a person with whom they do not have a personal relationship, but whom they seek to understand. For example, one mathematician described mathematics as a virtuoso whom the mathematician aspired to be more like. Another mathematician described mathematics as a knowledgeable wandering Jew who points out the flaws in peoples thinking and in turn promotes societal progress. We interpret this diversity in mathematicians' descriptions of mathematics as a reflection of the different natures and experiences of mathematicians and pre-service teachers, respectively. The added complexities in the types of relationships forged with personified mathematics, as well as the varied details about the character of mathematics (e.g., religious, sexy, scandalous), are brought to light with our method and add insight into the varied complexities and characteristics that draw or deter career mathematicians.

Analyzing mathematicians' elicited personification using conceptual blending

We now turn to an excerpt from one mathematician's personification interview. As with the Kukla excerpt, we analyze this excerpt using conceptual blending.

> *While I'm actually engaged in proving a theorem a lot of the time there is joy. However the time I'm actually proving theorems is very small. So my relationship with mathematics is not just my relationship with proving theorems. It's also my relationship with grading papers, my relationship with going to committee meetings, my relationship with advising students, writing papers, which is kind of tedious and is very different from writing proofs. So all of this sort of comes along with a career in mathematics. So even though there maybe that child's heart that I still have that takes joy in doing it when I do have an hour or two to sit down and do math. I have pleasure in that. But I recognize that that isn't all of mathematics… Its like when you first take a lover and that intense rush you feel. That's fantastic. But after a while you realize that that's not the entirety of a relationship. It's not just that physical rush. There is also a lot of other things that go along with that. So you may still have that feeling. But it's only one piece of a much much large tapestry…. I love the wife, it's not that I don't love the wife. But there's a lot of groceries to buy, and taking out the trash, and stuff like that…it's not all proving theorems.*

The excerpt describes mathematics as a lover that eventually becomes a wife. The mathematician's love of proving theorems maps to his enjoyment of intimate moments with this lover/wife. This mapping provides a great level of detail with respect to how much this mathematician enjoys proving theorems – it provides a physical rush. However, as he points out there a lot of additional parts to his relationship with this wife; proving theorems is one piece of a much larger tapestry. In particular, for this mathematician, "mathematics" is no longer simply the discipline itself, but now carries with it all of the attributes of an academic career in the discipline. These necessary but not enjoyable parts of his relationship with this wife (e.g., buying groceries, taking out the trash) map to the less enjoyable parts of his (new and broader) relationship with mathematics (grading, meetings, advising students). This mapping provides detail regarding how he feels about other parts of his relationship with mathematics.

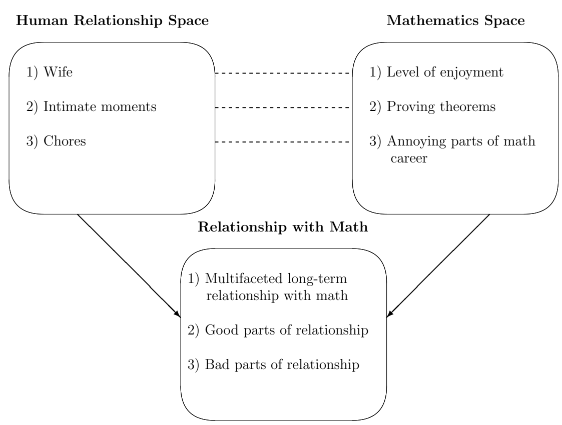

Figure 3. Wife blending diagram.

Advantages of personification data

Let us compare the Kukla and wife excerpts above in relation to hypothetical responses to a Likert-scale item that asks for one's level of agreement with the statement "Mathematics is enjoyable and stimulating to me". This item is borrowed from Bessant's (1995) factors influencing mathematics anxiety (FIMA) assessment. This is a fairly common instrument used to assess relationships with mathematics. The author of the Mathonious excerpt could have likely answered "strongly disagree", given her horrible beast characterization, whereas the mathematician might have answered "strongly agree", given his lover characterization. However, both responses provide only a snap shot – in the former example, the assessment is a current one, lacking the background illustrating how the relationship came to sour, in the latter example, the assessment is narrowly construed, extracted from the broader context in which the mathematician now engages with mathematics. This points to the Likert-scale question's limitations for capturing the tangible emotions involved, and their connection to the broader experiences and history of the individual. Further, more nuanced details are captured in the picture of a lover who inspires a physical rush or a sensible friend who was once a favorite companion than are afforded by the Likert-scale item. Certainly, eliciting personification is not a replacement for survey methods, but it does have particular advantages in terms of detailing emotional relationships that add an important dimension to research on affect and mathematics education.

We suggest one of these advantages is empathy. When reading both the discussed excerpts, one might notice similarities to his or her own experiences and feelings. The experience of having a friendship turn sour in spite of repeated attempts to salvage it is common, eliciting empathy for the author. Similarly, settling into a long-term relationship is also a common human experience to which many of us not only relate but also strive for. However, one might also empathize with the personified mathematics – as a friend who has been shunned, a wife who feels neglected, or a 'monster' whom nobody understands. Most people have either undergone these experiences first hand or been exposed to someone who has. By relating these experiences to their relationships with mathematics, participants in this study helped us understand the nature of their relationship with mathematics and how it evolved. In short, the data provides an avenue for our original stated goal of allowing one to empathize with others' relationships with mathematics, and it sheds new insight on what mathematics "looks like" from the eyes of our participants.

We propose the eliciting personification method as an indirect way of examining someone's relationship with mathematics, which can paint a particularly vivid image of this relationship and the characters involved. The conceptual blend of personified mathematics is certainly a dramatized version of participants' relationship with mathematics itself due to the very nature of personification. However, we do not view this dramatization as a detriment. Rather, we view this dramatization as a useful means through which to foster empathy, as well as to elicit and distill the essence of individuals' relationships with mathematics.

Discussion

In this article, we introduced *eliciting personification* as a method in which participants describe their relationship with mathematics by describing mathematics as if it were a person. Two personification excerpts, one from a pre-service elementary school teacher and one from a professional mathematician, were used to illustrate the lens into participants' affect provided by the method. Several approaches to analyzing the data were also discussed. These approaches included using character summaries for summarizing the data, and conceptual blending, which was used for deeper analysis. The use of these techniques helped to distill a rich image of two participants' dispositions toward mathematics and how their relationships had evolved over time. Interestingly, both excerpts highlighted the complexities of human relationships that are not necessarily accessible through quantitative approaches to data collection. Both relationships involved issues of trust and caring (friend, loving wife), that grew overtime to include frustrations and even resentment (horrible monster, nagging wife). We note that conceptual blending and character summaries are by no means a complete list of appropriate, applicable approaches to personification analysis, and we encourage other researchers interested in using the eliciting personification method in their studies to experiment with other analysis approaches.

The eliciting personification approach offers a particularly vivid window into study participants' relationships with mathematics, and we view it as an important complement to the case study and assessment instrument methodologies used in the past. We suggest that our approach offers a novel lens through which to research mathematical disposition and the nuances involved in individuals' affective experiences with the discipline. Eliciting personification is proposed as an innovative research tool that affords participants creative ways to describe aspects of their relationship with mathematics that might otherwise remain tacit. Additionally, as discussed in Zazkis (2015), personification can be used in teacher education as a method for facilitating pre-service teachers' self-reflection about their relationship with mathematics and for fostering empathy toward their future students.

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ON SYMBOLS, RECIPROCALS, AND INVERSE FUNCTIONS

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In mathematics the same symbol – superscript (-1) – is used to indicate an inverse of a function and a reciprocal of a rational number. Is there a reason for using the same symbol in both cases? We analyze the responses to this question of prospective secondary school teachers presented in a form of a dialogue between a teacher and a student. The data show that the majority of participants treat the symbol $\square^{-1}$ *as a homonym, that is, the symbol is assigned different and unrelated meanings depending on a context. We exemplify how knowledge of advanced mathematics can guide instructional interaction*

Keywords: Scripting, inverse element, inverse function, reciprocal, homonymy, polysemy

There is an ongoing conversation in mathematics education research on teacher knowledge and its various facets (e.g., Rowland & Ruthven, 2011). One important focus within this discussion is secondary teachers' "advanced mathematical knowledge" (AMK), defined as knowledge acquired during tertiary education (Zazkis & Leikin, 2010). Is this knowledge essential, or even useful, in teaching? Research demonstrated that teachers' opinions on the matter differ considerably, ranging from "irrelevant" to "extremely important" (ibid.)

However, even teachers who claim that AMK is essential for their teaching have difficulty in providing particular examples or recalling teaching scenarios where their AMK was utilized. Our study provides an example where a teacher's knowledge of advanced mathematics can shape an instructional interaction.

The Study

Twenty two prospective secondary school mathematics teachers participated in the study. The participants held degrees in mathematics or science and at the time of data collection were enrolled in a problem-solving course, in the last term of their teacher education program. One of the goals of the course was to draw connections between undergraduate mathematics and school mathematics, and in doing so deepen their knowledge of school mathematics.

During the course the participants had several experiences with script writing assignments – assignments in which they are asked to compose an imagined conversation between a teacher and a student (or students), following a given prompt (e.g., Zazkis, 2014). Our data consists of participants' responses to the Scripting Task, presented below. The task invites participants to write a dialogue for an imaginary interaction between a teacher and a student, related to the appearance of (-1) as a superscript, that is, symbol $\square^{-1}$.

The Scripting Task

The Task is presented in Figure 1. In Part 1 of the Task, the participants were asked to extend the dialogue following the presented prompt. They were explicitly asked to imagine themselves in the teacher's role. In Part 2 the participants were asked to explain their particular choices, which may not be evident from the scripts themselves. While Parts 1 and 2 could be overshadowed by pedagogical considerations, in Part 3 of the assignment we sought the explanation for a "mathematically mature" colleague. This was aimed at liberating the participants from considering the mathematical constraints of the audience and enabling them to project their personal mathematical knowledge. We assumed that participants' personal understanding of the situation and the chosen explanation for students could be different.

Task analysis

In mathematics, a character followed by a subscripted (-1) – such as in f^{-1} and 5^{-1} – represents an inverse element in a group structure. That is, a group element (A^{-1}) is considered to be an inverse of another group element (A), if a binary operation (*) involving the two element results in an identity element (I). Symbolically, this relationship is given as $A * A^{-1} = A^{-1} * A = I$

The two cases presented in the Scripting Task, that is, the cases of f^{-1} and 5^{-1}, differ in terms of the set and the binary operation. For the set of (rational without zero) numbers, the implied binary operation is multiplication, and the multiplicative inverse of 5 is 5^{-1}, as $5 \times 5^{-1} = 5^{-1} \times 5 = 1$, where 1 is a multiplicative identity. For the set of bijective functions, an inverse function satisfies $f \circ f^{-1} = f^{-1} \circ f = id$, where the binary operation here is the composition of functions, and *id(x)=x* is the identity element.

Scripting Task

Part 1: You are given the beginning of an interaction between a teacher and a student and your task is to extend this imaginary interaction in a form of a dialogue between a teacher and a student (or several students). You may also wish to explain the setting, that is, the circumstances in which the particular interaction takes place.

T: So today we will continue our exploration of how to find an inverse function for a given function. Consider for example f(x) = 2x+5 Yes, Dina?

S: So you said yesterday that f^{-1} stands for an inverse function

T: This is correct.

S: But we learned that this power (-1) means 1 over, that is, $5^{-1} = \frac{1}{5}$, right?

T: Right.

S: So is this the same symbol, or what?

Part 2: You are also asked to explain your choice of approach, that is, why did you choose a particular example, what student difficulties do you foresee, why do you find a particular explanation appropriate, etc.

Part 3 (optional): The way you understand the idea yourself could be different from the way you explain it to a student. If this is the case, please indicate how you could clarify the issue for yourself, or for a "mathematically mature" colleague.

Figure 1: Scripting Task

Theoretical Constructs

We rely on the theoretical construct focus of attention (Mason, 2010) and on the linguistic constructs of homonymy and polysemy (Durkin & Shire, 1991). In what follows we briefly introduce each construct and describe how it relates to the work of teachers.

According to Mason (2010) learning involves transformation of attention. In particular, "learning has taken place when people discern details, recognize relationships and perceive properties not previously discerned, through attending in fresh or distinct ways, and when they have fresh possibilities for action from which to choose. Learning necessarily involves shifts in the form as well as the *focus of attention*" (p. 24, our italics). In line with this view, we claim that teachers' work is geared towards focusing students' attention in any given instructional interaction. This choice of focus is intended to draw students' attention to similarities and differences, to stress some aspects of the mathematical concepts and procedures, necessarily ignoring other aspects. For example, in the case of the two appearances of superscript (-1), a teacher may choose to highlight the differences in the procedures implied by the common symbol, or to focus on the unifying idea of 'inverse' element.

Among variously nuanced definitions for homonymy and polysemy, we follow the definitions of Durkin and Shire (1991). According to these authors, homonymy denotes the property of some words to share the same form but point to distinct meanings. Polysemy refers to the property of some words to have different but related meanings, and some shared sense. The context in which the words appear determine their intended meaning.

Durkin and Shire exemplified homonymy and polysemy between the mathematical register and the everyday register. For example, 'volume' is an example of homonymy, as it refers to distinct meanings: intensity of sound or a measure of a 3-dimentional object. 'Continuous' is an example of polysemy, where the everyday meaning of "no breaks" is related to the mathematical definition, such as in "continuous functions". Notably, polysemy may results in learners' misinterpretations of mathematical concepts by assigning the everyday meaning of the related mathematical terms (e.g., Tall & Vinner, 1991, Pimm,1987).

Zazkis (1998) extended the idea of polysemy, relating it to the use of terms *within* the mathematical register (rather than between mathematical and everyday registers). Her example focused on the term 'divisor', that can mean a role of the number in a number sentence (in $12\div4 = ?$ 4 is the divisor and 12 is the dividend ; in $25\div4=?$, 4 is the divisor and 25 is the dividend) or a number-theoretic relationship (4 is a divisor of 12, 4 is not a divisor of 25). Mamolo (2010) further extended the idea of polysemy within the mathematics register to mathematical symbols. She discussed different but related meanings of the '+' symbol denoting a binary operation among elements in different sets.

Of our interest here is another symbol, $\square^{-1}$, its meaning, as determined by different mathematical contexts, and its homonymous or polysemous interpretations by the participants in our study. We posed the following research question: In the two appearances of superscript (-1), what similarities and what differences do teachers identify and focus students' attention?

Data analysis

In analyzing each script (Part 1 of the Task) we identified what is stressed and consequently what is ignored in considering the appearance of superscripts (-1) in the contexts of numbers and functions. In other words, we considered what similarities and what differences the participants identify in the two uses and how they chose to communicate these issues to their students. We confirmed our analysis of the scripts with participants declared pedagogical intentions outlined in their responses to Part 2 of the Task, and identified the intellectual needs of students that the script writers aimed to address. We further attended to Part 3 if it was included in the submission in an attempt to identify whether their choice of pedagogical approach differed from their personal mathematical understanding of the situation.

In the beginning of data analysis, following our mathematical analysis of the Task and some informal conversations with teachers, we considered two extremes:

- A group theory approach, where $\square^{-1}$ stands for the same notion of inverse in a group structure and 5^{-1} and f^{-1} are particular instantiation of inverses in this structure (pointing to different sets of elements and different operation).
- The common symbol $\square^{-1}$ is seen as a homonym, signaling different, context dependent, interpretations for numbers and for functions.

The attempts to classify each response to the Task in terms of association with either these approaches resulted in adding the third abundant possibility, in which similarities beyond the common symbol are sought and exposed through different means. We explained these approaches in terms of *polysemy*, that is, signaling to different but related interpretations of the symbol $\square^{-1}$.

Results and Analysis

In the data we identified the main focus of attention in each script. We first describe the approaches that resulted from considering $\square^{-1}$ as a homonymous symbol. We then turn to scripts in which $\square^{-1}$ was seen as pointing to related meanings, which we describe in terms of polysemy. No script adopted a group-theoretic approach, focusing on the mathematical meaning of "inverse element" with respect to an operation. Surprisingly, there were no noticeable differences between the approaches of students who majored in mathematics and those who majored in science. We exemplify participants' responses via excerpts from their scripts and the accompanying explanations

Focusing on homonymy

Fourteen (out of 22) participants interpreted the appearance of superscript (-1) as a "homonymous symbol", that is, the same symbol applied to different unrelated ideas, the meaning of which is determined by the context. For example, Alan (Excerpt 1) after attending to the terminology, focused on the "neighbor" of the (-1).

Excerpt 1 (Alan): "depending on what it's beside"

Teacher:	It's important to recognize that constants and variables are different from functions. A function takes in a constant or variable and outputs something new based on certain rules. It's like a recipe book. When a function such as $f(x)$ has a power -1 beside the f, it becomes the inverse function. If the power -1 is beside a constant or variable, it means reciprocal.
Student:	So even though it's the same little -1, <u>depending on what it's beside</u>, it can mean either a reciprocal or an inverse?
Teacher:	Exactly.

Alan commented:

"Yes, the two symbols are the same. They both look like exponents, but if you look to what the "exponent" is being applied, it will tell you the meaning of the -1. If the -1 is found above a variable or a constant, then it is understood as an exponent and means a reciprocal. When the -1 is found above a function, it is understood as an inverse function.

$f^{-1}(x)$- *inverse* $\quad$ 5^{-1} - *reciprocal*

$sin^{-1}(x)$ – *inverse* $\quad$ x^{-1} – *reciprocal*

It is important that students understand the difference between the cases"

Alan's explanation highlights the context in which the symbol appears in order to determine whether it refers to a reciprocal or to an inverse function. For a student, "what is beside" serves an indicator for determining the meaning intended by the mathematical context. In a similar way, highlighting the differences, another participant focused on the letters that determine the "neighbour":

To help students cope with the perceived problematics in dealing with a homonymous symbol, eight participants appealed to analogies of other context-dependent notions. The goal of the analogies appears to convince students that context-dependency is a common phenomenon, which is not unique the superscript (-1). In Rob's script a teacher draws an analogy between an inverse and the word 'set'.

Excerpt 2 (Rob): Analogy to 'set', "context changes its meaning"

Student:	So is this the same symbol, or what?
Teacher:	Yes, it's the same symbol, but it doesn't mean exactly the same thing.
Student:	That doesn't make sense.
Teacher:	Think of it this way: ($^{-1}$) means 'inverse' and your examples are different kinds of inverses. This symbol is used in both contexts, and the <u>context changes its meaning</u>.
Student:	That's so confusing.

Teacher:	I'm going to give you an example. Consider the word 'set.' I could say to you, "Dina, could you 'set' the table?" or I could say "Dina, did you see the sun 'set' last night?" Do you see the difference?
Student:	Yes…
Teacher:	The word 'set' is used in different contexts and those contexts show you which meaning I am using. It's the same with the symbol for 'inverse.'

In this excerpt the teacher acknowledged that in both cases the symbol (-1) points to an inverse. In considering the different meanings of this word, the script-writer builds an analogy with a homonymous word 'set', a word that has different meanings in different contexts. We mentioned above that a majority of participants considered $\square^{-1}$ as a homonymous symbol. Here we see a homonymous word-symbol pair. That is, not only does the symbol get its meaning from the context, but so does the word 'inverse'.
Rob commented:

> *"By relating the idea of mathematical language to language that the student is more comfortable with, I was able to show the importance of context and the flexibility of the notation we use.*

We note that rather than considering the definition of the mathematical term inverse (that is, a binary operation performed on an element and its inverse results in the identity), Rob considers the English language word in its different uses in a mathematical situation. In the next section we examine the linguistic connection further – pointing to the interpretations assigned to the word by considering its synonyms.

Focusing on Polysemy

Seven (out of 22) scripts focused students' attention on a common word, inverse, and the way it is interpreted. (This is in contrast with associating the symbol $\square^{-1}$ with two different words, inverse and reciprocal.) These participants focused on similar features within the two appearances of $\square^{-1}$. To reiterate, the property of a word to point to different but related meanings is referred to as polysemy. As exemplified in Cathy's response, the polysemy is seen in the implied action.

Excerpt 3 (Cathy): Inverse as 'switch'

Student:	So is this the same symbol, or what?
Teacher:	They are the same symbol. Now let's take a step back and investigate this. Dina, can you grab the dictionary at the side of the room for me please and look up the word 'inverse' in a non-mathematical setting
[…]	
Student:	Well, I read that inverse means opposite or reverse, so in a fraction would it mean that we are switching the top and bottom.
Teacher:	Yes, the inverse of a fraction is what we get when we switch the numerator and denominator. Now let's get back to what we are learning about today the inverse of functions. When we are talking functions what are two parts do you think of?
Student:	Left side and right side
Teacher:	Let's look at that what happens to the equation $y = 3x + 4$ if we switch the left side with the right side we get $3x + 4 = y$ what do you notice about these two equations?
Student:	They are exactly the same.
Teacher:	So if switching the left side and right side did not give us what we want, any other suggestions?
Student2:	What about switching the letters x, y?
Teacher:	That is correct; an inverse of the function $y = 3x + 4$ would be $x = 3y + 4$ now let's look at this in more detail.

While the dictionary meaning is assigned to 'inverse' as a noun or an adjective, it is interpreted in case of a fraction as an action of switching between the numerator and denominator. Then the question becomes, what can be "switched" in a function. The first student suggestion – that is considered and consequently rejected – is switching between the left and right sides of an equation. The next suggestion – which is accepted and later examined in further detail is to switch x and y.

Cathy wrote:

> *I choose this method in order to clarify this question to the students because there are a number of terms that are used in multiple ways in math. If we are able to discover what the term actually means then the students may be able to see why the same term is used for both the things identified. I think looking at the definition of inverse and looking at the parts of fractions and equations then the students will hopefully be able to see why the word inverse is used for both."*

Another action, that of "undoing", is featured in the approach presented by Joe, Excerpt 4. In this approach an inverse is associated with an operation that cancels the previous operation and "returns to the starting point". The approach is illustrated in the following excerpt.

Excerpt 4 (Joe): "Get back to the starting point"

Teacher:	Let's say I pushed the wrong button on a calculator, and instead of multiplying by 3, I multiplied by 5. Can anyone give me suggestions to what should do next? Should I hit clear and start a long calculation from the beginning?
Student:	No! You should just divide by 5.
Teacher:	Yes Dina, good instincts. Would dividing by 5 return me to where I was just before I made the mistake?
[The class confirms, with yes and nods.]	
Teacher:	But what is another way of writing dividing by 5? Yes, Dina.
Student:	Putting 5 under 1, so $\frac{1}{5}$?
Teacher:	Exactly! So Dina, what do you think it means when I say that the inverse of 5 is $\frac{1}{5}$?
Student:	That if I multiplied by five and I want to <u>get back to the starting point</u>, I would multiply by $\frac{1}{5}$ because their multiplication cancels the effects of each other.
[… the dialogue turns to an inverse of a function]	
Teacher:	So what do you think it means when I say to find the inverse of a function called f?
Student:	It means you're trying to find another function related to f, so that it would undo what f did. Return to the starting point.
Teacher:	Good! Let's try an example. […]

Joe elaborated on his chosen approach in the following way:

> *"I chose to explain the relations because I felt that trying to convince a student that the power (-1) means two different things was harder than explaining the real reason (that it simply means an inverse). Even though ideas like this might be complex, I think that students should understand that they aren't nonsensical. The people who chose what symbols to use did so for specific reasons. I personally kept the idea of different rules applying to numbers and functions when dealing with inverses. It was not until university Mathematics did I reconcile the two ideas in one overarching idea of an inverse."*

Joe acknowledged that he "kept the idea of different rules" when first introduced to the symbolic notation for a function inverse. However, his study of advanced mathematics helped him adopt a different perspective. Joe's reference to the "overarching idea of an inverse" points to a strong connection he sees in the two contexts of using the symbol.

Summary and Discussion

We described each pedagogical approach in the scripts in terms of the chosen emphasis on the similarities and/or differences in the interpretations of $\square^{-1}$. We attempted to place each approach within the two extremes: common structure of inverses related to different sets and operations, or homonymous symbol referring to unrelated meanings in different contexts.

Participants who considered $\square^{-1}$ as a homonymous symbol amplified the differences by focusing on different context (fractions or functions), different terminology (reciprocal or inverse), and different procedures. Several participants appealed to students familiar experiences and linked different meanings of $\square^{-1}$ to other homonymous words and symbols. Though no participant attempted to connect the script to the mathematical meaning of the term 'inverse', expressing a group-theoretic perspective, several approaches identified polysemy, that is related meanings in the two interpretations of $\square^{-1}$. This was seen by focusing on a common word (inverse) and on a common action implied by the word (swap, undo).

On contextualized knowledge

As mentioned above, the Scripting Task was administered to the participants as part of their work in one of the final courses in their teacher education program. Following the completion of the Task different explanations for the "curious case of superscript (-1)" were discussed in class. Most participants readily recalled or accepted the connection via group theory, referring to the concept of inverse of an element with respect to a particular operation. However, only four participants mentioned the connection in the mathematical meaning of "inverse" in their responses to Part 3 of the Task. (Most participants have not addressed this part or noted that their explanation to students reflected their personal understanding.)

This discrepancy can be explained from the perspective of situated cognition (e.g., Greeno, 1998): Teachers' knowledge is situated within the mathematics classroom and mathematics curricula. In school mathematics there is attention to procedures related to finding an inverse function or multiplicative inverses of fractions. Therefore, the majority of participants have chosen to focus on procedural knowledge expected from students, situating their scripts relative to school curricula, and possibly mimicking how they were taught in school.

Same name to different things

Henri Poincaré is often quoted in saying that "mathematics is the art of giving the same name to different things". He further commented that "It is enough that these things, though differing in matter, should be similar in form, to permit of their being, so to speak, run in the same mold. When language has been well chosen, one is astonished to find that all demonstrations made for a known object apply immediately to many new objects: nothing requires to be changed, not even the terms, since the names have become the same." (Poincaré, 1908)

In English, the multiplicative inverse is usually denoted as 'reciprocal'. Echoing Poincaré, we recognize here "different names for the same thing", which obscures the speakers of English ability to recognize the connection in their different views of $\square^{-1}$. In several languages, where reciprocal for a fraction and inverse for a function are denoted by similar or close words the connection is easier detected.

We agree with Zaslavsky (2009) that "identification of similarities and differences between objects along several dimensions [...] is fundamental mathematical thinking". The Scripting Task discussed in this paper helps highlight similarities and differences in terminology, in structure and in procedures. It provides an impetus for strengthening teachers' personal mathematical knowledge by connecting the ideas of advanced mathematics to a classroom situation.

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